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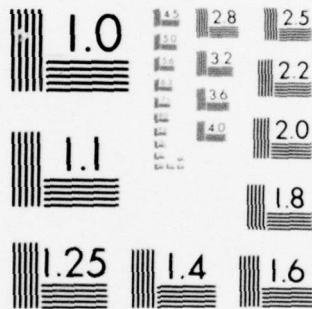
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CONFIDENCE INTERVALS RELATED TO SEQUENTIAL  
TESTS FOR THE EXPONENTIAL DISTRIBUTION

BY

D. SIEGMUND

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TECHNICAL REPORT NO. 3

OCTOBER 10, 1977

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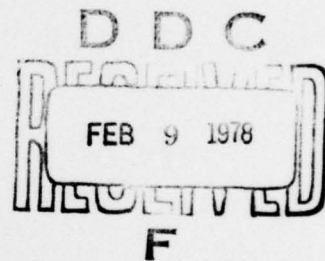
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by

D. Siegmund

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DEPARTMENT OF STATISTICS  
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Abstract

One sided sequential tests for the mean of an exponential distribution are proposed, and the related confidence intervals are computed. The tests behave like the classical sequential probability ratio test when the mean is small and like a fixed time test when the mean is large and accurate estimation is important.

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CONFIDENCE INTERVALS RELATED TO SEQUENTIAL  
TESTS FOR THE EXPONENTIAL DISTRIBUTION

1. Introduction

An important example of hypothesis testing for the purpose of decision making is the quality control of industrial production, for which the required decision frequently is acceptance or rejection of a production lot. In such cases sequential testing offers certain advantages over fixed sample size tests. Generally speaking sequential tests of the same Type I and Type II error probabilities (producer's risk and consumer's risk) require on the average less data to reach a decision than their fixed sample counterparts, and this shorter (average) testing time is usually economically desirable.

There may be cases in which one would like to supplement the decision provided by a test of hypothesis with a point or interval estimate for some parameter. An example is the testing of a prototype with the goal of deciding whether to start production or to continue development. In the case that a decision to start production is made, it may be advisable to have reliability estimates which are accurate enough to permit subsequent comparisons of the field reliability of production items with the laboratory reliability of the prototypes. In such cases the advantages of sequential tests are not so apparent; for the early termination of a sequential test, which seems advantageous from a decision making point of view, may result in insufficient data for accurate estimation.

The goal of this note is to show by a concrete example that sequential tests can be adapted to maintain some of their advantages while minimizing their concomitant disadvantages from the viewpoint of estimation.

## 2. A sequential test and the related confidence intervals

Assume that the times to failure of nominally identical items form a sequence  $x_1, x_2, \dots$  of independent exponentially distributed random variables with mean time between failures (mean lifetime) equal to  $\theta$ . Suppose that to determine the acceptability of these items one desires to test the hypothesis  $H_0: \theta \geq \theta_0$  v.s.  $H_1: \theta \leq \theta_1$  with some prespecified error probabilities  $\alpha = P_{\theta=\theta_0}(\text{Reject } H_0)$  and  $\beta = P_{\theta=\theta_1}(\text{Accept } H_0)$ . If  $H_0$  is rejected, the items are deemed unacceptable, and thus  $\alpha$  denotes the producer's risk. Similarly,  $\beta$  is the consumer's risk. By a change of scale it may be assumed that  $\theta_1 = 1$  and indeed will be in the rest of this note. To simplify the discussion it will be convenient to assume that one item is put on test, replaced when it fails by a second item, etc. until the test is terminated.

Let  $s_n = x_1 + \dots + x_n$  denote the time of the  $n^{\text{th}}$  failure. The customary fixed time test of  $H_0$  against  $H_1$  rejects  $H_0$  if and only if for appropriately chosen  $m_0$  and  $t_0$   $s_{m_0} \leq t_0$ . The test parameters  $m_0$  and  $t_0$  are chosen to give the desired values of  $\alpha$  and  $\beta$ . It is sometimes more convenient to describe the test in terms of  $X(t)$ , the number of failures prior to time  $t$ , which is a Poisson process of intensity  $\lambda = \theta^{-1}$ . The rejection region is given in terms of  $X(t)$  by  $X(t_0) \geq m_0$ . Operationally the test is usually censored, i.e., one

observes the process of failures until the  $m_0^{\text{th}}$  failure or for  $t_0$  units of time, whichever occurs first. Censoring has no effect on the accept-reject decision and hence no effect on the error probabilities. It does, however, mean that the actual time on test is a random variable  $\leq t_0$ . Nevertheless, the term "fixed time test" will be used to describe the censored version in order to distinguish it easily from the truly sequential tests described below. It is important to note that the confidence intervals obtained from a (censored) fixed time test are not in general the same as from a true fixed time test unless an accept decision is reached.

Suppose now that in addition to testing  $H_0$  one desires to give a confidence interval for the mean time between failures  $\theta$  (or equivalently for the Poisson intensity  $\lambda$ ). Suppose also that this confidence interval is of primary interest when  $H_0$  is accepted. When  $H_0$  is rejected, further development and testing will be necessary; so it is more important to reach a reject decision as soon as possible than to provide an extremely accurate estimate.

