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BEST MEAN APPROXIMATION BY SPLINES
SATISFYING GENERALIZED CONVEXITY
CONSTRAINTS

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BEST MEAN APPROXIMATION BY SPLINES SATISFYING
GENERALIZED CONVEXITY CONSTRAINTS

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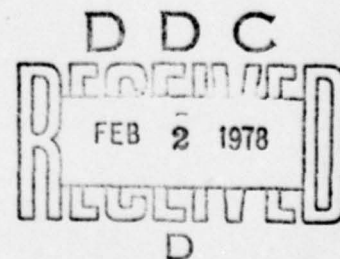
ABSTRACT

A characterization of the best L_1 -approximation to a continuous function by classes of fixed-knot polynomial splines which satisfy generalized convexity constraints is presented and uniqueness is shown. Included is the possibility of specifying the positivity, monotonicity, or convexity of the class. The proof of uniqueness uses recently developed results for Hermite-Birkhoff interpolation by splines.

AMS(MOS) Subject Classifications: 41-A15, 41-A50.

Key Words: Monotone splines, Convex splines, Hermite-Birkhoff spline interpolation, Best L_1 -approximation.

Work Unit Number 6 - Spline Functions and Approximation Theory.



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SIGNIFICANCE AND EXPLANATION

In practical applications of approximation theory, it often occurs that the function being approximated is known to have some additional properties such as positivity, monotonicity, and/or convexity. A best approximation may not preserve any of these, even though such properties may be very important for the application. In this case one should search only among approximations which satisfy the desired side conditions of positivity, monotonicity, or convexity. These are special cases of generalized convexity constraints.

This paper begins the study of best approximation in the mean or L_1 -norm by fixed-knot polynomial spline functions which satisfy generalized convexity constraints. As one particular example, it is shown that the best L_1 -approximation to any continuous function by nonnegative, nonincreasing cubic splines with fixed knots exists and is unique.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

BEST MEAN APPROXIMATION BY SPLINES SATISFYING
GENERALIZED CONVEXITY CONSTRAINTS

Dennis D. Pence

Introduction

The concept of monotone approximation by polynomials was introduced by O. Shisha [12]. It has been studied by many authors including R. A. Lorentz [4], who demonstrated uniqueness for best approximation by monotone polynomials in the uniform and L_1 -norm. J. A. Roulier and G. D. Taylor [9] have generalized the concept to include more arbitrary restrictions on the ranges of derivatives. If the restrictions are all of nonnegativity or nonpositivity, they are called generalized convexity constraints.

Classes of splines satisfying generalized convexity constraints were introduced and studied in the author's thesis [6]. Best uniform approximations were characterized and partial uniqueness was established. This paper continues the study of such splines by considering the L_1 -norm. A characterization for best L_1 -approximation by these classes of splines is given and uniqueness is demonstrated. Previous papers considering best L_1 -approximation by splines include the work of M. P. Carroll and D. Braess [2], R. V. Galkin [3], and A. Pinkus [8].

We use results for Hermite-Birkhoff interpolation by splines which were developed by the author [6], [7]. For completeness the required results are given in the following section.

1. Hermite-Birkhoff Interpolation by Splines

Suppose $-\infty < a \equiv \xi_0 < \xi_1 < \dots < \xi_q < \xi_{q+1} \equiv b < \infty$ and integers R_v with $0 < R_v \leq m$, $v = 1, \dots, q$, are given. Let $S_p^m = S_p^m(\{\xi_v\}_1^q; \{R_v\}_1^q)$ denote the space of polynomial spline functions of order m with fixed knots $\{\xi_v\}_1^q$, each with multiplicity R_v , respectively, where $p = \sum_{v=1}^q R_v$. Thus $g \in S_p^m$ is piecewise a polynomial of degree at most $m - 1$ with $g^{(j)}$ discontinuous only at a knot ξ_v where $j \geq m - R_v$. We adopt the convention that all elements of S_p^m and all derivatives of elements of S_p^m are

defined everywhere by assuming continuity from the right. Notice that $\dim S_p^m = m + p$.

We review needed facts about Hermite-Birkhoff interpolation (HBI) by polynomial splines.

Let interpolation points

$$(1.1) \quad X = \{a \leq x_1 < x_2 < \dots < x_k \leq b\}$$

be given. A matrix

$$(1.2) \quad E = \{e_{ij}\}, i = 1, \dots, k; j = 0, 1, \dots, m-1$$

is called a spline incidence matrix for X and S_p^m provided $e_{ij} = 0, \pm 1$, or 2 and $e_{ij} = -1$ or 2 only if $x_i = \xi_v$ for some v and $j \geq m - R_v$. The HBI problem defined by (E, X, S_p^m) is:

Given any values $\{\gamma_{ij} : e_{ij} = 1 \text{ or } 2\}$ and

$\{\gamma_{ij}^- : e_{ij} = -1 \text{ or } 2\}$, find $g \in S_p^m$ with

$$(1.3) \quad g^{(j)}(x_i) = \gamma_{ij} \text{ whenever } e_{ij} = 1 \text{ or } 2,$$

$$(1.4) \quad g^{(j)}(x_i^-) = \gamma_{ij}^- \text{ whenever } e_{ij} = -1 \text{ or } 2.$$

