

AD-A051 938

NAVAL RESEARCH LAB WASHINGTON D C  
ABSORPTION AND TRANSFORMATION OF ELECTROMAGNETIC WAVES IN THE V--ETC(U)  
NOV 77 T M ANTONSEN, W M MANHEIMER

F/G 20/9

E(49-20)-1006

UNCLASSIFIED

NRL-MR-3583

SBIE-AD-E000 126

NL

|OF|  
AD  
A051938



END  
DATE  
FILMED  
5-78  
DDC

AD A 051938

R

1  
12

15 Mar 78

ad 000126

NRL Memorandum Report 3583

# Absorption and Transformation of Electromagnetic Waves in the Vicinity of Electron Cyclotron Harmonics

T. M. ANTONSEN, Jr. and W. M. MANHEIMER

*Plasma Physics Division*

November 1977

AD No. \_\_\_\_\_  
DDC FILE COPY



DDC  
RECEIVED  
MAR 29 1978  
B

NAVAL RESEARCH LABORATORY  
Washington, D.C.

18 SBIE 19 AD-E 126

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Memorandum Report 3583	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ABSORPTION AND TRANSFORMATION OF ELECTRO- MAGNETIC WAVES IN THE VICINITY OF ELECTRON CYCLOTRON HARMONICS (U)	5. TYPE OF REPORT & PERIOD COVERED Interim report, a continuing NRL Problem	
7. AUTHOR(s) Thomas M. Antonsen, Jr. Wallace M. Manheimer	8. CONTRACT OR GRANT NUMBER(s) 12/35p.	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, D. C. 20375	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NRL Problem No. 77HO2-37 Project ERDA No. E(49-20):1006	
11. CONTROLLING OFFICE NAME AND ADDRESS NRL-MR-3583	12. REPORT DATE Nov 77	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) E(49-20)-1006	13. NUMBER OF PAGES 35	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.	15. SECURITY CLASS. (of this report) UNCLASSIFIED	
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A	
18. SUPPLEMENTARY NOTES This research was sponsored by the U. S. Energy Research and Development Administration, Project No. E(49-20):1006.		
19. KEY WORDS (Continue on reverse side if necessary; and identify by block number) Electron Cyclotron Harmonic Mode Conversion Wave Absorption		
20. ABSTRACT (Continue on reverse side if necessary; and identify by block number) (U) The recent development of multikilowatt millimeter wave sources has motivated the study of electron cyclotron heating of tokamaks. The direct absorption of the ordinary and extraordinary modes at the first and second harmonics of the electron cyclotron frequency are examined. For K-B=O WKB theory is inapplicable and a full wave treatment is necessary to determine transmission, absorption, reflection and mode conversion coefficients. The ordinary wave is efficiently absorbed at the first harmonic. The extraordinary wave is efficiently absorbed at the second harmonic due to mode conversion of the incident wave to a highly damped electrostatic wave.		

251 950- All

PRECEDING PAGE NOT FILMED  
BLANK

### CONTENTS

I. INTRODUCTION .....	1
II. REVIEW OF COLD PLASMA WAVE PROPAGATION .....	2
III. WKB ANALYSIS OF WAVE INTERACTIONS AT THE CYCLOTRON HARMONICS .....	4
A. Ordinary Wave .....	4
B. Extra Ordinary Mode .....	6
IV. PERTURBED CURRENT DENSITY IN AN INHOMOGENEOUS MAGNETIC FIELD .....	9
V. EXTRAORDINARY WAVE AT THE SECOND HARMONIC .....	13
A. Wave Incident from the High Field Side .....	18
B. Wave Incident from the Low Field Side .....	19
VI. ORDINARY WAVE AT THE FIRST HARMONIC .....	20
VII. ACCESSIBILITY OF CYCLOTRON RESONANCE AND APPLICATIONS TO HEATING IN TOROIDAL DEVICES .....	22
ACKNOWLEDGMENT .....	23
APPENDIX - Asymptotic Evaluation of $E_+(x)$ .....	24
REFERENCES .....	26

ACCESSION for	
NTIS	White Section <input checked="" type="checkbox"/>
DDC	Buff Section <input type="checkbox"/>
UNANNOUNCED	<input type="checkbox"/>
JUSTIFICATION _____	
BY _____	
DISTRIBUTION/AVAILABILITY CODES	
Dist. AVAIL. and/or SPECIAL	
A	

# ABSORPTION AND TRANSFORMATION OF ELECTROMAGNETIC WAVES IN THE VICINITY OF ELECTRON CYCLOTRON HARMONICS

## I. INTRODUCTION

Recent advances in the development of multi-kilowatt RF sources in the  $10^{10}$  Hz frequency range<sup>1</sup> have led to interest in the possibility of electron cyclotron heating in tokamaks. Electron cyclotron frequency heating in plasmas has been demonstrated in smaller (lower magnetic field) devices<sup>2</sup> and preliminary experiments in tokamaks have yielded encouraging results.<sup>3</sup> In this paper we will investigate the linear propagation and absorption of electromagnetic waves in the vicinity of cyclotron harmonics, that is regions of space where the wave frequency is an integer multiple of the electron cyclotron frequency.

Previous analyses have derived electron heating rates by integrating the equations of motion for the electrons in given electromagnetic fields. This method neglects the effects of the perturbed charge and current densities in determining the fields themselves and can lead to error.

Our approach is to analyze the propagation of electromagnetic waves in an inhomogeneous magnetic field. We assume the scale length for variation of the magnetic field and electron density is greater than both the electron gyroradius and the vacuum wavelength. Thus, a WKB analysis is expected to be accurate except in the immediate vicinity of a cyclotron resonance. Near the resonant surface a full wave theory is required that will connect the WKB

---

Manuscript submitted: August 5, 1977.

solutions valid on either side of the resonance. The full wave solution yields transmission, reflection, absorption and linear mode conversion coefficients from which the electron heating rate can be determined. Such a theory is presented in this paper.

The organization of the paper is as follows. In Section II we briefly review the propagation of electromagnetic waves in a cold plasma. In Section III we present a simplified analysis of wave absorption at a cyclotron harmonic based on WKB theory. That is we calculate wave absorption by integrating the imaginary part of the wave vector across the resonance. Strictly speaking, this method is invalid if  $dk_x/dx > k_x^2$  where  $k_x$  is the wave number and  $x$  the distance from the resonance. However, this procedure is found in later sections to give the correct transmission coefficient.

In Section IV we derive from the Vlasov equation an expression for the perturbed current density resulting from arbitrary wave fields near a cyclotron resonance. In Section V and Section VI we use the result of Section IV to analyze the propagation of the extraordinary wave near the second harmonic and the ordinary wave near the first harmonic respectively. Section VII contains our conclusions concerning the applicability of these mechanisms to the heating of tokamaks.

## II. REVIEW OF COLD PLASMA WAVE PROPAGATION

There are two electromagnetic waves that propagate in a cold, magnetized plasma. If these waves propagate in a direction nearly perpendicular to the applied magnetic field they are called the ordinary and extraordinary waves.

The ordinary wave is characterized by the alignment of the wave electric field nearly parallel to the applied magnetic field. For purely perpendicular propagation the dispersion relation for the ordinary wave in a cold plasma (immobile ions) is given by,

$$k_{\perp}^2 c^2 = \omega^2 - \omega_p^2 \quad (1)$$

where  $k_{\perp}$  is the wave vector perpendicular to  $\mathbf{B}_0$ , the applied magnetic field,  $c$  is the speed of light,  $\omega$  is the wave frequency and  $\omega_p^2 = 4\pi e^2 n_0 / m_e$  is the square of the plasma frequency. Notice that Eq. (1) does not involve  $B_0$  explicitly. This is a consequence of the parallel alignment of the wave electric field and the applied magnetic field.

The extraordinary wave is characterized by the alignment of the wave electric field perpendicular to the applied magnetic field. For purely perpendicular propagation the dispersion relation for the extraordinary wave is given by,

$$k_{\perp}^2 c^2 (\omega^2 - \omega_p^2 - \Omega^2) = (\omega_p^2 - \omega^2)^2 - \Omega^2 \omega^2, \quad (2)$$

where  $\Omega = eB_0 / m_e c$  is the electron cyclotron frequency. The extraordinary wave is resonant (i.e.  $k_{\perp} \rightarrow \infty$ ) as  $\omega^2 \rightarrow \omega_p^2 + \Omega^2$ , (this is known as the upper hybrid resonance) and it is cut off ( $k_{\perp} \rightarrow 0$ ) as  $\omega_p^2 \rightarrow \omega^2 + \Omega \omega$ .

Including a finite  $k_z$  (parallel to  $B_0$ ) modifies these dispersion relations slightly. Figure 1 shows an Allis<sup>4</sup> diagram (assuming  $k_z$  is fixed) which details the cut-offs and resonances for the two electromagnetic modes in a cold plasma.

We wish to point out that neither of the waves experience a resonance or cut-off when the wave frequency is equal to the cyclotron frequency and  $k_z$  is fixed. Thus in an inhomogeneous plasma where the density is nearly uniform along a field line cold plasma theory predicts that electromagnetic waves will propagate through regions where  $\omega = \Omega(\mathbf{x})$  without loss of energy. Linear absorption of wave energy by the particles is a kinetic effect that can only be determined by considering the behavior of the plasma on a microscopic level and it must result from the plasma having a nonzero temperature.