The following test of  $H_0$  against  $H_1$  is designed to perform like a sequential probability ratio test when  $H_1$  is true and early termination is desired and like a fixed length test when  $H_0$  is true and accurate estimation is of primary concern. Let  $\ell(t) = X(t) \log(\lambda_1/\lambda_0) - (\lambda_1 - \lambda_0)t$  be the log likelihood ratio for testing the simple hypothesis  $\lambda = \lambda_0 (= \theta_0^{-1})$  against  $\lambda = \lambda_1 (= \theta_1^{-1} = 1)$ . Define the stopping rule  $T =$  smallest value of  $t$  such that  $\ell(t) \geq c$ . Given  $t_1$ , stop testing at  $\min(T, t_1)$  and reject  $H_0$  if and only if  $T \leq t_1$ .

It is convenient to put  $a = c/\log(\lambda_1/\lambda_0)$  and  $b = (\lambda_1 - \lambda_0)/\log(\lambda_1/\lambda_0)$ , so that

$$(1) \quad T = \text{smallest value of } t \text{ such that } X(t) \geq a + bt .$$

(See Figure 1.) The Type I and Type II error probabilities are  $P_{\lambda=\lambda_0}\{T \leq t_1\}$  and  $P_{\lambda=\lambda_1}\{T > t_1\}$ . The parameters  $a$  and  $t_1$  should be chosen to make these equal to the desired  $\alpha$  and  $\beta$ .

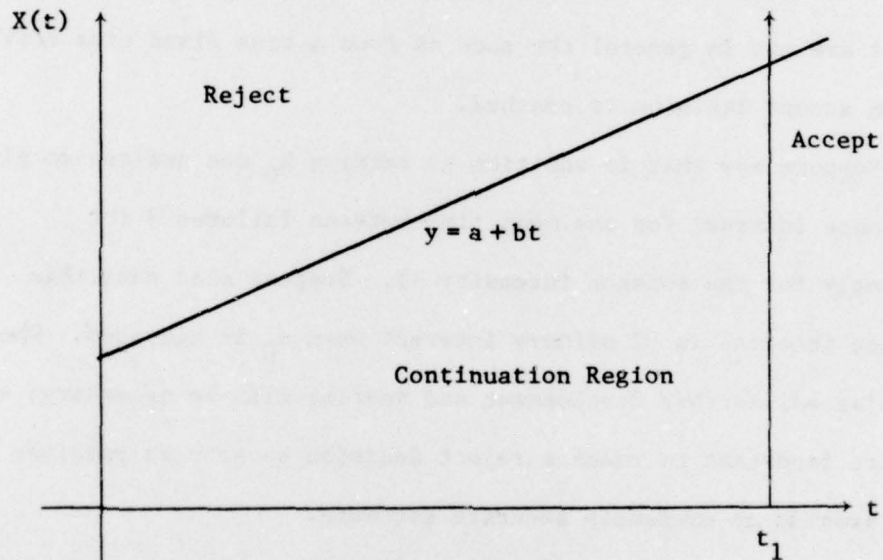


Figure 1. Stopping rule  $T$  and sequential test.

Computation of the error probabilities for this test is more complicated than for the customary fixed length test. A method which gives adequate approximations for practical purposes is described in the Appendix. The resulting approximations to the Type II and Type I error probabilities are respectively

$$(2) \quad P_1\{T > t_1\} \approx P_1\{s_m > t_1\} - (\lambda''/\lambda')^{a+bt_1} \exp\{t_1(\lambda'/\lambda'' - 1)\} P_{\lambda'/\lambda''}\{s_m > t_1\}$$

and

$$(3) \quad P_{\lambda_0}\{T \leq t_1\} \approx \lambda_0^a [(b - \lambda_0)/(1 - b) - \lambda_0 b^{-1} P_1\{T > t_1\}]$$

In equation (2)  $\lambda'$  and  $\lambda''$  are defined by

$$(4) \quad \lambda' t_1 = a + b t_1 \quad \text{and} \quad b = (\lambda' - \lambda'')/\log(\lambda'/\lambda'') ,$$

and  $m$  is the smallest integer  $\geq a + b t_1$ . Also  $P_\lambda$  denotes probability when the true intensity of  $X(t)$  is  $\lambda$ , so  $P_\lambda\{s_m > t_1\}$  is the probability that a  $\chi^2$  random variable with  $2m$  degrees of freedom exceeds  $2\lambda t_1$ . To attain given values  $\alpha = P_{\lambda_0}\{T \leq t_1\}$  and  $\beta = P_1\{T > t_1\}$  using (2) and (3), first set  $P_1\{T > t_1\}$  equal to the desired value  $\beta$  on the right hand side of (2) and then solve for  $a$  in terms of  $\alpha$  and  $\beta$ . The value of  $t_1$  may easily be determined from (2) by trial and error. Table 1 compares two (censored) fixed length tests with the corresponding sequential tests. For both examples  $\alpha = \beta = .10$ . The sequential tests require on the average about 30 percent less testing time than the fixed length tests when  $\lambda = 1$ , and it is desirable to minimize testing time. They pay for this by requiring about 30 percent more maximum testing time. The maximum testing time is attained whenever  $H_0$  is accepted, and in this case one desires to have an accurate estimate of  $\lambda$ . The fact that more time on test generally leads to more accurate estimation of  $\lambda$  tends to minimize the disadvantage of a larger maximum test duration. Nevertheless, a modified sequential test in which the maximum time on test is the same as in the fixed length test will be described in Section 3.

