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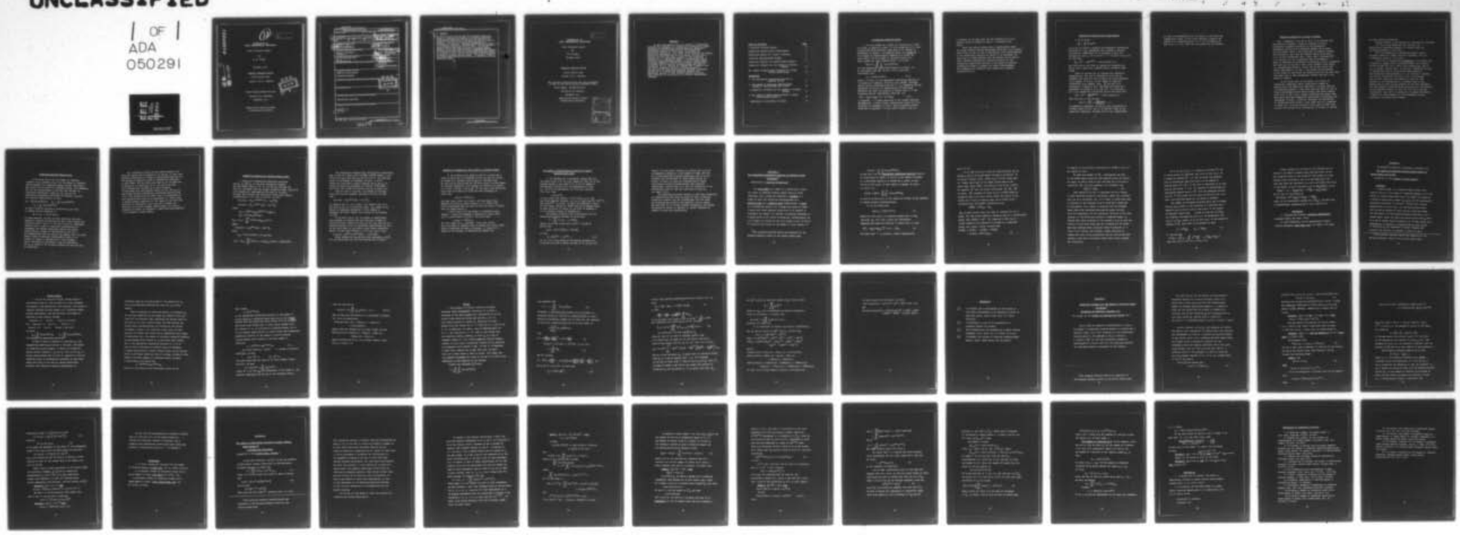

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ENUMERATION OF LARGE COMBINATORIAL STRUCTURES

Final Technical Report

By

E. M. Wright

November 1977

EUROPEAN RESEARCH OFFICE

United States Army

London, N.W.1., England

Grant Number DA-ERO-75-G-014

University of Aberdeen

Aberdeen, U.K.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ENUMERATION OF LARGE COMBINATORIAL STRUCTURES.		5. TYPE OF REPORT & PERIOD COVERED FINAL TECHNICAL REPORT, DEC [redacted]74 - NOV [redacted]77
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) [redacted] Edward M. Wright		8. CONTRACT OR GRANT NUMBER(s) DAERO-75-G-014
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Aberdeen, Aberdeen, U.K.		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 6.11.02A - 1T161102B1BC - 00485
11. CONTROLLING OFFICE NAME AND ADDRESS U.S. Army R&S Group (EUR) Box 65 FPO NEW YORK 09510		12. REPORT DATE NOV[redacted]77
		13. NUMBER OF PAGES 52
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE

DISTRIBUTION UNLIMITED

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Combinatories; Graph Theory.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

SEE OVER

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20 Abstract

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Abstract

We give new results about the number of 2-connected labelled (n,q) graphs, i.e. graphs on n points and q lines, including a combinatorial interpretation of Temperley's differential equation satisfied by the exponential generating function of this number (applicable in Statistical Mechanics). This leads to two methods for finding asymptotic approximations to this number. We also find a curious (but not very deep) paradox in the asymptotic enumeration of unlabelled graphs. I continue my work on connected, sparsely-edged graphs of various kinds, again including asymptotic results with possible applications. I complete Wille's work on oriented graphs and prove a conjecture of his. Finally Dr. Sheehan and I describe our new "ghost expansion" method to obtain asymptotic results from tha Exclusion-Inclusion Theorem by applying the method to a particular graphical example. The appendices consist of four research papers which have been submitted for publication to different mathematical journals.

Table of Contents.

2-connected labelled graphs.

1. An (n,q) graph is a graph on n points and q lines (no loops, no parallel lines). It is k-connected if at least k points and their adjacent edges have to be removed to disconnect the graph. A 2-connected graph is a block. The (2,1) graph is conventionally regarded as a block. We write b(n,q) for the number of labelled blocks (i.e. whose points are (say) numbered from 1 to n) and

$$B(X,Y) = \sum_{n=2}^{\infty} \sum_{q} b(n,q) X^n Y^q / n!$$

for the exponential generating function of b(n,q). It is well-known (see [4], pp.10,11 for a proof and references) that

$$C_X = X \partial B(Z,Y)/\partial Z, \qquad (1.1)$$

where $C = C(X,Y)$ is the exponential generating function of the number of connected, labelled (n,q) graphs, suffixes denote partial differentiation and $Z = C_X$. Temperley [6] was interested in B(X,Y) for applications in Statistical Mechanics and used the calculus to deduce from (1.1) the following partial differential equation, viz.

$$X^2\{1 + B_{XX}(1-XB_{XX})^{-1}\} = 2(1+Y)B_Y. \qquad (1.2)$$

In Appendix 1 (a paper submitted to the journal Discrete Mathematics), I find a direct combinatorial proof of (1.2). My method is analagous to that I used in another problem described in Appendix 1 of last year's Report, that is, I consider in how many ways one can construct an (n,q+1) labelled block by adding an edge to a suitable (n,q) graph.

While one cannot always find a combinatorial proof for every identity between generating functions, it is usually worth while to try. If one is successful, one's understanding of the underlying structure is likely to be increased. In this case, it seems possible that the method can be used to find a partial differential equation satisfied by the exponential generating function of the number of labelled 3-connected graphs or perhaps for higher connectedness. The arguments are certainly more complicated, perhaps prohibitively so.

Asymptotic results about connectedness

2. If we write

$$\beta_n = \sum_q b(n,q)Y^q$$

in (1.2), we obtain a sequence of recurrence differential equations for β_n, reasonably well adapted to machine calculation, which enable us to determine b(n,q) for successive n and q. If we write $Z = 1 + Y$ and $N = n(n-1)/2$, we find that

$$\beta_n = Z^N - \alpha_2 Z^{N-n+2} + \text{lower powers of } Z$$

and from this we can find an asymptotic expansion for b(n,q) for n large and q greater than a suitable lower bound.

