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It takes of the order of N^3 operations to solve a set of N linear equations in N unknowns. When the underlying physical problem has some time- or shift-invariance properties, the coefficient matrix is of Toeplitz (or difference or convolution) type and the equations can be solved with $O(N^2)$ operations. We have shown that with any nonsingular N x N matrix, we can associate an integer α between 1 and N such that it takes $O(N^2\alpha)$ operations to invert the matrix. The number α may be small for many non-Toeplitz matrices of physical interest. Some aspects of this result are discussed here, including extensions to continuous-

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Alpha

20. Abstract

time kernels and integral equations.

SOME NEW METHODS FOR SOLVING LINEAR EQUATIONS[f]

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ABSTRACT. It takes of the order of N^3 operations to solve a set of N linear equations in N unknowns. When the underlying physical problem has some time- or shift-invariance properties, the coefficient matrix is of Toeplitz (or difference or convolution) type and the equations can be solved with $O(N^2)$ operations. We have shown that with any non-singular $N \times N$ matrix, we can associate an integer α between 1 and N such that it takes $O(N^2\alpha)$ operations to invert the matrix. The number α may be small for many non-Toeplitz matrices of physical interest. Some aspects of this result are discussed here, including extensions to continuous-time kernels and integral equations.

1. INTRODUCTION. Problems in many fields lead ultimately to the solution of linear matrix equations

$$Ra = m ,$$

where R is a given $N \times N$ matrix, say, and m is a given $N \times 1$ vector. The number of operations required to solve such an equation, or to find R^{-1}, is of the order of N^3 (multiplications and additions). This can be prohibitive if N is large (500 or 1000 or 3000, as can arise in many power system or econonometric calculations). For this, and other reasons, we must often try to bring in any special features or structures that may be present in the original physical problem. In many applications

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[f] This report is a summary of a talk given at the 22nd Conference of Army Mathematicians, Watervliet Arsenal, New York, May 1976. It was based on work done jointly with B. Friedlander, L. Ljung and M. Morf (see the references).

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we have the property

$$R = [r_{ij}] = [r_{i-j}]$$

That is, the phenomena are invariant to a change in the time- or space-origin (e.g., as with stationary random processes, or homogeneous media, etc.). In this case, the matrix R is said to be a Toeplitz matrix and has the nice feature that its inverse can be found with only $O(N^2)$ multiplications. Moreover the inverse can be computed recursively, i.e., the $N \times N$ inverse can be easily updated to yield the $(N + 1) \times (N + 1)$ inverse, and Toeplitz matrices also have other useful properties.

Unfortunately, most operations on Toeplitz matrices destroy the Toeplitz property. For example, the inverse of a Toeplitz matrix is not Toeplitz, unless the matrix is also lower- or upper-triangular. So also the product of two Toeplitz matrices is not Toeplitz, unless the matrices are also both lower-triangular or both upper-triangular. However, some reflection will show that in various ways one can regard certain matrices as being "less non-Toeplitz" than others, though present solution methods cannot take advantage of this--they require $O(N^2)$ operations in the Toeplitz case, and $O(N^3)$ otherwise.

By a long process of abstraction and simplification of results originally obtained for certain nonlinear differential equations [1], [2], we have been able to show essentially the following (more precise results are stated later): with any invertible $N \times N$ matrix R we can associate an integer α, $1 \leq \alpha \leq N$, such that it takes $O(N^2\alpha)$ operations to compute its inverse. The integer α may be called the displacement rank (or index of nonstationarity) of the matrix and has the property that it is low for matrices that are Toeplitz or near to Toeplitz, while it is high for arbitrary matrices. For example,

i) $\alpha = 1$ for $R = L$ or U or LU or UL, where L and U denote lower- and upper-triangular Toeplitz matrices.

ii) $\alpha = 2$ for $R = (L + U)$ and $R = (L + U)^{-1}$

iii) $\alpha \leq 4$ for $R = (L_1 + U_1)(L_2 + U_2)$

iv) $\alpha \leq 3$ for $R = [L_1 + U_1 \vdots L_2 + U_2]$

v) $\alpha \leq n$, if R is the covariance matrix of a linear combination of the components of any n-vector wide-sense Markov random process.

In such cases, $O(N^2\alpha)$ can often be significantly less than $O(N^3)$, thus yielding many advantages, not just for solving a given large set of equations, but also for interactive adjustment of the mathematical model (i.e., of R and m) based on actual examination of the now-more-easily determined solution a.

