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AUG 77 R J SERFLING  
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ON THE STRONG LAW OF LARGE NUMBERS AND  
RELATED RESULTS FOR QUASI-STATIONARY SEQUENCES

by R. J. Serfling

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Department of Statistics  
The Florida State University  
Tallahassee, Florida 32306

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ABSTRACT

ON THE STRONG LAW OF LARGE NUMBERS AND  
RELATED RESULTS FOR QUASI-STATIONARY SEQUENCES

Under second moment assumptions and weak dependence conditions on a sequence of random variables  $\{X_i\}$ , Gaposkin (1975) has established almost sure convergence of the series  $\sum_1^\infty \lambda_k X_k$  under certain restrictions on the rate of convergence to 0 of the constants  $\{c_k\}$ . Similarly, Móricz (1977) has established conditions for the almost sure convergence to 0 of the sequence  $\lambda_n \sum_1^n X_k$ . In the present paper, some extensions of these results are obtained.

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1. Main results and discussion. Consider a sequence of random variables  $\{X_i\}$  satisfying

$$EX_i \equiv 0, \quad EX_i^2 \equiv 1$$

and, for a sequence of constants  $\{\phi_i\}$ ,

$$|EX_j X_k| \leq \phi_{k-j}, \quad \text{all } j \leq k.$$

Such a sequence  $\{X_i\}$  is called *quasi-stationary* with respect to the sequence  $\{\phi_i\}$ . The almost sure asymptotic behavior of the sum  $\sum_{i=1}^n X_i$  may be characterized by an assertion of the form

$$(A) \quad \lambda_n \sum_{i=1}^n X_i \rightarrow 0, \quad n \rightarrow \infty, \quad \text{w.p.1.}$$

where  $\{\lambda_n\}$  is a sequence of positive constants tending to 0. It is of interest to establish (A) under mild restrictions on the constants  $\{\phi_i\}$  and  $\{\lambda_i\}$ . A related problem concerns the almost sure behavior of the sum  $\sum_{i=1}^n \lambda_i X_i$  for such a sequence of constants  $\{\lambda_n\}$ . In this case the desired assertion is

$$(a) \quad \sum_{i=1}^{\infty} \lambda_i X_i \quad \text{converges w.p.1.}$$

By the well-known Kronecker lemma, (a) implies (A) in the case of  $\lambda_n$  non-increasing.

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Rademacher (1922) and Mensov (1923) independently established that

( $\alpha$ ) holds if

$$(1.1a) \quad \phi_i = 0, \quad i > 0,$$

and

$$(1.1b) \quad \sum_1^\infty \lambda_n^2 \log^2 n < \infty.$$

Kac, Salem and Zygmund (1948) relaxed (1.1a) to  $\phi_n = O(n^{-1-\epsilon})$  for an  $\epsilon > 0$ .

Gaposkin (1975) proved the following much broader result. Put

$$w(n) = \sum_{i=1}^n \phi_i.$$

**THEOREM  $\alpha$  (Gaposkin).** *If*

$$(1.2) \quad \sum_1^\infty w(n) \lambda_n^2 \log^2 n < \infty,$$

*then ( $\alpha$ ) holds.*

This theorem allows the possibility of  $w(n) \rightarrow \infty$ , whereas the earlier results are confined to the case  $w(\infty) < \infty$ .

Returning to (A), we have

**COROLLARY  $\alpha$ .** *If (1.2) is satisfied and  $\lambda_n$  is nonincreasing, then (A) holds.*

On the other hand, a direct approach — bypassing ( $\alpha$ ) — offers the possibility of obtaining (A) under weaker restrictions than (1.2). In this direction, Móricz (1977) has obtained the following result.

THEOREM A (Móricz). *If*

$$(1.3a) \quad \sum_1^{\infty} w(n) \lambda_n^2 < \infty,$$

$$(1.3b) \quad w(n) \lambda_n^2 \text{ is nonincreasing,}$$

*and*

$$(1.3c) \quad w(2n)/w(n) \geq q > 1, \text{ all } n,$$

*then (A) holds.*

Note that (1.3a) relaxes (1.2). However, (1.3c) requires  $w(n)$  to grow at a fast rate. For example, (1.3c) is satisfied by  $w(n)$  of the form  $w(n) = cn^a$ , but not by  $w(n)$  of the form  $w(n) = \exp(2\sqrt{\log n})$ . For the latter, Theorem A is inapplicable, whereas Theorem  $\alpha$  does yield a conclusion.

The present note provides an alternate to Theorem A which essentially removes condition (1.3c). As in [5], put  $W(1) = w(1)$  and, for  $n \geq 2$ , define  $W(n)$  by

$$W^{\frac{1}{2}}(n) = W^{\frac{1}{2}}([1/2]n) - 1 + w^{\frac{1}{2}}([1/2]n).$$

THEOREM B. *If*

$$(1.4a) \quad \sum_1^{\infty} W(n) \lambda_n^2 < \infty$$

*and*

(1.46)  $W(n) \lambda_n^2$  is nonincreasing,

then (A) holds.

