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THE LANDAU PROBLEM FOR MOTIONS ON CURVES.(U)

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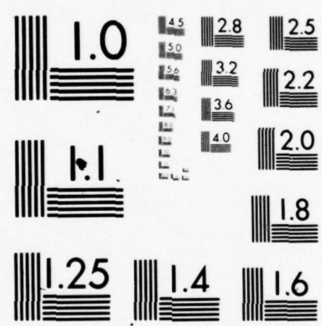

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THE LANDAU PROBLEM FOR MOTIONS ON CURVES

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ABSTRACT

Let Γ be a curve in the complex plane (Γ is not an infinite straight line), which is rectifiable with arc-length s, having a continuously turning tangent, as function of s, and having a radius of curvature $R = R(s)$ (finite or infinite) that is piecewise continuous, and satisfying Dirichlet's condition on every finite subarc. Let $f(t)$ $(-\infty < t < \infty)$ denote a "motion" on Γ, i.e. $f(t) \in \Gamma$ for all t, having a continuous velocity vector $\dot{f}(t)$, and a bounded and piecewise continuous acceleration vector $\ddot{f}(t)$.

Let $A > 0$ be given and let $(\Gamma)_A$ denote the totality of motions on Γ such that

(1) $$|\ddot{f}(t)| \leqq A \quad \text{for all } t .$$

It is shown (Theorem 2) that there is in $(\Gamma)_A$ a unique motion $\tilde{f}(t)$, called <u>the Landau motion for the constant</u> A, having the following three properties:

(i) $|\ddot{\tilde{f}}(t)| = A$ for all t,

(ii) Whenever $\tilde{f}(t)$ is at, or returns to, a point P of Γ, then it has at P the same speed $|\dot{\tilde{f}}|$, which we denote by $|\dot{\tilde{f}}_P|$.

(iii) If $f(t) \in (\Gamma)_A$, and t_0 is such that $f(t_0) = P$, then
$$|\dot{f}(t_0)| \leqq |\dot{\tilde{f}}_P| .$$

In Part II the existence and uniqueness of $\tilde{f}(t)$ is demonstrated. In Part III $\tilde{f}(t)$ is explicitly determined for a few special curves Γ. For the case when Γ is the segment $\{-a \leqq x \leqq a\}$, the motion $\tilde{f}(t)$ was determined by Landau.

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SIGNIFICANCE AND EXPLANATION

If a particle moves with bounded acceleration on a bounded section of a straight line, then its velocity is bounded. As a specific example, if the particle moves along the segment of the n-axis given by $|x| \leq a$, with $|\ddot{x}| \leq b$, then $|\dot{x}| \leq (2ab)^{1/2}$, and this velocity is attainable. The choice of acceleration and starting position and velocity required to achieve this velocity is a simple problem in control theory.

The present paper generalizes the above problem to the case where a curve is given in a plane, and a maximum allowable value is given for the acceleration of a particle moving on this curve. It turns out that then the magnitude of the velocity of the particle is bounded by a certain function of position on the curve.

More precisely, let Γ be a smooth curve in the plane, and let $f(t)$ $(-\infty < t < \infty)$ be a "motion" on Γ (i.e. $f(t)$ lies on Γ for all t) having the velocity vector $\dot{f}(t)$, and acceleration vector $\ddot{f}(t)$. Let $A > 0$ be given. It is shown that among all motions on Γ, such that

(1) $$|\ddot{f}(t)| \leqq A \quad \text{for all} \quad t ,$$

there is a uniquely defined motion $\tilde{f}(t)$, called the Landau motion on Γ, for the constant A, such that at every point P of Γ, the speed $|\dot{\tilde{f}}|$ is $\geqq$ the speed $|\dot{f}|$ at P, of any other motion f satisfying (1).

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE LANDAU PROBLEM FOR MOTIONS ON CURVES

I. J. Schoenberg

In memory of Edmund Landau on the Centenary
of his birth, February 14, 1977.

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Introduction

Let $f(t)$ be real-valued, $-\infty < t < \infty$. As we think of t as time, we denote its derivatives by $\dot{f}(t)$ and $\ddot{f}(t)$. We assume $f \in C^1(\mathbb{R})$, hence $\dot{f}(t) \in C(\mathbb{R})$, while $\ddot{f}(t)$ is piecewise continuous, with discontinuities only of the first kind, and <u>bounded</u>. Using the supremum norm on $\mathbb{R}$, we have Landau's inequality (see [3], [4])

$$\|\dot{f}\| \leqq \sqrt{2\cdot\|f\|\cdot\|\ddot{f}\|}\,, \tag{1}$$

where $\sqrt{2}$ is the best constant. An equivalent formulation is as follows:

If $a > 0$ and $f(t)$ satisfies

$$-a \leqq f(t) \leqq a \quad (t \in \mathbb{R}) \tag{2}$$

then

$$\|\dot{f}\| \leqq \sqrt{2a}\cdot\|\ddot{f}\|^{1/2} \tag{3}$$

and $\sqrt{2a}$ is here the best constant. If we interpret (2) as saying that $f(t)$ describes a motion on the segment

$$I_a = \{-a \leqq x \leqq a\}\,, \tag{4}$$

and define the functional

$$F(f) = \frac{\|\dot{f}\|}{\sqrt{\|\ddot{f}\|}}\,, \tag{5}$$

then

$$\sup_f F(f) = \sqrt{2a}\,, \tag{6}$$

the supremum being taken among all motions on I_a.

A rather natural generalization is as follows. Let S denote a closed and connected set in the complex plane $\mathbb{C}$, such that S <u>contains no infinite straight line</u>. Furthermore, let

$$(S) = \{f(t);\ f(t) \in S \text{ for all } t,\ f(t) \not\equiv \text{const.}\}\,. \tag{7}$$

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Now $f(t)$ is complex-valued and $\dot{f}(t)$, $\ddot{f}(t)$, are the velocity and acceleration vectors of the motion f. If, besides $f(t) \not\equiv$ const., we also assume that $|\ddot{f}(t)|$ is bounded, then $F(f)$ is seen to be well defined ($\leqq \infty$) for all $f \in (S)$. We now formulate

Landau's problem for S: To determine

$$L(S) = \sup_{f \in (S)} F(f) . \tag{8}$$

We call $L(S)$ the Landau constant of S.

Part I presents a few examples where $L(S)$ has been determined. The results of §2 were stated in [5] without proofs. The main results are found in Parts II and III. The problems here discussed belong to the subject of Optimal Control, but we use no results from its theory, as our special problems are elementary and can be attacked directly. For each curve Γ our Landau problem depends on what Ernesto Cesàro called the natural (or intrinsic) equation of Γ, giving its radius of curvature $R(s)$ as a function of the arc-length s. Many interesting and instructive examples of such natural equations are found in his book [2].

I wish to thank C. Vargas, of the MRC Computing Staff, for carrying out successfully the difficult numerical integrations of §11.

I. The Landau problem for motions in sets

1. **Examples**. The Landau constant $L(S)$ is known for a few simple choices of S.

(i) $S = I_a$. In terms of the definitions (8) and (4), we may rewrite Landau's result (6) as

$$L(I_a) = \sqrt{2a} . \tag{1.1}$$

An extremizing motion $\tilde{f}(t)$ such that

$$F(\tilde{f}) = \sqrt{2a} \tag{1.2}$$

is readily described. In terms of the quadratic Euler spline $E_2(t)$ defined by

$$E_2(t) = 1 - 4t^2, \text{ if } -\frac{1}{2} \leqq t \leqq \frac{1}{2} , \tag{1.3}$$

and extended to all t by

$$E_2(t+1) = -E_2(t), \quad (t \in \mathbb{R}) , \tag{1.4}$$

we find that

$$\tilde{f}(t) = aE_2(t) \tag{1.5}$$

satisfies (1.2). This because $\|\dot{\tilde{f}}\| = 4a$, $\|\ddot{\tilde{f}}\| = 8a$. For a discussion of the Euler splines and their role in Landau's problem and its generalizations see [4], where further references are found.

(ii) S <u>is the circular ring</u>

$$R_{a,b} = \{b \leqq |z| \leqq a;\ 0 < b < a\} . \tag{1.6}$$

In [5, Corollary 1 of §8] I determined $L(R_{1,r})$ $(0 < r < 1)$ in terms of an intermediate angular parameter u. By expressing cos u in terms of r, we easily find that

$$L(R_{1,r}) = \sqrt{1 + \sqrt{1 - r^2}} . \tag{1.7}$$

Writing $r = b/a$, we find that $f(t) \in (R_{a,b})$ if and only if $f_1(t) = a^{-1}f(t) \in (R_{1,r})$. As this implies that $F(f) = \sqrt{a}\, F(f_1)$, we find by (1.7) on taking suprema, that

$$L(R_{a,b}) = \sqrt{a}\, L(R_{1,b/a}) = \sqrt{a}\sqrt{1 + \sqrt{1 - (b/a)^2}}$$

and finally that

$$L(R_{a,b}) = \sqrt{a + \sqrt{a^2 - b^2}} . \tag{1.8}$$

(iii) S is the circle

(1.9) $$C_a = \{|z| = a\} .$$

The Landau constant $L(S)$ enjoys an obvious monotonicity property:

(1.10) If $S_1 \subset S_2$ then $L(S_1) \leqq L(S_2)$.

From $C_a \subset R_{a,b}$ we therefore conclude that

$$L(C_a) \leqq L(R_{a,b}) = \sqrt{a + \sqrt{a^2 - b^2}} \quad (0 < b < a) .$$

Letting $b \to a$ we obtain that

(1.11) $$L(C_a) \leqq \sqrt{a} .$$

On the other hand, for the uniform circular motion $f(t) = ae^{it}$ we find that $\|\dot{f}\| = a$, $\|\ddot{f}\| = a$ and therefore $F(f) = \sqrt{a}$. This implies that $L(C_a) \geqq \sqrt{a}$ and now (1.11) shows that

(1.12) $$L(C_a) = \sqrt{a} .$$

(iv) S is the circular disk

(1.13) $$D_a = \{|z| \leqq a\} .$$

From (1.10) we obtain that

$$L(D_a) \geqq L(R_{a,b}) = \sqrt{a + \sqrt{a^2 - b^2}}$$

and letting $b \to 0$ we obtain that $L(D_a) \geqq \sqrt{2a}$. From Theorem 1 below, concerning convex sets S, to be established in §2, we conclude that here we have equality, hence

(1.14) $$L(D_a) = \sqrt{2a} .$$

2. A lemma and the case when S is convex.

Lemma 1. If the motion

(2.1) $$\tilde{f}(t) \in (S)$$

has the

Property A. If

(2.2) $$f \in (S) \quad \text{and} \quad \|\ddot{f}\| \leqq \|\ddot{\tilde{f}}\|$$

then

(2.3) $$\|\dot{f}\| \leqq \|\dot{\tilde{f}}\| ,$$

then it also has the

Property B.

(2.4) $$F(\tilde{f}) = L(S) ,$$

and conversely, if $\tilde{f}$ enjoys the Property B, then it also has Property A.

Proof: 1. Property A implies Property B. We assume (2.1) and Property A and we are to establish (2.4).

If $f \in (S)$ then for the motion $g(t) = f(t\sqrt{\|\ddot{\tilde{f}}\| / \|\ddot{f}\|})$ we find that $g(t) \in (S)$ and also

$$\|\ddot{g}\| = \|\ddot{f}\|(\|\ddot{\tilde{f}}\|/\|\ddot{f}\|) \quad \text{hence} \quad \|\ddot{g}\| = \|\ddot{\tilde{f}}\| .$$

By Property A we conclude that $\|\dot{g}\| \leqq \|\dot{\tilde{f}}\|$, and by the definition of $g(t)$, that

$$\|\dot{f}\| \sqrt{\|\ddot{\tilde{f}}\|/\|\ddot{f}\|} \leqq \|\dot{\tilde{f}}\| , \quad \text{and so} \quad F(f) \leqq F(\tilde{f}) .$$

But then

$$L(S) = \sup_{f \in S} F(f) \leqq F(\tilde{f}) \quad \text{and therefore (2.4) holds.}$$

2. Property B implies Property A. Now we assume (2.4) and that f satisfies (2.2), and we are to show that

(2.5) $$\|\dot{f}\| \leqq \|\dot{\tilde{f}}\| .$$

From (2.4) we obtain $\|\dot{f}\|/\sqrt{\|\ddot{f}\|} \leqq \|\dot{\tilde{f}}\|/\sqrt{\|\ddot{\tilde{f}}\|}$ whence

$$\|\dot{f}\| \leqq \sqrt{\|\ddot{f}\|/\|\ddot{\tilde{f}}\|} \cdot \|\dot{\tilde{f}}\| \leqq \|\dot{\tilde{f}}\|$$

because of the second relation (2.2). Thus (2.5) is established.

