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# LIMIT THEOREMS FOR THE SIMPLE BRANCHING PROCESS ALLOWING IMMIGRATION: A REVIEW AND NEW RESULTS

## I. THE CASE OF FINITE OFFSPRING MEAN

by

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## ABSTRACT

This paper reviews known limit theorems for the population sizes of a Bienaymé-Galton-Watson process allowing immigration. For the non-critical cases it is known that the limit distribution is non-defective iff a logarithmic moment of the immigration distribution is finite. The new results of this paper are concerned with the situation where this moment is infinite and give limit theorems for a certain slowly varying function of the population size. A parallel discussion is given for the critical case and also for the continuous time process.

The methods of the paper are used to give some results on the rate of decay of the transition probabilities and on the growth rate of the stationary measure. These in turn are used to obtain some limit theorems for a reversed time process. Known results and applications are reviewed.



Key words: Bienaymé-Galton-Watson, immigration, limit theorem, non-linear norming, Markov branching process, slow variation, stationary measure, reversed time.

## 1. Introduction

We shall consider the Bienaymé-Galton-Watson process allowing immigration (BPI) which is a Markov chain, $\chi = \{X_n : n=0,1,\ldots\}$, with minimal state space $\mathcal{S} \in \mathbb{N} = \{0,1,\ldots\}$ and one step transition probabilities given by

$$p_{ij} = \text{coeff. of } s^j \text{ in } h(s)(f(s))^i \qquad (i,j \in \mathbb{N})$$

where $h(s) = \sum h_j s^j$ and $f(s) = \sum p_j s^j$ $(\sum(\cdot)_j = \sum_{j \geq 0} (\cdot)_j)$ are probability generating functions (pgf's) and $h_0 < 1$, $p_j \neq 1$. The process $\chi$ is usually thought as describing the evolution of the size of a population in which each individual produces $j$ progeny with probability $p_j$ at (or by) the end of its life and time is measured in generations. With probability $h_j$, $j$ immigrants enter the nth generation and contribute to the (n+1)th generation in the same way as natives. All individuals reproduce independently of each other and of the immigration process, and the numbers immigrating into successive generations are independent. Thus $X_n$ represents the total number of individuals in the nth generation. This intuitive picture implies the recursive relationship

$$X_{n+1} = \sum_{i=1}^{X_n} \xi_i^{(n)} + I_{n+1} \qquad (n \in \mathbb{N}) \tag{1}$$

where $\{\xi_i^{(n)} ; i \in \mathbb{N}_+, n \in \mathbb{N}\}$ $(\mathbb{N}_+ = \{1,2,\ldots\})$ are independent and have pgf $f(\cdot)$, $\{I_n : n \in \mathbb{N}_+\}$ are independent with pgf $h(\cdot)$ and both sequences are independent.

The process $\chi$ has been studied in some detail since the mid 1960s but its history goes back much earlier; special cases have been used from time to time since the early days of stochastic processes. In 1915 Smoluchowski set up the following model for the number of particles in a small region $\Omega$ of a volume of fluid whose molecules subject the particles to independent Brownian motions. He assumed that the number of particles in $\Omega$ has an equilibrium Poisson distribution with parameter $\nu$. The population of $\Omega$ is observed at time instants spaced $\tau$ apart and there is a probability P, depending on the detailed physics of the situation, that a given particle in $\Omega$ at $n\tau$ has departed by $(n+1)\tau$. Independence and statistical equilibrium then imply that the number of particles diffusing into $\Omega$ during $(n\tau, (n+1)\tau)$ has pgf $h(s) = \exp[-\nu P(1-s)]$ and hence if $X_n$ denotes the particle population size in $\Omega$ at $n\tau$, then $\chi$ is a stationary BPI with $f(s) = P + (1-P)s$. Experimental observations have confirmed that the model is reliable and moreover they gave an independent estimate of Avogadro's number. The model was also used to resolve certain paradoxes which arose in the early development of statistical mechanics. We refer the reader to [3, §3] for a detailed account and further references. Situations similar to that here have arisen in other contexts, for example, in estimating the speed of spermatozoa [41]; see [20] [21] for further discussion and references. Smoluchowski's model is also referred to in an exercise in [12, Chap. 15].

The BPI has also been suggested as a model for the population size of rare mutant genes or bacteria that are produced by an effectively infinite population of normal individuals [16][50]. Another biological application is to the release of quanta of transmitter from a nerve terminal under the stimulus of an externally applied electrical pulse [53]. This model is very similar to

Smoluchowski's. The quanta of transmitter are viewed as being contained in vesicles which diffuse into the region of the nerve terminal and the numbers entering the region between pulse applications are assumed to be independent Poisson variates. When the field is applied, each vesicle is assumed to discharge its transmitter through the terminal independently of other vesicles and with a constant probability. If $X_n$ is the number of vesicles in the terminal region after the nth pulse, $\{X_n\}$ is a BPI.

Dunne and Potts [7] proposed the following model of a computer controlled intersection consisting of two signalized entries. The time domain is discrete and it is assumed that traffic approaches the intersection along each entry according to independent Bernoulli processes. If an entry has the green light, this remains until the waiting vehicle queue is empty, at which time the lights change. Each crossing phase consists of an initial fixed period during which no crossings occur and a random crossing period during which one car crosses the intersection at each time unit until there are no more waiting vehicles. The stochastic process $\{Q^{(1)}(n), Q^{(2)}(n)\}$, where $Q^{(i)}(n)$ is the number of waiting vehicles at entry i at time unit n, is a two dimensional random walk. If $N_n^{(i)}$ is the time unit at which the light changes to green for the nth time at entry i, then the imbedded processes $\{Q^{(i)}(N_n^{(i)})\}$ $(i=1,2)$ are BPI's. So also are $\{L_n^{(i)}\}$ where $L_n^{(i)}$ is the length of the crossing portion of the nth green phase at entry i. The BPI also occurs as an imbedded process in the one channel version of a gated queue arrangement considered by Leibowitz [26].

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Electrical Engineering Laboratories at Monash University. Consider a Poisson process and an independent renewal process, each of whose events are registered by separate counters. The Poisson process events are accumulated in its counter in the normal way but the renewal process subtracts from its counter. When this counter empties, the device emits an output pulse, adds $r > 0$ counts to the accumulating counter and then interchanges the input processes so that the renewal process now subtracts counts from what was previously the accumulating counter. If $X_n$ denotes the total count in the accumulating counter at the time of the nth output pulse, then $\{X_n\}$ is a BPI. In practice both input processes are Poisson which are realized by using radioactive sources and Geiger counters. Lee [25] has given an exhaustive treatment of this particular case.

Finally we mention two rather more theoretical instances of the BPI. It has been suggested in the dam theory literature [36] that if $\{J_n\}$ represents successive inflows to a dam, then there are good reasons for requiring that this sequence should possess the linear regression property

$$E(J_{n+1} | J_n)) = aJ_n + b \qquad (0 < a < 1).$$

Most of the models used to realize this in [36] are particular BPI's. A more general analysis was subsequently given in [30] and some further aspects are considered in [52]. Let $\{S_n\}$ be a simple random walk with $p > 1/2$ being the probability of a jump to the right. We say that an over-crossing of height $a$ takes place at time $n$ if $S_n = a$, $S_{n-1} = a+1$ and $S_i = a$ for some $i < n$. If $N(a)$ is the total number of over-crossings of height $a$ then Dwass [8] has shown that $\{N(0), N(1), \ldots\}$ is a stationary BPI.

The recent interest in the BPI began with Heathcote's investigations [18][19]. These papers seem to be the first attempt at a general treatment in the open literature. It is of interest to note that a number of results on the existence of a limiting distribution and total progeny of multitype BPI's were obtained some twenty years earlier by Everett and Ulam [11]. Our interest will center on the limit behavior of the sequence $\chi$ and we shall review the known results in the following section. However, we shall give here the following comments in order to describe the new results in this paper.

Let $m = f'(1-)$ be the offspring mean. We shall consider three cases; $m < 1$ (subcritical); $m = 1$ (critical); and $1 < m < \infty$ (supercritical). We shall defer the explosive case $(m = \infty)$ to a sequel. Typically we try to norm the sequence $\chi$ so that it converges in some sense and in the sub- and supercritical cases it is known that this can be done if the following condition on $\mathcal{H} = \{h_j\}$ holds.

<u>Condition A:</u> $E(\log^+ I) < \infty$

where $I$ is a random variable with the distribution $\mathcal{H}$. When $m < 1$ the norming sequence is constant, thus $\mathcal{S}$ is positive if Condition A holds. If Condition A fails, then the normed version of $\chi$ still converges, but the limit distribution is defective. There is a similar but slightly more complicated situation for the critical case; namely there is a condition on $\mathcal{H}$ ensuring that $\mathcal{S}$ is positive. The question arises of discovering something about the growth rate of $\chi$ when, for example, Condition A does not hold.

The author has previously made some progress in this direction [33][34][35]. When $m \neq 1$ the author obtained a limit theorem under rather complicated conditions involving both $\mathcal{K}$ and $\mathcal{P} = \{p_0, p_1, \cdots\}$. He also quoted examples which indicated that the hypotheses could be modified to yield alternative limit theorems. The main task of this paper will be to unify and simplify these results. We shall distinguish four different modes of behavior of the function

$$G(x) = 1 - h(1 - e^{-x}).$$

For all results that we obtain we assume that Condition A does not hold and this is equivalent to the condition

$$\int_0^\infty G(x)dx = \infty. \tag{2}$$

The four modes of behavior just referred to are simply conditions on the rate of convergence to zero of $G(\cdot)$:

(i) $xG(x) \longrightarrow 0 \qquad (x \longrightarrow \infty)$;

(ii) $xG(x) \longrightarrow a \in (0, \infty) \qquad (x \longrightarrow \infty)$;

(iii) $G(x) = (x^\Delta L(x))^{-1}$

where $L(\cdot)$ is slowly varying (sv) at infinity and <u>either</u> $0 < \Delta < 1$ <u>or</u> $\Delta = 1$ <u>and</u> $L(x) \longrightarrow 0 \quad (x \longrightarrow \infty)$;

(iv) as in (iii) but $\Delta = 0$ <u>and</u> $L(x) \longrightarrow \infty$.

The conditions assumed by the author for his earlier results (Theorem 4 in [33] and Theorem 1 in [35]) are equivalent to Condition (i) and the quoted examples are specific instances of Conditions (ii) and (iii). The contribution of this paper will be to provide much tidier versions of the theorems cited in the previous references and with greatly simplified proofs. We shall also prove limit theorems for Conditions (ii) - (iv) and give examples which, hopefully, will illuminate the situation. When $m = 1$ we shall define a function, also to be denoted by $G(x)$, which involves both $\chi$ and $\varrho$ and prove limit theorems for Conditions (i) - (iv). The cases (i) - (iii) have been dealt with in [35]. The contribution of the present paper will be a simplified proof for (i) and a new corollary, improved versions of (ii) and (iii), and (iv) is new. The present version of the limit theorem for (ii), Theorem 10 below, is notable since it trivially contains the important Theorem D below.