As in [6], [7], when we display such a matrix E , we indicate the relationship between the interpolation points X and the knots of the spline space S_p^m by drawing the following lines:

- (i) If $x_i < \xi_v < x_{i+1}$, we draw a solid line between the i -th and $(i+1)$ -th rows extending from the $(m - R_v)$ -th column to the $(m-1)$ -th column. If more than one knot lies between x_i and x_{i+1} , then draw several lines.
- (ii) If $x_i = \xi_v$, we enclose in a box the entries in the i -th row from the $(m - R_v)$ -th column to the $(m-1)$ -th column.

Thus an entry of E may be -1 or 2 only if it is boxed.

Define

$$(1.5) \quad \|E\| = \sum_{i,j} |e_{ij}|.$$

We say (E, X, S_p^m) is full when $\|E\| = \dim S_p^m = m + p$. If (E, X, S_p^m) has a unique solution for any given data values or, equivalently, if the only solution to the homogeneous problem is the zero spline, the problem is called poised. Obviously (E, X, S_p^m) must be full for this to happen. When $\|E\| \leq m + p$, we say (E, X, S_p^m) is quasi-poised if the dimension of the solution space for the homogeneous problem is exactly $m + p - \|E\|$.

We now define what are essentially submatrices of E . For $n = 0, 1, \dots, m-1$ and $0 \leq \ell < s \leq q+1$, let $k_1 = \min\{i : \xi_\ell \leq x_i\}$, $k_2 = \max\{i : x_i \leq \xi_s\}$, and

$$(1.6) \quad E(n : \ell, s) = \{e_{ij}^*, i = k_1, \dots, k_2; j = n, \dots, m-1\}$$

where

$$(1.7) \quad e_{ij}^* = \begin{cases} 1, & \text{if } i = k_1, x_i = \xi_\ell, \text{ and } e_{ij} = 1 \text{ or } 2, \\ e_{ij}, & \text{if } x_i \in (\xi_\ell, \xi_s) \text{ or if } i = k_2, x_i = \xi_s, \text{ and } j < m - R_s, \\ 1, & \text{if } i = k_2, x_i = \xi_s, \text{ and } e_{ij} = -1 \text{ or } 2, \\ 0, & \text{otherwise.} \end{cases}$$

By a simple dimension argument, it is easy to see that the following called the local Polya conditions (LPC) for (E, X, S_p^m) are necessary for quasi-poisedness:

$$(1.8) \quad \|E(n : \ell, s)\| \leq m - n + p(n : \ell, s),$$

for all $n = 0, 1, \dots, m-1; 0 \leq \ell < s \leq q+1$

where

$$(1.9) \quad p(n : \ell, s) = \begin{cases} \sum_{v=\ell+1}^{s-1} \min[R_v, m-n], & \text{if } \ell+1 < s, \\ 0, & \text{if } \ell+1 = s. \end{cases}$$

It is also easily verified that all of the (LPC) are satisfied if we have that

$$(1.10) \quad \|E(n : \ell, s)\| \leq m - n + p(n : \ell, s),$$

for all $\{(n, \ell, s) : R_v < m - n, \text{ when } \ell < v < s\}$.

In particular when (E, X, S_p^m) is full, the (LPC) imply that

$$(1.11) \quad \|E\| - \|E(0 : \ell, q+1)\| \geq \sum_{v=1}^{\ell} R_v, \quad \ell = 1, \dots, q.$$

LEMMA 1.1. If $\|E(0 : \ell, s)\| = m + p(0 : \ell, s)$ for some $0 < \ell < s \leq q+1$ or $0 \leq \ell < s < q+1$, then (E, X, S_p^m) can be split vertically into two or three HBI problems, each defined on a spline space of order still m but with fewer knots than S_p^m . The "central" one of the decomposed problems has incidence matrix $E(0 : \ell, s)$.

LEMMA 1.2. If $x_i = \xi_v$ for some i and v , and $e_{ij} = 1$ for all $j = 0, 1, \dots, m - R_v - 1$, or if $R_v = m$ for some v , then (E, X, S_p^m) can also be split vertically into two HBI problems considering fewer knots. $E(0 : 0, v)$ and $E(0 : v, q+1)$ will be the incidence matrices for these two smaller problems.

We further note that the above decompositions preserve the (LPC) and that if the original problem has a full matrix, then so do all of the smaller problems. Quasi-poisedness of (E, X, S_p^m) is equivalent to quasi-poisedness of all of the split problems. Similar decompositions have been noted by several authors, see [5], for example. The complete details are tedious but not hard and can be found in [6].