As an application we establish

Theorem 1. If S is a closed and bounded convex set, then

(2.6) $$L(S) = \sqrt{\text{diameter of } S} .$$

Proof: Let A and B be points of S such that

(2.7) $$|A - B| = \text{diameter of } S = d .$$

On the segment $[A,B]$ we consider the motion

(2.8) $$\tilde{f}(t) = \frac{1}{2}(B + A) + \frac{1}{2}(B - A)E_2(t) ,$$

where $E_2(t)$ is defined by (1.3) and (1.4). From the convexity of S it is clear that $\tilde{f} \in (S)$. For this to-and-fro motion on $[A,B]$ we find that

(2.9) $$\|\dot{\tilde{f}}\| = \frac{1}{2}|B - A|\cdot 4 = 2d, \quad \|\ddot{\tilde{f}}\| = \frac{1}{2}|B - A|\cdot 8 = 4d$$

and therefore

(2.10) $$F(\tilde{f}) = \sqrt{d} = L([A,B]) .$$

<u>We are to show that also</u>

(2.11) $$L(S) = \sqrt{d} .$$

Let $f \in (S)$ be such that

(2.12) $$\|\ddot{f}\| \leq \|\ddot{\tilde{f}}\| = 4d$$

and let us show that these assumptions imply that

(2.13) $$\|\dot{f}\| \leq \|\dot{\tilde{f}}\| = 2d .$$

To derive this, we consider an arbitrary but fixed t_0 such that $\dot{f}(t_0) \neq 0$. Let L be a fixed straight line such that

(2.14) L <u>is parallel to the vector</u> $\dot{f}(t_0)$.

Let the segment $[P,Q]$ be the orthogonal projection of S onto the line L (the reader is asked to draw a diagram). Clearly

(2.15) $$|Q - P| \leq |A - B| = d .$$

Let $f_p(t)$ denote the orthogonal projection of $f(t)$ on L. Clearly $f_p(t) \in ([P,Q])$, because $f_p(t) \in [P,Q]$ for all t. Since vectors are only shortened by projecting them, we have by (2.12) that

(2.16) $$\|\ddot{f}_p\| \leq 4d .$$

From Landau's theorem we know that

$$\|\dot{f}_p\|/\sqrt{\|\ddot{f}_p\|} \leq \sqrt{|Q - P|} \leq \sqrt{d}, \quad \text{or} \quad \|\dot{f}_p\| \leq \sqrt{d}\sqrt{\|\ddot{f}_p\|} ,$$

and, by (2.16), this shows that

(2.17) $$\|\dot{f}_p\| \leq \sqrt{d}\cdot\sqrt{4d} = 2d .$$

However, the assumption (2.14) insures that on the one hand

$$|\dot{f}_p(t_0)| = |\dot{f}(t_0)| ,$$

while on the other, by (2.17), we have

$$|\dot{f}_p(t_0)| \leq 2d .$$

It clearly follows that $|\dot{f}(t_0)| \leq 2d$. Since t_0 was arbitrary we conclude that

$$\|\dot{f}\| \leq 2d = \|\tilde{f}\| .$$

This establishes (2.13) and the motion $\tilde{f}(t)$ is seen to have the Property A of Lemma 1. By Lemma 1 we conclude that f has Property B which states that

$$L(S) = F(\tilde{f}) = \sqrt{d} .$$

Therefore (2.11) is established.

Remark. Clearly Theorem 1, and its proof, remains valid if the convex set S belongs to a euclidean space of any finite number of dimensions. It does seem curious that the motion $f(t)$ on the diameter $[A,B]$ can not use the greater freedom offered by S to increase the value $\sqrt{d}$ of the Landau constant.

II. The Landau problem for motions on curves

3. Statement of main result. As indicated by its title, the chief purpose of the present article is to study the Landau problem for motions on a set S, when

(3.1) $$\text{the set } S \text{ reduces to a curve } \Gamma .$$

The curve Γ, of the complex plane, is assumed to be rectifiable, its arc-length being denoted by s. We also assume that Γ has a continuously varying tangent, as function of s, and that the radius of curvature $R = R(s)$, at the point s, is a piecewise continuous function of s, having at most discontinuities of the first kind, and that $R(s)$ satisfies Dirichlet's condition in every finite s-interval.

We distinguish four types of curves Γ.

Case 1. Γ is a biinfinite arc; a parabola is an example. As we assumed in our Introduction that the set S contains no infinite straight line, we exclude the case when Γ is a straight line. Measuring s from a suitable point 0 of Γ, we see that the range of values of s is

(3.2) $$-\infty < s < \infty .$$

Case 2. Γ is a half-infinite arc. If 0 is its endpoint, we think of Γ as being a biinfinite arc that is doubled-up on itself at the point 0. Hence (3.2) again holds, with $\pm s$ denoting the same point of Γ. Examples are a half-line, or one-half of a parabola.

Case 3. Γ is a closed curve of total length 2ℓ. A circle, or an ellipse, are examples. Again (3.2) holds, and s_1 and s_2 denote the same point of Γ if $s_1 \equiv s_2 (\text{mod } 2\ell)$.

Case 4. Γ is a finite arc of length ℓ. We think of this as a special case of Case 3, when the two parts $0 \leqq s \leqq \ell$ and $\ell \leqq s \leqq 2\ell$, of Γ, coincide geometrically. As s increases, the corresponding point is seen to describe Γ infinitely often in a to-and-fro motion.

In all four cases s ranges over all reals.

As in (7), we are concerned with the class of motions

(3.3) $$(\Gamma) = \{f(t);\ f(t) \in \Gamma \text{ for all } t,\ f(t) \not\equiv \text{const.}\} .$$

We select an arbitrary, but fixed constant

(3.4) $$A > 0 ,$$

and define the subclass of motions

(3.5) $$(\Gamma)_A = \{f(t);\ f \in (\Gamma),\ |\ddot{f}(t)| \leqq A \text{ for all real } t\} .$$

In words: $(\Gamma)_A$ is the class of motions on Γ having at all times accelerations whose moduli do not exceed the value A.

The following seems obvious: If $f(t) \in (\Gamma)$ and $f(t_0)$ coincides with an end-point of Γ (in Cases 2 and 4), then $\dot{f}(t_0) = 0$. This follows from $\dot{f}(t) \in C(\mathbb{R})$.

Our main result is as follows.

Theorem 2. There is a unique motion $\tilde{f}(t) = \tilde{f}_A(t)$ on Γ, called the Landau motion on Γ, corresponding to A, having the following three properties:

(i)

(3.6) $$|\ddot{\tilde{f}}(t)| = A \text{ for all } t .$$

(ii) Let $\tilde{f}(t)$ be at the point of arc-length

(3.7) $$s = \tilde{s}(t) .$$

This function of t is uniquely defined by $\tilde{f}(t)$, if we require that $\tilde{s}(t) \in C(\mathbb{R})$ and that

(3.8) $$\tilde{s}(0) = 0 .$$

(3.9) The function $\tilde{s}(t)$ increases steadily from $-\infty$ to $+\infty$ as t increases from $-\infty$ to $+\infty$.

Further properties of $\tilde{s}(t)$, hence of $\tilde{f}(t)$, are as follows.

In Case 1, by (3.9), $\tilde{f}(t)$ sweeps out the entire curve Γ just once.

In Case 2, again by (3.9), $\tilde{f}(t)$ sweeps out Γ twice. Moreover,

(3.10) $$\tilde{s}(-t) = -\tilde{s}(t), \text{ hence } \tilde{f}(-t) = \tilde{f}(t), \text{ for all } t .$$

In Case 3, let $2T$ be the least positive value such that

(3.11) $$\tilde{s}(2T) = 2\ell ,$$

then

(3.12) $$\tilde{s}(t + 2T) = \tilde{s}(t) + 2\ell, \text{ or } \tilde{f}(t + 2T) = \tilde{f}(t) \text{ for all } t .$$

In Case 4, let T be the least positive value such that

$$\tilde{s}(t) = \ell . \tag{3.13}$$

Then

$$\tilde{s}(-t) = -\tilde{s}(t) \quad \text{and} \quad \tilde{s}(t + 2T) = \tilde{s}(t) + 2\ell , \tag{3.14}$$

or equivalently

$$\tilde{f}(-t) = \tilde{f}(t) \quad \text{and} \quad \tilde{f}(t + 2T) = \tilde{f}(t) \quad \text{for all } t . \tag{3.15}$$

It follows (3.10), (3.12), and (3.15), by differentiation, that if P is a point of Γ, and if

$$\tilde{f}(t) = P , \tag{3.16}$$

then the corresponding speed $|\dot{\tilde{f}}(t)|$ has always the same value, which we denote by

$$|\dot{\tilde{f}}_P| , \tag{3.17}$$

whenever $f(t)$ returns to the point P.

(iii) This is the decisive characterizing property of $\tilde{f}$: If

$$f(t) \in (\Gamma)_A , \tag{3.18}$$

and t_0 is such that

$$f(t_0) = P , \tag{3.19}$$

then

$$|\dot{f}(t_0)| \leqq |\dot{\tilde{f}}_p| . \tag{3.20}$$

In words: The speed $|\tilde{f}_p|$ of the Landau motion at a point $P \in \Gamma$, can not be exceeded by the speed at P, of any motion $f(t)$, on Γ, such that $|\ddot{f}(t)| \leqq A$ for all t. We may also say that $\tilde{f}$ maximizes the speed at every $P \in \Gamma$, within the class of motions in $(\Gamma)_A$.

We have the following

Corollary 1. If $\tilde{f}(t)$ is the Landau motion on Γ, corresponding to A, then

$$L(\Gamma) = F(\tilde{f}) = \frac{\|\dot{\tilde{f}}\|}{\sqrt{A}} . \tag{3.21}$$

Proof: We claim that $\tilde{f}$ enjoys the Property A of Lemma 1. Indeed, let us assume that $f(t) \in (\Gamma)$ and that

$$\|\ddot{f}\| \leqq \|\ddot{\tilde{f}}\| . \tag{3.22}$$

From (3.6) we conclude that $\|\ddot{f}\| \leqq A$. Now (3.20) shows that

$$\|\dot{f}\| \leqq \|\dot{\tilde{f}}\| ,$$

and this is the desired Property A of $\tilde{f}$. By Lemma 1 we conclude that $\tilde{f}$ has Property B, and (3.21) is established.

Of course, in Cases 1 and 2, we may well have that $L(\Gamma) = \infty$.

Remark. The dependence of $\tilde{f}(t) = \tilde{f}_A(t)$ on the constant A is a trivial one. For if we select another constant $B > 0$, then we claim that the motion defined by

$$\tilde{f}_B(t) = \tilde{f}_A(t\sqrt{B/A}) \tag{3.23}$$

is the corresponding Landau motion. For, in the first place, (3.23) implies that

$$\|\ddot{\tilde{f}}_B\| = (B/A)\|\ddot{\tilde{f}}_A\| = B \quad \text{for all} \quad t .$$

As the same change of scale in t transforms $(\Gamma)_A$ into $(\Gamma)_B$, we find that $\tilde{f}_B$ also enjoys the characteristic point wise maximal property (3.20) within the class $(\Gamma)_B$.