Conditions (1.3a) and (1.3c) together imply (1.4a), as evident from the Lemma below. Also, the mild constraints (1.3b) and (1.4b) are mere variants of each other. Thus Theorem B has somewhat broader application than Theorem A. In particular, it yields

EXAMPLE. Consider  $w(n) = \exp(2\sqrt{\log n})$ . In this case (by the Lemma below)

$$W(n) = O(w(n) \log n),$$

so that (A) holds if  $\lambda_n$  satisfies (1.4b) for this  $w(n)$  and if

$$(1.5) \quad \sum_1^{\infty} w(n) \lambda_n^2 \log n < \infty. \quad \square$$

In the preceding example, the use of Corollary  $\alpha$  would be less effective than Theorem B, since (1.5) is weaker than (1.2). The gain in effectiveness of Theorem B over Corollary  $\alpha$  occurs when  $w(n)$  grows sufficiently fast.

LEMMA. (i) In general,  $W(n) = O(w(n) \log^2 n)$ .

(ii) If  $w(n) = \exp(2\sqrt{\log n})$ , then  $W(n) = O(w(n) \log n)$ .

(iii) If  $w(n)$  satisfies (1.5c), then  $W(n) = O(w(n))$ .

As a complement to Theorem B, the following generalization of Theorem  $\alpha$  will be established.

THEOREM  $\beta$ . If (1.4a) and

$$(1.6a) \quad \sum_1^{\infty} w(n) \lambda_n^2 (\log n) (\log \log n)^{1+\epsilon} < \infty, \text{ for some } \epsilon > 0,$$

are satisfied, then (a) holds.

Since (1.2) implies each of (1.4a) and (1.6a), Theorem  $\beta$  generalizes Theorem  $\alpha$ .

## 2. Proofs.

PROOF OF THE LEMMA. Note that, for  $2^k \leq n < 2^{k+1}$ ,

$$(2.1) \quad W^{\frac{1}{2}}(n) < W^{\frac{1}{2}}(2^{k+1} - 1) = \sum_{j=0}^k w^{\frac{1}{2}}(2^j).$$

Thus  $W^{\frac{1}{2}}(n) \leq kw^{\frac{1}{2}}(n) = O(w^{\frac{1}{2}}(n) \log n)$ , which gives (i). Now, for  $j \leq k$ ,

$$\begin{aligned} \exp\sqrt{j} &= \exp\sqrt{k} \exp\left(\frac{j-k}{\sqrt{j} + \sqrt{k}}\right) \\ &\leq \exp\sqrt{k} \exp\left(\frac{j-k}{2\sqrt{k}}\right) \\ &= \exp \frac{1}{2}\sqrt{k} \left(\exp \frac{1}{2\sqrt{k}}\right)^j. \end{aligned}$$

Thus by (2.1) we obtain, for the case  $w(n) = \exp(2\sqrt{\log n})$ , that

$$W^{\frac{1}{2}}(n) \leq (\exp \frac{1}{2}\sqrt{k}) \frac{\exp\sqrt{k}}{\exp\left(\frac{1}{2\sqrt{k}}\right) - 1} \leq 2\sqrt{k} \exp\sqrt{k},$$

i.e.,  $W^{\frac{1}{2}}(n) = O(\sqrt{\log n} w^{\frac{1}{2}}(n))$ , so that (ii) is proved. Finally, for  $w(n)$  satisfying (1.3c), the use of (2.1) yields

$$W^{\frac{1}{2}}(n) \leq w^{\frac{1}{2}}(2^k) \sum_{j=0}^k \left(\frac{1}{q}\right)^{\frac{1}{2}(k-j)},$$

from which (iii) follows.  $\square$

In proving Theorems B and  $\beta$ , the following maximal inequality will be used.

LEMMA 2.1. For  $m \geq 1$ ,  $n \geq 1$ ,

$$E \left\{ \max_{1 \leq k \leq n} \left( \sum_{i=m+1}^{m+k} a_i X_i \right)^2 \right\} \leq 2W(n) \sum_{i=m+1}^{m+k} a_i^2.$$

This was proved by Móricz (1976), extending an earlier result of Serfling (1970a). For Theorem  $\beta$ , we will also need the following easily proved parallel result [5], [8].