Let (E, X, S_p^m) indicate a given HBI problem. If $x_i \notin \{\xi_v\}_1^q$, then we say that we have a regular sequence beginning with e_{ij} of order μ when $e_{ij} = e_{i,j+1} = \dots = e_{i,j+\mu-1} = 1$ with $e_{i,j-1} = 0$ and $e_{i,j+\mu} = 0$ if either is defined. Also if $x_i = \xi_v$, then we say that we have a regular sequence beginning with e_{ij} of order μ when $e_{ij} = e_{i,j+1} = \dots = e_{i,j+\mu-1} = 1$ with $j + \mu \leq m - R_v$, $e_{i,j-1} = 0$ and $e_{i,j+\mu} = 0$ if either is defined. Further a regular sequence $e_{ij}, \dots, e_{i,j+\mu-1}$ is called strongly regular if $e_{i,j+\mu}$ is defined, zero, and, in the case where $x_i = \xi_v$, $j + \mu < m - R_v$. A sequence is even if it has even order and odd otherwise.

We say that a regular sequence $e_{ij}, \dots, e_{i,j+\mu-1}$ is supported provided there exist integers i_1, j_1, i_2, j_2 with $i_1 < i < i_2$, $e_{i_1, j_1} = 1$ or 2 ,

$$(1.12) \quad j_1 < \min\{j, \{m-R_v : x_{i_1} < \xi_v < x_{i_1}\}\} ,$$

$$(1.13) \quad j_2 < \min\{j, \{m-R_v : x_{i_1} < \xi_v < x_{i_2}\}\} , \text{ and}$$

$$(1.14) \quad e_{i_2, j_2} = \begin{cases} 1, & \text{if } x_{i_2} \notin \{\xi_v\}_1^q , \\ 1, & \text{if } x_{i_2} = \xi_v \text{ and } j_2 < m-R_v , \\ -1 \text{ or } 2, & \text{if } x_{i_2} = \xi_v \text{ and } j_2 \geq m-R_v . \end{cases}$$

The problem (E, X, S_p^m) is called weakly conservative (C) if every supported strongly regular sequence is even.

THEOREM 1.1. Suppose (E, X, S_p^m) satisfies the (LPC) and (C). Then it is quasi-poised.

This theorem generalizes the sufficiency theorem of Atkinson and Sharma for HBI by polynomials [1]. The proof can be found in [6], [7].

We shall need the following technical lemmas. All three lemmas concern attempts to add conditions of some sort to a given HBI problem.

LEMMA 1.3. Suppose (E, X, S_p^m) satisfies the (LPC) but when some strongly regular sequence in E is extended to have an additional one to the right giving the matrix $\tilde{E}$, then the (LPC) are violated. There exists $\xi \notin \{\xi_v\}_1^q \cup X$ so that when ξ is added as a simple knot to the spline space, then $(\tilde{E}, X, S_{p+1}^m(\{\xi_v\}_1^q, \xi; \{R_v\}_1^q, 1))$ satisfies the (LPC).

Proof. Suppose $e_{i, j-\mu} = \dots = e_{i, j-1} = 1$ is the strongly regular sequence of E and that e_{ij} is changed from a zero to a one to obtain $\tilde{E}$. Then $\exists \eta \leq j, 0 \leq \ell < s \leq q+1$ with $x_i \in [\xi_\ell, \xi_s]$ so that for S_p^m ,

$$(1.15) \quad \begin{aligned} \|\tilde{E}(\eta : \ell, s)\| &= \|E(\eta : \ell, s)\| + 1 \\ &= m - \eta + p(\eta : \ell, s) + 1 . \end{aligned}$$

Without loss of generality assume that (1.15) cannot happen first for any $\tilde{\eta} > \eta$ and secondly with η for any $\tilde{\ell}$ and $\tilde{s}$ with $[\xi_{\tilde{\ell}}, \xi_{\tilde{s}}] \subsetneq [\xi_\ell, \xi_s]$.

Let

$$(1.16) \quad \epsilon = \min \{ (x_i - x_{i-1}), (x_{i+1} - x_i), \{ |x_i - \xi_v| : \xi_v \neq x_i \} \}.$$

We choose $\xi \in (x_i - \epsilon, x_i + \epsilon) \setminus \{x_i\}$ in such a way that when ξ is added to the knot set, i.e. $\{\xi'_v\}_1^{q+1} = \{\xi_v\}_1^q \cup \{\xi\}$ properly ordered, $\xi = \xi'_\ell$, $R'_\ell = 1$, and $\ell < \tilde{\ell} < s+1$, we have

$$(1.17) \quad \|\tilde{E}(\eta : \ell, \tilde{\ell})\| < \|\tilde{E}(\eta : \ell, s+1)\| \quad \text{and}$$

$$(1.18) \quad \|\tilde{E}(\eta : \tilde{\ell}, s+1)\| < \|\tilde{E}(\eta : \ell, s+1)\|,$$

with respect to $S_{p+1}^m(\{\xi'_v\}_1^{q+1}; \{R'_v\}_1^{q+1})$.

It is easily seen that this can be done. The proof is completed by checking the various ways $\tilde{E}$ might violate the (LPC) with respect to the new spline space. If this happens, then (E, X, S_p^m) must violate the (LPC), contrary to hypothesis.

