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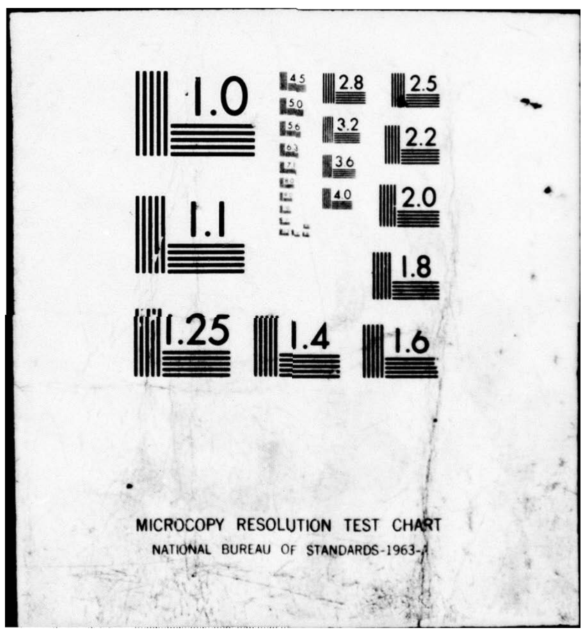
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very simple form when the boundary conditions are translatory. The results are applied to several well-known boundary treatments, and overall stability is achieved without effort.

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(6) # BOUNDARY-DEPENDENT STABILITY CRITERIA FOR DIFFERENCE APPROXIMATIONS OF HYPERBOLIC PROBLEMS. I.

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ABSTRACT. We study one-step, explicit, dissipative approximations to scalar hyperbolic problems in the quarter plane $x \geq 0$, $t \geq 0$. In both, outflow and inflow cases, Kreiss' theory is used to derive stability criteria which are <u>independent</u> of the basic difference scheme, and are stated entirely in terms of the boundary conditions. The results are applied to several well-known boundary treatments, and overall stability is achieved without effort.



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## 0. Introduction.

In this paper we study one-step finite difference approximations of scalar, well-posed, hyperbolic problems in the quarter plane $x \geq 0$, $t \geq 0$. We assume that the basic scheme is explicit and dissipative (hence stable in the usual Von Neumann sense), and that the complimentary boundary conditions are of a general two-level type.

Kreiss' theory in [5] provides a sufficient stability criterion for the above approximation. Briefly speaking, stability is assured if the operator which represents the overall numerical algarithm has no eigenvlues on the boundary or in the exterior of the unit disc. That is, Kreiss' criterion is (indirectly) stated in terms of the basic scheme as well as the boundary conditions. Our analysis uses Kreiss' theory and provides stability criteria which do not take into account the basic scheme, but instead, are given solely in terms of the boundary conditions. This is done for the outflow as well as the inflow cases.

We begin, in Section 1, by discussing the (somewhat simpler) outflow problem together with the set-up which allows us to quote Kreiss' criterion. This leads, in Section 2, to our first boundary-dependent conditions.

In Section 3 we use the new results to derive simple criteria for the common case where the boundary conditions are translatory; i.e., where boundary conditions at different grid points are determined by the same coefficients. Among other results we show that if the boundary conditions are generated by an explicit stable scheme, then overall stability is assured. Indeed, we find it very easy to apply the criteria obtained in this section to popular boundary treatments such as extrapolation, explicit Euler, implicit Euler, and the Box-Scheme. All those examples are found to be unconditionally satisfactory in the sense that when augmenting arbitrary (stable) dissipative schemes, they always maintain stability.

Finally, in Section 4, we slightly modify the previous criteria in order to fit the inflow problem. As an example we show in this case that homogeneous summation at the boundary preserves stability.

In a forthcoming paper, we discuss an extension of the above theory to nondissipative and multi-step approximations. The main key to these cases is given in the theory of Gustafsson, Kreiss and Sundström, [3].

It is hoped that the present work can also be extended to include systems of equations where outflow and inflow unknowns interact at the boundary. Such extensions should be possible at least for certain families of difference approximations.

## 1. The Outflow Problem and Kreiss' Criterion.

Consider the scalar, hyperbolic, $L^2(0,x)$ well-posed initial value problem

$$\partial u(x,t)/\partial t = a\partial u(x,t)/\partial x; \quad a > 0; \quad x \geq 0, \ t \geq 0; \quad u(x,0) = f(x) \ . \tag{1.1}$$

To approximate (1.1), we introduce a mesh size $\Delta x > 0$, $\Delta t > 0$; a grid function $v_\nu(t) = v(\nu\Delta x,t)$, $\nu = 0, \pm 1, \pm 2, \ldots$ ; and a consistent finite difference scheme

$$v_\nu(t + \Delta t) = Qv_\nu(t) \ , \quad \nu = 1, 2, 3, \ldots \ , \tag{1.2}$$

$$Q = \sum_{j=-r}^{p} a_j E^j; \quad Ev_\nu = v_{\nu+1} \ .$$

Here, the $a_j$ are constants depending on the coefficient $a$ and on the fixed ratio $\lambda \equiv \Delta t/\Delta x$, such that $a_{-r}$ and $a_p$ do not vanish. Initial values are given by

$$v_\nu(0) = f_\nu \quad , \quad \nu = 1, 2, 3, \ldots \quad .$$

Throughout the entire paper two assumptions will be taken for granted.

ASSUMPTION 1. The difference scheme is dissipative, i.e., for some $\delta > 0$ and natural $\omega$, the amplification factor

$$\hat{Q}(\xi) = \sum_{j=-r}^{p} a_j e^{ij\xi} \quad , \quad -\pi \leq \xi < \pi \quad ,$$

satisfies

$$|\hat{Q}(\xi)| \leq 1 - \delta|\xi|^{2\omega} \quad , \quad -\pi \leq \xi < \pi \quad .$$

Clearly, in our scalar case, dissipativity implies that $\hat{Q}(\xi)$ is power bounded (by 1), which is equivalent to the (strong) Von Nuemann stability of the scheme should it be applied to the pure Cauchy problem for $-\infty < x < \infty$.