LEMMA 2.2. For  $n \geq 1$ ,

$$E \left\{ \left( \sum_{i=m+1}^{m+n} a_i X_i \right)^2 \right\} \leq 2w(n) \sum_{i=m+1}^{m+n} a_i^2.$$

PROOF OF THEOREM B. In order to show (A), it is equivalent to show that for every  $\epsilon > 0$ ,

$$(2.2) \quad P\{|\lambda_n S_n| > \epsilon \text{ infinitely often}\} = 0.$$

Now observe that the nonincreasingness of  $\lambda_n^2 W(n)$ , combined with the non-decreasingness of  $W(n)$ , implies that  $\lambda_n$  is nondecreasing. Thus, by the Borel-Cantelli lemma, (2.2) holds if

$$(2.3) \quad \sum_{k=0}^{\infty} P\{\lambda_{2^k} \max_{2^k \leq n < 2^{k+1}} |S_n| > \epsilon\} < \infty .$$

Now

$$\begin{aligned} P\{\lambda_{2^k} \max_{2^k \leq n < 2^{k+1}} |S_n| > \epsilon\} \\ \leq \epsilon^{-2} \lambda_{2^k}^2 E\{\max_{2^k \leq n < 2^{k+1}} S_n^2\} \\ \leq 2\epsilon^{-2} \lambda_{2^k}^2 E\{S_{2^k}^2 + \max_{2^k \leq n < 2^{k+1}} (S_n - S_{2^k})^2\} . \end{aligned}$$

A two-fold application of Lemma 2.1 gives

$$P\{\lambda_{2^k} \max_{2^k \leq n < 2^{k+1}} |S_n| > \epsilon\} \leq 4\lambda_{2^k}^2 W(2^k)2^k .$$

Thus the sum in (2.3) is bounded by  $4 \sum_{k=0}^{\infty} 2^k \lambda_{2^k}^2 W(2^k)$ , which in turn

in view of (1.3b) is clearly bounded by  $4 \sum_{n=1}^{\infty} \lambda_n^2 W(n)$ . By (1.3a), the required (2.3) thus holds.  $\square$

PROOF OF THEOREM  $\beta$ . Following the approach of [1], and using a standard elementary argument, we first establish that  $T_{2^k}$  converges w.p.1 to a limit  $T_{\infty}$ , by showing that

$$(2.4) \quad \sum_{k=0}^{\infty} \|\Delta_k\| < \infty ,$$

where  $\Delta_k = S_{2^{k+1}} - S_{2^k}$  and  $\|\Delta_k\|$  denotes  $(E \Delta_k^2)^{\frac{1}{2}}$ . By the Cauchy-

Schwarz inequality,

$$\sum ||\Delta_k|| \leq \left( \sum d_k^2 ||\Delta_k||^2 \right) \left( \sum d_k^{-2} \right)$$

for positive constants  $d_k$ . Choose  $d_k = k^{\frac{1}{2}} (\log k)^{\frac{1}{2}(1+\epsilon)}$ ,  $\epsilon > 0$ . Then, applying Lemma 2.2 and (1.6a), we obtain for an appropriate constant  $C$ ,

$$\begin{aligned} \sum ||\Delta_k|| &\leq C \sum k (\log k)^{1+\epsilon} [2w(2^k) \sum_{j=2^{k+1}}^{2^{k+1}} \lambda_j^2] \\ &\leq 2C \sum_{n=1}^{\infty} (\log n) (\log \log n)^{1+\epsilon} w(n) \lambda_n^2 < \infty . \end{aligned}$$

Next we establish that  $T_n$  converges w.p.1 to  $T_{\infty}$ , by showing that

$$\max_{2^k \leq n < 2^{k+1}} |T_n - T_{2^k}| \longrightarrow 0 \text{ w.p.1.}$$

This follows, by an argument similar to the proof of Theorem B, if

$$\sum_{k=0}^{\infty} E \left\{ \max_{2^k \leq n < 2^{k+1}} (T_n - T_{2^k})^2 \right\} < \infty ,$$

which in turn is established by Lemma 2.2 and (1.4a), via

$$2 \sum_{k=0}^{\infty} w(2^k) \sum_{n=2^k}^{2^{k+1}-1} \lambda_n^2 \leq \sum_{n=1}^{\infty} w(n) \lambda_n^2 < \infty . \quad \square$$

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20. ABSTRACT  
Under second moment assumptions and weak dependence conditions on a sequence of random variables  $\{X_i\}$ , Gaposkin (1975) has established almost sure convergence of the series  $\sum_{k=1}^{\infty} \lambda_k X_k$  under certain restrictions on the rate of convergence to 0 of the constants  $\{c_k\}$ . Similarly, Móricz (1977) has established conditions for the almost sure convergence to 0 of the sequence  $\lambda_n \sum_{k=1}^n X_k$ . In the present paper, some extensions of these results are obtained.  
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 < c sub k >  
 (lambda sub n) (sum from 1 to n of X sub k)  
 sum from 1 to infinity of lambda sub k X sub k  
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