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EXTREME VALUE DISTRIBUTION FOR NORMALIZED SUMS FROM STATIONARY --ETC(U)

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Extreme Value Distribution for Normalized Sums From Stationary Gaussian Sequences

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Institute of Statistics Mimeo Series No. 1160

March, 1978

DEPARTMENT OF STATISTICS
Chapel Hill, North Carolina

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Extreme Value Distribution for Normalized Sums From Stationary Gaussian Sequences[1]

Short Title: Maxima of Normalized Sums

by

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[1]AMS subject classification: Primary 60G15, 60G10, Secondary 60J65

[2]Research supported in part by ONR, Contract No. N00014-75C-0809 and grant to SIMS from ERDA and Rockefeller Foundation

Key Words and Phrases: Normalized sums, extreme value distribution, stationary Gaussian sequences, Brownian motion.

Summary; Let $\{X_i, i \geq 0\}$ be a sequence of independent identically distributed random variables with finite absolute third moment. Then Darling and Erdös have shown that

$$\underset{n\to\infty}{\ell t}\, P\left(\mu_n \leq \chi_n + \frac{\ell n_3 n - \ell n\, 4\Pi}{2\chi_n} + \frac{t}{\chi_n}\right) = \exp(-e^{-t})$$

for $-\infty < t < \infty$ where $\mu_n = \max\limits_{0\leq k\leq n} \frac{\sum_{i=0}^{k} x_i}{k^{\frac{1}{2}}}$ and $\chi_n = (2\ \ell n\ell n\ n)^{\frac{1}{2}}$. The result is extended to dependent sequences but assuming that $\{X_i\}$ is a standard stationary Gaussian sequence with covariance function $\{r_i\}$. When $\{X_i\}$ is moderately dependent (e.g. when $v(\sum_{i=1}^{n} X_i) \sim n^{\alpha}$ $0 < \alpha < 2$) we get

$$\underset{n\to\infty}{\ell t}\, P\left(\mu_n \leq \chi_n + \left(\frac{1}{\alpha} - \frac{1}{2}\right) \frac{\ell n\left(\frac{H_\alpha}{2\Pi^{\frac{1}{2}}}\ \ell n\ell n\ n\right)}{\chi_n} + \frac{t}{\chi_n}\right) = \exp(-e^{-t})$$

where H_α is a constant. In the strongly dependent case (e.g. when $v(\sum_{i=1}^{n} X_i) \sim n^2 r(n)$) we get

$$\underset{n\to\infty}{\ell t}\, P\left(\mu_n \leq \chi_{1/r_n} + \frac{1}{2}\, \frac{\ell n_3\ 1/r_n - \ell n\ 4\Pi}{\chi_{1/r_n}}\ \frac{t}{\chi_{1/r_n}}\right) = \exp(-e^{-t})$$

for $-\infty < t < \infty$.

1. <u>INTRODUCTION</u>: Let $\{X_i, i \geq 0\}$ be a sequence of independent identically distributed random variables with mean zero and variance one. Many classical problems have come out of the study of the partial sums $S_k = \sum_{i=1}^{k} X_i$. The celebrated law of the iterated logarithms (LIL) gives the first order terms in the growth of S_k. It states

$$\lim_{k\to\infty} \frac{S_k}{(2k\ell n\ell n\, k)^{1/2}} = 1 \quad \text{a.s.} \quad . \tag{1.1}$$

Sometimes the LIL is stated in the following Feller form, giving information about the second order terms. For $\phi(n)$ positive and nondecreasing

$$P(S_k > k^{1/2}\phi(k) \text{ i.o.}) = \begin{matrix} 0 \\ \text{or} \\ 1 \end{matrix} \text{ according as} \tag{1.2}$$

$$\sum^{\infty} \frac{\phi(n)}{n} \exp(-\tfrac{1}{2}\phi^2(n)) \quad \begin{matrix} <\infty \\ \text{or} \\ \infty \end{matrix} \quad .$$