### III. WKB ANALYSIS OF WAVE INTERACTIONS AT THE CYCLOTRON HARMONICS

The dispersion relations for the ordinary and extraordinary waves obtained from the Vlasov-Maxwell set of equations<sup>4,5</sup> indicate that these waves will have resonances and cut-offs near each of the cyclotron harmonics. If the gradients in applied magnetic field and density are gentle then a WKB description is expected to be valid everywhere in the plasma except in the immediate vicinity of resonance or cut-off.

In the later sections of this paper we will present a "full wave" calculation that is valid in the immediate vicinity of a cyclotron harmonic resonance. The results of our analysis is a solution for the wave electromagnetic field, valid in the resonance region, which asymptotically matches on to the WKB solutions far from the resonance. Hence, we obtain a set of connection formulas which determine the fractions of wave energy that are transmitted, reflected, absorbed, and converted to other wave types.

It is instructive to apply WKB theory first to the resonance region to determine whether a wave will suffer significant energy loss in propagating through the resonance. WKB theory of course is not completely accurate in that it will not predict reflection or mode conversion although it is surprisingly accurate in determining the wave energy transmission coefficient.<sup>6</sup>

#### A. Ordinary Wave

The dispersion relation for the ordinary wave propagating perpendicularly to an applied magnetic field, obtained from the Vlasov Maxwell set of equations, is given by,

$$\frac{k_{\perp}^2 c^2}{\omega^2} = 1 + \sum_n \frac{\omega_p^2}{\omega} \int d^3v \frac{J_n(k_{\perp} v_{\perp} / \Omega) v_z}{\omega - n\Omega} \frac{\partial f}{\partial v_z}, \quad (3)$$

where  $f(\mathbf{v})$  is the velocity distribution function for electrons and  $J_n$  is the ordinary Bessel function.

To determine the total amount of wave energy damping in the resonant region we assume that  $\Omega$  is a weak function of position,  $\Omega = \Omega(x)$  and integrate the imaginary part of  $k_{\perp}(x)$  as determined by Eq. (3) through the resonant region. The effect of the resonance will be treated as a perturbation to the cold dispersion relation. For example near the first cyclotron harmonic we obtain from Eq. (3) by expanding the Bessel functions for small arguments

$$\frac{k_{\perp}^2 c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \left[ 1 + \frac{k_{\perp}^2 v_e^2}{4\Omega^2} \frac{\omega}{\omega - \Omega} \right]. \quad (4)$$

Taking  $\Omega(x)$  to be  $\Omega(x) = \omega \left( 1 - \frac{x}{L} \right)$  where  $L$  is the scale length for variation of the magnetic field, we find

$$\frac{k_{\perp r}^2 c^2}{\omega^2} \approx 1 - \frac{\omega_p^2}{\omega^2},$$

and

$$\frac{2k_{\perp r} k_{\perp i}(x) c^2}{\omega^2} \approx -\text{Im} \left\{ \frac{\omega_p^2}{\omega^2} \frac{1}{4} \frac{k_{\perp r}^2 v_e^2}{\omega^2} \frac{1}{\frac{x}{L} + i\epsilon} \right\}$$

where  $k_{\perp r}$  ( $k_{\perp i}$ ) is the real (imaginary) part of  $k_{\perp}$ , and  $\epsilon$  is a positive infinitesimal resulting from the causality condition  $\text{Im}(\omega) > 0$ . The wave energy transmission coefficient  $T$  is given by

$$T = \exp \left\{ \int_{-\infty}^{\infty} dx 2k_{\perp i}(x) \right\} = \exp(-2\pi\eta_a) \quad (5)$$

where

$$\eta_a = \frac{L}{4} \frac{\omega_p^2}{\omega^2} \left[ 1 - \frac{\omega_p^2}{\omega^2} \right]^{1/2} \frac{\omega}{c} \frac{T_e}{m_e c^2}$$

and  $T_e = \frac{1}{2} m_e v_e^2$ . The wave transmission coefficient at first glance would appear to be nearly unity (no wave absorption) owing to the smallness of  $T_e/(m_e c^2)$ . However, the quantity  $L\omega/c$  can be

quite large indicating small wave transmission and potentially high wave absorption. In a plasma with the following parameters  $L = 130$  cm,  $B = 35$  KG,  $T_e = 2$  KeV,  $n = 0.5 \times 10^{14} \text{cm}^{-3}$ , we find  $T = 5.3 \times 10^{-3}$  indicating negligible wave transmission.

If the same analysis is applied to the ordinary wave at the second harmonic we find a similar expression

$$T = \exp(-2\pi\eta_b),$$

where

$$\eta_b = \frac{L\omega}{c} \left[ 1 - \frac{\omega_p^2}{\omega^2} \right]^{3/2} \frac{\omega_p^2}{\omega^2} \left( \frac{T_e}{m_e c^2} \right)^2 \quad (6)$$

the expression for  $\eta_b$  in Eq. (6) is smaller than that in Eq. (5) for  $n_a$  by a factor of order  $T_e/m_e c^2$  thus the transmission coefficient at the second harmonic is nearly unity and wave absorption and transformation are not expected to be important in this case.

### B. Extra Ordinary Mode

For purely perpendicular propagation the dispersion equation for the extraordinary mode is

$$\begin{bmatrix} \frac{1}{2} \frac{k_x^2 c^2}{\omega^2} - K_r & -\frac{1}{2} \frac{k_x^2 c^2}{\omega^2} - K_{rl} \\ -\frac{1}{2} \frac{k_x^2 c^2}{\omega^2} - K_{rl} & \frac{1}{2} \frac{k_x^2 c^2}{\omega^2} - K_l \end{bmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} = 0 \quad (7)$$

where  $E_{\pm} = 2^{-1/2}(E_x \pm iE_y)$ , the wave electric field has Cartesian components  $\mathbf{E} = (E_x, E_y, 0)$ , the wave vector has components  $\mathbf{k} = (k_x, 0, 0)$  and the applied magnetic field is in the  $\hat{z}$  direction. The matrix elements  $K_{r,l}$  and  $K_{rl}$  contain the response of the electron plasma and are given by,<sup>4</sup>

$$K_{r,l} = 1 - i \frac{\partial}{\partial \mu} \left\{ \frac{\omega_p^2}{\omega \Omega} \mu \int_0^\infty d\eta \exp \left[ -i \left( \frac{\omega}{\Omega} \pm 1 \right) \eta - \mu \frac{b^2}{2} (1 - \cos \eta) \right] \right\} \Big|_{\mu=1}$$

$$K_{rl} = -i \frac{\partial}{\partial \mu} \left\{ \frac{\omega_p^2}{\omega \Omega} \int_0^\infty d\eta \exp \left[ -i \frac{\omega}{\Omega} \eta - \mu \frac{b^2}{2} (1 - \cos \eta) \right] \right\} \Big|_{\mu=1}, \quad (8)$$

where  $b = k_x v_e / \Omega$ , ( $T_e = \frac{1}{2} m v_e^2$ ), and we have assumed a Maxwellian distribution of electron velocities. For our purposes here it is only necessary to have approximations for  $K_{r,l}$  and  $K_{rl}$  valid for  $k^2 v_e^2 \ll \omega^2$ .

These approximations are found to be,<sup>4</sup>

$$K_{r,l} = 1 - \frac{\alpha^2}{1 \pm \beta} \left[ 1 + \frac{k_x^2 c^2}{\omega^2} \frac{\delta}{1 \pm 2\beta} \right] - \frac{3}{32} \left[ \frac{k_x^4 c^4}{\Omega^4} \frac{\delta^2}{1 \pm \beta} \right] \quad (9a)$$

$$K_{rl} = -\frac{1}{2} \alpha^2 \left( \frac{k_x c}{\omega} \right)^2 \frac{\delta}{1 - \beta^2}, \quad (9b)$$

where  $\alpha^2 = \omega_p^2 / \omega^2$ ,  $\beta = \Omega / \omega$ , and  $\delta = 2T_e / (m_e c^2)$ . The first terms of Eq's. (9a) and (9b) are correct to first order in  $\delta$ . When the dispersion equation, Eq. (7), is solved near the first harmonic resonance due to cancellations it is necessary to retain terms that are second order in  $\delta$ . Thus we have included the last term in Eq. (9a) which is second order in  $\delta$ .

The analyses of the preceding section can be carried out in a similar fashion to determine the wave energy transmission coefficient for the extraordinary wave at the first and second electron cyclotron harmonic. We find  $T = \exp(-2\pi n)$  where at the first harmonic

$$n = \frac{5}{8} \frac{\omega_p^2}{\omega^2} \left( \frac{T_e}{m_e c^2} \right)^2 \frac{L\omega}{c} \left( 2 - \frac{\omega_p^2}{\omega^2} \right)^{3/2}$$

and at the second harmonic,

$$\eta = \frac{1}{3} \frac{L\omega}{c} \left( \frac{3 - 2\alpha^2}{1 - 2\alpha^2} \right) \left( \frac{4(1 - \alpha^2)^2 - 1}{3 - 4\alpha^2} \right)^{3/2} \frac{T_e}{m_e c^2},$$

where  $\alpha^2 = \omega_p^2/\omega^2$ .

In contrast to the case of the ordinary wave the wave transmission for perpendicular propagation is nearly unity at the first harmonic and possibly low at the second harmonic.