TABLE 1  
COMPARISON OF (CENSORED) FIXED LENGTH AND SEQUENTIAL TESTS

	$\theta_0 = 2(\lambda_0 = 1/2)$		$\theta_0 = 1.5(\lambda_0 = 2/3)$	
	Fixed Length	Sequential	Fixed Length	Sequential
Test Parameters	$m_0 = 14$	$a = 2.86,$ $b = .721,$ $m = 20$	$m_0 = 40$	$a = 5.1,$ $b = .822,$ $m = 58$
	$t_0 = 18.9$	$t_1 = 23.8,$ $\lambda' = .842,$ $\lambda'' = .613$	$t_0 = 48.3$	$t_1 = 64.3,$ $\lambda' = .901,$ $\lambda'' = .748$
Expected Testing Time When $\lambda = 1$	13	8.9	36	25

To compute confidence intervals for  $\lambda$  based on observing  $X(t)$  for  $0 \leq t \leq \min(T, t_1)$ , it is convenient to define several auxiliary functions. For fixed  $0 < \gamma \leq \frac{1}{2}$  let

$$\bar{\lambda}_1(t) = \sup\{\lambda: P_\lambda\{T \geq t\} \geq \gamma\} ,$$

$$\bar{\lambda}_2(n) = \sup\{\lambda: P_\lambda\{T > t_1, X(t_1) \leq n\} \geq \gamma\} ,$$

$$\underline{\lambda}_1(t) = \inf\{\lambda: P_\lambda\{T \leq t\} \geq \gamma\} ,$$

and

$$\underline{\lambda}_2(n) = \inf\{\lambda: P_\lambda\{T \leq t_1\} + P_\lambda\{T > t_1, X(t_1) \geq n\} \geq \gamma\} .$$

It is easy to see that  $\bar{\lambda}_i \geq \underline{\lambda}_i$  ( $i=1,2$ ). Moreover, by an argument similar to that given by Siegmund (1977) for a similar problem, it may be shown that

$$(5) \quad \bar{\lambda} = \begin{cases} \bar{\lambda}_1(T) & \text{if } T \leq t_1 \\ \bar{\lambda}_2(X(t_1)) & \text{if } T > t_1 \end{cases}$$

is a  $(1 - \gamma)$  100% upper confidence bound for  $\lambda$  and

$$(6) \quad \underline{\lambda} = \begin{cases} \underline{\lambda}_1(T) & \text{if } T \leq t_1 \\ \underline{\lambda}_2(X(t_1)) & \text{if } T > t_1 \end{cases}$$

is a  $(1 - \gamma)$  100% lower confidence bound for  $\lambda$ . Also  $[\underline{\lambda}, \bar{\lambda}]$  is a  $(1 - 2\gamma)$  100% confidence interval.

Computation of the probabilities entering into the definitions of  $\underline{\lambda}$  and  $\bar{\lambda}$  involves the same problems as computation of the error probabilities, and is discussed in detail in the Appendix. For many of the cases of greatest interest, i.e., when  $H_0$  is accepted,  $\bar{\lambda}$  and  $\underline{\lambda}$  are almost the same as the customary confidence limits based on the same data. For example, suppose that the sequential test defined above terminated at time  $t_1$  with  $X(t_1) = n$ . Then finding an upper confidence bound for  $\lambda$  involves computing

$$(7) \quad P_{\lambda}\{T > t_1, X(t_1) \leq n\} = P_{\lambda}\{X(t_1) \leq n\} - P_{\lambda}\{T < t_1, X(t_1) < n\} .$$

If a fixed time test were to lead to this same data, the appropriate  $(1 - \gamma)$  100% upper confidence bound for  $\lambda$  would be  $\lambda^*(X(t_1))$ , where  $\lambda^*(n) = \sup\{\lambda: P_{\lambda}\{X(t_1) \leq n\} \geq \gamma\}$ . Whenever the second probability on the right hand side of (7) is small compared to the first probability,  $\bar{\lambda}_2(n)$  will be about equal to  $\lambda^*(n)$ ; and an appropriate confidence bound based on the sequential test will be about the same as the usual confidence bound. This is presumably the

case unless  $n$  is quite close to  $a + bt_1$ . As an example, consider the first sequential test described in Table 1, for which  $a + bt_1 = 20$ , and suppose that  $X(t_1) = 17$ . The usual 80 percent confidence interval for  $\lambda$  would be  $[\underline{\lambda}, \bar{\lambda}] = [.50, .945]$ , while the sequential test based confidence interval is  $[\underline{\lambda}, \bar{\lambda}] = [.47, .92]$ . Although the usual interval is based on the hypothesis that a fixed length test was being used and is not strictly speaking correct under the present conditions, it is probably accurate enough for practical purposes. For  $X(t_1) < 17$  it will be even more accurate. If  $T \leq t_1$ , there is no reason to believe that the usual confidence interval based on the same data will approximate the interval  $[\underline{\lambda}, \bar{\lambda}]$  defined in (5) and (6).