The quantity $S_k/k^{1/2}$ in both (1.1) and (1.2) can very easily be replaced by $\max_{1\leq i\leq k} S_i/i^{1/2}$. In this paper we will be interested in the behavior of this maximum. Besides the fixed growth rate given so precisely by (1.2), the randomness of $\max_{1\leq i\leq k} S_i/k^{1/2}$ is also of interest. Darling and Erdös[3] prove that if $\{X_i\}$ has finite absolute third moment then

$$\ell t_{n\to\infty} P\left\{\max_{1\leq k\leq n} \frac{S_k}{k^{1/2}} \leq (2\,\ell n_2 n)^{1/2} + \frac{\ell n(\ell n_2 n/4\Pi)}{2(2\ell n_2 n)^{1/2}} + \frac{t}{(2\ell n_2 n)^{1/2}}\right\} = \exp(-e^{-t}) \tag{1.3}$$

for all $-\infty < t < \infty$. As usual $\ell n_q n = q^{th}$ iterated logarithm of n.

The invariance principle of Erdös and Kac offers an elegant tool to prove these results. The required results are first proved in the special case when X_i are normally distributed. The Central Limit Theorem lets us approximate partial sums S_k/k by normal variables. The results then follow by essentially concluding that this approximation is, in fact, very good. Thus the study of the partial sums from the stationary Gaussian sequences is more significant than any other particular case in this area.

From now on, let $\{X_i\}$ be assumed to be Gaussian. It is interesting to see how (1.2) and (1.3) are affected when the assumption of independence among X_i is relaxed. Theorem 4 of Lai and Stout [5] states the following.

<u>Theorem</u> (Lai and Stout). Suppose there exists $0 < \alpha \le 2$, $\gamma > \alpha/2$, $M \ge 1$, $\delta > 0$, $\beta > 0$ and positive sequence $\{L(n)\}$ satisfying the following conditions

(i) $n^{\alpha}L(n) << v(n) << n^{\alpha}L(n)$ as $n \to \infty$, $v(n) = \text{variance}(\sum_{i=1}^{n} \; _{i})$.

(ii) $\left|\frac{v(n+m)}{v(n)} - 1\right| \le M(\frac{m}{n})^{\gamma}$ if $\delta n \ge m \ge M$.

(1.4)

(iii) $\limsup_{n\to\infty} \left\{ \max_{n(\ell n \ell n\, n)^{-\beta} \le j \le n} \frac{L(j)}{L(n)} \right\} < \infty$

and

$\liminf_{n\to\infty} \left\{ \min_{n(\ell n \ell n\, n)^{-\beta} \le j \le n} \frac{L(j)}{L(n)} \right\} > 0$

(iv) $\forall\ \rho > 0$ there exists m_ρ and $1 \ge \lambda_\rho > 0$ such that if $\lambda_\rho n \ge m \ge m_\rho$ then $(\frac{m}{n})^{\rho} \le \frac{L(n)}{L(m)} \le (\frac{n}{m})^{\rho}$.

Let $\phi(t)$ be a positive nondecreasing function on $[1,\infty)$. Then

$$P(S_n > (v(n))^{\frac{1}{2}}\phi(n) \text{ i.o.}) = \begin{matrix} 0 \\ \text{or} \\ 1 \end{matrix} \quad \text{according as} \tag{1.5}$$

$$\int_1^\infty \frac{(\phi(t))^{2/\alpha - 1}}{t} \exp\left(-\frac{\phi^2(t)}{2}\right) dt \quad \begin{matrix} <\infty \\ \text{or} \\ =\infty \end{matrix} \; .$$

In Sections 2 and 3 we extend (1.3) in the cases of moderately and strongly dependent stationary Gaussian sequences. We prove that if $\mu_n = \max_{0 \le i \le n} S_i/(v(i))^{\frac{1}{2}}$

and

$$(1.6) \qquad v(n) \sim n^{\alpha} L(n) \text{ as } n \to \infty$$

with some additional conditions on the slowly varying function L(n) to make it well behaved then

$$(1.7) \qquad \lim_{n\to\infty} P\left(\mu_n \le (2\ln_2 n)^{\frac{1}{2}} + \left(\frac{1}{\alpha} - \frac{1}{2}\right)\frac{\ln(c_\alpha \ln_2 n)}{(2\ln_2 n)^{\frac{1}{2}}} + \frac{x}{(2\ln_2 n)^{\frac{1}{2}}}\right) = \exp(-e^{-x})$$