The WKB treatment of wave absorption at a cyclotron harmonic for arbitrary angle of incidence, i.e. arbitrary  $k_z$ , can be formulated in the following way. We wish to solve the matrix equation  $\mathbf{D}(\mathbf{k}, x) \cdot \mathbf{E} = 0$  for  $k_x(x)$ , where  $\mathbf{D}(\mathbf{k}, x)$  is the local dielectric tensor  $\mathbf{D} = \mathbf{k}_x \mathbf{k}_x - (\omega^2/c^2)\mathbf{I} + (4\pi i\omega/c^2)\sigma$  and  $\sigma$  is the conductivity tensor.<sup>4</sup> In the limit  $v_e^2/c^2 \ll 1$ ,  $\mathbf{D}$  may be approximated by the cold dielectric tensor  $\mathbf{D}_0$  plus a resonant term  $\mathbf{D}_1$ , which results from finite Larmor radius effects. The tensor  $\mathbf{D}_1$  will only be important near a resonance (it contains terms that vary as  $x^{-1}$  where  $x = 0$  is the location of the resonance) and we will treat  $\mathbf{D}_1$  as a perturbation. The solution for the perpendicular wave number  $k_x(x)$ , and the electric field,  $\mathbf{E}(x)$ , will differ slightly from their values predicted by cold theory  $k_x = k_0 + k_1$ ,  $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ , where  $\mathbf{D}_0(k_0, x) \cdot \mathbf{E}_0 = 0$ .

We Taylor expand  $\mathbf{D}$  in  $k_x$  about  $k_0$  and multiply the dispersion equation on the left by  $\mathbf{E}_0^+$  and obtain an expression for  $k_1$

$$k_1(x) = \mathbf{E}_0^+ \cdot \mathbf{D}_1(k_0, x) \cdot \mathbf{E}_0 \left[ \mathbf{E}_0^+ \cdot \frac{\partial}{\partial k_0} \mathbf{D}_0(k_0) \cdot \mathbf{E}_0 \right]^{-1}.$$

the wave transmission will then be given by  $T = \exp(-2\pi\eta)$ , where

$$\eta = \frac{1}{\pi i} \int_{-\infty}^{\infty} dx k_1(x).$$

Calculations of this type have shown that the extraordinary wave can be absorbed at the first harmonic for sufficiently large  $k_z^7$ .

#### IV. PERTURBED CURRENT DENSITY IN AN INHOMOGENEOUS MAGNETIC FIELD

In this section we begin the "full wave" analysis of electromagnetic wave propagation near a cyclotron harmonic resonance that will provide us with connection formulas, linking the various WKB solutions that are valid far from a resonance. Our approach will be to assume the wave electric field can be expressed as a complex Fourier integral, then determine the response of a plasma from the linearized Vlasov equation.

Substituting the given form of the electric field and the perturbed current density in Maxwell's equations will yield an integral equation in  $k$ -space which we will solve in later sections. Once the integral equation in  $k$ -space is solved for the Fourier representation of the wave electric field the connection formulas are obtained by asymptotically expanding the Fourier inversion of the  $k$ -space representation of the electric field.

Our model of the plasma is the following. We assume that the applied magnetic field points in the  $z$  direction and depends on  $x$ . The density in general is also a function of  $x$ ; however, near a cyclotron harmonic resonance the important spatial dependence is that of the magnetic field and hence the variation of the density near a resonance will be neglected. We take the electric field to be of the form,

$$\hat{\mathbf{E}}(x) = \int_L \frac{dk_x}{2\pi} \mathbf{E}(k_x) \exp(ik_x x + ik_z z - i\omega t), \quad (10)$$

where we have neglected variations of the wave quantities in the  $y$  direction, and  $L$  is a contour in the complex  $k_x$ -plane that will be specified later.

In order to determine the response of the electrons we solve the Vlasov equation using the method of characteristics,<sup>5</sup> that is,

$$f_1 = - \int_{-\infty}^t \frac{q}{m} \hat{\mathbf{E}} \cdot \frac{\partial f_0}{\partial \mathbf{v}} dt'$$

where  $f_1$  ( $f_0$ ) is the perturbed (unperturbed) distribution function,  $q$  ( $m$ ) is the particle charge (mass) and the quantities in the integrand are evaluated along the unperturbed orbits of the charged particles. The applied magnetic field is assumed to have a linear dependence on  $x$  in the immediate vicinity of the cyclotron resonance,  $B \simeq B_0(1 - x/L)$ , where  $L$  is the scale length of the magnetic field and is assumed to be much larger than either the gyroradius,  $v_e/\Omega$ , or the vacuum wavelength  $c/\omega$ .

The particle orbits to lowest order in  $L^{-1}$  are given by,

$$x'(\tau) - x \simeq (v_{\perp}/\Omega) [\sin(\Omega\tau - \theta) + \sin\theta], \quad (11a)$$

$$y'(\tau) - y \simeq (v_{\perp}/\Omega) [\cos(\Omega\tau - \theta) - \cos\theta] + \frac{v_{\perp}^2\tau}{2\Omega L}, \quad (11b)$$

$$v'_x(\tau) \simeq v_{\perp} \cos(\Omega\tau - \theta), \quad (11c)$$

$$v'_y(\tau) \simeq v_{\perp} \sin(\Omega\tau - \theta), \quad (11d)$$

where  $\Omega = \Omega_0[1 - x/L - v_{\perp}/(\Omega_0 L) \sin\theta]$  and  $\Omega_0 = qB_0/(mc)$ .  $\Omega$  is the cyclotron frequency of a particle with its guiding center located at  $x + (v_{\perp}/\Omega_0) \sin\theta$ . It may appear at first that it is permissible to neglect the  $(v_{\perp}/\Omega_0 L) \sin\theta$  dependence of  $\Omega$  owing to the smallness of the gyroradius compared to the scale length. However, this term is found to have an important effect near a resonance,  $\omega = n\Omega_0(1 - x/L)$ .

To simplify our analysis we switch from fixed Cartesian coordinates to rotating<sup>4</sup> coordinates,

$$E_x = 2^{-1/2}(E_+ + E_-),$$

$$E_y = -i2^{-1/2}(E_+ - E_-),$$

$$v'_x = v_{\perp} \cos\phi', v'_y = v_{\perp} \sin\phi', \phi' = \theta - \Omega\tau,$$

where we have neglected the diamagnetic drift in Eq. (11b). Of interest is  $J = i4\pi q\bar{v}/\omega$ , where  $\bar{v}$  is the perturbed current density,

$$\hat{j} = -i \frac{\omega_p^2}{\omega} \int d^3v \int_{-\infty}^t dt \hat{\mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{v}}.$$

Expressed in rotating coordinates we have

$$\hat{j}(x) = \int_L \frac{dk_x}{2\pi} \hat{\Sigma}(k_x, x) \cdot \mathbf{E}(k_x) \exp[ik_x x + ik_z z - \omega t], \quad (12)$$

where,

$$\hat{\Sigma} = -\frac{i\omega_p^2}{2\omega} \int v_{\perp} dv_{\perp} du d\theta \int_{-\infty}^0 d\tau \exp[i\hat{b}(\sin(\Omega\tau - \theta) + \sin\theta) - i\omega\tau + ik_z u\tau]$$

$$\times \begin{bmatrix} v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} \exp(i\Omega\tau) & v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} \exp[i(2\theta - \Omega\tau)] & 2^{1/2} v_{\perp} \frac{\partial f_0}{\partial u} \exp(i\theta) \\ v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} \exp(i\Omega\tau - 2i\theta) & v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} \exp(-i\Omega\tau) & 2^{1/2} v_{\perp} \frac{\partial f_0}{\partial u} \exp(-i\theta) \\ 2^{1/2} u \frac{\partial f_0}{\partial v_{\perp}} \exp[i(\Omega\tau - \theta)] & 2^{1/2} u \frac{\partial f_0}{\partial v_{\perp}} \exp[-i(\Omega\tau - \theta)] & 2u \frac{\partial f_0}{\partial u} \end{bmatrix}$$

and  $\hat{b} = k_x v_{\perp} / \Omega$ . Here, the  $x$  dependence of  $\hat{\Sigma}$  is a result of the  $x$  dependence of  $\Omega$ , otherwise the conductivity tensor  $\Sigma$  is the same as for a homogeneous plasma.<sup>4</sup>

We adopt the notation of Allis<sup>4</sup> for the various elements of the tensor  $\hat{\Sigma}$

$$\hat{\Sigma} = \begin{bmatrix} \hat{\Sigma}_r & \hat{\Sigma}_{rl} & \hat{\Sigma}_{rp} \\ \hat{\Sigma}_{lr} & \hat{\Sigma}_l & \hat{\Sigma}_{lp} \\ \hat{\Sigma}_{pr} & \hat{\Sigma}_{pl} & \hat{\Sigma}_p \end{bmatrix}$$

We now evaluate  $\hat{\Sigma}_r$ ,

$$\hat{\Sigma}_r = -\frac{i\omega_p^2}{2\omega} \int v_{\perp} dv_{\perp} d\theta du v_{\perp} \frac{\partial f_0}{\partial v_{\perp}} \int_{-\infty}^0 d\tau \exp[i\hat{b}(\sin(\Omega\tau - \theta) + \sin\theta) + (\Omega + k_z u - \omega)\tau]$$