ASSUMPTION 2. The operator $Q$ is not right-sided, i.e., we have $r > 0$.

Actually, using dissipativity, it was shown in Corollary 1 of [2] that $Q$ must be two-sided if it is to be consistent with $\partial u/\partial t = a\partial u/\partial x$ for arbitrary $a \neq 0$.

The fact that $r > 0$ implies that in order to approximate the differential problem in the quarter plane $x \geq 0$, $t \geq 0$, the difference scheme has to be augmented by numerical boundary conditions. More precisely, we have to specify, at each time step, boundary values at the $r$ grid points $\mu = 0, -1, \ldots, -r+1$. We do it by setting boundary conditions of the form

$$(1.3) \qquad \sum_{j=0}^{s} c_{\mu j}^{(1)} v_{\mu+j}(t + \Delta t) = \sum_{j=0}^{s} c_{\mu j}^{(0)} v_{\mu+j}(t) \quad ,$$

where the $c_{\mu j}$ are fixed coefficients and $s \geq 1$.

We choose to write (1.3) in its present form because it allows us the following two options which we use later in various examples:

(i) The boundary conditions are of a one-level type; that is, all the $c_{\mu j}^{(1)}$ vanish, and

$$c_{\mu 0}^{(0)} \neq 0 \quad , \quad \mu = 0, \ldots, -r + 1 \quad .$$

(ii) The conditions are two-leveled, i.e., not all the $c_{\mu j}^{(0)}$ vanish, and

$$c_{\mu 0}^{(1)} \neq 0 \quad , \quad \mu = 0, \ldots, -r + 1 \quad .$$

In any event it is understood of course, that the required boundary values must be computed in the specified order, namely, $\mu = 0, -1, \ldots, -r + 1$.

The difference approximation is completely defined now by the stable scheme (1.2) together with the boundary conditions (1.3), and we raise the question of overall stability. For this purpose we consider the space $\ell_2 = \ell_2(\Delta x)$ of all grid functions $w = \{w_\nu\}_{\nu=-r+1}^{\infty}$ which satisfy

$$\sum_{\nu=-r+1}^{\infty} |w_\nu|^2 < \infty \quad .$$

Upon defining inner product and norm by

$$(w,v) = \Delta x \sum_{\nu=-r+1}^{\infty} \bar{w}_\nu v_\nu \quad , \quad \|w\|^2 = (w,w) \quad ,$$

$\ell_2$ becomes a Hilbert space which is a discrete analogue of $L^2(0,\infty)$.

Now, the difference approximation may be written as

$$v(t + \Delta t) = Gv(t); \quad v(t) \quad , \quad v(t + \Delta t) \in \ell_2 \quad ,$$

where $G : \ell_2 \to \ell_2$ is a linear bounded operator, defined by the basic scheme (1.2) together with the boundary conditions (1.3). Thus, the numerical algorithm is embeded in $\ell_2$, and is said to be stable if there exists a constant K, independent of $\Delta t$, such that

$$\|v(t)\| \leq K\|v(0)\| \text{ for all } t = m\Delta t \quad , \quad m = 0, 1, 2, \ldots, \text{ and } v(0) \in \ell_2 \; .$$

The above formulation with Option 1, is equivalent to the scalar version of Kreiss' problem in [5]. In fact, it is not hard to see that the various results of [5] hold also when the boundary conditions are those of Option 2 (e.g. [2]). We assume therefore that the reader is familiar with the details of [5], and conclude this section by quoting Kreiss' stability criterion.

THEOREM 1 (Kreiss). The approximation defined by (1.2), (1.3) is stable if

(i) the corresponding operator G has no eigenvalues z with $|z| \geq 1$, $z \neq 1$;

(ii) $z = 1$ is not a (possibly generalized) eigenvalue of G.

## 2. Boundary-Dependent Stability Criteria.

Associated with the difference scheme (1.2) is the characteristic equation

$$\sum_{j=-r}^{p} a_j \kappa^j - z = 0 \quad , \tag{2.1}$$

whose $r + p$ roots, $\kappa_i$, are continuous functions of $z$. Kreiss has used dissipativity and consistency to show, in Lemma 2 and in part of the proof of Lemma 7 of [5], the following.

LEMMA 1 (Kreiss). (i) For $z$ with $|z| \geq 1$, $z \neq 1$, the $r+p$ roots of (2.1) split: $p$ with $|\kappa_i| > 1$ and $r$ with $0 < |\kappa_i| < 1$.

(ii) Suppose that the coefficient $a$ in (1.1) is positive (negative), and let $z$ with $|z| \geq 1$, $z \neq 1$, tend to 1. Then, the above splitting is maintained, except for a single $\kappa_i$ which tends to 1 from the exterior (interior) of the unit disc.

