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A "Sinc - Galerkin" Method of Solution of  
Boundary Value Problems\*

by

Frank Stenger\*\*

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A SINC-GALERKIN METHOD OF SOLUTION OF BOUNDARY VALUE PROBLEMS

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Abstract

This paper illustrates the application of a "Sinc-Galerkin" method to the approximate solution of linear and nonlinear second order ordinary differential equations, and to the approximate solution of some linear elliptic and parabolic partial differential equations in the plane. The method is based on approximating functions and their derivatives by use of the Whittaker cardinal function. The DE is reduced to a system of algebraic equations via new accurate explicit approximations of the inner products, the evaluation of which does not require any numerical integration. Using  $n$  function evaluations the error in the final approximation to the solution of the DE is  $O(e^{-cn^{1/2d}})$  where  $c$  is independent of  $n$ , and  $d$  denotes the dimension of the region on which the DE is defined. This rate of convergence is optimal in the class of  $n$ -point methods which assume that the solution is analytic in the interior of the interval, and which ignore possible singularities of the solution at the end-points of the interval.

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## 1. Introduction and Summary

The function  $\text{sinc}(x)$  is defined on the real line by

$$(1.1) \quad \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

The whittaker cardinal function of an arbitrary function  $f$  is defined for any  $h > 0$  by

$$(1.2) \quad C(f, h, x) = \sum_{k=-\infty}^{\infty} f(kh) \text{sinc}\left[\frac{x-kh}{h}\right], \quad h > 0,$$

whenever this series converges.

The approximation of  $f$  using a finite number of terms of (1.2) has been extensively studied. The paper [8] contains a review of the properties of  $C(f, h, x)$  which were discovered by E.T. Whittaker [15], J.M. Whittaker [16], Hartly [5], Nyquist [9] and Shannon [12]. In [13] new approximations are derived by means of  $C(f, h, x)$ , for interpolating, integrating and approximating the Fourier (over  $(-\infty, \infty)$  only) and Hilbert transforms over  $(-\infty, \infty)$ ,  $(0, \infty)$  and  $(-1, 1)$ . In [7] the function  $C(f, h, x)$  is used to obtain formulas for approximating the derivatives of functions over  $(-\infty, \infty)$ ,  $(0, \infty)$  and  $(-1, 1)$ .

In the present paper we use results of [7, 13] to derive basis functions  $\{\psi_k/g\}$  for Galerkin schemes of solving second order problems, and we derive explicit and highly accurate expressions for inner products such as  $(f \frac{d^2 u}{dx^2}, \psi_k)$ ,  $(f \frac{du}{dx}, \psi_k)$ ,  $(fu, \psi_k)$ . All of these are expressed in terms of the function values of  $u$ , and not the derivatives of  $u$ . We then study the application of the derived

approximations on the approximate solution of some ordinary and partial differential equations, via the Galerkin method. The combined method thus yields a system of algebraic equations, without the use of any quadrature.

Let us briefly compare the present method of approximate solution of linear differential equations with currently popular finite difference methods, or with finite element methods that use piecewise linear elements. The finite difference or finite element methods lead to a sparse system of equations. The use of  $n$  solution evaluations usually leads to a linear system of  $n$  algebraic equations having non-singular coefficient matrix. The error in the resulting approximation is  $O(n^{-p})$ , where  $p$  is usually 1 or 2. The "Sinc-Galerkin" method of the present paper also leads to a system of order  $n$  on the basis of  $n$  solution evaluations. This system has a nonsingular full matrix. The error in the

resulting approximate solution is  $O(e^{-cn^{1/(2d)}})$ , where  $d$  denotes the dimension of the problem. The advantage of the present method is that due to its rapid convergence it does not require the solution of a very large system of equations in order to achieve a desired accuracy, if more than two significant figures of accuracy are required in the approximate solution. In addition, the rate of convergence of the present method is the same, regardless of possible singularities of the solution of an equation on the boundary of the region.

The approximate methods of [7,13] have previously been effectively applied to the approximate solution of integral equations via Galerkin-

type methods in [2,10,11]. In [6] an effective Galerkin-type method is derived which uses approximations derived in [13] to obtain an approximate solution to the problem

$$(1.3) \quad y'' = y - y^3/x^2, \quad y(0) = y(\infty) = 0$$

via the minimization of a certain nonlinear functional. In all of the above cases the error of an  $n$ -point approximate solution is  $O(e^{-cn^{1/2}})$ .

In Sec.2 of the present paper we review the relevant known approximation properties of Whittaker's cardinal function, and we then use these to derive explicit approximate inner products, in general as well as for the important special cases of the intervals  $[0,1]$ ,  $[-1,1]$ ,  $[0,\infty]$  and  $[-\infty,\infty]$ . In Sec.3 we illustrate the application of the previously derived formulas to the approximate solution of some simple "model" problems, such as  $u'' = -2$ ,  $u'' = u - u^3/x^2$ ,  $u_t = u_{xx}$  and  $u_{xx} + u_{yy} = f$ , with appropriate boundary conditions. In Sec.4 we carry out an error analysis, proving the  $O(e^{-cn^{1/(2d)}})$  rate of convergence referred to above.

It is worthwhile, for purposes of better understanding of the results of the paper, to consider the solution of the one-dimensional problems

$$(1.4) \quad \begin{aligned} L(f)(x) &= f''(x) + q(f(x), f'(x), x) = 0, \quad 0 < x < 1 \\ f(0) &= f(1) = 0 \end{aligned}$$

or

$$(1.5) \quad \begin{aligned} L(f)(x) &= r_0(x)f''(x) + r_1(x)f'(x) + r_2(x)f(x) = r_3(x), \quad 0 < x < 1 \\ f(0) &= f(1) = 0 \end{aligned}$$

We seek an approximate solution of (1.4) or (1.5) of the form

$$(1.6) \quad f(x) = \sum_{k=-N}^N f_k S(k, h) \phi(x)$$

where

$$(1.7) \quad S(k, h)(x) = \frac{\sin\left[\frac{\pi}{h}(x-kh)\right]}{\frac{\pi}{h}(x-kh)}, \quad \phi(x) = \log \frac{x}{1-x}.$$

We introduce a convenient function  $g$  which plays the role of a weight function, set

$$(1.8) \quad S_k(x) = g(x) S(k, h) \phi(x)$$

and reduce the solution of (1.4) or (1.5) to the solution of the Galerkin system

$$(1.9) \quad (L(f), S_k) = \int_0^1 L(f)(x) S_k(x) dx = 0, \quad k = -N, -N+1, \dots, N.$$

The results of thm. 2.11 together with Eq. (2.54) may then be used to get an explicit approximation of (1.9) for the case of (1.5), whereas the combination of Thm. 2.7, 2.11 and Eqs. (2.54) may be used to get an explicit approximation of (1.9) for the case of (1.4).

The author is grateful to A. Adler and J. Varah at the U.B.C. for valuable discussions involving Thm. 2.12 of this paper. The author also wishes to thank Bob Burke of the Univ. of Utah for carrying out the computations in Sec. 3.

## 2. Preliminaries and Fundamentals

In this section we shall recall some known properties [8] and derive some new properties of Whittaker's cardinal function, which we shall require in this paper.

Definition 2.1. Let  $R$  denote the real line,  $C$  the complex plane, and let  $B(h)$  denote the family of all functions defined on  $C$  that are entire, such that  $f \in L^2(R)$  and such that

$$(2.1) \quad |f(z)| \leq Ce^{\pi|y|/h}, \quad z = x + iy \in C,$$

for some constant  $C$ . Set

$$(2.2) \quad S(j,h)(x) = \text{sinc}\left[\frac{x-jh}{h}\right]$$

and

$$(2.3) \quad \delta_{jk}^{(n)} = s^{(n)}(j,1)(k) = \left(\frac{d}{dx}\right)^n S(j,1)(x) \Big|_{x=k}.$$

In particular, we have

$$(2.4) \quad \left\{ \begin{array}{l} \delta_{jk}^{(0)} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \\ \delta_{jk}^{(1)} = \begin{cases} 0 & \text{if } j = k \\ \frac{(-1)^{k-j}}{k-j} & \text{if } j \neq k \end{cases} \\ \delta_{jk}^{(2)} = \begin{cases} -\frac{\pi^2}{3} & \text{if } j = k \\ \frac{-2(-1)^{k-j}}{(k-j)^2} & \text{if } j \neq k \end{cases} \end{array} \right.$$