A condition corresponding to $e_{i,0} = 1$ is called a Lagrange condition and we say that we are adding a Lagrange condition at t to an HBI problem (E, X, S_p^m) if a zero in the $j = 0$ column is changed to a one, possibly by adding a new row to E if $t \notin X$.

LEMMA 1.4. Suppose (E, X, S_p^m) satisfies the (LPC) but is not full. Then there exists a point $t \notin X \cup \{\xi_v\}_1^q$ so that a Lagrange condition can be added to (E, X, S_p^m) without violating the (LPC).

Proof. If $\|E(0 : \ell, s)\| = m + p(0 : \ell, s)$ for some $0 < \ell < s \leq q+1$ or $0 \leq \ell < s < q+1$, then we can decompose according to Lemma 1.1 and consider one of the split problems which is not full. Thus without loss of generality we assume this never happens. But then any Lagrange condition can be added without violating the (LPC).

LEMMA 1.5. Assume that (E, X, S_p^m) satisfies the (LPC) but is not full. Without loss of generality, assume that $\{(\xi_v)_1^q \cup \{b\}\} \subset X$, possibly by having some rows with all zero entries in E . Then we can "fill" E in such a way that the (LPC) remain valid by

- (i) changing some "boxed" zeros to ones,
- (ii) changing some "boxed" ones to twos, and/or
- (iii) changing some zeros to ones in the last row corresponding to b .

Proof. Inductively for $\ell = 1, 2, \dots, q$ we make changes of type (i) or (ii) for "boxed" entries corresponding to the interpolation point and knot ξ_ℓ so that (1.11) will be valid for that integer after the changes are made. Further we make changes one at a time for entries with j -index as large as possible without violating the (LPC). To show that this is always possible, suppose we have done this for $\ell = 1, 2, \dots, \ell_* - 1$ (if any) and have

$$(1.19) \quad \|E\| - \|E(0 : \ell, q+1)\| \geq \sum_{v=1}^{\ell} R_v, \quad \ell = 1, 2, \dots, \ell_* - 1.$$

Suppose $x_{i_*} = \xi_{\ell_*}$. If

$$(1.20) \quad \|E\| - \|E(0 : \ell_*, q+1)\| \geq \sum_{v=1}^{\ell_*} R_v,$$

then there is no need to make any changes in the i_* -th row. If $e_{i_*, j} = -1$ or 2 for all $j = m - R_{\ell_*}, \dots, m - 1$, then (1.20) holds. Suppose

$$(1.21) \quad e_{i_*, j_*} = 0 \text{ or } 1 \text{ where } m - R_{\ell_*} \leq j_*,$$

$$(1.22) \quad e_{i_*, j} = -1 \text{ or } 2 \text{ for all } j = (m - R_{\ell_*}), \dots, j_* - 1 \text{ (if any)},$$

but e_{i_*, j_*} cannot be changed to -1 or 2 , respectively, without violating the (LPC). By

(1.10), there exist integers $\tilde{n}$, $\tilde{\ell}$, and $\tilde{s}$ with $\tilde{\ell} < \ell_* \leq \tilde{s}$, $\tilde{n} \leq j_*$, $R_v < m - \tilde{n}$ for all $\tilde{\ell} < v < \tilde{s}$ (if any), and

$$(1.23) \quad \|E(\tilde{n} : \tilde{\ell}, \tilde{s})\| = m - \tilde{n} + p(\tilde{n} : \tilde{\ell}, \tilde{s}).$$

Then

$$(1.24) \quad \begin{aligned} \|E\| - \|E(0 : \ell_*, q+1)\| &\geq (\|E\| - \|E(0 : \ell, q+1)\|) + \|E(\tilde{n} : \tilde{\ell}, \tilde{s})\| \\ &- \left\{ \begin{array}{l} \|E(\tilde{n} : \ell_*, \tilde{s})\|, \text{ if } \ell_* < \tilde{s}, \\ \max[0, m - R_{\ell_*} - \tilde{n}], \text{ if } \ell_* = \tilde{s} \end{array} \right\} + \max[0, \tilde{n} - m + R_{\ell_*}] \end{aligned}$$

$$\geq \sum_{v=1}^{l_*} R_v .$$

Again (1.20) holds. Thus we can always accomplish the induction step.

If the matrix is still not full after all of these changes, then we make changes of type (iii) at entries with j -index as large as possible without violating the (LPC). We argue in a similar manner that if it is not possible to change some such entry, then it is unnecessary to do so.

Example 1.1.

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{0} \\ 0 & 0 & \boxed{0} & \boxed{2} & \boxed{1} \\ 0 & 0 & \boxed{0} & \boxed{0} & \boxed{0} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

becomes via the procedure given for Lemma 1.5

$$\hat{E} = \begin{pmatrix} 0 & 0 & 0 & 0 & \boxed{-1} \\ 0 & 0 & \boxed{-1} & \boxed{2} & \boxed{2} \\ 0 & 0 & \boxed{-1} & \boxed{0} & \boxed{0} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The display E is quasipoised and $\hat{E}$ is poised by Theorem 1.1.