Now let $z$ with $|z| \geq 1$, be given. If $z$ is an eigenvalue of the operator $G$, then there exists a corresponding (nontrivial) eigenvector, $g \in \ell_2$, such that $Gg = zg$; thus by the definition of $G$, $g$ must first satisfy the resolvent equation

$$(2.2) \qquad zg_\nu = \sum_{\nu=-r}^{p} a_j g_{\nu+j} \ , \quad \nu = 1, 2, 3, \ldots \quad .$$

Equation (2.2) is an ordinary difference equation with constant coefficients; hence its most general solution in $\ell_2$ is

$$g_\nu = \sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell - 1} \sigma_{\ell,k} P_{\ell,k}(\nu) \kappa_\ell^\nu \ , \quad \nu \geq -r+1 \ ,$$

where $\kappa_\ell = \kappa_\ell(z)$, $1 \leq \ell \leq n$, are the distinct roots of (2.1) which satisfy $0 < |\kappa_\ell| < 1$, each with multiplicity $m_\ell$; and the $P_{\ell,k}(\nu)$ are arbitrary polynomials in $\nu$ with degree$[P_{\ell,k}(\nu)] = k$. The $\sigma_{\ell,k}$ are parameters to be determined, and by Lemma 1, their precise number is

$$\sum_{\ell=1}^{n} m_\ell = r \quad .$$

In particular we choose

$$P_{\ell,k}(\nu) = \kappa_\ell^{-k}\, k!\binom{\nu}{k} \quad ,$$

which leads to a general solution of (2.1) of the form

(2.3) $$g_\nu = \sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell-1} \sigma_{\ell,k}\, k!\binom{\nu}{k} \kappa_\ell^{\nu-k} \quad , \quad \nu \geq -r+1 \ .$$

To determine the $\sigma_{\ell,k}$ we remember that being an eigenvector of $G$, $g$ must also satisfy the boundary relations

(2.4) $$(Gg)_\mu = zg_\mu \quad , \quad \mu = 0 \quad , \ldots, -r+1 \ .$$

The definition of $G$ at the boundary is given by (1.3); hence (2.4) becomes

(2.5) $$z \sum_{j=0}^{s} c_{\mu j}^{(1)} g_{\mu+j} = \sum_{j=0}^{s} c_{\mu j}^{(0)} g_{\mu+j} \quad , \quad \mu = 0 \quad , \ldots, -r+1 \ .$$

Thus, substituting (2.3) in (2.5), we finally obtain

(2.6) $$\sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell-1} \sum_{j=0}^{s} [zc_{\mu j}^{(1)} - c_{\mu j}^{(0)}]k!\binom{\mu+j}{k}\kappa_\ell^{\mu+j-k}\sigma_{\ell,k} = 0 \quad ,$$

$$\mu = 0, \ldots, -r+1 \quad ,$$

which constitutes a linear homogeneous system of $r$ equations in the $r$ unknowns $\sigma_{\ell,k}$. Clearly, $g$ is an eigenvector of $G$ if and only if not all the $\sigma_{\ell,k}$ in (2.3) vanish, that is, (2.6) has a nontrivial solution.

At this point we associate with the boundary conditions (1.3) a set of rational boundary-functions

$$(2.7)\quad R_\mu(z,\kappa) = \sum_{j=0}^{s} (zc_{\mu j}^{(1)} - c_{\mu j}^{(0)})\kappa^{\mu+j} \quad , \quad \mu = 0, \; \ldots, \; -r+1 \quad ,$$

which are uniquely determined by the boundary coefficients $c_{\mu j}^{(0)}$, $c_{\mu j}^{(1)}$. Since

$$\frac{\partial^k R_\mu(z,\kappa)}{\partial k^k} = \sum_{j=0}^{s} (zc_{\mu j}^{(1)} - c_{\mu j}^{(0)})k!\binom{\mu+j}{k}\kappa^{\mu+j-k} \quad ,$$

then system (2.6) may be written as

$$(2.8)\qquad \sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell-1} \frac{\partial^k R_\mu(z,\kappa_\ell)}{\partial \kappa_\ell^k} \sigma_{\ell,k} = 0 \quad , \qquad \mu = 0, \ldots, -r+1 \; .$$

It follows that the coefficient matrix of this system, which we denote by

$$(2.9a)\qquad D = D(z \; ; \; \kappa_1,\ldots,\kappa_n \; ; \; m_1,\ldots,m_n) \quad ,$$

is of the form

$$(2.9b)\qquad D = [B(z,\kappa_1,m_1),\ldots,B(z,\kappa_n,m_n)] \quad ,$$

where the $B(z,\kappa_\ell,m_\ell)$, $1 \le \ell \le n$, are $r \times m_\ell$ blocks given by

$$(2.9c)\quad B(z,\kappa_\ell,m_\ell) = \left[\begin{bmatrix} R_0(z,\kappa) \\ R_{-1}(z,\kappa) \\ \vdots \\ R_{-r+1}(z,\kappa) \end{bmatrix}, \frac{\partial}{\partial\kappa}\begin{bmatrix} R_0(z,\kappa) \\ R_{-1}(z,\kappa) \\ \\ R_{-r+1}(z,\kappa) \end{bmatrix}, \ldots, \frac{\partial^{m_\ell-1}}{\partial\kappa^{m_\ell-1}}\begin{bmatrix} R_0(z,\kappa) \\ R_{-1}(z,\kappa) \\ \vdots \\ R_{-r+1}(z,\kappa) \end{bmatrix}\right]_{\kappa=\kappa_\ell} .$$

We recall that $g$ is an eigenvector of $G$ if and only if (2.8) has a nontrivial solution, i.e., if $D$ is singular. This gives us

LEMMA 2. Let $z$ with $|z| \geq 1$ be given, and let $\kappa_\ell$ with $0 < |\kappa_\ell| < 1$, $1 \leq \ell \leq n$, be the corresponding distinct roots of the characteristic equation (2.1), each with multiplicity $m_\ell$. Then $z$ is an eigenvalue of $G$ if and only if

$$\det D(z\ ;\ \kappa_1,\ldots,\kappa_n\ ;\ m_1,\ldots,m_n) = 0 \quad .$$

Defining a partition of $r$ to be any set of integers, $\{m_\ell\}_{\ell=1}^q$, which satisfy

$$1 \leq m_1 \leq m_2 \leq \cdots \leq m_q \quad \text{and} \quad m_1 + m_2 + \cdots + m_q = r \quad ,$$

we are now ready to state our basic boundary-dependent criterion.