### 3. Other Sequential Tests

The sequential test suggested above has one important disadvantage vis a vis the customary fixed length test, to wit its maximum test duration is greater. One can argue that this is actually an advantage, because the maximum test time is attained when  $H_0$  is accepted and accurate estimation of  $\lambda$  is of primary importance. Nevertheless, it is interesting to note that one can find tests similar in spirit to the sequential tests of Section 2 with no increase in the maximum testing time over the usual fixed length tests.

Suppose that  $t_0$  and  $m_0$  define a fixed length test as in Section 2 and let the stopping rule  $T$  be defined by (1) with  $b = -(1 - \lambda_0)/\log \lambda_0$  as before, but with  $a$  to be determined. Consider the class of sequential tests which stop testing at  $\min(T, t_0)$  and for some  $m_0^* < a + bt_0$  reject  $H_0$  if and only if  $T \leq t_0$  or  $T > t_0$  and

$X(t_0) \geq m_0^*$ . The Type I and Type II error probabilities are respectively

$$P_{\lambda_0} \{T \leq t_0\} + P_{\lambda_0} \{T > t_0, X(t_0) \geq m_0^*\}$$

and

$$P_1 \{T > t_0, X(t_0) < m_0^*\} ;$$

and the two free test parameters,  $a$  and  $m_0^*$ , are to be chosen to make these error probabilities equal to the desired  $\alpha$  and  $\beta$ . Approximate computation of these probabilities presents essentially the same problems as the determination of the confidence intervals of the preceding section and may be handled by the methods of the Appendix. Unfortunately, a trial and error determination of  $a$  and  $m_0^*$  is a little more complicated than determination of  $a$  and  $t_1$  in Section 2. Nevertheless, it can be accomplished fairly quickly by starting from the values of  $a$  for the sequential test and  $m_0$  for the fixed length test in Section 2 and increasing both slightly to obtain trial values of  $a$  and  $m_0^*$ .

Corresponding to the first test in Table 1 ( $\lambda_0 = 1/2$ ,  $\alpha = \beta = .1$ , and  $t_0 = 18.9$ ), for  $a = 3$  and  $m_0^* = 15$  the test defined above has approximate error probabilities  $\alpha = .099$  and  $\beta = .109$ , which are probably close enough to their nominal values for practical purposes. The expected testing time when  $\lambda = 1$  is about 9.5, which is only slightly larger than that for the sequential test of Section 2 and still considerably smaller than that for the fixed time test. Computation of confidence intervals for this test is exactly the same as for the sequential test of Section 2. It is interesting to note

that whenever both this test and the fixed time test accept  $H_0$  (i.e.,  $T > t_0$  and  $X(t_0) < m_0$ ) the confidence limits for the sequential test are shifted slightly towards smaller values of  $\lambda$  (larger values of  $\theta$ ) than the customary intervals, but the difference is probably not important for practical purposes.

Until now it has been assumed that the desire to estimate  $\lambda$  accurately when  $H_0$  is true is of primary consideration and that early termination of the test under  $H_0$  is relatively unimportant. Hence, the proposed sequential tests have been designed to behave like a sequential probability ratio test under  $H_1$  and like a fixed time test under  $H_0$ . It is undoubtedly possible to obtain somewhat earlier termination for small values of  $\lambda$  without an appreciable loss of estimation accuracy, but whether the benefits are worth the additional complication remains for a future study.

One possibility is to introduce a lower stopping boundary  $-c + bt$ , so that testing terminates if  $X(t) \geq a + bt$ ,  $X(t) \leq -c + bt$ , or  $t = t_2$  for some suitably chosen  $a$ ,  $c$ , and  $t_2$ . This stopping rule is similar to the usual (truncated) sequential probability ratio test. However, by choosing  $c$  fairly large, one does not reduce the sampling time under  $H_0$  "too much" and accurate estimation is still possible. Confidence intervals may be defined similarly to those above, and it is hoped that the approximate computational methods of the Appendix may be adapted to this case as well.

## Appendix

The following Theorem is the theoretical basis for (2) and the calculations required to determine  $\bar{\lambda}_i$  ( $i=1,2$ ). It is related to Proposition 1 of Siegmund (1977). (See also Woodroffe, 1976.) The approximation (3) is based on a heuristic argument which is given following the proof of Theorem 1. An approximation to the expected sample size for  $\lambda > \lambda'$  is suggested at the end of this Appendix.