4. The condition $|\ddot{f}(t)| \leqq A$ amounts to a differential inequality. Let $f(t) \in (\Gamma)$ and let s denote the arc-length corresponding to the point $f(t)$ of Γ. Evidently $s = s(t)$ is a function of t. Also the speed of $f(t)$,

$$v = \frac{ds}{dt} = \pm|\dot{f}(t)| , \tag{4.1}$$

is a function of t. Let us assume that

$$\ddot{f}(t) \text{ and } R(s) \text{ are continuous if } \alpha < t < \beta . \tag{4.2}$$

At the point $f = f(t) = (x,y)$ we draw a half line tangent to Γ in the direction of increasing s, and denote by φ its angle with the x-axis. Then the following classical relations hold:

$$\frac{d\varphi}{dt} = \frac{d\varphi}{ds} \cdot \frac{ds}{dt} = \frac{1}{R} v , \tag{4.3}$$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt} = v \cos \varphi , \tag{4.4}$$

$$\dot{y} = v \sin \varphi . \tag{4.5}$$

Differentiating again the last two relations, we obtain by (4.3), that

$$\begin{aligned} \ddot{x} &= \dot{v} \cos \varphi - \frac{v^2}{R} \sin \varphi , \\ \ddot{y} &= \dot{v} \sin \varphi + \frac{v^2}{R} \cos \varphi . \end{aligned} \tag{4.6}$$

These give the well-known components of the acceleration $\ddot{f}$ in the direction of the tangent and the normal to Γ at the point $f = (x,y)$. Now

$$|\ddot{f}(t)|^2 = \ddot{x}^2 + \ddot{y}^2 = \dot{v}^2 + \frac{v^4}{R^2}$$

and we immediately conclude the following: The inequality

$$|\ddot{f}(t)|^2 \leqq A^2 \tag{4.7}$$

holds, if and only if

$$\dot{v}^2 \leqq A^2 - \frac{v^4}{R^2} \tag{4.8}$$

is satisfied.

Since

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}, \tag{4.9}$$

we may also rewrite (4.8) as

$$v^2\left(\frac{dv}{ds}\right)^2 \leqq A^2 - \frac{v^4}{R^2}. \tag{4.10}$$

This we multiply by $16\,v^4$ to obtain

$$\left(4\,v^3\frac{dv}{ds}\right)^2 \leqq 16\,v^4\left(A^2 - \frac{v^4}{R^2}\right). \tag{4.11}$$

A glance at this shows that we should introduce the new variable

$$u = v^4 \tag{4.12}$$

to obtain the inequality

$$\left(\frac{du}{ds}\right)^2 \leqq 16\,u\left(A^2 - \frac{u}{R^2}\right). \tag{4.13}$$

Clearly, as shown, (4.10) implies (4.13), but the converse is true only if $v \neq 0$. We have therefore established

Lemma 2. We assume $R(s)$ and $\ddot{f}(t)$ to be continuous at the time t, and that

$$|v| = |\dot{f}(t)| > 0. \tag{4.14}$$

In terms of the new variable u, defined by (4.12), the inequality

(4.15) $$|\ddot{f}(t)| \leqq A$$

<u>is equivalent with the differential inequality</u>

(4.16) $$\left(\frac{du}{ds}\right)^2 \leqq 16\, u\left(A^2 - \frac{u}{R^2(s)}\right).$$

<u>Moreover, the equation</u>

(4.17) $$|\ddot{f}(t)| = A$$

<u>is equivalent with the differential equation</u>

(4.18) $$\left(\frac{du}{ds}\right)^2 = 16\, u\left(A^2 - \frac{u}{R^2(s)}\right).$$

This lemma has far reaching consequences, some of which are as follows.

1°. Observe that the right side of (4.16) is a quadratic polynomial in u, while (4.16) implies that this quadratic is non-negative. It follows that <u>if</u> (4.16) <u>holds, then the corresponding values of</u> s <u>and</u> $u = v^4$ <u>must satisfy the inequalities</u>

(4.19) $$0 \leqq u \leqq A^2R^2(s) .$$

2°. <u>If in the time interval</u> $\alpha < t < \beta$ <u>both</u> (4.14) <u>and</u> (4.15) <u>hold, then the corresponding function</u> $u = u(s)$ <u>must satisfy the inequalities</u>

(4.20) $$-4\sqrt{u\left(A^2 - \frac{u}{R^2(s)}\right)} \leqq \frac{du}{ds} \leqq 4\sqrt{u\left(A^2 - \frac{u}{R^2(s)}\right)} .$$

3°. <u>If for</u> $t \in (\alpha,\beta)$ <u>we have</u>

(4.21) $$|\ddot{f}(t)| = A, \quad |\dot{f}(t)| > 0 .$$

<u>then for such</u> t (4.16) <u>becomes, an equality, hence</u>

(4.22) $$\left(\frac{du}{ds}\right)^2 = 16\, u\left(A^2 - \frac{u}{R^2(s)}\right).$$

This differential equation is the key to a proof of Theorem 2.

5. <u>On the solutions of the differential equation</u> (4.22). To simplify notations, we write

(5.1) $$U(u,s) = 4\sqrt{u\left(A^2 - \frac{u}{R^2(s)}\right)} .$$

It is clear that a solution of (4.22) must satisfy one or the other of the differential equations

(5.2) $$\frac{du}{ds} = U(u,s) ,$$

or

(5.3) $$\frac{du}{ds} = -U(u,s) .$$

It is important that we get a clear idea of the families of curves which are solutions of these two differential equations. To fix the ideas, we assume in the present section that Γ is a closed curve of length 2ℓ, hence that we are in Case 3, and that Γ has everywhere a positive and continuous radius of curvature $R(s)$. Therefore $R(s)$ $(-\infty < s < \infty)$ is a positive and continuous function of period 2ℓ. We may as well assume that

(5.4) $$R(0) = \min_{s} R(s) .$$

In the (s,u)-plane we consider the curve

(5.5) $$\gamma : u = A^2(R(s))^2 \quad (-\infty < s < \infty)$$

and Fig. 1 represents a period $0 \leqq s \leqq 2\ell$ of this function. The direction fields of the equations (5.2) and (5.3) are easily described: For every fixed value of s, and every u such that $0 \leqq u \leqq A^2R^2(s)$, we have the two slopes $\pm U(u,s)$ *of opposite signs*. The angle between these line elements is $= 0$ if $u = 0$, or if $u = A^2R^2(s)$, while increasing to a maximal value halfway between these endpoints. This is so because $U^2(u,s)$ is a quadratic function of u.

The u-axis, $u = 0$, is a singular solution of both equations (5.2), (5.3).

Let

(5.6) V_+ <u>denote the class of solutions of</u> $\frac{du}{ds} = + U(u,s)$,

(5.7) V_- <u>denote the class of solutions of</u> $\frac{du}{ds} = - U(u,s)$.

An example of an element of V_+ is the arc P_+Q_+ of Fig. 1. Its left endpoint P_+ is on $u = 0$, while its right endpoint Q_+ is on the curve γ. The slopes of P_+Q_+ vanish at both endpoints and are positive in between. These we call <u>elements of</u> V_+ <u>of type</u> I <u>and denote their class by</u> V_+^I.

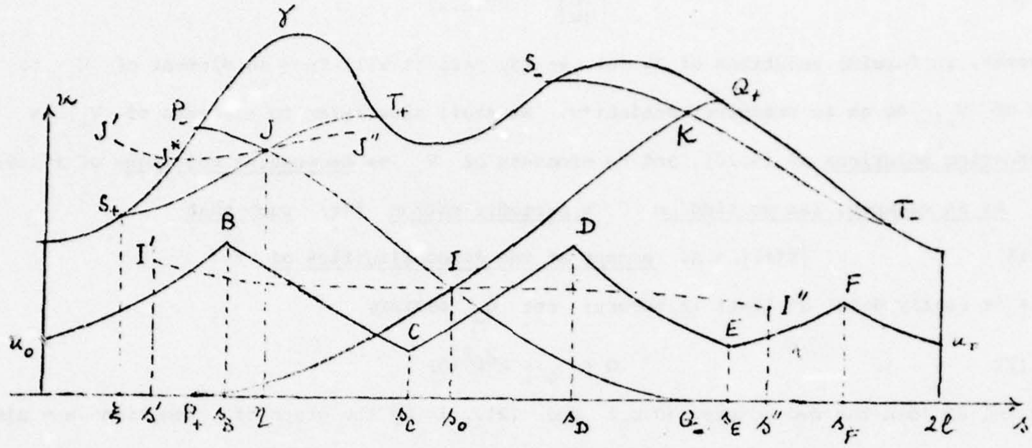

Fig. 1

There is yet another type of elements of V_+: An example is the arc S_+T_+ (Fig. 1) having its left endpoint S_+ on γ. We call them elements of V_+ of type II and denote their class by V_+^{II}.

There is a similar dichotomy within V_-: Its elements, like P_-Q_-, having its right endpoint on $u = 0$, are called of type I and their class denoted by V_-^I. Finally, elements S_-T_-, having their right endpoint on γ, are called of type II and their class denoted by V_-^{II}. We have just described the partitions

(5.8) $$V_+ = V_+^I \cup V_+^{II}, \quad V_- = V_-^I \cup V_-^{II} .$$

The elements of V_+ form a field of increasing curves that covers simply the region

(5.9) $$\Omega : 0 < u < A^2R^2(s), \quad -\infty < s < \infty ,$$

between γ and the u-axis. Also the elements of V_- form a field of decreasing curves covering Ω simply.

The reason is that within any compact subset of Ω, each of the equations (5.2), and (5.3), satisfy a Lipschitz condition $|U(u_1,s) - U(u_2,s)| < K|u_1 - u_2|$, with a constant K depending on the compact set.

Evidently, the elements of V_+, or of V_-, satisfy the equation

$$\left(\frac{du}{ds}\right)^2 = U^2(u,s) \ . \tag{5.10}$$

However, in forming solutions of (5.10), we may pass at will from an element of V_+ to one of V_-, so as to preserve continuity. We shall also refer to elements of V_+ as increasing solutions of (5.10), and to elements of V_- as decreasing solutions of (5.10).

As an example, let us find on Γ a periodic motion $f(t)$ such that

$$|\ddot{f}(t)| = A, \text{ except at the discontinuities of } \ddot{f} \ . \tag{5.11}$$

This is easily done, at least in theory: Let u_0 satisfy

$$0 < u_0 \leqq A^2R^2(0) \ , \tag{5.12}$$

and let us join the two points $(0,u_0)$ and $(2\ell,u_0)$ by the *graph of a function* $u = u(s)$ that will generate a motion $f(t)$ satisfying (5.11).

1. We start from $(0,u_0)$ an increasing solution $\widehat{u_0B}$, and from $(2\ell,u_0)$ a decreasing solution $\widehat{Fu_0}$.

2. At the point B we switch to a decreasing solution $\overline{BC}$. Again, at C we pass to the *increasing solution* $\overline{CD}$, to be followed by the decreasing $\overline{DE}$. Finally, let $\overline{EF}$ be an increasing solution that intersects $\widehat{Fu_0}$ at the point F. In this construction we should take care not to ever touch the line $u = 0$, so as to satisfy the basic assumption (4.14) ($u = 0$ would imply $v = 0$).

Let the curve $u_0BCDEFu_0$ represent the graph of one period of the periodic function

$$u = u(s), \quad (-\infty < s < \infty) \ . \tag{5.13}$$

This being a solution of (4.18) for all t, it follows by Lemma 2, that $|\ddot{f}(t)| = A$ holds for all continuity points of $\ddot{f}$. To determine the basic variable t we use the relations (4.1) and (4.12) to obtain that

$$t = \int_0^s \frac{ds}{(u(s))^{1/4}} \ , \tag{5.14}$$

which gives a $1 - 1$ map of the t-axis onto the s-axis. At the points $s = 0, s_B, s_C, s_D, s_E, s_F$ of the curve Γ (see Fig. 1) we have the jump discontinuities of the acceleration vector $\ddot{f}(t)$. Fig. 1 also shows nicely the character of the motion $f(t)$

within one revolution: $f(t)$ is accelerated on $[0,s_B]$, $[s_C,s_D]$ and $[s_E,s_F]$, and decelerated on $[s_B,s_C]$, $[s_D,s_E]$ and $[s_F,2\ell]$.