We shall now sketch the nature of the results to be proved and for definiteness we shall focus attention on the supercritical case. In this case Cohn [5] has shown that if (2) holds then it is not possible to norm $\chi$ so as to produce a sequence which has a limit distribution which is non-defective but not degenerate at zero. A simpler proof is presented in the next section. The kernel of these proofs is to show that if such a norming sequence existed, denote it by $\{c_n\}$, then $c_{n+1}/c_n \longrightarrow m$, that is $c_n = O(m^{n(1+\varepsilon)})$. This shows that $\chi$ is growing faster than some geometric rate. At this point it is worth pointing out that another recent result of Cohn [6] shows that if a Markov chain can be normed by a sequence $\{a_n\}$ such that the normed process has a non-defective limit distribution, then necessarily $\limsup_{n\to\infty} a_{n+1}/a_n < \infty$, that is, it can grow at most geometrically fast. A possible approach is to "slow down" the growth rate of $\chi$ by considering a suitable nonlinear function of $X_n$ and then norming the resulting sequence. In fact the following version of this idea works. First we fix $\{c_n\}$ to be the sequence of Seneta constants, to be defined in §2. In Theorem C below we point out that if (2) holds then $X_n/c_n \xrightarrow{a.s.} \infty$. We shall show that under each of Conditions (i) - (iv) there is a sv function $\Lambda(\cdot)$ and a sequence $\{a_n\}$ such that $\{\Lambda(X_n/c_n)/a_n\}$ has a non-defective limit distribution. In all cases the limit distribution is given explicitly. When (2) holds we shall refer to the "immigration regime" since the growth rate of $\chi$ is being influenced mainly by $\mathcal{Y}$, whereas if (2) does not hold the immigration component of $\chi$ serves only to prevent extinction; the growth rate is determined predominantly by $\varrho$. We shall then refer to the "branching regime."

Limit theorems of this nature occur in other areas of probability theory. The simplest example occurs when we take the logarithm of a product of positive independent random variables and then invoke the central limit theorem. Erickson [9] has proved similar results for the forward and backward recurrence times of a renewal process whose mean lifetime is "just infinite." Finally such limit theorems occur quite naturally in the theory of explosive Galton-Watson process [2][22].

The format of the paper is as follows. In §2 we will review the known results for the branching regime, while in §3 we shall consider the immigration regime. This section shall be divided in three parts which deal respectively with the subcritical, supercritical and critical cases. In §4 we show that the methods used in §3 will easily yield analogous results for the Markov branching process allowing immigration (MBI). In §5 we shall use some of the results of §3 to obtain information on the rate at which the n-step transition probability, $p_{ij}^{(n)}$, decays to zero ($n \to \infty$) when $m \leq 1$. The case $m > 1$ has been covered in [27]. We shall then show how these results provide some information on the asymptotic behavior of the (unique) invariant measure. Finally, in §6, we shall use these results to prove some limit theorems for a reversed time BPI. These complement results obtained by Esty [10] for the non-immigration situation.

The numbering of equations is sequential within (sub) sections and multi-place labels will be used only for references to equations from other (sub) sections.

## 2. The branching regime

Let $[p_{ij}^{(n)}]$ $(n \in \mathbb{N}_+)$ denote the n-step transition matrix of $\chi$, let $P_i^{(n)}(s) = \sum p_{ij}^{(n)} s^j$ $(0 \leq s \leq 1)$ and let $f_0(s) = s$, $f_{n+1}(s) = f(f_n(s))$ $(n \in \mathbb{N})$. Equation (1.1) implies that

$$P_i^{(n)}(s) = (f_n(s))^i \prod_{k=0}^{n-1} h(f_k(s)). \tag{1}$$

If $p_0 = 0$, then $\chi$ is supercritical, $X_n \uparrow \infty$ $(n \to \infty)$ and hence all states are transient. If $p_0 > 0$ then [38] $\mathcal{S}$ is irreducible, aperiodic and contains $i^* = \min\{i \mid h_i > 0\}$. It is of interest to classify $\mathcal{S}$. When $m > 1$ and $p_0 > 0$ it follows from (1) that

$$p_{i^*i^*}^{(n)} \leq q^{i^*}(h(q))^n$$

where $q$ is the least positive solution of $f(s) = s$. Thus in this case $\mathcal{S}$ is transient, a fact first pointed out under slightly less general conditions in [27].

The situation is more complicated when $m \leq 1$. Heathcote [18][19] proved that when $m < 1$ and $\mathcal{S} = \mathbb{N}$, then $\mathcal{S}$ is positive iff Condition A holds and when this is the case, the pgf of the limiting distribution is given by

$$\prod(s) = \prod_{k=0}^{\infty} h(f_k(s)). \tag{2}$$

The condition $\mathcal{S} = \mathbb{N}$ was relaxed first by Seneta [43] and then to the present assumption by Quine [38]. We summarize this as

Theorem A. When $m < 1$, $\mathcal{S}$ is positive iff $E(\log^+ I) < \infty$.

Seneta [42] first pointed out through an example that when $m = 1$, it is possible for $\mathcal{S}$ to be positive. Foster and Williamson [15] proved that in all cases $\mathcal{S}$ is positive iff

$$\int_0^1 \frac{1-h(s)}{f(s)-s}\, ds < \infty. \tag{3}$$

We shall be concerned with the following specialization of the critical case:

$$f(s) = s + (1-s)^{1+\nu} \mathcal{L}(1-s) \tag{4}$$

where $0 < \nu \leq 1$ and $\mathcal{L}(\cdot)$ is sv at the origin. In this case we have the following analogue of Theorem A [34].

Theorem B. Suppose (4) holds and in addition if $\nu = 1$ then $\mathcal{L}(s) \longrightarrow \infty$ $(s \downarrow 0)$ and $\int_0^{\varepsilon} ds/s\mathcal{L}(s) < \infty$ for some $\varepsilon > 0$. Then $\mathcal{S}$ is positive iff $E(I^{\nu}/\mathcal{L}(I^{-1})) < \infty$.

It has been shown [33] that when $m \leq 1$ and (3) does not hold then $\mathcal{S}$ can be either null or transient. In particular if $m = 1$, $\beta = h'(1-) < \infty$ and $\gamma = f''(1-)/2 < \infty$ then $\mathcal{S}$ is null if $\sigma = \beta/\gamma < 1$ and transient if $\sigma > 1$ [28][29]. The case $\sigma = 1$ is a boundary: if also $\sum p_j j^2 \log^+ j$, $\sum h_j j \log^+ j < \infty$ then $\mathcal{S}$ is null, and a necessary and sufficient condition for transience is

$$\sum_{n=1}^{\infty} \exp\left(- \sum_{k=0}^{n-1} (1 - h(f_k))\right) < \infty$$

where $f_k = f_k(0)$ [55][29]. These possibilities are not always appreciated. For example in [21] a method of parameter estimation is given for the critical case which depends on the recurrence of $\mathcal{S}$. It is erronously asserted that when $\gamma$, $h''(1-) < \infty$, $\mathcal{S}$ is null.

We need the following notation to describe the supercritical case. Let $\tau(t)$ $(0 < t < -\log p_0)$ be the inverse function of $k(\theta) = -\log f(e^{-\theta})$ $(\theta > 0)$ and let $\tau_n(\cdot)$ be the nth functional iterate of $\tau(\cdot)$. Fix $t_0 \in (0, -\log q)$ and let $c_n = 1/\tau_n(t_0)$; $\{c_n\}$ are called the Seneta constants. It is known [1] that $c_n \uparrow \infty$, $c_{n+1}/c_n \longrightarrow m$ and $Z_n/c_n \xrightarrow{a.s.} W$ $(n \longrightarrow \infty)$, where $\{Z_n\}$ is the simple branching process, $Z_0 = 1$, and $W$ is a non-defective random variable whose DF has an atom of size $q$ at the origin and which is absolutely continuous on the set of positive numbers. Let $\phi(\theta) = E(e^{-\theta W})$. The following theorem describes the growth of $\chi$.

<u>Theorem C</u>. <u>If</u> $X_0 = 0$, $1 < m < \infty$ <u>then</u>:

(i) $X_n/c_n \xrightarrow{a.s.} V$ <u>and if Condition A holds,</u> V <u>is a non-defective random variable with a continuous DF which is defined by</u>

$$E(e^{-\theta V}) = \prod_{n=1}^{\infty} h(\phi(\theta m^{-n})).$$

(ii) In addition the DF of V is absolutely continuous. If $(-\log h(q))/(\log m) > 1$, V has a bounded continuous density and if $E(I)$, $E(Z_1 \log^+ Z_1) < \infty$ <u>then</u> V <u>has a density which is continuous on the set of positive numbers</u>.

(iii) <u>If Condition A does not hold then</u> $P(V = \infty) = 1$, <u>and</u>

(iv) <u>there is no sequence of constants</u> $\{a_n\}$ <u>such that</u> $a_n \to \infty$ <u>and</u> $\{X_n/a_n\}$ <u>converges in law to any non-defective DF which is not degenerate at the origin</u>.

Parts (i) and (iii) are due to Seneta [44][45], (ii) to Pakes [32] and (iv) to Cohn [5]. Seneta [45] also proved there is no sequence $\{a_n\}$ such that $a_{n+1}/a_n \to m$ and the other contingencies under (iv) hold. Cohn's proof uses only the strong law of large numbers and a representation of $p_{ij}^{(1)}$. We now give a shorter proof generalizing the argument used in [45].

Suppose that there is a sequence $\{a_n\}$ satisfying the contingencies in (iv) and let $\psi(\theta)$ be the Laplace-Stieltjes transform (LST) of the limiting DF, $F(\cdot)$. If $\psi_n(\theta) = E(\exp(-\theta X_n/a_n))$ it follows from (1.2) that

$$\psi_n(\theta) = \prod_{k=0}^{n-1} h(f_k(\exp(-\theta/a_n))) \to \psi(\theta).$$

Clearly

$$\psi_n(\theta) = h(f_{n-1}(\exp(-\theta/a_n)))\,\psi_{n-1}(\theta a_{n-1}/a_n). \tag{5}$$

Suppose there is a subsequence $\{n_j\}$ such that $a_{n_j-1}/a_{n_j} \to \infty$. If $F_n(x)$ is the DF of $X_n/a_n$ then

$$\psi_{n_j-1}(\theta a_{n_j-1}/a_{n_j}) \le F_{n_j-1}(\epsilon) + \int_\epsilon^\infty \exp(-\theta a_{n_j-1}/a_{n_j})\, dF_{n_j}(x)$$

where $\epsilon > 0$ is a continuity point of $F(\cdot)$.