THEOREM 2. The approximation (1.2), (1.3) of the initial value problem (1.1) is stable if for any $z$ with $|z| \geq 1$, any partition $\{m_\ell\}_{\ell=1}^q$ of $r$ and any distinct values $\{\kappa_\ell\}_{\ell=1}^q$ with $0 < |\kappa_\ell| < 1$,

$$\det D(z\ ;\ \kappa_1,\ldots,\kappa_q\ ;\ m_1,\ldots,m_q) \neq 0 \quad .$$

Proof. Take an arbitrary $z$ with $|z| \geq 1$ and let $\kappa_\ell = \kappa_\ell(z)$, $1 \leq \ell \leq n$, be the corresponding distinct roots of (2.1), each with $0 < |\kappa_\ell| < 1$ and multiplicity $m_\ell$. We may assume that the $m_\ell$ are ordered, so by the hypothesis,

$$\det D(z\ ;\ \kappa_1,\ldots,\kappa_n\ ;\ m_1,\ldots,m_n) \neq 0 \quad .$$

Hence, by Lemma 2, $z$ is not an eigenvalue of $G$. Since $z$ was arbitrary, Theorem 1 assures stability, and the proof is complete.

Theorem 2 is simplified when the boundary conditions are of the one-level type

$$(2.10)\qquad \sum_{j=0}^{s} c_{\mu j} v_{\mu+j}(t) = 0 \quad , \quad c_{\mu 0} \neq 0 \; , \quad \mu = 0, \ldots, -r+1 \quad .$$

In this case the associated functions (2.7) are

$$R_\mu(\kappa) = \sum_{j=0}^{s} c_{\mu j} \kappa^{\mu+j} \quad , \quad \mu = 0, \ldots, -r+1 \quad ,$$

and $D$ of (2.9) is given by

$$(2.11a)\qquad D \equiv D(\kappa_1, \ldots, \kappa_n \; ; \; m_1, \ldots, m_n) \equiv [B(\kappa_1, m_1), \ldots, B(\kappa_n, m_n)]$$

with

$$(2.11b)\qquad B(\kappa_\ell, m_\ell) = \left[ \begin{bmatrix} R_0(\kappa) \\ \cdot \\ \cdot \\ \cdot \\ R_{-r+1}(\kappa) \end{bmatrix}, \frac{d}{d\kappa} \begin{bmatrix} R_0(\kappa) \\ \cdot \\ \cdot \\ \cdot \\ R_{-r+1}(\kappa) \end{bmatrix}, \ldots, \frac{d^{m_\ell - 1}}{d\kappa^{m_\ell - 1}} \begin{bmatrix} R_0(\kappa) \\ \cdot \\ \cdot \\ \cdot \\ R_{-r+1}(\kappa) \end{bmatrix} \right]_{\kappa = \kappa_\ell} \quad , \; 1 \leq \ell \leq n \; .$$

The matrix $D$ of (2.11) no longer depends on $z$, hence Theorem 2 becomes z-independent, and we have

COROLLARY 1. The approximation (1.2), (2.10) is stable if for any partition $\{m_\ell\}_{\ell=1}^{q}$ of $r$ and distinct values $\{\kappa_\ell\}_{\ell=1}^{q}$ with $0 < |\kappa_\ell| < 1$,

$$\det D(\kappa_1, \ldots, \kappa_q \; ; \; m_1, \ldots, m_q) \neq 0 \quad .$$

## 3. Translatory Boundary Conditions.

In practice, the boundary values $v_\mu$, $\mu = 0, \ldots, -r+1$, are usually determined by a repeated procedure; that is, the boundary conditions are of the translatory form

$$\text{(3.1)} \quad \sum_{j=0}^{s} c_j^{(1)} v_{\mu+j}(t+\Delta t) = \sum_{j=0}^{s} c_j^{(0)} v_{\mu+j}(t) \quad , \quad \mu = 0, \ldots, -r+1 \quad ,$$

where the coefficients $c_j$ are independent of $\mu$.

The rational boundary-functions associated with (3.1) are

$$R_\mu(z,\kappa) = \sum_{j=0}^{s} (zc_j^{(1)} - c_j^{(0)})\kappa^{\mu+j} \quad , \quad \mu = 0, \quad \ldots, -r+1 \quad .$$

Consequently

$$R_\mu(z,\kappa) = \kappa^\mu R_0(z,\kappa) \quad , \quad \mu = 0, \ldots, -r+1 \quad ,$$

so the $r \times r$ matrix

$$D \equiv D(z \; ; \; \kappa_1,\ldots,\kappa_q \; ; \; m_1,\ldots,m_q) \equiv [B(z,\kappa_1,m_1),\ldots,B(z,\kappa_q,m_q)]$$

of (2.9), is given by the $r \times m_\ell$ blocks

$$B(z,\kappa_\ell,m_\ell) = \left[\begin{bmatrix} R_0(z,\kappa) \\ \kappa^{-1}R_0(z,\kappa) \\ \cdot \\ \cdot \\ \cdot \\ \kappa^{-r+1}R_0(z,\kappa) \end{bmatrix}, \frac{\partial}{\partial\kappa}\begin{bmatrix} R_0(z,\kappa) \\ \kappa^{-1}R_0(z,\kappa) \\ \cdot \\ \cdot \\ \cdot \\ \kappa^{-r+1}R_0(z,\kappa) \end{bmatrix}, \ldots, \frac{\partial^{m_\ell-1}}{\partial\kappa^{m_\ell-1}}\begin{bmatrix} R_0(z,\kappa) \\ \kappa^{-1}R_0(z,\kappa) \\ \\ \\ \kappa^{-r+1}R_0(z,\kappa) \end{bmatrix}\right]_{\kappa=\kappa_\ell} ,$$

$$1 \leq \ell \leq q \quad .$$

The fact that $D$ is determined now by the single boundary-function $R_0(z,\kappa)$, implies the following significant simplification of Theorem 2.