6. Where and what are the jumps of the acceleration $\ddot{f}(t)$? To answer this question we return to the relations (4.6). Let us denote by $\Delta\ddot{f}(t_0) = \ddot{f}(t_0+) - \ddot{f}(t_0-)$ the jump of $\ddot{f}$ at $t = t_0$, and use this notation for other functions as well. Since v and φ are continuous, and assuming $R(s)$ continuous, we find from (4.6) that

$$(6.1) \qquad \Delta\ddot{x} = (\Delta\dot{v}) \cos\varphi, \quad \Delta\ddot{y} = (\Delta\dot{v}) \sin\varphi .$$

From (4.9), and writing $v' = dv/ds$, $u' = du/ds$, we find that $\Delta\dot{v} = v\Delta v'$. Now $u = v^4$ shows that $u' = 4v^3v'$, and therefore $\Delta u' = 4v^3\Delta v'$. It follows that

$$\Delta\dot{v} = [v/(4v^3)]\Delta u' = \frac{1}{4v^2}\Delta u' = \frac{1}{4\sqrt{u}}\Delta u' .$$

From this and (6.1) we conclude that

$$(6.2) \qquad \Delta\ddot{f}(t_0) = e^{i\varphi}\frac{1}{4\sqrt{u(s_0)}}\Delta u'(s_0) .$$

This establishes

Lemma 3. If $f(t)$ is a motion on Γ such that

$$(6.3) \qquad |\ddot{f}(t)| = A \quad \text{at the continuity point of } \ddot{f} ,$$

and if $u = u(s)$ is the corresponding (s,u)-diagram, then the jumps of $\ddot{f}(t)$ correspond to the corners of the graph of $u(s)$, as long as

$$(6.4) \qquad u(s) > 0, \quad \text{or equivalently} \quad \dot{f}(t) \neq 0 .$$

Moreover, at a discontinuity $t = t_0$, the vector

$$(6.5) \qquad \Delta\ddot{f}(t_0) = \ddot{f}(t_0+) - \ddot{f}(t_0-) \qquad (\dot{f}(t_0) > 0)$$

is parallel to the tangent to Γ at the point $f(t_0)$.

7. Some global properties of the graph of $u = u(s)$ corresponding to a motion $f(t) \in (\Gamma)_A$. So far we have discussed motions $f(t)$ satisfying the condition (5.11); their corresponding curves $u = u(s)$ were solutions of the differential (5.10). However, as we shall see, Lemma 2 also gives complete information on the curves $u = u(s)$ corresponding to motions in $(\Gamma)_A$.

Let $f(t)$ satisfy

(7.1) $$|\ddot{f}(t)| \leqq A \quad \text{wherever } \ddot{f}(t) \text{ is continuous,}$$

and let a portion of the corresponding

(7.2) $$u = u(s) \quad (s' \leqq s \leqq s''), \; (u(s) > 0) \;,$$

be represented by the arc I' I I" of Fig. 1. The assumption that the arc I'I" does not intersect the u-axis, implies that $u(s)$ is continuous and $u'(s)$ piecewise continuous.

Let $s' < s_0 < s''$, $I = (s_0, u(s_0))$, and let P_+Q_+ and P_-Q_- be the elements of V_+ and V_-, respectively, that pass through the point I.

Lemma 4. The arc I'I" can not cross either of the two arcs P_+Q_+ and P_-Q_-, except, of course, at the point I.

This is a global corollary of Lemma 2. We know that (4.20) holds along I'I". Let us even assume that $-U(u,s) < du/ds < U(u,s)$ at the point I. If the arc I I" could dip down and cross the arc IQ_-, then, at the first point where it crosses IQ_- (or a neighboring decreasing solution) it would violate (4.20) of Lemma 2.

Let P_+ and Q_- also denote the s-coordinates of these points. An important consequence is that we may assume in Fig. 1 that

(7.3) $$s' \leqq P_+, \; Q_- \leqq s'' \;.$$

Indeed, assuming that $v = ds/dt = u^{1/4}$ is positive along I'I" (rather than negative), then we find, by Lemma 4, that s is a strictly increasing function of t in the range

(7.4) $$P_+ \leqq s \leqq Q_- \;.$$

But then surely u admits a single-valued representation $u = u(s)$ in the interval (7.4).

We come now to a crucial point of our discussion. Again we assume that (7.1) holds, but let now the graph of (7.2) be the arc J' J J" of Fig. 1. Now the elements of V_+ and V_- that pass through the point J are the arcs S_+T_+ and P_-Q_-, respectively. Observe that S_+T_+ is an element of V_+^{II}. We also assume that the arc S_+T_+ is not minimal in the sense that there are arcs $S_+'T_+'$ in V_+^{II} below it and arbitrarily close to it.

Lemma 5. Our assumption that

(7.5) $$S_+T_+ \in V_+^{II},$$

contradicts our basic assumption (7.1).

We know, by Lemma 4, that the arc $J'J$ can not cross either of the arcs S_+J and P_-J. Let ξ and η be the s-coordinates of the points S_+ and J, respectively. Since $s = s(t)$ must be strictly monotone in the range $\xi \leqq s \leqq \eta$ (for an appropriate t-interval) because $v = ds/dt \neq 0$, it follows that the arc $J'J$ must cross the curve γ at some point J^*, on the arc S_+P_-, with vanishing slope, because at J^* we have two equality signs in (4.20). Therefore the arc $J'J^*$ is forced to enter the region above the curve γ, where the inequality (7.1) is known to be reversed by (4.16). This contradiction proves the lemma.

A similar contradiction is reached if the arc $u = u(s)$ should pass through a point K of an arc S_-T_-, of V_-^{II}, where we assume, as above, that the arc S_-T_- is not minimal.

8. Construction of the Landau motion $\tilde{f}(t)$ and a proof of Theorem 2. In the proof of Lemma 5 we have assumed that the arcs S_+T_+ and S_-T_- (Fig. 1) were not minimal. Such arcs of $V_+^{II} \cup V_-^{II}$, which are minimal, will now solve our problem.

Again we argue in the case of Fig. 1, when Γ is a closed curve of positive and continuous radius of curvature $R(s)$ exhibiting no intervals of constancy. In Fig. 2 we draw again the graph of one period of the curve

(8.1) $$\gamma : u = A^2R^2(s) \qquad (-\infty < s < \infty),$$

where $R(0) = \min R(s)$, and $R(s) > R(0)$ if s is positive and close to 0.

1. We start from the point $S_+ = (0, A^2R^2(0))$ and draw the arc $S_+T_+ \in V_+^{II}$, which necessarily terminates at a point T_+ of a descending branch of γ, with zero slope at that point. We proceed from T_+ along this descending branch of γ until we reach the next minimum point S_+', where we construct the arc $S_+'T_+' \in V_2^{II}$ terminating at T_+' with zero slope, again on a descending branch of γ. We continue in like manner obtaining the finite sequence of arcs

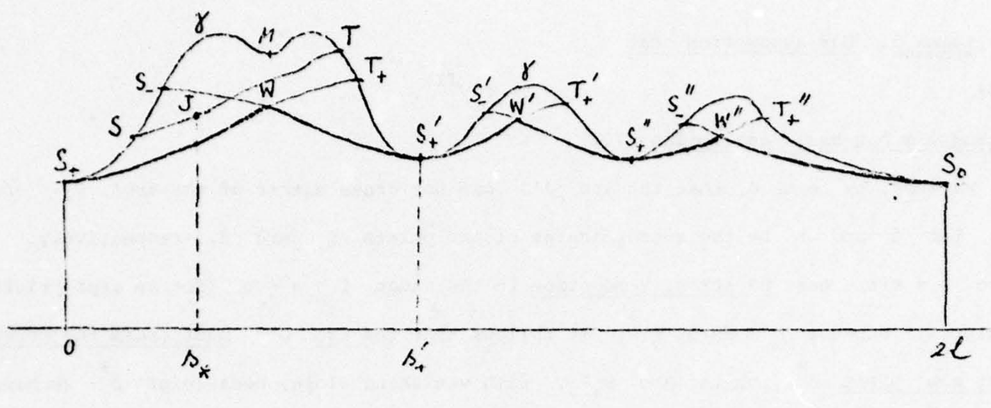

Fig. 2

(8.2) $$S_+T_+,\ S'_+T'_+,\ S''_+T''_+ \ \text{in}\ V_+^{II},$$

such that the first minimum point on γ, beyond T''_+, is the point S_0 at the end-point $s = 2\ell$ of the period.

2. Now we draw the arcs

(8.3) $$S_-S'_+,\ S'_-S''_+,\ S''_-S_0 \ \text{in}\ V_-^{II}$$

which are determined by the minimum points S'_+, S''_+, s_0 of γ. The choice of the successive minima $S'_+, S''_+, \ldots,$ insures that each of the arcs (8.3) intersects some arc among the arcs (8.2) (usually the corresponding one in (8.2), as in Fig. 2, but it could also be an earlier one) at the points W, W', W".

3. Of these arcs we now select that portion which has least ordinate, for each value of s, obtaining the (heavily drawn) connected curve

(8.4) $$S_+W\ S'_+W'S''_+W''S_0$$

of Fig. 2. Let this curve be the graph of one period of the periodic function

(8.5) $$u = \tilde{u}(s) \qquad (-\infty < s < \infty)\ .$$

Furthermore, let

(8.6) $$t = \int_0^s \frac{ds}{\tilde{u}(s)^{1/4}} \qquad (-\infty < s < \infty)\ ,$$

like (5.14), define the dependence of s on t. Finally, we denote by

(8.7) $$\tilde{f}(t)$$

the point of Γ corresponding to the arc-length s.

I claim that the function (8.7) is the Landau motion on Γ, corresponding to the constant A, and having the properties stated in Theorem 2 for the Case 4 of a closed curve Γ.

Proof of Theorem 2: The property (3.6) holds for $\tilde{u}(s)$ for the same reason that it held for the function (5.13) defined by the curve $u_0BCDEFu_0$ of Fig. 1: Both are defined by alternate arcs of V_+ and V_- which do not touch the u-axis. The properties (3.14) to (3.17) are evident by construction.

There remains to establish the crucial extremum property (iii). Assuming (3.18), let

(8.8) $$u = u(s)$$

be the (s,u)-diagram corresponding to f(t) and let us show that

(8.9) $$u(s) \leqq \tilde{u}(s) \quad \text{for all} \quad s \; .$$

This inequality is equivalent to (3.20), if we recall that we may write

$$u(s) = |\dot{f}(t_0)|^4, \; \tilde{u}(s) = |\dot{\tilde{f}}_p|^4 \; .$$

To establish (8.9), we assume that

(8.10) $$u(s_*) > \tilde{u}(s_*) \quad \text{for some} \quad s_* \; ,$$

and let us reach a contradiction. Let

(8.11) $$J = (s_*, u(s_*)) \; .$$

We loose no generality in assuming that $0 < s_* < 2\ell$, and even that the point J is above the arc S_+T_+ of Fig. 2, and of course below the curve γ. This means that there is an arc $ST \in V_+^{II}$, passing through J, which is not minimal. We now reach the contradiction with (7.1), as stated in Lemma 5. A similar contradiction is reached if J should be above any other arc of the curve (8.4). This establishes the inequality (8.9), and therefore (3.20) holds.

Remarks. 1. To better understand the way the Landau motion $\tilde{f}$ is put together, let us consider its arc

(8.12) $$WS_+'W' \quad \text{of Fig. 2} \; ,$$

corresponding to a point $s = s'_+$ of maximal curvature. Observe

1°. That the speed $\tilde{u}(s'_+)^{1/4}$ at $s = s'_+$ is the greatest speed that a motion in $(\Gamma)_A$ may reach at that point.