Using the Bessel identity  $\exp(i\hat{b} \sin(\Omega\tau - \theta)) = \sum_n J_n(\hat{b}) \exp(in(\Omega\tau - \theta))$  the integration over  $\tau$  may be performed to yield,

$$\hat{\Sigma}_r = -\frac{\omega_p^2}{2\omega} \int v_\perp^2 dv_\perp d\theta du \frac{\partial f_o}{\partial v_\perp} \sum_n \frac{J_n(\hat{b}) \exp[i(\hat{b} \sin \theta - m\theta)]}{(n+1)\Omega + k_z u - \omega}. \quad (13)$$

Here  $\Omega$  is a weak function of  $x, v_\perp$ , and  $\theta$ . We will treat  $\Omega$  as a constant everywhere except when it appears in a resonant denominator. That is, if  $\omega = (m+1)\Omega_o$  we will retain the  $x, v_\perp$ , and  $\theta$  dependences of  $\Omega$  in the denominator of the  $m^{\text{th}}$  term of the sum in Eq. (13), and everywhere else we will replace  $\Omega$  by  $\Omega_o$ . With this approximation  $\hat{\Sigma}_r$  divides naturally into two parts; a part which depends on  $x, \hat{\Sigma}_{r1}$ , (the  $m^{\text{th}}$  term of the sum) and a part which is independent of  $x, \hat{\Sigma}_{r0}$  (the remaining terms of the sum).  $\hat{\Sigma}_{r0}$  may be evaluated in the same fashion as one would evaluate an element of the homogeneous plasma conductivity tensor. The evaluation of  $\Sigma_{r1}$  proceeds as follows:

$$\Sigma_{r1} = +\frac{\omega_p^2}{2\omega^2} \int v_\perp^2 dv_\perp du d\theta \frac{\partial f_o}{\partial v_\perp} \frac{J_m(\hat{b}) L \exp[i(\hat{b} \sin \theta - m\theta)]}{x + (v_\perp/\Omega_o) \sin \theta + Lk_z u/\omega + i\epsilon L}, \quad (14)$$

where  $\epsilon$  is a positive infinitesimal resulting from the causality condition.

The  $x$  dependence of  $\hat{\Sigma}_{r1}$  can be expressed as the inversion of a Fourier transform,

$$\hat{\Sigma}_{r1} = \int_0^\infty dk_1 \exp(ik_1 x) \Sigma_{r1}(k_x, k_1)$$

where

$$\begin{aligned} \Sigma_{r1}(k_x, k_1) &= \frac{i\omega_p^2}{2\omega^2} \int v_\perp^2 dv_\perp du d\theta \frac{\partial f_o}{\partial v_\perp} L J_m(\hat{b}) \\ &\times \exp[+ik_1((v_\perp/\Omega_o) \sin \theta + Lk_z u/\omega) + i(\hat{b} \sin \theta - m\theta)]. \end{aligned}$$

Using the Bessel identity we find,

$$\Sigma_{r1} = \frac{\pi L \omega_p^2 i}{\omega^2} \int v_\perp^2 dv_\perp du \frac{\partial f_o}{\partial v_\perp} \exp(ik_1 Lk_z u/\omega) J_m(\hat{b}) J_m(\hat{b} + \hat{b}_1)$$

where  $\hat{b} = k_1 v_{\perp} / \Omega_o$ .  $\Sigma_{r1}$  can be further reduced when  $f_o$  is specified. If  $f_o$  is Maxwellian ( $f_o = \pi^{-3/2} v_e^{-3} \exp(-v^2/v_e^2)$ ) we find

$$\Sigma_{r1} = \frac{\pi i L \omega_p^2}{\omega^2} \frac{\partial}{\partial \mu} \mu \exp \left[ -\frac{1}{4} \mu (b_1^2 + 2bb_1 + 2b^2) \right] I_m \left[ \frac{1}{2} \mu b (b + b_1) \right] \times \exp(-a) \quad (15)$$

where  $b = k_x v_e / \Omega_o$ ,  $b_1 = k_1 v_e / \Omega_o$ , and  $a = \frac{1}{4} (k_1 k_z L v_e / \omega)^2$ . If  $f_o$  has a Maxwellian dependence on perpendicular velocity and a Lorentzian dependence on parallel velocity, ( $f_o = \pi^{-2} v_e^{-1} \exp(-v_{\perp}^2/v_e^2) (u^2 + v_e^2)^{-1}$ ) we find the  $\Sigma_{r1}$  is given by Eq. (15) with  $a = k_1 k_z L v_e / \omega$ . This latter form of  $\Sigma_{r1}$  will be useful in determining the effects of nonzero  $k_z$  on wave absorption and mode conversion.

Finally, it remains to write down an expression for the perturbed current,

$$J = \int_L \frac{dk_x}{2\pi} \exp(ik_x x) [\hat{\Sigma}_o(k_x) \cdot \mathbf{E}(k_x) + \hat{\Sigma}_1(k_x, x) \cdot \mathbf{E}(k_x)]$$

writing  $\hat{\Sigma}_1$  in terms of its Fourier inversion, and changing variables of integration we find

$$J = \int_L \frac{dk_x}{2\pi} \exp(ik_x x) \left[ \hat{\Sigma}_o(k_x) \cdot \mathbf{E} + \int_{-\infty}^{k_x} dk' \Sigma_1(k', k_x - k') \cdot \mathbf{E}(k') \right] \quad (16)$$

where the first elements of  $\Sigma_o$  and  $\Sigma_1$  have been determined in this section. The remaining elements can be obtained using the same technique.

## V. EXTRAORDINARY WAVE AT THE SECOND HARMONIC

The full matrix equation for wave propagation near a cyclotron harmonic in an inhomogeneous magnetic field is obtained by substituting  $J$  as given by Eq. (16) in Maxwell's equations. The result is three coupled integral equations for the three components of  $\mathbf{E}(k_x)$ . To analyze these equations we assume nearly perpendicular propagation  $k_z c / \omega \ll 1$ . In this limit

the ordinary and extraordinary waves decouple. That is, the propagation of the ordinary wave is governed by a single integral equation for  $E_z(k_x)$  and the propagation of the extraordinary wave is governed by two coupled integral equations for  $E_+(k_x)$  and  $E_-(k_x)$ . The effects of small  $k_z$  are retained in the extraordinary wave calculation by including  $k_z$  in the evaluation of the matrix elements of  $\Sigma$ . What we find is that for sufficiently large  $k_z, 1 \gg k_z c / \omega > c / (v_e k_\perp L)$  the results of simple WKB theory become valid. This occurs when the spatial width of the cyclotron resonance ( $|\omega - (m + 1)\Omega(x)| \approx k_z v_e$ ) is comparable to the perpendicular wave length ( $k_\perp$ ) predicted by cold plasma theory. Thus, departures from simple WKB theory occur only for the case of nearly perpendicular propagation.

We consider the propagation of the extraordinary wave near the second harmonic in the limit  $v_e^2 \ll c^2$ . In this limit the only resonant term is  $\Sigma_{r1}$ . The remaining nonresonant terms  $\Sigma_o$  can be approximated by their cold plasma limits. Thus Maxwell's equations reduce to

$$0 = \int_L \frac{dk_x}{2\pi} \exp(ik_x x) \left\{ \left[ \frac{1}{2} \frac{k_x^2 c^2}{\omega^2} - 1 - \Sigma_{r0} \right] E_+(k_x) - \frac{1}{2} \frac{k_x^2 c^2}{\omega^2} E_-(k_x) - 2iL\alpha^2 \delta \int_{-\infty}^{k_x} dk' \frac{k_x k' c^2}{\omega^2} \exp[d' - d] E_+(k') \right\} \quad (17a)$$

$$0 = \int_L \frac{dk_x}{2\pi} \exp(ik_x x) \left\{ -\frac{1}{2} \frac{k_x^2 c^2}{\omega^2} E_+(k_x) + \left[ \frac{1}{2} \frac{k_x^2 c^2}{\omega^2} - 1 - \Sigma_{l0} \right] E_-(k_x) \right\} \quad (17b)$$

where  $\Sigma_{r0} = -\alpha^2 / (1 - \beta)$ ,  $\Sigma_{l0} = -\alpha^2 / (1 + \beta)$ ,  $d = k_x k_z L v_e / \omega$ , and  $d' = k' k_z L v_e / \omega$ , and we have assumed a Lorentzian distribution in axial velocity. Equations (17a) and (17b) are satisfied when the integrands vanish. Thus, we can determine  $E_-(k_x)$  in terms of  $E_+(k_x)$  from Eq. (17b) and substitute in (17a) to obtain the single equation,

$$\epsilon(k_x) E_+(k_x) \exp(d) / k_x - iL \int_{-\infty}^{k_x} dk' G(k') E_+(k') \exp(d') / k' = 0, \quad (18)$$

where

$$\epsilon(k_x) = \left( \frac{1}{2} k_x^2 c^2 / \omega^2 - 1 - \Sigma_{ro} \right) - \left( \frac{1}{2} k_x^2 c^2 / \omega^2 \right)^2 / \left( \frac{1}{2} k_x^2 c^2 / \omega^2 - 1 - \Sigma_{lo} \right),$$

and

$$G(k') = 2^2 \alpha^2 \delta k'^2 c^2 / \omega^2$$

By differentiating Eq. (18) with respect to  $k_x$  a first order differential equation is obtained, its solution is found to be,

$$E_+(k_x) = k_x \exp \left[ -d + iL \int_0^{k_x} dk' G(k') / \epsilon(k') \right] / \epsilon(k_x) \quad (19)$$

which may be inverted to yield,

$$E_+(x) = \int_L \frac{k_x dk_x}{2\pi} \exp \left[ ik_x x - d + iL \int_0^{k_x} dk' G(k') / \epsilon(k') \right] / \epsilon(k_x). \quad (20)$$

Here we see the quantity  $d = Lk_x k_z v_e / \omega$  in the exponent of Eq. (20) serves to shift the  $x$  axis by an amount  $ik_z v_e L / \omega$ . So, we can eliminate  $d$  by making the transformation  $x \rightarrow x + ik_z v_e L / \omega$ . In what follows we will let  $d \rightarrow 0$  knowing that we can recover the effects of finite  $d$  by shifting the  $x$  axis.