There is another way of finding this asymptotic expansion and this can be extended to find such an expansion not only for (n,q) blocks, but for k-connected (n,q) graphs. If d(n,q) is the number of labelled k-connected (n,q) graphs, then $\sum_q d(n,q)Y^q$ is a polynomial of degree N in Y. This may be written as a polynomial of degree N in Z, say P(Z). If $q \geq N-n+k+1$, all (n,q) graphs are k-connected and we can prove that

$$P(Z) = Z^N - k\binom{n}{k}Z^{N-n+k} + \text{lower powers of } Z$$

and from this we can see that

$$d(n,q) = \binom{N}{q} - k\binom{n}{k}\binom{N-n+k}{q-k+1} + \ldots,$$

an asymptotic expansion for d(n,q). The calculation of further terms in P(Z), and so of further terms in the asymptotic expansion, becomes increasingly complicated, as does the determination of the range of q in which the expansion is valid. It all looks possible, however, and should give a generalisation to $k > 1$ of the results of [9] for $k = 1$, but I have not yet worked out the details.

Unlabelled graphs on q lines; a paradox

3. As I remarked at the end of § 1, it is possible that the method of Appendix 1 might be extended to find a differential equation satisfied by the exponential generating function for 3-connected graphs. But a paper by T.R.S.Walsh (which I have been sent to referee) tackles this problem by the "core and mantle" method which gives (1.1) in the 2-connected case. I have seen a reference [8] to a solution for the 3-connected case by N.Wormald, a pupil of Professor R.W.Robinson at Newcastle, N.S.W., Australia, and have sent to ask for a copy. Since Wormald (like Walsh) is a postgraduate student, I hope that he has had my idea rather than Walsh's, so that I can retire from the matter in his favour. If both Walsh and Wormald have had the same idea, they must come to some arrangement between them and I will pursue my method.

In the second part of his paper, as a result of substantial computing, Walsh conjectures a number of results about the asymptotic behaviour of certain families of unlabelled graphs. I could show that the truth of some of these conjectures follows fairly simply from known results in [1,2,10,11,12]; then I became interested and proved some more which were less immediate, essentially by extending the techniques of [12]. I could not recommend the publication of conjectures whose truth I could prove and so I suggested to Walsh that we should publish a joint paper giving these results with proofs while leaving him, of course, to publish his main results separately.

Amongst others, I have proved the following two theorems:

(I) If k is any fixed positive integer, the proportion of unlabelled graphs on n points which are k-connected tends to 1 as $n \to \infty$.

(II) The proportion of unlabelled graphs without isolated points on q edges which are connected, but not 2-connected, tends to 1 as $q \to \infty$.

Since (I) is clearly true <u>a fortiori</u> if we confine our attention to graphs without isolated points, the contrast between (I)and (II) is at first sight paradoxical. But it is, of course, obvious that, if one has a countable infinity of objects of which an infinity has some property P and an infinity lack property P, it is possible to arrange the objects in mutually exclusive sets of increasing size so that almost all the members of each set have property P or, alternatively, to arrange them so that almost none of each set has property P. But this "paradox" has been created deliberately and artificially. The interest in the contrast between (I) and (II) lies in it arising "naturally", i.e. not by a selection to produce the result. I know of no other example of this phenomenon arising naturally, but others may be known.

Connected sparsely-edged graphs

4. We write $f(n,q)$ for the number of labelled connected (n,q) graphs, $v(n,q)$ for the number of labelled smooth graphs (i.e. connected graphs without end points) and $u(n,q)$ for the number of labelled blocks. (See Temperley [6] for the applications of $f(n,n+k)$ and $u(n,n+k)$ in Statistical Mechanics.) Again W_k, V_k, U_k are the exponential generating functions for $f(n,n+k)$, $v(n,n+k)$ and $u(n,n+k)$ respectively, so that

$$W_k = \sum_n f(n,n+k)X^n/n!, \quad V_k = \sum_{n\geq 3} v(n,n+k)X^n/n!$$
$$U_k = \sum_{n\geq 3} u(n,n+k)X^n/n!$$

In [13] I used two methods to calculate W_k as a sum of powers (mostly negative) of

$$\theta = 1-G = 1-\sum n^{n-1}X^n/n!$$

Of these methods, one was entirely practical and well-adapted to machine calculation (in fact, Gray, Murray and Young [3] used it up to $k = 24$) but it was not inherently obvious that it could be continued indefinitely as k increased. The second (reduction) method was highly inefficient as a means of calculating W_k for $k > 3$, but demonstrated that the form of W_k wassuch that the first method could be continued indefinitely (that is, in theory; of course, a machine with a finite memory would eventually be saturated). From the expression for W_k in powers of θ an exact formula for $f(n,n+k)$ can be calculated.

As I described briefly in last year's report, the reduction method can be applied to V_k and U_k, in each case proving inefficient as a means of calculating them for $k > 3$ but giving an essential piece of information (different in each case). For V_k, this shows that V_k is the same sum of powers of 1-X that W_k is of θ and so we can read off a formula for v(n,n+k) as a sum of binomial coefficients from the computer result for W_k. For U_k, we can deduce another differential recurrence equation from (1.2). The reduction method shows that, for successive k, this equation must lead to an expression as a finite sum of powers (mostly negative) of 1-X. Again we can solve the equation for successive k and read off a formula for u(n,n+k) as a sum of binomial coefficients. All this is described in detail in Appendix 2, which has been submitted to the Journal of Graph Theory.

Asymptotic results for sparsely-edged graphs

5. Using the differential-recurrence equations for W_k, V_k and U_k, we can find recurrence formulae for the lowest power of θ or 1-X (as the case may be) that is, the power with the largest negative index. This gives us an asymptotic approximation for each of f(n,n+k), v(n,n+k) and u(n,n+k) for successive fixed k and large n. In particular, if $k = O(1)$ as $n \to \infty$, we have

$$f(n,n+k) = \alpha_k \, n^{n+\frac{1}{2}(3k-1)}\{1 + O(n^{-\frac{1}{2}})\},$$

$$v(n,n+k) = \beta_k \, n^{(3k-1)} n!\{1 + O(n^{-1})\},$$

where

$$\alpha_k = \pi^{\frac{1}{2}} 3^{k+1} k! a_k / 2^{\frac{1}{2}(5k+1)} \Gamma(\tfrac{3}{2}k+1),$$

$$\beta_k = 3^{k+1} k! a_k / 2^k (3k)!,$$

$$a_{k+1} = a_k + \sum_{h=1}^{k-1} h!(k-h)! a_h a_{k-h} / (k+1)!$$

Again

$$u(n,n+k) = \gamma_k n^{3k-1} n! \{1 + O(n^{-1})\},$$

where

$$\gamma_k = 2(3k-1)(3k-4)\ldots 5.2 b_k / (3k)!$$

$$b_{k+1} = b_k + \sum_{h=1}^{k-1} (3h-1)\ldots 5.2 b_h b_{k-h} / (3k+2)\ldots\{3(k-h)+2\}.$$

The recurrence sequences $\{a_k\}$ and $\{b_k\}$ are interesting and, so far as I know, have not been studied before. I can prove that $a_k \to a$ and that $b_k \sim bk^{\frac{1}{3}}$ as $k \to \infty$ and evaluate a and b to 6 places of decimals. In fact, I can do better and find fairly simple formulae for a_k and b_k in terms of k which are accurate to 6 places of decimals for $k \geq 20$. But I have not written this up in detail yet.