Theorem 1. Let  $T$  be defined by (1). Let  $m$  denote the least integer  $\geq a + bt_1$ . Assume that  $a \rightarrow \infty$  and  $t_1 \rightarrow \infty$  in such a way that  $\Delta = m - (a + bt_1)$  remains constant and for some  $\lambda' > b$

$$(8) \quad \lambda' t_1 = a + bt_1 \quad .$$

Define  $\lambda'' < b$  by

$$(9) \quad b = (\lambda' - \lambda'') / \log(\lambda' / \lambda'') \quad .$$

Then for each  $\lambda > 0$ ,  $k=1,2,\dots$ , as  $a \rightarrow \infty$

$$P_\lambda \{T < t_1, X(t_1) = m - k\} \sim (\lambda'' / \lambda')^{a + bt_1} \exp\{\lambda t_1 (\lambda' / \lambda'' - 1)\} P_{\lambda \lambda' / \lambda''} \{X(t_1) = m - k\} \quad .$$

Proof. By the definition of conditional probability

$$(10) \quad P_\lambda \{T < t_1, X(t_1) = m - k\} = P_\lambda \{T < t_1 | X(t_1) = m - k\} P_\lambda \{X(t_1) = m - k\} \quad .$$

It follows from Lemmas 1 and 3 below that the conditional probability in (10) converges to  $(\lambda'' / \lambda')^{k - \Delta}$  as  $a \rightarrow \infty$ . The theorem follows by substitution of the values of  $P_\lambda \{X(t_1) = m - k\}$ .

Lemma 1.  $P_\lambda\{T < t_1 | X(t_1) = m - k\}$  does not depend on  $\lambda$ , and under the conditions of Theorem 1 converges to  $P_\lambda\{X(t) \leq -k + \Delta + bt \text{ for some } t > 0\}$  as  $a \rightarrow \infty$ .

Proof. That  $P_\lambda\{T < t_1 | X(t_1) = m - k\}$  does not depend on  $\lambda$  is an immediate consequence of the sufficiency of  $X(t_1)$ . Hence, it suffices to give the proof for some particular value of  $\lambda$ . Also for  $0 < r < t_1$

$$\begin{aligned} P_\lambda\{X(t) \geq a + bt_1 \text{ for some } t_1 - r \leq t < t_1 | X(t_1) = m - k\} &\leq P_\lambda\{T < t_1 | X(t_1) = m - k\} \\ (11) \quad &\leq P_\lambda\{T_1 < t_1 - r | X(t_1) = m - k\} + P_\lambda\{X(t_1) \geq a + bt \text{ for some } t_1 - r \leq t < t_1 | X(t_1) = m - k\}. \end{aligned}$$

By symmetry the first probability in (11) equals

$$\begin{aligned} P_\lambda\{X(t_1) - X(t) \leq -k + \Delta + b(t_1 - t) \text{ for some } t_1 - r \leq t < t_1 | X(t_1) = m - k\} \\ = P_\lambda\{X(t) \leq -k + \Delta + bt \text{ for some } 0 < t \leq r | X(t_1) = m - k\}. \end{aligned}$$

It is easy to see by direct calculation that conditional on  $X(t_1) = m - k \sim \lambda' t_1$ , the process  $X(t)$ ,  $0 \leq t \leq r$ , converges to a Poisson process of intensity  $\lambda'$ . Hence, the preceding conditional probability converges to

$$P_\lambda\{X(t) \leq -k + \Delta + bt \text{ for some } 0 < t \leq r\}.$$

Since  $r$  is arbitrary, to complete the proof it suffices by (11) to show that

$$(12) \quad P_\lambda\{T < t_1 - r | X(t_1) = m - k\} \leq \varepsilon(r),$$

where  $\varepsilon(r)$  does not depend on  $a$  (or  $t_1$ ) and converges to 0 as  $r \rightarrow \infty$ .

First observe that

$$P_{\lambda}\{X(t_1) = m - k\} = \exp\{(\lambda' - \lambda)t_1\}(\lambda/\lambda')^{m-k} P_{\lambda'}\{X(t_1) = m - k\},$$

and by the local limit theorem and (8)

$$P_{\lambda'}\{X(t_1) = m - k\} \sim (2\pi m)^{-1/2}.$$

Hence, to prove (12) it suffices to show for some  $\lambda > 0$

$$(13) \quad P_{\lambda}\{T \leq t_1 - r\} \leq \varepsilon(r) \exp\{(\lambda' - \lambda)t_1\}(\lambda/\lambda')^{m-k} m^{-1/2}.$$

The proof of (13) is simplified notationally by assuming that  $m = a + bt_1$  and taking  $r$  to be such that  $br = j$  is an integer. The general case requires only slight changes. Then

$$\begin{aligned} P_{\lambda}\{T \leq t_1 - r\} &= P_{\lambda}\{X(t) \geq a + bt \text{ for some } t \leq t_1 - r\} \\ &= P_{\lambda}\{n \geq a + bs_n \text{ for some } n \leq m - j\} \\ &\leq \sum_{a \leq n \leq m-j} P_{\lambda}\{s_n \leq b^{-1}(n - a)\} = \sum_{a \leq n \leq m-j} P_{\lambda}\{X(b^{-1}(n - a)) \geq n\} \\ &= \sum_{a \leq n \leq m-j} \sum_{i \geq n} \exp\{(\lambda' - \lambda)b^{-1}(n - a)\}(\lambda/\lambda')^i P_{\lambda'}\{X(b^{-1}(n - a)) = i\}. \end{aligned}$$

