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AN ALGORITHM FOR THE ELECTROMAGNETIC SCATTERING DUE TO AN AXIALLY SYMMETRIC BODY WITH AN IMPEDANCE BOUNDARY CONDITION *

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ABSTRACT

Let B be a body in R^3, and let S denote the boundary of B. The surface S is described by $S = \{(x,y,z) : \sqrt{x^2 + y^2} = f(z), -1 \leq z \leq 1 \}$ where f is an analytic function that is real and positive on $(-1,1)$ and $f(\pm 1) = 0$. An algorithm is described for computing the scattered field due to a plane wave incident field, under Leontovich boundary conditions. The Galerkin method of solution used here leads to a block diagonal matrix involving $2M + 1$ blocks, each block being of order $2(2N+1)$. If e.g. $N = 0(M^2)$, the computed scattered field is accurate to within an error bounded by $Ce^{-cN^{\frac{1}{2}}}$ where C and c are positive constants depending only on f.

1. INTRODUCTION AND SUMMARY

Let B be a bounded body in R^3, having surface S which is given by

$$(1.1) \qquad S = \{(x,y,z) : \sqrt{x^2+y^2} = f(z), \ -1 \leq z \leq 1\},$$

where f is an analytic function that is real and positive on $(-1,1)$ and

$$(1.2) \qquad C_1 (1+z)^{\alpha_1} (1-z)^{\beta_1} \leq f(z) \leq C_2 (1+z)^{\alpha_2} (1-z)^{\beta_2}, \ -1 \leq z \leq 1,$$

where C_j, $\alpha_j < 1$ and $\beta_j < 1$ are positive constants. In this paper we describe as algorithm for computing the field scattered from B due to an incident field $\overline{E}^o(\overline{r})$ of the form

$$(1.3) \qquad \overline{E}^o(\overline{r}) = \overline{\epsilon} \, e^{i\overline{k}_o \cdot \overline{r}}$$

where $\overline{\epsilon}$ and $\overline{k}_o$ ($|\overline{k}_o| = k_o = \omega/c = 2\pi/\lambda$) denote the polarization and propogation vectors respectively, and $\overline{r} = x\hat{x} + y\hat{y} + z\hat{z}$, where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors pointing in the direction of the x,y and z axes respectively.

Let the body B (rsp. free space) be homogeneous, with permittivity ϵ (rsp. ϵ_o) permeability μ (rsp. μ_o) and conductivity σ (rsp. σ_o), so that the refractive index of the body is

$$(1.4) \qquad N = \left\{ \frac{\mu}{\mu_o} \left(\frac{\epsilon}{\epsilon_o} + \frac{i\sigma}{\omega\epsilon_o} \right) \right\}^{\frac{1}{2}}$$

where ω denotes the frequency of the incident field. We shall furthermore assume that

$$(1.5) \qquad |N| \, |k_o| \, \rho \gg 1,$$

where ρ is the smallest radius of curvature of S. This assumption enables us to apply the Leontovich boundary conditions [15, 21]

$$(1.6) \qquad (\hat{n} \times \overline{E}) \times \hat{n} = \eta Z \, \hat{n} \times \overline{H}$$

on the surface of the body, where

$$(1.7) \qquad \eta = \mu/(\mu_o N), \quad Z = (\mu_o /\varepsilon_o)^{\frac{1}{2}}$$

and where $\hat{n}$ denotes the outward unit normal to S, and $\overline{E}$ and $\overline{H}$ denote the total electric and magnetic fields on S. The conditions (1.5) and (1.6) are satisfied automatically if the body B is perfectly conducting, i.e. if $\sigma = \infty$.

The condition (1.6) makes it possible to obtain a singular vector integral equation over S for the surface current $\overline{K}$ on S. In the present paper we describe an algorithm for solving this integral equation via the Galerkin method, using as basis functions

$$(1.8) \quad \psi_{mn}(z,\varphi) = e^{im\varphi} \, f^{\frac{1}{2}}(z) \, (1-z^2) \, \frac{\sin[\pi \, N^{\frac{1}{2}}\{\log(\frac{1+z}{1-z}) - \frac{n}{N^{\frac{1}{2}}} \}],}{\pi \, N^{\frac{1}{2}}\{\log(\frac{1+z}{1-z}) - \frac{n}{N}1/2\}}$$

$$m = 0, \pm 1, \ldots, \pm N, \quad n = 0, \pm 1, \ldots, \pm M.$$

These basis functions effectively handle singularities of $\overline{K}$
and $\frac{\partial \overline{K}}{\partial z}$ as a function of z, which occur at z = ±1; they are similary very
effective approximants in the ϕ variable, since the Fourier series of
$\overline{E}^o$ (and therefore $\overline{K}$) converges very rapidly. The singularities occuring
in the kernel of the integral equation for $\overline{K}$ are of the type $1/(z' - z)$
or $\log |z' - z|$ at z = z', and of the type of $f^m(z)$, m = 0, 1, ... at
z = ±1. The first of these is effectively handled by substracting out
the principal value. The remaining ones are effectively handled by
means of the quadrature formula (see [22])

$$(1.9) \qquad \int_1^1 g(t)dt \cong h \sum_{k=-L}^{L} \frac{2e^{kh}}{(1+e^{kh})^2} \; g\left(\frac{e^{kh}-1}{e^{kh}+1}\right)$$

after transforming the intervals (-1,z) and (z,1) to (-1,1).
    The integrations over S involve two variables, ϕ' and z'. While
the integrations with respect to z' must be carried out numerically, due
to singularities of the kernel in the region of integration, the integrations
with respect so ϕ are carried out explicitly, the results being expressed
via hypergeometric functions. The hypergeometric functions have logarith-
mic singularities which were not present in the kernel; for this reason
explicit integration and later evaluation of the hypergeometric function as
described in the Appendix has an advantage over direct numerical integration,
since any known direct numerical integration procedure such as the trap-
ezoidal rule would poorly handle this type of transformation of singularity.
    The use of the basis function (1.8) thus leads to a block
diagonal Galerkin system of equations, one system of order 2(2N+1) for
each m. By forming 2M+1 such blocks, and taking $N=M^2$, we arrive an
approximation $\overline{K}_N$ of $\overline{K}$ which is accurate to within an error $\overline{\epsilon}$, where
$|\overline{\epsilon}| = 0 \; (e^{-cN^{\frac{1}{2}}})$ as N → ∞ with c>0 and independent of N. The use of
(1.8) furthermore makes it possible to evaluate the scattered field
$\overline{E}^s$ by means of simple one-dimensional trapezoidal rule integrations.
The error $\overline{\delta}_N$ in our approximation $\overline{E}_N^s$ of the scattered field $\overline{E}^s$ satisfies
the relation

$$(1.10) \qquad |\overline{\delta}_N(\overline{r})| = |\overline{E}^s(\overline{r}) - \overline{E}^s_N(\overline{r})| = 0(e^{-cN^{\frac{1}{2}}}) \text{ as } N \to \infty,$$

for all $\overline{r}$ on the exterior of B and not arbitrarily close to B.

The above problem of computing $\overline{E}^s$ given $\overline{E}^o$ as in (1.2) and S as in (1.1) was studied in [20] for the perfectly conducting case and in [5] for the case as described above. The Galerkin approximating basis function used in [5,20] are of the form $\psi_{mn} = \cos(m\varphi) \, S_n(z)$ and $\sin(m\varphi) \, S_n(z)$, where S_n is the linear spline which is zero at z_{n-1} and z_{n+1}, and 1 at z_n. Thus the resulting rate of convergence is $0(1/N^2)$ if f has no singularities at $z = \pm 1$ and $0(1/N^\alpha)$ if e.g. $f(z) \sim C(1-z)^\alpha$ as $z \to 1$, where it is assumed that the interval $(-1,1)$ is divided into N subintervals. In addition, the quadratures used in [5,20] converge very slowly as a result of the singularities present in the integral equation. Furthermore the expression for the gradient of $G = e^{ik_0|\overline{r}-\overline{r}'|}/(4\pi|\overline{r}-\overline{r}'|)$ obtained in [5,20] is incorrect.

The algorithm of the present paper has been checked out on a computer for the case of a sphere of radius 1 for which the surface current $\overline{K}$ and the scattered field $\overline{E}^s$ can be expressed explicitly. Using $M = 4$ and $N=M^2= 6$, we find that $\overline{K}$ is accurate to 4 significant figures, and for $r > 2$, $\overline{E}^s$ is also accurate to 4 significant figures.

The paper is organized as follows.

In Sec. 2 we describe the geometry of the surface. In Sec. 3 we derive a representation on the surface S for the incident plane wave. Sec. 4 contains a derivation of the integral equation for the surface current, as well as an integral expression for the scattered field in terms of this surface current. In Sec. 5 we describe the basis functions

to be used in the Galerkin method of Sec. 7. Sec. 6 contains an approximate representation of the incident electric field, in terms of the basis functions of Sec. 5. In Sec. 7 we derive the Galerkin equations for the surface current, and we describe a method of computing the coeffieicents of this system, and for solving this system. In Sec. 8 we describe a procedure for evaluating the scattered field. In Sec. 9 we discuss the rate convergence of the procedure. Finally, Sec. 10 contains a numerical example, illustrating the algorithm. Appendix A contains a study of the functions G_m derived in Sec. 7 as well as of their derivatives. The results of this appendix illustrate the type of singular behavior of the functions G_m and thus they dictate the type of approximate methods to be used in order to achieve high accuracy, and they simplify our proof of convergence.

The rate of convergence of the method of this paper, namely $O(e^{-cN^{\frac{1}{2}}})$ using an approximation of the form

$$(1.11) \qquad \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{mn} \theta_n(z) e^{im\varphi} \quad (N=M^2)$$

for each component of the surface current is optimal, in a certain sense. By the results of [25], given any approximation method of the type (1.11) which is to converge for all f analytic on (-1.1) and satisfying (1.2), the resulting error of this method cannot converge to zero faster than $e^{-\gamma N^{\frac{1}{2}}}$, for some $\gamma > 0$.

Similar numerical methods have been considered but it is believed that the order of convergence obtained is not as good as that demonstrated here. For further discussion of such methods see for example Andreasen [2] whose proposed method considers a maximum period of 20

wave lengths.   Barber and Yeh [3] and Waterman [27] have considered

extended boundary methods, while Kennaugh [13] and Schultz, et. al.

[19] have discussed other implementations using a product z, ϕ basis.

2. GEOMETRY OF THE SURFACE

A point on the surface S is represented by

$$(2.1) \qquad \bar{r} = f(z) \cos\varphi \, \hat{x} + f(z) \sin\varphi \, \hat{y} + z\hat{z},$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ denote the unit vectors in the direction of the x, y and z axes respectively.

It is convenient to introduce three unit vectors on the surface, $\hat{n}$, $\hat{\varphi}$ and $\hat{t}$ where:

$$
\begin{aligned}
\hat{n} &= \alpha(z) \cos\varphi\hat{x} + \alpha(z) \sin\varphi \, \hat{y} - f'(z)\alpha(z)\hat{z} \\
(2.2) \qquad \hat{\varphi} &= -\sin\varphi \, \hat{x} + \cos\varphi \, \hat{y} \\
\hat{t} &= f'(z)\alpha(z) \cos\varphi \, \hat{x} + f'(z)\alpha(z) \sin\varphi \, \hat{y} + \alpha(z)\hat{z},
\end{aligned}
$$

where;

$$(2.3) \qquad \alpha(z) = [1 + f'(z)^2]^{-\frac{1}{2}}.$$

The vector $\hat{n}$ is the unit normal to the surface, $\hat{\varphi}$ is the unit vector at $\bar{r}$, pointing in the direction of increasing φ, and $\hat{t}$ is the unit longitudinal vector, pointing in the direction of increasing arc length. Thus $\hat{n}$, $\hat{\varphi}$, $\hat{t}$ and $\hat{x}$, $\hat{y}$, $\hat{z}$ are related by means of the equations

$$(2.4) \quad \begin{pmatrix} \hat{n} \\ \hat{\varphi} \\ \hat{t} \end{pmatrix} = \begin{pmatrix} \alpha(z)\cos\varphi & \alpha(z)\sin\varphi & -f'(z)\alpha(z) \\ -\sin\varphi & \cos\varphi & 0 \\ f'(z)\alpha(z) & \cos\varphi\, f'(z)\alpha(z) & \sin\varphi\, \alpha(z) \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

and

$$(2.5) \quad \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} \alpha(z)\cos\varphi & -\sin\varphi & f'(z)\alpha(z)\cos\varphi \\ \alpha(z)\sin\varphi & \cos\varphi & f'(z)\alpha(z)\sin\varphi \\ -f'(z)\alpha(z) & 0 & \alpha(z) \end{pmatrix} \begin{pmatrix} \hat{n} \\ \hat{\varphi} \\ \hat{t} \end{pmatrix}$$

3. THE INCIDENT ELECTRIC FIELD

Let the incident radiation be a plane wave, given by

$$(3.1) \qquad E^o(\bar{r}) = \bar{\varepsilon}\, e^{i\bar{k}_o \cdot \bar{r}},$$

where $\bar{\varepsilon}$ points in the direction of polarization, and $\bar{k}_o$ is the propogation vector which satisfies the relations

$$k_o \equiv |\bar{k}_o| = \frac{\omega}{c} = \frac{2\pi}{\lambda}$$

$$(3.2)$$

$$\bar{k}_o \cdot \bar{\varepsilon} = 0$$

We shall furthermore assume that $\bar{k}_o$ lies in the xz plane and makes an angle θ_o with the z-axis. Thus

$$(3.3) \qquad \bar{k}_o = k_o \sin\theta_o\, \hat{x} + k_o \cos\theta_o\, \hat{z}.$$

It is convenient to set

$$(3.4) \qquad \bar{\varepsilon} = a_1 \bar{\varepsilon}_1 + a_2 \bar{\varepsilon}_2$$

where a_1 and a_2 are scalars, while

$$(3.5) \qquad \bar{\varepsilon}_1 = \hat{y}, \quad \bar{\varepsilon}_2 = -\hat{x}\cos\theta_o + \hat{z}\sin\theta_o.$$

Borison [4] has shown that if $a_2 = 0$ (rsp. $a_1 = 0$) then the back scattered electric field $E^s(\overline{r})$ is polarized only in the direction $\overline{\epsilon}_1$ (rsp. $\overline{\epsilon}_2$).