When we turn to s(n,n+k), the number of strongly connected (n,n+k) digraphs, discussed in [14], we find that

$$s(n,n+k) = \delta_k n^{3k-1} n! \{1 + O(n^{-1})\},$$

for large n and k = O(1), where δ_k depends only on k, but this is all we can discover about δ_k, except by an amount of computing equivalent to that needed to calculate the exact formula for s(n,n+k) by the methods of [14].

If we write h(n,q) for the number of Hamiltonian (n,q) graphs, we cannot find any recurrence formula to give us an exact formula for h(n,n+k) and can only use the reduction method, with its inevitable inefficiency for $k > 3$. On the other hand, we can prove a much better asymptotic result. We need only restrict k to be o(n) and we have

$$h(n,n+k) = \{n^{2k-1} n!/2^{k-1} k!\} \{1 + O(k^2/n^2)\}.$$

These asymptotic results have been announced in [15] but the proofs have not yet been written up in detail.

Asymptotic formulae for the number of oriented graphs

6. An oriented (n,q) graph is one on n points and q lines in which any two different points A,B are not joined or joined by a directed line AB <u>or</u> by a directed line BA, but not by both AB and BA. We write $r(n,q)$ for the number of oriented unlabelled (n,q) graphs, $N = \frac{1}{2}n(n-1)$ as before, $\mu = \mu(q) = 2(q/n) - \log n$ and $\Lambda = \Lambda(n,q) = N!/\,q!(N-q)!n!$ Wille used the results of [10] and other arguments to prove that

$$r(n,q) \sim 2^q \Lambda(n,q),$$

provided that $\mu \to \infty$ as $n \to \infty$. For any fixed n the maximum of $2^q \Lambda(n,q)$ occurs exactly at $q = [2(N+1)/3]$. Wille conjectures that the same is true for $r(n,q)$. I prove this for large enough n.

I write $T(n,q)$ for the number of unlabelled (n,q) graphs. I found the asymptotic behaviour of $T(n,q)$ when $\mu < A$ in some detail in [11]. I now prove that

$$r(n,q) \sim 2^q T(n,q)$$

for this range of μ, provided that $q \to \infty$ with n, and so am able to complete Wille's results for $r(n,q)$. This work is described in detail in Appendix 3, which is to appear in the Journal of Combinatorial Theory (B).

The number of Hamiltonian circuits in a large, heavily-edged graph

7. Dr. Sheehan and I published a paper [5] with the above title. G is an (n,q) graph, $\alpha = q/n$, β is the maximum degree of any point of G, H is the number of Hamiltonian circuits in $\overline{G}$, the complement of G, and $M = (n-1)!/2$. We provedthat, if $\alpha \to a$ as $n \to \infty$ and $\beta = o(n)$, then

$$H/M \to e^{-2a}.$$

We also stated the result:- (III) If A_1, A_2, ε are any fixed positive numbers, $A_1 < \alpha < A_2 \log n$ and $\beta = O(n^{1-\varepsilon})$, then $H \sim Me^{-2\alpha}$ as $n \to \infty$. We have now submitted a proof of this result (which Dr. Erdős told me he thought would be very difficult to prove) in the form of Appendix 4 to the Glasgow Journal of Mathematics.

The primary interest lies in the method. By the Exclusion-Inclusion Theorem, we prove that

$$H = \sum_{r=0}^{x-1} (-1)^r L_r + (-1)^x \theta L_x,$$

where x is at our choice and $0 \leq \theta \leq 1$. In [5] we proved that

$$L_r/M = \{(2\alpha)^r/r!\}\{1 + r^2 o(1)\},$$

so that

$$H = M\{e^{-2\alpha} + o(e^{2\alpha}) . \qquad (7.1)$$

If $\alpha = O(1)$. this gives us our earlier theorem. But, if $\alpha \to \infty$ as $n \to \infty$, however slowly, we do not get our result. Our new method consists in proving that L_r/M has an asymptotic expansion to any number of terms and that the coefficients have certain properties. But we do not evaluate the coefficients of these terms (except the first). In fact, we could not evaluate any of these coefficients without putting severe, and otherwise unnecessary, restrictions on G. But the properties of the coefficients that we find are sufficient to give us our result (III).

The Exclusion-Inclusion Theorem is used very frequently in Combinatorics and, from the point of view of getting asymptotic results, has the disadvantage that the terms are alternatively positive and negative. Thus we cannot get more than (7.1) without further information. Our method (described in detail in Appendix 4) should therefore be capable of much wider application.

Appendix 1

The exponential generating function of labelled blocks

E.M.Wright †

(Submitted to Discrete Mathematics)

An (n,q) graph is a graph on n points and q lines (no loops, no parallel lines); except where we state otherwise, the n points are labelled. A network is a graph in which two points are distinguished as a positive pole and a negative pole respectively. A block is a 2-connected graph (i.e. a graph from which at least 2 points and their adjacent lines have to be removed to disconnect the graph) or a maximal 2-connected sub-graph of a graph which is not itself 2-connected; conventionally the (2,1) graph is a block and the (1,0) graph is not. We write $N = n(n-1)/2$ and $b(n,q)$ is the number of (n,q) blocks. If

† The research reported herein was supported by the European Research Office of the United States Army.

$$F(X,Y) = \sum_n \sum_q f(n,q)X^nY^q/n!,$$

we say that F is the exponential generating function (e.g.f.) of f and write $F = E(f)$. If $f(n,q)$ is the number of graphs of a particular family on n points and q lines, we say that F is the e.g.f. of that family of graphs. We write $B = E(b)$, i.e.

$$B(X,Y) = \tfrac{1}{2}X^2Y + \sum_{n=3}^{\infty} \sum_{q=n}^{N} b(n,q)X^nY^q/n!,$$

so that B is the e.g.f. of the family of blocks. We use suffixes to denote partial differentiation.