Theorem 2.2 [8]: Let  $f \in B(h)$ . Then  $f(z) = C(f, h, z)$ . Moreover

$$(2.5) \quad f(z) = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} g(t) e^{izt} dt \quad \text{for some } g \in L^2\left(-\frac{\pi}{h}, \frac{\pi}{h}\right);$$

$$(2.6) \quad f(z) = \frac{1}{h} \int_{\mathbb{R}} \text{sinc}\left[\frac{z-t}{h}\right] f(t) dt;$$

$$(2.7) \quad \int_{\mathbb{R}} |f(x)|^2 dx = h \sum_{k=-\infty}^{\infty} |f(kh)|^2$$

and the sequence  $\{h^{-\frac{1}{2}} S(k, h)\}_{k=-\infty}^{\infty}$  is therefore a complete orthonormal sequence in  $B(h)$ ;

$$(2.8) \quad f \in B(h) \Rightarrow f' \in B(h).$$

Theorem 2.3: Let  $\delta_{jk}^{(n)}$  be defined as in (2.3). Then

$$(2.9) \quad \int_{\mathbb{R}} \left\{ \text{sinc}\left[\frac{x-jh}{h}\right] \right\}^{(n)} \text{sinc}\left[\frac{x-kh}{h}\right] dx = h^{1-n} \delta_{jk}^{(n)},$$

$$n = 0, 1, 2, \dots$$

**Proof:** Let us set

$$(2.10) \quad f(t) = S(j, h)(t)$$

and let us note that  $f \in B(h)$ . By Eq. (2.8) it thus follows that  $f^{(n)} \in B(h)$ ,  $n = 0, 1, 2, \dots$ . Eq. (2.9) thus follows by taking  $f = S(j, h)^{(n)}$  in (2.6), and noting by (2.3) that

$$(2.11) \quad S^{(n)}(j, h)(kh) = h^{-n} \delta_{jk}^{(n)}.$$

**Definition 2.4:** Let  $d > 0$ , and let  $B(\mathcal{D}'_d)$  denote the family of all functions  $f$  that are analytic in

$$(2.12) \quad \mathcal{D}'_d = \{z = x + iy : |y| < d\},$$

such that

$$(2.13) \quad \int_{-d}^d |f(x+iy)| dy \rightarrow 0 \text{ as } x \rightarrow \pm\infty$$

and such that  $N(f, \mathcal{D}'_d) < \infty$ , where

$$(2.14) \quad N(f, \mathcal{D}'_d) = \lim_{y \rightarrow d^-} \left\{ \int_{\mathbb{R}} |f(x+iy)| dx + \int_{\mathbb{R}} |f(x-iy)| dx \right\}$$

**Theorem 2.5 [13]:** Let  $h$  and  $d$  be positive, let  $f \in B(\mathcal{D}'_d)$ , and let  $\epsilon(f)$  be defined by

$$(2.15) \quad \epsilon(f)(x) = f(x) - C(f, h, x), \quad x \in \mathbb{R}.$$

Then

$$(2.16) \quad \epsilon(f)(x) = \frac{\sin(\pi x/h)}{2\pi i} \int_{\mathbb{R}} \left[ \frac{f(t-id^-)}{(t-x-id)\sin[(t-id)\pi/h]} - \frac{f(t+id^-)}{(t-x+id)\sin[(t+id)\pi/h]} \right] dt$$

Moreover

$$(2.17) \quad \|\epsilon(f)\|_{\infty} = \sup_{x \in \mathbb{R}} |\epsilon(f)(x)| \leq \frac{N(f, \mathcal{D}'_d)}{2\pi d \sinh(\pi d/h)}.$$

**Definition 2.6:** Let  $\mathcal{D}$  be a simply-connected domain in the complex plane  $\mathbb{C}$ , and let  $\mathcal{D}'_d$  be defined as in (2.12). Let  $\phi$  be a conformal map of  $\mathcal{D}$  onto  $\mathcal{D}'_d$ , and let  $\psi = \phi^{-1}$  denote the inverse map. Let  $a = \psi(-\infty)$  and  $b = \psi(\infty) \neq a$  be boundary points of  $\mathcal{D}$ , and let us take

$$(2.18) \quad \Gamma = \{w \in \mathcal{D} : w = \psi(x), \quad -\infty \leq x \leq \infty\} .$$

Let  $B(\mathcal{D})$  denote the family of all functions that are analytic in  $\mathcal{D}$ , such that for  $u$  real

$$(2.19) \quad \int_{\psi(L+u)} |f(z)dz| \rightarrow 0 \quad \text{as } u \rightarrow \pm\infty$$

where

$$(2.20) \quad L = \{iy : -d \leq y \leq d\},$$

and such that

$$(2.21) \quad N(f, \mathcal{D}) = \liminf_{C \rightarrow \partial\mathcal{D}, C \subset \mathcal{D}} \int_C |f(z)dz| < \infty$$

(Note that if  $f \in B(\mathcal{D})$ , then  $f \circ \psi \in B(\mathcal{D}'_d)$ .) Set

$$(2.22) \quad x_k = \psi(kh) , \quad k = 0, \pm 1, \pm 2, \dots,$$

and let  $g$  be a function which is analytic in  $\mathcal{D}$ , which plays the role of a weight function in the inner products, and whose properties we shall determine in the sequel. Finally, we set

$$(2.23) \quad S_j(z) = g(z) \operatorname{sinc}\left[\frac{\phi(z) - 1}{h}\right] = g(z) S(j, h) \circ \phi(z) .$$

The following result was established in [7].

**Theorem 2.7:** Let  $m$  be a nonnegative integer, and let  $f\phi'/g \in B(\mathcal{D})$  .

Let there exist positive constants  $\alpha$  ,  $C_0$  depending only on  $m, d$  and  $g$ ,  $C_1$  depending only on  $m$  and  $g$ , and  $C_2$  depending only on  $m, g$  and  $f$ , such that

$$(2.24) \quad \left| \frac{f(x)}{g(x)} \right| \leq C_2 e^{-\alpha|\phi(x)|} \quad \text{for all } x \in \Gamma$$

$$(2.25) \quad \left. \begin{aligned} \left| \left( \frac{d}{dx} \right)^n S_k(x) \right| &\leq C_1 h^{-n} \text{ for all } x \in \Gamma \\ \left| \left( \frac{d}{dx} \right)^n \left\{ \frac{g(x) \sin[\pi\phi(x)/h]}{\phi(z) - \phi(x)} \right\} \right| &\leq C_0 h^{-n} \text{ for all } x \in \Gamma, z \in \partial D \end{aligned} \right\} n=0,1,\dots,m.$$

Then there exists a constant  $K$  depending only on  $m, d, \alpha, g$  and  $f$  such that if  $h = [\pi d / (\alpha N)]^{\frac{1}{2}}$  then

$$(2.27) \quad \left| f^{(n)}(x) - \sum_{j=-N}^N \frac{f(x_j)}{g(x_j)} S_j^{(n)}(x) \right| \leq K N^{\frac{n+1}{2}} \exp[-(\pi d \alpha N)^{\frac{1}{2}}]$$

for all  $x \in \Gamma$ , and for  $n = 0, 1, \dots, m$ .

**Theorem 2.8** [13]: If  $f \in B(D)$ , then the identity\*

$$(2.28) \quad \begin{aligned} \frac{f(x)}{\phi'(x)} - \sum_{j=-\infty}^{\infty} \frac{f(x_j)}{\phi'(x_j)} S(j, h) \phi(x) \\ = \frac{\sin[\pi\phi(x)/h]}{2\pi i} \int_{\partial D} \frac{f(z) dz}{[\phi(z) - \phi(x)] \sin[\pi\phi(z)/h]} \end{aligned}$$

is valid for all  $x \in \Gamma$ . Moreover

$$(2.29) \quad \begin{aligned} \int_{\Gamma} f(x) dx - h \sum_{j=-\infty}^{\infty} \frac{f(x_j)}{\phi'(x_j)} \\ = \frac{1}{2} \int_{\partial D} \frac{\exp\left[\frac{i\pi\phi(z)}{h} \operatorname{sgn} \operatorname{Im} \phi(z)\right]}{\sin\left[\frac{\pi}{h} \phi(z)\right]} f(z) dz \end{aligned}$$

The results of this theorem may be conveniently combined with those of the formulas obtained above, to yield explicit approximate expressions for inner products. The results of the following lemma are useful for

\*Here and henceforth  $\int_{\partial D} f(z) dz$  is defined by  $\lim_{\substack{1 \rightarrow f \\ C \rightarrow \partial D, C \subset D}} \int_C f(z) dz$ , for any  $f \in B(D)$ .