Using (2.5) and (3.5) we can express $\overline{\epsilon}$ in components of $\hat{\varphi}$, $\hat{t}$ and $\hat{n}$. We get

$$\overline{\epsilon} = a_1[\alpha(z) \sin\varphi \; \hat{n} + \cos\varphi \cdot \hat{\varphi} + \alpha(z) \; f'(z) \sin\varphi \; \hat{t}]$$

$$(3.6) \quad + a_2[-\alpha(z)(\cos\theta_0 \cos\varphi + \sin\theta_0 f'(z)) \; \hat{n} + \cos\theta_0 \sin\varphi \; \hat{\varphi}$$

$$-\alpha(z)(\cos\theta_0 \cos f'(z) - \sin\theta_0) \; \hat{t}].$$

Next, let us find the Fourier expansion of $E^o(\overline{r})$ on S. To this end we use the identity

$$(3.7) \qquad\qquad e^{ix\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m J_m(x) e^{im\varphi}$$

where $J_m(x)$ denotes the Bessel function, $J_m(x) = \sum_{n=0}^{\infty} \dfrac{(-1)^n (x^2/4)^{m+n}}{n! \; (n+m)!}$. Using (2.1) and (3.3) we get

$$(3.8) \qquad \overline{k}_o \cdot \overline{r} = k_o \; f(z) \sin\theta_0 \cos\varphi + k_o \; z \cos\theta_0.$$

Hence combining (3.1), (3.7) and (3.8) we find that the incident field on S is given by

$$(3.9) \qquad \overline{E}^o(\overline{r}) = \overline{\epsilon} \; e^{ik_o z \cos\theta_0} \sum_{m=-\infty}^{\infty} i^n J_m(k_o f(z) \sin\theta_0) e^{im\varphi}.$$

where $\overline{\epsilon}$ is given in (3.6). Combining (3.6) and (3.9) we get

$$\overline{E}^o(\overline{r}) = \{[\, a_1\alpha(z)\, \sin\varphi - a_2\alpha(z)\, (\cos\theta_o\, \cos\varphi + f'(z)\, \sin\theta_o)\,]\,\hat{n}$$

$$+ [\, a_1\cos\varphi + a_2\cos\theta_o\, \sin\varphi\,]\,\hat{\varphi}$$

$$(3.10)$$

$$+ [\, a_1\alpha(z)f'(z) - a_2\alpha(z)\, (\cos\theta_o\, \cos\varphi f'(z) - \sin\theta_o)\,]\,\hat{t}\}$$

$$\cdot e^{ik_o z \cos\theta_o} \sum_{m=-\infty}^{\infty} i^m J_m(k_o f(z)\, \sin\theta_o) e^{im\varphi}$$

4. THE SCATTERED FIELD AND THE INTEGRAL
EQUATION FOR THE SURFACE CURRENT

The scattered field $\overline{E}^s = \overline{E}^s(\overline{r})$ is given in terms of the total electric $(\overline{E})$ and magnetic $(\overline{H})$ fields on S by [26]

$$(4.1) \qquad \overline{E}^s = - \int_S [\, i\omega\mu(\hat{n}\times\overline{H})G + (\hat{n}\times\overline{E})\times\nabla G + (\hat{n}\cdot\overline{E})\nabla G \,]\, dS$$

where it is assumed that the field vectors $\overline{E}$ and $\overline{H}$ have time dependence of the factored form $e^{-i\omega t}$, and

$$(4.2) \qquad G = G(\overline{r},\overline{r}') = \frac{1}{4\pi} \frac{\exp[\,ik_0|\overline{r}-\overline{r}'|\,]}{|\overline{r}-\overline{r}'|} \quad .$$

The remaining vectors in the integrand (4.1) are expressed in terms of $\overline{r}$, and ∇ is expressed in terms of $\overline{r}$.

Let $\overline{K}$ denote the surface current on S, due to the fields $\overline{E}$ and $\overline{H}$. In view of the Leontovich boundary condition (1.4), $\overline{K}$ satisfies the relations

$$(4.3) \qquad \overline{K} = -\hat{n}\times\overline{H} \; ; \; \hat{n}\times\overline{E} = -\eta Z \, \hat{n}\times\overline{K}$$
$$\sigma = -\varepsilon\hat{n}\cdot\overline{E} = \frac{1}{i\omega} \nabla\cdot\overline{K}$$

Our development of the integral equation for $\overline{K}$ follows that in [5]. We include the derivation in [5] for this equation, for sake of completeness.

Let $\overline{A}$ be a continuous vector field tangent to S. Then the following results are valid, if S satisfies the Lyapunov conditions [17, p. 90] (a) the integral

$$I(\overline{r}') \equiv \int_S \overline{A}(\overline{r}) \; G(\overline{r},\overline{r}') \; dS$$

is a continuous function of $\overline{r}'$ in R^3.

(b) As $\overline{r}' \to r_o' \in S$, the relations

$$n(\overline{r}_o) \times \lim_{\overline{r}' \to \overline{r}_o'} \int_S \overline{A}(\overline{r}) \times \nabla G \; (\overline{r},\overline{r}')dS$$

$$= \pm\tfrac{1}{2} A(\overline{r}_o') + \int_S \hat{n}(\overline{r}_o) \times [\overline{A}(\overline{r}) \times \nabla G \; (\overline{r}_o,\overline{r}) \;] \, dS$$

are satisfied where the plus (resp. minus) sign corresponds to an approach from the outside (resp. inside) of B.

(c) The term [17, p. 95]

$$\hat{n}(\overline{r}') \cdot \int_S \overline{A}(\overline{r}) \times \nabla G \; (\overline{r},\overline{r}')dS$$

is a continuous function of $\overline{r}'$ on S. The term

$$\overline{E}_3(\overline{r}') \equiv \int_S \hat{n}(\overline{r}) \cdot \overline{E}(\overline{r}) \; \nabla G \; (\overline{r},\overline{r}')dS$$

suffers a discontinuity on transition through S equal to $\hat{n} \cdot \Delta\overline{E}_3$, where $\Delta\overline{E}_3$ is the difference of the values outside and inside. The third term in $\overline{E}^s$, therefore does not affect the tangential component, but reduces the normal component of $\overline{E}$ to zero.

Since the total electric field $\overline{E} = \overline{E}^o + \overline{E}^s$ is zero inside the

scatterer, (4.1) yields

$$0 = \hat{n} \times \overline{E}^0 - \lim_{\overline{r}' \to \overline{r}_0' \varepsilon S} \hat{n}(\overline{r}_0') \times \int_S [i\omega\mu(\hat{n}\times\overline{H})G$$

$$+ (\hat{n}\times E) \times \nabla G + (\hat{n}\cdot\overline{E}) \nabla G] dS$$

where the approach is from the inside.

Application of (b) gives

$$(4.4) \quad \begin{aligned} 0 &= \hat{n}(\overline{r}') \times \overline{E}^0 (\overline{r}') + \tfrac{1}{2} \hat{n} (\overline{r}')\times\overline{E}(\overline{r}') \\ &- \hat{n}(\overline{r}') \times \int_S [i\omega\mu(\hat{n}\times\overline{H})G + (\hat{n}\times\overline{E}) \times \nabla G + (\hat{n}\cdot\overline{E})\nabla G] dS \end{aligned}$$

Next, taking the limit as $\overline{r}\to\overline{r}_0'$ from the outside of B in (4.1) and using the relation

$$\hat{n}(\overline{r}_0') \times E(\overline{r}_0') = \hat{n}(\overline{r}_0') \times \{\overline{E}^0(\overline{r}_0') + E^S (\overline{r}')_0\}$$

we get

$$\hat{n}\times\overline{E} = \hat{n}\times\overline{E}^0 - \lim_{\overline{r}' \to \overline{r}_0'} \hat{n}(\overline{r}_0') \times \int_S [i\omega\mu(\hat{n}\times\overline{H})G]$$

$$+ (\hat{n}\times\overline{E}) \times \nabla G + (\hat{n}\cdot\overline{E}) \nabla G] dS$$

Application of (b) yields

$$(4.5) \quad \tfrac{3}{2} \hat{n}(\overline{r}') \times \overline{E}(\overline{r}') = \hat{n}(\overline{r}') \times \overline{E}^0 (\overline{r}) - \hat{n} (\overline{r}') \times \int_S [i\omega\mu(\hat{n}\times\overline{H})G +$$

$$+ (\hat{n}\times\overline{E}) \times \nabla G + (\hat{n}\cdot\overline{E})\nabla G] dS$$

Adding (4.4) and (4.5), we get

$$\hat{n}(\overline{r'}) \times \overline{E}(\overline{r'}) + 2\hat{n}(\overline{r'}) \times \int_S [\ i\omega\mu(\hat{n}\times\overline{H})G +$$

$$+ (\widehat{n\times\overline{E}}) \times \nabla G + (\hat{n}\cdot\overline{E}) \nabla G\ ] dS = 2\hat{n}(\overline{r'}) \times E^o(\overline{r})$$

The definitions (4.3) now yield

$$- \eta Z\hat{n}(\overline{r'}) \times \overline{K}(\overline{r'}) + 2\hat{n}(\overline{r'}) \times \int_S \{\ -i\omega\mu\overline{K}G - \eta Z(\hat{n} \times \overline{K}) \times \nabla G$$

$$- \frac{1}{i\omega\mu} (\nabla\cdot\overline{K})\nabla G\ \} dS = 2\hat{n}(\overline{r'}) \times \overline{E}^o(\overline{r'})$$

This equation can be written in equivalent form

(4.6)
$$[\ \tfrac{1}{2}\eta\ Z\overline{K}(\overline{r})' + \int_S \ i\omega\mu\overline{K}G + Z(\hat{n} \times \overline{K}) \times \nabla G$$

$$+ \frac{1}{i\omega\epsilon} (\nabla\cdot\overline{K})\nabla G\ ] dS + \overline{E}^o(\overline{r})'\ ] = 0$$
$$\begin{array}{c} \text{tan} \\ \overline{r}'\in S \end{array}$$

Using Eqs. (4.3), the scattered field $\overline{E}^s$ given by (4.1) is expressed in terms of $\overline{K}$ by means of the integral

(4.7)
$$\overline{E}^s = \int_S [\ i\omega\mu\overline{K}G + \eta Z(\hat{n} \times \overline{K}) \times \nabla G + \frac{1}{i\omega\epsilon} (\nabla\cdot\overline{K})\nabla G\ ] dS$$

5. THE BASIS FUNCTIONS FOR APPROXIMATING SURFACE CURRENT AND ELECTRIC FIELD

Let $d>0$ and $d'>1$, and let Ω_d and $A_{d'}$ be defined by

$$\Omega_d = \left\{ \, z \in \mathbb{C} \, : \, \left| \arg\left[\frac{(1+z)}{(1-z)} \right] \right| \, < d \, \right\}$$

(5.1)

$$A_{d'} = \left\{ \, w \in \mathbb{C} \, : \, 1/d' < |w| < d' \, \right\}$$
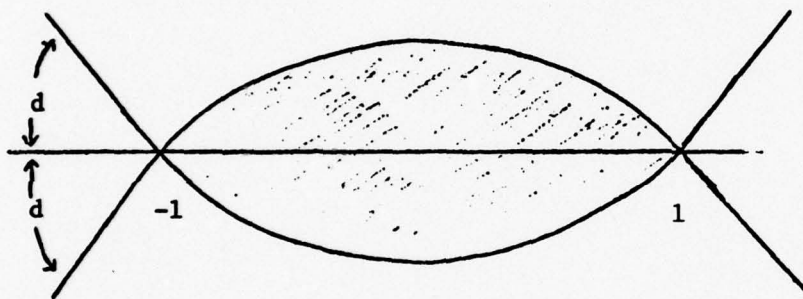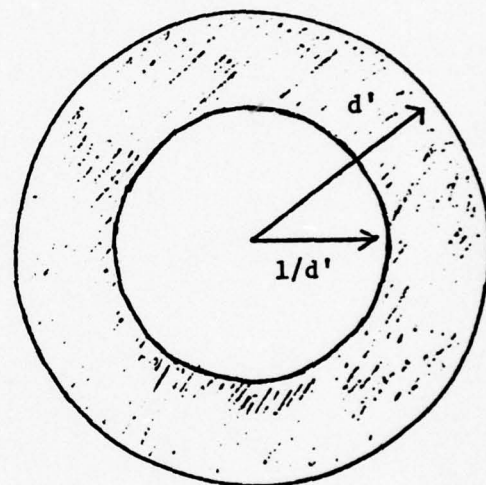
Figure 5.1.   The Region Ω_d

Figure 5.2.   The Region $A_{d'}$

Let $H(\Omega_d)$ (rsp. $H(A_{d'})$) denote the family of all functions g that are analytic in Ω_d (rsp. $A_{d'}$) such that

$$(5.2) \quad \|g\|_{\Omega_d} \equiv \sup_{z \in \Omega_d} |g(z)| < \infty \quad (rsp. \|g\|_{A_{d'}} = \sup_{z \in A_{d'}} |g(z)| < \infty).$$

Let us set

(5.3)
$$\nu(z) = f(z)(1-z^2)$$

where $f \in H(\Omega_d)$ satisfies (1.2), $f \neq 0$ in Ω_d. Let $H = H(d,d')$ denote the family of all functions $F = F(z,w)$ such that

i) $F(z,e^{i\varphi})/\nu(z)$ belongs to $H(\Omega_d)$ as a function of z for all $\varphi \in [0,2\pi]$;

ii) $F(z,w)$ belongs to $H(A_{d'})$ as a function of w for all $z \in [-1,1]$.