It follows from (5) that $\psi(\theta) \le F(\epsilon) < 1$ for all $\theta > 0$ and where $\epsilon$ is chosen to be sufficiently small. This contradicts the non-defectiveness of

$F(\cdot)$. Now choose a sub-sequence, again denoted by $\{n_j\}$, such that $a_{n_j-1}/a_{n_j} \to \alpha$ where $0 \le \alpha < \infty$. The uniform convergence property of LST's [13, p. 252] shows that

$$\psi_{n_j-1}(\theta a_{n_j-1}/a_{n_j}) \to \phi(\alpha\theta)$$

and it follows that $h(f_{n_j-1}(\exp(-\theta/a_{n_j}))$ converges and that the limit is the LST of a non-defective DF. The convergence theorem for the process without immigration and Kinchin's theorem on convergence to types then shows that $a_{n_j} \sim k c_{n_j-1} \sim k m^{-1} c_{n_j}$ where $0 < k < \infty$. But we know that when Condition A does not hold, $E(\exp(-\theta X_{n_j}/c_{n_j})) \to 0$, and using Kinchin's theorem again, this contradicts the assumed existence of $\{a_n\}$.

In the next section we shall prove some results describing the growth of $\chi$ when the necessary and sufficient condition in each of Theorems A-C is relaxed. When $m = 1$, one such result has been known for some time:

Theorem D. If $m = 1$, $\beta = h'(1-)$, $\gamma = f''(1-) < \infty$ and $\sigma = \beta/\gamma$ then

$$P(X_n/n\gamma \le x) \to (\Gamma(\sigma))^{-1} \int_0^x y^{\sigma-1} e^{-y} dy.$$

This result is important since its conditions will normally be satisfied in applications. It was found independently by Foster [14], Pakes [28] and Seneta [46].

## 3. The immigration regime

3.1 $m < 1$. We begin our treatment of the subcritical case by reviewing some results about $\{Z_n\}$ that will be needed. The proofs in the sequel will make repeated use of the uniform convergence theorem for regularly varying functions and we shall not refer to its use on each occasion; see [48] for an account of this and other properties that we shall use about regularly varying functions.

Let $Q(s) = \lim_{n\to\infty} E(s^{Z_n} | Z_n > 0, Z_0 = 1)$; this is a pgf satisfying the functional equation [1]

$$Q(f(s)) = mQ(s) + 1 - m.$$

Iteration yields

$$1 - f_n(s) = \omega(m^n(1-Q(s))) \tag{1}$$

where $\omega(.)$ is the inverse function of $1 - Q(1-s)$ and has the following properties [47].

The function $\gamma(x) = x^{-1}\omega(x)$ is sv at the origin and $\downarrow 1/Q'(1-) \geq 0$ $(x \downarrow 0)$. It follows that $1 - Q(1-s)$ is 1-varying at the origin (equivalently, $1-Q(1-x^{-1})$ is -1-varying at infinity) and in particular this gives the following result [48, p.18].

<u>Lemma 1</u>. $\log(1-Q(1-s)) \sim \log\omega(s) \sim \log s \qquad (s \downarrow 0).$

Since $Q(1-1/x)$ has a monotone derivative with respect to $x^{-1}$ it follows [48, pp.60, 88] that

$$(1-s)\,Q'(s)/Q(s) \longrightarrow 1 \tag{2}$$

Observing that $\omega(\cdot)$ is non-decreasing it follows that

$$h(s)/h(f_n(s)) \leq P_0^{(n)}(s)\exp\left(-\int_0^n \log h(1-\omega(m^x(1-Q(s))))ds\right) \leq 1$$

and hence if $s_n \longrightarrow 1-$

$$P_i^{(n)}(s_n) \sim \exp\zeta\int_{s_n}^{f_n(s_n)} \frac{Q'(s)\log h(s)}{1-Q(s)}\,ds \qquad (n \longrightarrow \infty)$$

where $\zeta = 1/\log m^{-1}$ and the last integral arises by a change of variables in the preceding one. It now follows from (2) that

$$
\begin{aligned}
-\log P_i^{(n)}(s_n) &\sim \zeta \int_{s_n}^{f_n(s_n)} \frac{1-h(s)}{1-s}\,ds \\
&= \zeta \int_{-\log(1-s_n)}^{-\log(1-f_n(s_n))} G(y)\,dy .
\end{aligned} \tag{3}
$$

In the sequel we shall always assume that $\delta$ is not positive, that is, that (1.2) holds. The proofs of the following theorems all follow by assuming one of Conditions (i) - (iv), choosing $\{s_n\}$ so that the limit of the right-hand side of (3) exists and then invoking the following proposition [22].

<u>Proposition 1</u>. <u>Let</u> $T(\cdot)$ <u>be a strictly increasing continuous function,</u> $T(1) = 0$, $T(\infty) = \infty$ <u>and</u> $T(\cdot)$ <u>is sv at infinity. Let</u> $w(y) = T^{-1}(y)$, $\{a_n\}$ <u>a sequence of positive constants such that</u> $a_n \to \infty$ <u>and</u> $u(x)$ <u>be a continuous non-decreasing function on</u> $(0, \infty)$. <u>If for a sequence of non-negative random variables</u> $\{\xi_n\}$,

$$E[(1-1/w(a_n x))^{\xi_n}] \to u(x) \qquad (n \to \infty;\ x > 0)$$

<u>then</u>

$$P(a_n^{-1} T(1+\xi_n) \leq x) \to u(x).$$

<u>Remark</u>. Some changes in the proof of Proposition 1 allow us to assume that $T(0) = 0$ and conclude that $P(a_n^{-1} T(\xi_n) \leq x) \to u(x)$.

For $x \geq 1$ let

$$\Lambda(x) = \exp\left(\int_0^{\log x} G(y)\,dy\right) \tag{4}$$

and for $0 \leq x \leq 1$ define $\Lambda(\cdot)$ so that on $[0,\infty)$ it is continuous, strictly increasing and $\Lambda(0) = 0$. Equation (1.2) implies that $\Lambda(x) \to \infty$ and a change of variables shows that $\log \Lambda(x) = \int_1^x G(\log y) y^{-1}\,dy$, and hence that $\Lambda(\cdot)$ is sv.

<u>Theorem 1</u>. <u>Suppose that</u> $m < 1$ <u>and both</u> (1.2) <u>and Condition (i) hold. Then</u> $\{\Lambda(X_n)/\Lambda(m^{-n})\}$ <u>has the limit DF</u>

$$A(x) = (1 \wedge x)^{\zeta} \qquad (x \geq 0).$$

<u>Remark</u>. Condition (i) can be rewritten as $(\log(1-s))(1-h(s)) \to 0$.

<u>Proof</u>. Let $1-s_n = 1/\Lambda^{-1}(x\Lambda(m^{-n}))$. Clearly $s_n \to 1-$ and hence (3) applies. Furthermore the integrals in (3) can be written as

$$\begin{aligned} &\log \Lambda((1-f_n(s_n))^{-1}) - \log \Lambda((1-s_n)^{-1}) \\ &= \log \Lambda((1-f_n(s_n))^{-1}) - \log x\Lambda(m^{-n}) \\ &= -\log x + \int_{-\log m^n}^{-\log(1-f_n(s_n))} G(y)\,dy \end{aligned}$$

Let $J_n$ denote the integral. Then

$$\begin{aligned} 0 \leq J_n &\leq G(-\log m^n)[-\log(1-f_n(s_n)) + \log m^n] \\ &= \delta_n \left( \frac{-\log(1-f_n(s_n))}{-\log m^n} - 1 \right), \end{aligned}$$

where $\delta_n \to 0$ $(n \to \infty)$ and we have here used Condition (i). However, (1) and Lemma show that

$$-\log(1-f_n(s_n)) \sim -\log(m^n(1-Q(s_n)))$$

and $-\log(1-Q(s_n)) \sim -\log(1-s_n) \leq \log m^{-n}$ where in the last step we assume $0 < x \leq 1$. It follows that the lim sup of the coefficient of $\delta_n$ is finite and hence $J_n \longrightarrow 0$. Thus we have shown that $P_i^{(n)}(s_n) \longrightarrow A(x)$ and Proposition 1 (Remark) is now applicable with $T(x) = \Lambda(x)$ and $a_n = m^{-n}$.

It was shown in [33] that if $m \leq 1$ then

$$\lim_{n\to\infty} n(1-h(f_n)) < 1 \,(>1) \Longrightarrow \mathcal{S} \text{ is null (transient).} \tag{5}$$

Condition (i) implies that

$$n(1-h(f_n)) = \varepsilon(f_n)\, n/(-\log(1-f_n))$$

where $\varepsilon(s) \longrightarrow 0$ $(s \longrightarrow 1-)$ and (1) and Lemma 1 show that $\log(1-f_n) \sim \log m^n$. Thus $\mathcal{S}$ is null under the conditions of Theorem 1.

Example 1. The conditions of Theorem 1 are satisfied if

$$h_j = c[j(\log j)^2 \prod_{k=2}^{r} \log_k j]^{-1} \qquad (j \geq J)$$

where $\log_1 j = \log j$, $\log_k j = \log(\log_{k-1} j)$, $r \geq 2$, $J$ is chosen so large that $h_j$ is well-defined and $c$ is chosen so that $\mathcal{H}$ is a probability distribution. It follows that

$$\sum_{i=j}^{\infty} h_i \sim c\left(\prod_{k=1}^{r} \log_k j\right)^{-1} \qquad (j \longrightarrow \infty)$$

and an Abelian theorem for power series [13, p. 452] yields

$$1-h(s) \sim c\left(\prod_{k=1}^{r} \log_k (1-s)^{-1}\right)^{-1} \qquad (s \longrightarrow 1-).$$

Theorem 2. If $m < 1$ and Condition (ii) holds then $\{(\zeta/n)\log(1+X_n)\}$ has the limiting DF

$$B(x) = (x/(1+x))^{a\zeta} \qquad (x \geq 0).$$

Proof. Let $s_n = 1 - m^{xn}$ in (3). Condition (ii) implies that

$$-\log P_i^{(n)}(s_n) \sim a\zeta \int_{-\log(1-s_n)}^{-\log(1-f_n(s_n))} dy/y.$$

However, Lemma 1 yields $\log(1-f_n(s_n)) \sim \log m^n(1-Q(s_n))$ and $\log(1-Q(s_n)) \sim \log(1-s_n)$. Thus if $x > 0$ it is now obvious that $P_i^{(n)}(s_n) \to B(x)$ and Proposition 1 now applies with $T(x) = \zeta \log(1+x)$ and $a_n = n$.

We remark that $B(a\zeta x)$ defines an F-distribution. Condition (ii) and Lemma 1 show that $n(1-h(f_n)) \to a\zeta$ and hence, from (5), $\mathcal{S}$ is null if $a\zeta < 1$ and transient if $a\zeta > 1$. Further details of the structure of $\mathcal{X}$ are required to decide the case $a\zeta = 1$.