It is well known that

$$\log C_X = \partial B(Z,Y)/\partial Z, \qquad (1)$$

where C is the e.g.f. of connected graphs and $Z = XC_X$. (See [1], pp. 10,11 for a proof and references). Temperley [2] used the calculus to deduce from (1) that

$$X^2\{1 + B_{XX}(1-XB_{XX})^{-1}\} = 2(1 + Y)B_Y. \qquad (2)$$

My object here is to produce a direct combinatorial proof of (2).

If we form an (n,q+1) block in every possible way by adding a line to an (n,q) graph, we have a collection $\mathcal{C}$ of (n,q+1) blocks. In $\mathcal{C}$, every (n,q+1) block occurs just q+1 times, since each of its edges occurs once as the added edge. Hence $|\mathcal{C}| = (q+1)b(n,q+1)$ and so $B_Y = E(|\mathcal{C}|)$. We separate $\mathcal{C}$ into the three collections $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$. Of these, $\mathcal{C}_1$ consists of those (n,q+1) blocks formed by adding an edge to an (n,q) block. There are b(n,q) of the latter and to each of them an edge can be edded in N-q different ways. Hence $|\mathcal{C}_1| = (N-q)b(n,q)$ and

$$E(|\mathcal{C}_1|) = \tfrac{1}{2}X^2B_{XX} - YB_Y.$$

$\mathcal{C}_2$ is empty except when n=2, when it contains the (2,1) graph, the only block formed by adding an edge to a disconnected graph; thus $E(|\mathcal{C}_2|) = \tfrac{1}{2}X^2$. $\mathcal{C}_3$ consists of the (n,q+1) blocks formed by the addition of an edge to a connected graph, not itself a block. We have then

$$E(|\mathcal{C}_3|) = E(|\mathcal{C}|) - E(|\mathcal{C}_1|) - E(|\mathcal{C}_2|)$$

$$= (1+Y)B_Y - \tfrac{1}{2}X^2(1+B_{XX}). \qquad (3)$$

It remains to find another expression for $E(|\mathcal{C}_3|)$, which we can equate to this.

We take each member of $\mathcal{C}_3$, distinguish the ends of the added edge as positive and negative poles and remove the edge. We can do this in just two ways and so we have a collection $\mathcal{C}_4$ of (n,q) networks, all different, and

$$|\mathcal{C}_4| = 2\,|\mathcal{C}_3|. \qquad (4)$$

Each network M in $\mathcal{C}_4$ is connected but not a block. It must therefore contain s cut-points, where $s \geqslant 1$. Neither pole can be a cut-point, for, if it were, it would have been a cut-point in the original (n,q+1) block and a block has no cut-points. If we remove a cut-point and its adjacent edges from M, the resulting disconnected graph can have only two components, for the subsequent addition of the line joining the two poles must produce a connected graph. It follows that each cut-point of the network M lies on just two blocks of M and that every path in M joining the two poles must pass through every cut-point. Hence M consists of a chain of s+1 blocks, each having a single cut-point in common with each of its neighbours. The two end blocks each contain a pole and a cut-point; every other block contains two cut-points.

Let F be the e.g.f. of a family of networks $\mathcal{F}$, all of whose points are labelled, and let G be the e.g.f. of a family of networks $\mathcal{G}$, in each of which the negative pole is unlabelled. Then the e.g.f. of the number of ordered pairs ($\mathcal{F}$, $\mathcal{G}$) is FG. This is unaltered if in each pair we now fasten $\mathcal{F}$ and $\mathcal{G}$ together by identifying the unlabelled negative pole of $\mathcal{G}$ with the labelled positive pole of $\mathcal{F}$ and regard the new point as labelled but not a pole. The resulting graph is, of course, a network.

The number of different networks which can be formed from an (n,q) graph by the selection of a positive and a negative pole is n(n−1) and so the e.g.f. of the family of networks formed from blocks in this way is X^2B_{XX}. If, however, the negative pole is to be unlabelled, the e.g.f. is XB_{XX}. Hence, if D_s is the e.g.f. of the number of members of $\mathcal{C}_4$ which have s cut-points, we have

$$D_1 = X^3B_{XX}^2, \qquad D_{s+1} = XB_XD_s. \tag{5}$$

It follows that

$$E(|\mathcal{C}_4|) = \sum_{s=1}^{\infty} D_s = \sum_{s=1}^{\infty} X^{s+2}B_{XX}^{s+1} = X^3B_{XX}^2(1-XB_X)^{-1}.$$

From this and (3) and (4), we have (2).

A minor variant on the above is to consider what we obtain if we attach a single block with two poles in the way described above to our network M. The result is a new network (with different n and q) of the same kind, but with more than one cut-point, i.e. with $s \geq 2$. Hence the e.g.f. of the family of all M for which $s \geq 2$ is $XB_{XX}E(|\mathcal{C}_4|)$ and so

$$E(|\mathcal{C}_4|) = D_1 + XB_{XX}E(|\mathcal{C}_4|) = X^3B_{XX}^2 + XB_{XX}E(|\mathcal{C}_4|).$$

By (3) and (4) this gives us

$$\{2(1+Y)B_Y - X^2(1+B_{XX})\}(1-XB_{XX}) = X^3B_{XX}^2,$$

which is (2), multiplied through by $(1-XB_{XX})$. We have thus a combinatorial interpretation of this form of (2).

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1. F.Harary and E.Palmer, Graphical Enumeration, (Academic Press, New York, 1973).

2. H.N.V.Temperley, On the enumeration of Mayer cluster integrals, Proc. Phys. Soc. 72 (1959), 1141-1144.

Appendix 2

E.M.Wright†, University of Aberdeen, Aberdeen, U.K.

The number of connected sparsely-edged graphs II

Smooth graphs and blocks.

(Submitted to the *Journal of Graph Theory*.)

Abstract.

A smooth graph is a connected graph without end-points; $f(n,q)$ is the number of connected graphs, $v(n,q)$ the number of smooth graphs and $u(n,q)$ the number of blocks on n labelled points and q edges: W_k, V_k, U_k are the exponential generating functions of $f(n,n+k)$, $v(n,n+k)$ and $u(n,n+k)$ respectively. For any $k \geq 1$, our reduction method shows that V_k can be deduced at once from W_k, which was found for successive k by the computer method described in our former paper. Again the reduction method shows that U_k must be a sum of powers (mostly negative) of $1-X$ and, given this information, we develop a recurrence method well suited to calculate U_k for successive k. Exact formulae for $v(n,n+k)$ and $u(n,n+k)$ for general n follow at once.

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† The research reported herein was supported by the European Research Office of the United States Army.

Smooth graphs.