We define a norm on $H(d,d')$ by

(5.4)
$$\|F\|_H = \max\{\max_{\varphi \in [0,2\pi]} \|F\|_{\Omega_d}, \max_{z \in [-1,1]} \|F\|_{A_{d'}}\}$$

If $F \in H(d,d')$, we approximate F on $S = [-1,1] \times [0,2\pi]$ as follows

(5.5)
$$F(z,e^{i\varphi}) \cong \ell_{MN}(z,\varphi) \equiv \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_{mn} \psi_{mn}(z,\varphi)$$

where the a_{mn} are numbers and the ψ_{mn} are basis functions given by

$$\psi_{mn}(z,\varphi) \equiv e^{im\varphi} \theta_n(z)$$

$$\theta_n(z) \equiv f^{\frac{1}{2}}(z)(1-z^2) \operatorname{sinc}[\frac{\omega(z)-nh}{h}]$$

(5.6)
$$\omega(z) \equiv \log(\frac{1+z}{1-z}); \quad \operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x};$$
$$h > 0.$$

The numbers a_{mn} are given by

$$(5.7) \qquad a_{mn} = \frac{1}{2\pi f^{\frac{1}{2}}(z_n)(1-z_n^2)} \int_0^{2\pi} F(z_n, e^{i\varphi})e^{-im\varphi} \, d\varphi$$

$$z_n = \tanh \frac{(nh)}{2}$$

Next, recalling the definition of f and the relation (1.2), let us set

$$(5.8) \qquad \gamma_2 = \frac{1}{2}\min(\alpha_2, \beta_2) \, , \quad \gamma = \min[(\pi d\gamma_2)^{\frac{1}{2}}, \log d']$$

Theorem 5.1: Let $F \in H$, let $N = M^2$, and let $h = [\pi d/(\gamma_2 N)]^{\frac{1}{2}}$. Then there exist constants C, C_1 and C_2 which are independent of N, such that

$$(5.9) \qquad \max_{(z,\varphi)\in S} |F(z, e^{i\varphi}) - \ell_{MN}(z,\varphi)| \le CN^{\frac{1}{2}}e^{-\gamma N^{\frac{1}{2}}}$$

$$(5.10) \qquad \max_{(z,\varphi)\in S} |\frac{\partial}{\partial z} F(z,\varphi) - \frac{\partial}{\partial z} \ell_{MN}(z,\varphi)| \le C_1 N e^{-\gamma N^{\frac{1}{2}}}$$

$$(5.11) \qquad \max_{(z,\varphi)\in S} |\frac{\partial}{\partial\varphi} F(z,\varphi) - \frac{\partial}{\partial\varphi} \ell_{MN}(z,\varphi)| \le C_2 N^{\frac{1}{2}}e^{-\gamma N^{\frac{1}{2}}}$$

Proof: It is shown in [16] that if $g/\nu \in H(\Omega_d)$, then by taking $h = [\pi d/(\gamma_2 N)]^{\frac{1}{2}}$ there are positive constants C' and C'' such that

$$(5.12) \qquad \max_{z\in[-1,1]} |g(z) - \sum_{n=-N}^{N} \frac{f^{\frac{1}{2}}(z_n)}{\nu(z_n)} \theta_n(z)| \le C'e^{-\gamma N^{\frac{1}{2}}}$$

and

$$(5.13) \qquad \max_{z\in[-1,1]} |g'(z) - \sum_{n=-N}^{N} \frac{f'(z_n)}{\nu(z_n)} \theta_n'(z)| \le C'' N^{\frac{1}{2}}e^{-\gamma N^{\frac{1}{2}}}$$

Similarly, it follows from Cauchy's theorem, that if $G \in H(A_{d'})$, and if a_m is defined by

$$(5.14) \qquad a_m = \frac{1}{2\pi} \int_0^{2\pi} G(e^{i\varphi}) \, e^{-im\varphi} \, d\varphi, \quad m = -M, -M+1, .., M,$$

then there are constants K' and K'' such that for all $\varphi \in [0, 2\pi]$,

$$(5.15) \qquad \left| G(e^{i\varphi}) - \sum_{m=-M}^{M} a_m e^{im\varphi} \right| \leq K' \, (d')^{-M}$$

and

$$(5.16) \qquad \left| \frac{\partial}{\partial \varphi} G(e^{i\varphi}) - \sum_{m=-M}^{M} im \, a_m e^{im\varphi} \right| \leq K'' \, (d')^{-M} \; .$$

On noting that $M = N^{\frac{1}{2}}$, the inequality (5.9) is obtained if we use (5.11) and (5.15) in thm 6.2 in [24]. Similarly, (5.10) is a consequence of (5.13) and (5.15), while (5.11) follows from (5.12) and (5.16).

Theorem 5.2 [16]. If $\nu g / f^{\frac{1}{2}} \in H(\Omega)_d$, then there exists a constant K such that

$$(5.17) \qquad \left| \int_{-1}^{1} g(z) \, \theta_n(z) \, dz - h \frac{\nu(z_n) g(z_n)}{f^{\frac{1}{2}}(z_n) \omega'(z_n)} \right| \leq K \, e^{-\pi d/h}$$

This theorem shows that if g is any function such that $\nu g / f^{\frac{1}{2}} \in H(\Omega_d)$, then for h sufficienty small, the sequence $\{\theta_n\}_{-\infty}^{\infty}$ may be considered to be an orthogonal sequence, for practical purposes, and g may be expanded with respect to this sequence. The coefficients of this expansion take on the very simple form

$$(5.18) \qquad \frac{f^{\frac{1}{2}}(z_n)}{\nu(z_n)} g(z_n) \ , \ z_n = \tanh(nh/2).$$

For purposes of numerical integration, we state the following.

<u>Theorem 5.3</u> [23] : Let $g \in H(\Omega_d)$, and let $|g(z)| \leq C(1-z^2)^{\alpha-1}$ on $(-1,1)$, where $C, \alpha > 0$. If $N > 0$ is an integer, and $h = (2\pi d/\alpha N)^{\frac{1}{2}}$, then there exists a constant C', independent of N, such that

$$(5.19) \qquad |\int_{-1}^{1} g(z)dz - h \sum_{n=-N}^{N} \frac{g(z_n)}{\omega'(z_n)} | \leq C' e^{-(2\pi d\alpha N)^{\frac{1}{2}}}$$

Finally, thm. 5.4 which follows describes the accuracy of the midordinate rule used in Secs. 7 and 8 for integrating periodic functions.

<u>Theorem 5.4</u>: Let $d' > 0$ and let $g \in H(A_{d'})$ be bounded on $A_{d'}$. Then there exists a constant C depending only on g such that for $M = 1,2,3,\ldots,$

$$(5.20) \qquad |\int_{0}^{2\pi} g(e^{i\varphi}) \, d\varphi - \frac{2\pi}{M} \sum_{k=1}^{M} g(e^{(2k-1)\pi i/M}) | \leq C(d')^{-M}.$$

6. APPROXIMATION OF THE INCIDENT ELECTRIC FIELD

We shall make the approximation

$$(6.1) \qquad \overline{E}^o(\overline{r}) \cong \overline{E}^o(\overline{r}) \equiv \sum_{m=-M}^{M} \sum_{n=-N}^{N} [\alpha'_{mn}\hat{\varphi} + \beta'_{mn}\hat{t}] \, \psi_{mn}, \overline{r} \in S,$$

of the incident electric field, where ψ_{mn} are defined in Sec. 5. For the sake of convenience we shall use the notations

$$z_n = \tanh(nh/2)$$

$$(6.2)$$

$$d_{mn} = \frac{i^m \exp\{ik_o z_n \cos\theta_o\} J_m(k_o f(z_n)\sin\theta_o)}{\nu(z_n)}.$$

As will be seen in Sec. 7, we will approximate a slightly altered field $\overline{J}^o \equiv \nu\overline{E}^o$ where $\nu(z) = f(z)(1-z^2)$. Thus

$$\overline{J}^o(\overline{r}) \cong \overline{J}^o \equiv \sum_{m=-M}^{M} \sum_{n=-N}^{N} [\alpha_{mn}\hat{\varphi} + \beta_{mn}\hat{t}] \, \psi_{mn}.$$

The results of Secs. 3 and 5 show that if $\overline{r} \in S$ then each component of $\overline{J}^o$ is in $H(d,d')$. Using the formulas (5.6), we have

$$\alpha_{mn} = \frac{f^{1/2}(z_n)}{2\pi} \int_0^{2\pi} \overline{E}^o(\overline{r}) \cdot \hat{\varphi} \, e^{-im\varphi} \, d\varphi \, \Big|_{z=z_n}$$

$$(6.3)$$

$$\beta_{mn} = \frac{f^{1/2}(z_n)}{2\pi} \int_0^{2\pi} \overline{E}^o(\overline{r}) \cdot \hat{t} \, e^{-im\varphi} \, d\varphi \, \Big|_{z=z_n}$$

If this is taken together with (3.10), we get

$$\alpha_{mn} = f^{\frac{1}{2}}(z_n)[\tfrac{1}{2}a_1(d_{m+1,n} + d_{m-1,n}) + \frac{\cos\theta_o}{2i}a_2(d_{m-1,n} - d_{m+1,n})]$$

(6.4)

$$\beta_{mn} = f^{\frac{1}{2}}(z_n)\{[a_1 f'(z_n) + a_2\sin\theta_o]\,\alpha(z_n)d_{mn} -$$

$$- \tfrac{1}{2}a_2\cos\theta_o\,\alpha(z_n)\,f'(z_n)\,[d_{m+1,n} + d_{m-1,n}]\}.$$

7.(a) THE GALERKIN EQUATION FOR THE SURFACE CURRENT
DERIVATION OF THE EQUATIONS

Rather than solve (4.11) for $\overline{K}$, it is numerically more convenient to define $\overline{J}$ by

$$(7.1) \qquad \overline{J}(\overline{r}) = \nu(z)\ \overline{K}(\overline{r})$$

where

$$(7.2) \qquad \nu(z) = f\ (z)\ (1-z^2),$$

and to solve the resulting integral equation for $\overline{J}$. This transformation helps to take account of the unknown* singular behavior of $\overline{K}$ at $z = \pm 1$, and enables us to effectively approximate both $\overline{J}$ and its first derivative with respect to z, by methods of Sec. 5.

The substitution (7.1) replaces (4.6) by the equation

$$(7.2) \qquad \{\tfrac{1}{2}\eta\, z\ \overline{J}\ + \nu \int_S [\frac{i\omega\mu}{\nu}\ \overline{J}\ G + \frac{\eta z}{\nu}\ (\hat{n}\times\overline{J})\ \times\nabla G$$
$$+ \frac{1}{i\omega\varepsilon}\ (\nabla\cdot\tfrac{1}{\nu}\overline{J})\ \nabla G\ ]\ dS + \overline{J}^o\ \}_{tan} = 0$$

where

$$(7.3) \qquad \overline{J}^o = \nu\overline{E}^o.$$

---

*IF S satisfies Liapunov conditions (see Sec. 4), then $\overline{K}$ is bounded on S. It is difficult to determine the exact singular behavior of the derivatives of $\overline{K}$ at $z = \pm 1$.

We now make the Galerkin approximation

$$(7.4) \qquad \bar{J} \cong J = \sum_{m=-M}^{M} \sum_{n=-N}^{N} [a_{mn} \hat{\varphi} + b_{mn} \hat{t}] \psi_{mn}$$

in (7.2), where the ψ_{mn} are defined as in (5.16), and the a_{mn} and b_{mn} are unknown numbers.

Let us now recall that if $\bar{r}, \bar{r}' \in S$, then

$$R = |\bar{r} - \bar{r}'| = \{f^2 + f*^2 - 2ff* \cos (\varphi - \varphi') + (z - z')^2\}^{\frac{1}{2}}$$

$$(7.5) \qquad G = \frac{1}{4\pi} \frac{e^{ik_o R}}{R}$$

where here and henceforth

$$(7.6) \qquad f \equiv f(z) \; , \; f* \equiv f(z') \; , \; \alpha = \alpha(z), \; \alpha* = \alpha(z').$$

We shall also use the notations

$$(7.7) \qquad \begin{cases} R^f = \dfrac{\partial R}{\partial f} = \dfrac{f - f* \cos(\varphi - \varphi')}{R} \\[2mm] R^z = \dfrac{\partial R}{\partial z} = \dfrac{z - z'}{R} \\[2mm] R^\varphi = \dfrac{ff* \sin (\varphi - \varphi')}{R} \\[2mm] (G^f, G^z, G^\varphi) = (R^f, R^z, R^\varphi) \dfrac{d}{dR} G. \end{cases}$$

Moreover, writing $\displaystyle\sum_{m,n}$ for $\displaystyle\sum_{m=-M}^{M} \sum_{n=-N}^{N}$, we have

$$\hat{n} \times \mathcal{J} = \sum_{m,n} \begin{vmatrix} \hat{\varphi} & \hat{t} & \hat{n} \\ 0 & 0 & 1 \\ a_{mn} & b_{mn} & 0 \end{vmatrix} \psi_{mn}$$

(7.8)

$$= \sum_{m,n} [-b_{mn}\hat{\varphi} + a_{mn}\hat{t}] \psi_{mn},$$

and also, that

$$\nabla G = \frac{1}{f} G^{\varphi} \hat{\varphi} + \alpha(f'G^f + G^z) \hat{t} + \alpha(G^f - f'G^z) \hat{n}$$

(7.9)

$$\nabla \cdot (\frac{1}{\nu} \mathcal{J}) = \frac{1}{f} \sum_{m,n} [\frac{a_{mn}}{\nu} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z} (\frac{f}{\nu} \psi_{mn})].$$

Using these results, we get

$$(\hat{n} \times \mathcal{J}) \times \nabla G = \sum_{m,n} \begin{vmatrix} \hat{\varphi} & \hat{t} & \hat{n} \\ -b_{mn} & a_{mn} & 0 \\ \frac{1}{f}G^{\varphi} & \alpha(f'G^f + G^z) & \alpha(G^f - f'G^z) \end{vmatrix} \psi_{mn}$$

(7.10)

$$= \sum_{m,n} [a_{mn} \alpha(G^f - f'G^z)\hat{\varphi} + b_{mn} \alpha(G^f - f'G^z) \hat{t}$$

$$- \{a_{mn} \cdot \frac{1}{f} G^{\varphi} + b_{mn} \alpha(f'G^f + G^z)\} \hat{n}] \psi_{mn},$$

as well as

$$\nabla \cdot (\frac{1}{\nu}\mathcal{J}) \nabla G$$

(7.11)

$$= \{\frac{1}{f} G^{\varphi} \hat{\varphi} + \alpha(f'G^f + G^z) \hat{t} + \alpha (G^f - f'G^z) \hat{n} \}.$$

$$\cdot \frac{1}{f} \sum_{m,n} [\frac{a_{mn}}{\nu} \frac{\partial \psi_{mn}}{\partial \varphi} + \alpha b_{mn} \frac{\partial}{\partial z} (\frac{f}{\nu} \psi_{mn})].$$

Substituting (7.4), (7.10), (7.11) as well as the approximation $\overline{J}^0$

(7.12)
$$\overline{J}^0 \cong J^0 \equiv \sum_{m,n} [\ \alpha_{mn}\ \hat{\varphi} + \beta_{mn}\ \hat{t}\ ]\ \psi_{mn}$$

into (7.2), and recalling that

(7.13)
$$dS = \frac{f}{\alpha}\ d\varphi\ dz,$$

we get

$$\{\ \tfrac{1}{2}\eta\ Z\ \sum_{m,n} [\ a_{mn}\hat{\varphi}* + b_{mn}\ \hat{t}*\ ]\ \psi^*_{mn}$$

$$+\ \nu*\ \int_1^1 \int_0^{2\pi} \{\frac{i\omega\mu f}{\alpha\nu}\ G\ \sum_{m,n} [\ a_{mn}\hat{\varphi} + b_{mn}\ \hat{t}\ ]\ \psi_{mn}$$

$$+\ \frac{\eta Z}{\nu}\ \sum_{m,n} [\ a_{mn}\ f(G^f - f'G^z)\hat{\varphi} + b_{mn}\ f(G^f - f'G^z)\ \hat{t}$$

(7.14)

$$-\ \{\ a_{mn}\ \tfrac{1}{\alpha}\ G^\varphi + b_{mn}\ f(f'G^f + G^z)\ \}\ \hat{n}\ ]\ \psi_{mn}$$

$$+\ \frac{1}{i\omega\epsilon}\ \{\ \tfrac{1}{\alpha f}\ G^\varphi\ \hat{\varphi} + (f'G^f + G^z)\ \hat{t} + (G^f - f'G^z)\ \hat{n}\ \}$$

$$\cdot\ \sum_{m,n} [\ \frac{a_{mn}}{\nu}\ \frac{\partial \psi_{mn}}{\partial\varphi} + \alpha\ b_{mn}\ \frac{\partial}{\partial z'}\ (\tfrac{f}{\nu}\ \psi_{mn})\ ]\ \}\ d\varphi\ dz$$

$$+\ \sum_{m,n} [\ \alpha_{mn}\ \hat{\varphi}* + \beta_{mn}\ \hat{t}*\ ]\ \psi^*_{mn}\ \}_{tan} = 0$$

where the starred variables denote functions of z' and φ'.

