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ON THE BERRY-ESSEEN THEOREM FOR  
SIMPLE LINEAR RANK STATISTICS

by

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ABSTRACT

ON THE BERRY-ESSEEN THEOREM FOR  
SIMPLE LINEAR RANK STATISTICS

The rate of convergence  $O(N^{-\frac{1}{2}+\delta})$  for any  $\delta > 0$  is established for two theorems of Hájek (1968) on asymptotic normality of simple linear rank statistics. These pertain to smooth and bounded scores, arbitrary regression constants, and broad conditions on the distributions of individual observations. The results parallel those of Bergström and Puri (1977), which appeared in print just as this paper was completed. Whereas Bergström and Puri provide explicit constants of proportionality in the  $O(\cdot)$  terms, the present development is in closer touch with Hájek (1968), provides some alternative arguments of proof, and provides explicit application to relax the conditions of a theorem of Jurečková and Puri (1975) giving the above rate for the case of location-shift alternatives.

1. Introduction and main results. Hájek (1968) established the asymptotic normality of simple linear rank statistics under broad conditions on the regression constants, the distribution functions of individual observations and the scores-generating function. Corresponding to his theorems for the case of smooth and bounded scores, the rate of convergence  $O(N^{-\frac{1}{2}+\delta})$ ,  $N \rightarrow \infty$ , for any  $\delta > 0$  is obtained in the present paper (Theorems 2 and 3). Recently, Bergström and Puri (1977) have already established a version of part of Theorem 2, exhibiting explicit constants in the  $O(\cdot)$  term. Previously, Jurečková and Puri (1975) have shown the above rate for the cases (a) iden-

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tical distributions, and (b) location-shift alternatives. In case (a), their score-smoothness condition is slightly milder than that of the present results. However, for case (b) their stringent conditions on the shift parameters and on the score-smoothness are considerably reduced here (Corollary).

Our method of proof consists in approximating the simple linear rank statistic by a sum of independent random variables and establishing, for arbitrary  $\nu$ , a suitable bound on the  $\nu$ -th moment of the error of approximation (Theorem 1).

Let  $X_{N1}, \dots, X_{NN}$  be independent random variables with ranks  $R_{N1}, \dots, R_{NN}$ . The simple linear rank statistic to be considered is

$$(1.1) \quad S_N = \sum_{i=1}^N c_{Ni} a_N(R_{Ni}),$$

where  $c_{N1}, \dots, c_{NN}$  are arbitrary "regression constants" and  $a_N(1), \dots, a_N(N)$  are "scores". Throughout, the following condition will be assumed.

CONDITION A. (i) The scores are generated by a function  $\phi(t)$ ,  $0 < t < 1$ , in either of the following ways:

$$(1.2) \quad a_N(i) = \phi\left(\frac{i}{N+1}\right), \quad 1 \leq i \leq N,$$

$$(1.3) \quad a_N(i) = E\phi(U_N^{(i)}), \quad 1 \leq i \leq N,$$

where  $U_N^{(i)}$  denotes the  $i$ -th order statistic in a sample of size  $N$  from the uniform distribution on  $(0, 1)$ .

(ii)  $\phi$  has a bounded second derivative.

(iii) The regression constants satisfy

$$(1.4) \quad \sum_{i=1}^N c_{Ni} = 0, \quad \sum_{i=1}^N c_{Ni}^2 = 1,$$

$$(1.5) \quad \max_{1 \leq i \leq N} c_{Ni}^2 = O(N^{-1} \log N), \quad N \rightarrow \infty. \quad \square$$

Note that (1.4) may be assumed without loss of generality.

The  $X_{Ni}$ 's are assumed to have continuous distribution functions  $F_{Ni}$ ,  $1 \leq i \leq N$ . Put  $H_N(x) = N^{-1} \sum_{i=1}^N F_{Ni}(x)$ . The derivatives of  $\phi$  will be denoted by  $\phi'$ ,  $\phi''$ , etc. Also, put  $\mu_\phi = \int_0^1 \phi(t) dt$  and  $\sigma_\phi^2 = \int_0^1 [\phi(t) - \mu_\phi]^2 dt$ . Finally, denote by  $\phi$  the standard normal cdf. Hereafter the suffix  $N$  will be omitted from  $X_{Ni}$ ,  $R_{Ni}$ ,  $c_{Ni}$ ,  $S_N$ ,  $F_{Ni}$ ,  $H_N$  and other notation.