We shall outline our major results in Section 2, for matrix equations. A similar, and somewhat simpler, development can be carried out for integral equations, as noted in Section 3. Section 4 contains some concluding remarks on possible extensions and generalizations.

2. THE MATRIX CASE. More details can be found in the paper [3], but we note the key definitions and results here.

Definition 1. The (+)-displacement rank of an $N \times N$ matrix R is the smallest integer $\alpha_+(R)$ such that we can write

$$R = \sum_{1}^{\alpha_+(R)} L_i U_i$$

for some lower-triangular Toeplitz matrices $\{L_i\}$ and some upper-triangular Toeplitz matrices $\{U_i\}$.

Definition 2. The (-)-displacement rank of an $N \times N$ matrix R is the smallest integer $\alpha_-(R)$ such that we can write

$$R = \sum_{1}^{\alpha_-(R)} U_i \mathcal{L}_i$$

for some lower-triangular Toeplitz matrices $\{\mathcal{L}_i\}$ and upper-triangular Toeplitz matrices $\{U_i\}$.

Definition 3. Let

Z = the lower-shift matrix

$$= \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & \ddots & \\ & & \ddots & 0 \end{bmatrix}$$

Lemma 1. Computation of Displacement Ranks

$$\alpha_+(R) = \rho\{\nabla(R)\}$$

where

$$\nabla(R) = R - ZRZ', \tag{1}$$

and

$\rho\{A\}$ = the rank of the matrix A.

Also

$$\alpha_-(R) = \rho\{\Gamma(R)\}$$

where

$$\Gamma(R) = R - Z'RZ \ . \tag{2}$$

The proof follows by using the result of Lemma 2.

Lemma 2. Given two column vectors x, y there is one and only one solution of the functional equation

$$\square(R) = xy' \ , \tag{3a}$$

and this is

$$R = L(x)U(y') \ , \tag{3b}$$

where $'$ denotes transpose, $L(x)$ is a lower-triangular Toeplitz matrix whose first column is x, and $U(y')$ is an upper-triangular Toeplitz matrix with first row y'.

Proof. For uniqueness, note that

$$\square(R_1) = \square(R_2)$$

implies

$$R_1 - ZR_1Z' = R_2 - ZR_2Z'$$

or

$$R_1 - R_2 = Z(R_1 - R_2)Z' \ ,$$

whose only solution is clearly zero.

The rest amounts to verifying that $\square L(x)U(y') = xy'$, which the reader may find amusing to check by direct computation for 3×3 matrices. ∎

Lemma 1 now follows easily from the observation that

$$R = \sum_1^{\alpha} L(x_i)U(y_i') \iff \square(R) = \sum_1^{\alpha} x_i y_i' \tag{4}$$

We can now state a first simple, but apparently new, result.

Theorem 1.

$$\alpha_-(R^{-1}) = \alpha_+(R) \ . \tag{5}$$

Therefore,

$$R = \sum_{1}^{\alpha_+(R)} L_i U_i \qquad (6a)$$

implies that R^{-1} has the form

$$R^{-1} = \sum_{1}^{\alpha_+(R)} \mathcal{U}_i \mathcal{L}_i \qquad (6b)$$

Proof. We give the simple proof (suggested by S-Y. Kung) because it shows that the result is quite general and depends very little on the nature of the entries of R--for example, they could themselves be matrices. We note that

$$\begin{aligned} \alpha_-(R^{-1}) &= \rho\{R^{-1} - Z'R^{-1}Z\} \\ &= \rho\{(R^{-1} - Z'R^{-1}Z)R\} \\ &= \rho\{I - Z'R^{-1}ZR\} \end{aligned}$$

since rank is unaffected by multiplication by a nonsingular matrix. Now by a well-known matrix result that

$$\rho\{I - AB\} = \rho\{I - BA\}$$

we can continue the above chain as

$$\begin{aligned} \alpha_-(R^{-1}) &= \rho\{I - ZRZ'R^{-1}\} \\ &= \rho\{(I - ZRZ'R^{-1})R\} \\ &= \rho\{R - ZRZ'\} \\ &= \alpha_+(R) \ . \ \blacksquare \end{aligned}$$

Example. If T is a symmetric Toeplitz matrix, then $\alpha_+(T) = 2 = \alpha_-(T)$ since we have the representations

$$\begin{aligned} T &= T_+ \cdot I + I \cdot T_+' \\ &= I \cdot T_+ + T_+' \cdot I \ , \end{aligned}$$

where

T_+ = the lower-triangular part of the matrix T.