It is convenient to introduce several identities in order to reduce (7.14) to a system of linear algebraic equations. The equations (2.4) yield the identities

$$\hat{\varphi} \cdot \hat{\varphi}* = \cos (\varphi - \varphi')$$

$$\hat{t} \cdot \hat{\varphi}* = - \alpha(z) \, f'(z) \, \sin (\varphi' - \varphi)$$

$$\hat{n} \cdot \hat{\varphi}* = - \alpha(z) \, \sin (\varphi' - \varphi)$$

(7.15)

$$\hat{\varphi} \cdot \hat{t}* = \alpha(z') \, f'(z') \, \sin (\varphi' - \varphi)$$

$$\hat{t} \cdot \hat{t}* = \alpha(z') \, \alpha(z) \, [\, 1 + f'(z) \, f'(z') \, \cos (\varphi - \varphi') \,]$$

$$\hat{n} \cdot \hat{t}* = \alpha(z') \, \alpha(z) \, [\, f'(z') \, \cos (\varphi - \varphi') - f'(z) \,] \, .$$

These identities are useful for taking components of $\hat{\varphi}*$ and $\hat{t}*$ in (7.14).

The identities (7.16) - (7.23) which follow serve to achieve further simplicity by enabling us to symbolically eliminate the integrations with respect to φ .

Upon setting

(7.16)
$$G(\bar{r}, \bar{r}') = \frac{e^{ik_o |\bar{r} - \bar{r}'|}}{4\pi |\bar{r} - \bar{r}'|} \equiv \sum_{m=-\infty}^{\infty} G_m \, e^{im(\varphi' - \varphi)}$$

it follows that

(7.17)
$$e^{im\varphi'} \, G_m = \frac{1}{2\pi} \int_0^{2\pi} G(\bar{r}, \bar{r}') \, e^{im\varphi} \, d\varphi \, .$$

Notice furthermore, that $G_{-m} = G_m$. Therefore

$$(7.18) \quad \frac{1}{2\pi} \int_0^{2\pi} G(\overline{r},\overline{r}') \, e^{im\varphi} \cos(\varphi'-\varphi) \, d\varphi = \frac{e^{im\varphi'}}{2} [G_{m+1} + G_{m-1}]$$

and

$$(7.19) \quad \frac{1}{2\pi} \int_0^{2\pi} G(\overline{r},\overline{r}') \, e^{im\varphi} \sin(\varphi-\varphi') \, d\varphi = \frac{e^{im\varphi'}}{2i} [G_{m+1} - G_{m-1}].$$

By means of integration by parts, we furthermore find that

$$(7.20) \quad \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \frac{\partial G(\overline{r},\overline{r}')}{\partial \varphi} \, d\varphi = \frac{-im}{2\pi} \int_0^{2\pi} e^{im\varphi} G(\overline{r},\overline{r}') \, d\varphi$$

$$= - \, im \, e^{im\varphi'} \, G_m,$$

$$(7.21) \quad \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \cos(\varphi-\varphi') \frac{\partial G}{\partial \varphi} \, d\varphi$$

$$= - \frac{i \, e^{im\varphi'}}{2} [(m+1) \, G_{m+1} + (m-1) \, G_{m-1}],$$

$$(7.22) \quad \frac{1}{2\pi} \int_0^{2\pi} e^{im\varphi} \sin(\varphi-\varphi') \frac{\partial G(\overline{r},\overline{r}')}{\partial \varphi} \, d\varphi$$

$$= - \frac{e^{im\varphi'}}{2} [(m+1) \, G_{m+1} - (m-1) \, G_{m-1}].$$

We use (7.13) in (7.12) to take components of $\hat{\varphi}$ and $\hat{t}$. Then, recalling that

(7.23)
$$\psi_{mn}(z,\varphi) = e^{im\varphi} \theta_n(z)$$

where

(7.24)
$$\theta_n(z) = \frac{\nu(z)}{f(z)^{\frac{1}{2}}} \operatorname{sinc}\left[\frac{\omega(z)-nh}{h}\right]$$

and where $\nu(z)$ is given in (7.2), and

(7.25)
$$\omega(z) = \log\left(\frac{1+z}{1-z}\right),$$

we can symbolically carry out the integrations with respect to φ' in the equation resulting from (7.12), by using (7.16) - (7.21). Upon equating coefficients of $e^{im\varphi}$ in the resulting equations, we obtain

$$\tfrac{1}{2}\eta z \sum_n a_{mn} \theta_n + \nu \sum_{n=-N}^{N} [P^{mn} a_{mn} + Q^{mn} b_{mn}] = -\sum_{n=-N}^{N} \alpha_{mn} \theta_n$$

(7.26)
$$\tfrac{1}{2}\eta z \sum_n b_{mn} \theta_n + \nu \sum_{n=-N}^{N} [R^{mn} a_{mn} + S^{mn} b_{mn}] = -\sum_{n=-N}^{N} \beta_{mn} \theta_n$$

$$m = -M, -M+1, \ldots, M.$$

The coefficients P^{mn}, Q^{mn}, R^{mn} and S^{mn} depend on z', and are given by

$$P^{mn} = \pi \int_{-1}^{1} \frac{\theta_n}{\nu} \{ G_{m+1} [\frac{i\omega\mu f}{\alpha} + (m+1)\eta Z + \frac{m(m+1)}{i\omega\epsilon\alpha f}]$$

$$+ G_{m-1} [\frac{i\omega\mu f}{\alpha} - (m-1)\eta Z + \frac{m(m-1)}{i\omega\epsilon\alpha f}]$$

$$(7.27) \qquad + G_{m+1}^{f} [\eta Z f + \frac{m}{i\omega\epsilon\alpha}]$$

$$+ G_{m-1}^{f} [\eta Z f - \frac{m}{i\omega\epsilon\alpha}]$$

$$- (G_{m+1}^{z} + G_{m-1}^{z})\eta Z ff' \} dz$$

$$Q^{mn} = \pi \int_{-1}^{1} \left[\frac{\theta_n}{\nu} \{ (G_{m+1} - G_{m-1}) \omega\mu ff' \right.$$

$$- (G_{m+1}^{z} - G_{m-1}^{z}) \frac{\eta Z f}{i\alpha} \}$$

$$(7.28) \qquad + (\frac{f\theta_n}{\nu})' \{ - \frac{1}{\omega\epsilon f} [(m+1) G_{m+1} + (m-1) G_{m-1}]$$

$$\left. - \frac{1}{\omega\epsilon} (G_{m+1}^{f} - G_{m-1}^{f}) \} \right] dz$$

$$R^{mn} = \pi \int_{-1}^{1} \frac{\theta_n}{\nu} \left\{ G_{m+1} \, \alpha^* f^{*\prime} \left[\frac{-\omega\mu f}{\alpha} + i(m+1)\eta \, z + \frac{m(m+1)}{\omega\epsilon\alpha f} \right] \right.$$

$$+ G_{m-1} \, \alpha^* f^{*\prime} \left[\frac{\omega\mu f}{\alpha} + i(m-1)\eta \, z - \frac{m(m-1)}{\omega\epsilon\alpha f} \right]$$

$$+ G_m \, \alpha^* \left[\, - 2im f^\prime \, \eta \, z \, \right]$$

(7.29)
$$+ G_{m+1}^{f} \, \alpha^* f^{*\prime} \left[\frac{-\eta z f}{i} + \frac{m}{\omega\epsilon\alpha} \right]$$

$$+ G_{m-1}^{f} \, \alpha^* f^{*\prime} \left[\frac{\eta z f}{i} + \frac{m}{\omega\epsilon\alpha} \right]$$

$$+ \alpha^* f^{*\prime} \, \frac{\eta z f f^\prime}{i} \left[\, G_{m+1}^{z} - G_{m-1}^{z} \, \right]$$

$$\left. + \frac{2m\alpha^*}{\omega\epsilon\alpha} \, G_{m}^{z} \right\} \, dz$$

$$S^{mn} = \pi \int_{-1}^{1} \left[\left(\frac{\theta_n}{\nu}\right)\left\{ \alpha^* f^{*\prime} \, i\omega\mu \, f f^\prime \left[\, G_{m+1} + G_{m-1} \, \right] + 2\alpha^* \, i\omega\mu f \, G_m \right. \right.$$

$$\left. + 2\eta \, \frac{z f \alpha^*}{\alpha} \, G_{m}^{f} - \eta \frac{z f \alpha^* f^{*\prime}}{\alpha} \left[\, G_{m+1}^{z} + G_{m-1}^{z} \, \right] \right\}$$

(7.30)
$$+ \left(\frac{f\theta_n}{\nu}\right)^\prime \left\{ \frac{\alpha^* f^{*\prime}}{i\omega\epsilon f} \left[\, (m+1) \, G_{m+1} - (m-1) \, G_{m-1} \, \right] \right.$$

$$+ \frac{\alpha^* f^{*\prime}}{i\omega\epsilon} \left[\, G_{m+1}^{f} + G_{m-1}^{f} \, \right]$$

$$\left. \left. + \frac{2\alpha^*}{i\omega\epsilon} \, G_{m}^{z} \right\} \right] \, dz$$

In these equations $f^{*\prime} = f^\prime(z^\prime)$, $\alpha^* = \left[\, 1 + (f^{*\prime})^2 \, \right]^{-\frac{1}{2}}$.

Next, setting $z^\prime = z_\ell = \tanh(\ell h/2)$ in (7.26) and using the relations

(7.31)
$$\theta_n(z_\ell) = \begin{cases} 0 & \text{if } n \neq \ell \\ \dfrac{\nu(z_\ell)}{f^{\frac{1}{2}}(z_\ell)} & \text{if } n = \ell \end{cases}$$

we arrive at the system

$$\frac{1}{2}\eta Z a_{m\ell} + f^{\frac{1}{2}}(z_\ell) \sum_{n=-N}^{N} [P_\ell^{mn} a_{mn} + Q_\ell^{mn} b_{mn}] = - \alpha_{m\ell}$$

(7.32)

$$\frac{1}{2}\eta Z b_{m\ell} + f^{\frac{1}{2}}(z_\ell) \sum_{n=-N}^{N} [R_\ell^{mn} a_{mn} + S_\ell^{mn} b_{mn}] = - \beta_{m\ell}$$

where we have used the notation

$$P_\ell^{mn} = P^{mn}(z_\ell) \ , \ Q_\ell^{mn} = Q^{mn}(z_\ell)$$

(7.33)

$$R_\ell^{mn} = R^{mn}(z_\ell) \ , \ S_\ell^{mn} = S^{mn}(z_\ell)$$

The system (7.32) is a block diagonal system of the form

(7.34)
$$
\begin{bmatrix} B_{-M} & & & \\ & B_{-M+1} & & \\ & & \ddots & \\ & & & B_M \end{bmatrix}
\begin{bmatrix} \bar{a}_{-M} \\ \bar{a}_{-M+1} \\ \vdots \\ \bar{a}_M \end{bmatrix}
=
\begin{bmatrix} \bar{\alpha}_{-M} \\ \bar{\alpha}_{-M+1} \\ \vdots \\ \bar{\alpha}_M \end{bmatrix}
$$

where each B_m is a complex matrix of order $2(2N+1)$, and (since $G_{-m} = G_m$)
$B_m = B_{-m}$. The m'th system

(7.35)
$$B_m \bar{a}_m = \bar{\alpha}_m$$

in (7.33) corresponds to all of the equations (7.35), for fined m.
Thus if we denote by A_ℓ^{mn} the 2x2 matrix

$$(7.36) \qquad A_\ell^{mn} = f^{\frac{1}{2}}(z_\ell) \begin{bmatrix} P_\ell^{mn} & Q_\ell^{mn} \\ R_\ell^{mn} & S_\ell^{mn} \end{bmatrix} + \tfrac{1}{2}\eta \ Z \ \delta_{n\ell} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$(7.37) \qquad B_m = \begin{bmatrix} A_{-N}^{m,-N} & A_{-N}^{m,-N+1} & \cdots & A_{-N}^{mN} \\ A_{-N+1}^{m,-N} & A_{-N+1}^{m,-N+1} & \cdots & A_{-N+1}^{mN} \\ & \cdots & & \\ A_{N}^{m,-N} & A_{N}^{m,-N+1} & \cdots & A_{N}^{mN} \end{bmatrix}$$

and

$$(7.38) \qquad \bar{a}_m = \begin{bmatrix} a_{m,-N} \\ b_{m,-N} \\ a_{m,-N+1} \\ b_{m,-N+1} \\ \vdots \\ a_{mN} \\ b_{mN} \end{bmatrix} \qquad \bar{\alpha}_m = - \begin{bmatrix} \alpha_{m,-N} \\ \beta_{m,-N} \\ \alpha_{m,-N+1} \\ \beta_{m,-N+1} \\ \vdots \\ \alpha_{mN} \\ \beta_{mN} \end{bmatrix}$$

7.(b) EVALUATION OF P^{mn}, Q^{mn}, R^{mn} and S^{mn}

It is convenient to set

(7.39)
$$\theta_n/\nu = \gamma_n \ , \ \left(\frac{\theta_n f}{\nu}\right)' = \delta_n$$

The following integrals appear in (7.27) - (7.30).