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bounding the error of these approximate expressions.

Lemma 2.9: If  $|\operatorname{Im} z| = d > 0$  and if  $k$  is an integer, then

$$(2.30) \quad \frac{1}{2} \left| \frac{\operatorname{sinc}[(z-kh)/h]}{\sin(\pi z/h)} \right| \leq C_1(h,d) \equiv \frac{h}{2\pi d}$$

$$(2.31) \quad \frac{1}{2} \left| \frac{(d/dz)\{\operatorname{sinc}[(z-kh)/h]\}}{\sin(\pi z/h)} \right| \leq C_2(h,d) \equiv \frac{d+(h/\pi)\tanh(\pi d/h)}{2d^2 \tanh(\pi d/h)} ;$$

$$(2.32) \quad \frac{1}{2} \left| \frac{\frac{d^2}{dz^2}\{\operatorname{sinc}[(z-kh)/h]\}}{\sin(\pi z/h)} \right| \leq C_3(h,d) \equiv \frac{[(2h/\pi)+\pi^2 d/h]d \tanh(\pi d/h)+2d}{2 d^3 \tanh(\pi d/h)} ;$$

$$(2.33) \quad |\sin(\pi z/h)| \geq \sinh(\pi d/h), \quad |\cos(\pi z/h)| \leq \cosh(\pi d/h) .$$

Proof: We shall only prove (2.31), since the proofs of the remaining cases are similar, and we omit them. We have

$$w = \frac{d}{dz} \{S(k,h)(z)\} = \frac{\cos[\pi(z-kh)/h]}{z-kh} - \frac{h}{\pi} \frac{\sin[\pi(z-kh)/h]}{(z-kh)^2}$$

Now if  $|\operatorname{Im} z| = d$ , then  $|z-kh| \geq d$ ,  $|\cos[\pi(z-kh)]| \leq \cosh(\pi d/h)$  and  $|\sin(\pi z/h)| \geq \sinh(\pi d/h)$ ; hence

$$\left| \frac{w}{\sin(\pi z/h)} \right| \leq \frac{1}{d \tanh(\pi d/h)} + \frac{h}{\pi d^2} = C_2(h,d) .$$

Theorem 2.10: Let  $\delta_{jk}^{(n)}$  be defined as in (2.3) and (2.4), let  $C_j(h,d)$  be defined as in Lemma 2.9, let  $x_k$  be defined as in (2.22),  $S_k$  as in (2.23), set  $F_k = F(x_k)$  for an arbitrary function  $F$ , and let  $r$  and  $f$  be functions which are analytic in  $\mathcal{D}$ .

(a) Let  $rfg \in B(\mathcal{D})$ . Then

$$(2.34) \quad \left| \int_{\Gamma} r(x)f(x)S_k(x)dx - h \frac{f_k r_k g_k}{\phi_k'} \right| \leq C_1(h,d)N(rfg, \mathcal{D})e^{-\pi d/h}.$$

(b) Let  $[rfg/\phi](x) \rightarrow 0$  as  $x \rightarrow a$  and as  $x \rightarrow b$  along  $\Gamma$ , and let  $(rg)'f$  and  $rg\phi'f \in B(\mathcal{D})$ . Then

$$(2.35) \quad \left| \int_{\Gamma} r(x)f'(x)S_k(x)dx + h \sum_{j=-\infty}^{\infty} f_j \left\{ \frac{(rg)_j'}{\phi_j'} \delta_{kj}^{(0)} + (rg)_j \frac{\delta_{kj}^{(1)}}{h} \right\} \right| \\ \leq [C_1(h,d)N(f(rg)', \mathcal{D}) + C_2(h,d)N(frg\phi', \mathcal{D})]e^{-\pi d/h}.$$

(c) Let  $[frg'/\phi](x)$ ,  $[frg\phi'/\phi](x)$  and  $[f'rg/\phi](x) \rightarrow 0$  as  $x \rightarrow a$  and as  $x \rightarrow b$  along  $\Gamma$ , and let  $f(rg)''$ ,  $f[2(rg)'\phi' + rg\phi'']$  and  $frg(\phi')^2 \in B(\mathcal{D})$ . Then

$$(2.36) \quad \left| \int_{\Gamma} r(x)f''(x)S_k(x)dx - h \sum_{j=-\infty}^{\infty} f_j \left\{ \frac{(rg)_j''}{\phi_j'} \delta_{kj}^{(0)} + \frac{[2(rg)_j'\phi_j' + (rg)_j\phi_j'']}{\phi_j'} \frac{\delta_{kj}^{(1)}}{h} + (rg)_j\phi_j' \frac{\delta_{kj}^{(2)}}{h^2} \right\} \right| \\ \leq [C_1(h,d)N(f(rg)'', \mathcal{D}) + C_2(h,d)N(f[2(rg)'\phi' + rg\phi''], \mathcal{D}) + C_3(h,d)N(frg(\phi')^2, \mathcal{D})] \\ \cdot e^{-\pi d/h}.$$

Proof: We shall only prove the (b)-part of Theorem 2.9, since the proofs of the (a) and (c)-parts are similar.

We find, upon integration by parts, that

$$(2.37) \quad \int_{\Gamma} r(x)f'(x)S_k(x)dx = r(x)f(x)S_k(x) \Big|_a^b \\ - \int_{\Gamma} f(x)[r(x)S_k'(x) + r'(x)S_k(x)]dx$$

The first term on the right-hand side vanishes, by assumption of the (b)-part of the theorem, while by expansion of the second part of (2.37), we have

$$(2.38) \quad \int_{\Gamma} r(x)f'(x)S_k(x)dx \\ = - \int_{\Gamma} f(x)[(rg)'(x)S(k,h) \circ \phi(x) + (rg\phi')(x)S'(k,h) \circ \phi(x)]dx$$

Hence by replacing  $f$  in (2.29) by the integrand on the right-hand side of (2.38), and noting that if  $z \in \partial\mathcal{D}$ , then  $|\text{Im}\phi(z)| = d$  and  $|\exp[\frac{i\pi}{h}\phi(z)\text{sgn Im}\phi(z)]| = e^{-\pi d/h}$  we find by (2.29), Lemma 2.9 and Theorem 2.3, that

$$\left| \int_{\Gamma} r(x)f'(x)S_k(x)dx \right. \\ \left. + h \sum_{j=-\infty}^{\infty} f_j \left\{ \frac{(rg)_j}{\phi_j'} \delta_{kj}^{(0)} + (rg)_j \frac{\delta_{kj}^{(1)}}{h} \right\} \right| \\ \leq e^{-\pi d/h} \int_{\partial\mathcal{D}} [C_1(h,d)|[f(rg)'](z)| + C_2(h,d)|(frg\phi')(z)|] |dz|,$$

which is just (2.35).

**Theorem 2.11:** Let  $N$  be a positive integer,  $\alpha$  a positive constant, and take  $h = [\pi d/(\alpha N)]^{\frac{1}{2}}$ .

(a) Under the assumptions of Theorem 2.9 (a),

$$(2.39) \quad \left| \int_{\Gamma} r(x)f(x)S_k(x)dx - h \frac{f_k r_k g_k}{\phi_k'} \right| \leq \frac{K_1}{N^{\frac{1}{2}}} e^{-(\pi d \alpha N)^{\frac{1}{2}}},$$

where  $K_1$  depends only on  $f, r, g, d$  and  $\alpha$ ;

(b) If  $|\text{[rgf]}(x)| \leq K_2' \exp[-\alpha|\phi(x)|]$  on  $\Gamma$ , then under the assumptions of Theorem 2.9 (b),

$$(2.40) \quad \left| \int_{\Gamma} r(x) f'(x) S_k(x) dx + h \sum_{j=-N}^N f_j \left\{ \frac{(rg)_j'}{\phi_j'} \delta_{kj}^{(0)} + (rg)_j \frac{\delta_{kj}^{(1)}}{h} \right\} \right| \\ \leq K_2 e^{-(\pi d \alpha N)^{\frac{1}{2}}}, \quad k = -N, -N+1, \dots, N,$$

where  $K_2$  depends only on  $f, r, g, d$  and  $\alpha$ ;

(c) If  $\{f\{2(rg)' + rg\phi''/\phi'\}\}(x)$  and  $\{rgf\phi'\}(x)$  are bounded by  $K_3' \exp[-\alpha|\phi(x)|]$  on  $\Gamma$ , then under the assumptions of Theorem 2.9 (c),

$$(2.41) \quad \left| \int_{\Gamma} r(x) f''(x) S_k(x) dx - h \sum_{j=-N}^N f_j \left\{ \frac{(rg)''_j}{\phi_j'} \delta_{kj}^{(0)} + \frac{[2(rg)'_j \phi_j'' + (rg)_j \phi_j''']}{\phi_j'} \frac{\delta_{kj}^{(1)}}{h} + (rg)_j \phi_j' \frac{\delta_{kj}^{(2)}}{h^2} \right\} \right| \\ \leq K_3 N^{\frac{1}{2}} e^{-(\pi d \alpha N)^{\frac{1}{2}}}, \quad k = -N, -N+1, \dots, N,$$

where  $K_3$  depends only on  $f, r, g, d$  and  $\alpha$ .