THEOREM 3. The approximation (1.2), (3.1) of (1.1) is stable if for any $z$ with $|z| \geq 1$ and $\kappa$ with $0 < |\kappa| < 1$,

$$R_0(z,\kappa) \equiv \sum_{j=0}^{s} (zc_j^{(1)} - c_j^{(0)})\kappa^j \neq 0 \quad . \tag{3.2}$$

<u>Proof</u>. Take an arbitrary $z$ with $|z| \geq 1$, a partition $\{m_\ell\}_{\ell=1}^{q}$ of $r$, and distinct values $\kappa_\ell$, $1 \leq \ell \leq q$, with $0 < |\kappa_\ell| < 1$. In order to prove stability, it suffices, by Theorem 2, to show that

$$\det D(z\ ;\ \kappa_1,\ldots,\kappa_q\ ;\ m_1,\ldots,m_q) \neq 0 \quad . \tag{3.3}$$

For this purpose, let

$$\sum_{\mu=-r+1}^{0} \alpha_\mu \begin{bmatrix} \kappa_1^\mu R_0(z,\kappa_1) \\ \vdots \\ \dfrac{\partial^{m_q-1}}{\partial\kappa_q^{m_q-1}}[\kappa^\mu R_0(z,\kappa_q)] \end{bmatrix}' = 0 \quad , \tag{3.4}$$

be a vanishing linear combination of the rows of $D$ (where we denote the transpose of a vector by a prime). The vector relation in (3.4) consists of $r$ scalar equations,

$$\sum_{\mu=-r+1}^{0} \alpha_\mu \frac{\partial^k}{\partial\kappa_\ell^k} [\kappa_\ell^\mu R_0(z,\kappa_\ell)] = 0 \quad , \quad 1 \leq \ell \leq q \quad , \quad 0 \leq k \leq m_\ell - 1 \quad ,$$

which we write as

$$\frac{\partial^k}{\partial\kappa_\ell^k}\left[\kappa_\ell^{-r+1} R_0(z,\kappa_\ell) \cdot \sum_{\mu=r+1}^{0} \alpha_\mu \kappa_\ell^{r+\mu-1}\right] = 0 \quad , \quad 1 \leq \ell \leq q \quad , \quad 0 \leq k \leq m_\ell - 1 \ . \tag{3.5}$$

Since $0 < |\kappa_\ell| < 1$, then by hypothesis, the left member in the above brackets satisfies

$$\kappa_\ell^{-r+1} R_0(z,\kappa_\ell) \neq 0 \quad , \quad 1 \leq \ell \leq q \quad .$$

Thus, expending by Leibniz' rule and using induction on $k \geq 0$, we find that the right member in (3.5) has vanishing derivatives at $\kappa = \kappa_\ell$, i.e.,

$$\frac{d^k}{d\kappa_\ell^k}\left[\sum_{\mu=-r+1}^{0} \alpha_\mu \kappa_\ell^{r+\mu-1}\right]_{\kappa=\kappa_\ell} = 0 \quad , \quad 1 \leq \ell \leq q \quad , \quad 0 \leq k \leq m_\ell - 1 \quad .$$

We conclude that the polynomial

$$P(\kappa) \equiv \sum_{\mu=-r+1}^{0} \alpha_\mu \kappa^{r+\mu-1} \quad ,$$

which is of degree $r - 1$ at most, has $r$ roots; $\kappa_\ell$, $1 \leq \ell \leq q$, each with multiplicity $m_\ell$. Hence, $P(\kappa) \equiv 0$ and the coefficients $\alpha_\mu$ must vanish. By (3.4), therefore, the rows of $D$ are linearly independent, so (3.3) follows, and stability is established.

As was realized in the previous section, if (3.1) is reduced to the one-level case

$$(3.6) \qquad \sum_{j=0}^{s} c_j v_{\mu+j}(t) = 0 \quad , \quad c_0 \neq 0 \quad , \quad \mu = 0, \ldots, -r+1 \quad ,$$

then the boundary-functions

$$R_\mu(\kappa) = \sum_{j=0}^{s} c_j \kappa^{\mu+j} \quad , \quad \mu = 0, \ldots, -r+1 \quad ,$$

cease to depend on $z$, and Theorem 3 provides us with

COROLLARY 2. The approximation (1.2), (3.6) is stable if

$$R_0(\kappa) \equiv \sum_{j=0}^{s} c_j \kappa^j \neq 0 \quad , \forall \kappa \text{ with } 0 < |\kappa| < 1 \quad .$$

Before turning to examples, we remark that the case $r = 1$ (which is important if only because it applies to the dissipative Lax-Wendroff scheme, [6]) is translatory by definition. In this case, however, the results of Theorems 3 and Corollary 2 are given, respectively, by Theorems 2 and Corollary 1 since for $r = 1$ the matrix $D$ is reduced to the scalar function $R_0$.

EXAMPLE 1. Let the boundary values be determined by an interpolation polynomial of arbitrary degree $s - 1$. More specifically, we extrapolate at each time step from $v_1(t), \ldots, v_s(t)$ to $v_0(t)$; then from $v_0(t), \ldots, v_{s-1}(t)$ to $v_{-1}(t)$, etc., to obtain the one-level translatory procedure

$$(3.7) \qquad v_\mu(t) = \sum_{j=1}^{s} \binom{s}{j}(-1)^{j+1} v_{\mu+j}(t) \quad , \quad \mu = 0, \ldots, -r+1 \quad .$$

The associated boundary-function is

$$R_0(\kappa) = 1 - \sum_{j=1}^{s} \binom{s}{j}(-1)^{j+1}\kappa^j = (1-\kappa)^s \quad ;$$

thus $R_0(\kappa) \neq 0$ for $\kappa$ with $0 < |\kappa| < 1$, and Corollary 2 implies the stability of (1.2), (3.7).