$$I_{mn}^{(1)} = \int_{-1}^{1} \gamma_n \ (f/\alpha) \ G_m \ dz \qquad I_{mn}^{(4)} = \int_{-1}^{1} \gamma_n ff' \ G_m \ dz$$

$$I_{mn}^{(2)} = \int_{-1}^{1} \gamma_n \ G_m \ dz \qquad I_{mn}^{(5)} = \int_{-1}^{1} \gamma_n f' \ G_m \ dz$$

$$I_{mn}^{(3)} = \int_{-1}^{1} [\ \gamma_n/(\alpha f)\ ] \ G_m \ dz \ \ m \neq 0 \qquad I_{mn}^{(6)} = \int_{-1}^{1} \gamma_n f \ G_m \ dz$$

$$J_{mn}^{(1)} = \int_{-1}^{1} \gamma_n f \ G_m^f \ dz \qquad K_{mn}^{(1)} = \int_{-1}^{1} \gamma_n ff' \ G_m^z \ dz$$

$$(7.40) \quad J_{mn}^{(2)} = \int_{-1}^{1} \gamma_n \ (1/\alpha) \ G_m^f \ dz \ \ (m \neq 0) \qquad K_{mn}^{(2)} = \int_{-1}^{1} \gamma_n \ (1/\alpha) \ G_m^z \ dz$$

$$J_{mn}^{(3)} = \int_{-1}^{1} \gamma_n \ (f/\alpha) \ G_m^f \ dz \qquad K_{mn}^{(3)} = \int_{-1}^{1} \gamma_n \ (f/\alpha) \ G_m^z \ dz$$

$$L_{mn}^{(1)} = \int_{-1}^{1} \delta_n \ (1/f) \ G_m \ dz \ \ (m \neq 0)$$

$$M_{mn}^{(1)} = \int_{-1}^{1} \delta_n \ G_m^f \ dz$$

$$N_{mn}^{(1)} = \int_{-1}^{1} \delta_n \ G_m^{z'} \ dz$$

The notations (7.40) enable us to express (7.29) - (7.30) in the "more computable" form

$$P^{mn} = \pi \{ i\omega\mu (I^{(1)}_{m+1,n} + I^{(1)}_{m-1,n}) + \eta Z [(m+1) I^{(2)}_{m+1,n} - (m-1)I^{(2)}_{m-1,n}]$$

$$+ \frac{m}{i\omega\epsilon} [(m+1) I^{(3)}_{m+1,n} + (m-1) I^{(3)}_{m-1,n}]$$

(7.41)

$$+ \eta Z (J^{(1)}_{m+1,n} + J^{(1)}_{m-1,n} - K^{(1)}_{m+1,n} - K^{(1)}_{m-1,n})$$

$$+ \frac{m}{i\omega\epsilon} (J^{(2)}_{m+1,n} - J^{(2)}_{m-1,n}) \}$$

$$Q^{mn} = \pi \{ \omega\mu (I^{(4)}_{m+1,n} - I^{(4)}_{m-1,n}) - \frac{\eta Z}{i} (K^{(3)}_{m+1,n} - K^{(3)}_{m-1,n})$$

(7.42)

$$- \frac{1}{\omega\epsilon} [(m+1) L^{(1)}_{m+1,n} + (m-1) L^{(1)}_{m-1,n} + M^{(1)}_{m+1,n} - M^{(1)}_{m-1,n}]$$

$$R^{mn} = \pi \, \alpha^* f^{*'} \{ - \omega\mu (I^{(1)}_{m+1,n} - I^{(1)}_{m-1,n}) + i\eta Z [(m+1) I^{(2)}_{m+1,n} + (m-1) I^{(2)}_{m+1,n}]$$

$$+ \frac{m}{\omega\epsilon} (m+1) I^{(3)}_{m+1,n} - (m-1) I^{(3)}_{m-1,n}] - \frac{2i\eta Zm}{f^{*'}} I^{(5)}_{mn}$$

(7.43)

$$- \frac{\eta Z}{i} (J^{(1)}_{m+1,n} - J^{(1)}_{m-1,n} - K^{(1)}_{m+1,n} + K^{(1)}_{m-1,n})$$

$$+ \frac{m}{\omega\epsilon} (J^{(2)}_{m+1,n} + J^{(2)}_{m-1,n} + \frac{2}{f^{*'}} K^{(2)}_{mn}) \}$$

and

$$S^{mn} = \pi \, \alpha^* f^{*'} \{ i\omega\mu \, (I^{(4)}_{m+1,n} + I^{(4)}_{m-1,n} + \frac{2}{f^{*'}} I^{(6)}_{mn})$$

$$- Z \, (K^{(3)}_{m+1,n} + K^{(3)}_{m-1,n} - \frac{2}{f^{*'}} J^{(3)}_{mn})$$

(7.44)

$$= + \frac{1}{i\omega\varepsilon} [(m+1) \, L^{(1)}_{m+1,n} - (m-1) \, L^{(1)}_{m-1,n} +$$

$$+ M^{(1)}_{m+1,n} + M^{(1)}_{m-1,n} + \frac{2}{f^{*'}} N^{(1)}_{mn}] \}.$$

The integrals (7.40) can be simultaneously evaluated for all
m,n by evaluating the quantities f, f', $\alpha = (1+f'^2)^{-\frac{1}{2}}$, $f^{*'}$ and $\alpha^* = (1+f^{*'2})^{-\frac{1}{2}}$,
θ_n ($n = - N, - N+1, \ldots, N$) and G_m ($m = - M, -M+1, \ldots, N$). The integrals
$I^{(1)}_{mn}$ and $L^{(1)}_{mn}$ are evaluated simultaneously by means of the formulas

(7.45)

$$\int_{-1}^{1} H \, (z,z') \, dz = \int_{-1}^{z'} H \, (z,z') \, dz + \int_{z'}^{1} H \, (z,z') \, dz$$

$$\tilde{=} \sum_s u_s \, H \, (z',\mu_s) + \sum_s v_s \, H \, (z',\nu_t)$$

The definitions for the exact form of u_s and v_s follow in (7.48).
On the other hand the integrals J_{mn}, K_{mn}, M_{mn} and N_{mn} are evaluated by
means of the following formulas.

$\underline{J_{mn}, M_{mn}}$ Here we use the results (A.2).

$$\int_{-1}^{1} G^f_m \, (z,z') \, g \, (z) \, dz$$

(7.46)

$$= - \frac{1}{4\pi^2} \cdot \frac{1}{f^*} \, f^* \alpha^{*2} \, g(z') \, \log \, (\frac{1-z'}{1+z'}) +$$

$$+ \int_{-1}^{1} [G^f_m \, (z,z') \, g(z) + \frac{1}{4\pi^2} \cdot \frac{f^{*'}}{f^*} \frac{\alpha^{*2}}{z-z'} \, g(z')] \, dz$$

and the integral on the extreme right (7.46) is evaluated by method (7.45).

$\underline{K_{mn}, N_{mn}}$. Using (A.2),

$$
\begin{aligned}
\text{(7.47)} \quad \int_{-1}^{1} & G_m^z(z,z')\, g(z)\, dz = -\frac{1}{4\pi^2} \cdot \frac{\alpha*^2}{f*}\, g(z')\, \log\left(\frac{1-z'}{1+z'}\right) \\
& + \int_{-1}^{1} \left[\, G_m^z(z,z')\, g(z) + \frac{1}{4\pi^2}\, \frac{\alpha*^2}{f*} \cdot \frac{1}{z-z'}\, g(z')\, \right]\, dz,
\end{aligned}
$$

and the integral on the right hand side is again evaluated by means of (7.45).

We split the integration in (7.47) since $G_m(z,z')$ has a singularity at $z = z'$. The formula (5.18) is then used to evaluate each integral. Thus

$$
\begin{aligned}
u_s &= \frac{z'+1}{2}\, h*\, \mathrm{sech}^2\left(\frac{sh*}{2}\right), \quad \mu_s = \frac{z'+1}{2} + \frac{z'+1}{2}\, \tanh\left(\frac{sh*}{2}\right) \\
\text{(7.48)} \quad v_t &= \frac{1-z'}{2}\, h*\, \mathrm{sech}^2 \frac{sh*}{2}, \quad \nu_t = \frac{1+z'}{2} + \frac{1-z'}{2}\, \tanh\left(\frac{th*}{2}\right)
\end{aligned}
$$

for suitably chosen $h* > 0$.

8.   APPROXIMATION OF THE SCATTERED FIELD

The scattered field $\overline{E}^s$ is expressed in terms of $\overline{K}$ by means of the integral (4.7). Once $\overline{J}$ has been obtained by means of Secs. 7, 8 and 9, we form an approximation $\overline{K}$ of K by means of the equation (see Eq. (7.1))

$$(8.1) \qquad\qquad \overline{K} = \overline{K}(z,\varphi) = \frac{1}{\nu(z)}\, \overline{J}(z,\varphi),$$

and we substitute $\overline{K}$ for $\overline{K}$ in (4.12) to get an expression for an approximation $\overline{E}^s$ of $\overline{E}^s$. In this section we shall give a detailed description of the evaluation of $\overline{E}^s$.

We shall approximate $\overline{E}^s(\overline{r})$ for

$$(8.2) \qquad\qquad \overline{r}' = \rho\, \cos\varphi'\, \hat{x}' + \rho\, \sin\varphi'\, \hat{y}' + z'\hat{z}'$$

where $\rho > f(z')$. If ρ is close to $f(z)$ i.e., if $\overline{r}'$ is close to the surface, (say $|\overline{r}'-S| \leq .2$ if $N = 10$) then we recommend the integration methods of Sec. 9 to evaluate the integrals. This would involve splitting the integrals from -1 to 1 into integrals from -1 to z' and from z' to 1, as in (7.45). For sake of simplicity, we shall describe an algorithm for evaluating $\overline{E}^s$ which is valid if $\overline{r}'$ is not arbitrarily close to S (say $|\overline{r}'-S| > .2$ if $N \geq 10$). In this latter case it is convenient to integrate by parts in the integrals with respect to z which involve θ'_n, so that the resulting integrals involve θ_n. This

latter procedure enables us to avoid numerical integration, by means of the approximation

$$(8.3) \qquad \int_{-1}^{1} H(z)\, \omega_n(z)\, dz \;\tilde{=}\; h\, \frac{H(z_n)}{f^{\frac{1}{2}}(z_n)\omega'(z_n)}$$

which we know to be accurate, by Thm. 5.2, where $\omega_n = \theta_n/\nu$.
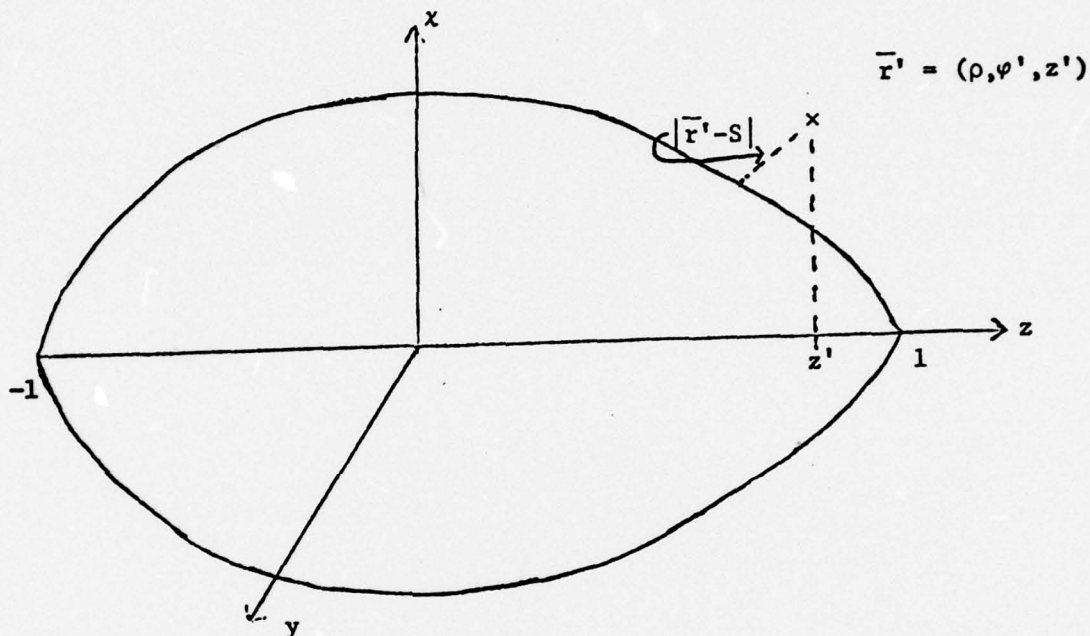
Using (4.7), the scattered field may be expressed in the form

$$(8.4) \qquad \overline{E}^s(\overline{r}) = E_x^s\,\hat{x} + E_y^s\,\hat{y} + E_z^s\,\hat{z}$$

where E_x^s, E_y^s and E_z^s are scalar quantities. Upon substituting the approximation (7.4) into (4.7) we get

$$(8.5) \qquad E_x^s \;\tilde{=}\; \sum_{m,n} [\, a_{mn}\, P^{mn} + b_{mn}\, Q^{mn}\,]$$

$$(8.6) \qquad E_y^s \cong \sum_{m,n} [\; a_{mn}\, R^{mn} + b_{mn}\, S^{mn} \;]$$

$$(8.7) \qquad E_z^s \cong \sum_{m,n} [\; a_{mn}\, T^{mn} + b_{mn}\, U^{mn} \;]$$

The relations (7.7) – (7.11) and (7.15) – (7.22) enable us to obtain explicit expression for $P^{mn}, \ldots, U^{mn}$. Setting

$$(8.8) \qquad \omega_n = \omega_n(z) = \frac{\theta_n(z)}{\nu(z)},$$

we get

$$\bar{E}^s = \int_{-1}^{1} \int_{0}^{2\pi} \{ \frac{i\omega\mu}{\alpha} fG \sum_{m,n} [a_{mn} (-\sin\varphi \ \hat{x} + \cos\varphi \ \hat{y})$$

$$+ b_{mn} (f' \alpha \cos\varphi \ \hat{x} + f' \alpha \sin\varphi \ \hat{y} + \alpha \hat{z})] e^{im\varphi} \omega_n$$

$$+ \eta Z \sum_{m,n} [a_{mn} f(G^f - f'G^z) (-\sin\varphi \ \hat{x} + \cos\varphi \ \hat{y})$$

$$+ b_{mn} f(G^f - f'G^z) (f' \alpha \cos\varphi \ \hat{x} + f' \alpha \sin\varphi \ \hat{y} + \alpha\hat{z})$$

$$- \{ a_{mn} \frac{1}{\alpha} G^\varphi (\alpha \cos\varphi \ \hat{x} + \alpha \sin\varphi \ \hat{y} - f'\alpha\hat{z})$$

(8.9)

$$+ b_{mn} f(f'G^f + G^z) (\alpha \cos\varphi \ \hat{x} + \alpha \sin\varphi \ \hat{y} - f'\alpha\hat{z})\}] e^{im\varphi}\omega_n$$

$$+ \frac{1}{i\omega\epsilon} \{ \frac{1}{\alpha f} G^\varphi (-\sin\varphi \ \hat{x} + \cos\varphi \ \hat{y})$$

$$+ (f'G^f + G^z) (f' \alpha \cos\varphi \ \hat{x} + f' \alpha \sin\varphi \ \hat{y} + \alpha \hat{z})$$

$$+ (G^f - f'G^z) (\alpha \cos\varphi \ \hat{x} + \alpha \sin\varphi \ \hat{y} - f'\alpha\hat{z}) \}$$

$$\cdot \sum_{m,n} [a_{mn} im \omega_n + \alpha b_{mn} (f\omega_n)'] e^{im\varphi} \} d\varphi \, dz$$

Here $G = \frac{e^{ik_o \bar{R}}}{4\pi R}$; $R = \{ (z-z')^2 + f^2 + \rho^2 - 2f'\rho\cos(\varphi-\varphi') \}^{\frac{1}{2}}$.