In order to determine the wave transmission, reflection, and conversion coefficients it is necessary to asymptotically evaluate Eq. (20) for large positive and negative  $x$ . Before specifying the contour it is helpful to see what sort of asymptotic behavior can occur. There are two branch points of  $E_+(k_x)$  both where  $\epsilon = 0$ . They are at

$$\frac{k_o^2 c^2}{\omega^2} = \frac{\beta^2 - (1 - \alpha^2)^2}{\beta^2 + \alpha^2 - 1}, \quad (21)$$

which are the two extraordinary modes in a cold plasma. In addition there are two saddle point contributions from saddle points at

$$\frac{k_{xs}c}{\omega} = \pm \frac{\left(1 - \frac{4}{3}\alpha^2\right)^{1/2}}{\alpha\delta^{1/2}} \left(\frac{x + ik_zLv_e}{\omega}\right)^{1/2} \quad (22)$$

The steepest descent path makes an angle of  $135^\circ$  with the positive  $x$ -axis. These are electrostatic waves which can propagate only for  $x > 0$ .

For large  $k_x$ ,  $\epsilon$  is independent of  $k_x$  and  $G$  is proportional to  $k_x^2$ . Thus the integrand is evanescent as  $|k_x| \rightarrow \infty$  and  $Im(k_x^3) < 0$ .

Two possible contours for the evaluation of Eq. (20) are shown in Fig. 2 and are labeled  $L_1$  and  $L_2$ . Both contours terminate at  $|k_x| = \infty$  in the sector of the  $k_x$ -plane for which  $0 > arg(k_x) > -\pi/3$ . The contour  $L_1$  loops around the branch point at  $+k_o$  and the contour  $L_2$  loops around the branch point at  $-k_o$ . For  $x > 0$  both contours can be deformed so as to pass through the saddle point at  $+k_s$  along the path of steepest descent. Thus, as  $|x| \rightarrow \infty$  the solution obtained from  $L_1$  which we call  $E_1(x)$ , will contain a contribution from the branch point at  $k_o$  (an extraordinary wave with  $v_{gx} = \partial\omega/\partial k_x > 0$ ) and a contribution from the saddle point (an electrostatic wave with  $v_{gx} > 0$ ). The solution obtained from  $L_2$ , which we call  $E_2$ , will contain a contribution from the branch point at  $-k_o$  (an extraordinary wave with  $v_{gx} < 0$ ) and a contribution from the saddle point (an electrostatic wave with  $v_{gx} > 0$ ).

For  $x < 0$  the two contours are deformed as shown in Fig. 3. Contour  $L_2$  passes through the saddle point now located on the imaginary  $k_x$  axis. However, the contribution from this saddle point is evanescent as  $x \rightarrow -\infty$  and need not be considered. Thus, only contributions from the branch points need be considered for large negative  $x$ .  $E_1(x)$  will contain a contribution from  $k_o$  (an extraordinary wave with  $v_{gx} > 0$ ) and  $E_2(x)$  will contain contributions from both branch points (extraordinary waves with  $v_{gx} < 0$ ). Therefore the solution  $E_1(x)$  applies for the case in which an extraordinary wave is incident from the high

magnetic field side ( $x = -\infty$ ) of the resonance. The solution  $E_2(x)$  contains two waves incident on the resonance and in itself does not apply to a case of physical interest. However an appropriate linear combination of  $E_1$  and  $E_2$  can be formed which applies to the case of an extraordinary wave incident from the low field side of the resonance.

The details of the asymptotic evaluation of  $E_1(x)$  and  $E_2(x)$  are given in the Appendix. We quote here the results: For  $x > 0$  and  $|x| \rightarrow \infty$ ,

$$E_1(x) \sim k_o [1 - \exp(-2\pi\eta_o)] \Gamma(i\eta_o) \exp(\pi\eta_o/2) \Psi_{+o} / \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] - k_s \sinh \pi\eta_o \left[ 2 / (\pi\epsilon(k_s)) \frac{\partial}{\partial k_s} (\epsilon(k_s)x + G(k_s)L) \right]^{1/2} \exp(i\theta) \quad (23a)$$

$$E_2(x) \sim k_o [1 - \exp(2\pi\eta_o)] \Gamma(-i\eta_o) \exp(-3\pi\eta_o/2) \Psi_{-o} / \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] + k_s \sinh \pi\eta_o \left[ 2 / (\pi\epsilon(k_s)) \frac{\partial}{\partial k_s} (\epsilon(k_s)x + G(k_s)L) \right]^{1/2} \exp(i\theta), \quad (23b)$$

and for  $x < 0$  and  $|x| \rightarrow \infty$

$$E_1 \sim k_o [1 - \exp(-2\pi\eta_o)] \Gamma(i\eta_o) \exp(3\pi\eta_o/2) \Psi_{+o} / \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] \quad (24a)$$

$$E_2 \sim k_o [1 - \exp(-2\pi\eta_o)] \Gamma(-i\eta_o) \exp(-5\pi\eta_o/2) \Psi_{-o} / \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] + [1 - \exp(-2\pi\eta_o)] E_1, \quad (24b)$$

where  $\Gamma(x)$  is the gamma function,  $\eta_o = LG(k_o) / \left[ \frac{\partial}{\partial k_o} \epsilon(k_o) \right]$ ,  $\Psi_{\pm o} = \exp(\pm ik_o x \pm iH(k_o) \pm i\eta_o \ln |k_o x|)$   $\theta = i \int_0^x dx' k_s(x')$ , and  $k_o$  and  $k_s$  are given in Eq. (21) and Eq. (22).

The real function  $H(k_o)$  is defined in the appendix.

The asymptotic values of  $E_1(x)$  and  $E_2(x)$  are seen to be linear combinations of the various WKB solutions of the local dispersion relation. Thus, Eq.'s (23) and (24) provide connection formulas linking the WKB solutions on either side of a resonant surface. To determine the energy transmission and reflection coefficients we must determine the energy flux of each of the WKB solutions in the asymptotic expansions. The energy flux of a wave in a weakly inhomogeneous plasma is given by Stix<sup>5</sup>,

$$P_x = -\frac{\omega}{16\pi} \mathbf{E} \cdot \frac{\partial}{\partial k_x} \mathbf{D} \cdot \mathbf{E}, \quad (25)$$

where  $\mathbf{D} = \frac{c^2}{\omega^2} \mathbf{k}_x \mathbf{k}_x - \mathbf{I} + \Sigma$ . In our case Eq. (25) reduces to

$$P = -\frac{\omega}{16\pi} |E_+(x)|^2 \frac{\partial}{\partial k_1} \left[ \epsilon(k_x) + \frac{L}{x} G(k_x) \right], \quad (26)$$

where  $k_x = k_o$  for an extraordinary wave and  $k_x = k_s(x)$  for an electrostatic wave.

#### A. Wave Incident from the High Field Side

With the definition of the wave energy flux given by Eq. (26) we can calculate the energy transmission and reflection coefficients for the solution  $E_1(x)$ , which applies in the situation of an extraordinary wave incident on the resonance from the high field side. The ratio of transmitted to incident power is found from Eq. (23a) and Eq. (24a) to be,

$$T_1 = \exp(-2\pi\eta_o). \quad (27)$$

This is precisely the result predicted by WKB theory. The ratio of mode converted electrostatic wave energy to incident wave energy is found, after some algebra, to be,

$$F_1 = (1 - T_1). \quad (28)$$

Thus, all the energy that WKB theory predicts to be absorbed is in fact mode converted to the electrostatic wave. If we now reinsert the effects of finite  $k_z$  by shifting the  $x$ -axis, per the

discussion following Eq. (20), we find that Eq.'s (27) and (28) remain valid. However, the electrostatic wave is heavily damped (cf. Eq. (22)) so the net effect is the absorption by the electrons of a fraction  $F_1$  of the incident energy. In this sense WKB theory is seen to give the correct answer for  $Lk_z k_0 v_e / \omega > 1$ .

### B. Wave Incident from the Low Field Side

In order to obtain a solution which contains a single incident wave (incident from  $x = \infty$ ) we form the linear combination,

$$E_3(x) = E_2(x) - [1 - \exp(-2\pi\eta_0)]E_1(x). \quad (29)$$

From Eq.'s (24) it is seen that  $E_3(x)$  contains only a transmitted extraordinary wave for  $x < 0$ . From Eq.'s (23) it is seen that  $E_3(x)$  contains an incident and a reflected extraordinary wave and a backscattered electrostatic wave for  $x > 0$ . We define the coefficients  $T_2$ ,  $R_2$ , and  $F_2$  to be the fractions of energy transmitted, reflected, and mode converted to an electrostatic wave respectively. By application of Eq. (26) these coefficients are found to be

$$T_2 = \exp(-2\pi\eta_0), \quad (29a)$$

$$R_2 = (1 - T_2)^2, \quad (29b)$$

and

$$F_2 = T_2(1 - T_2) \quad (29c)$$

which apply for  $k_z = 0$ . It can be verified  $T_2 + R_2 + F_2 = 1$ .