2°. That (8.12) is a part of the motion $S_- S'_+ T'_+$, which at S_- and T'_+ starts having accelerations that exceed in modulus the constant A. However, before reaching the point T'_+, the next arc $W'S''_+W''$ takes over.

3°. That along $W S'_+$ the motion $\tilde{f}$ is decelerated because $\tilde{u}(s)$ decreases, and accelerated along S'_+W': Minimal speeds occur only at some of the points of Γ of maximal curvature.

4°. That $\ddot{\tilde{f}}(t)$ is continuous between W and W', even at the point S'_+, according to the relation (6.2). Also by (6.2) we observe that the jumps of $\ddot{\tilde{f}}$ are at the points W, W', and W''.

It therefore appears that $\tilde{f}$ is composed of a succession of smooth motions corresponding to some (see 6° below) of the minima of $R(s)$, each having the maximal speed at such a minimum, and still going around that minimum point and yet satisfying $|\ddot{\tilde{f}}(t)| \leqq A$.

5°. Let τ be the unit vector tangent to Γ at s, and pointing in the direction of increasing s. For the inner product $(\ddot{f},\tau)$ we find, by (4.6), the value

$$\ddot{x}\cos\varphi + \ddot{y}\sin\varphi = \dot{v} = |\ddot{f}|\cos\theta, \quad \theta = \angle(\ddot{f},\tau) .$$

This shows that $\theta > 90°$ between W and S'_+ because $\dot{v} < 0$, and $\theta < 90°$ between S'_+ and W' because $\dot{v} > 0$, while $\theta = 90°$ at S'_+. Therefore $\ddot{\tilde{f}}(t)$ is normal to Γ at the points of maximal curvature.

6°. Notice that our construction of $\tilde{f}$ did not use the minimum point M, where the speed $|\dot{\tilde{f}}|$ would be too large to be consistent with $f \in (\Gamma)_A$.

2. By relation (3.21) of Corollary 1 we find that

$$L(\Gamma) = (\max_s \tilde{u}(s))^{1/4}/\sqrt{A} . \tag{8.13}$$

The quantity $\max \tilde{u}(s)$ equals the largest ordinate of the vertices W, W', W'', of Fig. 2, and can be determined approximately by numerical integration of the differential equations (5.2) and (5.3), in case that exact integrals are not available.

3. We have established in §8 the Theorem 2 for Case 3 under restrictive assumptions on $R(s)$. The modifications needed for Cases 1, 2, 4, or if $R(s)$ has discontinuities, or intervals of constancy, will become fairly obvious in our discussion of special curves Γ.

III. The Landau motion for special curves Γ

9_1. The curve Γ is a circle C_R. Now $R(s) = R$ is a constant. By the remark 1° (following Lemma 2 of §4) we gather that for a motion $f(t) \in (C_R)_A$, the speed $|v| = |\dot{f}(t)|$ may never exceed the value $\sqrt{AR}$. On the other hand, for the uniform circular motion

$$\tilde{f}(t) = Re^{it\sqrt{A/R}}, \tag{9.1}$$

we find that

$$\dot{\tilde{f}}(t) = i\sqrt{AR}\, e^{it\sqrt{A/R}}, \quad \ddot{\tilde{f}}(t) = -Ae^{it\sqrt{A/R}},$$

hence

$$|\dot{\tilde{f}}(t)| = \sqrt{AR}, \quad |\ddot{\tilde{f}}(t)| = A \quad \text{for all } t . \tag{9.2}$$

This establishes

Theorem 3. The Landau motion $\tilde{f}(t)$ on the circle C_R, for the constant A, is the uniform circular motion (9.1).

In particular, by (3.21) of Corollary 1, we obtain from (9.2) that

$$L(C_R) = \sqrt{AR}/\sqrt{A} = \sqrt{R}, \tag{9.3}$$

in agreement with our earlier relation (1.12).

9_2. The curve Γ is a circular arc. To discuss our next case when Γ is a circular arc, we need the solutions of the equation (4.22) which, in our case, becomes

$$\left(\frac{du}{ds}\right)^2 = 16\, u(A^2 - uR^{-2}) . \tag{9.4}$$

Variables separate and we find its solutions to be composed of the function

$$u = g(s) = \frac{1}{2} A^2R^2(1 - \cos\frac{4s}{R}) = A^2R^2\sin^2\frac{2s}{R}, \quad (-R\pi/4 \leqq s \leqq R\pi/4), \tag{9.5}$$

and all its horizontal translates $g(s - s_0)$. A graph of $g(s)$ is the arc BOD of Fig. 3. Moreover, $u = 0$ and $u = A^2R^2$ are singular solutions of (9.4).

Let now Γ be

$$\Gamma = C_{R,\ell}, \tag{9.6}$$

an arc of C_R of length ℓ. By our convention, described following (3.2), again s roams over all reals, and we find that

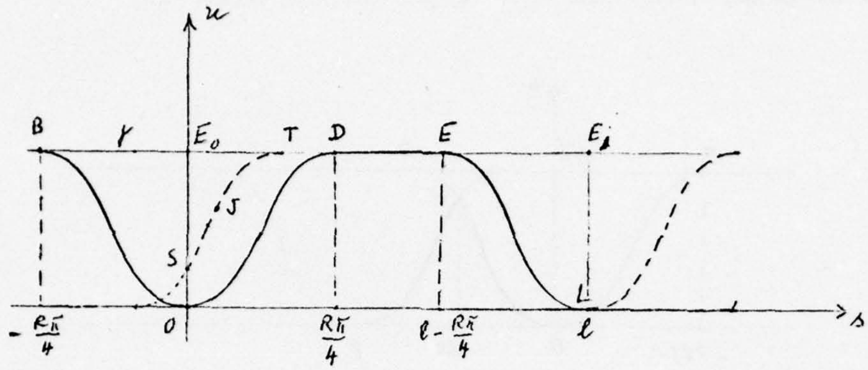

Fig. 3

(9.7) $$R(s) = \begin{cases} +R & \text{if} \quad 0 < s < \ell \quad (\text{mod } 2\ell) \\ -R & \text{if} \quad -\ell < s < 0 \quad (\text{mod } 2\ell) \ . \end{cases}$$

Moreover, we must set

(9.8) $$R(s) = 0 \quad \text{if} \quad s \equiv 0 \ (\text{mod } \ell) \ ,$$

i.e. at the endpoints of $C_{R,\ell}$.

The function $u = \tilde{u}(s)$ corresponding to the Landau motion $\tilde{f}(t)$ for $C_{R,\ell}$ depends on the size of ℓ. We will show that <u>if</u>

(9.9) $$\ell \geqq R\pi/2 \ ,$$

<u>i.e. the arc</u> $C_{R,\ell}$ <u>is a quarter-circle or longer, then a period</u> $[0,\ell]$ <u>of</u> $\tilde{n}(s)$ <u>is given by</u>

(9.10) $$\tilde{u}(s) = \begin{cases} g(s) & \underline{\text{if}} \quad 0 \leqq s \leqq R\pi/4 \ , \\ A^2R^2 \ \text{if} & \underline{\text{if}} \quad R\pi/4 \leqq s \leqq \ell - R\pi/4 \ , \\ g(s - \ell) & \underline{\text{if}} \quad \ell - R\pi/4 \leqq s \leqq \ell \ . \end{cases}$$

<u>If</u>

(9.11) $$\ell < \frac{R\pi}{2} \ ,$$

<u>then</u>

(9.12) $$\tilde{u}(s) = \begin{cases} g(s) & \underline{\text{if}} \quad 0 \leqq s \leqq \ell/2 \ , \\ g(s - \ell) & \underline{\text{if}} \quad \ell/2 \leqq s \leqq \ell \ . \end{cases}$$

For graphs of the two functions (9.10) and (9.12) see Figures 3 and 4, respectively. They show the period $[0,\ell]$, and $u(s)$ is periodic with period ℓ.

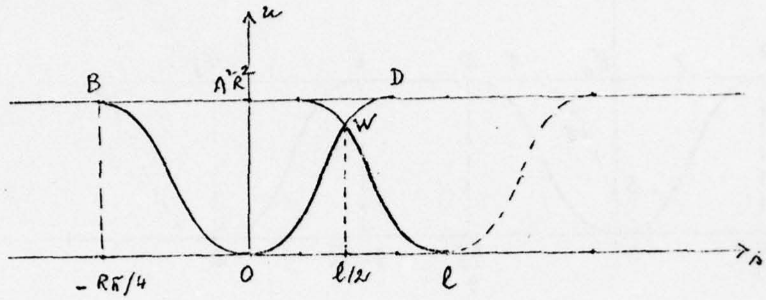

Fig. 4

Proof: For the dependence of s on t, we again use the relation (8.6). The novel situation is that $A^2R^2(s)$ is constant, except for its discontinuities at the multiples of ℓ, by (9.8).

The validity of (3.6) and (3.12) follows by construction. A proof of the extremum property (iii) is again based on Lemmas 4 and 5 of §7. The old proof of §7 will apply again if we use the following device. Mark in Fig. 3 the equidistant points

$$E_\nu = (\nu\ell, A^2R^2) \quad (\nu = 0, 1, \ldots) ,$$

and think of the curve γ, defined by (8.1), as consisting of the succession of segments

$$\ldots, E_{-1}E_0, E_0 0, 0E_0, E_0E_1, E_1L, LE_1, E_1E_2, \ldots .$$

We may now think of $0E_0$ as an ascending branch of γ. But then the arc ST, of Fig. 3, belongs to the class V_+^{II}, and the old reasoning of §7 applies.

Remarks. 1. Figures 3 and 4 show the character of the motions $u = \tilde{u}(s)$ in the two cases. If the arc exceeds a quarter-circle, Fig. 3 shows that the motion from $s = 0$ to $R\pi/4$ is accelerated; also that it is uniform with speed $\sqrt{AR}$ from $s = R\pi/4$ to $\ell - R\pi/4$, and descelerated from $s = \ell - R\pi/4$ to ℓ. The acceleration $\ddot{\tilde{f}}$ is continuous at all times. Fig. 4 shows that for arcs shorter than a quarter-circle the uniform middle section has disappeared, and that $\ddot{\tilde{f}}$ has a jump at $s = \ell/2$. All this easily leads to

Theorem 4. *The Landau constant of the arc* $C_{R,\ell}$ *is given by*

$$L(C_{R,\ell}) = \begin{cases} \sqrt{R \sin \frac{\ell}{R}} & \text{if } \ell \leqq R\pi/2 , \\ \sqrt{R} & \text{if } \ell \geqq R\pi/2 . \end{cases} \tag{9.13}$$

The most remarkable seems to be the case $\ell = R\pi/2$ of a quarter-circle. *What is the duration* T *of one complete oscillation?*

From (9.10) we find that $\frac{ds}{dt} = (g(s))^{1/4} = \sqrt{AR \sin(2s/R)}$, whence

$$\frac{1}{4} T = \frac{1}{\sqrt{AR}} \int_0^{R\pi/4} \frac{ds}{\sqrt{\sin(2s/R)}} .$$

Setting $x = \sqrt{\sin(2s/R)}$, we get that

$$\frac{1}{4} T = \sqrt{R/A} \int_0^1 \frac{dx}{\sqrt{1 - x^4}} .$$

This is the integral which equals one quarter of the length of the Lemniscate $r^2 = \cos 2\theta$. The change of variable $x = \cos \theta$ shows that

$$\frac{1}{4} T = \sqrt{R/A} \frac{1}{\sqrt{2}} \int_0^1 \frac{d\varphi}{\sqrt{1 - (1/2)\sin^2\varphi}} .$$

Using the value of this complete elliptic integral of the first kind as given in [1, 608], we obtain the final result:

$$T = (5.244\ 115\ 108)\sqrt{R/A} . \tag{9.14}$$

2. The acceleration pattern. Suppose that we are in Case 4 of Theorem 2. By differentiating twice by t the relations (3.15), it follows that $\tilde{f}(t)$ associates to every point $P \in \Gamma$ a *unique* acceleration $\ddot{\tilde{f}}_P$, such that $|\ddot{\tilde{f}}_P| = A$, except at the discontinuities of $\ddot{\tilde{f}}$. In the present case of (9.9), there are no such points. We call the correspondence $P \rightarrow \ddot{\tilde{f}}_P$ the acceleration pattern of $\tilde{f}$. For the present case when $\Gamma = C_{R,\ell}$, and assuming (9.9), we can describe the (continuous) acceleration pattern $\ddot{\tilde{f}}_P$ as follows.