The statistic  $S$  will be approximated by the same sum of independent random variables introduced by Hájek (1968), namely

$$(1.6) \quad T = \sum_{i=1}^N \lambda_i(X_i),$$

where

$$(1.7) \quad \lambda_i(x) = N^{-1} \sum_{j=1}^N (c_j - c_i) \int [u(y-x) - F_i(y)] \phi'(H(y)) dF_j(y),$$

with

$$(1.8) \quad u(x) = 1, x \geq 0; u(x) = 0, x < 0.$$

**THEOREM 1.** *Assume Condition A. Then, for every integer  $r$ , there exists a constant  $M = M(\phi, r)$  such that*

$$(1.9) \quad E(S - ES - T)^{2r} \leq MN^{-r}, \text{ all } N.$$

The case  $r = 1$  was proved by Hájek (1968). The extension to higher order is needed for the present purposes.

**THEOREM 2.** *Assume Condition A. (i) If*

$$(1.10) \quad \text{Var } S > B > 0, N \rightarrow \infty,$$

*then for every  $\delta > 0$ ,*

$$(1.11) \quad \sup_x |P\{S - ES < x(\text{Var } S)^{1/2}\} - \phi(x)| = O(N^{-1/2+\delta}), N \rightarrow \infty.$$

(ii) *The assertion remains true with Var S replaced by Var T.*

(iii) *Both assertions remain true with ES replaced by*

$$(1.12) \quad \mu = \sum_{i=1}^N c_i \int \phi(H(x)) dF_i(x).$$

Compare Theorem 2.1 of Hájek (1968) and Theorem 1.2 of Bergström and Puri (1977).

**THEOREM 3.** *Assume Condition A and that*

$$(1.13) \quad \sup_{i,j,x} |F_i(x) - F_j(x)| = O(N^{-\frac{1}{2}} \log N), \quad N \rightarrow \infty.$$

*Then for every  $\delta > 0$*

$$(1.14) \quad \sup_x |P\{S - ES < x\sigma_\phi\} - \phi(x)| = O(N^{-\frac{1}{2} + \delta}), \quad N \rightarrow \infty.$$

*The assertion remains true with  $\sigma_\phi^2$  replaced by either Var S or Var T, and/or ES replaced by  $\mu$ .*

Compare Theorem 2.2 of Hájek (1968). As a corollary of Theorem 3, the case of local location-shift alternatives will be treated. The following condition will be assumed.

**CONDITION B.** (i) *The cdf's  $F_i$  are generated by a cdf  $F$  as follows:*

$$F_i(x) = F(x - \Delta d_i), \quad 1 \leq i \leq N, \quad \text{with } \Delta \neq 0.$$

(ii)  *$F$  has a density  $f$  with bounded derivative  $f'$ .*

(iii) *The shift coefficients satisfy*

$$(1.15) \quad \sum_{i=1}^N d_i = 0, \quad \sum_{i=1}^N d_i^2 = 1,$$

$$(1.16) \quad \max_{1 \leq i \leq N} d_i^2 = O(N^{-1} \log N), \quad N \rightarrow \infty. \quad \square$$

Note that (1.15) may be assumed without loss of generality.

**COROLLARY.** *Assume Conditions A and B and that*

$$(1.17) \quad \sum_{i=1}^N c_i^2 d_i^2 = O(N^{-1} \log N), N \rightarrow \infty.$$

Then for every  $\delta > 0$

$$(1.18) \quad \sup_x |P\{S - \bar{\mu} < x\sigma_\phi\} - \phi(x)| = O(N^{-1+\delta}), N \rightarrow \infty,$$

where

$$(1.19) \quad \bar{\mu} = \Delta \left( \sum_{i=1}^N c_i d_i \right) \int \phi'(F(x)) f^2(x) dx.$$

(The corresponding result of Jurečková and Puri (1975) requires  $\phi$  to have four bounded derivatives and requires further conditions on the  $c_i$ 's and  $d_i$ 's. On the other hand, their result for the case of all  $F_i$ 's identical requires only a single bounded derivative for  $\phi$ .)

2. The proofs. The main development will be carried out for the case of scores given by (1.2). In Lemma 7 it will be shown that the case of scores given by (1.3) may be reduced to this case.