This result was stated by Kreiss in [4] and was proved later by Goldberg, [1].

In the remaining examples of this section we analyse two-level boundary conditions. Examples 2, 3 and 5 were studied by Gustafsson et al. [3]; and Example 4 by Sköllermo [7]. In the above references, however, the authors did not consider general basic schemes, and attention was restricted to specific approximations such as Lax-Wendroff, Leap-Frog and Crank-Nicolson.

**EXAMPLE 2.** Consider boundary values which are defined by extrapolating along the mesh-digonal. The two-level version of this procedure is of zero-order accuracy, and is given by

$$(3.8) \qquad v_\mu(t + \Delta t) = v_{\mu+1}(t) \quad , \quad \mu = 0, \ldots, -r + 1 \ .$$

The corresponding boundary-function is simply

$$R_0(z,\kappa) = z - \kappa \ .$$

Hence,

$$R_0(z,\kappa) \neq 0 \quad , \quad \forall |z| \geq 1 \quad , \quad |\kappa| < 1 \quad ,$$

and by Theorem 3 the approximation (1.2), (3.8) is stable.

Note that (3.8) is inconsistent with the differential equation (1.1), unless $\lambda \equiv \Delta t/\Delta x$ satisfies $\lambda a = 1$, where (3.8) coincides with our next example.

**EXAMPLE 3.** Here, the boundary conditions are generated by the right-sided, first-order accurate, explicit Euler scheme, which yields

$$(3.9) \; v_\mu(t + \Delta t) = v_\mu(t) + \lambda a[v_{\mu+1}(t) - v_\mu(t)] \; , \; \lambda = \Delta t/\Delta x \; , \; \mu = 0, \ldots, -r + 1 \ .$$

We have

$$R_0(z,\kappa) = z - 1 + \lambda a(1 - \kappa) \quad ,$$

where in order to secure stability of the basic scheme (1.2), one may assume that

$$0 < \lambda a \leq 1 \quad .$$

It follows that for all $z, \kappa$ with $|z| \geq 1, |\kappa| < 1,$

$$|R_0(z,\kappa)| = |z - \lambda a\kappa - (1 - \lambda a)| > |z| - |\lambda a\kappa| - |1 - \lambda a| =$$

$$= |z| - \lambda a|\kappa| - 1 + \lambda a = (|z| - 1) + \lambda a(1 - |\kappa|) > 0 \quad .$$

Thus, the hypothesis of Theorem 3 is fulfilled, and stability of (1.2), (3.9) follows.

EXAMPLE 4. Take

(3.10) $v_\mu(t + \Delta t) - \lambda a[v_{\mu+1}(t + \Delta t) - v_\mu(t + \Delta t)] = v_\mu(t) \; , \; \mu = 0, \ldots, -r + 1 \; ,$

which is determined by the right-sided implicit Euler scheme. Since

$$R_0(z,\kappa) = z(1 + \lambda a - \lambda a\kappa) - 1 \quad , \qquad (\lambda a > 0) \quad ,$$

then for $|z| \geq 1, \quad |\kappa| < 1,$

$$|R_0(z,\kappa)| \geq |z| \cdot |1 + \lambda a - \lambda a\kappa| - 1 \geq |1 + \lambda a - \lambda a\kappa| - 1 \geq$$

$$\geq 1 + \lambda a - |\lambda a\kappa| - 1 \geq \lambda a(1 - |\kappa|) > 0 \quad ,$$

and Theorem 3 implies stability.

Although the above examples follow without difficulty, it is clear that the more complex the boundary conditions are,the harder it is to verify the hypotheses of Theorem 3 or Corollary 2. This is particularly true in the two-level case

$$(3.11) \quad \sum_{j=0}^{s} c_j^{(1)} v_{\mu+j}(t+\Delta t) = \sum_{j=0}^{s} c_j^{(0)} v_{\mu+j}(t) \ , \ c_0^{(1)} \neq 0, \ \mu = 0, \ldots, -r+1 \ ,$$

where the relevant stability criterion in Theorem 3 involves <u>two</u> independent variables, $z$ and $\kappa$.

In order to provide a z-independent alternative to Theorem 3, we consider the <u>boundary-scheme</u>

$$(3.12) \quad \begin{aligned} & S^{(1)} v_\nu(t+\Delta t) = S^{(0)} v_\nu(t) \ , \quad \nu = 0, \pm 1, \pm 2, \ldots, \\ & S^{(\ell)} \equiv S^{(\ell)}(E) = \sum_{j=0}^{s} c_j^{(\ell)} E^j \ , \quad \ell = 1, 2 \ , \end{aligned}$$

which generates (3.11) by restricting $\nu$ to the values $0, -1, \ldots, -r+1$.

We prove

THEOREM 4. <u>Let the boundary-scheme</u> (3.12)

(i) <u>*satisfy the solvability-condition*</u>

$$(3.13) \quad S^{(1)}(\kappa) \equiv \sum_{j=0}^{s} c_j^{(1)} \kappa^j \neq 0 \ , \quad \forall |\kappa| \leq 1 \ ;$$

(ii) <u>be stable (in the usual Von Neumann sense</u>.)

<u>Then</u> (1.2), (3.11) <u>is a stable approximation to</u> (1.1).