1. We use the notation of [4], except where we specifically vary it. Our aim here is to find formulae for general n and particular k for v(n,n+k), the number of "smooth" labelled (n,n+k) graphs (i.e. connected graphs without end-points), and for u(n,n+k), the number of labelled (n,n+k) blocks. Clearly

$$v(n,n+k) = 0 \quad (k<0), \qquad v(n,n) = (n-1)!/2$$

and $$u(n,n+k) = 0 \quad (k<-1), \qquad u(2,1) = 1,$$

$$u(n,n-1) = 0 \quad (n>2), \qquad u(n,n) = (n-1)!/2.$$

We write

$$V_k = \sum_{n=3}^{\infty} v(n,n+k)X^n/n!, \qquad U_k = \sum_{n=3}^{\infty} u(n,n+k)X^n/n!,$$

the exponential generating functions.

In [4] we used two methods to calculate W_k, the exponential generating function of f(n,n+k), the number of connected (n,n+k) graphs. Of these, the reduction method readily extende to V_k and U_k. Just as for W_k, it proves impracticable for $k>3$, but it supplies us with an essential piece of information, different in each case; armed with this, we are able to devise practicable methods, well adapted to machine computation, to determine V_k and U_k for particular k: the methods for V_k and U_k are entirely different and that for V_k is very simple.

When we applied the reduction method to determine W_k, we took any connected (n,n+k) graph and removed every end-point and its adjacent edge, repeating the process until we had what we now call a smooth graph. We then reduced the smooth graph homeomorphically by eliding all the points of degree 2 except a few carefully chosen ones and then removed the labels. The result was an unlabelled basic graph. We considered j(n), the number of different connected labelled (n,n+k) graphs which reduced to a particular basic graph, and J, the exponential generating function of j. To do this we had to consider all possible ways of arranging t points (including the points already there) on the special paths in the basic graph and then of rooting t trees at these points, the total number of (labelled) points on these t trees being n. We found that

$$J = G^{r+2a+c}(1-G)^{-a-b-c}/g,$$

where g is the order of the automorphic group of the basic graph,

$$G = \sum n^{n-1}X^n/n!$$

is the exponential generating function of the number of rooted trees on n labelled points and r,a,b,c are numbers derived from the particular basic graph. If we go back from our particular basic graph only to those smooth graphs which are reduced to it, we distribute n labelled points, not t, on the special paths and add no rooted trees. The exponential generating function of the number of smooth labelled (n,n+k) graphs which reduce to the basic graph is therefore

$$J_1 = X^{r+2a+c}(1-X)^{-a-b-c}/g. \qquad (1)$$

If we now write $W_k = W_k(\theta)$ (a change of notation from [4]), we have

$$W_k = \sum J, \qquad V_k = \sum J_1,$$

the sums being over the same set of basic graphs. Hence, for $k \geq 1$, we have

$$V_k = W_k(1-X) = \sum_{s=-3k}^{2} c_{ks}(1-X)^s,$$

since $\theta = 1-G$. The c_{ks} are determined in the course of the machine computation for W_k and we can therefore easily read off v(n,n+k) as

$$v(n,n+k) = n! \sum_{s=1}^{3k} c_{k,-s} B(n+s-1,\ s-1) \qquad (n>2).$$

Thus we see that v(n,n+k)/n! is a polynomial of degree 3k-1 in n. In particular,

$$24v(n,n+1)/n! = 26 - 19B(n+1,1) + 5B(n+2,2)$$
$$= (n-3)(5n-8)/2.$$

Again, from the formula for W_2 in §3 of [4], we have

$$48v(n,n+2)/n! = 15B(n+1,5) - 5B(n,4) - 2B(n-1,3)$$
$$-5B(n-2,2) - B(n-3,1),$$

again divisible by (n-3), as we should expect, since v(3,4) = v(3,5) = 0.

Blocks

2. The property of being a block is obviously invariant under homeographic reduction. If we have a catalogue of the basic graphs for a particular k, we can select those which are blocks and sum the J_1 of (1) over those selected to obtain U_k. There are two basic graphs which are blocks for k = 1, nine for k = 2 and 48 for k = 3. In practice, of course, this is impracticable for $k > 3$ (and tedious for k = 3). What we can deduce from the reduction method, however, is that U_k is a sum of integral powers of (1-X), mostly negative. This is useful, because we shall find a recurrence formula for U_k involving an integration with respect to $\phi = 1-X$ and it is important to know that, however far we may take this, we shall (for $k \geq 1$) never obtain a term in $\log \phi$. This means that our computer programme will only be brought to an end by the exhaustion of the capacity of the machine.

Riddell and Uhlenbeck [1] wrote

$$F = \sum_{n=1}^{\infty} \sum_{q=0}^{\infty} f(n,q) Z^n Y^q / n!$$

and remarked that

$$e^F = 1 + \sum_{n=1}^{\infty} Z^n(1+Y)^N/n! \qquad (2)$$

Professor C.C.Rousseau [2] pointed out to me that, by carrying further an idea due to Temperley [3], he could deduce my recurrence formula for W_k from (2); the converse is also true. Temperley [3] goes on to show that, if

$$S = \sum_{n=2}^{\infty}\sum_{q=1}^{\infty} u(n,q)Z^nY^q/n!,$$

then

$$Z^2 + Z^2\frac{\partial^2 S}{\partial Z^2}\left(1-Z\frac{\partial^2 S}{\partial Z^2}\right)^{-1} = 2(1+Y)\frac{\partial S}{\partial Y}. \qquad (3)$$

We put $Z = X/Y$ and $S = (X^2/2Y) + T$, so that

$$T = \sum_{k=0}^{\infty} Y^kU_k \qquad (4)$$

and (3) becomes

$$X^3 + YX^2(1+X)\frac{\partial^2 T}{\partial X^2} = 2(1+Y)\left(1-X-XY\frac{\partial^2 T}{\partial X^2}\right)\left(X\frac{\partial T}{\partial X} + Y\frac{\partial T}{\partial Y}\right). \qquad (5)$$

If we put $Y = 0$ in (5), we find that

$$U_0' = X^2\{2(1-X)\}^{-1}, \qquad (6)$$

where a dash denotes differentiation with respect to X. We write

$$\zeta_k = XU_k' + kU_k + (-1)^k\{X(1+X)/2\}, \qquad (7)$$

so that

$$X\frac{\partial T}{\partial X} + Y\frac{\partial T}{\partial Y} + \frac{X(1+X)}{2(1+Y)} = \sum_{k=0}^{\infty} Y^k \zeta_k.$$

If we multiply both sides of (5) by $(1+Y)^{-1} = \sum_{s=0}^{\infty}(-1)^s Y^s$ and thenequate the coefficients of Y^k, we have

$$\zeta_k = X(1-X)^{-1}\left\{\sum_{s=0}^{k-1} U_s'' \zeta_{k-1-s} + (-1)^k \tfrac{1}{2}\right\} \; (k \geq 1). \qquad (8)$$