Assuming  $\phi''$  bounded, put

$$(2.1) \quad K_1 = \sup_{0 < t < 1} |\phi'(t)|, K_2 = \sup_{0 < t < 1} |\phi''(t)|.$$

By Taylor expansion the statistic  $S$  may be written as

$$S = U + V + W,$$

where, with  $\rho_i = R_i/(N+1)$ ,  $1 \leq i \leq N$ ,

$$(2.2) \quad U = \sum_{i=1}^N c_i \phi(E(\rho_i | X_i)),$$

$$(2.3) \quad V = \sum_{i=1}^N c_i \phi'(E(\rho_i | X_i)) [\rho_i - E(\rho_i | X_i)]$$

and

$$(2.4) \quad W = \sum_{i=1}^N c_i K_2 \varepsilon_i [\rho_i - E(\rho_i | X_i)]^2,$$

the random variables  $\xi_i$  satisfying  $|\xi_i| \leq 1$ ,  $1 \leq i \leq N$ . It will first be shown that  $W$  may be neglected. To see this, note that

$$(2.5) \quad R_i = \sum_{j=1}^N u(X_i - X_j), \quad 1 \leq i \leq N,$$

where  $u(\cdot)$  is given by (1.8). Thus

$$(2.6) \quad E(\rho_i | X_i) = [\sum_{\substack{j=1 \\ j \neq i}}^N F_j(X_i) + 1] / (N+1)$$

and

$$(2.7) \quad \rho_i - E(\rho_i | X_i) = \frac{1}{N+1} \sum_{\substack{j=1 \\ j \neq i}}^N [u(X_i - X_j) - F_j(X_i)].$$

Observe that, given  $X_i$ , the summands in (2.7) are conditionally independent random variables centered at means. Hence the following classical result, due to Marcinkiewicz and Zygmund (1937), is applicable.

**LEMMA 1.** *Let  $Y_1, Y_2, \dots$  be independent random variables with mean 0. Let  $v$  be an integer. Then*

$$(2.8) \quad E \left| \sum_{i=1}^N Y_i \right|^v \leq A_v n^{1/2 v - 1} \sum_{i=1}^n E |Y_i|^v,$$

where  $A_v$  is a universal constant depending only on  $v$ .

**LEMMA 2.** *Assume (1.4). For each positive integer  $r$ ,*

$$(2.9) \quad E W^{2r} \leq K_2^{2r} A_{4r} N^{-r}, \quad \text{all } N.$$

**PROOF.** Write  $W$  in the form  $W = K_2 \sum_{i=1}^N c_i W_i$ . By the Cauchy-Schwarz inequality and (1.4),

$$(2.10) \quad W^{2r} \leq K_2^{2r} \left( \sum_{i=1}^N c_i^2 \right)^r \left( \sum_{i=1}^N W_i^2 \right)^r = K_2^{2r} \left( \sum_{i=1}^N W_i^2 \right)^r.$$

Minkowski's inequality then yields

$$(2.11) \quad E W^{2r} \leq K_2^{2r} \left[ \sum_{i=1}^N (E W_i^{2r})^{1/r} \right]^r.$$

By Lemma 1,

$$(2.12) \quad E\{[\rho_i - E(\rho_i | X_i)]^{4r} | X_i\} \leq (N+1)^{-4r} A_{4r} (N-1)^{2r-1/N}$$

so that

$$(2.13) \quad E W_i^{2r} \leq E[\rho_i - E(\rho_i | X_i)]^{4r} \leq A_{4r} N^{-2r}.$$

Thus (2.9) follows.  $\square$

Thus  $S$  may be replaced by  $Z = U + V$ , in the sense that  $E(S-Z)^{2r} = O(N^{-r})$ ,  $N \rightarrow \infty$ , each  $r$ . It will next be shown that, in turn,  $Z$  may be replaced in the same sense by a sum of independent random variables, namely by its *projection*

$$(2.14) \quad \hat{Z} = \sum_{i=1}^N E(Z | X_i) - (N-1) E(Z).$$

Clearly,  $\hat{Z} = \hat{U} + \hat{V}$  and  $\hat{U} = U$ . Thus  $Z - \hat{Z} = V - \hat{V}$ .