Inclusion of finite  $k_z$  modifies only  $R_2$ . It is found, by shifting the  $x$ -axis, that  $R_2$  is reduced by a factor  $\exp(-2k_0 L k_z v_e / c)$  for small but finite  $k_z$ . Thus, we see that for  $k_z = 0$  all the incident energy remains in waves, but if  $k_z K_0 L v_e / c \sim 1$  a substantial portion of this energy can be transferred to particles by cyclotron damping.

## VI. ORDINARY WAVE AT THE FIRST HARMONIC

Our analysis in this section parallels that of the previous section. Here we consider the propagation of the ordinary mode in the vicinity of the first cyclotron harmonic. With  $k_z = 0$  a single integral equation describes the propagation of the ordinary mode,

$$\int_L \frac{dk_x}{2\pi} \left[ \left( \frac{k_x^2 c^2}{\omega^2} - 1 + \frac{\omega_p^2}{\omega^2} \right) E_z - iL \frac{\omega_p^2}{\omega^2} \int_{-\infty}^{k_x} dk' \frac{k_x k' v_e^2}{4\Omega_0^2} E_z(k') \right] \exp(ik_x x) = 0, \quad (30)$$

where we have assumed  $v_e^2 \ll c^2$  in evaluating the matrix element  $\Sigma_p$ . Equation (30) can be cast in the form of Eq. (18) and hence yields the solution,

$$E_z(x) = \int_L \frac{k_x dk_x}{2\pi} \exp \left[ ik_x x + iL \int_0^{k_x} dk' G(k')/\epsilon(k') \right] / \epsilon(k_x) \quad (31)$$

where

$$G(k') = \frac{1}{4} \alpha^2 v_e^2 k'^2 / \Omega_0^2,$$

and

$$\epsilon(k_x) = k_x^2 c^2 / \omega^2 - 1 + \alpha^2.$$

Two contours for the asymptotic evaluation of Eq. (31) are shown in Fig. 4. These contours are similar to those of Fig. 2 for the evaluation of Eq. (20). However, in the present case there are no saddle points, and the evanescence of the integrand for large  $|k_x|$  is determined by the  $ik_x x$  term rather than the  $\int_0^{k_x} dk' G/\epsilon$  term. Thus, for  $x > 0$  we must end the contours in the upper half of the  $k_x$ -plane and for  $x < 0$  the contours and branch cuts must be rotated to the lower half of the  $k_x$ -plane.

Figure 5 shows the contours for  $x < 0$  that are obtained when the branch cuts and contours are rotated in the clockwise sense. The proper sense of rotation is determined from the

analytic properties of  $E_z(x)$ . We see from Eq. (14) that the causality condition  $\text{Im}(\omega) > 0$  is equivalent to requiring the conductivity elements be analytic in the upper half of the complex  $x$ -plane. Thus, we seek a solution  $E_z(x)$  with this same property. That is, the analytic continuation of  $E_z(x)$  for  $x < 0$  is obtained from the solution for  $x > 0$  by increasing the argument of  $x$  from 0 to  $\pi$ . Hence, the branch cuts and contours of Fig. 4 are rotated in the clockwise sense to obtain the analytic continuation of  $E_z(x)$ .

For large  $x$  only the portions of the contours near the branch points contribute to the integrals. Thus, we may write down the asymptotic expressions for  $E_{1z}(x)$  and  $E_{2z}(x)$  by analogy to Eq.'s (23) and (24),

$$E_1(x) \sim k_o [1 - \exp(-2\pi\eta_o)] \Gamma(i\eta_o) \exp(\pi\eta_o/2) \psi_{+o} \left/ \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] \right. \quad (31a)$$

$$E_2(x) \sim k_o [1 - \exp(2\pi\eta_o)] \Gamma(-i\eta_o) \exp(-3\pi\eta_o/2) \psi_{-o} \left/ \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] \right. \quad (31b)$$

for  $x > 0$ , and

$$E_1 \sim k_o [1 - \exp(-2\pi\eta_o)] \Gamma(i\eta_o) \exp(3\pi\eta_o/2) \psi_{+o} \left/ \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] \right. \quad (32a)$$

$$E_2 \sim k_o [1 - \exp(2\pi\eta_o)] \Gamma(-i\eta_o) \exp(-5\pi\eta_o/2) \psi_{-o} \left/ \left[ 2\pi \frac{\partial}{\partial k_o} \epsilon(k_o) \right] \right. \quad (32b)$$

for  $x < 0$ , where  $k_o^2 c^2 / \omega^2 = 1 - \omega_p^2 / \omega^2$ .

The determination of the transmission, and reflection coefficients proceeds as before. We find for the case of an ordinary wave incident from the high magnetic field side a fraction  $T = \exp(-2\pi\eta_a)$ , where  $\eta_a = LG(k_o) / (\partial\epsilon/\partial k_o)$  is transmitted, and no energy is reflected. This is equivalent to the WKB result, Eq. (5).

In the case of a wave incident from the low field side a fraction of energy  $T = \exp(-2\pi\eta_a)$  is transmitted, and a fraction  $R = (1 - T)^2$  is reflected.

In both of these cases the energy contained in the scattered waves is less than the energy contained in the incident wave indicating the absorption of energy by the electrons. That this absorbed energy goes into increasing the motion of the electrons perpendicular to the applied magnetic field rather than the parallel motion can be seen by considering the component of canonical momentum parallel to the magnetic field. For the case of purely perpendicular propagation, which we consider here, the electromagnetic wave field can be generated by the vector potential  $\mathbf{A} = \mathbf{a}_z A(x, t)$ . Thus the  $z$  component of canonical angular momentum is conserved,  $p_z = mv_z - eA(x, t)/c$ .  $A(x, t)$  is an oscillating, bound function. Thus,  $v_z$  must be oscillating and bound as well indicating that on the average the electrons do not gain parallel energy. Energy is fed at the perpendicular motion by the  $e\mathbf{v}_z B_y/c$  force where  $B_y$  is the wave magnetic field.

If we allow  $k_z$  to be sufficiently large ( $k_z v_e/\omega \geq (k_L L)^{-1}$ ) then the width of the resonance becomes large enough that WKB theory applies. In this case the transmission coefficients remain the same, but the reflection that occurs for a wave incident from the low field side with  $k_z = 0$  should disappear. Thus the ordinary wave can be effectively absorbed at the first cyclotron harmonic ( $\omega = \Omega(x)$ ) regardless of whether it is incident from the high or low field side of the resonance.

## VII. ACCESSIBILITY OF CYCLOTRON RESONANCES AND APPLICATIONS TO HEATING IN TOROIDAL DEVICES

In the preceding sections of this paper we found two situations in which electromagnetic wave energy could be absorbed directly at a cyclotron harmonic. These situations were: an ordinary wave incident on the first cyclotron harmonic, and an extraordinary wave incident on the second harmonic. It was also found that there could be appreciable energy absorption regardless of whether the wave was incident on the resonance from the high or low magnetic field side.

The accessibility conditions of the first cyclotron harmonic for the ordinary wave are quite simple. If  $k_z = 0$  all that is required is that the plasma density be low enough so that  $\omega_{pe}^2 < \omega^2 = \Omega_e^2$  (c.f. Eq. (1)). It is anticipated that  $\omega_{pe}^2 < \Omega_e^2$  for most tokamaks so the ordinary wave shows great promise for plasma heating purposes. Allowing for nonzero  $k_z$  the accessibility condition becomes  $(\omega_{pe}^2/\omega^2) < \text{the minimum of } 1 \text{ and } 2(1 - k_z^2 c^2/\omega^2)$ .

In order for the second harmonic to be accessible to the extraordinary wave we must have  $\omega_p^2/\omega^2 = \frac{1}{4} \omega_p^2/\Omega^2 < \frac{1}{2}$  (c.f. Eq. (2)), or  $\omega_p^2 < 2\Omega^2$ . Thus the second harmonic is available for heating in higher density plasmas. Inclusion of finite  $k_z$  alters this condition slightly,  $\omega_p^2 < 2\Omega^2(1 - k_z^2 c^2/\omega^2)$ .

In Fig. 6 the three proposed methods of heating using waves in the electron cyclotron frequency range are summarized. The ray paths through the parameter plane  $(\omega_{pe}^2(x)/\omega^2, \Omega^2(x)/\omega^2)$  are traced for the ordinary wave incident on the first harmonic, the extraordinary wave incident on the second harmonic, and the extraordinary wave incident on the upper hybrid resonance. The paths of incidence from both high and low magnetic field sides are shown for the two cases of direct absorption at a cyclotron resonance considered in this paper.

#### ACKNOWLEDGMENT

This work was supported by ERDA and The National Research Council.