In Fig. 5 let O D E L be the arc $C_{R,\ell}$, divided according to the graph of $\tilde{u}(s)$ of Fig. 3. At O and L, where $\dot{\tilde{f}}$ vanishes, $\ddot{\tilde{f}}_O$ and $\ddot{\tilde{f}}_L$ must be tangent to the arc, hence

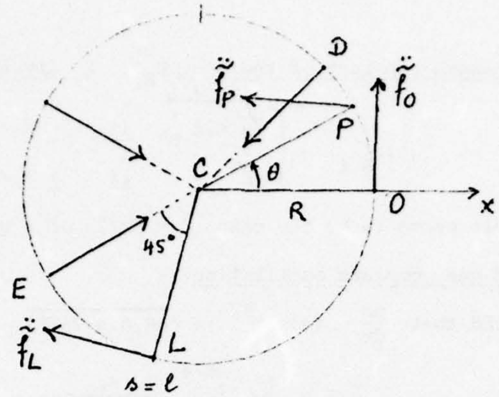

Fig. 5

$$\ddot{\tilde{f}}_0 = Ai, \quad (i = \sqrt{-1}) \tag{9.15}$$

At a variable point P on the arc O D, with $\theta = \angle$ OCP, we have

$$\ddot{\tilde{f}}_p = \ddot{\tilde{f}}_0 e^{3\theta i}, \qquad (0 \leqq \theta \leqq \pi/4) . \tag{9.16}$$

This shows that

(9.17) $\ddot{\tilde{f}}_p$ turns three times as fast as the radius CP .

As P reaches D, (9.16) shows that $\ddot{\tilde{f}}_D$ points towards the center C. This radial pattern holds along the arc DE, because of the constant speed (Fig. 3). Finally, along EL, the rule (9.17) again holds.

These results follow from (9.10) and (9.5), combined with the relations (4.6), describing the components of $\ddot{\tilde{f}}_p$. We omit the details.

9_3. Letting $R \to \infty$: Γ is a straight segment. Letting $R \to \infty$ in (9.5) we obtain

$$u = g_\infty(s) = 4A^2s^2 \qquad (-\infty < s < \infty) , \tag{9.18}$$

and its translates $g_\infty(s - s_0)$ are the solutions of the differential equation (9.4), where we let $R \to \infty$. For the straight segment $\Gamma = [0,\ell]$ the relation (9.12) goes over into

$$\tilde{u}(s) = \begin{cases} 4A^2s^2 & 0 \leqq s \leqq \ell/2 , \\ 4A^2(s-\ell)^2 \text{ if } & \ell/2 \leqq s \leqq \ell , \end{cases} \tag{9.19}$$

which is to be extended periodically with period ℓ. The arcs O W and W L of Fig. 4, are seen to go over into the two parabolic arcs of (9.19). Letting $R \to \infty$ in the (first) relation (9.13) we obtain that

$$L([0,\ell]) = \sqrt{\ell} , \tag{9.20}$$

which is Landau's original result (1.1).

9_4. The Landau motion on a racetrack. Using the results of §§9_1, 9_2 and 9_3, we could discuss our problem on any open or closed circle-spline. By this I mean a curve composed of arcs of circles of different radii (finite or infinite) that join with a continuously turning tangent. As a matter of fact, any Γ, as described in the opening paragraph of §3, could be closely approximated by an appropriate circle-spline S, and the Landau motion on S would approximate the motion on Γ.

However, we prefer to discuss the simplest closed circle-spline, having the shape of a racetrack $RT_{a,b}$, as shown in Fig. 6: A $2a \times 2b$ rectangle capped by two half-circles, having total perimeter $2\ell = 4a + 2\pi b$. By using the experience gained in §§9_1 to 9_3, we conclude that the function $\tilde{u}(s)$, giving the Landau motion $\tilde{f}(t)$ (for the constant A) on $RT_{a,b}$, has the period $2\ell = 4a + 2\pi b$, and that its graph within this period is as given in Fig. 6'. Here we use the shifted version $g_\infty(s + \frac{1}{2}b)$ of the function (9.18) and define

$$\tilde{u}(s) = 4A^2(s + \tfrac{1}{2}b)^2 \quad \text{if} \quad 0 \leqq s \leqq a . \tag{9.21}$$

This gives the parabolic arc B C, while D C, E F, and G F, are parabolic arcs congruent to B C.

Observe that $\tilde{u}(s)$ is maximal at the vertices C and F, hence

$$\max \tilde{u}(s) = 4A^2(a + \tfrac{1}{2}b)^2 .$$

It follows that

$$L(RT_{a,b}) = \frac{1}{\sqrt{A}} (\tilde{u}(a))^{1/4} = \frac{1}{\sqrt{A}} \sqrt{2A(a + \tfrac{1}{2}b)}$$

or

$$L(RT_{a,b}) = \sqrt{2a + b} . \tag{9.22}$$

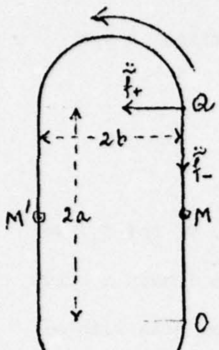

Fig. 6

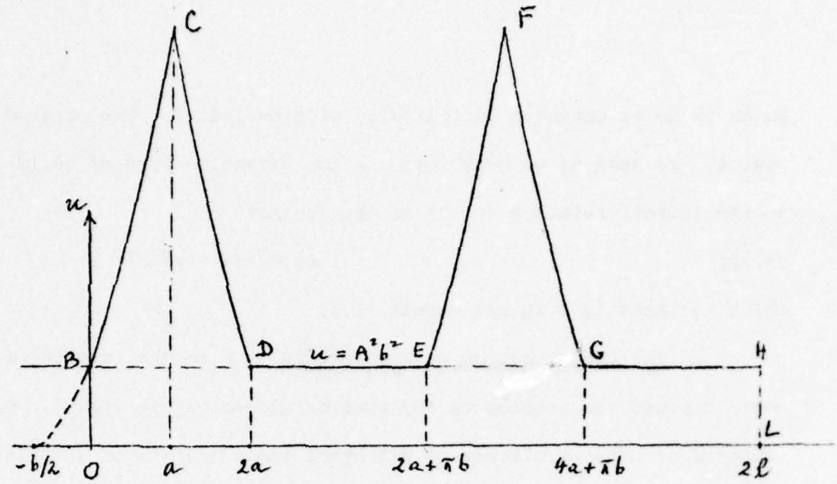

Fig. 6'

Remarks. 1. Notice that if $b = 0$ we obtain

$$L(RT_{a,0}) = \sqrt{2a} ,$$

which is again Landau's result (9.20). That if $a = 0$ we get

$$L(RT_{0,b}) = \sqrt{b} ,$$

which is the old result (1.12) of our Introduction.

2. Notice that $\ddot{\tilde{f}}$ has discontinuities at the midpoints M and M' of the straight side of Fig. 6, where the speed $|\dot{\tilde{f}}(t)|$ reaches its maximal value

$$\max|\dot{\tilde{f}}(t)| = \sqrt{A(2a + b)} .$$

However, $\ddot{\tilde{f}}$ has four further discontinuities at the four vertices of the rectangular portion of Γ. One such discontinuity is explicitly indicated at the vertex Q, and *we see that the jump in* $\ddot{\tilde{f}}$ does not satisfy the relation (6.2), since $\Delta\ddot{\tilde{f}}_Q$ is not parallel to the (vertical) tangent to Γ at Q. Reason: The relation (6.2) assumed $R(s)$ to be continuous at Q, which is not true in our case.

10. A comparison theorem for arcs Γ. Observe that the inequality (4.16), of Lemma 2, remains valid if we replace $R^2(s)$ by a larger function of s. This is the source of

Theorem 5. Let Γ_1 and Γ_2 be two arcs (Case 4) having the same length ℓ, and such that for their respective radii of curvature we have

$$(R_1(s))^2 \leqq (R_2(s))^2 \quad \text{for} \quad 0 \leqq s \leqq \ell . \tag{10.1}$$

Then

$$L(\Gamma_1) \leqq L(\Gamma_2) . \tag{10.2}$$

Proof: Let $\tilde{f}_1(t)$ and $\tilde{f}_2(t)$ be their respective Landau motions, and $\tilde{u}_1(s)$ and $\tilde{u}_2(s)$ their (s,u)-diagrams. We now use Lemma 2:

Since $|\ddot{\tilde{f}}_1(t)| = A$, we conclude that

$$\left(\frac{d\tilde{u}_1}{ds}\right)^2 = 16\,\tilde{u}_1\left(A^2 - \frac{\tilde{u}_1}{R_1^2(s)}\right) ,$$

and by (10.1) a fortiori

$$\left(\frac{d\tilde{u}_1}{ds}\right)^2 \leqq 16\,\tilde{u}_1\left(A^2 - \frac{\tilde{u}_1}{R_2^2(s)}\right) .$$

By Lemma 2 this implies that $\tilde{f}_1(t) \in (\Gamma_2)_A$ and therefore

$$L(\Gamma_2) \geqq \|\dot{\tilde{f}}_1\|/\sqrt{A} = L(\Gamma_1) ,$$

which proves (10.2).

As a special case we obtain

Corollary 2. If Γ_ℓ is an arc of length ℓ, then

$$L(\Gamma_\ell) \leqq \sqrt{\ell} , \tag{10.3}$$

with equality if $\Gamma_\ell = [0,\ell]$.

11. The curve Γ is a parabola. Let the parabola be

$$\Pi : y = \frac{1}{2p} x^2 . \tag{11.1}$$

On Π we measure the arc-length s from its vertex O. As we wish to apply our general approach of Part II, we select $A > 0$ and draw in the (s,u)-plane (Fig. 7) the curve

$$\gamma : u = A^2R^2(s) \quad (-\infty < s < \infty) . \tag{11.2}$$

For Π the function $R(s)$ is a complicated elementary function of s; accordingly, we shall not use its explicit expression. However, observe that $R(s)$ is an even

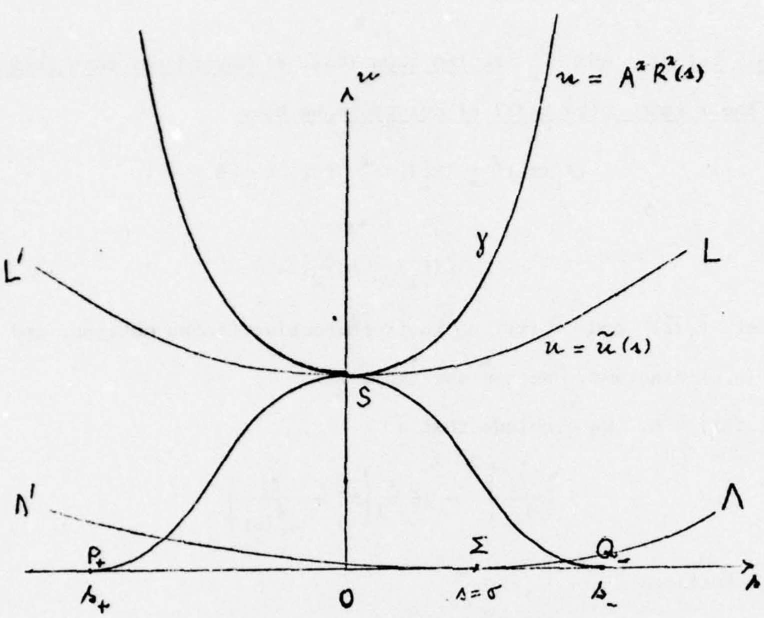

Fig. 7

function, that it decreases if $s < 0$, and therefore increases if $s > 0$. (The curve γ is also convex, but we shall not use this fact). Finally $R(0) = p$.