The fact that

$$\alpha_+(T) = 2 = \alpha_-(T)$$

can also be seen by checking that

$$\Delta(T) = T - ZTZ'$$

$$= \begin{bmatrix} t_1 & t_2 & \cdots & t_N \\ t_2 & & & \\ \vdots & & \bigcirc & \\ t_N & & & \end{bmatrix}, \quad \text{for all } N \geq 2$$

and

$$\Gamma(T) = \begin{bmatrix} & & & t_N \\ & & & t_{N-1} \\ & & \bigcirc & \vdots \\ & & & t_2 \\ t_N & \cdots & t_2 & t_1 \end{bmatrix}, \quad N \geq 2 .$$

Now it turns out to have been well-known in many contexts (see the discussion in [4]) that there exist two lower-triangular Toeplitz matrices A and B such that

$$T^{-1} = A'A - B'B \tag{7}$$

so that

$$\alpha_-(T) = 2 = \alpha_+(T) . \blacksquare$$

Remark. Notice that the displacement ranks seem to identify a better property of matrices than their being Toeplitz. The class of Toeplitz matrices is not closed under inversion, unlike the $(\pm)$-displacement ranks and the corresponding representations (6).

Theorem 2. The inverse of an $N \times N$ matrix R can be found with $O(N^2\alpha)$ multiplications, where α is an integer such that $\alpha_+ \leq \alpha \leq \alpha_+ + 2$. This

can be done via certain recursive formulas called the generalized Szegö Levinson recursions.

The recursions are a bit too complicated to describe here, but we may note that for Toeplitz matrices they are equivalent to the well-known recursions for the Szegö polynomials orthogonal on the unit circle (see, e.g., [5, Ch. 11] or [6]). These were rediscovered in the statistics literature by Levinson [7] and by Durbin [8] for recursively solving the so-called Yule-Walker normal equations [9].

For other results in the matrix case, we refer to [3], [10]-[11], and instead turn briefly here to an examination of the integral operator case.

3. INTEGRAL EQUATIONS. The Fredholm integral equation of the second kind

$$a(t) + \int_0^T K(t,s)a(s)\,ds = m(t), \quad 0 \le t \le T \tag{8}$$

has been extensively studied, see, e.g., the recent monograph [12]. Except for the handful of cases where explicit analytic solution is possible, the generic technique is to replace the integral equation by some approximating set of N linear equations

$$Ra = m .$$

This can be done in various ways--use of degenerate kernels, projection (Galerkin and collocation) methods, etc. For example in the degenerate kernel method we replace $K(t,s)$ by the function

$$K_N(t,s) = \sum_1^N \phi_i(t)\psi_i(s) \tag{9}$$

for some suitably chosen functions $\{\phi_i(\cdot),\psi_i(\cdot)\}$. In any case, the resulting set of linear equations will in general require $O(N^3)$ operations for their solution and this may be prohibitively large. More significant however is the observation that such approximation methods will generally destroy any nice structure that might have been present in the original problem.

For example, if the kernel was of Toeplitz (also called displacement or convolution) type,

$$K(t,s) = K(t - s) , \text{ say}$$

then in general

$$K_N(t,s) \neq \text{Toeplitz for } N < \infty .$$

This is bad, because the Toeplitz property can be exploited to find a nice solution of the integral equation. Briefly, first define

$$H(t,s) = \text{the Fredholm resolvent of } K(t,s)$$

as the solution of the integral equation

$$H_T(t,s) + \int_0^T H_T(t,r)K(r,s)\,dr = K(t,s) , \quad 0 \leq t,s \leq T .$$

In operator notation, we can write this as

$$H + HK = K$$

or as

$$(I - H)(I + K) = I$$

which shows that the original equation

$$(I + K)a = m$$

can be resolved as

$$a = (I + K)^{-1}m = (I - H)m$$

or

$$a(t) = m(t) - \int_0^T H_T(t,s)m(s)\,ds .$$

Therefore the basic problem is to find $H(t,s)$. Now even though $K(t - s)$ may be Toeplitz, this will not in general be true of its resolvent $H_T(t,s)$ (for $T < \infty$). Nevertheless $H_T(t,s)$ is not a completely arbitrary kernel, but should in some sense be close to a Toeplitz kernel (after all, its resolvent is Toeplitz).