We write $\phi = 1-X$ and find U_0'' and ζ_0 from (6) and (7) as sums of integral powers of ϕ. If we have done the same for ζ_h and U_h'' for $0 \leq h \leq k-1$, then we can repeat this for ζ_k by (8). From (7), we have

$$U_k = (1-\phi)^{-k}\int_{\phi}^{1}(1-\phi)^{k-1}\zeta_k d\phi + \frac{(-1)^{k+1}}{2}\left(\frac{(1-\phi)}{k+1} + \frac{(1-\phi)^2}{k+2}\right) \qquad (9)$$

and so we can calculate U_k. We know from our reduction method that U_k is a sum of powers of ϕ, i.e. the integrand in (9) has no term in ϕ^{-1}. Thus we find U_k and U_k'' from (9) as sums of powers of ϕ and so can repeat the process to determine U_k for successive k. It is easily seen that ζ_k

and U_k'' contain no positive powers of ϕ. Thus we have

$$U_k = \sum_{t=-2}^{3k} u_{kt}\phi^{-t},$$

where the u_{kt} can be determined by machine calculation based on the above. It follows that

$$u(n,n+k) = n! \sum_{t=1}^{3k} u_{kt} B(n+t-1,t-1),$$

provided $k > 0$ and $n > 2$.

It is convenient to extract the factor $(1-\phi)^4$ from U_1 and U_2 and the factor $(1-\phi)^5$ from U_3. We have then

$$12U_1 = (1-\phi)^4(\phi^{-3} + 2\phi^{-2}),$$

$$48U_2 = (1-\phi)^4(5\phi^{-6} + 6\phi^{-5} + \phi^{-4} - 4\phi^{-3} - 6\phi^{-2}),$$

$$720U_3 = (1-\phi)^5(220\phi^{-9} + 275\phi^{-8} + 120\phi^{-7} - 30\phi^{-6} - 117\phi^{-5} - 126\phi^{-4} - 72\phi^{-3}).$$

and so

$$12u(n,n+1)/n! = B(n-2,2) + 2B(n-1,1) = (n-3)(n+2)/2,$$

$$48u(n,n+2)/n! = 5B(n+1,5) + B(n,4) + B(n-1,3) - 4B(n-2,2) - 6B(n-3,1),$$

$$720u(n,n+3)/n! = 220B(n+3,8) + 275B(n+2,7) + 120B(n+1,6) - 30B(n,5) - 117B(n-1,4) - 126B(n-2,3) - 72B(n-3,2),$$

so that, as we should expect, $u(n,n+2)$ is divisible by

$n-3$ and $u(n,n+3)$ by $(n-3)(n-4)$. In fact,

$$1152\, u(n,n+2)/n! = (n-3)(n^4 + 4n^3 - 15n^2 - 46n - 40)$$

and

$$1451520\, u(n,n+3)/n! = (n-3)(n-4)(11n^6 + 146n^5 + 281n^4 - 1891n^3 - 8608n^2 - 12614n - 7560).$$

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Appendix 3

Asymptotic Formulae for the Number of Oriented Graphs

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(To appear in the Journal of Combinatorial Theory (B).)

Wille found an asymptotic approximation to $r(n,q)$, the number of unlabelled oriented graphs on n points and q directed lines, for a wide interval of q and conjectured that, for given n, the maximum of $r(n,q)$ occurs at $q = [2(N+1)/3]$. We find the (different) asymptotic approximation to $r(n,q)$ valid for the remaining interval of q and prove Wille's conjecture for all large n.

† The research reported herein was supported by the European Research Office of the United States Army.

We write T(n,q) for the number of non-isomorphic unlabelled graphs (no loops,no multiple linws) on n points and q lines and r(n,q) for the corresponding number of oriented unlabelled graphs, i e. graphs in which any two points are not joined, or joined by a directed line in one direction or by a directed line in the other direction. Wewrite also $N = n(n-1)/2$, $\Lambda = \Lambda(n,q) = N!/\{q!(N-q)!n!\}$, $\mu = \mu(q) = 2(q/n) - \log n$.

In what follows, A,C and η are numbers, not always the same at each occurrence. Of these,A and C are positive and independent of n and q. A denotes any positive number we may choose, while C is a suitable positive number which may depend on any A present or implied. All our statements carry the implied condition that n and q are large enough, that is, that $n > C$ and $q > C$. The O( ) notation refers to the passage of n and q to infinity and the constant implied is a C. An η is a number which is $O(q^{-C})$ for some C.

Wille [1] has proved that

$$r(n,q) \sim 2^{q}\Lambda(n,q), \qquad (1)$$

provided that $\mu \to \infty$ as $n \to \infty$. I have proved [2] that

$$T(n,q) \sim \Lambda(n,q), \qquad (2)$$

provided that $\min\{\mu(q), \mu(N-q)\} \to \infty$ as $n \to \infty$. I found the asymptotic behaviour of $T(n,q)$ when $\mu < A$ in some detail in [3]; Theorem 1 enables us to deduce that of $r(n,q)$.

<u>Theorem 1</u>. <u>If</u> $q \to \infty$ <u>and</u> $\mu < A$ <u>as</u> $n \to \infty$, <u>then</u>

$$r(n,q) = 2^q T(n,q)\{1 + O(q^{-C})\}.$$

For any fixed n the maximum of $\Lambda(n,q)$ occurs at $q = [2(N+1)/3]$. Wille conjectures that the same is true for $r(n,q)$. I prove this for large enough n.

<u>Theorem 2</u>. <u>If</u> $n > C$, <u>then the maximum of</u> $r(n,q)$ <u>occurs when</u> $q = [2(N+1)/3]$.

Willeremarks that

$$2^q \Lambda(n,q) \leq r(n,q) \leq 2^q T(n,q) \qquad (3)$$

for all n and q and so deduces (1) from my (2) for those q for which the latter holds. From Theorem 3 of [2] we have the following lemma.

<u>Lemma 1</u>. <u>If</u>

$$AN < q < (1-A)N, \qquad (4)$$

<u>then</u>

$$T(n,q) = \Lambda(n,q)\{1 + O(e^{-Cn})\}.$$

If (4) is satisfied, it follows from (3) and Lemma 1, that

$$r(n,q) = 2^q \Lambda(n,q)\{1 + O(e^{-Cn})\}.$$

Hence

$$r(n,q+1)/r(n,q) = \{2(N-q)/(q+1)\}\{1+O(e^{-Cn})\}$$
$$= (1+3\{(q-Q)/(q+1)\})\{1+O(e^{-Cn})\},$$

where $3Q = 2N-1 = n^2 - n - 1 \equiv \pm 1 \pmod 3$. Since $e^{-Cn} = o(1/(q+1))$, the maximum of $r(n,q)$ in the range (4) occurs at

$$q = [(Q+1)] = [2(N+1)/3].$$

It follows easily from Wille's result (1) that this is the maximum in the interval in which $\mu \to \infty$. The interval in which $\mu < A$ is covered by Theorem 1 and the results of [3] and so it only remains to prove Theorem 1.