Proof: The proof is similar to that of Theorem 8.1 of [13], and we omit it.

The results of Theorem 2.10 are especially suited to the solution of linear differential equations via a Galerkin method, for which the functions  $\{S_k/g\}$  are the approximating basis functions. We remark that we could have obtained alternate expressions of  $\int_{\Gamma} r(x) f^{(n)}(x) S_k(x) dx$ , by combining equations (2.29) and (2.27), i.e., if  $rgf^{(n)} \in B(\mathcal{D})$ , then by Eq. (2.29)

$$(2.42) \quad \left| \int_{\Gamma} r(x) f^{(n)}(x) S_k(x) dx - \frac{h r_k g_k f^{(n)}(x_k)}{\phi_k'} \right| \\ \leq C_1(h, d) N(rgf^{(n)}, \mathcal{D}) e^{-\pi d/h}$$

and we could now use (2.29) to approximate  $f^{(n)}(x_k)$  on  $\Gamma$ . However, the resulting expressions are not as accurate as those of Theorem 2.10. Nevertheless the pair of equations (2.27) and (2.29) do form a powerful combination for purposes of solving nonlinear equations. For example, if  $g \in B(D)$ , where  $G = G(x, f(x), f'(x))$ , then

$$(2.43) \quad \left| \int_{\Gamma} G(x, f(x), f'(x)) S_k(x) dx - h \frac{G(x_k, f(x_k), f'(x_k))}{\phi'(x_k)} g(x_k) \right|$$

$$\leq C_1(h, d) N(G, D) e^{-\pi d/h};$$

if the conditions of Theorem 2.7 are satisfied for  $m = 1$ , we may now replace  $f'(x_k)$  in (2.43) by the approximation

$$(2.44) \quad f'(x_k) \approx \sum_{j=-N}^N \frac{f_j}{g_j} [g_j' \delta_{jk}^{(0)} + g_j \phi_j' \frac{\delta_{jk}^{(1)}}{h}]$$

given by Eq. (2.27).

The approximating expressions of Theorem 2.10 may be more compactly expressed by means of matrices. To this end, let  $m = 2N + 1$ , and let  $\underline{S}_m$  and  $\underline{f}_m$  be column vectors defined by

$$(2.45) \quad \underline{S}_m(x) = \begin{pmatrix} S_{-N}(x) \\ S_{-N+1}(x) \\ \vdots \\ S_N(x) \end{pmatrix}, \quad \underline{f}_m = \begin{pmatrix} f_{-N} \\ f_{-N+1} \\ \vdots \\ f_N \end{pmatrix}$$

Corresponding to a function  $u = u(x)$ , let  $\underline{A}_m(u)$  denote a diagonal

matrix, whose diagonal elements are  $u(x_{-N}), u(x_{-N+1}), \dots, u(x_N)$  and whose off-diagonal elements are zero. Let  $\tilde{I}_m^{(1)}$  and  $\tilde{I}_m^{(2)}$  denote the matrices

$$(2.46) \quad \tilde{I}_m^{(1)} = \begin{bmatrix} 0 & -1 & \frac{1}{2} & -\frac{1}{3} & \dots & \frac{1}{2N} \\ 1 & 0 & -1 & -\frac{1}{2} & \dots & -\frac{1}{2N-1} \\ -\frac{1}{2} & 1 & 0 & \dots & & \frac{1}{2N-2} \\ & & \dots & & & \\ \frac{1}{2N} & \frac{1}{2N-1} & -\frac{1}{2N-2} & \frac{1}{2N-3} & \dots & 0 \end{bmatrix} = [\delta_{jk}^{(1)}]$$

$$(2.47) \quad \tilde{I}_m^{(2)} = \begin{bmatrix} -\frac{\pi^2}{3} & \frac{2}{1^2} & -\frac{2}{2^2} & \dots & \frac{-2}{(2N)^2} \\ \frac{2}{1^2} & -\frac{\pi^2}{3} & \frac{2}{1^2} & \dots & \frac{2}{(2N-1)^2} \\ & & \dots & & \\ -\frac{2}{(2N)^2} & \frac{2}{(2N-1)^2} & -\frac{2}{(2N-2)^2} & \dots & -\frac{\pi^2}{3} \end{bmatrix} = [\delta_{jk}^{(2)}]$$

With this notation, Eqs. (2.39), (2.40) and (2.41) take the approximating form

$$(2.48) \quad \begin{aligned} \int_{\Gamma} r(x) f(x) S_m(x) dx &= h A_m \left( \frac{rg}{\phi} \right) f_m \\ \int_{\Gamma} r(x) f'(x) S_m(x) dx &= -h \left[ A_m \left( \frac{(rg)'}{\phi} \right) + \frac{1}{h} \tilde{I}_m^{(1)} A_m(rg) \right] f_m \\ \int_{\Gamma} r(x) f''(x) S_m(x) dx &= h \left[ A_m \left( \frac{(rg)''}{\phi} \right) + \frac{1}{h} \tilde{I}_m^{(1)} A_m(2(rg)' + rg\phi''/\phi') \right. \\ &\quad \left. + \frac{1}{h^2} \tilde{I}_m^{(2)} A_m(rg\phi') \right] f_m \end{aligned}$$

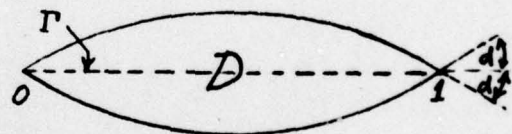
By [4] pp. 67-72, the matrix  $I_m^{(1)}$  is simply related to a Toeplitz-type matrix, by considering the Fourier series expansion of  $-it$  on  $(-\pi, \pi)$ . The matrix  $I_m^{(2)}$  is a Toeplitz matrix, obtainable by considering the Fourier series expansion of  $-t^2$  on  $[-\pi, \pi]$ . Thus [4 p.65], the eigenvalues of  $I_m^{(1)}$  are  $i\lambda_k^{(1)}$ ,  $k = -N, -N+1, \dots, N$ , where  $-\pi < \lambda_k^{(1)} < \pi$  while the eigenvalues of  $I_m^{(2)}$  are  $-\lambda_k^{(2)}$ ,  $k = -N, -N+1, \dots, N$ , where  $0 < \lambda_k^{(2)} < \pi^2$ . Indeed, let  $0 < \lambda_{-N}^{(2)} \leq \lambda_{-N+1}^{(2)} \leq \dots \leq \lambda_N^{(2)} < \pi^2$ . Then by [4 pp.64 and 67], since  $x^2 \geq 2 - 2 \cos x$  on  $[-\pi, \pi]$ , it follows that  $\lambda_{-N}^{(2)} > 2 - 2 \cos[\pi/(2N+2)] = 4 \sin^2[\pi/(4N+4)]$ . That is,  $|\lambda_N^{(2)}/\lambda_{-N}^{(2)}|$ , the condition number of  $I_m^{(2)}$  is bounded by  $\pi^2/[4 \sin^2[\pi/(4N+4)]] \sim (N+1)^2$ .

Summing up, we have

**Theorem 2.12:** (a)  $I_m^{(1)}$  is a skew-symmetric matrix having determinant zero. The eigenvalues  $i\lambda_k^{(1)}$  of  $I_m^{(1)}$  satisfy the inequality  $-\pi < \lambda_k^{(1)} < \pi$ ,  $k = -N, -N+1, \dots, N$ . (b)  $I_m^{(2)}$  is a negative definite matrix having eigenvalues  $-\lambda_k^{(2)}$ , where  $4 \sin^2[\pi/(4N+4)] \leq \lambda_k^{(2)} < \pi^2$ ,  $k = -N, -N+1, \dots, N$ .