Example 2. Let $h_0 = 1 - a/\log 2 > 0$ and

$$h_j = a[(\log(j+2))^{-1} - (\log(j+3))^{-1}] \qquad (j=1,2,\ldots).$$

It follows that

$$\sum_{i \geq j} h_i \sim a/\log j\,. \tag{6}$$

and Condition (ii) follows from an Abelian theorem. Conversely it follows from a Tauberian theorem for power series [13, p.452] that the condition of Theorem 2 implies (6). For this example $\mathcal{S}$ is null when $a\zeta = 1$. To see this observe that $\sigma(s) = (1-h(s))/(1-s) = a \sum s^j/\log(j+2)$ and that $a^{-1}\sigma(s) - I(s) = O(1) \quad (0 \leq s \leq 1)$ where

$$I(s) = \int_0^\infty \frac{s^x}{\log(x+2)}\, dx = \tau^{-1} \int_0^\infty \frac{\exp(-y)\,dy}{\log(y+2\tau) - \log \tau}$$

and $\tau = -\log s$. Moreover

$$I(s) - (\tau(-\log \tau))^{-1} = (\tau(-\log \tau))^{-1} J(\tau)$$

where

$$J(\tau) = \int_0^\infty \frac{(\log(y+2\tau))\exp(-y)\,dy}{\log(y+2\tau) - \log\tau}$$

$$= (\log\tau^{-1})^{-1}\int_0^\infty \frac{(\log(y+2\tau))\exp(-y)\,dy}{1-(\log(y+2\tau))/(\log\tau)}$$

and it can be shown that the last integral converges as $\tau \to 0$ and the limit is $\int_0^\infty (\log y)e^{-y}\,dy$ which is minus one times Euler's constant. By putting these estimates together it follows that

$$(\log n)(n(1-h(f_n)) - 1) \to 0$$

and hence Bertrand's test shows that $\sum_{n\geq 0}\prod_{k=0}^{n} h(f_k) = \infty$. Thus $\mathcal{S}$ is null.

Define the sequence $\{a_n\}$ by $G(a_n) = \zeta/n$. If Condition (iii) holds then $\{a_n\}$ is $\Delta^{-1}$-varying.

<u>Theorem 3.</u> <u>If</u> $m < 1$ <u>and Condition (iii) holds then</u> $\{a_n^{-1}\log(1+X_n)\}$ <u>has the limiting extreme value DF</u>

$$C(x) = \exp(-\zeta x^{-\Delta}) \qquad (x > 0).$$

<u>Proof</u>. Let $s_n = 1-\exp(-xa_n)$. The right-hand term in (3) can be written as

$$\zeta b_n \int_1^{[\log(1-f_n(s_n))]/[\log(1-s_n)]} G(b_n y)\,dy$$

where $b_n = -\log(1-s_n) = xa_n$. Using Lemma 1 it is not difficult to show that the upper terminal is of the form $1+(n/\zeta x a_n)(1+o(1))$ and of course $n/a_n \to 0$. A mean value theorem then shows that $-\log P_1^{(n)}(s_n) \sim \zeta b_n G(b_n)(n/\zeta x a_n)$

$$\sim \zeta(n/\zeta)x^{-\Delta}G(a_n) = \zeta x^{-\Delta}$$

and the theorem readily follows.

<u>Theorem 4</u>. <u>If</u> $m < 1$ <u>and Condition (iv) holds then</u> $\{n[1-h(\exp(-X_n^{-1}))]\}$ <u>has the limiting DF</u>

$$D(x) = 1 - e^{-x} \qquad (x > 0).$$

<u>Remark</u>. The assertion may be re-expressed by stating that the reciprocal of the sequence above has the limiting extreme value DF $\exp(-x^{-1})$.

<u>Proof</u>. Let $1-s_n = [1+1/(-\log h^{-1}(1-(1-h_0)/(1+xn(1-h_0))))]^{-1}$

$$\sim -\log h^{-1}(1-(1-h_0)/(1+xn(1-h_0))).$$

Since $G(\cdot)$ is sv at infinity, $x^{\delta}G(x) \to \infty$ $(x \to \infty)$ for any $\delta > 0$ and this allows us to infer that $-\log(1-s_n) \geq K n^{\delta'}$, $0 < K < \infty$, $\delta' > 0$ and hence that

$$[\log(1-f_n(s_n))]/[\log(1-s_n)] = 1 - (n/\zeta \log(1-s_n))(1+o(1))$$

and $n/\log(1-s_n) \to 0$. By proceeding in a similar manner to the proof of Theorem 3, we obtain

$$-\log P_i^{(n)}(s_n) \sim nG(-\log(1-s_n))$$

$$\sim nG[-\log(1-h^{-1}(1-(1-h_0)/(1+xn(1-h_0))))]$$

$$= n(1-h_0)/(1+xn(1-h_0)) \to x^{-1}.$$

Proposition 1 (Remark) may now be applied with $T(x) = (1-h(e^{-1/x}))^{-1} - (1-h_0)^{-1}$, $a_n = n$ and $u(x) = \exp(-x^{-1})$, and the theorem follows.

Under the conditions of Theorems 3 and 4,

$$1-h(f_n) = G(-\log(1-f_n)) \sim (n/\zeta)^{-\Delta}/L(n)$$

where we have used (1) and Lemma 1 and it follows from (5) that $\mathcal{S}$ is transient.

Example 3. Let

$$h_j = cj^{-1}(\log j)^{-\delta} \qquad (j=2,3,\ldots;\ 1<\delta<2)$$

where $c$ is a constant. It was shown in [33] that

$$1 - h(s) \sim c(\delta-1)(-\log(1-s))^{1-\delta}$$

and it follows that the conditions of Theorem 3 are fulfilled with $\Delta = \delta-1$. Moreover, it is clear that we can replace $a_n$ by $a'_n = c(\delta-1)(n/\zeta)^{(\delta-1)^{-1}}$ since $a_n \sim a'_n$.

Example 4. We modify the previous example by letting

$$h_j = c[j\Big(\prod_{k=1}^{r-1}\log_k j\Big)(\log_r j)^{1+b}]^{-1}$$

for $j \geq J$, $r \geq 2$ and $b > 0$. It follows that

$$1-h_j \sim (c/b)(\log_r j)^{-b}$$

and hence

$$1-h(s) \sim (c/b)(\log_r (1-s)^{-1})^{-b} \qquad (s \to 1-).$$

Thus

$$G(x) \sim (c/b)(\log_{r-1} x)^{-b} \qquad (x \to \infty)$$

and this satisfies Condition iv. Furthermore since $X_n \xrightarrow{a.s.} \infty$ we may replace $1-h(\exp(-1/X_n))$ by $(c/b)(\log_r^+ X_n)^{-b}$ in the assertion of Theorem 4 and the remark then shows that $\{n^{-1/b}(\log_r^+ X_n)\}$ has the limiting extreme value DF. $\exp[-(c/b)x^{-b}]$ $(x > 0)$.

It is interesting to note the role of Lemma 1 in each of the proofs above, namely, that because of it the offspring distribution figures in the results only through the offspring mean $m$. We can make some comparison of the growth rate of $\chi$ for each of Theorems 1-4. First note that the tail of $\gamma$ grows longer as we go from Conditions (i)-(iv) and hence we expect the growth rate to increase as we go from (i)-(iv). This is clearly so for Theorems 2 and 3 since after normalising by $\log(\cdot)$ a linear norming works in Theorem 2, but the norming sequence $\{a_n\}$ in Theorem 3 grows at most algebraically fast, but $a_n/n \to \infty$. If Condition (iv) is in force then $1-h(\exp x^{-1}) = G(-\log(1-\exp x^{-1})) \sim G(\log(1+x))$. Thus we can always re-express Theorem 4 as: $\{n^{-1}L(\log(1+X_n))\}$ has the limit DF $\exp(-x^{-1})$. In this case the non-linear norming is much more severe than for Theorems 2 and 3. We have the same situation for Theorem 1 since under Condition (i) it is easily shown that $\Lambda(e^x)$ is sv at infinity. However, it is also true that the norming sequence $\{\Lambda(m^{-n})\}$ is sv. It does not seem easy to exhibit specific examples of $\Lambda(\cdot)$. However, we can derive bounds in the case of Example 1 as follows. It follows from the definition that $x\Lambda'(x)/\Lambda(x) \sim 1-h(1-x^{-1})$ and hence for Example 1 we obtain

$$x\Lambda'(x)/\Lambda(x) \sim c\left(\prod_{k=1}^{r} \log_k x\right)^{-1}.$$

Let $\epsilon > 0$ be given. If $x_0$ is large enough it follows that

$$\begin{aligned} c(1-\epsilon)(\log_{r+1}x - \log_{r+1}x_0) &\le \log(\Lambda(x)/\Lambda(x_0)) \\ &\le c(1+\epsilon)(\log_{r+1}x - \log_{r+1}x_0) \end{aligned}$$

or

$$K_1(\log_r x)^{c(1-\epsilon)} \le \Lambda(x) \le K_2(\log_r x)^{c(1+\epsilon)}. \tag{7}$$

Cohn [4] has given a theorem which gives conditions which permit almost sure convergence of an in-homogeneous Markov chain to be inferred from its convergence in law. He showed that this was applicable to the chain $\{T(1+Z_n)/a_n\}$ where $\{Z_n\}$ is an explosive branching process, $T(\cdot)$ is a certain sv function and $\{a_n\}$ a certain sequence of constants. Despite the similarity of this situation and those considered in this paper, convergence in law is the most we can assert for our results. This follows because each of the sequences in Theorems 1-4 is mixing [40, Theorem 2] and hence cannot even converge in probability [39].

<u>3.2. $1 < m < \infty$.</u> Recall the notation introduced before Theorem C and let $K(\theta) = -\log E(e^{-\theta W})$. The following facts are demonstrated in [47]:

$$K(\theta) = \theta\lambda(\theta)$$

where $\lambda(\cdot)$ is strictly decreasing, sv at the origin and $\uparrow EW$ $(\theta \downarrow 0)$. Let $\omega(\cdot)$ be the inverse function of $K(.)$. Then

$$\tau_n(t) = K(m^{-n}\omega(t)) \qquad (0 < t < -\log q) \tag{1}$$

and it is clear that

$$K(1) = t_0, \ \omega(t_0) = 1 \tag{2}$$

and hence that

$$\omega(\rho_n) = m^{-n}, \ K(m^{-n}) = \rho_n \tag{3}$$

where $\rho_n = 1/c_n$. Let $\psi(s)$ be the inverse function of $\phi(\cdot)$; it is defined for $q < s \leq 1$. Clearly

$$K(\theta) = -\log\phi(\theta), \ \omega(t) = \psi(e^{-t}) \tag{4}$$

and hence the regular variation properties of $K(\cdot)$ and $\omega(\cdot)$ are inherited by $1-\phi(\theta)$ and $\psi(1-s)$, respectively. In particular

$$(1-s)\psi'(s)/\psi(s) \longrightarrow -1 \qquad (s \longrightarrow 1-). \tag{5}$$

Finally, since $\phi(\cdot)$ satisfies the functional equation $\phi(\theta m) = f(\phi(\theta))$ [1], it follows that

$$f_n(s) = \phi(m^n \psi(s)) \qquad (q < s \leq 1). \tag{6}$$

The following lemma is a consequence of the regular variation property [48, p.18].