LEMMA 3. *The projection of  $V$  is*

$$(2.15) \quad \hat{V} = \frac{1}{N+1} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_j k_{ji}(X_i),$$

where

$$(2.16) \quad k_{ji}(x) = \int [u(y-x) - F_j(y)] \phi'(E(\rho_j | X_j=y)) dF_j(y).$$

PROOF. Put

$$Y_{ij} = \phi'(E(\rho_i | X_i)) [u(X_i - X_j) - F_j(X_i)].$$

For  $j \neq i$ ,  $j \neq k$ , we have

$$(2.17) \quad \begin{aligned} E(Y_{ij} | X_k) &= E\{E[Y_{ij} | X_i, X_k] | X_k\} \\ &= E\{\phi'(E(\rho_i | X_i)) E[u(X_i - X_j) - F_j(X_i) | X_i, X_k] | X_k\} \\ &= E\{\phi'(E(\rho_i | X_i)) \cdot 0\} = 0. \end{aligned}$$

For  $j \neq i$ ,

$$(2.18) \quad E(Y_{ij} | X_j) = l_{ij}(X_j),$$

where  $l_{mn}(x)$  is defined by (2.16). Also, by (2.17),

$$(2.19) \quad EY_{ij} = 0, \text{ if } i \neq j.$$

Therefore, the projection of  $Y_{ij}$ , for  $i \neq j$ , is  $\hat{Y}_{ij} = l_{ij}(X_j)$ . Since, by (2.3) and (2.7),

$$(2.20) \quad V = \frac{1}{N+1} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_j Y_{ji},$$

the projection  $\hat{V}$  is given by (2.15).  $\square$

LEMMA 4. Assume (1.4). For each positive integer  $r$ , there exists a constant  $B_r$  such that

$$(2.21) \quad E(V - \hat{V})^{2r} \leq K_1^{2r} B_r N^{-r}, \text{ all } N.$$

PROOF. By (2.15) and (2.20),

$$(2.22) \quad E(V - \hat{V})^{2r} = (N+1)^{-2r} \sum_{i_1=1}^N \cdots \sum_{i_{2r}=1}^N c_{i_1} \cdots c_{i_{2r}} \sum_{\substack{j_1=1 \\ j_1 \neq i_1}}^N \cdots \\ \sum_{\substack{j_{2r}=1 \\ j_{2r} \neq i_{2r}}}^N \delta_{i_1, j_1, \dots, i_{2r}, j_{2r}},$$

where

$$(2.23) \quad \delta_{i_1, j_1, \dots, i_{2r}} = E \prod_{k=1}^{2r} [Y_{i_k j_k} - l_{i_k j_k}(X_{j_k})].$$

Consider a typical term of the form (2.23). If the  $i_k$  index in the  $k$ -th factor occurs only in that factor, then the entire product of factors has expectation 0, for

$$E[Y_{i_k j_k} - \ell_{i_k j_k}(X_{j_k}) | X_m, m \in \{i_1, \dots, i_{k-1}, i_{k+1}, \dots, i_{2r}, j_1, \dots, j_{2r}\}] = 0.$$

By a similar argument, the same conclusion holds if the  $j_k$  index in the  $k$ -th factor occurs only in that factor. Thus the expectation in (2.23) is possibly nonzero only if each factor has both indices repeated in other factors.

Among such cases, consider now only those terms corresponding to a given pattern of the possible identities  $i_a = i_b$ ,  $i_a = j_b$ ,  $j_a = j_b$  for  $1 \leq a \leq 2r$ ,  $1 \leq b \leq 2r$ . For example, for  $r = 3$ , one such specific pattern is:  $i_2 = i_1$ ,  $i_3 \neq i_1$ ,  $i_4 = i_1$ ,  $i_5 = i_3$ ,  $i_6 \neq i_1$ ,  $i_6 \neq i_3$ ,  $j_2 = j_1$ ,  $j_3 = j_1$ ,  $j_4 \neq j_1$ ,  $j_5 = j_4$ ,  $j_6 = j_4$ ,  $j_1 = i_3$ ,  $j_4 \neq i_1$ . In general, there are at most  $2^{6r}$  such patterns. For such a pattern, let  $q$  denote the number of distinct values among  $i_1, \dots, i_{2r}$  and  $p$  the number of distinct values among  $j_1, \dots, j_{2r}$ . Let  $p_1$  denote the number of distinct values among  $j_1, \dots, j_{2r}$  *not* appearing among  $i_1, \dots, i_{2r}$  and put  $p_2 = p - p_1$ . Within the given constraints, and after selection of  $i_1, \dots, i_{2r}$ , the number of choices for  $j_1, \dots, j_{2r}$  clearly is of order