We close this section with a derivation of the formulas of Theorem 2.10 for the case of the important intervals  $[0,1]$ ,  $[-1,1]$ ,  $[0,\infty)$ , and  $[-\infty,\infty)$ .

**Ex.1:**  $\Gamma = [0,1]$ . In this case



$$(2.49) \quad \begin{cases} \phi(z) = \log\left(\frac{z}{1-z}\right), & \phi'(z) = \frac{1}{z(1-z)}, \\ D = \{z: |\arg\left(\frac{z}{1-z}\right)| < d\}. \end{cases} \quad \begin{array}{l} \text{Fig. 2.1} \\ D \text{ of Ex. 1} \end{array}$$

Let us assume that the coefficients  $r$  of a second order equation are analytic and bounded in  $D$ , and that the same is true of  $r'$  and  $r''$ . It is then convenient to take

$$(2.50) \quad g(x) = \frac{1}{\phi'(x)} = x(1-x).$$

The conditions of Theorem 2.10 are satisfied if  $f$  is analytic and bounded

on  $D$ , and if on  $[0,1]$ ,  $|f(x)| \leq C[x(1-x)]^\alpha$ , where  $C$  and  $\alpha$  are positive constants. If  $f$  does not vanish at  $0$  and  $1$ , we replace  $f$  by  $F$  in the differential equation, where

$$(2.51) \quad F(x) = f(x) - a(1-x) - bx$$

and where  $a = f(0)$ ,  $b = f(1)$ . The functions  $S_k$  are

$$(2.52) \quad \{S_k(x)\}_{k=-N}^N = \{x(1-x)S(k,h)\phi(x)\}_{k=-N}^N.$$

To  $\{S_k/g\}$  it may be necessary to adjoin  $1-x$  if  $a$  is unknown, and  $x$  if  $b$  is unknown. Differentiating  $g$  and  $\phi'$ , we get

$$(2.53) \quad g'(x) = 1 - 2x, \quad g''(x) = -2, \quad \phi''(x) = -\frac{1-2x}{x^2(1-x)^2}.$$

Hence

$$(2.54) \quad \left\{ \begin{array}{l} (rg)(x) = x(1-x)r(x) \quad \left(\frac{rg}{\phi}\right)(x) = x^2(1-x)^2r(x) \\ \left(\frac{(rg)'}{\phi}\right)(x) = x(1-x)[x(1-x)r'(x) + (1-2x)r(x)] \\ \left(\frac{(rg)''}{\phi}\right)(x) = x(1-x)[x(1-x)r''(x) + 2(1-2x)r'(x) - 2r(x)] \\ \left(\frac{2(rg)'\phi' + rg\phi''}{\phi^2}\right)(x) = 2x(1-x)r'(x) + (1-2x)r(x) \\ (rg\phi')(x) = r(x) \end{array} \right.$$

Hence we get the approximations (2.48), in which  $x_k = \frac{1}{2} + \frac{1}{2} \tanh(kh/2)$ .

Ex.2.  $\Gamma = [-1,1]$ . In this case

$$(2.55) \quad \phi(z) = \log\left(\frac{1+z}{1-z}\right), \quad \phi'(z) = \frac{2}{1-z^2}$$

$$D = \{z: |\arg\left(\frac{1+z}{1-z}\right)| < d\}.$$

Under assumptions on  $r$  similar to those of Ex. 1, we take

$$(2.56) \quad g(x) = \frac{1}{\phi'(x)} = \frac{1}{2}(1-x^2)$$

The conditions of Theorems 2.9 and 2.10 are satisfied if  $f$  is analytic and bounded on  $D$ , and if on  $(-1,1)$ ,  $|f(x)| \leq C(1-x^2)^\alpha$ , where  $C$  and  $\alpha > 0$ . If  $f$  does not vanish on  $-1$  and  $1$ , we set  $f = F + p$  in the differential equation, where

$$(2.57) \quad p(x) = a \frac{1-x}{2} + b \frac{1+x}{2}$$

and where  $a = f(-1)$ ,  $b = f(1)$ . The functions  $S_k$  are

$$(2.58) \quad \{S_k(x)\}_{k=-N}^N = \left\{ \frac{1}{2}(1-x^2) S(k, h) \phi(x) \right\}_{k=-N}^N;$$

to  $\{S_k/g\}$  it may be necessary to adjoin  $(1-x)/2$  and/or  $(1+x)/2$  if  $a$  and/or  $b$  are unknown. Differentiating  $g$  and  $\phi'$ , we get

$$(2.59) \quad g'(x) = -x, \quad g''(x) = -1, \quad \phi''(x) = x \left( \frac{2}{1-x^2} \right)^2,$$

so that

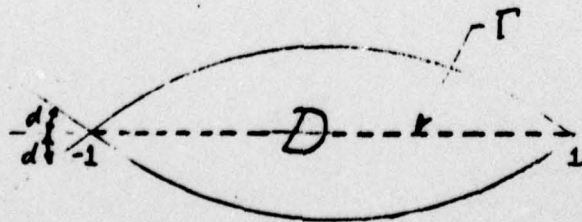


Fig. 2.2.  
D of Ex. 2.

$$(2.60) \quad \begin{cases} (rg)(x) = \frac{1}{2}(1-x^2)r(x) ; & \left(\frac{rg}{\phi'}\right)(x) = \left(\frac{1-x^2}{2}\right)^2 r(x) \\ \left(\frac{(rg)'}{\phi'}\right)(x) = \left(\frac{1-x^2}{2}\right)^2 r'(x) - x\left(\frac{1-x^2}{2}\right)r(x) \\ \left(\frac{(rg)''}{\phi'}\right)(x) = \left(\frac{1-x^2}{2}\right)^2 r''(x) - x(1-x^2)r'(x) - \left(\frac{1-x^2}{2}\right)r(x) \\ (2(rg)' + rg\phi''/\phi')(x) = (1-x^2)r'(x) - xr(x) \\ (rg\phi')(x) = r(x) \end{cases}$$

Hence, with  $x_k = \tanh(kh/2)$ , the approximations of Theorem 2.10 take the form (2.48).

Ex.3: The case  $\Gamma = [0, \infty]$ . In this case

$$(2.61) \quad \phi(z) = \log z, \quad \phi'(z) = \frac{1}{z}, \quad \mathcal{D} = \{z: |\arg z| < d\}.$$

Suppose that the coefficients  $r$  and the derivatives of  $r$  are analytic and bounded in  $\mathcal{D}$ . If on  $\mathcal{D}$ ,  $|f(z)| \leq C|z|^\alpha/(1+|z|)^{2\alpha}$  where  $C, \alpha$  are positive then it is convenient to take  $g(z) = z/(1+z)^2$ , in order that the conditions of Theorems 2.9 and 2.10 are satisfied. However, if  $|f(z)| \leq C|z|^\alpha/(1+|z|)^{2+\alpha}$  in  $\mathcal{D}$ , where  $C$  and  $\alpha$  are positive, then it is possible to choose a simpler form for  $g$ , and  $S_k(x)$ , namely

$$(2.62) \quad g(x) = \frac{1}{\phi'(x)} = x ; \quad S_k(x) = g(x)S(k, h) \circ \phi(x).$$

In this latter case

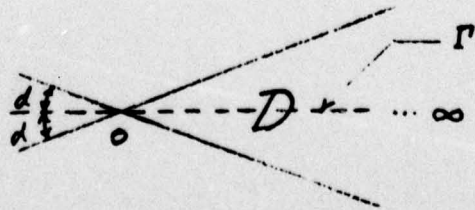


Fig. 2.3  
D of Ex. 3

$$(2.63) \quad \begin{cases} (rg)(x) = xr(x) & \left(\frac{rg}{\phi'}\right)(x) = x^2 r(x) \\ \left(\frac{(rg)'}{\phi'}\right)(x) = x^2 r'(x) + xr(x) & \left(\frac{(rg)''}{\phi'}\right)(x) = x^2 r''(x) + 2xr'(x) \\ 2(rg)' + rg\phi''/\phi'(x) = 2xr'(x) + r(x) \\ (rg\phi')(x) = r(x) \end{cases}$$

The approximations now take the form (2.48), in which  $x_k = e^{kh}$ .