The so-called Burnside lemma tells us that

$$n!\, T(n,q) = \sum_{\pi} F_{\pi},$$

where F_{π} is the number of labelled (n,q) graphs invariant under the permutation π of the labels of the n points and the summation is over all possible π. If π leaves the labels of just p of the points unaltered and if F_{π}' is the number of labelled (n,q) graphs in which all the other n-p points are isolated, we have $F_{\pi}' = P!/\{q!(P-q)!\}$, where $P = p(p-1)/2$. The fundamental Lemma 3 of [3] tells us that

$$n!\,T(n,q) = \left(\sum_{\pi} F'_{\pi}\right)\{1 + O(q^{-C})\}, \qquad (5)$$

provided

$$An < q < An \log n. \qquad (6)$$

If we apply the arguments of the proof of the fundamental lemma to the case of the oriented graph, we find that

$$n!\,r(n,q) = \left(2^{q} \sum_{\pi} F'_{\pi}\right)\{1 + O(q^{-C})\},$$

when (6) is satisfied and the result of Theorem 1 above follows at once. We can extend this to the range $q < \frac{1}{2}n$ just as in [3].

Once we had a result equivalent to (5) above in [3], the problem of summing $\sum_{\pi} F'_{\pi}$, and so finding the approximation to T(n,q) for $\mu < A$, was complicated rather than difficult. We give two theorems which follow immediately from Theorem 1 and the results of [3].

<u>Theorem 3</u>. <u>If</u> $A < \mu < A$, <u>then</u>

$$r(n,q) = 2^{q} \Lambda(n,q) \exp(-e^{-\mu})(1 + \eta)/(1 - e^{-\mu}).$$

We take v to be the positive real number such that $v \log v = 2q$ and write $V = [v]$ and

$$K(v) = e^{-1}\{2\pi v/(1 + \log v)\}^{1/2}.$$

<u>Theorem 4</u>. <u>If</u> $\mu < -An^{-1/2}$, <u>then</u>

$$r(n,q) = 2^{q} K(V) \Lambda(V,q)(1 + \eta).$$

We see that the approximation in Theorem 4 depends only on q and not on n. In the narrow range of q between the intervals covered by Theorems 3 and 4 slightly more complicated results hold good; these may readily be deduced from those of 3 by Theorem 1.

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Appendix 4

The number of Hamiltonian circuits in large, heavily edged graphs II

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(Submitted to the *Glasgow Math. Journal*.)

We now prove Theorem 2 of [1]. We use the notation of [1] without further definition. By the Exclusion-Inclusion Theorem, we found that

$$H = \sum_{r=0}^{x-1} (-1)^r L_r + (-1)^x \theta L_x, \qquad (1)$$

where x is at our choice and $0 \leq \theta \leq 1$. Next we proved that

$$L_r/M = \{(2\alpha)^r/r!\}\{1+r^2 o(1)\} \qquad (2)$$

and so that

$$H = M\{e^{-2\alpha} + o(e^{2\alpha})\}.$$

This tells us that $H \sim Me^{-2\alpha}$, provided that $\alpha = O(1)$.

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† This author's research reported herein was supported by the European Research Office of the United States Army.

This limitation arises, of course, from the alternations of sign in (1). If we wish to extend our result to larger α, we must learn more about the error term in (2) and obtain an asymptotic expansion for it. But,as we show here, it is not necessary to determine the coefficients in the asymptotic expansion but only to show that they satisfy certain requirements. Not only does this simplify our work very greatly; it also enables us to prove our result under very general conditions, which would not otherwise be remotely possible. This fairly trivial idea may not be new, but we have not seen it elsewhere. It would seem capable of much wider application, in view of the importance of the Exclusion-Inclusion Theorem. It will be best understood by its application to our present problem.

At the end of this paper we state two results for 1-factors and sketch their proof.

We require a few further definitions. A and C are positive numbers independent of n, q and r; of these, each A is at our choice, while C depends on any A present or implied. The constant implied in the O( ) notation is a C. We write d for a number which does not depend on r, but may depend on n and q and on the structure of G, but which satisfies $|d| < C$. We write D(t) for any polynomial in r of degree t, each of whose coefficients is a d. The numbers A,C,d and the polynomial D(t) are not always the same at each occurrence; when we want to fix one of them, we add a suffix. Thus A_1, C_1, d_1, $D_1(t)$ always have the same meaning. We write $\gamma = \beta/q$ and

$$F = F(\gamma) = 1 + \sum_{w=1}^{v-1} D(2w)\gamma^{w} + O(r^{2v}\gamma^{v}),$$

where again F is not always the same at each occurrence. In what follows, v and V are integers which we choose later but v, V, u, s, t, w are all O(1) and independent of r. We suppose henceforth that the conditions of Theorem 2 are satisfied, i.e. $A_1 < \alpha < A_2 \log n$ and $\beta < An^{1-A_3}$. We shall first deduce Theorem 2 from the following lemma, which we prove later.

Lemma 1. *If* $r \leq y = [(\log n)^2]$, *then*

$$L_\lambda = (2\alpha)^\lambda MF/r!$$

We have

$$\log \{(2\alpha)^y/y!\} = y \log (2\alpha e/y) + O(\log y)$$
$$\leq -\tfrac{1}{2}\log n \log \log n$$

and

$$D_1(2w) = d_0 + \sum_{s=1}^{2w} d_s r(r-1)\ldots(r-s+1).$$

Hence, if $x = y+2v \geq y+2w$, we have

$$\sum_{\lambda=0}^{x-1} (-2\alpha)^\lambda D_1(2w)/r!$$
$$= e^{-2\alpha} \sum_{s=0}^{2w} (-1)^s d_s (2\alpha)^s + O(n^{-\frac{1}{4}\log \log n}).$$

Again

$$\sum_{\lambda=0}^{x-1} (2\alpha)^\lambda O(r^{2v})/r! = O(e^{2\alpha} \alpha^{2v}).$$

By (1) and Lemma 1,

$$H/M = e^{-2\alpha}\Big\{1 + \sum_{w=1}^{v-1} O(\alpha^{2w}\gamma^{w}) + O(e^{4\alpha}(\alpha^2\gamma)^v)\Big\}$$
$$+ O((\log n)^{-\frac{1}{4}\log n}).$$

But

$$e^{4\alpha}(\alpha^2\gamma)^v < Cn^{4A_2-(v-1)A_3} = o(1),$$

if we choose $v > 1 + (4A_2/A_3)$. Theorem 2 follows.