It may be shown for all  $t \leq t_1$  and  $i \geq a$ --by standard large deviation theory for  $t \leq \varepsilon t_1$  (Bahadur and Rao, 1960) and by a local limit theorem for  $\varepsilon t_1 \leq t \leq t_1$ --that  $P_{\lambda'}\{X(t) = i\} \leq Cm^{-1/2}$ . Hence, for all  $\lambda < \lambda'$

$$\begin{aligned} P_{\lambda}\{T \leq t_1 - r\} &\leq C(1 - \lambda/\lambda')^{-1} m^{-1/2} \exp\{(\lambda - \lambda')b^{-1}a\} \sum_{a \leq n \leq m-j} [\exp\{(\lambda' - \lambda)b^{-1}\}(\lambda/\lambda')]^n \\ &= C(1 - \lambda/\lambda')^{-1} m^{-1/2} (\lambda/\lambda')^{m-j} \\ &\quad \cdot \exp\{(\lambda - \lambda')(-t_1 + b^{-1}j)\} \sum_{a \leq n \leq m-j} [\exp\{(\lambda' - \lambda)b^{-1}\}(\lambda/\lambda')]^{n-m+j}. \end{aligned}$$

By (9), for  $\lambda'' < \lambda < \lambda'$ ,  $\exp\{(\lambda' - \lambda)b^{-1}\}(\lambda/\lambda') > 1$  and hence the indicated sum above can be extended as a convergent series to  $n = -\infty$ . Comparing the resulting inequality with (13) and recalling that  $j = br$  completes the proof.

The following lemmas are known, but for lack of a convenient reference and for completeness their proofs are sketched.

Lemma 2. Let  $\mu_1, \mu_2$  be positive numbers and  $\tau$  a stopping rule for  $X(t)$ ,  $0 \leq t < \infty$ . Then for arbitrary  $t > 0$

$$(14) \quad P_{\mu_1} \{ \tau \leq t \} = \int_{\{ \tau \leq t \}} \exp\{(\mu_2 - \mu_1)\tau\} (\mu_1/\mu_2)^{X(\tau)} dP_{\mu_2}$$

Proof. Let  $P_{\lambda}^{(t)}$  denote the restriction of  $P_{\lambda}$  to the space of  $X(s)$ ,  $0 \leq s \leq t$ , and let  $Z(t) = \exp\{(\mu_2 - \mu_1)t\} (\mu_1/\mu_2)^{X(t)}$  be the likelihood ratio of  $P_{\mu_1}^{(t)}$  relative to  $P_{\mu_2}^{(t)}$ . Then

$$P_{\mu_1} \{ \tau > t \} = \int_{\{ \tau > t \}} Z(t) dP_{\mu_2} .$$

Since  $Z(t)$ ,  $0 \leq t < \infty$ , is a  $P_{\mu_2}$ -martingale,

$$1 = E_{\mu_2} [Z\{\min(\tau, t)\}] = \int_{\{ \tau < t \}} Z(\tau) dP_{\mu_2} + \int_{\{ \tau > t \}} Z(t) dP_{\mu_2} ,$$

and (14) follows by subtraction.

Lemma 3. For each  $x > 0$  and  $\lambda' > b$

$$P_{\lambda} \{ X(t) \leq -x + bt \text{ for some } t > 0 \} = (\lambda''/\lambda')^x ,$$

where  $\lambda'' < b$  is defined by (9).

Proof. Let  $\tau = \inf\{t: X(t) \leq -x + bt\}$ , and observe that with probability 1

$$(15) \quad X(\tau) = -x + bt \quad \text{on} \quad \{\tau < \infty\} .$$

Letting  $t \rightarrow \infty$  in (14) and appealing to (9) and (15) yields

$$\begin{aligned} P_{\lambda'}\{\tau < \infty\} &= \int_{\{\tau < \infty\}} \exp\{(\lambda'' - \lambda')\tau\} (\lambda'/\lambda'')^{X(\tau)} dP_{\lambda''} \\ &= (\lambda''/\lambda')^x P_{\lambda''}\{\tau < \infty\} . \end{aligned}$$

Since  $\lambda'' < b$ ,  $P_{\lambda''}\{\tau < \infty\} = 1$ , which completes the proof.

Corollary 1. For  $\lambda > \lambda'$  and  $k = 1, 2, \dots$ , under the conditions of Theorem 1

$$(16) \quad \begin{aligned} P_{\lambda}\{T \leq t_1, X(t_1) \leq m - k\} &\sim (\lambda''/\lambda')^{a+bt_1} \\ &\cdot \exp\{\lambda t_1 (\lambda'/\lambda'' - 1)\} P_{\lambda\lambda'/\lambda''}\{X(t_1) \leq m - k\} . \end{aligned}$$

Proof. Obviously for all  $j > 0$

$$(17) \quad \begin{aligned} P_{\lambda}\{T \leq t_1, m - k - j < X(t_1) \leq m - k\} &\leq P_{\lambda}\{T \leq t_1, X(t_1) \leq m - k\} \\ &\leq P_{\lambda}\{T \leq t_1, m - k - j < X(t_1) \leq m - k\} + P_{\lambda}\{X(t_1) \leq m - k - j\} . \end{aligned}$$

For fixed  $j$  the probability on the left and the first probability on the right of (17) may be evaluated asymptotically by appealing to Theorem 1. The second term on the right hand side of (17) may be shown to be of smaller order of magnitude for large  $j$  by standard large deviation arguments. See Bahadur and Rao (1960) or Siegmund (1975b) for the details of similar arguments.