2. Best L_1 -approximation by Splines with Generalized Convexity Constraints

Let integers $0 \leq k_0 < k_1 < \dots < k_w \leq m-1$ and $\varepsilon_v = \pm 1$, $v = 0, 1, \dots, w$ be given. Suppose $S_p^m \subset C[a, b]$, i.e. $R_v < m$, $v = 1, \dots, q$ and $m > 1$.

Define

$$(2.1) \quad G = \{ \psi \in S_p^m : \varepsilon_v \psi^{(k_v)}(t) \geq 0, \\ a \leq t \leq b, v = 0, 1, \dots, w \} .$$

Recall we assume right continuity of all spline derivatives. Also for every $g \in G$ we have

$$(2.2) \quad \varepsilon_v g^{(k_v)}(\xi_v^-) \geq 0 \text{ if } k_v \geq m - R_v .$$

For an integrable function h , let $\|h\|_1 = \int_a^b |h(t)| dt$. Suppose f is in $C[a,b]$, the space of continuous functions defined on $[a,b]$, but is not in G . Then a best L_1 -approximation to f from G is a spline $g_* \in G$ such that

$$(2.3) \quad \|g_* - f\|_1 = \inf_{g \in G} \|g - f\|_1.$$

Denote by $P_G(f)$ the collection of all such best approximations. $P_G(f) \neq \emptyset$ because G is closed, convex, and finite-dimensional. We have the following characterization theorem.

THEOREM 2.1. Assume $f \in C[a,b]$ and G is defined as in (2.1). Then there exist

- (i) functions $\varphi_1, \dots, \varphi_r$, $r \geq 1$ where $|\varphi_i(t)| = 1$ for almost every $t \in [a,b]$, $i = 1, \dots, r$,
- (ii) an HBI problem (E, X, S_p^m) , $\|E\| + r \leq m + p + 1$ where $e_{ij} \neq 0$ only if $j = k_v$ for some v ,
- (iii) positive scalars $\lambda_1, \dots, \lambda_r$, and
- (iv) scalars $\{\mu_{ij} : e_{ij} = 1 \text{ or } 2\}$, $\{\mu_{ij}^- : e_{ij} = -1 \text{ or } 2\}$ where $\text{sgn } \mu_{ij} = -\text{sgn } e_v$ whenever $j = k_v$

with

$$(2.4) \quad \sum_{i=1}^r \lambda_i \int_a^b \varphi_i(t) \psi(t) dt + \sum \{\mu_{ij} \psi^{(j)}(x_i) : e_{ij} = 1 \text{ or } 2\} \\ + \sum \{\mu_{ij}^- \psi^{(j)}(x_i^-) : e_{ij} = -1 \text{ or } 2\} = 0,$$

for all $\psi \in S_p^m$,

such that $g \in G$ is in $P_G(f)$ if and only if

$$(2.5) \quad \int_a^b \varphi_i(t) [g(t) - f(t)] dt = \|g - f\|_1, \quad i = 1, \dots, r,$$

$$(2.6) \quad g^{(j)}(x_i) = 0, \quad \text{whenever } e_{ij} = 1 \text{ or } 2, \text{ and}$$

$$(2.7) \quad g^{(j)}(x_i^-) = 0, \quad \text{whenever } e_{ij} = -1 \text{ or } 2.$$

Proof. This theorem is a specific case of a theorem of Rozema and Smith [10, Theorem 4.1]

once we note that it is easy to find a polynomial $\psi \in S_P^m$ satisfying

$$(2.8) \quad \varepsilon_v \psi^{(k_v)}(t) > 0; \quad a \leq t \leq b; \quad v = 0, 1, \dots, w.$$

LEMMA 2.1. If $f \in C[a, b]$, g_1 and g_2 are elements of $P_G(f)$, and $g_0 = \frac{1}{2}(g_1 + g_2)$, then $g_1 - g_2$ vanishes at the zeros of $g_0 - f$.

This lemma can be found in [2] and is a special case of [10, Lemma 6.1].

THEOREM 2.2. For every $f \in C[a, b]$, there is a unique best L_1 -approximation from G defined as in (2.1).

Proof. Let $\varphi_1, \dots, \varphi_r, \lambda_1, \dots, \lambda_r, (E, X, S_P^m), \{u_{ij}\}$, and $\{u_{ij}^-\}$ be as guaranteed by Theorem 2.1. Without loss of generality we may assume (E, X, S_P^m) is quasi-poised since any dependency in these conditions could be used to accomplish (2.4), (2.6), and (2.7) with a smaller HBI problem made up of independent conditions.