It remains to prove Lemma 1. We write $\Lambda_r = \Lambda_r(G)$ for the number of sets of r independent edges in G, i.e. the number of r-sets (sets of r edges) in G which do not contain a two-arc (a pair of adjacent edges). By the Exclusion-Inclusiod Theorem we have

$$\Lambda_r(G) = B(q,r) + \sum_{u=1}^{V} (-1)^u S(u) + O(S(V)), \qquad (3)$$

where S(u) is the sum over all possible sets of u two-arcs in G of the number of r-sets in G, each of which contains that set of u two-arcs. We prove first

<u>Lemma 2</u>. <u>If</u> $r^2\beta = o(q)$, <u>then</u>

$$\Lambda_r(G) = B(q,r)F.$$

We write Γ_{st} to denote a graph with s connected components, each having two or more edges, and a total of t edges. A set of u two-arcs form a graph Γ_{st} such that

$$2s \leq t \leq 2u \leq t(t-1).$$

We call w = t-s the weight of Γ_{st} and have

$$\sqrt{u} \leq t \leq 2w \leq 4u.$$

For a given u, two sets of u two-arcs are said to be <u>equivalent</u> if the two graphs they form are isomorphic.

Since $u = O(1)$, the sets of u two-arcs in G are thus separated into O(1) equivalence classes. There are $O(q^s \beta^{t-s})$ sub-graphs in G isomorphic to Γ_{st}, since we can choose an edge in each of the s components in $O(q^s)$ ways and the remaining t-s edges in $O(\beta^{t-s})$ ways. Each such sub-graph occurs in B(q-t,r-t) of the r-sets of G. Hence each equivalence class of sets of u two-arcs contributes

$$dq^s \beta^{t-s} B(q-t,r-t) = D(2w) \gamma^w B(q,r) \qquad (4)$$

to S(u).

If $V = 4v^2$, then $w \geqslant v$ for all sets of V two-arcs and so, if $r^2 = o(q)$, we have

$$S(V) = O(B(q,r) r^{2v} \gamma^v).$$

If we sum the contributions (4) over all the equivalence classes for a given u and over all u such that $1 \leqslant u \leqslant V-1$ and substitute in (3), we have Lemma 2.

<u>Lemma 3.</u> <u>If</u> $r^2\beta = o(q)$, <u>then</u>

$$B(q,r)/\{(n-1)\dots(n-r)\} = \alpha^r F/r!$$

We have

$$B(q,r)/\{(n-1)\dots(n-r)\} = \alpha^r Q/r! \qquad (r < n),$$

where

$$\log Q = \sum_{s=0}^{r-1} \{\log(1-(s/q)) - \log(1-((s+1)/n)\}$$

$$= \sum_{m=1}^{v} D(m+1)n^{-m} + O(r^{v+2}n^{-v-1})$$

and so

$$Q = 1 + \sum_{r=1}^{v} D(2v)n^{-v} + O(r^{2v+2}n^{-v-1}).$$

The lemma follows since $1/n = O(\gamma)$.

We recall that J = 0 unless the r-set consists of an independent set of R arcs, where $R \leq r$. For such an r-set,

$$J = 2^{R-1}(n-r-1)! \qquad (5)$$

by the argument of Wright [2].

Let us take a particular r-set of this kind and suppose that $r-R = w \leq v$, so that the graph formed of those arcs which are each of length more than one is a Γ_{st}, where $2 \leq 2s \leq t \leq 2w$. By our earlier argument, there are

$$dq^s \beta^{t-s} = dq^t \gamma^w \qquad (6)$$

such Γ_{st} with given arc lengths. For each such Γ_{st} we have to choose the remaining r-t independent edges from those edges in G not belonging to Γ_{st} and not adjacent to any node of Γ_{st}. These form a sub-graph G' containing q' edges, where $q' = q - d\beta - t$, and so can be chosen in $\Lambda_{r-t}(G')$ ways.

By Lemma 2, we have

$$\Lambda_{r-t}(G') = B(q', r-t)F(\gamma_1),$$

where $\gamma_1 = \beta/q'$ and so $\gamma_1^a = \gamma^a F(\gamma)$. Hence

$$\Lambda_{r-t}(G') = B(q', r-t)F(\gamma) = B(n-t-1, r-t)\alpha^{r-t}F(\gamma)$$

by Lemma 3. We now sum this over all the Γ_{st} corresponding to a given set of arc lengths, of which the number is (6). Hence the number of r-sets with the given set of arc lengths is

$$B(n-1, r)\alpha^r D(t)\gamma^w F.$$

If we now multiply by J (see (5)) and sum over all Γ_{st} for which $t-s = w = r-R$ and so $t \leq 2w$, we find that these contribute to L_r the number

$$\{M(2\alpha)^r/r!\}\{\sum_{W=w}^{v-1} D(2W)\gamma^W + O(r^{2v}\gamma^v)\}. \qquad (7)$$

Those r-sets for which $w \geq v$ all have as sub-graph a Γ_{st} of weight v and so are not more in number than

$$Cq^t \gamma^v B(q-t, r-t) \leq C\gamma^v r^{2v} B(q,r),$$

since $t \leq 2v$. Using this and summing (7) over all r-sets for which $w < v$, we have Lemma 1.

The number of 1-factors in G. We now suppose n even and put $n = 2m$. We write H_1 for the number of 1-factors (i.e. sets of m independent edges) in G and M_1 for the number of 1-factors in the complete graph K_n, so that

$$M_1 = (2m)!/(n!2^m).$$

We write $J_1(e_1,\ldots,e_r)$ for the number of different 1-factors in K_n which include the edges $e_1,\ldots,e_r$ of G and

$$L_{r1} = \sum J_1(e_1,\ldots,e_r),$$

where the sum is over all r-sets in G, and $L_{01} = M_1$. We have, as before,

$$H_1 = \sum_{r=0}^{x-1} (-1)^r L_{r1} + (-1)^x \theta L_{x1}.$$

Clearly

$$J_1 = (2m-2r)!/ (m-r)!2^{m-r},$$

if $e_1,\ldots,e_r$ are an independent set of edges and otherwise

$J_1 = 0$. Hence

$$L_r = (2m-2r)!\Delta_r(G)/\{(m-r)!2^{m-r}.$$

By the same method of proof as that of Lemma 3, we show that if $\alpha > A$ and $r^2\beta = o(q)$, then

$$\frac{B(q,r)2^r m(m-1)\ldots(m-r+1)}{(2m)\ldots\ldots(2m-2r+1)} = \frac{\alpha^r F}{r!2^r}.$$

We have only to use Lemma 2 to deduce the following theorem.

<u>Theorem 3</u>. <u>If</u> $A < \alpha < A \log n$ and $\beta = O(n^{1-A})$, <u>then</u> $H_1 \sim M_1 e^{-\alpha}$ <u>as</u> $n \to \infty$.

Again, by the method of [1], we can prove

<u>Theorem 4</u>. <u>If</u> $\alpha \to a < \infty$ <u>and</u> $\beta = o(n)$ <u>as</u> $n \to \infty$, <u>then</u> $H_1/M_1 \to e^{-a}$.

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