Remarks. (i) Corollary 1 may be used to evaluate approximately the probability in (2) and the probabilities appearing in the definitions of  $\bar{\lambda}_1$  by virtue of the relation

$$P_{\lambda}\{T > t_1, X(t_1) \leq n\} = P_{\lambda}\{X(t_1) \leq n\} - P_{\lambda}\{T \leq t_1, X(t_1) \leq n\} .$$

The approximation can be expected to yield accurate results only when  $n$  is close to  $a + bt_1$ , i.e.,  $n = m - k$  for a fairly small  $k$ , and  $\lambda > \lambda'$ . But these are essentially the only values of interest, as was observed in Section 2.

(ii) A slightly more sophisticated argument similar to that given by Siegmund (1977) shows that (16) holds for all  $\lambda > \lambda''$ ,  $\lambda \neq \lambda'$ .

(iii) It is tempting to evaluate the probability appearing in the definition of  $\lambda_2(n)$  by re-writing it as

$$P_{\lambda}\{X(t_1) \geq n\} + P_{\lambda}\{T \leq t_1, X(t_1) \leq n\}$$

and then applying Corollary 1. However, this evaluation is of interest for small  $\lambda$ , when Corollary 1 is not applicable.

Now suppose that  $\lambda_0 < \lambda''$ , and let  $\lambda_1 > \lambda'$  be defined by

$$(18) \quad (\lambda_1 - \lambda_0) = b \log (\lambda_1 / \lambda_0) .$$

These values  $\lambda_0$  and  $\lambda_1$  may be the values specified by the test of hypothesis of Section 2 but need not be. The approximation (3) and an approximate evaluation of the probabilities appearing in the definitions of  $\lambda_1$  ( $i = 1, 2$ ) may be obtained by writing

$$(19) \quad P_{\lambda_0}\{T \leq t_1\} = P_{\lambda_0}\{T < \infty\} - P_{\lambda_0}\{t_1 < T < \infty\}$$

and giving separate approximations for the two terms on the right hand

side of (19). Letting  $t \rightarrow \infty$  in Lemma 2 (note that  $P_{\lambda_1} \{T < \infty\} = 1$ ) and appealing to (18) gives the representations

$$P_{\lambda_0} \{T < \infty\} = (\lambda_0/\lambda_1)^a E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T)-a-bt}\}$$

and

$$(20) \quad P_{\lambda_0} \{t_1 < T < \infty\} = (\lambda_0/\lambda_1)^a \int_{\{T > t_1\}} E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T)-a-bT} | X(t_1)\} dP_{\lambda_1}.$$

The limiting value of  $E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T)-a-bt}\}$  as  $a \rightarrow \infty$  is given in Lemma 4 below leading to the asymptotic relation

$$(21) \quad P_{\lambda_0} \{T < \infty\} \sim \{(b - \lambda_0)/(\lambda_1 - b)\} (\lambda_0/\lambda_1)^a \quad \text{as } a \rightarrow \infty.$$

The integral in (20) may be re-written as

$$(22) \quad \sum_{k=1}^m P_{\lambda_1} \{T_a > t_1, X(t_1) = m - k\} E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T_{k-\Delta}) - k + \Delta - bT_{k-\Delta}}\},$$

where  $T_x = \inf\{t: t > 0, X(t) \geq x + bt\}$  and  $\Delta = m - (a + bt_1)$  as before.

To obtain a simple approximation to (22), one might replace the expectations there by their limits as  $k \rightarrow \infty$ , or one might replace all these expectations by that for  $T_0$ , which is also given in Lemma 4.

The latter alternative seems more attractive for two reasons:

(a) the values of  $P_{\lambda_1} \{T > t_1, X(t_1) = m - k\}$  are larger for smaller values of  $k$ , and (b) using  $k = 0$  rather than  $k$  infinitely large gives a smaller approximation to (22) and hence a larger (more conservative) approximation to  $P_{\lambda_0} \{T \leq t_1\}$ . Replacing the  $T_{k-\Delta}$  by  $T_0$  in (22) leads by Lemma 4 below to the approximation

$$P_{\lambda_0} \{t_1 < T < \infty\} \approx \lambda_0 b^{-1} P_{\lambda_1} \{T > t_1\} (\lambda_0/\lambda_1)^a .$$

Substituting this result and (21) into (19) gives

$$P_{\lambda_0} \{T \leq t_1\} \approx (\lambda_0/\lambda_1)^a [(b - \lambda_0)/(\lambda_1 - b) - \lambda_0 b^{-1} P_{\lambda_1} \{T > t_1\}] ,$$

which is (3) in the special case  $\lambda_1 = 1$ .