Suppose g_1 and g_2 are both in $P_G(f)$. Then $g_0 = \frac{1}{2}(g_1 + g_2)$ is also in $P_G(f)$ since it is convex. Applying (2.5) to g_0 , we see that almost everywhere that $g_0(t) \neq f(t)$ we have

$$(2.9) \quad \varphi_i(t) = \text{sgn}[g_0(t) - f(t)], \quad i = 1, \dots, r.$$

Let ϕ be the right continuous function defined by

$$(2.10) \quad \phi(t) = \lim_{\varepsilon \rightarrow 0} \text{sgn}[g_0(t+\varepsilon) - f(t+\varepsilon)].$$

Define

$$(2.11) \quad T = \text{closure} \{t \in [a, b] : \phi(t) = 0 \text{ or } \phi \text{ changes sign at } t\}.$$

Thus $g_0(t) = f(t)$, for every $t \in T$ (and possibly at other isolated points where $g_0 - f$ doesn't change sign).

Suppose $g \in P_G(f)$. If $e_{ij} = 1$, then $j = k_v$ for some v and $g^{(j)}(x_i) = 0$. By the definition of G in (2.1), $g^{(j)}$ does not change sign. Using the zero counting procedure for splines devised by Schumaker [11] (see also [6], [7]), this means that x_i is in some interval (possibly degenerate) where $g^{(j)}$ is identically zero and that this

interval is either an even zero for $g^{(j)}$ or it contains one of $\{a\}$ or $\{b\}$. We conclude that if $a < x_i < b$, $j < m-1$, $e_{ij} = 1$, and $e_{i,j+1}$ is not "boxed", then

$$(2.12) \quad g^{(j+1)}(x_i) = 0.$$

One by one, we add enough of the zero conditions designated in (2.12) to the HBI problem (E, X, S_P^m) to assure that it has no odd strongly regular sequences in rows for which $a < x_i < b$ and, if the sequence begins with $e_{i,0}$, for which $x_i \notin T$. We add conditions from (2.12) to (E, X, S_P^m) , if necessary, to assure that it has no even strongly regular sequences beginning with $e_{i,0}$ if $x_i \in T \cap (a,b)$. If it is necessary at each step in order to preserve the (LPC), a simple knot is added to the spline space as described in Lemma 1.3. Then we define scalars $\tilde{u}_{i,j}$, possibly zero and having no particular sign, so that (2.4) remains valid, even for the enlarged spline space. The fact that such constants $\tilde{u}_{i,j}$ can always be chosen follows from elementary linear algebra and Theorem 1.1. Thus we obtain $(\tilde{E}, X, S_P^m)$, $\{\tilde{u}_{ij}\}$, and $\{\tilde{u}_{ij}^-\}$ satisfying the (LPC) of the desired form with

$$(2.13) \quad \sum_{i=1}^r \lambda_i \int_a^b \varphi_i(t) \psi(t) dt + \sum \{\tilde{u}_{ij} \psi^{(j)}(x_i) : \tilde{e}_{ij} = 1 \text{ or } 2\} \\ + \sum \{\tilde{u}_{ij}^- \psi^{(j)}(x_i^-) : \tilde{e}_{ij} = -1 \text{ or } 2\} = 0,$$

$$\text{for all } \psi \in S_P^m$$

and for any $g \in P_G(f) \subset S_P^m \subset S_P^m$, (in particular for g_1 and g_2),

$$(2.14) \quad g^{(j)}(x_i) = 0, \text{ whenever } \tilde{e}_{ij} = 1 \text{ or } 2, \text{ and}$$

$$(2.15) \quad g^{(j)}(x_i^-) = 0, \text{ whenever } \tilde{e}_{ij} = -1 \text{ or } 2.$$

Choose a maximal subset $t_1, \dots, t_\alpha$ from $T \setminus \{x_i : e_{i,0} = 1\}$ so that when Lagrange conditions at $t_1, \dots, t_\alpha$ are added to $(\tilde{E}, X, S_P^m)$ to obtain (E_1, X_1, S_P^m) , the (LPC) are still satisfied. Note that (E_1, X_1, S_P^m) satisfies (C) so by Theorem 1.1 it is quasi-poised.

Claim 1:

$$\|E_1\| = m + \tilde{p}.$$

If not, then by Lemma 1.4 there exists a point $t_* \notin X_1 \cup \{\xi_v\}_1^q$ so that a Lagrange condition at t_* can be added to get (E_2, X_2, S_P^m) . $t_* \notin T$ or else it would have been added in before using the maximality of the chosen $t_1, \dots, t_q$.

We obtain a full $(\bar{E}, \bar{X}, S_P^m)$ by applying Lemma 1.5 to (E_2, X_2, S_P^m) , if necessary. The only Lagrange condition that might possibly be added in this step is one at b . Note that this full HBI problem will still satisfy (C) as well as the (LPC) so by Theorem 1.1 it is poised. There exists a unique $\psi_* \in S_P^m$ with

$$(2.17) \quad \begin{cases} \psi_*(t_*) = \phi(t_*) \neq 0, \\ \psi_*^{(j)}(\bar{x}_i) = 0, \text{ whenever } \bar{e}_{ij} = 1 \text{ or } 2 \text{ and } \bar{x}_i \neq t_*, \\ \psi_*^{(j)}(\bar{x}_{i-}) = 0, \text{ whenever } \bar{e}_{ij} = -1 \text{ or } 2. \end{cases}$$

First $\psi_*(t) = 0$ for all $t \in T$. If $t \in T \setminus \{t_1, \dots, t_q\}$, then $t \in [\xi_\ell, \xi_s]$ where equality occurred in the (LPC) indexed by $(0, \ell, s)$ for (E_1, X_1, S_P^m) , hence in $(\bar{E}, \bar{X}, S_P^m)$ and $t_* \notin [\xi_\ell, \xi_s]$. $(\bar{E}, \bar{X}, S_P^m)$ can be decomposed according to Lemma 1.1 and the split problem involving t will have all zero data values from (2.17). Thus ψ_* is identically zero on $[\xi_\ell, \xi_s]$.