We apply our discussion of §5 of the solutions of the differential equation

(11.3) $$\left(\frac{du}{ds}\right)^2 = 16\, u\left(A^2 - \frac{u}{R^2(s)}\right)$$

to the present situation. <u>Of particular interest is its solution</u>

(11.4) $$u = \tilde{u}(s) \qquad (-\infty < s < \infty) ,$$

<u>passing through the minimum point</u> $S = (0, A^2 p^2)$ <u>of</u> γ. Its graph L'L is composed of two symmetric arcs L'S and S L. In our terminology of §5, the branch S L is an element of V_+^{II}, while L'S belongs to V_-^{II}. Both arcs are tangent to γ at S, and it follows that

(11.5) $$\tilde{u}'(s) \in C(\mathbb{R}) .$$

Also, clearly,

(11.6) $$\tilde{u}(s) > 0 \quad \text{for all } s .$$

Lemma 6. The function (11.4) is the only solution of (11.3) having the properties (11.5) and (11.6).

Proof: A solution $u = u(s)$ of (11.3), satisfying (11.6), is composed of a sequence of arcs of elements of V_+ and V_- (see the example (5.13) of the last two paragraphs of §5. In our present case the condition of periodicity drops out, and we may therefore have an infinite sequence of arcs that alternate between V_+ and V_-). However, all of these solutions have corners (hence violate (10.5)), and the only way that corners can be avoided is to have only one arc of V_-, and one arc of V_+, joining together at the point S. This gives the solution (11.4).

We now turn to the parabola Π and consider on Π motions $f(t)$ such that

$$|\ddot{f}(t)| = A \quad \text{for all } t , \tag{11.7}$$

and

$$\dot{f}(t) \neq 0 \quad \text{for all } t . \tag{11.8}$$

Such motions are in a 1 - 1 correspondence with the positive solutions $u(s)$ of (11.3). Among such motions $f(t)$ there is one outstanding motion $f_G(t)$, that we call the Galilean motion. It is given parametrically by

$$f_G(t) = x + iy, \text{ where } \begin{cases} x = t\sqrt{Ap} , \\ y = \frac{1}{2} At^2 . \end{cases} \tag{11.9}$$

That $f_G(t) \in (\Pi)$ is clear, because on eliminating t we obtain (11.1). That it satisfies (11.7) is seen from

$$\ddot{f}_G(t) = \ddot{x} + i\ddot{y} = iA . \tag{11.10}$$

Also (11.8) follows from $|\dot{f}(t)|^2 = Ap + A^2t^2 > 0$. Finally, observe that (11.10) implies trivially that

$$\ddot{f}_G(t) \in C(\mathbb{R}) . \tag{11.11}$$

These properties of f_G show that the corresponding function $u_G(s)$ shares, with $\tilde{u}(s)$, the properties (11.5) and (11.6). Now Lemma 6 shows that $u_G(s) = \tilde{u}(s)$. This proves

Theorem 6. For the parabola Π, the Landau motion $\tilde{f}(t)$, corresponding to A, is identical with the Galilean motion (10.9).

A generalization of the Galilean motion. Let the curve

(11.12) $$\Gamma : x = x(s),\ y = y(s), \quad (-\infty < s < \infty),$$

be given parametrically in terms of its arc-length s. We assume that $x(s) \in C^2(\mathbb{R})$ and $y(s) \in C^2(\mathbb{R})$. For its radius of curvature

(11.13) $$R(s) = \frac{1}{x'(s)y''(s) - x''(s)y'(s)}$$

we assume the following:

(11.14) $$R(s) > 0 \text{ for all } s,$$

(11.15) $R(s)$ decreases strictly for $s \leqq 0$, and increases strictly for $s \geqq 0$.

We choose $A > 0$ and may state

Theorem 7. Among all motions $f(t)$ on Γ, such that $f(t)$ reaches all points of Γ, there is a unique motion $f(t)$ such that

(11.16) $$|\ddot{f}(t)| = A \text{ for all } t,$$

and

(11.17) $\ddot{f}(t)$ is continuous for all t.

Proof: We refer again to Fig. 7, where the (upper) curve γ corresponds to (11.13). We have seen that the only solution of (11.3) that satisfies (11.5) and (11.6), is the $\tilde{u}(s)$ defined by the curve $L'\ S\ L$.

Let now $u = u(s)$ correspond to an $f(t)$ satisfying (11.16) and (11.17). We know that (11.16) and (11.17) imply that

(11.18) $$u'(s) \in C(\mathbb{R}),$$

provided that

(11.19) $$u(s) > 0 \text{ for all } s,$$

holds.

We claim:

(11.20) The assumptions (11.16) and (11.17) do imply (11.19).

Indeed, let us assume that there is a σ such that

(11.21) $$u(\sigma) = 0.$$

(This is the case if $u = u(s)$ has, e.g. the graph $\Lambda'\Sigma\Lambda$, where $\Lambda'\Sigma \in V_{-}^{I}$ and

$\Sigma\Lambda \in V_+^I$), and let us show that our assumption (11.17) is violated: At the point P of Γ, corresponding to $s = \sigma$, we have, by (11.21), a vanishing speed

$$|\dot{f}(t_0)| = 0 . \qquad (11.22)$$

Therefore f(t) decelarates for $s < \sigma$ (or $t < t_0$), and accelerates for $s > \sigma$. Since accelerations $\ddot{f}(t_0-)$ and $\ddot{f}(t_0+)$ are both tangent to Γ at P, by (11.22), we conclude that the vectors $\ddot{f}(t_0-)$ and $\ddot{f}(t_0+)$ are directly opposite vectors of common magnitude A. Thus (11.17) is violated.

We conclude that (11.19) holds, and now Lemma 6 shows that $u(s) = \tilde{u}(s)$. It follows that f(t), satisfying (11.16) and (11.17), must be the Landau motion $\tilde{f}(t)$, for A, and Theorem 7 is established.

Have we used the assumption of Theorem 7, that f(t) reaches all points of Γ? Indeed, we have used it, for if we remove it, we find other motions satisfying (11.16) and (11.17), as follows. We assume that Fig. 7 represents the curve

$$\gamma : u = A^2R^2(s)$$

corresponding to (11.13). Through S we draw the arcs

$$P_+S \in V_+^I \quad \text{and} \quad SQ_- \in V_-^I ,$$

and let s_+ and s_- denote the s-coordinates of P_+ and Q_-, respectively. Let

$$u = u_0(s), \qquad (s_+ \leqq s \leqq s_-) ,$$

be the function defined by the curve P_+SQ_-, and let Γ_0 be the subarc of (11.12) for $s_+ \leqq s \leqq s_-$. It is then found that $u_0(s)$ defines the (to-and-fro) Landau motion $f_0(t)$ on the arc Γ_0 (Case 4). Also that $f_0(t)$ satisfies the conditions (11.16) and (11.17).

Remarks. 1. We have shown that the Landau motion $\tilde{f}(t)$ on Γ, is the only motion on Γ satisfying the two conditions (11.16) and (11.17). Evidently $\tilde{f}(t)$ reduces to the Galilean motion (11.9) if Γ is the parabola (11.1), and therefore generalizes the latter. Galileo's constant force-field (11.10) is now replaced by the variable continuous force-field $\ddot{\tilde{f}}(t)$ of constant magnitude A.

Examples of curves (11.12), satisfying (11.14) and (11.15), are numerous. The Catenary $y = a\cosh(x/a)$ is one such. For this curve we have $R = y^2/a$, $s = a\sinh(x/a)$ and therefore

$$R(s) = a + \frac{1}{a}s^2 .$$

Can anybody integrate the corresponding equation

$$\left(\frac{du}{ds}\right)^2 = 16\,u\left(A^2 - \frac{a^2 u}{(a^2+s^2)^2}\right) ? \tag{11.23}$$

From Theorem 6 we know that it has a unique solution $\tilde{u}(s)$ $(s \in \mathbb{R})$ satisfying (11.5) and (11.6). Also that $\tilde{u}(0) = A^2a^2$, and $\tilde{u}(-s) = \tilde{u}(s)$.

2. Following the procedure of §9_2 we could also discuss the case when the curve Γ is a finite subarc of the parabola (11.1). We omit this discussion because of our lack of quantitative information concerning the solutions of the differential equation (11.3).

12. The curve Γ is an ellipse E defined by

$$E = E_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad (0 < b < a) , \tag{12.1}$$

shown in Fig. 8. Denoting by 2ℓ the perimeter of E, we find the periodic function

$$\gamma : u = A^2R^2(s) \tag{12.2}$$

to be represented by a graph as sketched in Fig. 9. Our general Fig. 2 now reduces to Fig. 9; it shows that the Landau motion $\tilde{f}$ on E, is derived from the graph, of period 2ℓ, defined by the curve $S_+W\,S'_+W'S_-$ representing the function

$$u = \tilde{u}(s) . \tag{12.3}$$

Of course, the arc S_+W is the solution of

$$\frac{d\tilde{u}}{ds} = 4\sqrt{\tilde{u}\left(A^2 - \frac{\tilde{u}}{R^2(s)}\right)} , \tag{12.4}$$

satisfying the initial condition

$$u(0) = A^2R^2(0) = A^2b^4/a^2 . \tag{12.5}$$

Because $R(s)$ is a complicated (elementary) function of the arc-length s (measured from the vertex M of Fig. 8), we prefer to use the representation

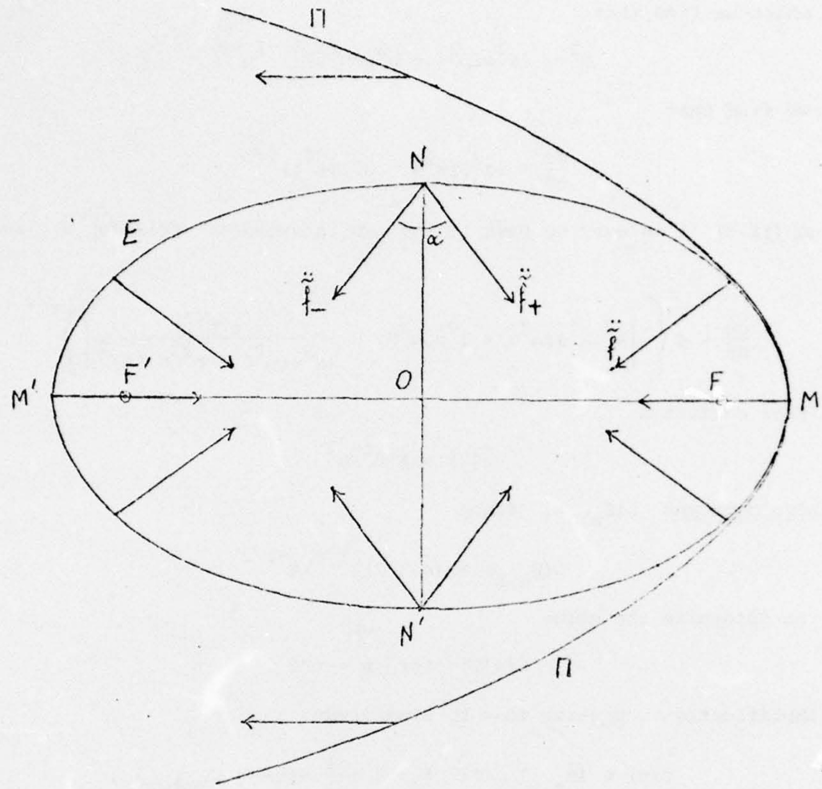

Fig. 8

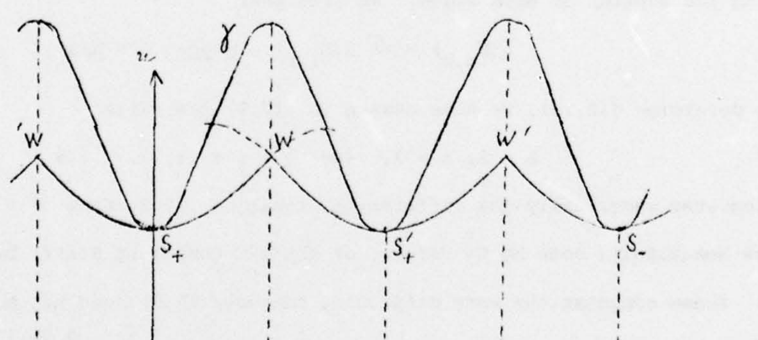

Fig. 9

(12.6) $$E : x = a \cos \theta,\ y = b \sin \theta ,$$

in terms of which we find that

(12.7) $$R^2 = (a^2\sin^2\theta + b^2\cos^2\theta)^3 a^{-2} b^{-2} .$$

From (12.6) we find that

(12.8) $$\frac{ds}{d\theta} = (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2} .$$