$$(2.24) \quad O(N^{p_1}).$$

Now clearly  $2p_1 \leq 2r - p_2$ , i.e.,

$$(2.25) \quad p_1 \leq r - \frac{1}{2}p_2.$$

Now let  $q_1$  denote the number of  $i_1, \dots, i_{2r}$  used only *once* among  $i_1, \dots, i_{2r}$ .

Then obviously

$$(2.26) \quad q_1 \leq p_2.$$

By (2.24), (2.25) and (2.26), it is seen that the contribution to (2.22) from summation over  $j_1, \dots, j_{2r}$  is of order at most

$$O(N^{r-\frac{1}{2}q_1}),$$

since the quantity in (2.23) is of magnitude  $\leq K_1^{2r}$ . It follows that

$$(2.27) \quad E(V-\hat{V})^{2r} \leq (N+1)^{-2r} K_2^{2r} [O(N^{r-\frac{1}{2}q_1})] \sum_{\ell_1=1}^N \dots \sum_{\ell_q=1}^N |c_{\ell_1}^{a_1} \dots c_{\ell_q}^{a_q}|,$$

where  $a_1, \dots, a_q$  are integers satisfying  $a_i \geq 1$ ,  $a_1 + \dots + a_q = 2r$ , and exactly  $q_1$  of the  $a_i$ 's are equal to 1. Now, for  $a \geq 2$ ,

$$(2.28) \quad \sum_{i=1}^N |c_i|^a \leq (\sum_{i=1}^N c_i^2)^{\frac{1}{2}a} = 1,$$

by (1.4). Further,

$$(2.29) \quad \sum_{i=1}^N |c_i| \leq N \left( \frac{1}{N} \sum_{i=1}^N c_i^2 \right)^{\frac{1}{2}} = N^{\frac{1}{2}}.$$

Thus

$$(2.30) \quad \sum_{\ell_1=1}^N \dots \sum_{\ell_q=1}^N |c_{\ell_1}^{a_1} \dots c_{\ell_q}^{a_q}| \leq N^{\frac{1}{2}q_1}.$$

Combining (2.27) and (2.30), we obtain (2.21).  $\square$

Next it is shown that  $Z$  may be replaced by  $\tilde{Z} = \tilde{U} + \tilde{V}$ , where

$$(2.31) \quad \tilde{U} = \sum_{i=1}^N c_i \phi(H(X_i))$$

and

$$(2.32) \quad \tilde{V} = \frac{1}{N} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N c_j \tilde{\ell}_{ji}(X_i),$$

with

$$(2.33) \quad \tilde{\ell}_{ji}(x) = \int [u(y-x) - F_i(y)] \phi'(H(y)) dF_j(y).$$

LEMMA 5. Assume (1.4). Then

$$(2.34) \quad |\hat{Z} - \tilde{Z}| \leq (K_2 + 3K_1)N^{-1/2}.$$

PROOF. By (2.6),

$$(2.35) \quad E(\rho_i | X_i) = H(X_i) + \frac{1 - F_i(X_i) - H(X_i)}{N+1}.$$

Hence, by the Mean Value Theorem,

$$(2.36) \quad |\phi(E(\rho_i | X_i)) - \phi(H(X_i))| \leq K_1 N^{-1}.$$

Therefore, by (2.29),

$$(2.37) \quad |U - \tilde{U}| \leq K_1 N^{-1/2}.$$

Now

$$\hat{V} - \tilde{V} = \sum_{i=1}^N \sum_{j=1}^N \frac{1}{N} c_j [\ell_{ji}(X_i) - \tilde{\ell}_{ji}(X_i)] - \frac{1}{N} \hat{V} - \frac{1}{N} \sum_{i=1}^N c_i \ell_{ii}(X_i).$$