Lemma 4. For  $\lambda_0$  and  $\lambda_1$  defined by (18)

$$\lim_{a \rightarrow \infty} E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T_a) - a - bT_a}\} = (b - \lambda_0)/(\lambda_1 - b)$$

and

$$E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T_0) - bT_0}\} = \lambda_0/b$$

Proof. Let  $\tau_a = \inf\{n: n - bs_n \geq a\}$ , so  $X(T_a) = \tau_a$  and  $T_a = s_{\tau_a}$ . Also let  $\tau_- = \inf\{n: n - bs_n < 0\}$ . By Lemma 2 and random walk theory (Feller, 1966, Chapter XII)

$$\begin{aligned} E_{\lambda_1} \{(\lambda_0/\lambda_1)^{X(T_0) - bT_0}\} &= P_{\lambda_0} \{T_0 < \infty\} = 1 - P_{\lambda_0} \{\tau_0 = \infty\} \\ &= 1 - 1/E_{\lambda_0} \tau_- = 1 - (1 - b\lambda_0^{-1})/E_{\lambda_0} (\tau_- - bs_{\tau_-}) = 1 + (1 - b\lambda_0^{-1})/b\lambda_0^{-1} = \lambda_0/b . \end{aligned}$$

By considering the renewal process defined by  $\tau_0 - bs_{\tau_0}$ , the renewal theorem gives the  $P_{\lambda_1}$  limiting distribution of  $\tau_a - bs_{\tau_a} = X(T_a) - bT_a$  as  $a \rightarrow \infty$  (Feller, 1966, Chapter XI). Combined with (18) and random walk theory (Feller, 1966, Chapter XII), this leads to

$$\begin{aligned}
\lim_{a \rightarrow \infty} E_{\lambda_1} \{ (\lambda_0/\lambda_1)^{X(T_a) - a - bT_a} \} &= \lim_{a \rightarrow \infty} E_{\lambda_1} \exp\{-b^{-1}(\lambda_1 - \lambda_0)(\tau_a - bs_{\tau_a})\} \\
&= \{E_{\lambda_1}(\tau_0 - bs_{\tau_0})\}^{-1} \int_0^\infty \exp\{-b^{-1}(\lambda_1 - \lambda_0)x\} P_{\lambda_1}\{\tau_0 - bs_{\tau_0} > x\} dx \\
&= \{(1 - b/\lambda_1)E_{\lambda_1}\tau_0\}^{-1} b(\lambda_1 - \lambda_0)^{-1} P_{\lambda_0}\{\tau_0 = \infty\} \\
&= b(\lambda_1 - \lambda_0)^{-1} (1 - b/\lambda_1)^{-1} (E_{\lambda_0}\tau_-)^{-1} P_{\lambda_1}\{\tau_- = \infty\} \\
&= b(\lambda_1 - \lambda_0)^{-1} (1 - b/\lambda_1)^{-1} (1 - \lambda_0/b)(1 - \lambda_0/\lambda_1) \\
&= (b - \lambda_0)/(\lambda_1 - b) .
\end{aligned}$$

There remains the task of computing approximations to  $E_\lambda\{\min(T, t_1)\}$ , especially for  $\lambda \geq 1$ . With the help of Theorem 1, this becomes an easy matter. It is easy to see that for  $\lambda > b$

$$(23) \quad E_\lambda\{\min(T, t_1)\} = E_\lambda(T) - \int_{\{T > t_1\}} E_\lambda\{T - t_1 | X(t_1)\} dP_\lambda .$$

The usual Wald approximation to  $E_\lambda(T)$  obtained by using Wald's identity and ignoring the excess  $X(T) - (a + bT)$  is

$$(24) \quad E_\lambda(T) \approx a/(\lambda - b) .$$

The same argument gives the approximation

$$(25) \quad E_\lambda\{T - t_1 | X(t_1) = m - k\} \approx (k - \Delta)/(\lambda - b) .$$

Hence the second term on the right hand side of (23) may be re-written

$$\sum_{k=1}^m E_\lambda\{T - t_1 | X(t_1) = m - k\} [P_\lambda\{X(t_1) = m - k\} - P_\lambda\{T \leq t_1, X(t_1) = m - k\}]$$

and evaluated approximately with the help of (25) and Theorem 1 to obtain

$$\int_{\{T > t_1\}} E_{\lambda} \{T - t_1 | X(t_1)\} dP_{\lambda} \cong (\lambda - b)^{-1} [(a + bt_1) P_{\lambda} \{T > t_1\} - \lambda t_1 P_{\lambda} \{s_{m-1} > t_1\} + (\lambda''/\lambda')^{m-1} \lambda t_1 \exp\{\lambda t_1 (\lambda'/\lambda'' - 1)\} P_{\lambda \lambda' / \lambda''} \{s_{m-1} > t_1\}].$$

(26)

Substituting (24) and (26) into (23) gives an approximation to  $E_{\lambda} \{\min(T, t_1)\}$ . It should be fairly accurate at least whenever  $\lambda > \lambda'$ , so  $P_{\lambda} \{T > t_1\}$  is small and the second term on the right hand side of (23) is small compared with the first. It is possible to approximate the expected excess over the boundary  $E_{\lambda} \{X(T) - a - bT\}$ , and hence give a slightly more accurate approximation. However, the increase in accuracy is rather modest in contrast to the problem of approximating error probabilities, where the excess over the boundary is more important. (See Siegmund, 1975a.)

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