## Appendix

### ASYMPTOTIC EVALUATION OF $E_+(x)$

The contours for the evaluation of Eq. (20) are shown in Fig. 2. Branch cuts in the  $k_x$ -plane originate from each of the solutions of  $\epsilon(k) = 0$ , ( $= \pm k_0$ ). These branch cuts are associated with different possible paths of integration for determining the function,

$$M(k_x) = \int_0^{k_x} dk' G(k') / \epsilon(k'). \quad (\text{A1})$$

#### A. Saddle Point Contribution

The contour  $L_1$  loops around the branch point at  $k_0$  and for  $x > 0$  both the contour and branch cut are made to pass through the saddle point along the path of steepest descent. The contour passes through the saddle point twice. The value of the integrand on the upper (lower) leg of the contour is determined by choosing the path of integration in  $M(k_x)$  to pass above (below) the singularity at  $\epsilon(k_0) = 0$ .

To evaluate the saddle point contribution we define  $F$  to be the exponent of the integrand,

$$F(k_x) = ik_x x + iLM(k_x), \quad (\text{A2})$$

which we will Taylor expand to second order in  $k_x$  at the saddle point,  $k_x = k_{xs}$ . The contribution from the saddle point is therefore given approximately,

$$E_s \sim \frac{k_{xs}}{2\pi} \epsilon^{-1}(k_{xs}) \exp(F(k_{xs})) \int dk' \exp\left\{\frac{1}{2} F''(k_{xs})(k - k_{xs})^2\right\},$$

where the vanishing of the first derivative of  $F(k_x)$  defines the saddle point,

$$\frac{dF}{dk_x} = ix + iLG(k_{xs})/\epsilon(k_{xs}) = 0. \quad (\text{A3})$$

Equation (A3) can be rearranged in such a way so that it is seen to be identical to the local

dispersion relation. We define the solution of Eq. (A3) to be  $k_x = k_{xs}(x)$ .

The value of  $F(k_{xs}(x))$  must be determined for both legs of the contour,

$$F_{\pm}(k_{xs}(x)) = ik_{xs}(x)x + iLP \int_0^{k_{xs}} dk' G(k')/\epsilon(k') \pm \pi\eta_0,$$

where  $\eta_0 = LG(k_0) [\partial\epsilon(k_0)/\partial k_0]^{-1}$ , and  $+$  ( $-$ ) applies to the upper (lower) leg of the contour. By simple manipulation, and use of Eq. (A3) we find,

$$F_{\pm}(k_{xs}(x)) = Pi \int_0^x dx' k_{xs}(x') \pm \pi\eta_0.$$

Thus, the saddle point contribution will match on to the WKB solution of an electrostatic wave obtained from the local dispersion relation.

Upon expanding  $F(k_x)$  to second order in  $k_x - k_{xs}$ , performing the indicated integrations, and summing the contributions from both legs of the contour; we obtain the saddle point contribution to  $E_1(x)$  for  $x > 0$ ,

$$E_{c1} = -k_0 \sinh \pi\eta_0 \exp [i(\theta + 3\pi/4)] \times (2/(\pi\epsilon\partial(\epsilon x + LG)/\partial k_{xs}))^{1/2} \quad (\text{A5})$$

where all quantities are defined in the text. The saddle point contribution to  $E_2(x)$  is found in a similar manner to be  $-E_{c1}(x)$ .

### B. Branch Point Contribution

Let us define  $H(k_x)$  in the following way,

$$iL \int_0^{k_x} dk' G/\epsilon = iH(k_x) + i\eta_0 \int_0^{k_x} dk'(k - k')^{-1},$$

where,

$$iH(k_x) \equiv iL \int_0^{k_x} dk' \left[ \frac{G(k')}{\epsilon(k')} - \frac{G(k_0)}{\epsilon'(k_0)(k' - k_0)} \right]$$

is an analytic function of  $k_x$  near the branch point  $k_0$ . We now transform variables of integration by letting,

$$k - k_0 = u \exp(-3i\pi/2)/x. \quad (\text{A6})$$

The contour of integration in the  $u$ -plane is shown in Fig. 7. The transformation, Eq. (A6), is such that  $2\pi > \arg(u) > 0$ . Thus for  $x > 0$  and  $|x| \rightarrow \infty$  we obtain,

$$E_{B1} \approx \frac{1}{2\pi} k_0 \exp\{ik_0 x + iH_1(k_0) - i\eta_0 \ln(k_0 x) + \pi\eta_0/2\} c(\eta_0) / (\partial\epsilon/\partial k_0), \quad (\text{A7})$$

where

$$c(\eta_0) \equiv \int_c du u^{-1} \exp[-u + i\eta_0 \ln u],$$

$$c(\eta_0) = (1 - \exp(-2\pi\eta_0)) \Gamma(i\eta_0).$$

For  $x < 0$  it is necessary to let  $x = |x|e^{i\pi}$ .

In a similar manner the branch point contributions for  $E_2(x)$  can be obtained. However, there are contributions from the portions of the contour  $L_2$  that pass near the branch points at  $k_0$ . These contributions are found to be linearly proportional to the asymptotic value of  $E_1(x)$ . The final asymptotic formulas are given in Eq. (23) and Eq. (24).

#### ACKNOWLEDGMENTS

This work was supported under Dept. of Energy contract E(49-20)-1006.

#### REFERENCES

1. See for example J.L. Hirshfield and V.L. Granatstien, IEEE Transactions on Microwave Theory and Techniques, *MTT-25*, 513 (June 1977), and references contained therein.
2. R.A. Dandl, H.O. Easton, G.E. Guest, C.L. Hendrick, H. Ikegami, and D.B. Nelson, in Plas-

ma Physics and Controlled Nuclear Fusion Research (Proc. Conf. Tokyo, Japan, 1974) II, IAEA, Vienna (1975) 141.

3. V.V. Alikaev, G.A. Bobrovskii, V.I. Poznyak, K.A. Razumova, V.V. Sannikov, Yu.A. Sokolov, and A.A. Shmarin, *Sov. J. Plasma Phys.* **2**, 212 (1977).
4. W.P. Allis, S.J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas*. MIT Press, Cambridge, Mass. (1963).
5. T.H. Stix, *The Theory of Plasma Waves*, McGraw-Hill, New York, N.Y. (1962)
6. K.G. Budden, *Radio Waves in the Ionosphere*, Cambridge University Press, 1961, Ch. 21.15.
7. O. Eldridge and Won Namkung, *Bull. Am. Phys. Soc.*, **21**, 1173 (Oct. 1976).

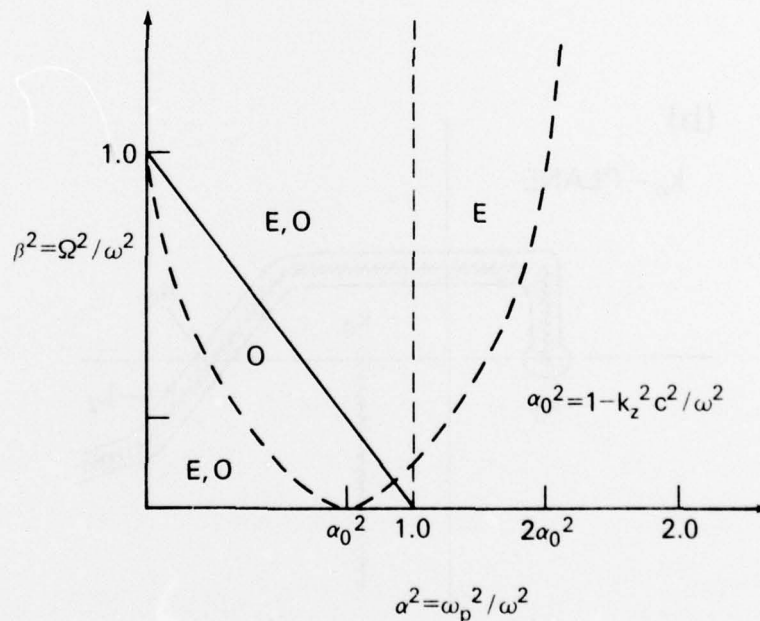


Figure 1 — Allis diagram for electromagnetic wave propagation with fixed  $k_z$  ( $k_z \parallel \mathbf{B}$ ) in a cold plasma. The solid line is the upper hybrid resonance. The dashed lines are various cut-offs. In regions containing the letter E(O) the extraordinary (ordinary) wave propagations.



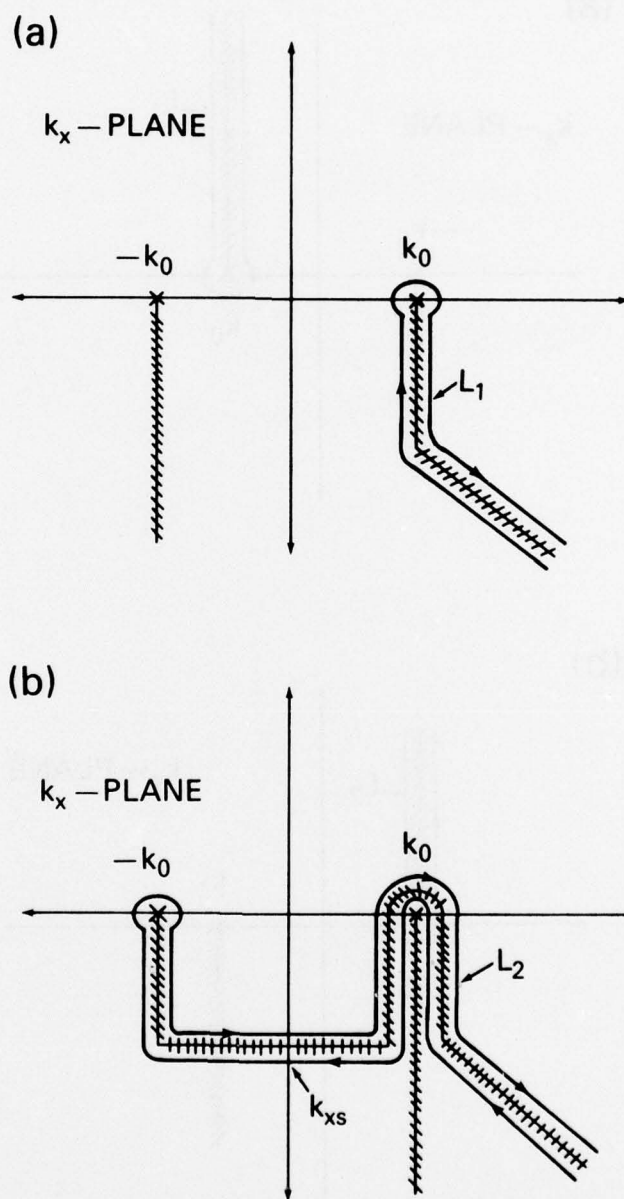


Figure 3 — The deformation of the two contours of Fig. 2 for the evaluation of Eq. (20) for  $x < 0$ .