But

$$|\hat{V}| \leq K_1 \sum_{i=1}^N |c_i| \leq K_1 N^{1/2},$$

i.e.,

$$(2.38) \quad \frac{1}{N} |\hat{V}| \leq K_1 N^{-1/2}.$$

Similarly,

$$(2.39) \quad \left| \frac{1}{N} \sum_{i=1}^N c_i \ell_{ii}(X_i) \right| \leq K_1 N^{-1/2}.$$

Finally,

$$|\ell_{ji}(X_i) - \tilde{\ell}_{ji}(X_i)| \leq K_2 N^{-1}$$

so that

$$(2.40) \quad \left| \sum_{i=1}^N \sum_{j=1}^N \frac{1}{N} c_j [\ell_{ji}(X_i) - \tilde{\ell}_{ji}(X_i)] \right| \leq K_2 N^{-1/2}.$$

Thus (2.34) follows.  $\square$

Now we connect with the random variable  $T$  of Theorem 1.

LEMMA 6. *Let  $T$  be defined by (1.6) and  $\mu$  by (1.12). Then*

$$(2.41) \quad \tilde{Z} - \mu = T$$

and there exists a constant  $K_4 = K_4(\phi)$  such that

$$(2.42) \quad |ES - \mu| \leq K_4 N^{-1/2}.$$

PROOF. (2.42) is shown by Hájek (1968), p. 340. To obtain (2.41), check that

$$(2.43) \quad \begin{aligned} \tilde{Z} - \mu - T &= \sum_{i=1}^N c_i \{ \phi(H(X_i)) - E\phi(H(X_i)) \\ &+ \int [u(x-X_i) - F_i(x)] \phi'(H(x)) dH(x) \}. \end{aligned}$$

Now, by integration by parts, for any distribution function  $G$  we have

$$\int \phi'(H(x)) G(x) dH(x) = -\int \phi(H(x)) dG(x) + \text{constant},$$

where the constant may depend on  $\phi$  and  $H(\cdot)$  but not on  $G(\cdot)$ . Thus the sum in (2.43) reduces to 0.  $\square$

Up to this point, only the scores given by (1.2) have been considered. The next result provides the basis for interchanging with the scores given by (1.3).

LEMMA 7. *Denote  $\sum_{i=1}^N c_i a_{iN}(R_i)$  by  $S$  in the case corresponding to (1.2) and by  $S'$  in the case corresponding to (1.3). Assume (1.4). Then there exists  $K_5 = K_5(\phi)$  such that*

$$(2.43) \quad |S - ES - (S' - ES')| \leq K_5 N^{-1/2}.$$

PROOF. It is easily found (see Hájek (1968), p. 341) that

$$(2.44) \quad \left| \phi\left(\frac{i}{N+1}\right) - E\phi(U_N^{(i)}) \right| \leq K_0 N^{-1},$$

where  $K_0$  does not depend on  $i$  or  $N$ . Thus, by (2.29),

$$(2.45) \quad |S - S'| \leq K_0 N^{-1/2}$$

and hence also

$$(2.46) \quad |ES - ES'| \leq K_0 N^{-1/2}.$$

Thus (2.43) follows with  $K_5 = 2K_0$ .  $\square$

PROOF OF THEOREM 1. Consider first the case (1.2). By Minkowski's inequality,

$$(2.47) \quad \begin{aligned} & [E(S - ES - T)^{2r}]^{1/2r} \leq [E(S - Z)^{2r}]^{1/2r} + [E(Z - \hat{Z})^{2r}]^{1/2r} \\ & + [E(\hat{Z} - \bar{Z})^{2r}]^{1/2r} + [E(\bar{Z} - \mu - T)^{2r}]^{1/2r} \\ & + |ES - \mu|. \end{aligned}$$

By Lemmas 2, 4, 5 and 6, each term on the right-hand side of (2.47) may be bounded by  $KN^{-1/2}$  for a constant  $K = K(\phi, r)$  depending only on  $\phi$  and  $r$ . Thus (1.9) follows. In the case of scores given by (1.3), we combine Lemma 7 with the preceding argument.  $\square$