Collecting coefficients as indicated in (8.5) to (8.7), we get

$$P_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \{ \frac{-\omega\mu f}{\alpha} [e^{i\phi'} G_{m+1} - e^{-i\phi'} G_{m-1}] +$$

$$+ i\eta Z [f \{ (G^f_{m+1} - f'G^z_{m+1}) e^{i\phi'} - (G^f_{m-1} - f'G^z_{m-1}) e^{-i\phi'} \} +$$

(8.10)

$$+ (m+1) e^{i\phi'} G_{m+1} + (m-1) e^{-i\phi'} G_{m-1}] +$$

$$+ \frac{m}{\omega\epsilon\alpha} [\frac{1}{f} \{ (m+1) e^{i\phi} G_{m+1} - (m-1) e^{-i\phi} G_{m-1} \} +$$

$$+ \{ e^{i\phi} G^f_{m+1} + e^{-i\phi} G^f_{m-1} \}] \} \omega_n dz$$

$$Q_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \left\{ \left[i\omega\mu ff' \left[e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1} \right] + \right.\right.$$

$$- \frac{\eta Z f}{\alpha} \left[e^{i\phi'} G_{m+1}^{z} + e^{-i\phi'} G_{m-1}^{z} \right] \right] \omega_{n} +$$

$$\text{(8.11)}$$

$$+ \frac{1}{i\omega\epsilon} \left[\frac{1}{f} \left((m+1) e^{i\phi'} G_{m+1} - (m-1) e^{-i\phi'} G_{m-1} \right) + \right.$$

$$\left.\left. + e^{i\phi'} G_{m+1}^{f} + e^{-i\phi'} G_{m-1}^{f} \right] (f\omega_{n})' \right\} dz$$

$$R_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \left\{ \frac{i\omega\mu f}{\alpha} (e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1}) + \right.$$

$$+ \eta Z \left[f(e^{i\phi'} (G_{m+1}^{f} - f'G_{m+1}^{z}) + e^{-i\phi'} (G_{m-1}^{f} - f'G_{m-1}^{z})) + \right.$$

$$\text{(8.12)}$$

$$\left. + (m+1) e^{i\phi'} G_{m+1} - (m-1) e^{-i\phi'} G_{m-1} \right] +$$

$$- \frac{im}{\alpha\omega\epsilon} \left[\frac{1}{f} \left((m+1) e^{i\phi'} G_{m+1} + e^{-i\phi'} G_{m-1} \right) + \right.$$

$$\left.\left. + e^{i\phi'} G_{m+1}^{f} - e^{-i\phi'} G_{m-1}^{f} \right] \right\} \omega_{n} dz$$

$$S_{mn} = \pi e^{im\phi'} \int_{-1}^{1} \left\{ \left[\omega\mu ff' (e^{i\phi'} G_{m+1} - e^{-i\phi'} G_{m-1}) + \right.\right.$$

$$\frac{i\eta Z f}{\alpha} \left[e^{i\phi'} G_{m+1}^{z} - e^{-i\phi'} G_{m-1}^{z} \right] \right] \omega_{n} +$$

$$\text{(8.13)}$$

$$- \frac{1}{\omega\epsilon} \left[\frac{1}{f} \left((m+1) e^{i\phi'} G_{m+1} + (m-1) e^{-i\phi'} G_{m-1} \right) + \right.$$

$$\left.\left. + e^{i\phi'} G_{m+1}^{f} - e^{-i\phi'} G_{m-1}^{f} \right] (f\omega_{n})' \right\} dz$$

$$\text{(8.14)} \qquad T_{mn} = - 2\pi m e^{im\phi'} \int_{-1}^{1} \left\{ i\eta Z f' G_{m} - \frac{1}{\omega\epsilon\alpha} G_{m}^{z} \right\} \omega_{n} dz$$

$$(8.15) \quad U_{mn} = 2\pi \, e^{im\phi'} \int_{-1}^{1} \left\{ \left[i\omega\mu f \, G_m + \frac{\eta Z f}{\alpha} \, G_m^f \right] \omega_n + \frac{1}{i\omega\varepsilon} \, G_m^z \, (f\omega_n)' \right\} dz$$

Let us next eliminate the $(f\omega_n)'$ terms which appear in Q_{mn}, S_{mn} and U_{mn} above. Setting

$$(8.14) \quad J_{mn} = \int_{-1}^{1} \frac{1}{f} \left(\frac{1}{f} \, G_m \right) (f\omega_n)' \, dz$$

we have, upon integration by parts,

$$(8.15) \quad J_{mn} = \frac{1}{f} \, G_m \, f\omega_n \bigg|_{-1}^{1} - \int_{-1}^{1} \left(\frac{1}{f} \, G_m \right)' f\omega_n \, dz$$

Under the assumption made on S in Sec. 1, $\overline{K}$ is bounded on S, and therefore it follows, upon replacing ω_n by $\overline{K}$ in (8.15), that the first term on the right hand side of (8.15) vanishes, provided that $m \neq 0$ (see (A.1)). However, inspection of Q_{mn} and S_{mn} shows that we need never evaluate J_{mn} if $m = 0$. Thus

$$(8.16) \quad J_{mn} = \int_{-1}^{1} \frac{f'}{f} \, G_m \, \omega_n \, dz - \int_{-1}^{1} G_m' \, \omega_n \, dz, \quad m \geq 1,$$

where $G_m' = d \, G_m / dz$.

Similarly, we have, for all $m \geq 0$,

$$(8.17) \quad K_{mn} \equiv \int_{-1}^{1} G_m^f \, (f\omega_n)' \, dz = - \int_{1}^{1} (G_m^f)' \, f\omega_n \, dz$$

and

$$(8.18) \qquad L_{mn} \equiv \int_{-1}^{1} G_m^z (f\omega_n)' \, dz = - \int_{-1}^{1} (G_m^z)' \, f\omega_n \, dz,$$

these being the only remaining terms requiring integration by parts in (8.10) to (8.15).

Hence, we make the definitions

$$I_{mn}^1 = \int_{-1}^{1} \frac{f}{\alpha} G_m \omega_n \, dz \, , \; m \geq 0 \, ; \qquad J_{mn}^2 = \int_{-1}^{1} \frac{1}{\alpha} G_m^f \omega_n \, dz \, , \; m \geq 0 \, ;$$

$$I_{mn}^2 = \int_{-1}^{1} G_m \omega_n \, dz \, , \; m \geq 1 \, ; \qquad J_{mn}^3 = \int_{-1}^{1} \frac{f}{\alpha} G_m^f \, dz \, , \; m \geq 0 \, ;$$

$$I_{mn}^3 = \int_{-1}^{1} \frac{1}{f\alpha} G_m \omega_n \, dz \, , \; m \geq 1 \, ; \quad K_{mn}^1 = \int_{-1}^{1} ff' \, G_m^z \omega_n \, dz \, , \; m \geq 0 \, ;$$

$$I_{mn}^4 = \int_{-1}^{1} ff' \, G_m \omega_n \, dz \, , \; m \geq 0 \, ; \quad K_{mn}^2 = \int_{-1}^{1} \frac{f}{\alpha} G_m^z \omega_n \, dz \, , \; m \geq 0 \, ;$$

$$(8.19)$$

$$I_{mn}^5 = \int_{-1}^{1} f' \, G_m \omega_n \, dz \, , \; m \geq 1 \, ; \quad K_{mn}^3 = \int_{-1}^{1} \frac{1}{\alpha} G_m^z \omega_n \, dz \, , \; m \geq 1 \, ;$$

$$I_{mn}^6 = \int_{-1}^{1} f \, G_m \omega_n \, dz \, , \; m \geq 0 \, ; \quad IP_{mn} = \int_{-1}^{1} G_m' \omega_n \, dz \, , \; m \geq 1 \, ;$$

$$I_{mn}^7 = \int_{-1}^{1} \frac{f'}{f} G_m \omega_n \, dz \, , \; m \geq 1 \, ; \quad JP_{mn} = \int_{-1}^{1} (G_m^f)' \, f\omega_n \, dz \, , \; m \geq 0 \, ;$$

$$J_{mn}^1 = \int_{-1}^{1} f \, G_m^f \omega_n \, dz \, , \; m \geq 0 \, ; \quad KP_{mn} = \int_{-1}^{1} (G_m^z)' \, f\omega_n \, dz \, , \; m \geq 0 \, .$$

Some of the quantities in (8.19) are not required for $m = 0$, since their coefficient in (8.10) to (8.15) is zero. In some cases this is fortunate, since some of the above integrals do not exist when m is taken to be zero. In terms of the above integrals (8.19) we may express the terms(8.10) to (8.15) as follows.

$$
\begin{aligned}
P_{mn} = \pi\, e^{im\varphi} \{ & -\omega\mu\, [\ e^{i\varphi}\, I^1_{m+1,n} - e^{-i\varphi}\, I^1_{m-1,n}\ ] \\
& + i\eta Z\, [\ e^{i\varphi}\,(J^1_{m+1,n} - K^1_{m+1,n}) - e^{-i\varphi}\,(J^1_{m-1,n} - K^1_{m-1,n}) \\
& \qquad + (m+1)\, e^{i\varphi}\, I^2_{m+1,n} + (m-1)\, e^{-i\varphi}\, I^2_{m-1,n}\ ] \\
& + \frac{m}{\omega\epsilon}\, [\ (m+1)\, e^{i\varphi}\, I^3_{m+1,n} - (m-1)\, e^{i\varphi}\, I^3_{m-1,n} \\
& \qquad + e^{i\varphi}\, J^2_{m+1,n} + e^{-i\varphi}\, J^2_{m-1,n}\ ]\ \} \ ;
\end{aligned}
$$

(8.20)

$$
\begin{aligned}
Q_{mn} = \pi\, e^{im\varphi} \{ & i\omega\mu\, [\ e^{i\varphi}\, I^4_{m+1,n} + e^{-i\varphi}\, I^4_{m-1,n}\ ] \\
& - \eta Z\, [\ e^{i\varphi}\, K^2_{m+1,n} + e^{-i\varphi}\, K^2_{m-1,n}\ ] \\
& + \frac{1}{i\omega\epsilon}\, [\ (m+1)\, e^{i\varphi}\,(I^7_{m+1,n} - IP_{m+1,n}) \\
& \qquad - (m-1)\bar{e}^{i\varphi}\,(I^7_{m+1,n} - IP_{m-1,n}) \\
& \qquad - e^{i\varphi}\, JP_{m+1,n} - e^{-i\varphi}\, JP_{m-1,n}\ ]\ \} \ ;
\end{aligned}
$$

(8.21)

$$R_{mn} = \pi \, e^{im\varphi} \{ \, i\omega\mu \, [\, e^{i\varphi} \, I^1_{m+1,n} + e^{-i\varphi} \, I^1_{m-1,n} \,]$$

$$+ \eta Z \, [\, e^{i\varphi} \, (J^1_{m+1,n} - K^1_{m+1,n}) + e^{-i\varphi} \, (J^1_{m-1,n} - K^1_{m-1,n})$$

$$(8.22) \qquad + (m+1) \, e^{i\varphi} \, I^2_{m+1,n} - (m-1) \, e^{-i\varphi} \, I^2_{m-1,n} \,]$$

$$- \frac{im}{\omega\varepsilon} \, [\, (m+1) \, e^{i\varphi} \, I^3_{m+1,n} + (m-1) \, e^{-i\varphi} \, J^3_{m-1,n}$$

$$+ e^{i\varphi} \, J^2_{m+1,n} - e^{-i\varphi} \, J^2_{m-1,n} \,] \, \} \, ;$$

$$S_{mn} = \pi \, e^{im\varphi} \{ \, \omega\mu \, [\, e^{i\varphi} \, I^4_{m+1,n} - e^{-i\varphi} \, I^4_{m-1,n} \,]$$

$$+ i\eta Z \, [\, e^{i\varphi} \, K^2_{m+1,n} - e^{-i\varphi} \, K^2_{m-1,n} \,]$$

$$(8.23) \qquad - \frac{1}{\omega\varepsilon} \, [\, (m+1) \, e^{i\varphi} \, (I^7_{m+1,n} - IP_{m+1,n})$$

$$+ (m-1) \, e^{-i\varphi} \, (I^7_{m-1,n} - IP_{m-1,n})$$

$$- e^{i\varphi} \, JP_{m+1,n} + e^{-i\varphi} \, JP_{m-1,n} \,] \, \} \, ;$$

$$(8.24) \qquad T_{mn} = -2\pi \, m e^{im\varphi} \{ \, i\eta Z \, I^5_{mn} - \frac{1}{\omega\varepsilon} \, K^3_{mn} \, \} \, ;$$

$$(8.25) \qquad U_{mn} = 2\pi \, e^{im\varphi} \{ \, i\omega\mu \, I^6_{mn} + \eta Z \, J^3_{mn} - \frac{1}{i\omega\varepsilon} \, KP_{mn} \, \} \, .$$

Each of the integrals (8.19) may now be accurately evaluated using the one-term formula (8.3), provided that $\bar{r}'$ is not unduly close to S . We shall assume this to be the case. We shall, however, require G_m, G_m^f, G_m^z, dG_m/dz, dG_m^f/dz and dG_m^z/dz in order to evaluate the integrals (8.19). Let us now describe the evaluation of these quantities.