By (12.7) and (12.8) it is easy to pass to the new independent variable θ, and (12.4) becomes

(12.9) $$\frac{d\tilde{u}}{d\theta} = 4\left\{\tilde{u}\left[A^2(a^2\sin^2\theta + b^2\cos^2\theta) - \frac{a^2b^2\tilde{u}}{(a^2\sin^2\theta + b^2\cos^2\theta)^2}\right]\right\}^{1/2} ,$$

with the initial condition

(12.10) $$\tilde{u}(0) = A^2b^4/a^2 .$$

<u>The Landau constant</u> $L(E_{a,b})$. Since

(12.11) $$L(E_{a,b}) = (\tilde{u}(\ell/2))^{1/4}.A^{-1/2} ,$$

it suffices to determine the value

(12.12) $$\tilde{u}(\ell/2) \quad \text{for} \quad s = \ell/2 .$$

Here is a simplification: Observe that by similitude

$$f(t) \in (E_{a,b}) \quad \text{iff} \quad f_1(t) = \frac{1}{a} f(t) \in (E_{1,b/a}) ,$$

and therefore

$$F(f) = \sqrt{a}\, F(f_1) .$$

By taking the suprema of both sides, we find that

(12.13) $$L(E_{a,b}) = \sqrt{a}\, L(E_{1,r}), \quad \underline{\text{where}} \quad r = b/a .$$

To determine (12.12), we have chosen in (12.9) the values

(12.14) $$A = 1,\ a = 1,\ \text{and}\ b = r = .1, .2, \ldots, .9$$

and integrated numerically the differential equation (12.9) from $\theta = 0$ to $\theta = \pi/2$. This was beautifully done by C. Vargas, of the MRC Computing Staff, by a Runge-Kutta method. These computations were difficult, because (12.9) does <u>not</u> satisfy a Lipschitz condition in a neighborhood of the starting point $\theta = 0$, $\tilde{u} = A^2b^4/a^2$. The results are shown in Table 1.

Table 1

r	$L(E_{1,r})$	
.0	1.41421	$= \sqrt{2}$
.1	1.40978	
.2	1.39643	
.3	1.37394	
.4	1.34199	
.5	1.30011	
.6	1.24811	
.7	1.18609	
.8	1.11782	
.9	1.05409	
1.0	1.00000	

The values for the two endpoints of the table were known: For $E_{1,0} = I_1$, by (4) and (1.1), and for $E_{1,1} = C_1$ by (1.9) and (1.12).

A glance at Fig. 9 shows the Landau motion $\tilde{f}(t)$ to be accelerated on the arcs M N and M'N' of Fig. 8, and descelerated on N M', N'M, the maximal equal speeds occurring at N and N'. At these two points are the only discontinuities of $\ddot{\tilde{f}}$, as indicated in Fig. 8.

We mention the following three limiting cases in Fig. 8:

1. If b decreases and tends to zero, then the angle $\alpha = \angle(\ddot{\tilde{f}}_+, \ddot{\tilde{f}}_-)$ increases to 180°, and $\tilde{f}$ becomes the Landau motion on the segment M M'.

2. If b increases and tends to a, then α decreases to zero, and $\tilde{f}$ becomes the uniform motion on the circle C_a.

3. If we keep fixed the focus F of E, and also its vertex M, while we let its second focus F' tend to $-\infty$, then E approaches the parabola Π of Fig. 8. Also the acceleration pattern $\ddot{\tilde{f}}$ on the arc N'M N is fanning out and approaches

in the limit the horizontal pattern of the Galilean motion on Π.

An open question: Can the solution $\tilde{u}(\theta)$ of (12.9) and (12.10) be expressed in terms of elliptic functions?

13. The arc Γ is a cycloid (Fig. 10). We select

(13.1) $$A = a ,$$

and wish to show that the relations

(13.2) $$\tilde{f}(t) = x + iy, \quad \text{where} \quad \begin{cases} x = a(t - \sin t), \\ y = a(1 - \cos t), \quad (-\infty < t < \infty) , \end{cases}$$

describe the Landau motion on the cycloid C, corresponding to the constant (13.1).

We find that $\ddot{\tilde{f}}(t) = a(\sin t + i \cos t)$, which is also the complex number represented by the vector $\overrightarrow{PC}$. Therefore

(13.3) $$|\ddot{\tilde{f}}(t)| = a \quad \text{for all} \quad t .$$

Let us focus our attention on the interval $0 \leqq t \leqq 2\pi$, when $P = \tilde{f}(t)$ describes the first arch OPL of C. For s = length of arc OP we find

(13.4) $$s = 4a(1 - \cos \frac{t}{2}) \quad \text{and} \quad R = 2|\overrightarrow{PB}| = 4a \sin \frac{t}{2} .$$

Expressing R in terms of s, by the first relation (13.4), we find that

(13.5) $$a^2R^2(s) = a^2s(8a - s) \quad \text{in} \quad 0 \leqq s \leqq 8a .$$

Its periodic extension, with period $8a$, gives us the "upper curve"

(13.6) $$\gamma : u = a^2R^2(s) \quad (-\infty < s < \infty) ,$$

of Fig. 11. However, also the function $\tilde{u}(s)$, corresponding to $\tilde{f}(t)$, is readily found. Using again the first relation (13.4) we find that

(13.7) $$\tilde{u}(s) = v^4 = (\dot{x}^2 + \dot{y}^2)^2 = \frac{1}{16} s^2(8a - s)^2 \quad \text{in} \quad 0 \leqq s \leqq 8a .$$

Its periodic extension is shown in Fig. 11. By its construction we know that (13.7) is a solution of the differential equation

(13.8) $$\left(\frac{du}{ds}\right)^2 = 16\, u\left(a^2 - \frac{u}{s\cdot(8a - s)}\right) , \quad (0 \leqq s \leqq 8a) ,$$

as also easily verified directly. Let us point out, as readily shown, that the graphs

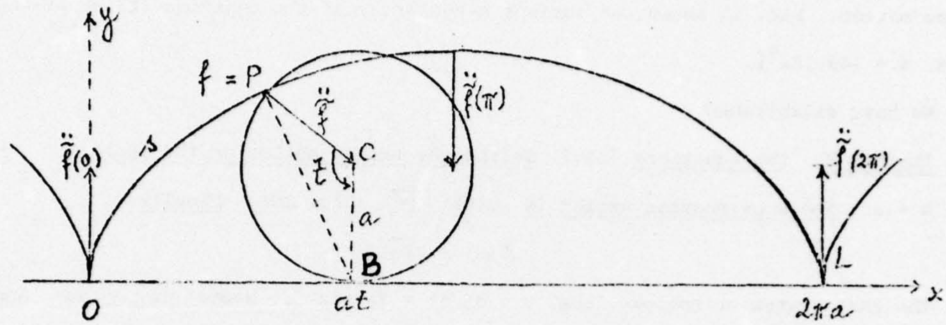

Fig. 10

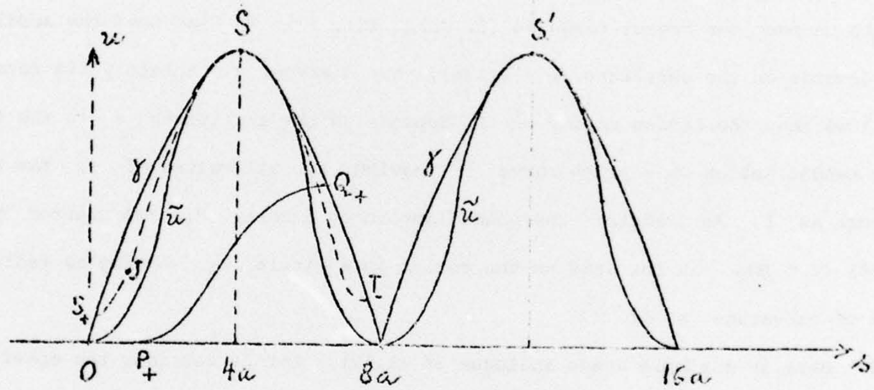

Fig. 11

of the functions (13.5) and (13.7) are tangent to each other at the point S of Fig. 11.

This fact has the following consequence: If we consider a solution of

(13.9) $$\frac{du}{ds} = 4\sqrt{u\left(a^2 - \frac{u}{s(8a - s)}\right)} \qquad 0 \leqq s \leqq 8a \ ,$$

of the kind that we classified as of type V_+^{II} in §5, originating in S_+, say, then we see that this increasing solution must terminate at the point S. Indeed, this increasing regular function has nowhere else to go!

In any case, the reasoning proving Lemma 5 applies and shows that $\tilde{f}(t)$ is the Landau motion. Fig. 11 shows the curious singularity of the equation (13.8) at the point $S = (4a, 16a^4)$.

We have established

Theorem 8. The equations (13.2) define the Landau motion on the cycloid C, for $A = a$. The acceleration vector is $\ddot{\tilde{f}}(t) = \overrightarrow{PC}$ (Fig. 10). Finally

$$L(C) = 2\sqrt{a}\ .$$

The last statement follows from $v = ds/dt = 2a \sin \frac{t}{2}$, hence $\|\dot{\tilde{f}}\| = 2a$, and so $L(C) = 2a/\sqrt{a} = 2\sqrt{a}$.

14. The case of skew curves. We conclude this already too long article with the following brief remarks.

1. If we apply our analysis of §4 to the case of a curve Γ in R^3, and apply for this purpose the Frenet formulae [5, Chap. III, §7], we find that the acceleration $\ddot{f}(t)$ depends on the curvature $\kappa = 1/R(s)$, but disregards completely its torsion τ. It follows that the Landau motion on Γ depends on the arc-length s in the same way as the Landau motion on a plane curve Γ^* having, for all values of s, the same curvature as Γ. An example: the Landau motion on a helix H, represented by $s = \tilde{s}(t)$ $(t \in \mathbb{R})$, is the same as the motion on a circle C_a having as radius the radius of curvature a of H.

2. Here is a simple space analogue of (1.12). Let us consider the spherical shell

$$(14.1) \qquad \Sigma_a : x^2 + y^2 + z^2 = a^2\ .$$

Its plane sections have radii that do not exceed a. By the first elements of surface theory [5, Chap. IV, §12] we know that this is also true for any smooth curve Γ drawn on Σ: The radius of curvature $R(s)$ of Γ satisfies $R(s) \leqq a$ for all s. This implies the following. If $f(t)$ is any motion on Σ_a, describing the curve Γ which we assume closed, say, then the curve

$$\gamma : u = A^2R^2(s)$$

of Fig. 2, will never rise above the horizontal $u = A^2a^2$, which corresponds to a great circle of Σ_a. But then surely $\tilde{u}(s) \leqq A^2a^2$, for all s, and we obtain

Theorem 9. *For the spherical shell* (14.1) *we have*

$$L(\Sigma_a) = \sqrt{a}\ .$$

More elusive and not yet determined is the Landau constant of the surface of an ordinary Torus.

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Among all motions $f(t)$ on a curve Γ such that $|\ddot{f}(t)| \le A$ for all t, there is just one, called $\tilde{f}(t)$, that maximizes the speed $|\dot{f}|$ at each point of Γ.

Gamma — abs. value of f''(t) — < or = — f-bar(t) — abs. value of f' — Gamma — JOB

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