<u>Lemma 2</u>. $\log K(\theta) \sim \log(1 - \phi(\theta)) \sim \log \theta \quad (\theta \downarrow 0)$

$$\log \omega(t) \sim \log t \ (t \downarrow 0), \ \log \psi(s) \sim \log(1-s) \qquad (s \longrightarrow 1-).$$

The results on the convergence of $\{Z_n/c_n\}$ mentioned in §2 tell us that $f_n(s^{\rho_n}) \longrightarrow \phi(-\log s)$ $(0 < s \leq 1)$ and hence if $s_n \longrightarrow 1-$, $f_n(s_n^{\rho_n}) \longrightarrow 1$. Thus it follows from (2.1) that $P_i^{(n)}(s_n^{\rho_n}) \sim P_0^{(n)}(s_n^{\rho_n})$. If $I_n(s) = \exp \int_0^n \log h(\phi(m^x \psi(s)))\,dx$ it follows using (6) that

$$1 \leq P_0^{(n)}(s)/I_n(s) \leq h(s)/h(f_n(s))$$

and hence if $s_n \longrightarrow 1-$, $P_0^{(n)}(s_n^{\rho_n}) \sim I_n(s_n^{\rho_n})$. Changing variables in $I_n(s)$ and using (5) we eventually obtain

$$-\log P_i^{(n)}(s_n^{\rho_n}) \sim \zeta \int_{-\log(1-f_n(s_n^{\rho_n}))}^{-\log(1-s_n^{\rho_n})} G(y)\,dy \tag{7}$$

where $\zeta = 1/\log m$.

We are now ready to obtain theorems for $Y_n = \rho_n X_n$ which are analogues of Theorems 1-4. For the next theorem we define $\Lambda(\cdot)$ exactly as in §3.1. It was mentioned in §3.1 that Condition (i) implies that $\Lambda(e^x)$ is sv at infinity. We shall need to use this fact in the proof of the following theorem.

<u>Theorem 5</u>. <u>If</u> $1 < m < \infty$, (1.2) <u>and Condition (i) hold, the sequence</u> $\{\Lambda(Y_n)/\Lambda(m^n)\}$ <u>has the limiting DF</u> $A(x)$.

<u>Proof</u>. Let $(1-s_n)^{-1} = \Lambda^{-1}(x\Lambda(m^n))$. The integral occurring in (7) can be written as

$$\int_{\log m^n}^{-\log(1-s_n^{\rho_n})} G(y)dy - \log[\Lambda((1-f_n(s_n^{\rho_n}))^{-1})/\Lambda(m^n)]. \tag{8}$$

By using the mean value theorem we can write

$$-\log(1-s_n^{\rho_n}) = -\log \delta_n - \log \rho_n - \log(1-s_n) \tag{9}$$

where $\delta_n \to 1$ $(n \to \infty)$. It follows from (3) and Lemma 2 that

$$(-\log \rho_n)/(\log m^n) \to 1. \tag{10}$$

If $0 < x \leq 1$, $-\log(1-s_n) \leq \log m^n$ and hence it follows as in the proof of Theorem 1 that the integral at (8) tends to zero. To proceed further we prove

<u>Lemma 3</u>. <u>Assuming only that</u> $s_n \to 1-$, $\log m^n \psi(s_n^{\rho_n}) \sim \log(1-s_n)$ $(n \to \infty)$.

<u>Proof</u>. First observe that (4) and the exponentiated form of (10) show that $\psi(s_n^{\rho_n}) \sim \omega(1-s_n^{\rho_n}) \sim \omega(\rho_n(1-s_n))$ and hence from (3)

$$m^n \psi(s_n^{\rho_n}) \sim \omega(\rho_n(1-s_n))/\omega(\rho_n) = (1-s_n)\gamma(\rho_n(1-s_n))/\gamma(\rho_n)$$

where $\gamma(t) = t^{-1}\omega(t)$ is sv at the origin. Since $\omega(x^{-1})$ has a monotone derivative with respect to $x^{-1}$ it follows that [48, pp. 60, 88] $t\omega'(t)/\omega(t) = 1 + \epsilon(t)$ where $\epsilon(t) \to 0$ $(t \downarrow 0)$. Integration yields the representation

$$\gamma(t) = t_0^{-1} \exp(-\int_t^{t_0} \epsilon(y)y^{-1}dy)$$

whence

$$\gamma(\rho_n(1-s_n))/\gamma(\rho_n) = \exp\left(-\int_{\rho_n(1-s_n)}^{\rho_n} \epsilon(y)y^{-1}dy\right)$$

$$= \exp(\eta_n \log(1-s_n))$$

where $\eta_n \longrightarrow 0$ and we have used a mean value theorem.

It now follows that $[\log m^n \psi(s_n^{\rho_n})]/[\log(1-s_n)] \sim 1 + \eta_n$ and the assertion follows.

Returning to the proof of Theorem 5, observe that Lemma 3 implies that $m^n \psi(s_n^{\rho_n}) \longrightarrow 0$ and hence, if $R(x) = \Lambda(e^x)$ it follows from (6) and Lemmas 2 and 3 that

$$R[-\log(1-f_n(s_n^{\rho_n}))] \sim R[-\log m^n \psi(s_n^{\rho_n})]$$

$$\sim R[-\log(1-s_n)] = \Lambda((1-s_n)^{-1}) = x\Lambda(m^n).$$

The proof is now easily completed since $P_i^{(n)}(s_n^{\rho_n}) = E(s_n^{Y_n})$. Observe that the slow variation of $R(x)$ was used in the last sequence of steps.

<u>Theorem 6</u>. <u>If</u> $1 < m < \infty$ <u>and Condition (ii) holds, then</u> $\{(\zeta/n)\log(1+Y_n)\}$ <u>has the limiting DF</u> $(B(x)$.

<u>Proof</u>. If $1 - s_n = m^{-xn}$ $(x > 0)$ it follows that

$$-\log E(s_n^{Y_n}) \sim a\zeta \log\left(\frac{\log(1-s_n^{\rho_n})}{\log(1-f_n(s_n^{\rho_n}))}\right).$$

It follows from (6), Lemmas 2 and 3, (9) and (10) that $E(s_n^{Y_n}) \longrightarrow B(x)$, whence the assertion.

Let Condition (iii) hold, define $a_n$ by $G(a_n) = \zeta/n$, let $b_n = -\log(1-s_n^{\rho_n})$ and $1-s_n = e^{-xa_n}$. It follows from (10) that $(\log \zeta_n)/a_n \longrightarrow 0$ and hence from (9), $b_n \sim xa_n$. Thus we finally obtain $G(b_n) \sim x^{-\Delta}(\zeta/n)$. We can rewrite (7) as

$$-\log E(s_n^{Y_n}) \sim \zeta \int_{[\log(1-f_n(s_n^{\rho_n}))]/[\log(1-s_n^{\rho_n})]}^{1} G(b_n y)\,dy. \tag{11}$$

Denote the lower terminal by $t_n$. Then since $1-\phi(\theta) = \theta\mu(\theta)$ and $\psi(s) = (1-s)g(1-s)$ where $\mu(\cdot)$ and $g(\cdot)$ are sv at the origin, it follows from (6) that

$$\log(1-f_n(s_n^{\rho_n})) = \log m^n + \log(1-s_n^{\rho_n}) + \log g(1-s_n^{\rho_n})$$
$$+ \log\mu(m^n\psi(s_n^{\rho_n})).$$

This yields

$$1-t_n = \frac{n}{\zeta\log(1-s_n^{\rho_n})}\left\{1 + \frac{\zeta\log g(1-s_n^{\rho_n})}{n\log(1-s_n^{\rho_n})} + \frac{\log\mu(m^n\psi(s_n^{\rho_n}))}{\log m^n\psi(s_n^{\rho_n})}\cdot\frac{\zeta\log m^n\psi(s_n^{\rho_n})}{n\log(1-s_n^{\rho_n})}\right\}.$$

Clearly the *second term* in parenthesis $\longrightarrow 0$ [48, p.18] and also using Lemma 3, it follows that the third term also $\longrightarrow 0$. Finally using (9) and (10) we obtain $1-t_n = (n/\zeta x a_n)(1+o(1))$. We can now proceed exactly as in the proof of Theorem 3 to obtain

<u>Theorem 7.</u> <u>If</u> $1 < m < \infty$ <u>and Condition (iii) holds then</u> $\{a_n^{-1}\log(1+Y_n)\}$ <u>has</u>

<u>the limiting DF</u> $C(x)$.

<u>Theorem 8.</u> <u>If</u> $1 < m < \infty$ <u>and Condition (iv) holds then</u> $\{n(1-h(\exp - Y_n^{-1}))\}$ <u>has the limiting DF</u> $D(x)$.

<u>Proof.</u> The relation (11) still holds if $b_n = -\log(1-s_n^{\rho_n})$ and

$$1-s_n = [1+1/[-\log h^{-1}(1-(1-h_0)/(1+xn(1-h_0)))]]^{-1}.$$

Using the last part of the proof of Theorem 7, we can proceed similarly to the proof of Theorem 4 once we show that $b_n \sim \log(1-s_n)$. To do this it suffices to show that $(-\log(1-s_n))/n \to \infty$.

Let $H(\cdot)$ be the inverse function of $G(\cdot)$; it is defined on $(0,1-h_0]$. Since $H(y) = -\log(1-h^{-1}(1-y))$ we have that $-\log(1-s_n) \sim H((1-h_0)/(1+xn(1-h_0)))$ and hence it suffices to show that $n^{-1}H(n^{-1}) \to \infty$. Let $0 < \epsilon < 1$ be given. Then for any $n$ large enough, $n > N(\epsilon)$ say, then $1-\epsilon < G(\epsilon^{-1}n)/G(n)$ and $(1-\epsilon)^{-1} < nG(n)$. These imply that $1 < nG(\epsilon^{-1}n)$ and hence that $n^{-1}H(n^{-1}) > \epsilon^{-1}$.

The form of Theorems 5-8 makes it clear that the examples and discussion in §3.1 apply here with obvious changes. Finally we mention that if $E(Z_1 \log^+ Z_1) < \infty$ then we can replace $Y_n$ by $m^{-n}X_n$ in Theorems 5-8.

<u>3.3. m = 1.</u>

Let $U(s)$ $(0 \leq s < 1)$ generate the invariant measure of the process $\{Z_n\}$, normalized so that $U(p_0) = 1$. Suppose now that (2.4) holds. It is known [51] that

$$U(s) = [\nu(1-s)^{\nu} M(1-s)]^{-1} \tag{1}$$

where $M(s) \sim \mathcal{L}(s)$ $(s \downarrow 0)$. The function $U(1-s)$ has an inverse $g(\cdot)$ defined on $(0,\infty)$ and hence

$$g(x) = x^{-1/\nu} K(x) \tag{2}$$

where $K(\cdot)$ is sv at infinity. The functional equation $U(f(s)) = 1 + U(s)$ [1, p. 68] may be iterated to yield

$$1 - f_n(s) = g(n + U(s)) \tag{3}$$

and this may be used [34] to show that if $s_n \longrightarrow 1-$

$$-\log P_i^{(n)}(s_n) \sim \int_{U(s_n)}^{U(f_n(s_n))} G(y)\,dy \tag{4}$$

$$= \int_{U(s_n)}^{n + U(s_n)} G(y)\,dy$$

where we now define

$$G(x) = 1 - h(1 - g(x)).$$

These relations reflect the fact that the results to be derived depend on a more intimate interaction between $\chi$ and $\varrho$ than is the case for $m \neq 1$. The roles played by $\log(\cdot)$ and $e^{-x}$ when $m \neq 1$ are taken here by $U(\cdot)$ and $g(\cdot)$. When $m < 1$ we made frequent use of the relation $-\log(1-f_n(s_n)) \sim n/\zeta - \log(1-Q(s_n))$ and a similar one was used when $m > 1$. In the critical case the analogous relation is the identity $U(f_n(s_n)) = n + U(s_n)$. Thus there are many similarities in the structure of the critical case and that of the non-critical cases. However, there are also great differences as our results will show.