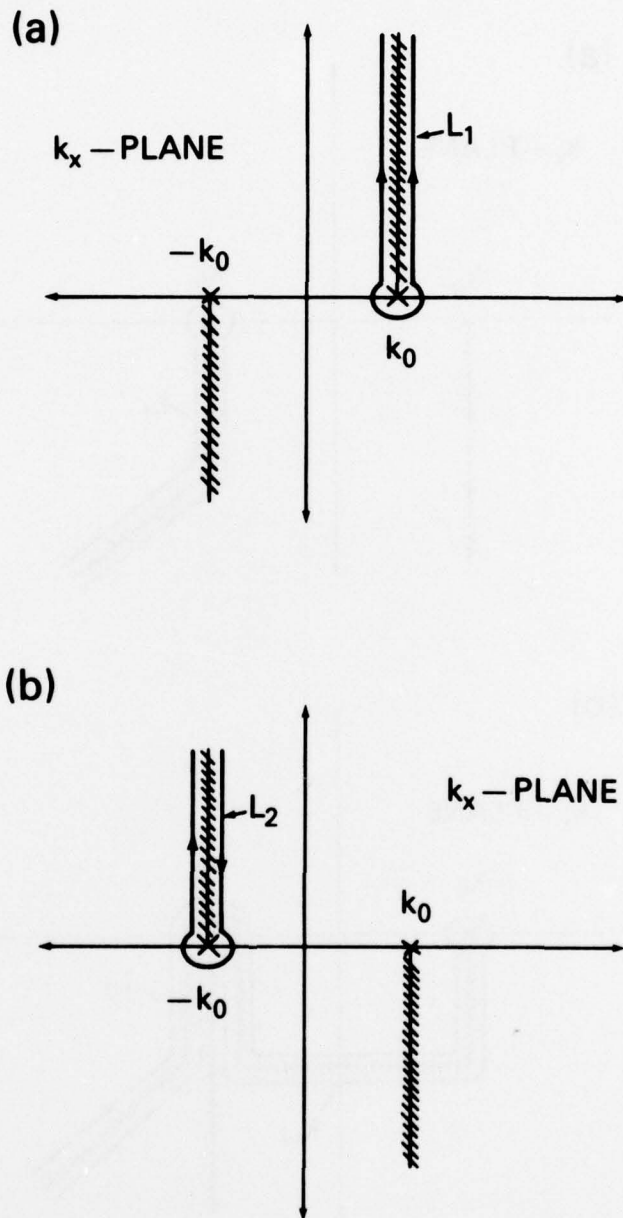


Figure 4 — Two contours in the  $k_x$ -plane for the evaluation of Eq. (31) for  $x > 0$ .

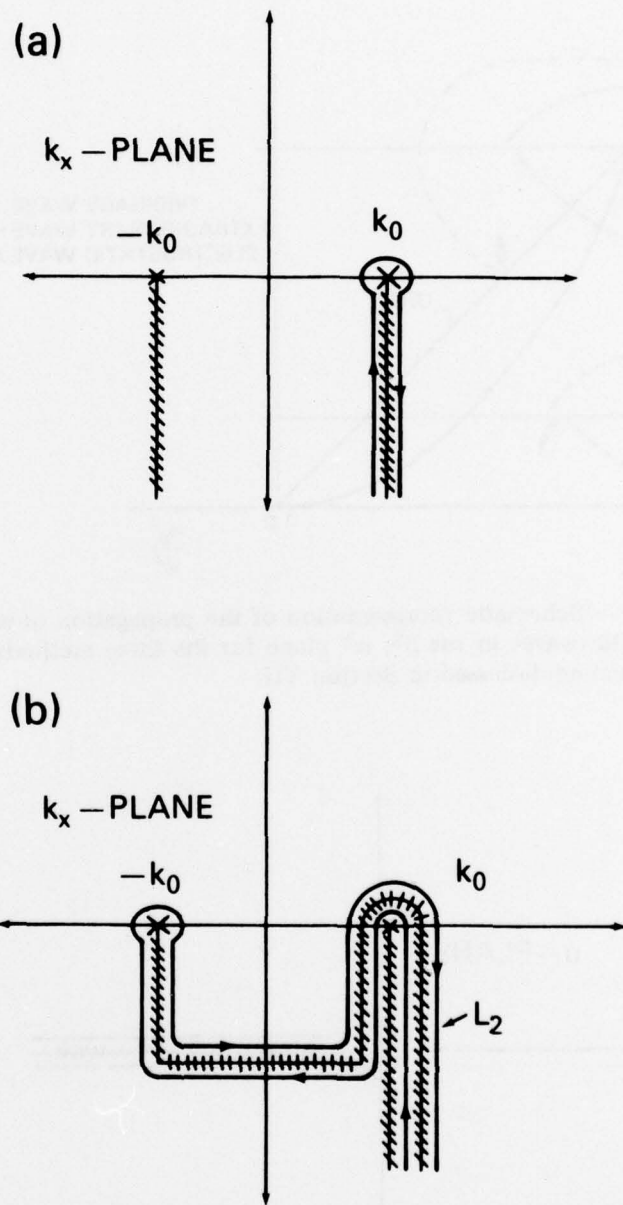


Figure 5 - The deformation of the two contours of Fig. 4 for the evaluation of Eq. (31) for  $x < 0$ .

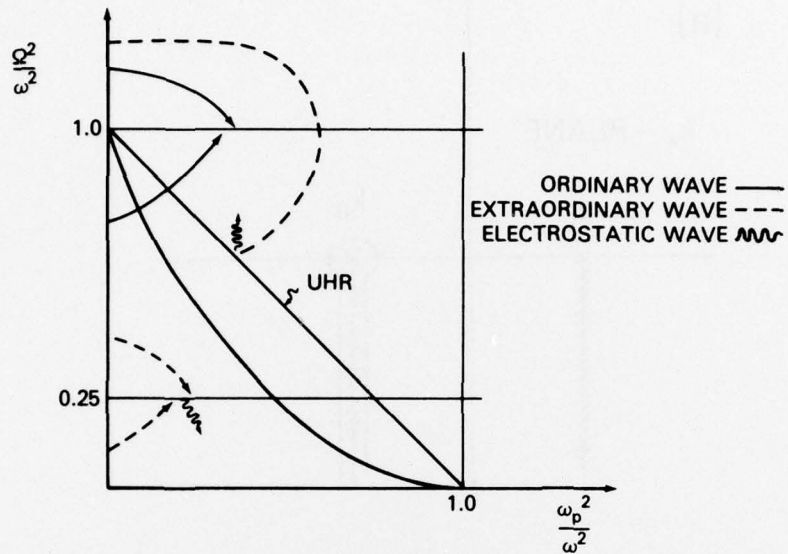


Figure 6 — Schematic representation of the propagation of electromagnetic waves in the  $\beta^2, \alpha^2$  plane for the three methods of plasma heating discussed in Section VII.

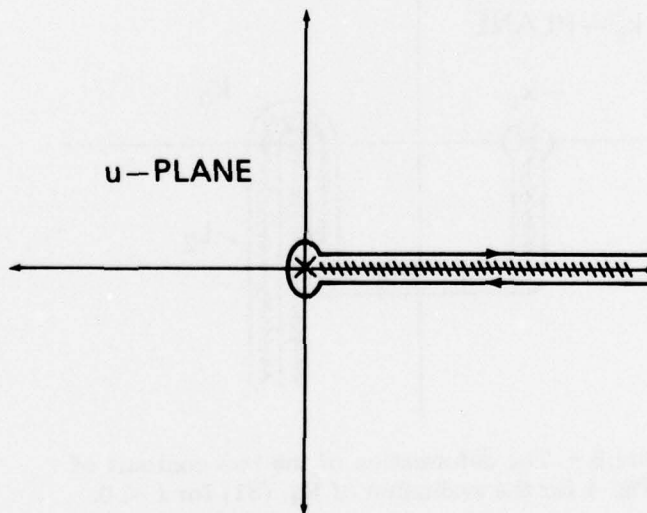


Figure 7 — The contour in the  $u$ -plane for the evaluation of  $C(\eta_0)$  appearing in Eq. (A7).