If  $f$  is merely bounded on  $\mathcal{D}$ , and if  $d = \lim_{(x \rightarrow \infty)} x^2 f'(x)$ , then we replace  $f$  by  $F$  in the differential equation, where

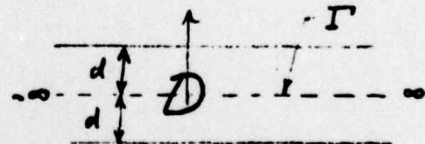
$$(2.64) \quad f(x) = F(x) + \frac{a}{1+x} + \frac{xb}{1+x} + \frac{xc}{(1+x)^2}$$

where

$$(2.65) \quad a = f(0), \quad b = f(\infty), \quad c = b - a - d.$$

If the limit  $\lim_{(x \rightarrow \infty)} x^2 f'(x)$  does not exist, it may be better to take  $g(x) = x/(1+x)$  or  $g(x) = x/(1+x)^2$ , depending upon the problem.

Ex. 4: The case  $\Gamma = [-\infty, \infty]$ . In this case



$$(2.66) \quad \phi(z) = z, \quad \phi'(z) = 1,$$

Fig. 2.4.  $\mathcal{D}$  of Ex. 4

and  $\mathcal{D} = \mathcal{D}'_d$  (see Eq. (2.12)). If the coefficients  $r$  of the differential equation are analytic and bounded in  $\mathcal{D}'_d$ , and if  $f \in B(\mathcal{D}'_d)$ , we simply take

$$(2.67) \quad g'(x) = 1, \quad \{S_k(x)\}_{k=-N}^N = \{S(k,h)(x)\}_{k=-N}^N$$

in order that the conditions of Theorem 2.9 become satisfied, and provided

that  $f$  vanishes at  $\pm\infty$ . The conditions of Theorem 2.10 also become satisfied if  $|f(x)| \leq Ce^{-\alpha|x|}$  on  $\Gamma$ . Then  $x_k = kh$ , and

$$(2.68) \quad \begin{cases} (rg)(x) = r(x), & \left(\frac{rg}{\phi'}\right)(x) = r(x), & \left(\frac{rg}{\phi'}\right)'(x) = r'(x); \\ \left(\frac{rg}{\phi'}\right)''(x) = r''(x), & (2(rg)' + rg\phi''/\phi')(x) = 2r'(x) \\ (rg\phi')(x) = r(x). \end{cases}$$

The approximating equations again take the form (2.48).

If  $f$  does not vanish at  $\pm\infty$ , we replace  $f$  by  $F$ , where

$$(2.69) \quad F(x) = f(x) - \frac{1}{e^{cx} + e^{-cx}} [e^{-cx}f(-\infty) + e^{cx}f(\infty)]$$

and where  $0 < c < \pi/(2d)$ .

### 3. Examples of Applications

In this section we shall illustrate the application of the formulas developed in Sec.2, on the solution of some simple ordinary and partial differential equations.

Ex.1: Consider the simple problem

$$(3.1) \quad f_{xx}(x) = -2, \quad 0 < x < 1; \quad f(0) = f(1) = 0$$

This has the solution  $f(x) = x(1-x)$ . By taking  $r(x) = 1$  in (2.54) and combining with (2.48), we arrive at the system of equations

$$(3.2) \quad h[-2A_m(x(1-x)) + \frac{1}{h} I_m^{(1)} A_m(1-2x) + \frac{1}{h^2} I_m^{(2)}] f_m \\ = -2hA_m(x^2(1-x)^2)e$$

where  $e = (1,1,\dots,1)^T$ ,  $T$  denoting the transpose. Solving this system for the case  $N = 16$ ,  $h = .75/N^{1/4}$ ,  $x_k = \frac{1}{2} + \frac{1}{2} \tanh(kh/2)$ , we get an approximate solution

$$f(x) \approx \sum_{k=-16}^{16} f_k S(k,h) \phi(x) \quad (\phi(x) = \log[x/(1-x)])$$

which is accurate to 5 dec. on  $[0,1]$ . Similar accuracy obtains if the  $-2$  in (3.1) is replaced by  $-2x^{\alpha-2}(1-x)^{\beta-2}$ ,  $\alpha, \beta > 0$ .

Ex.2.  $f'' = f - f^3/x^3$ ,  $f(0) = f(\infty) = 0$ . This problem was solved by different procedures in [1] and [6]. By taking  $x_k = e^{kh}$  and combining (2.63) and (2.48), we get the approximating system

$$(3.3) \quad \left[ I_m^{(1)} + \frac{1}{h} I_m^{(2)} \right] f_m = h A_m(x^2) [f_m - e_m]$$

where  $\underline{e}_m$  denotes the vector  $[x_{-N}^{-2} f^3, x_{-N+1}^{-2} f^3, \dots, x_N^{-2} f^3]^T$ .

The solution of (3.3) involves the solution of a system of nonlinear equations. By taking  $h = .5/N^{1/2}, N = 16$  we get an approximate solution\*

$$f(x) \approx \sum_{k=-16}^{16} f_k S(k,h) \circ \phi(x) \quad (\phi(x) = \log x)$$

which is accurate to 5 decimals on  $[0, \infty)$ .

Ex.3. 
$$\begin{cases} u_{xx} = u_t, & 0 < x < 1, & t > 0 \\ u(x,0) = \sin \pi x, & u(0,t) = u(1,t) = 0. \end{cases}$$

In order to get zero boundary conditions, we set

$$(3.5) \quad u = v + \sin(\pi x) e^{-4t}.$$

This yields the problem

$$(3.6) \quad \begin{cases} v_{xx} - v_t = (\pi^2 - 4) \sin(\pi x) e^{-4t}, & 0 < x < 1, & t > 0 \\ v(x,0) = 0, & v(0,t) = v(1,t) = 0. \end{cases}$$

We solve this by taking our approximating basis functions to be

$$(3.7) \quad \begin{cases} S_k(x) = x(1-x) S(k,h) \circ \phi(x), & \phi(x) = \log[x/(1-x)] \\ S_\ell^*(t) = t S(\ell,s) \circ \phi^*(t), & \phi^*(t) = \log t. \end{cases}$$

The problem (3.6) may now readily be reduced to a matrix problem, by proceeding as for (3.2) above. Setting

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\*The solution  $f$  satisfies  $f(x) \sim Ae^{-x}$  as  $x \rightarrow \infty$ , and consequently, it may be necessary, on some computers, to replace  $\sum_{-N}^N$  by  $\sum_{-M}^M$ , where  $M \ll N$ , in order to avoid underflow.

$$(3.8) \quad \underline{V} = \begin{bmatrix} v_{-N,N} & v_{-N,-N+1} & \dots & v_{-NN} \\ v_{-N+1,-N} & v_{-N+1,-N+1} & \dots & v_{-N+1,N} \\ \dots & \dots & \dots & \dots \\ v_{N,-N} & v_{N,-N+1} & \dots & v_{NN} \end{bmatrix},$$

$$(3.9) \quad \underline{B} = -2h\underline{A}_m(x(1-x)) + \underline{I}_m^{(1)} \underline{A}_m(1-\lambda x) + \frac{1}{h} \underline{I}_m^{(2)} \quad (x_k = \frac{1}{2} + \frac{1}{2} \tanh \frac{kh}{2}),$$

$$(3.10) \quad \underline{C} = \underline{A}_m(t) [\underline{I}_m^{(0)} - \underline{I}_m^{(1)}] (\underline{I}_m^{(0)}) = \text{unit matrix}, \quad t_k = e^{ks},$$

$$(3.11) \quad \underline{D} = h\underline{A}_m(x^2(1-x)^2), \quad (x_k = \frac{1}{2} + \frac{1}{2} \tanh(kh/2)),$$

$$(3.12) \quad \underline{E} = s\underline{A}_m(t^2), \quad (t_k = e^{ks}),$$

$$(3.13) \quad \underline{F} = \begin{bmatrix} F_{-N,-N} & F_{-N,-N+1} & \dots & F_{-N,N} \\ F_{-N+1,-N} & F_{-N+1,-N+1} & \dots & F_{-N+1,N} \\ \dots & \dots & \dots & \dots \\ F_{N,-N} & F_{N,-N+1} & \dots & F_{NN} \end{bmatrix},$$

where

$$F_{kl} = (\pi^2 - 4) \sin(\pi x_k) e^{-4t_l},$$

we arrive at the matrix system

$$(3.14) \quad D^{-1}BV + VCE^{-1} = F$$

Eq. (3.14) may be solved by diagonalizing  $\underline{D}^{-1}\underline{B}$  and  $\underline{C}\underline{E}^{-1}$ . If  $\lambda_{-N}, \lambda_{-N+1}, \dots, \lambda_N$  and  $\mu_{-N}, \mu_{-N+1}, \dots, \mu_N$  denote the eigenvalues of  $\underline{D}^{-1}\underline{B}$  and  $\underline{C}\underline{E}^{-1}$  respectively, obtained by taking  $\underline{X}^{-1}\underline{D}^{-1}\underline{B}\underline{X}$  and  $\underline{Z}\underline{C}\underline{E}^{-1}\underline{Z}^{-1}$  via e.g. the method of Gøub and Reinsch [3], and if  $\underline{G} = [g_{kl}] = \underline{X}^{-1}\underline{F}\underline{Z}^{-1}$ ,  $\underline{Y} = [y_{kl}] = \underline{X}^{-1}\underline{V}\underline{Z}^{-1}$  then  $y_{kl} = g_{kl}/(\lambda_k + \mu_l)$ , and  $\underline{V} = \underline{X}\underline{Y}\underline{Z}$ .