If $\|\bar{E}(0 : \ell, s)\| = m + \tilde{p}(0 : \ell, s)$ for some $0 \leq \ell < s \leq q+1$ where $t_* \notin [\xi_\ell, \xi_s]$, then the problem decomposes according to Lemma 1.1 again. Examining the part of (2.17) that each split problem must satisfy, we conclude that $\psi_*(t) = 0$ for all $t \in [a, \xi_s]$ if $\xi_s < t_*$ or $\psi_*(t) = 0$ for all $t \in [\xi_\ell, b]$ if $t_* < \xi_\ell$.

If $\bar{x}_i = \xi_v \neq t_*$ and $\bar{e}_{ij} = 1$ for all $j = 0, 1, \dots, m - R_v - 1$, then $(\bar{E}, \bar{X}, S_P^m)$ can be decomposed according to Lemma 1.2. As above the split problem not involving t_* will be homogeneous so ψ_* is identically zero either to the right or the left of x_i .

Suppose $\psi_*(t) = 0$ for all $t \in [\xi_{\ell_0}, \xi_{s_0}]$ for some $0 \leq \ell_0 < s_0 \leq q+1$, hence $t_* \notin [\xi_{\ell_0}, \xi_{s_0}]$. Let $(E^*, \bar{X}, S_P^m)$ denote $(\bar{E}, \bar{X}, S_P^m)$ with the Lagrange condition at t_* .

deleted. Let t_0 be any point in $(\xi_{\ell_0}, \xi_{s_0}) \setminus \bar{X}$. A Lagrange condition at t_0 cannot be added to $(E^*, \bar{X}, S_P^m)$ without violating the (LPC). If it could be, it would give a full poised HBI problem for which the nontrivial ψ_* satisfies all zero data values which is impossible. Thus there exist integers $0 \leq \ell < s \leq q+1$ with $\xi_\ell \leq t_0 \leq \xi_s$, $t_* \notin [\xi_\ell, \xi_s]$, and $\|E^*(0 : \ell, s)\| = m + \tilde{p}(0 : \ell, s)$, hence $\|\bar{E}(0 : \ell, s)\| = m + \tilde{p}(0 : \ell, s)$. As before either $\psi_*(t) = 0$ for all $t \in [a, \xi_{s_0}]$ if $\xi_{s_0} < t_*$ or $\psi_*(t) = 0$ for all $t \in [\xi_{\ell_0}, b]$ if $t_* < \xi_{\ell_0}$.

We conclude that ψ_* is identically zero except on some knot interval $[\xi_{\ell_*}, \xi_{s_*}]$ containing t_* (possibly $[a, b]$). On this interval there are only a finite number of points from T and only a finite number of points where ψ_* is zero. Any sequence beginning with $e_{i,0}$ for which $\xi_{\ell_*} < x_i < \xi_{s_*}$ and $x_i \neq t_*$ is strongly regular. Further

$$(2.18) \quad \|E^*(0 : \ell, s)\| < m + \tilde{p}(0 : \ell, s) \quad \text{if } 0 \leq \ell < s \leq q+1$$

$$\text{and } (\xi_\ell, \xi_s) \cap (\xi_{\ell_*}, \xi_{s_*}) \neq \emptyset.$$

Claim 2: $\text{sgn } \psi_*(t) = \phi(t)$, for a.e. $t \in [\xi_{\ell_*}, \xi_{s_*}]$.

Suppose ψ_* changes sign at some $t_0 \in (\xi_{\ell_*}, \xi_{s_*}) \setminus [T \cup \{t_*\}]$ where ϕ does not change sign. If $t_0 = \bar{x}_i$ for some i and $\bar{e}_{i,0} = 1$, then $\bar{e}_{i,0}$ begins a strongly regular even sequence $\bar{e}_{i,0}, \dots, \bar{e}_{i,u-1}$ and since t_0 must be an odd zero for ψ_* , $\psi_*^{(u)}(t_0) = 0$. Define (E_0, X_0, S_P^m) to be $(E^*, \bar{X}, S_P^m)$ with $\bar{e}_{i,u}$ changed from zero to one. Otherwise add a Lagrange condition at t_0 to $(E^*, \bar{X}, S_P^m)$ to obtain (E_0, X_0, S_P^m) . Either way (E_0, X_0, S_P^m) is full and by (2.18) must satisfy the (LPC). By construction (C) is satisfied so by Theorem 1.1 this is a poised problem. This is impossible since the nontrivial ψ_* satisfies the homogeneous problem. Similarly there cannot exist a point $t_0 \in (\xi_{\ell_*}, \xi_{s_*}) \cap T$ where ϕ changes sign but ψ_* does not. Thus Claim 2 is established.