PROOF OF THEOREM 2. First assertion (i) will be proved. Put

$$\alpha_N = \sup_x |P\{S - ES < x(\text{Var } S)^{1/2}\} - \phi(x)|,$$

$$\beta_N = \sup_x |P\{T < x(\text{Var } S)^{1/2}\} - \phi(x)|,$$

and

$$\gamma_N = \sup_x |P\{T < x(\text{Var } T)^{1/2}\} - \phi(x)|.$$

By a standard device, if

$$(2.48) \quad \beta_N = o(a_N), \quad N \rightarrow \infty,$$

for a sequence of constants  $\{a_N\}$ , then

$$(2.49) \quad \alpha_N = o(a_N) + P\{|S - ES - T|/(\text{Var } S)^{1/2} > a_N\}, \quad N \rightarrow \infty.$$

We shall obtain a condition of form (2.48) by first considering  $\gamma_N$ . By the classical Berry-Esséen theorem, as stated in Loève (1963), p. 288,

$$(2.50) \quad \gamma_N \leq C(\text{Var } T)^{-3/2} \sum_{i=1}^N E|\ell_i(X_i)|^3,$$

where  $C$  is a universal constant. Clearly,

$$|\ell_i(X_i)| \leq K_1 N^{-1} \sum_{j=1}^N |c_j - c_i|.$$

Now

$$(2.51) \quad \left(\sum_{j=1}^N |c_j - c_i|\right)^2 \leq N \left[\sum_{j=1}^N (c_j - c_i)^2\right] = N[1 + Nc_i^2].$$

By the elementary inequality (Loève (1963), p. 155)

$$(2.52) \quad |x+y|^m \leq \theta_m |x|^m + \theta_m |y|^m,$$

where  $m > 0$  and  $\theta_m = 1$  or  $2^{m-1}$  according as  $m \leq 1$  or  $m \geq 1$ , we thus have

$$\left(\sum_{j=1}^N |c_j - c_i|\right)^3 \leq N^{3/2} 2^{1/2} (1 + N^{3/2} |c_i|^3)$$

and hence

$$(2.53) \quad \sum_{i=1}^N E|\ell_i(X_i)|^3 \leq 2^{1/2} K_1^3 [N^{-1/2} + \sum_{i=1}^N |c_i|^3].$$

Now, by a double application of (2.52),

$$\begin{aligned}
 (2.54) \quad |\text{Var } S - \text{Var } T| &= |E(S - ES - T)(S - ES + T)| \\
 &\leq [E(S - ES - T)^2]^{\frac{1}{2}} [2 \text{Var } S + 2 \text{Var } T]^{\frac{1}{2}} \\
 &\leq [E(S - ES - T)^2]^{\frac{1}{2}} \sqrt{2} [(\text{Var } S)^{\frac{1}{2}} + (\text{Var } T)^{\frac{1}{2}}].
 \end{aligned}$$

Writing

$$|(\text{Var } S)^{\frac{1}{2}} - (\text{Var } T)^{\frac{1}{2}}| = \frac{|\text{Var } S - \text{Var } T|}{(\text{Var } S)^{\frac{1}{2}} + (\text{Var } T)^{\frac{1}{2}}}$$

and applying Theorem 1 in conjunction with (2.54), we have

$$(2.55) \quad |(\text{Var } S)^{\frac{1}{2}} - (\text{Var } T)^{\frac{1}{2}}| \leq M_0 N^{-\frac{1}{2}},$$

where the constant  $M_0$  depends only on  $\phi$ . It follows that if  $\text{Var } S$  is bounded away from 0, as per assumption (1.10), then the same holds for  $\text{Var } T$ , and conversely. Consequently, by (1.10), (2.50), (2.53) and (2.55), we have

$$\gamma_N = O(N^{-\frac{1}{2}}) + O\left(\sum_{i=1}^N |c_i|^3\right), \quad N \rightarrow \infty.$$

Therefore, by (1.4) and (1.5),

$$(2.56) \quad \gamma_N = O(N^{-\frac{1}{2}} \log N), \quad N \rightarrow \infty.$$

Now it is easily seen that

$$(2.57) \quad \beta_N \leq \gamma_N + O\left(\left|\frac{(\text{Var } S)^{\frac{1}{2}}}{(\text{Var } T)^{\frac{1}{2}}} - 1\right|\right).$$