By taking  $h = .75/N^{1/2}$ ,  $a = .5/N^{1/2}$ ,  $N=16$ , we get an approximation

$$u(x,t) \approx e^{-4t} \sin \pi x + \sum_{k,l=-16}^{16} v_{kl} S(k,h) \circ \phi(x) S(l,h) \circ \phi^*(t)$$

which is accurate to 4 dec. on  $[0,1] \times [0,\infty]$ .

Ex. 4

$$u_{xx} + u_{yy} = -1, \quad (x,y) \in S \equiv [0,1] \times [0,1]$$

(3.15)

$$u = 0 \quad \text{on } \partial S$$

Letting  $\underline{B}$  and  $\underline{D}$  be defined as in (3.9) and (3.11) we now get the approximating matrix system

$$(3.16) \quad \underline{D}^{-1}\underline{B}\underline{U} + \underline{U}(\underline{D}^{-1}\underline{B})^T = -\underline{H}$$

where  $\underline{U} = [u_{kl}]$ ,  $\underline{H} = [h_{kl}]$ ,  $h_{kl} = 1$ . This may now be readily solved via the diagonalization of  $\underline{D}^{-1}\underline{B}$ . By taking  $N = 16$ ,  $h = .75/N^{1/2}$  we get an approximate solution

$$u(x,y) \approx \sum_{k,l=-16}^{16} u_{kl} S(k,h) \circ \phi(x) S(l,h) \circ \phi(y)$$

which is accurate to 5 dec. on  $S$ .

#### 4. Error Analysis

For sake of simplicity, we shall restrict ourselves to the simpler case of the second order problem

$$(4.1) \quad u'' + f(x,u) = 0, \quad u(0) = u(1) = 0$$

The analysis for the case of other ordinary or partial differential equations is somewhat more complicated, but may be carried out similarly.

Throughout this section  $\alpha, C_1, C_2, \dots, C_{16}$  denote positive constants, and  $h = [\pi d / (\alpha N)^{\frac{1}{2}}]$ .

In the notation of the previous sections, we take  $\phi(z) = \log[z/(1-z)]$ , and we take the domain of analyticity to be  $\mathcal{D} = \{z: |\arg[z/(1-z)]| < d\}$ . We shall assume that (4.1) has a (locally) unique solution  $u_0$  which is analytic and bounded in  $\mathcal{D}$  and which satisfies the inequality

$$(4.2) \quad |u_0(x)| \leq C_1 x^\alpha (1-x)^\alpha, \quad 0 \leq x \leq 1.$$

Definition 4.1. Let  $M(d, \alpha)$  denote the family of all functions  $v$  that are analytic in  $\mathcal{D}$ , such that

$$(4.3) \quad \begin{cases} v(0) = v(1) = 0 \\ gv'' \in B(\mathcal{D}), \quad |g(x)v''(x)| \leq C_2 x^{\alpha-1} (1-x)^{\alpha-1} \text{ on } (0,1); \\ gf(\cdot, v) \in B(\mathcal{D}), \quad |g(x)f(x, v(x))| \leq C_3 x^{\alpha-1} (1-x)^{\alpha-1} \text{ on } (0,1); \end{cases}$$

where

$$(4.4) \quad g(x) = x(1-x).$$

We shall also assume that the solution of the Frechet derivative problem

$$(4.5) \quad \theta''(x) + f_u(x, u(x))\theta(x) = w(x), \quad \theta(0) = \theta(1) = 0$$

satisfies

$$(4.6) \quad |\theta(x)| \leq C_4 \|A^{-1}w\|$$

for all  $u \in M(d, \alpha)$  such that  $\|u - u_0\| \leq \epsilon$  where  $\|\cdot\|$  is defined by

$$(4.7) \quad \|f\| = \sup_{x \in (0,1)} |f(x)|,$$

where

$$(4.8) \quad (A^{-1}f)(x) = -\int_0^1 G(x,t)f(t)dt$$

and where for any  $x \in [0,1]$ ,

$$(4.9) \quad G(x,t) = \begin{cases} (1-x)t & \text{if } 0 \leq t \leq x \\ x(1-t) & \text{if } x \leq t \leq 1. \end{cases}$$

Moreover, we shall assume that if  $\|u - u_0\| \leq \epsilon$ , then

$$(4.10) \quad \| \{A^{-1}f(t, u(t))\} \| \leq C_5.$$

Let us assume that we have found an approximate solution

$$(4.11) \quad u_m(x) = \sum_{k=-N}^N u_k S(k,h) \phi(x) \quad (m = 2N + 1)$$

by the method of the previous sections, and let us set

$$(4.12) \quad \theta_m = u_m - u_0$$

Then

$$(4.13) \quad \theta_m''(x) + f_u(x, \bar{u}(x))\theta_m(x) = u_m''(x) + f(x, u_m(x))$$

for some  $\bar{u}$  between  $u_0$  and  $u_m$ , and therefore, by (4.5) and (4.6),

$$(4.14) \quad |\theta_m(x)| \leq C_4 \|u_m + A^{-1}f(\cdot, u_m)\|.$$

Now by Theorem 2.1, we find, by taking  $S_k(x) = g(x)S(k, h) \circ \phi(x)$ ,  $x_k = \frac{1}{2} + \frac{1}{2} \tanh(kh/2)$ , that

$$(4.15) \quad \int_0^1 [v''(x) + f(x, v(x))] S_k(x) dx = h \frac{g(x_k)}{\phi'(x_k)} [v''(x_k) + f(x_k, v(x_k))]$$

and

$$(4.16) \quad \int_0^1 v''(x) S_k(x) dx \\ = h \sum_{j=-N}^N v(x_j) \left[ \frac{g''(x_j)}{\phi'(x_j)} \delta_{kj}^{(0)} + \{2g'(x_j) + g(x_j)\phi''(x_j)/\phi'(x_j)\} \frac{\delta_{kj}^{(1)}}{h} \right. \\ \left. + g(x_j)\phi'(x_j) \frac{\delta_{kj}^{(2)}}{h^2} \right]$$

in which the error of either term on the right-hand side of (4.15) is bounded by  $C_6 N^{-\frac{1}{2}} e^{-(\pi \alpha N)^{\frac{1}{2}}}$  and the error of the right-hand side of (4.16) is bounded by  $C_7 N^{\frac{1}{2}} e^{-(\pi \alpha N)^{\frac{1}{2}}}$ . By our process of solution, the numbers  $u_k$  in (4.11) are determined such that

$$(4.17) \quad h \sum_{j=-N}^N u_j \left[ \frac{g_j''}{\phi_j'} \delta_{kj}^{(0)} + \{2g_j' + g_j \phi_j''/\phi_j'\} \frac{\delta_{jk}^{(1)}}{h} + g_j \phi_j' \frac{\delta_{kj}^{(2)}}{h^2} \right] \\ + h \frac{g_k}{\phi_k} f(x_k, u_k) = 0, \quad k = -N, -N+1, \dots, N.$$

Theorem 4.2: Let the numbers  $u_k$  ( $k = -N, -N+1, \dots, N$ ) be determined by (4.17), and let  $u_m(x)$  be defined as in (4.11). Then

$$(4.18) \quad |u_m(x) - u_0(x)| \leq C_{15} N^{3/2} e^{-(\pi d \alpha N)^{1/2}}, \quad 0 \leq x \leq 1.$$

where  $u_0$  is the solution of (4.1).