Then

$$(2.19) \quad \int_a^b \varphi_i(t) \psi(t) dt = \int_a^b |\psi_*(t)| dt > 0, \quad i = 1, \dots, r.$$

This together with (2.17) gives a spline ψ_* which contradicts (2.13). Thus Claim 1 is established.

Using Lemma 2.1, (2.14), and (2.15) we have that $g_1 - g_2$ satisfies the homogeneous poised problem problem (E_1, X_1, S_P^m) and $g_1 - g_2 \in S_P^m \subset S_P^m$. Thus $g_1 = g_2$. Since g_1 and g_2 were arbitrary elements of $P_G(f)$, the proof of the theorem is complete.

3. Monotonicity and Convexity

Definition (2.1) for G in the previous section is a natural generalization to splines of the notion of monotone polynomials introduced by O. Shisha [12] (see also [4]). Included is the possibility for requiring nonnegativity or nonpositivity by choosing $k_0 = 0$. Since we made the assumption that all of the elements of our spline space S_P^m were continuous, choosing some $k_v = 1$ requires the usual monotonicities, either nondecreasing or non-increasing.

If $R_v < m - 1$, $v = 1, \dots, q$, then some $k_v = 2$ implies that all of the elements of G are either convex or concave. However it is reasonable to ask for convexity or concavity even if some of the knots have multiplicity $m - 1$. Convexity is well defined (although not in terms of the second derivative) for linear splines, i.e. continuous piecewise linear functions, for example. Similarly monotonicity is well defined for discontinuous splines.

We briefly indicate how the preceding section would need to be modified to include the requirement of convexity when $R_v = m - 1$ for some of $v = 1, \dots, q$. We ask that

$$(3.1) \quad \psi^{(2)}(t) \geq 0, \quad a \leq t \leq b, \quad \text{and}$$

$$(3.2) \quad \psi^{(1)}(\xi_v+) - \psi^{(1)}(\xi_v-) \geq 0 \quad \text{whenever} \quad R_v = m - 1.$$

The conditions in (3.2) are also linear constraints on S_P^m . It is not difficult to show that there exists a spline in S_P^m which satisfies all of the constraints including these strictly so that the theorem of Rozema and Smith [10, Theorem 4.1] applies.

If none of the constraints of type (3.2) are chosen by the theorem of Rozema and Smith, then we proceed exactly as in the previous section. On the other hand if one of these "jump" constraints is active and is chosen, that implies that for $g \in P_G(f)$

$$(3.3) \quad g^{(1)}(\xi_{\nu+}) = g^{(1)}(\xi_{\nu-}) ,$$

i.e. the knot ξ_{ν} is really only of multiplicity $m - 2$ for all splines in $P_G(f)$. It is easy to show that $P_G(f) \subset P_{G \cap S_{p'}^m}(f)$ where $S_{p'}^m$ is S_p^m with the knots chosen in (3.3)

having multiplicity only $m - 2$ so that $p' < p$. In fact the above inclusion is an equality and $P_{G \cap S_{p'}^m}(f)$ can be characterized using the arguments of the previous section. In parti-

cular we can still conclude that uniqueness holds.

4. Further Extensions

With only minor modifications the work of this paper can be extended to the problem of finding a best global L_1 -approximation to a compact (in $L_1[a,b]$) set of continuous function F from G as defined in (2.1). Such best global approximations are also called restricted Chebyshev centers for F with respect to G or best approximations to the elements of F simultaneously. The methods of the paper by Rozema and Smith [10] apply in a straight-forward manner.

The techniques we have used can also be applied to the more general problem of best L_1 -approximation by splines with restricted ranges of their derivatives. In the uniform norm, the corresponding polynomial problem was introduced by Roulier and Taylor [9] and the spline problem was studied in [6]. We wish to point out the significant differences between uniform and L_1 -approximation by these restricted splines.

Examining the proofs in [9] and [6], only the functions which bound the ranges of the derivatives (other than the zero derivative) need to be assumed to be differentiable in order to guarantee uniqueness in the uniform norm. In L_1 , the functions which bound the range need to be differentiable as well in order to carry out the analog of Claim 2 in the proof of Theorem 2.1. The problem of one-sided L_1 -approximation of a differentiable function by splines which was studied by A. Pinkus [8] is a special case where the given function is also the range bound.

If there are bounds on the range, then for uniqueness in the uniform norm to be assured, the assumption is needed that the given function satisfies these range bounds at

least within some $\epsilon > 0$, where ϵ is strictly less than the distance from the given function to the set of restricted splines. If this is not the case, then a single linear functional (a point evaluation) in $C^*[a,b]$ may be a positive error-extremal and a negative constraint-extremal or vice versa in the terminology of [6]. No such assumption is needed in L_1 although the assumption that the given function is continuous is needed.

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