We shall assume in the sequel that $\delta$ is not positive, that is

$$\int_0^\infty G(y)\,dy = \infty. \tag{5}$$

Let

$$\Lambda(x) = \exp \int_0^{U(1-1/x)} G(y)\,dy \qquad (x \geq 1)$$

and extend the definition to $[0,\infty)$ so that $\Lambda(\cdot)$ is continuous, strictly increasing and $\Lambda(0) = 0$.

Theorem 9. *Let* $m = 1$ *and (5) and Condition (i) hold. If* $a_n = (1-f_n)^{-1}$ *then* $\{\Lambda(X_n)/\Lambda(a_n)\}$ *has a limiting uniform distribution on* $[0,1]$.

Proof. Let $(1-s_n)^{-1} = \Lambda^{-1}(x\Lambda(a_n))$. It follows as in the proof of Theorem 1, that

$$-\log P_i^{(n)}(s_n) \sim -\log x + \int_n^{n+U(s_n)} G(y)\,dy$$

and since if $x \leq 1$ $U(s_n) \leq U(f_n) = n$, the integral is dominated by $nG(n) \to 0$. The assertion readily follows.

There is a difference between this result and the corresponding results for $m \neq 1$, namely that $\Lambda(\cdot)$ is sv at infinity only as a consequence of the hypotheses of the theorem. To see this observe that

$$x\Lambda'(x)/\Lambda(x) = [x^{-1}U'(1-1/x)/U(1-1/x)]\,U(1-1/x)\,G(U(1-1/x))$$

and the term in square brackets $\to \nu$ $(x \to \infty)$. It is not true, without assumptions, that $\Lambda(e^x)$ is sv. If this were the case then the norming sequence in Theorem 9 could be replaced by $\{\Lambda(n^{1/\nu})\}$. It is easier to argue directly as follows. Suppose we let $(1-s_n)^{-1} = \Lambda^{-1}(x\Lambda(n^{1/\nu}))$ in the proof above. We would then be led to the integral

$$\int_{U(1-n^{-1/\nu})}^{n+U(1-n^{-1/\nu})} G(y)dy \leq nG(U(1-n^{-1/\nu}))$$

and since, by (1), $n/U(1-n^{-1/\nu}) = (\nu M(n^{-1/\nu}))^{-1}$ the following corollary is apparent.

<u>Corollary.</u> <u>If the conditions of Theorem 9 hold and if</u> $\liminf_{s\to 0} \mathcal{L}(s) > 0$ <u>then</u> $\{\Lambda(X_n)/\Lambda(n^{1/\nu})\}$ <u>has a limiting uniform distribution on</u> $(0,1)$.

If $s^{-\delta}(1-h(1-s))$ is sv at the origin it is readily checked that the conditions of Theorem 9 can hold only if $\delta = \nu$. If

$$h_j = c[j^{1+\nu} \prod_{k=1}^{r} \log_k j]^{-1}$$

for $j$ sufficiently large and $r \geq 1$ then if $0 < \nu < 1$

$$1-h(1-s) \sim (c/\nu)\Gamma(1-\nu)\, s^{\nu}\left(\prod_{k=1}^{r} \log_k s^{-1}\right)^{-1}.$$

Using (2), $\log g(x) \sim \log x^{-1/\nu}$ $(x \to \infty)$ and hence

$$x\,G(x) \sim (c/\nu)\Gamma(1-\nu)(K(x))^{\nu}\left(\prod_{k=1}^{r} \log_k x^{1/\nu}\right)^{-1}.$$

Thus the conditions of Theorem 9 will certainly be fulfilled if $K(x)$ is bounded and this is the case when, for example, $f(s) = s + (b/(1+\nu))(1-s)^{1+\nu}$ $(0 < b < 1,\ 1 < \nu < 2)$. This example satisfies inequality (3.1.7) provided $c$ is replaced by $(c\nu/b)(1+\nu)\Gamma(1-\nu)$. Moreover the conditions of the corollary are satisfied, but it is evident that if $r = 1$, $\Lambda(e^x)$ is not sv.

A further example was constructed in [35] for which $\beta = h'(1-) < \infty$ and $f(s) - s \sim K(1-s)^2 (\log(1-s)^{-1})^{1-c}$ where $0 < K < \infty$ and $0 < c < 1$. It can be shown that for this example

$$\exp[((1-\epsilon)\beta/Kc)(\log x)^c] \leq \Lambda(x) \leq \exp[((1+\epsilon)\beta/Kc)(\log x)^c]$$

so that $\Lambda(\cdot)$ increases much more rapidly than for the previous example. If

$\nu = 1$ the extra condition in the corollary is always satisfied since

$$\mathcal{L}(1-s) = (1-s)^{-1}\,(1-\tilde{f}(s))$$

where $\tilde{f}(s) = (1-f(s))/(1-s)$ is a pgf and hence $\mathcal{L}(1-s)$ generates the tail probabilities of the distribution defined by $\tilde{f}(.)$. Thus $\mathcal{L}(x)$ increases as $x \downarrow 0$. On the other hand, it is not difficult to construct examples for which the extra condition is not satisfied.

Although the two following theorems have a quite different character from our other results, they are in a sense, dual to Theorems 2 and 6, and 3 and 7, respectively.

<u>Theorem 10</u>. <u>If</u> $m = 1$ <u>and Condition (ii) holds then</u> $\{(1-f_n)X_n\}$ <u>has a a limiting distribution whose</u> LST <u>is</u> $(1+\theta^{\nu})^{-a}$ $(\theta \geq 0)$.

<u>Proof</u>. Let $s_n = \exp(-\theta(1-f_n))$ and use (4) and note that (1) implies that $U(s_n) \sim \theta^{-\nu} n$.

By using (2) we see that the hypotheses are equivalent to (2.4) and

$$1-h(s) \sim a\nu(1-s)^{\nu}\mathcal{L}(1-s) \qquad (s \to 1-). \tag{6}$$

Kawazu and Watanabe [23] started with these assumptions and obtained the conclusion of Theorem 10 by using a much more complicated method. Finally if $\gamma < \infty$ then $\nu = 1$, $1-f_n \sim (n\gamma)^{-1}$ [1] and since $\mathcal{L}(s) \to \gamma$ $(s \to 0)$ it follows that $g(x) \sim (x\gamma)^{-1}$. If, in addition, $\beta < \infty$ it follows that Condition (ii) is satisfied with $a = \sigma$ and hence we obtain Theorem D. This was shown in a more roundabout way in [34]. It was also shown in this reference that $\mathcal{S}$ is null when $a < 1$ and transient when $a > 1$.

Theorem 11. Let $m = 1$, Condition (iii) hold and let $a_n = 1 - h^{-1}(1-1/n)$. Then $\{a_n X_n\}$ has the limiting stable law whose LST is $\exp(-\theta^{\nu\Delta})$.

Proof. Let $s_n = \exp(-\theta a_n)$ and $b_n = U(s_n)$. Now

$$b_n \sim \theta^{-\nu} U(1-a_n) = \theta^{-\nu} G^{-1}(n^{-1}) \tag{7}$$

since $U(h^{-1}(1-x)) = G^{-1}(x)$. It follows from Condition (iii) that $x^{1/\Delta}G^{-1}(x)$ is sv at the origin and hence that $n/b_n \rightarrow 0$. Condition (iii) and (7) also show that $G(b_n) \sim \theta^{\Delta\nu} n^{-1}$. Equation (4) can be rewritten as

$$-\log P_i^{(n)}(s_n) \sim b_n \int_1^{1+n/b_n} G(b_n y)\,dy \sim nG(b_n) \rightarrow \theta^{\Delta\nu}$$

and the theorem follows.

The hypotheses of the theorem are equivalent to the requirements: Eq. (2.4) and

$$1 - h(s) = (1-s)^{\delta} \mu(1-s) \qquad (0 < \delta \leq \nu) \tag{8}$$

where $\mu(.)$ is sv at the origin and if $\delta = 2$, $\mu(s)/\mathcal{L}(s) \rightarrow \infty$ $(s \downarrow 0)$. For it is easily seen that Condition (iii) and (2) imply (8) with $\delta = \Delta\nu$ and $\mu(s) \sim (\nu\mathcal{L}(s))^{\Delta}/L((1-s)^{-\nu}/\mathcal{L}(s))$, and conversely (2) and (8) imply Condition (iii) with $\Delta = \delta/\nu$. When $\gamma < \infty$ Theorem 11 was given in [33, Theorem 8, Remark 2]. We see that the Conditions of Theorems 10 and 11 require $\chi$ to have a regularly varying tail with index in $[-\nu, 0)$ and this possibility is also allowed by Theorem 9. In particular the hypotheses of Theorems 9-11 imply that $\chi$ possesses moments of some algebraic order $\leq 1$. A boundary case, not yet covered, occurs if $\chi$ possesses no algebraic moments. This occurs if $\chi$ has a sv tail and the following theorem covers this contingency.

<u>Theorem 12.</u> <u>If</u> $m = 1$ <u>and Condition (iv) holds then</u> $\{n(1-h(\exp(-X_n^{-1})))\}$ <u>has the limiting DF</u> $D(x)$.

<u>Remark.</u> The hypotheses imply that (8) holds with $\delta = 0$ and conversely, if (2) and (8), with $\delta = 0$, hold then so does Condition (iv).

<u>Proof.</u> Let $s_n = h^{-1}(1 - (1-h_0)/(1+xn(1-h_0)))$ and $b_n = U(s_n)$. Using the argument at the end of the proof of Theorem 8 it follows that $n/b_n \to 0$. Furthermore, $G(b_n) = h(s_n) \sim (xn)^{-1}$ and hence, by proceeding exactly as in the last few lines of the proof of Theorem 11, we find that $P_i^{(n)}(s_n) \to \exp(-x^{-1})$. The assertion follows from Proposition 1.

Equation (3) implies that $1-h(f_n) = G(n)$ and hence it follows from (3.1.5) that $\mathcal{S}$ is transient under the conditions of Theorems 11 and 12. The remark shows that all the examples of $\mathcal{U}$ in §3.1 satisfy the hypotheses of Theorem 12. Thus it follows that for Examples 1-4, the following sequences converge in law to the extreme value DF $\exp(-x^{-1})$:

Example 1 ($r \geq 2$): $\{(cn)^{-1} \prod_{k=1}^{r} \log_k X_n\}$

Example 2: $\{(an)^{-1} \log X_n\}$

Example 3: $\{(cn(\delta-1))^{-1} (\log X_n)^{\delta-1}\}$ $(0 < \delta-1 < 1)$

Example 4: $\{(b/cn)(\log_r X_n)^b\}$ $(r \geq 2,\ b > 0)$.