<u>Proof</u>. The amplification factor of (3.12) is given by

$$\hat{S}(\xi) = \hat{S}^{(0)}(\xi)/\hat{S}^{(1)}(\xi) \ , \quad |\xi| \leq \pi \ .$$

where

$$\hat{S}^{(\ell)}(\xi) = S^{(\ell)}(e^{i\xi}) \ , \quad \ell = 1, 2 \ .$$

Since (3.13) implies

$$(3.14a) \quad \hat{S}^{(1)}(\xi) \neq 0 \ ,$$

then this amplification factor is well defined, and by hypothesis (ii) it satisfies

$$|\hat{S}(\xi)| \leq 1 \quad , \quad \forall \xi \quad ,$$

which we write as

(3.14b) $$|\hat{S}^{(0)}(\xi)| \leq |S^{(1)}(\xi)| \quad .$$

The boundary-function associated with (3.11) is

(3.15) $$R_0(z,\kappa) = \sum_{j=0}^{s} (zc_j^{(1)} - c_j^{(0)}) = zS^{(1)}(\kappa) - S^{(0)}(\kappa) \quad .$$

So, for $|z| > 1$, we use (3.14) to find that

$$|R_0(z,e^{i\xi})| = |zS^{(1)}(e^{i\xi}) - S^{(0)}(e^{i\xi})| = |z\hat{S}^{(1)}(\xi) - \hat{S}^{(0)}(\xi)| \geq$$

$$\geq |z| \cdot |\hat{S}^{(1)}(\xi)| - |\hat{S}^{(0)}(\xi)| > 0 \quad .$$

That is, the equation

(3.16) $$R_0(z,\kappa) = 0 \quad \text{with} \quad |z| > 1 \quad ,$$

has no roots $\kappa$ with $|\kappa| = 1$. Since the roots of (3.16) are continuous functions of $z$, we conclude that the number of $\kappa$ with $|\kappa| < 1$ is fixed for $|z| > 1$, and can be determined by considering large values of $|z|$. Writing (3.16) in the form

$$\hat{S}^{(1)}(\kappa) - z^{-1}\hat{S}^{(0)}(\kappa) = 0 \quad ,$$

We let $|z| \to \infty$ and use (3.13) to find that (3.16) has no roots in the unit disc. In other words, if $|z| > 1$ and $R_0(z,\kappa) = 0$, then $\kappa$ must satisfy $|\kappa| > 1$. By continuity therefore, if $|z| \geq 1$ and $R_0(z,\kappa) = 0$, then $|\kappa| \geq 1$; i.e.,

$$R_0(z,\kappa) \neq 0 \quad , \quad \forall |z| \geq 1 \quad , \quad |\kappa| < 1 \quad .$$

This implies (3.2), and Theorem 3 completes the proof.

We note that the stability properties of the boundary-scheme are often known in advance, so in applying Theorem 4 it remains to verify only the solvability condition (3.13). Moreover, if the boundary-scheme is explicit, i.e., the boundary condition are of the form

$$(3.17) \qquad v_\mu(t + \Delta t) = \sum_{j=0}^{s} c_j v_{\mu+j}(t) \quad , \quad \mu = 0, \ldots, -r + 1 \quad ,$$

then (3.13) is fulfilled automatically, and Theorem 4 amounts to

COROLLARY 3. If (3.17) is generated by a stable scheme (in the usual Von Neumann sense), then (1.2), (3.17) is stable.

Reviewing Example 2 in light of our new results, it is clear that the generating boundary-scheme,

$$v_\nu(t + \Delta t) = v_{\nu+1}(t) \quad , \quad \nu = 0, \pm 1, \pm 2, \ldots,$$

is unconditionally stable, hence Corollary 3 assures stability.

In Example 3, the explicit Euler scheme is known to be stable for $\lambda a \leq 1$, and Corollary 3 again implies stability.

Finally, the implicit Euler scheme of Example 4 is unconditionally stable, so to comply with Theorem 4, one has to verify that

$$S^{(1)}(\kappa) \equiv 1 - \lambda a(\kappa - 1) \neq 0 \quad , \quad \forall |\kappa| \leq 1 \quad .$$

Since $\lambda a > 0$, then $\kappa$ with $|\kappa| \leq 1$ gives

$$\text{Re } S^{(1)}(\kappa) = 1 + \lambda a(1 - \text{Re } \kappa) \geq 1 \quad ,$$

and the result holds.

In concluding this section we present a somewhat more intricate example due to Gustafsson et al. [3].

EXAMPLE 5. We use the Box-Scheme to generate

$$(3.18)\quad v_\mu(t+\Delta t) + v_{\mu+1}(t+\Delta t) - \lambda a[v_{\mu+1}(t+\Delta t) - v_\mu(t+\Delta t)] =$$

$$= v_\mu(t) + v_{\mu+1}(t) + \lambda a[v_{\mu+1}(t) - v_\mu(t)] \quad , \quad \mu = 0, \ldots, -r+1 \quad .$$

The amplification factor is $\hat{S}(\xi) \equiv 1$. Hence, our boundary-scheme is unconditionally stable, and by Theorem 4 it remains to check whether

$$S^{(1)}(\kappa) \equiv 1 + \kappa - \lambda a(\kappa - 1) \neq 0 \quad , \quad \forall |\kappa| \leq 1 \quad .$$

Indeed,

$$\text{Re } S^{(1)} = 1 + \text{Re } \kappa + \lambda a(1 - \text{Re } \kappa) > 0 \quad ,$$

and stability follows.

## 4. The Inflow case, $a < 0$.

This last section is devoted to remarks concerning the inflow problem

$$(4.1a)\quad \partial u/\partial t = a\partial u/\partial x \; ; \; a < 0 \; ; \; x \geq 0 \; , \; t \geq 0 \; ; \; u(x,t) = f(x) \quad ,$$

where

(4.1b) adequate conditions at $x = 0$ are given .

In order to approximate the new problem, we assume that the basic scheme (1.2) is consistent with the differential equation in (4.1a), and that the rest of the set-up in Section 1 is maintained. We are aware of course, of the fact that quite often conditions (4.1b) are nonhomogeneous, hence the homogeneous boundary conditions in (1.3) may not apply.

Our new situation does not affect Kreiss' criterion or Lemma 1.