By (1.10) and (2.55), the right-most term in (2.57) is  $O(N^{-\frac{1}{2}})$ . Hence

$$(2.58) \quad \beta_N = O(N^{-\frac{1}{2}} \log N).$$

Therefore, for any sequence of constants  $a_N$  satisfying  $N^{-\frac{1}{2}} \log N = o(a_N)$ , we have (2.48) and thus (2.49). A further application of Theorem 1, with Markov's inequality, yields for arbitrary  $r$

$$(2.59) \quad P\{|S - ES - T| / (\text{Var } S)^{\frac{1}{2}} > a_N\} \leq a_N^{-2r} (\text{Var } S)^{-r} MN^{-r}.$$

Hence (2.49) becomes

$$\alpha_N = o(a_N) + o(a_N^{-2r} N^{-\frac{1}{2}}).$$

Choosing  $a_N = o(N^{-r/(2r+1)})$ , we obtain

$$(2.60) \quad \alpha_N = o(N^{-r/(2r+1)}), \quad N \rightarrow \infty.$$

Since (2.60) holds for arbitrarily large  $r$ , the first assertion of Theorem 2 is established.

Assertions (ii) and (iii) are obtained easily from the foregoing arguments.  $\square$

PROOF OF THEOREM 3. It is shown by Hájek (1968), p. 342, that

$$(2.61) \quad |(\text{Var } T)^{\frac{1}{2}} - \sigma_\phi| \leq 2^{\frac{1}{2}}(K_1 + K_2) \sup_{i,j,x} |F_i(x) - F_j(x)|.$$

The proof is now straightforward using the arguments of the preceding proof.  $\square$

PROOF OF THE COROLLARY. By Taylor expansion,

$$(2.62) \quad |F_i(x) - F(x) - (-\Delta d_i f(x))| \leq A \Delta^2 d_i^2,$$

where  $A$  is a constant depending only on  $F$ . Hence, by (1.15) and (1.16),

$$(2.63) \quad \sup_{i,j,x} |F_i(x) - F_j(x)| = o(\max_i |d_i|) = o(N^{-\frac{1}{2}} \log N),$$

so that the hypothesis of Theorem 3 is satisfied. It remains to show that ES

may be replaced by the more convenient parameter  $\tilde{\mu}$ . A further application of (2.62), with (1.15), yields

$$|H(x) - F(x)| \leq A \Delta^2 N^{-1}$$

so that

$$|\phi(H(x)) - \phi(F(x))| \leq K_1 A \Delta^2 N^{-1}.$$

Hence, by (2.29),

$$(2.64) \quad \left| \mu - \sum_{i=1}^N c_i \int \phi(F(x)) dF_i(x) \right| \leq K_1 A \Delta^2 N^{-1}.$$

By integration by parts, along with (1.4) and (2.62),

$$(2.65) \quad \begin{aligned} \sum_{i=1}^N c_i \int \phi(F(x)) dF_i(x) &= -\sum_{i=1}^N c_i \int F_i(x) \phi'(F(x)) dF(x) \\ &= -\sum_{i=1}^N c_i \int (-\Delta d_i) f(x) \phi'(F(x)) dF(x) + \eta A \Delta^2 \sum_{i=1}^N c_i d_i^2, \\ &= \tilde{\mu} + \eta A \Delta^2 \sum_{i=1}^N c_i d_i^2, \end{aligned}$$

where  $|\eta| \leq 1$ . Now, by (1.15),

$$(2.66) \quad \sum_i |c_i| d_i^2 \leq \left( \sum_i c_i^2 d_i^2 \right)^{1/2}.$$

By (2.64), (2.65), (2.66) and (1.17),

$$|\mu - \tilde{\mu}| = O(N^{-1/2} \log N), \quad N \rightarrow \infty.$$

Thus  $\mu$  may be replaced by  $\tilde{\mu}$  in Theorem 3.  $\square$

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20. ABSTRACT

The rate of convergence  $O(N^{-\frac{1}{2}+\delta})$  for any  $\delta > 0$  is established for two theorems of Hájek (1968) on asymptotic normality of simple linear rank statistics. These pertain to smooth and bounded scores, arbitrary regression constants, and broad conditions on the distributions of individual observations. The results parallel those of Bergström and Puri (1977), which appeared in print just as this paper was completed. Whereas Bergström and Puri provide explicit constants of proportionality in the  $O(\cdot)$  terms, the present development is in closer touch with Hájek (1968), provides some alternative arguments of proof, and provides explicit application to relax the conditions of a theorem of Jurečková and Puri (1975) giving the above rate for the case of location-shift alternatives.

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