Many other examples could be constructed.

## 4. The Markov branching process allowing immigration.

In this section we shall consider the Markov branching process allowing immigration, denoted by $\{X_t; t \geq 0\}$. The intuitive picture for this process is as for the BPI except that each individual has a lifetime that has an exponential distribution with mean $\lambda^{-1}$, the inter-immigration times are exponentially distributed with mean $\delta^{-1}$ and these times are mutually independent and independent of the other elements of the process. The number of individuals living at time $t$ is $X_t$. Let $a(s) = \lambda(f(s) - s)$, $b(s) = \delta(h(s)-1)$ and $F(s,t)$ is the pgf of the process without immigration descended from a single individual. It is known that [1]

$$t = \int_s^{F(s,t)} dy/a(y).$$

We assume that $F(1-,t) \equiv 1$, that is, $\int_{1-\epsilon}^{1} dy/a(y) = \infty$ *for each sufficiently* small $\epsilon$.

It is known from [49] that

$$P_i(s,t) = E(s^{X_t} | X_0 = i) = (F(s,t))^i \exp\left(\int_0^t b(F(s,\tau))d\tau\right) \tag{1}$$

A change of variables yields $P_0(s,t) = \prod(s)/\prod(F(s,t))$ where $\prod(s) = \exp(-\int_0^s (b(y)/a(y))dy)$ is the generating function of the stationary measure of $\{X_t\}$ [31][54]. The stationary measure is unique up to constant multipliers. Since Theorems A and B are consequences of (2.3), it immediately follows from the form of $\prod(\cdot)$ that with a trivial rewording, Theorems A and B apply to $\{X_t\}$.

We shall work directly with (1) and deduce asymptotic expressions similar to (3.1.3), (3.2.7) and (3.3.4). Once this is done the proofs used in §3 carry through here with only trite changes. This approach seems to be more useful here than the embedding method that has been used for limit theorems describing the branching regime [31]. Let $\sigma = a'(1-) = \lambda(m-1) < \infty$.

$\sigma < 0$. Let $\{Z_t\}$ denote the Markov branching process and $Z_0 = 1$. It is known that $E(s^{Z_t} | Z_t > 0) \to Q(s)$ and that

$$1 - Q(1-s) = \exp\left(\sigma \int_0^{1-s} dy/a(y)\right)$$

$$= s \exp\left[\int_1^{1-s} \left(1 + \frac{\sigma}{xa(1-x^{-1})}\right)\frac{dx}{x}\right].$$

Since $(1+\sigma/xa(1-x^{-1})) \to 0$ $(x \to \infty)$ it is clear that the coefficient of $s$ is sv at the origin and hence both $1 - Q(1-s)$ and its inverse $\omega(\cdot)$ are 1-varying at the origin. The function $Q(\cdot)$ satisfies the functional equation $1 - Q(F(s,t)) = e^{\sigma t}(1-Q(s))$ whence

$$1 - F(s,t) = \omega(e^{\sigma t}(1-Q(s))$$

and it follows that if $s_t \to 1-$ then

$$-\log P_i(s_t, t) \sim \zeta \int_{-\log(1-s_t)}^{-\log(1-F(s_t,t))} G(y)\, dy$$

where $\zeta = \delta/\lambda(1-m)$ and $G(x) = 1 - h(1-e^{-x})$. Lemma 1 continues to apply and we now see that we have virtually the same situation as in §3.1. Define $\Lambda(\cdot)$ as at (3.1.4). The following result is obvious.

Theorem 13. Let $\sigma < 0$, $\int_0^\infty G(y)dy = \infty$ and $\zeta = \delta/\lambda(1-m)$. If Condition (i) holds then $\{\Lambda(X_t)/\Lambda(e^{-\sigma t})\}$ has the limiting DF $A(x)$.

If Condition (ii) holds then $\{(\lambda t(1-m))^{-1}\log(1+X_t)\}$ has the limiting DF $B(x)$.

If Condition (iii) holds and $a(t)$ is defined by $G(a(t)) = -t/\sigma$ then

$\{(1/a(t))\log(1+X_t)\}$ has the limiting DF $C(x)$.

If Condition (iv) holds then $\{-tb(\exp(-X_t^{-1}))\}$ has the limiting DF $D(x)$.

$0 < \sigma < \infty$. Let $k(\theta, t) = -\log F(e^{-\theta}, t)$ and denote its inverse function by $\tau(x, t)$; it exists if $0 \leq x \leq -\log F(0, t)$. Choose $x_0 \in (0, -\log q)$ and let $c(t) = 1/\tau(x_0, t)$. It is known [31] and references therein, that $c(t) \uparrow \infty$, $c(t+\tau)/c(t) \to e^{\sigma\tau}$ and $Z_t/c(t) \xrightarrow{a.s.} W$ where $W$ is a non-defective random variable whose DF has an atom of size $q$ at the origin and is absolutely continuous on the set of positive numbers. Let $\phi(\theta) = E(e^{-\theta W})$. Then

$$\phi(\theta) = F(\phi(\theta e^{-\sigma t}), t)$$

and if $\psi(s)$ is the inverse function of $\phi(\cdot)$ it is shown in [31] that

$$\psi(s) = A(1-s)\exp\left(\int_{\epsilon}^{s}\left(\frac{\sigma}{a(y)} + \frac{1}{1-y}\right)dy\right) \quad (\epsilon < s < 1)$$

where $A = \psi(\epsilon)/(1-\epsilon)$ and it follows that $\psi(1-s)$ is 1-varying at the origin and so also is $(1-\phi(\theta))$. Defining $K(.)$ and $\omega(\cdot)$ as in §3.2 we see that Lemma 2 holds.

Returning to the immigration process, it was shown in [31] that $X_t/c(t) \xrightarrow{a.s.} V$ where, if Condition A is satisfied, $V$ is a non-defective random variable with an absolutely continuous distribution and

$$E(e^{-\theta V}) = (\phi(\theta))^{x_0} \exp\int_1^{\phi(\theta)} (b(y)/a(y))\,dy.$$

If Condition A is not satisfied $V \overset{a.s.}{=} \infty$, and we now assume this to be the case. Let $\rho(t) = 1/c(t)$, $Y_t = \rho(t)X_t$ and $s_t \to 1-$. It is easily shown that

$$-\log E(s_t^{Y_t}) \sim \zeta \int_{-\log(1-F(s_t^{\rho_t}, t))}^{-\log(1-s_t^{\rho_t})} G(y)\,dy$$

where now $\zeta = \delta/\lambda(m-1)$. It should now be clear that we can write down a list of results with almost the same wording as Theorem 13; replace "$\sigma < 0$" by "$0 < \sigma < \infty$," "$X_t$" by "$Y_t$," and in the statement corresponding to Condition (i) replace "$\Lambda(e^{-\sigma t})$" by "$\Lambda(e^{\sigma t})$."

<u>$\sigma = 0$.</u> It is known [17] that

$$U(s) = \int_0^s (f(x)-x)^{-1}\,dx$$

generates a stationary measure for $\{Z_t\}$ and that $U(F(s,t)) = \lambda t + U(s)$ and hence that

$$1-F(s,t) = g(\lambda t + U(s))$$

where $g(\cdot)$ is again the inverse of $U(1-s)$. Assuming that (2.4) holds here, it immediately follows that (3.3.1) and (3.3.2) apply in the present context. Again defining $G(x) = 1 - h(1-g(x))$, (1) can be transformed to

$$-\log P_0(s,t) = (\delta/\lambda)\int_{U(s)}^{\lambda t+U(s)} G(y)\,dy.$$

The proofs of §3.3 apply here but there are some changes in parametrizations.

<u>Theorem 14.</u> <u>Let</u> $\sigma = 0$, $\int_0^\infty G(y)\,dy = \infty$ <u>and</u> (2.4) <u>hold</u>.

<u>If Condition (i) holds and</u> $a(t) = (1-F(0,t))^{-1}$ <u>then</u> $\{\Lambda(X_t)/\Lambda(a(t))\}$ <u>has the limiting DF</u> $(1\ \Lambda x)^{\delta/\lambda}$. <u>If, in addition,</u> $\liminf_{s\to 0} \mathcal{L}(s) > 0$, <u>we can replace</u> $\Lambda(a(t))$ <u>with</u> $\Lambda(t^{1/\nu})$.

<u>If Condition (ii) holds then</u> $\{(1-F(0,t))X_t\}$ <u>has the limiting DF whose LST is</u> $(1+\theta^{\nu})^{-\delta a/\lambda}$.

<u>If Condition (iii) holds and</u> $a(t) = 1 - h^{-1}(1-t^{-1})$ $(t \geq 1)$ <u>then</u> $\{a(t)X_t\}$ <u>has the limiting stable law whose LST is</u> $\exp(-\delta\theta^{\nu\Delta})$.

<u>If Condition (iv) holds then</u> $\{t[1-h(\exp(-X_t^{-1}))]\}$ <u>has the limiting DF</u> $D(\delta x)$.

When $\beta, \gamma < \infty$ it was shown in [31] that $\{X_t/t\}$ has a limiting gamma law. This follows immediately from the Condition (ii) statement above.

## 5. Asymptotic behavior of transition probabilities and the stationary measure.

A not insignificent portion of the literature on the simple branching process has been devoted to the determination of the rate of decay of its n-step transition probabilities. This problem has also been investigated for the BPI. If $m \leq 1$ and $h_0 > 0$ it can be inferred from results in [43] that

$$p_{ij}^{(n)} / p_{00}^{(n)} \longrightarrow \mu_j$$

where $\{\mu_j\}$ is the stationary measure (which is unique up to constant multipliers) normalized so that $\mu_0 = 1$ and hence it suffices to consider $\{p_{00}^{(n)}\}$. As above, we are interested in the case where $\delta$ is not positive. When $m = 1$ and $\beta, \gamma < \infty$ and some further regularity conditions hold, it has been shown [29] that $p_{00}^{(n)} \sim Kn^{-\sigma}$ $(\sigma = \beta/\gamma)$. When $1 < m \leq \infty$ a full analysis was given in [27]. Some general results and examples were given for $m \leq 1$ in [33] and here we shall consider the general results in conjunction with Conditions (i)-(iv).

If $m < 1$, Theorem 3 in [33] can be expressed as

$$p_{00}^{(n)} \sim K\, H(n) \tag{1}$$

where $0 < K < \infty$ and

$$\begin{aligned} H(x) &= \exp \int_0^x \log h(1 - \omega(m^x))\, dx \\ &= \exp \int_0^x \log [1 - G(-\log \omega(m^x))]\, dx. \end{aligned} \tag{2}$$

When $m = 1$, Theorem 4 of [33] can be written as at (1) but with

$$H(x) = \exp \int_0^x \log[1-G(x)]\,dx. \tag{3}$$

In the following theorem we define $\zeta = 1$ when $m = 1$.