To this end, let us set

$$c_j = \cos \left[\frac{2j-2}{4N+2} \pi \right]$$

$$\rho_j = \{ (z-z')^2 + \rho^2 + f^2 - 2f \rho c_j \}^{\frac{1}{2}}$$

$$\alpha_j = \frac{\partial \rho_j}{\partial z} = \frac{z-z'}{\rho_j}$$

$$\beta_j = \frac{\partial \rho_j}{\partial f} = \frac{f - 2\rho c_j}{\rho_j}$$

(8.26)
$$\gamma_j = \frac{d\rho_j}{dz} = \alpha_j + f' \beta_j$$

$$\delta_j = \frac{d}{dz} \alpha_j = \frac{1}{\rho_j} + \frac{z\gamma_j}{\rho_j^2}$$

$$\epsilon_j = \frac{d}{dz} \beta_j = \frac{f'}{\rho_j} + \frac{2\rho c_j}{\rho_j^2} \gamma_j$$

$$\omega_j = \frac{ik_o}{\rho_j} - \frac{1}{\rho_j^2}$$

$$\theta_j = \frac{-ik_o}{\rho_j^2} + \frac{2}{\rho_j^3} \qquad G_j^* = \frac{e^{ik_o\rho_j}}{\rho_j}$$

The relations

$$R \equiv \{ (z-z')^2 + \rho^2 + f^2 - 2f \rho \cos \theta \}^{\frac{1}{2}} \, , \quad G^* \equiv \frac{e^{ik_o R}}{R}$$

$$G_m = \frac{1}{4\pi^2} \int_0^\pi G^* \cos m\theta \, d\theta$$

$$\frac{dG_m}{dz} = \frac{1}{4\pi^2} \int_0^\pi (\frac{ik_o}{R} - \frac{1}{R^2}) \frac{dR}{dz} \, G^* \cos m\theta \, d\theta$$

$$G_m^f = \frac{\partial G_m}{\partial f} = \frac{1}{4\pi^2} \int_0^\pi (\frac{ik_o}{R} - \frac{1}{R^2}) \, R^f \, G^* \cos m\theta \, d\theta$$

$$G_m^z = \frac{\partial G_m}{\partial z} = \frac{1}{4\pi^2} \int_0^\pi (\frac{ik_o}{R} - \frac{1}{R^2}) \, R^z \, G^* \cos m\theta \, d\theta$$

$$\frac{d}{dz} G_m^f = \frac{1}{4\pi^2} \int_0^\pi \{ \, (- \frac{ik_o}{R^2} + \frac{2}{R^3}) \, R^f \frac{dR}{dz} + (\frac{ik_o}{R} - \frac{1}{R^2}) \frac{d}{dz} R^f$$

$$+ (\frac{ik_o}{R} - \frac{1}{R^2})^2 \, R^f \frac{dR}{dz} \quad G^* \cos m\theta \, d\theta$$

$$\frac{d}{dz} G_m^z = \frac{1}{4\pi^2} \int_0^\pi \{ \, (- \frac{ik_o}{R^2} + \frac{2}{R^3}) \, R^z \frac{dR}{dz} + (\frac{ik_o}{R} - \frac{1}{R^2}) \frac{d}{dz} R^z$$

$$+ (\frac{ik_o}{R} - \frac{1}{R^2})^2 \, R^z \frac{dR}{dz} \} \, G^* \cos m\theta \, d\theta$$

$$\cos m\theta = 2 \cos \theta \cos (m-1) \theta - \cos (m-2) \theta$$

then show that given z, z' and ρ, we can evaluate the six quantities
referred to in the title of this section, for $m = 0, 1, \ldots , M+1$ by
means of the following algorithm.

ALGORITHM 8.1   EVALUATION OF G_m, G_m^f, G_m^z, G_m', $(G_m^f)'$, $(G_m^z)'$

1. Evaluate each of the quantities (19.1), as well as $f = f(z)$ and $f' = f'(z)$ for $j = 1, 2, \ldots, 2N+2$.

2. $(G_m, G_m^f, G_m^z, (G_m)', (G_m^f)', (G_m^z)') \leftarrow (0,0,0,0,0,0)$

$$j \leftarrow 1$$

$$d_o \leftarrow 1$$

$$d_1 \leftarrow c_j$$

$$d_m \leftarrow 2c_j\, d_{m-1} - d_{m-2}, \quad m = 2, 3, \ldots, M+1$$

$$G_m \leftarrow G_m + G_j^* \, d_m$$

$$G_m^f \leftarrow G_m^f + \omega_j\, \beta_j\, G_j^*\, d_m$$

$$G_m^z \leftarrow G_m^{z'}, + \omega_j\, \alpha_j\, G_j^*\, d_m$$

$$G_m' \leftarrow G_m' + \omega_j\, \gamma_j\, G_j^*\, d_m$$

$$(G_m^f)' \leftarrow (G_m^f)' + [\,(\theta_j + \omega_j^2)\, \beta_j\, \gamma_j + \omega_j\, \varepsilon_j\,]\, G_j^*\, d_m$$

$$(G_m^z)' \leftarrow (G_m^{z'})' + [\,(\theta_j + \omega_j^2)\, \alpha_j\, \gamma_j + \omega_j\, \delta_j\,]\, G_j^*\, d_m$$

$m=0,1, \ldots, M+1.$

$$j \leftarrow j+1$$

$$j : 2N+2$$

(<)

$(=)$ $(G_m, G_m^f, G_m^z, G_m', (G_m^f)', (G_m^z)') \leftarrow \dfrac{1}{4\pi(2N+2)}\, (G_m, G_m^f, G_m^z, G_m', (G_m^f)', (G_m^z)')$
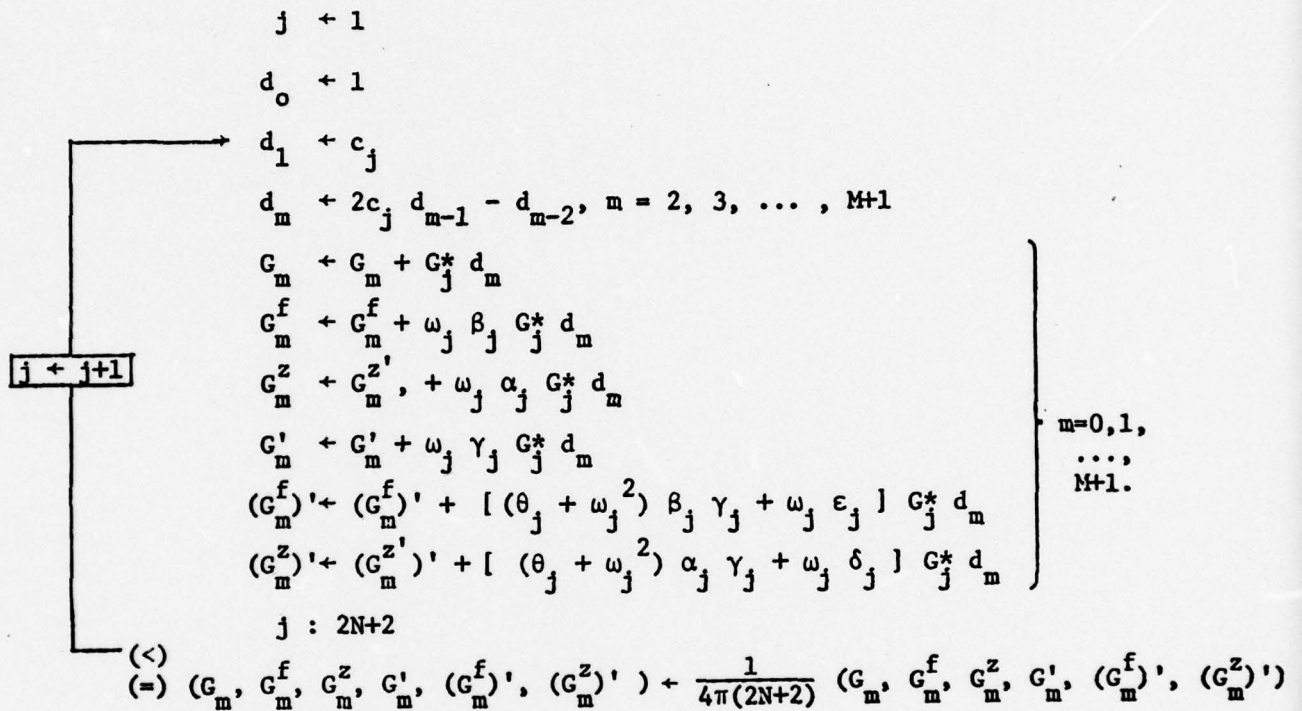
9. CONVERGENCE

The proof of convergence of the approximation scheme presented in Secs. 6 to 8 is quite simple, using thm. A.1 and the results of Sec. 5.

Let us denote the right hand side of (A.36) by $A\bar{J}_1$ and for $F \in H(d,d')$, let us denote ℓ_{MN} by $P_{MN}(F)$, where $\ell_{MN}(z,\varphi)$ is defined in (5.5).

In view of Thm. A.1, and Sec. 5, we have

    (a) $\|P_{MN} AJ - AJ\|_H \to 0$ for every $J \in H(d,d')$;

    (b) $\|P_{MN} J^\circ - \bar{J}^\circ\|_H \to 0$

    (c) $\sup \|P_{MN}\| \leq 1 + \|I - P_{MN}\| < \infty$

Hence according to [21 pp. 469-470] it follows that the approximation $J = J_{MN}$ produced by the algorithm of Sections 6 to 8 converges to the solution $\bar{J}$ of Eq. (7.2). Moreover, by taking $N = M^2$, it follows that

$$(9.1) \qquad \|\bar{J} - J_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}})$$

for some $\gamma > 0$. Due to quadratures and matrix solution involved in the actual algorithm we actually compute a perturbed solution $\tilde{J}_{MN}$; however, due to the accuracy of the quadrature schemes described in Sec. 5, and since the resulting Galerkin matrix is not ill-conditioned, we also have

$$(9.2) \qquad \|J_{MN} - \tilde{J}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}}), \quad N \to \infty$$

By combining (9.1) and (9.2), it thus follows that

$$(9.3) \qquad \|\overline{J} - \widetilde{J}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}}), \quad N \to \infty.$$

Finally, in computing the scattered field $\overline{E}^s = \overline{E}^s_{MN}$, as described in Sec. 8, we similarly have by Thm. 5.2 that

$$(9.4) \qquad \|\overline{E}^s - \widetilde{E}_{MN}\|_H = O(e^{-\gamma N^{\frac{1}{2}}})$$

where $\widetilde{E}_{MN}$ denotes the perturbed scattered field that we actually computed.

APPENDIX A: THE FUNCTIONS G_m, $\partial G_m/\partial f$ and $\partial G_m/\partial z$.

In this appendix we study the functions G_m, $G_m^f = \partial G_m/\partial f$ and $G_m^z = \partial G_m/\partial z$ which appear in Secs. 7 to 10. The results of this study will enable us to deduce the following:

i) Each component of the solution $\bar{J}$ of Eq. (7.2) is in $H(d,d')$;

ii) If $m \geqslant 0$,

(A.1)
$$\left\{
\begin{array}{l}
G_m(z,z') = 0\,(|f(z)|^m) \\[2ex]
G_m^f(z,z') = \left\{
\begin{array}{l}
0(1) \text{ if } m = 0 \\[1ex]
0(|f(z)|^{m-1}), \; m > 0
\end{array}
\right\} \\[3ex]
G_m^z(z,z') = 0(|f(z)|^m)
\end{array}
\right\}
\begin{array}{l}
\text{as } z \to \pm 1, \\
z' \in (-1,1);
\end{array}$$

and

(A.2)
$$\left\{
\begin{array}{l}
G_m(z,z') \sim \dfrac{1}{4\pi^2 f(z')} \log \dfrac{1}{|z-z'|} \\[3ex]
G_m^f(z,z') \sim \dfrac{-f'(z')\alpha(z')^2}{4\pi^2 f(z')(z-z')} \\[3ex]
G_m^z(z,z') \sim \dfrac{-\alpha(z')^2}{4\pi^2 f(z')(z-z')}
\end{array}
\right\}
\begin{array}{l}
\text{as } z \to z', \\
z' \in (-1,1).
\end{array}$$

The results (A.1) proved to be useful for choosing the basis functions, in order to be able to obtain a convergent Galerkin method, while the results (A.2) enabled us to choose the proper numerical

integration technique for evaluating the singular integrals to get the coefficients P^{mn}, Q^{mn}, R^{mn} and S^{mn} in Secs. 7-9.

Throughout this appendix, the following notation is being used.

$$f = f(z) \; , \; f* = f(z')$$

$$\nu = \{(z-z')^2 + (f+f*)^2\}^{\frac{1}{2}}$$

(A.3)

$$\kappa = \frac{4ff*}{\nu^2}$$

$$1-\kappa = \frac{(f-f*)^2 + (z-z')^2}{\nu^2}$$

(a) The Functions G_m

The functions G_m (see Eqs. (7.10) and (7.17)) are defined for given z, z' on $(-1,1)$ by

(A.4)
$$G_m = G_m(z,z') = \frac{1}{4\pi^2} \int_0^\pi \frac{e^{ik_o R}}{R} \cos(m\theta) \, d\theta,$$

where

(A.5)
$$R = \{(z-z')^2 + f^2 + f*^2 - 2ff* \cos\theta\}^{\frac{1}{2}}$$

In terms of (A.3), we therefore have

$$R = \nu \{1 - \kappa \cos^2 \frac{\theta}{2}\}^{\frac{1}{2}}$$

(A.6)
$$G_m = \frac{1}{2\pi^2} \int_0^{\pi/2} \frac{e^{ik_o \nu\{1-\kappa\cos^2\theta\}^{\frac{1}{2}}}}{\nu\{1-\kappa\cos^2\theta\}^{\frac{1}{2}}} \cos(2m\theta) \, d\theta$$

Expansion of the exponential in (A.6) and termwise integration

yields

(A.7) $\quad G_m = \dfrac{1}{2\pi^2} \displaystyle\sum_{s=o}^{\infty} \dfrac{(-1)^s \nu^{2s} k_o^{2s}}{(2s)!} \left\{ \dfrac{J_m^s}{\nu} + i \dfrac{k_o K_m^s}{2s+1} \right\}$

where

(A.8)

$$J_m^s = \int_0^{\pi/2} \{ 1 - \kappa \cos^2 \theta \}^{s-\frac{1}{2}} \cos (2m\theta) \, d\theta$$

$$K_m^s = \int_0^{\pi/2} \{ 1 - \kappa \cos^2 \theta \}^s \cos (m\theta) \, d\theta.$$

The relationship

(A.9)

$$\{ 1 - \kappa \cos^2 \theta \}^a \cos (2m\theta)$$

$$= \{ 1 - \kappa \cos^2 \theta \}^{a-1} \left[(1-\tfrac{\kappa}{2}) \cos (2m\theta) - \tfrac{\kappa}{4} \cos (2m+2) \theta - \tfrac{\kappa}{4} \cos (2m-2)\theta \right]$$

shows that

(A.10)

$$J_m^s = (1 - \tfrac{\kappa}{2}) J_m^{s-1} - \tfrac{\kappa}{4} J_{m+1}^{s-1} - \tfrac{\kappa}{4} J_{m-1}^{s-1}$$

$$K_m^s = (1 - \tfrac{\kappa}{2}) K_m^{s-1} - \tfrac{\kappa}{4} K_{m+1}^{s-1} - \tfrac{\kappa}{4} K_{m-1}^{s-1},$$

whereas, by (A.8)

(A.11) $\quad J_{-m}^s = J_m^s \, , \; K_{-m}^s = K_m^s.$

The relationships (A.10) show that in order to evaluate G_m using (A.7), we need only know J_m^o and K_m^o, for $m = 0, \pm 1, \pm 2, \ldots$; we can then get the remaining J_m^s and K_m^s for $s > 0$ using (A.10). To this end, integrating termwise in (A.8) and using the identity

(A.12)
$$\int_0^{\frac{\pi}{2}} \cos^{2n} \theta \, \cos(2m\theta) \, d\theta = \binom{2n}{n+m} \frac{\pi}{2^{2n+1}}$$

we get

(A.13)
$$J_m^o = \frac{\pi}{2} \frac{(\tfrac{1}{2})_m \kappa^m}{2^{2m} m!} \; F\left(\tfrac{1}{2}+m, \; \tfrac{1}{2}+m; \; 1+2m; \; \kappa\right)$$

$$K_m^o = \begin{cases} \dfrac{\pi}{2} \text{ if } m = 0 \\ \\ 0 \text{ if } m \neq 0. \end{cases}$$