Proof: In view of the errors in the approximations (4.15) and (4.16), the solution of (4.17) is equivalent to finding a function  $v \in M(d, \alpha)$ , such that

$$(4.19) \quad \frac{g(x_k)}{\phi'(x_k)} [v''(x_k) + f(x_k, v(x_k))] = \frac{\epsilon_k}{h}, \quad k = -N, -N+1, \dots, N,$$

where  $v(x_k) = u_k$ , and where

$$(4.20) \quad |\epsilon_k| \leq C_8 N^{1/2} e^{-(\pi d \alpha N)^{1/2}}, \quad k = -N, -N+1, \dots, N.$$

Since  $v \in M(d, \alpha)$ , it follows, for any  $t \in (0, 1)$ , that

$$(4.21) \quad \begin{aligned} & \frac{g(t)}{\phi'(t)} [v''(t) + f(t, v(t))] - \sum_{k=-\infty}^{\infty} \frac{g(x_k)}{\phi'(x_k)} [v''(x_k) + f(x_k, v(x_k))] S(k, h) \circ \phi(t) \\ &= \frac{\sin[\pi \phi(t)/h]}{2\pi i} \int_{\partial D} \frac{g(z) [v''(z) + f(z, v(z))] dz}{[\phi(z) - \phi(t)] \sin[\pi \phi(z)/h]} \end{aligned}$$

By multiplying (4.21) by  $\phi'(t)^2$ , taking  $\Lambda^{-1}$  of each side, and noting that  $g(t)\phi'(t) = 1$ , we get

$$(4.22) \quad \begin{aligned} & v(x) + (\Lambda^{-1} f(t, v(t)))(x) - \sum_{k=-\infty}^{\infty} \frac{g_k}{\phi_k} [v_k'' + f(x_k, v_k)] \Lambda^{-1} \{ \phi'(t)^2 S(k, h) \circ \phi(t) \}(x) \\ &= \Lambda^{-1} \left\{ \frac{\phi'(t)^2 \sin[\pi \phi(t)/h]}{2\pi i} \int_{\partial D} \frac{g(z) [v''(z) + f(z, v(z))] dz}{[\phi(z) - \phi(t)] \sin[\pi \phi(z)/h]} \right\}(x) \end{aligned}$$

Since  $\phi'(t) = 1/[t(1-t)]$ , it follows, by taking  $t = [1 + \tanh(u/2)]/2$ ,  $x = [1 + \tanh(w/2)]/2$ , and using (4.8) and (4.9), that

$$\begin{aligned}
 I_1(h,x) &\equiv A^{-1}\{\phi'(t)^2 \operatorname{sinc}[(\phi(t)-kh)/h]\}(x) \\
 &= - \int_{-\infty}^w \frac{1 - \tanh(w/2)}{1 - \tanh(u/2)} \operatorname{sinc}\left[\frac{u-kh}{h}\right] du \\
 &\quad - \int_w^{\infty} \frac{1 + \tanh(w/2)}{1 + \tanh(u/2)} \operatorname{sinc}\left[\frac{u-kh}{h}\right] du
 \end{aligned}
 \tag{4.23}$$

On the interval  $[-\infty, w]$ , the function  $[1 - \tanh(w/2)]/[1 - \tanh(u/2)]$  increases monotonically from  $[1 - \tanh(w/2)]/2$  to 1 while on  $[w, \infty]$ , the function  $[1 + \tanh(w/2)]/[1 + \tanh(u/2)]$  decreases monotonically from 1 to  $[1 + \tanh(w/2)]/2$ . For this reason, it may be shown by a somewhat lengthy, but simple argument, that

$$|I_1(h,x)| \leq 4\pi h . \tag{4.24}$$

Similarly if  $x \in [0, 1]$  and  $z \in \partial D$ , we can show that

$$|I_2(h,x)| \equiv \left| A^{-1} \left\{ \frac{\phi'(t)^2 \sin[\pi\phi(t)/h]}{2\pi i [\phi(z) - \phi(t)]} \right\} (x) \right| \leq \frac{2h}{d} \tag{4.25}$$

since  $\operatorname{Im} \phi(z) = td$ .

By means of (4.19), (4.22) and (4.25), Eq. (4.21) now yields

$$\begin{aligned}
 &|v(x) + \{A^{-1}f(t, v(t))\}(x)| \\
 &\leq |I_1(h,x)| \sum_{k=-N}^N \frac{|c_k|}{h} + |I_1(h,x)| \sum_{|k| > N} \frac{g_k}{\phi_k} |v_k'' + f(x_k, v_k)| \\
 &+ |I_2(h,x)| \int_{\partial D} \left| \frac{g(z)[v''(z) + f(z, v(z))]}{\sin[\pi\phi(z)/h]} \right| dz
 \end{aligned}
 \tag{4.26}$$

Using the bounds given in (4.20) and (4.24), we bound the first sum on the right-hand side of (4.26) by  $C_9 N^{3/2} e^{-(\pi d \alpha N)^{1/2}}$ ; using (4.3) and (4.24), and recalling that  $x_k = \frac{1}{2} + \frac{1}{2} \tanh(kh/2)$ , we bound the second sum on the right-hand side of (4.26) by  $C_{10} e^{-(\pi d \alpha N)^{1/2}}$ ; and using (4.25) and the fact that  $|\sin[\pi \phi(z)/h]| \geq \sinh[\pi d/h]$  if  $z \in \partial D$ , we bound the integral term on the right-hand side of (4.24) by  $2h N(g[v'' + f(\cdot, v)]/[d \sinh(\pi d/h)]) = C_{11} N^{-1/2} e^{-(\pi d \alpha N)^{1/2}}$ . Hence for all  $x \in [0, 1]$ ,

$$(4.27) \quad |v(x) + \{A^{-1} f(t, v(t))\}(x)| \leq C_{12} N^{3/2} e^{-(\pi d \alpha N)^{1/2}}.$$

Since  $v \in M(d, \alpha)$ , it follows from the first and second of (4.3) that

$$(4.28) \quad |v(x)| \leq C_{13} x^\alpha (1-x)^\alpha, \quad 0 \leq x \leq 1.$$

Furthermore, since  $v \in M(d, \alpha)$ , and since  $u_m$  and  $v$  coincide at  $x_{-N}, x_{-N+1}, \dots, x_N$ , it follows that [13, Theorem 8.2] for all  $x \in [0, 1]$ ,

$$(4.29) \quad |u_m(x) - v(x)| \leq C_{14} N^{1/2} e^{-(\pi d \alpha N)^{1/2}}.$$

In view of (4.5), (4.6) and (4.10), it now follows that for all  $x \in [0, 1]$ ,

$$(4.30) \quad |u_m(x) - \{A^{-1} f(t, u_m(t))\}(x)| \\ \leq |v(x) + \{A^{-1} f(t, v(t))\}(x)| + C_5 |u_m(x) - v(x)|$$

By (4.14), (4.27), (4.29) and (4.30), it thus follows that for all  $x \in [0, 1]$

$$(4.29) \quad |\theta_m(x)| = |u_m(x) - u_0(x)| \leq C_{15} N^{3/2} e^{-(\pi d \alpha N)^{1/2}}.$$

This completes the proof of Theorem 4.2.

Similarly, it may be shown that when using  $n = (2N+1)^2$  points to obtain an approximate solution of a partial differential equation, such as (3.15), the error is bounded by  $C_{16} N^{3/2} e^{-\gamma N^{1/2}} \leq 5C_{16} n^{3/4} e^{-\gamma n^{1/2}}$ .  
Indeed, for the case of (3.15), we may take  $C_{16} = 1$  and  $\gamma = \pi^2$ .

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**This paper illustrates the application of a "Sinc-Galerkin" method to the approximate solution of linear and nonlinear second order ordinary differential equations, and to the approximate solution of some linear elliptic and parabolic partial differential equations in the plane. The method is based on approximating functions and their derivatives by use of the Whittaker cardinal function. The DE is reduced to a system of**

→ algebraic equations via new accurate explicit approximations of the inner products, the evaluation of which does not require any numerical integration. Using  $n$  function evaluations the error in the final approximation to the

solution of the DE is  $O(e^{-cn^{1/d}})$  where  $c$  is independent of  $n$ , and  $d$  denotes the dimension of the region on which the DE is defined. This rate of convergence is optimal in the class of  $n$ -point methods which assume that the solution is analytic in the interior of the interval, and which ignore possible singularities of the solution at the end-points of the interval.

$\Phi(\exp(-c(n \text{ to the } \frac{1}{d} \text{ power})))$

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