The first deviation from the previous discussion, occurs in the argument leading to Lemma 2: The case $|z| \geq 1$, $z \neq 1$, goes over unchanged; but as $z \to 1$, Lemma 1 implies that one of inner roots, $\kappa_1$, of the characteristic equation (2.1), tends to 1 as well. According to Kreiss' theory this root has to be incorporated in the solution g of (2.3), thus g may fail to belong to $\ell_2$, and $z = 1$ is then a generalized eigenvalue of the operator G. As a result, Lemma 2 takes the form

LEMMA 2'. (i) Let z with $|z| \geq 1$, $z \neq 1$, be given, and let $\kappa_\ell$ with $0 < |\kappa_\ell| < 1$, $1 \leq \ell \leq n$, be the corresponding distinct roots of (2.1), each with multiplicity $m_\ell$. Then z is an eigenvalue of G if and only if the matrix D of (2.9) satisfies

$$\det D(z\ ;\ \kappa_1,\ldots,\kappa_n\ ;\ m_1,\ldots,m_n) = 0 \quad .$$

(ii) Let $\kappa_\ell = \kappa_\ell(1)$, $2 \leq \ell \leq n$, be the roots of (2.1) which corresponds to $z = 1$, each with multiplicity $m_\ell$. Then, $z = 1$ is a generalized eigenvalue of G if and only of

$$\det(1\ ;\ 1,\kappa_2,\ldots,\kappa_n\ ;\ 1,m_2,\ldots,m_n) = 0 \quad .$$

Having Lemma 2', the inflow version of Theorems 2 and 3 follow easily along the line of the original proofs. We first state

THEOREM 2'. The approximation (1.2), (1.2) of the initial boundary value problem (4.1) is stable if (i) for any z with $|z| \geq 1$, $z \neq 1$, any partition $\{m_\ell\}_{\ell=1}^{q}$ of r, and arbitrary distinct values $\{\kappa_\ell\}_{\ell=1}^{q}$ with $0 < |\kappa_\ell| < 1$,

$$\det D(z\ ;\ \kappa_1,\ldots,\kappa_q\ ;\ m_1,\ldots,m_q) \neq 0 \quad ;$$

(ii) <u>for any partition</u> $\{m_\ell\}_{\ell=2}^{q}$ <u>of</u> $r - 1$ <u>and distinct</u> $\{\kappa_\ell\}_{\ell=2}^{q}$ <u>with</u> $0 < |\kappa_\ell| < 1$,

$$\det D(1 ; 1,\kappa_2,\ldots,\kappa_q ; 1,m_2,\ldots,m_q) \neq 0 \quad .$$

As before, if the boundary conditions are of the one-level type (2.10), then $D$ is given by (2.11), and we get a z-independent version of Corollary 1 which is left to the reader.

Next, in the translatory case, we replace Theorem 3 by

THEOREM 3'. <u>The approximation</u> (1.2), (3.1) <u>of</u> (4.1) <u>is stable if the boundary-function in</u> (3.2) <u>satisfies</u> (i) $R_0(z,\kappa) \neq 0$ <u>for all</u> $|z| \geq 1$, $0 < |\kappa| < 1$; (ii) $R_0(1,1) \neq 0$.

REMARKS. (a) If $r = 1$, then Lemma 1 suggests that $\kappa = 1$ is the only relevant root for $z = 1$. Hence, hypothesis (i) of Theorem 3' can be restricted to $z$ with $|z| \geq 1$, $z \neq 1$.

(b) Had the boundary-scheme which generates (3.1) been consitent with (4.1a), we would have had the zero-moment condition

$$\sum_{j=0}^{s} c_j^{(1)} = \sum_{j=0}^{s} c_j^{(0)} \quad ,$$

which is equivalently written as

$$R_0(1,1) = 0 \quad .$$

We see that if hypothesis (ii) of Theorem 3' holds, then the boundary-scheme must be <u>inconsistent</u>, and in fact may have no accuracy with respect to the differential equation.

(c) If (3.1) reduces to the one-level case (3.6), then Theorem 3' simplifies and we obtain

COROLLARY 2'. The approximation (1.2), (3.6) is stable if $R_0(\kappa) \neq 0$ for all $\kappa$ with $0 < |\kappa| < 1$ or $\kappa = 1$.

EXAMPLE 6. Let the boundary conditions be of the simple form

$$\sum_{j=0}^{s} v_{\mu+j}(t) = 0 \quad , \quad \mu = 0, \ldots, -r+1 \quad . \tag{4.2}$$

These homogeneous sums are suitable in case that the analytic conditions (4.1b) are given, for instance, by $u(0,t) = 0$ or $\partial u(0,t)/\partial x = 0$. We have

$$R_0(\kappa) = \sum_{j=0}^{s} \kappa^j \quad ;$$

so $R_0(\kappa) \neq 0$ for all $\kappa \neq 0$, and again, by Corollary 2', the stability of (1.2), (4.2), is assured. This last result is also due to Kreiss, [4].

Last, we consider the boundary conditions in (3.11), and state the following inflow-counterpart of Theorem 4.

THEOREM 4'. Let the boundary-scheme (3.12)

(i) staisfy the solvability-criterion $S^{(1)}(\kappa) \neq 0$ for all $|\kappa| \leq 1$;

(ii) satisfy the inconsistency-condition $R_0(1,1) \neq 0$;

(iii) be stable(in the usual Von Neumann sense.)

Then (1.2), (3.11) is a stable approximation of (4.1).

Proof. Under hypotheses (i) and (iii), the proof of Theorem 4 implies that

$$R_0(z,\kappa) \neq 0 \quad , \quad \forall |z| \geq 1 \quad , \quad |\kappa| < 1 \quad . \tag{4.3}$$

Hypothesis (ii) (which in paractice can be easily checked by inspection) together with (4.3), constitute the assumptions of Theorem 3', and stability follows.

We conclude by recalling that when the boundary scheme is explicit and the boundary conditions are given by (3.17), hypothesis (i) of Theorem 4' becomes redundant.

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