Theorem 15. Let $m \le 1$ and (2.4) hold if $m = 1$.

If:

(a) Condition (i) holds then $\{p_{00}^{(n)}\}$ is sv;

(b) Condition (ii) holds then

$$p_{00}^{(n)} = n^{-a\zeta} R(n)$$

where $R(\cdot)$ is sv at infinity;

(c) Condition (iii)/(iv) hold then

$$p_{00}^{(n)} \sim K \exp(-\zeta n^{1-\Delta} \lambda(n)$$

where $\lambda(\cdot)$ is sv at infinity.

Proof. For Conditions (i) and (ii) it simply suffices to observe, using Lemma 1 when $m < 1$, that multiplying the integrands of (2) and (3) by $x$ gives functions which tend to $-a\zeta$ as $x \to \infty$; we define $a = 0$ for Condition (i). It follows that $H(x)$ is $-a\zeta$-varying at infinity.

For Conditions (iii)/(iv) it suffices to observe that the integrand of $-\log H(x)$ is $-\Delta$-varying at infinity whence the representation $H(x) = \exp(-\zeta x^{1-\Delta} \lambda(x))$ where $\lambda(\cdot)$ is sv at infinity.

Still assuming that $m \le 1$, let $A(s) = \sum \mu_j s^j$ which converges for $0 \le s < 1$ [43] and satisfies the equation

$$A(s) = h(s) A(f(s)).$$

Iteration and reference to (2.1) yields

$$A(f_n) = 1/P_0^{(n)}(0) = 1/p_{00}^{(n)}.$$

For $0 < s < 1$ choose the integer $n(s)$ for which $f_{n(s)} \leq s < f_{1+n(s)}$. Clearly $n(s) \uparrow \infty$ as $s \rightarrow 1-$ and

$$1 \leq A(s)/A(f_{n(s)}) < A(f_{1+n(s)})/A(f_{n(s)}) = 1/h(f_{n(s)}) \rightarrow 1$$

whence

$$A(s) \sim 1/p_{00}^{(n(s))} \qquad (s \rightarrow 1-). \tag{4}$$

Thus once we find the asymptotic dependence of $n(s)$ on $s$, we can appeal to (1)-(3) and /or Theorem 15 to obtain information on the growth of $A(s)$ and hence of $\sum_{j=0}^{n} \mu_j$.

When $m < 1$ we have $f_{n(s)} = 1 - \omega(m^{n(s)})$ whence

$$1 - m^{n(s)} \leq Q(s) \leq 1 - m^{1+n(s)} \tag{5}$$

giving

$$n(s) \sim \zeta \log(1-s)^{-1} \qquad (s \rightarrow 1-).$$

Similarly if $m = 1$ we obtain

$$n(s) \leq U(s) \leq 1 + n(s) \tag{6}$$

whence

$$n(s) \sim U(s).$$

These results may be applied to parts (a) and (b) of Theorem 15 to obtain, for example, for part (b): If $m < 1$ and Condition (ii) holds then

$$A(s) \sim [\zeta \log(1-s)^{-1}]^{a\zeta}/R(\log(1-s)^{-1})$$

and if $m = 1$

$$A(s) \sim (U(s))^{a\zeta}/R(U(s)).$$

Since $\log H(x)$ has a monotone derivative it follows that in Theorem 15, part (c), $\lambda(\cdot)$ has the integral representation

$$\lambda(x) = c \exp \int_1^x \varepsilon(y) y^{-1} dy$$

where $0 < c < \infty$ and $\varepsilon(x) \to 0$ $(x \to \infty)$. When $m = 1$ we see from (6) that $U(s) - n(s)$ is bounded and some algebra shows that

$$(n(s))^{1-\Delta} \lambda(U(s)) \to 0 \qquad (s \to 1-)$$

and hence that

$$A(s) \sim K^{-1} \exp((U(s))^{1-\Delta} \lambda(U(s))).$$

When $m < 1$ similar manipulations show that

$$A(s) \sim K^{-1} \exp\{\zeta[\log(1-s)^{-1}]^{1-\Delta} \lambda(\zeta \log(1-s)^{-1})\} \tag{7}$$

provided that $-\zeta \log(1-s) - n(s)$ remains bounded as $s \to 1-$. However, it follows from (5) that this is equivalent to the finitude of $Q'(1-)$. Thus if $m < 1$, Condition (iii) or (iv) holds and if $E(Z_1 \log Z_1) < \infty$ then (7) holds.

## 6. The reversed BPI

Esty [10] derived some limit theorems for a time reversed simple branching process. We now consider the corresponding reversed time BPI. For $0 < m \leq \infty$ let $h_0 > 0$ and $R = 1/h(q)$. When $m > 1$, $R > 1$ and it is known that $R^n p_{ij}^{(n)} \to q^i \gamma_j$ where $\gamma_j > 0$ if $j \in \mathcal{S}$, $\gamma(s) = \sum \gamma_j s^j$ converges if $s \leq q$, $\gamma(q) = 1$ and

$$R \sum \gamma_i p_{ij} = \gamma_j. \tag{1}$$

Thus if we define $\mu_j = \gamma_j/\gamma_0$ when $m > 1$ then it follows that for all $m$ the strong ratio limit property holds: if $i, j, k, \ell \in \mathcal{S}$ then

$$\lim_{n \to \infty} p_{ij}^{(n-m)}/p_{k\ell}^{(n)} = R^m q^{i-k} \mu_j/\mu_\ell .$$

If $0 < n_1 < \cdots < n_k$ and $0 \leq i_0, i_1, \ldots, i_k$ are integers and the i's are in $\mathcal{S}$ then

$$\lim_{n \to \infty} P(X_{n-n_1} = i_1, \ldots, X_{n-n_k} = i_k | X_n = i_0)$$

exists and is the joint probability

$$P(V_{n_1} = i_1, \ldots, V_{n_k} = i_k | V_0 = i_0)$$

of a Markov chain $\mathcal{V} = \{V_n\}$ having the n-step transition probabilities

$$Q_{ij}^{(n)} = R^n \mu_j p_{ji}^{(n)}/\mu_i \qquad (i, j \in \mathcal{S}). \tag{2}$$

We shall call $\mathcal{V}$ the reversed BPI.

When $m > 1$ the results quoted above show that $\mathcal{S}$ is positive for $\mathcal{V}$ and the limiting-stationary distribution is $\{q^j \gamma_j\}$. If $m \leq 1$, $R = 1$ and it follows that the classification of $\mathcal{S}$ for $\mathcal{V}$ is exactly that for $\chi$. In particular Theorems A and B hold for $\mathcal{V}$. We shall now consider cases where $\mathcal{S}$ is not positive for $\mathcal{V}$.

It follows from (2.1) and (1) that if $0 \leq t < 1$

$$\begin{aligned} \sum \mu_i t^i E_i(s^{V_n}) &= P_0^{(n)}(t) A(sf_n(t)) \\ &= A(t) A(sf_n(t)) / A(f_n(t)). \end{aligned} \tag{3}$$

If $m \leq 1$ and Condition (i) holds or if $m < 1$ and Conditions (ii)-(iv) hold and in the cases of (iii) and (iv) we also assume that $E(Z_1 \log Z_1) < \infty$, then the results of the last section show that $\alpha(x) = A(1 - x^{-1})$ $(x \geq 1)$ is sv at infinity.

If we choose $s = s_n$ in (3) so that $(1-f_n)/(1-s_n) \to 0$, the slow variation shows that

$$A(s_n f_n(t))/A(f_n(t)) \sim A(s_n)/A(f_n).$$

Vitali's theorem now yields

$$E_1(s_n^{V_n}) \sim A(s_n)/A(f_n).$$

Now choose $s_n$ so that $(1-s_n)^{-1} = \alpha^{-1}(x\alpha(a_n))$ where $a_n = (1-f_n)^{-1}$. Clearly $1 \le (1-f_n)/(1-s_n)$ and this quotient tends to infinity, for if not $\alpha((1-s_n)^{-1})/\alpha(a_n) \to 1$ as $n$ increases through some subsequence in $\mathbb{N}_+$ and this contradicts the definition of $s_n$. Thus $E_1(s_n^{V_n}) \to 1 \wedge x$ and application of Proposition 1 yields

<u>Theorem 16</u>. <u>If</u> $m \le 1$, $A(s) \to \infty$ $(s \to 1-)$ <u>and</u> $A(1-1/x)$ <u>is sv at infinity then</u> $\{[A(1-(1+V_n)^{-1})]/A(f_n)\}$ <u>has a limiting uniform distribution on</u> $(0,1)$.

<u>Theorem 17</u>. <u>If</u> $m = 1$ <u>and Condition (ii) holds then</u> $\{(1-f_n)V_n\}$ <u>has a limiting gamma distribution whose density is</u> $(x^{a\nu-1}e^{-x})/\Gamma(a\nu)$ $(x > 0)$.

<u>Remark</u>. If $\beta, \gamma < \infty$ then $\{X_n/\gamma n\}$ and $\{V_n/n\gamma\}$ have the same limiting distribution.

<u>Proof</u>. First observe that $A(1-x^{-1})$ is $a\nu$-varying at infinity and, from (3.3.2) and (3.3.3) $(1-f_n(t))/(1-f_n) \to 1$. Now set $s_n = \exp(-\theta(1-f_n))$; it is easily seen that $E(\exp(-\theta(1-f_n)V_n)) \to (1+\theta)^{-a\nu}$.

<u>Theorem 18</u>. <u>Let</u> $m = 1$, <u>Condition (iii) or (iv) hold and</u> $a_n = (1-f_n)/n\,G(n)$. <u>Then</u> $a_n V_n \xrightarrow{P} \nu$.

<u>Proof</u>. Let $s_n = \exp(-\theta a_n)$ and observe that $1-s_n f_n(t) \sim 1-f_n(t) \sim 1-f_n$ and hence that $U(s_n f_n(t)) \sim U(f_n)$. It follows from (5.1), (5.3) and (5.4), a mean value theorem and the regular variation of $G(\cdot)$ that

$$\begin{aligned}\log[A(s_n f_n(t))/A(f_n)] &\sim G(n)[U(s_n f_n(t)) - U(f_n)] \\ &\sim G(n)\, U'(f_n)(s_n f_n(t) - f_n) \\ &\sim -\theta G(n)\, U'(f_n) a_n (1-f_n)^{-1}\end{aligned}$$

where, in the penultimate step we have used the mean value theorem again. The theorem now follows on observing that $(1-s)^{-1}\, U'(s)/U(s) \longrightarrow \nu$ and $U(f_n) = n$.

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This paper reviews known limit theorems for population sizes of Bienaymé-Galton-Watson process allowing immigration. For non-critical cases it is known the limit distribution is non-defective iff a logarithmic moment of immigration distribution is finite. New results of this paper concern the situation where this moment is infinite & give limit theorems for a certain slowly varying function of the population size. A parallel discussion is given for the critical case & the continuous time process. Some results on rate of decay of transition probabilities & growth rate of stationary measure are given and used to obtain limit theorems for a reversed time process.

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