We can quantify this intuitive feeling in the following way (the analog of the method used in Section 2). Define the operator

$$\nabla K(t,s) = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)K(t,s) , \tag{10}$$

and note that

$$\nabla K(t - s) \equiv 0 .$$

If $K(t,s)$ is not Toeplitz $\nabla K(t,s) \not\equiv 0$, but it will be some function of two variables, which we can write as

$$\Delta K(t,s) = \sum_{1}^{\alpha} \phi_i(t)\psi_i(s) \qquad (11)$$

for some functions $\{\phi_i(\cdot),\psi_i(\cdot)\}$ and some integer α, possibly even infinite. However let us define the displacement rank of $K(t,s)$ as the smallest integer $\alpha(K)$ such that the representation (11) is possible.

Examples. i) K is Toeplitz, $\alpha(K) = 0$.

ii) $K(t,s) = \min(t,s)$, the covariance of the simplest nonstationary random process, the Wiener process. Clearly $\Delta K(t,s) \equiv 1$ and $\alpha = 1$.

iii) $K(t,s) = ts - \min(t,s)$, the covariance of the so-called Brownian bridge process. Now $\Delta K(t,s) = s + t - 1$ and $\alpha = 2$. ■

We can show the following result, analogous to Theorem 1 in the matrix case.

Theorem 3. $\alpha(H_T(t,s)) \leq \alpha(K(t,s)) + 2$.

Example. When K is Toeplitz, $\alpha(K) = 0$. However even though its resolvent $H_T(t,s)$ is not Toeplitz, there exist two functions $A_T(\cdot)$, $B_T(\cdot)$ such that

$$\Delta H_T(t,s) = A_T(t)A_T(s) - B_T(t)B_T(s), \qquad (12)^{\dagger}$$

so that

$$\alpha(H_T(t,s)) = 2 .$$

Moreover the functions $A_T(\cdot)$ and $B_T(\cdot)$, of one variable, can be determined more easily than functions of two variables. In fact they can be obtained via the differential equations

$$\left(\frac{\partial}{\partial T} + \frac{\partial}{\partial t}\right)A_T(t) = - B_T(t)B_T(T) , \quad 0 \leq t \leq T \qquad (13a)$$

$$\frac{\partial}{\partial T} B_T(t) = - A_T(t)B_T(t) , \quad 0 \leq t \leq T \qquad (13b)$$

with certain easily determined boundary conditions $A_T(0)$ and $B_T(T)$.

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†This is the analog of (7) in the matrix case.

The point is that these differential equations can be solved by a simple recursive procedure, which needs only proportional to N^2 operations, where N is the number of points in $[0,T]$ used in any discretization procedure.

We call (13) Krein-Szegö-Levinson equations because they are exactly the recursions found by Krein [13] for the continuous analogs of the Szegö polynomials on the unit circle.

Theorem 4. If $K(t,s)$ has displacement rank α, $H_T(t,s)$ can be found with α times as much computation as in the Toeplitz case. The solution is found recursively via a set of generalized Krein-Szegö-Levinson equations.

Proofs and further results can be found in the papers [14]-[15]. However, we might draw explicit attention to the fact that though we are using a degenerate-kernel representation in (11), this is for $\dot{\int} K(t,s)$ and not for $K(t,s)$. Even though $\dot{\int} K(t,s)$ is degenerate it can be seen by integration that, in operator notation,

$$K = \sum_{1}^{\alpha+1} L_i U_i$$

where the $\{L_i\}$ and $\{U_i\}$ are lower- and upper-Volterra operators. Therefore K can be very far from a degenerate kernel. The feature of our method is that it preserves any "Toeplitz-like" structure that may be present in $K(t,s)$. This thought is pursued a bit further in Section 4.

4. CONCLUDING REMARKS. We have taken Toeplitz kernels as basic because they, or things close to them, arise in many applications of interest to us. However in other problems, other "nice" kernels may be more basic. For example, we might have Hankel kernels

$$K(t,s) = K(t + s) \text{ , say .}$$

Integral equations with such kernels can be solved efficiently, and therefore it may be of interest to classify kernels in terms of their degree of "non-Hankelness". This can clearly be done as above by using the operator

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial s}\right)K(t,s) \text{ ,}$$

which gives zero when applied to Hankel kernels. Similar results can also be obtained for basic kernels of the form $K_1(t - s) + K_2(t + s)$.

Furthermore we could also define "second" and higher-order operators of the type

$$\mathcal{J}^2\{K(t,s)\} = \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^2 K(t,s)$$

and so on. It is easy to find examples where these are particularly appropriate.

As a final comment, we should express our feeling that the basic ideas described above should be adaptable to a variety of different situations. Also there is clearly some quite general algebraic structure lurking behind our results, which some of the people in this audience may be better equipped to identify than we can.

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