In (A.13) F denotes the hypergeometric function. Thus we may compute J_m^o for $0 \leq \kappa \leq .6$ by means of the formula

(A.14)
$$J_m^o = \frac{\pi}{2} \frac{(\tfrac{1}{2})_m \kappa^m}{2^{2m} m!} \sum_{n=0}^{\infty} \frac{[(\tfrac{1}{2}+m)_n]^2}{(1+2m)_n \, n!} \, \kappa^n$$

whereas, if $.6 < \kappa < 1$ (see [1, p.559]) it is preferable to use the formula

(A.15)
$$J_m^o = \kappa^m \sum_{n=0}^{\infty} \frac{[(\tfrac{1}{2}+m)_n]^2}{[n!]^2} \left[\psi(n+1) - \psi(\tfrac{1}{2}+m+n) - \tfrac{1}{2} \ln(1-\kappa) \right] (1-\kappa)^n$$

where

$$\psi(1) = -\gamma = -.57721 \quad 56649$$

(A.16)
$$\psi(\tfrac{1}{2}) = -\gamma - 2\ell n2 = - \quad 1.9635 \quad 10026$$

$$\psi(z+1) = \frac{1}{z} + \psi(z) .$$

Eq. (A.15) shows that

(A.17)
$$J_m^o \sim -\tfrac{1}{2} \ell n (1-\kappa) \text{ as } \kappa \to 1^-.$$

Now, by (A.3)

(A.18)
$$1-\kappa = \frac{\nu^2 - 4ff^*}{\nu^2} = \frac{(z-z')^2 + (f-f^*)^2}{\nu^2}$$

$$\sim \frac{[1+f^{*'2}](z-z')}{4f^{*2}} \text{ as } z \to z'$$

so that, by combining (A.17) and (A.18), we get

(A.19)
$$J_m^o \sim \ell n [\frac{2f^*}{(1+f^{*'2})^{\frac{1}{2}}} \frac{1}{|z-z'|}], \, z \to z'.$$

Using induction on (A.10), it thus follows that

(A.20)
$$J_m^s \sim 0 \text{ as } z \to z', \, s > 0 .$$

In view of (A.7), (A.13), (A.19) and (A.20) if therefore follows that

(A.21)
$$G_m \sim \frac{1}{2\pi^2} \frac{J_m^o}{\nu} \sim \frac{1}{4\pi^2 f^*} \ell n [\frac{2f^*}{(1+f^{*'2})^{\frac{1}{2}}} \frac{1}{|z-z'|}]$$

$$\text{as } z \to z',$$

which is the first of (A.2).

In view of (A.3), it follows that

(A.22) $$\kappa = O(f) \text{ as } z \to \pm 1,$$

for all $z' \in [-1,1]$. Hence by (A.13) and (A.14),

(A.23) $$G_m \sim O(f^{|m|}) \text{ as } z \to \pm 1, \text{ for all } z' \in [-1,1].$$

(b) The Functions G_m^f , $G_m^{z'}$

Upon defferentiating the expressions (A.3) we get

$$\nu^f \equiv \frac{\partial \nu}{\partial f} = \frac{f+f*}{\nu} \ , \quad \nu^z \equiv \frac{\partial \nu}{\partial z} = \frac{z-z'}{\nu}$$

(A.24)
$$\kappa^f = \frac{\partial \kappa}{\partial f} = \frac{4f*}{\nu^2} - \frac{8ff*(f+f*)}{\nu^4}$$

$$\kappa^z = \frac{\partial \kappa}{\partial z} = - \frac{8ff*}{\nu^3} \frac{(z-z')}{\nu}$$

We note therefore, that

(A.25)
$$2 \nu^f + \nu \frac{\kappa^f}{\kappa} = \frac{\gamma}{f}$$

$$2 \nu^z + \nu \frac{\kappa^z}{\kappa} = 0$$

By means of these solutions as well as (A.10), we get the identities

59

$$G_m^f \equiv \frac{\partial G_m}{\partial f} = \frac{1}{2\pi^2} \sum_{s=0}^{\infty} \{ \frac{(-1)^s \nu^{2s} ko^{2s}}{(2s)!} \ [\frac{s-\frac{1}{2}}{\nu} (\frac{1}{f} J_m^s - \frac{\kappa^f}{\kappa} J_m^{s-1})$$

(A.26)

$$\frac{-i\nu^2 ko^3}{2(2s+1)(2s+3)} \ (\frac{1}{f} K_m^{s+1} - \frac{\chi^f}{\chi} K_m^s) \ ] \}$$

and

(A.27)
$$G_m^z = \frac{z-z'}{\pi^2 \nu^3} \sum_{s=0}^{\infty} \{ \frac{(-1)^s \nu^{2s} ko^{2s}}{(2s)!} \ [\ (s-\tfrac{1}{2}) J_m^{s-1} + \frac{-i\nu^2 ko^3}{2(2s+1)(2s+3)} K_m^s] \}.$$

These series may be readily computed using the identities

$$J_m^{-1} = \frac{\pi}{2} \frac{(3/2)_m \kappa^m}{2^m m!} \ F(\frac{3}{2} + m, \frac{1}{2} + m; 2m+1; \kappa)$$

$$= \frac{\pi}{2} \frac{(3/2)_m \kappa^m}{2^m m!} \sum_{n=0}^{\infty} \frac{(3/2+m)_n (\frac{1}{2}+m)_n}{(2m+1)_n \ n!} \kappa^n, \ 0 \le \kappa \le .6$$

(A.28)

$$= \frac{\kappa^m}{1-\kappa} + \kappa^m (m^2 - \tfrac{1}{4}) \sum_{n=0}^{\infty} \frac{(\frac{3}{2}+m)_n (\frac{1}{2}+m)_n}{n! \ (n+1)!} (1-\kappa)^n.$$

$$\cdot [\ \ell n (1-\kappa) + 2\psi(\tfrac{1}{2}+m+n) - 2\psi(n+1) + \frac{1}{m+n+\frac{1}{2}} - \frac{1}{n+1}]$$

$$\text{if } .6 \le \kappa < 1$$

along with (A.10). These identities were obtained via a procedure similar to that which was used to get (A.14) and (A.15).

The expressions (A.26) and (A.28) enable us to deduce various growth properties of G_m^f and G_m^z as $z \to z'$ and as $z \to \pm 1$.

By (A.28) and (A.3),

$$J_m^{-1} \sim \frac{\overset{m}{\kappa}}{1-\kappa}$$

(A.29)

$$\sim \frac{4f*^2\alpha*^2}{(z-z')^2} \quad \text{as } z \to z' \in (-1,1)$$

and

(A.30)

$$J_m^{-1} = O(f^m) \quad \} \text{ as } z \to \pm 1$$

Combining these results with the asymptotic identities

(A.31)

$$\begin{cases} \dfrac{\overset{f}{\kappa}}{\kappa} \sim -\dfrac{f*'(z-z')}{2f*^2}, \quad z \to z' \\\\ \sim \dfrac{1}{f}, \quad z \to \pm 1, \end{cases}$$

and

$$\frac{\overset{z}{\kappa}}{\kappa} \sim -\frac{2f*^2(z-z')}{\nu^2}, \quad z \to z'$$

(A.32)

$$\frac{\overset{z}{\kappa}}{\kappa} = O(f), \quad z \to \pm 1$$

we get

$$G_m^f \sim -\frac{1}{4\pi^2} \frac{f*'\alpha*^2}{f*(z-z')}, \quad z \to z'$$

(A.33)

$$G_m^z \sim -\frac{1}{4\pi^2} \frac{\alpha*^2}{f*(z-z')}, \quad z \to z'$$

The relation (A.10), (A.13), (A.26), (A.28) and (A.31) yield

$$(A.34) \qquad G_m^f = \left\{ \begin{array}{l} 0(1) \text{ if } m=0 \\ \\ 0(f^{m-1}) \text{ if } m \end{array} \right\} \text{ as } z \to \pm 1,$$

while the relations (A.10), (A.13), (A.27), (A.28) and (A.32) yield

$$(A.35) \qquad G_m^z = 0(f^m) \text{ as } z \to \pm 1.$$

(c) Analyticity of $\overline{K}$

The above results show that

i) G_m is bounded as a function of z on $[-1,1]$, except at $z = z'$ where it becomes unbounded according to (A.21);

ii) $G_m(z,z')$ is an analytic function of $z \in \Omega_d$, except at $z = z'$, where it has a singularity of the form (A.21).

In the following theorem $H(d,d')$ is defined as in Sec. 5.

Theorem A.1: Let $\overline{K}$ be the solution of Eq. (4.6), and let $\overline{J}$ be defined by (7.1). Then each component of $\overline{J}$ is in $H(d,d')$.

Proof: Each component of the incident field $\overline{J}^o$ (see Eq. (7.3)) is in $H(d,d')$ where $d' > 0$ is arbitrary. In view of Eq. (7.2), we need only show that if $\overline{J}_1 = (J_{1t}, J_{1\varphi})$ denotes a pair of functions in $H(d,d')$, each of which is bounded on each of the sets

$$S_1 = \{ (z,\omega): z \in \Omega_d, |\omega| = 1 \}$$

$$S_2 = \{ (z,\omega) : -1 \leq z \leq 1, \frac{1}{d'} \leq |\omega| \leq d'\}$$

where $d' > 1$ is arbitrary,

then each of the components of

$$J_2 = (J_{2t}, \; J_{2\varphi})$$

$$(A.36) \qquad = \{ \; \nu \int_S \; [\; \frac{i\omega\mu}{\nu} \, \bar{J}_1 \, G + \eta \frac{Z}{\nu} \, (\hat{n} \times \bar{J}_1) \times \nabla G$$

$$+ \frac{1}{i\omega\varepsilon} \, \nabla \cdot (\frac{1}{\nu} \, \bar{J}_1) \, \nabla G \;] \; dS \; \}_{tan}$$

is in $H(d,d')$.

By our assumption on f in Sec. 3, the solution $\bar{K} = (K_t, K_\varphi)$ of Eq. (4.2) is bounded on $S = [-1,1] \times [0,2\pi]$. Hence for $z \in [-1,1]$, each component of $\bar{J}_1$ has the form

$$(A.37) \qquad F(z, e^{i\varphi}) = \sum_{-\infty}^{\infty} a_m(z) \, e^{im\varphi}$$

where $a_m / \nu \in H(\Omega_d)$; substituting this form of an expression into (A.36) for J_{1t} and $J_{1\varphi}$ and noting that $a_m / \nu = 0(e^{-d'|m|}) \; \forall \; z \in [-1,1]$ and for all $d' > 0$, we deduce, by inspection of (7.27) – (7.30) and (A.1) and (A.2) that $|P^{mn}|$, $|Q^{mn}|$, $|R^{mn}|$ and $|S^{mn}|$ are $0 \, (e^{-d'|m|})$ for all $d' > 0$ and for all $z' \in [-1,1]$. That is, $F(z,\omega) \in H(A_{d'})$ as a function of ω, for all $z \in [-1,1]$.

Hence in order to complete the proof, we need only show that if $\theta_n / \nu \in H(\Omega_d)$, then each of the right-hand sides of (7.27) to (7.30) is in $H(\Omega_d)$.

In view of the results of Appendix A.(a) and A.(b), the coefficients of θ_n / ν in the integrals (7.27) to (7.30) are of the following three types:

$$a(z,z') \ , \ a(z,z') \ \log |z-z'| \ , \ a(z,z')/(z-z'),$$

where $a(z,z')$ is a bounded function in $H(\Omega_d) \times H(\Omega_d)$. Hence, we need only show that given $g \in H(\Omega_d)$, each of the functions g_1, g_2 and g_3 are analytic in Ω_d, where

$$g_1(z') = \int_{-1}^{1} a(z,z') \ g(z) \ dz;$$

(A.38)
$$g_2(z') = \int_{-1}^{1} a(z,z') \ \log |z-z'| \ g(z) \ dz;$$

$$g_3(z') = \text{P.V.} \int_{-1}^{1} \frac{a(z,z')}{z-z'} \ g(z) \ dz$$

It is obvious that $g_1 \in H(\Omega_d)$.

Next if $z' \in \Omega_d \cap \{\text{Im } z' > 0\}$ then

(A.39)
$$g_3^*(z') = \int_{-1}^{1} \frac{a(z,z')}{z-z'} \ g(z) \ dz$$

is analytic in this region, and indeed, by altering the path of integration in (A.39) to the lower boundary of Ω_d, we see that g_3^* is in fact analytic in Ω_d. If we now return the path of integration to the interval $(-1,1)$ and let $\text{Im } z' \to 0$, we find that for $z' \in (-1,1)$,

(A.40)
$$g_3^*(z') = \pi i \ a(z',z') \ g(z') + \pi i \ g_3(z')$$

this expression shows that since both $g_3^*(z')$ and $a(z',z')$ have an analytic extension into Ω_d, so does g_3. Hence g_3 is analytic in Ω_d.

Finally, writing g_2 in the form of a convergent sum

$$(A.41) \qquad g_2(z') = \int_{-1}^{1} \sum_k \alpha_k(z) \, \beta_k(z') \log |z-z'| \, dz$$

where the functions α_n and β_n are in $H(\Omega_d)$, we need only show that g_k^* is analytic in Ω_d, where

$$(A.42) \qquad g_k^*(z') = \int_{-1}^{1} \alpha_k(z) \log |z-z'| \, dz,$$

Upon differentiating this expression <u>carefully</u>, we see that

$$(A.43) \qquad g_k^{*'}(z') = \text{P.V.} \int_{-1}^{1} \frac{\alpha_k(z)}{z-z'} \, dz.$$

By our argument involving g_3 above, it follows that $g_k^{*'}$ is analytic Ω_d, i.e., g_k^* and hence g_2 is analytic in Ω_d. This completes the proof of Theorem A.1.

By assumption for the case of finite conductivity of the body **B**, the function f is such that the surface S satisfies Liapunov conditions, in which case the surface current $\overline{K}$ is bounded on S. However, one notes that the integrals (7.27) - (7.30) converge so long as $f |\theta_n/\nu| = o(1)$ as $z \to \pm 1$, i.e., so long as $fK = o(1)$ as $z \to \pm 1$. Thus our method gives answers even if S is cone-shaped at one or both ends, although the Liapunov conditions are then violated, and our results may have no physical significance, except in the infinite conducting case.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A Galerkin integral equation method is described for solving the problem of the title. Each component of the scattered field is expressed in the form

$$(1) \quad E = \sum_{m=-M}^{M} \sum_{n=-M^2}^{M^2} A_{mn} e^{im\varphi} \frac{\sin\{\pi M(x - n/M)\}}{\pi M(x - n/M)}.$$

Under the assumption that the surface of the body is described by $\sqrt{x^2 + y^2} = f(x)$ where f is analytic and bounded in $\{z : |\arg[(1+z)/(1-z)]| < d\}$, the rate of convergence of (1) is

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20.

$O(e^{-cM})$; this rate cannot be improved.

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