for all $-\infty < x < \infty$. Also if

$$(1.8) \qquad v(n) \sim \left(1 + o\left(\frac{1}{(\ln n)^{\theta}}\right)\right) n^2 r(n)$$

for some $\theta > 0$ again with some additional conditions on r(n) then

$$(1.9) \quad \lim_{n\to\infty} P\left(\mu_n \le (2\ln_2 1/r_n)^{\frac{1}{2}} + \tfrac{1}{2}\frac{\ln_2 1/r_n - \ln 4\Pi}{(2\ln_2 1/r_n)^{\frac{1}{2}}} + \frac{x}{(2\ln_2 1/r_n)^{\frac{1}{2}}}\right) + \exp(-e^{-x})$$

for all $-\infty < x < \infty$.

The proofs as in [3], compare the maximum μ_n to that of a suitably chosen stationary Gaussian process on an appropriate set. The sketch of the main idea is as follows. We know that even though $\{S_k/(v(k))^{\frac{1}{2}}\}$ forms a standard Gaussian sequence, its no longer stationary. However, we could view this as coming from a stationary process being sampled with increasing frequency. If $v(k) \sim k^{\alpha}L(k)$, $0 < \alpha < 2$, then sample from an α-process (i.e., the correlation function near the origin is $\sim 1 - \frac{1}{2}|t|^{\alpha}$) at points $1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, \ldots, \sum_{i=1}^{n} 1/i, \ldots$. Thus we have n observations for this α-process in the interval about $[0, \ln n]$. The maximum of these n observations closely approximates μ_n. In this, what we call a "moderately dependent case", the frequency of sampling does not depend on the covariance function of $\{X_i\}$.

It is interesting to note that the above procedure fails when $\alpha = 2$. Oddly enough, we observe that in this case, the normalized sums process can be approximated by a Brownian motion with transformed time axis. If we have to carry the above analogy in this case as well, then we have to sample an $\alpha = 1$ process at a frequency that depends on the covariance function of the X-process. At any point k, instead of placing the next observation at a distance $1/k$ from the k^{th} observation, we place it at a distance approximately $|r'(k)/r(k)|$. We will call this the "strongly dependent" case. Thus, in this case, μ_n approximates the maximum of a $\alpha = 1$ process on the set $[0, \ell n\ 1/r_n]$. This allows arbitrarily slow growth rates of μ_n by choice of r_n. Though not surprising, it is interesting to note that in all cases, only the double exponential is obtained as the extreme value distribution.

There may seem a discripancy between (1.5) and (1.9). According to (1.5), if $v(k) = k^2 L(k)$ satisfying (1.4), then the first order term for μ_n is $(2\ \ell n_2 n)^{\frac{1}{2}}$ while as (1.9) places it at a much smaller number viz $(2\ \ell n_2\ 1/r_n)^{\frac{1}{2}}$. None of the covariances looked at in (1.8) satisfy (1.4). In fact, we could not find any examples of covariance functions satisfying (1.4) for $\alpha = 2$.

The next two sections give exact statements and the proofs of the moderately and the strongly dependent cases respectively. Examples of covariance functions satisfying the given conditions and some comments are in Section 4.

2. MODERATELY DEPENDENT SEQUENCES: Throughout the remaining, $\{X_i, i \geq 0\}$ will be a stationary Gaussian sequence with mean zero, variance one and $r_i = E\, X_0 X_i$. Let $v(k)$ = variance of $\sum_{i=0}^k X_i$; $Y_k = (v(k))^{-\frac{1}{2}} \sum_{i=0}^k X_i$ for $k = 0,1,2,\ldots$ and $\mu_n = \max_{0 \leq k \leq n} Y_k$.

Theorem 2.1: Suppose that for sufficiently large k,

$$\sum_{i=-k}^{k} r_i = k^{\delta} L(k) > 0 , \tag{2.1}$$

$-1 < \delta < 1$ and $L(k)$ is a slowly varying function satisfying

(2.2)
(i) for sufficiently large ℓ, $\frac{L(k)}{L(\ell)}$ is bounded if $k \geq \ell^{\theta}$, $\theta > 0$ and

(ii) if $\ell \geq k$, $k \to \infty$ such that $(\ell-k)/k^{1-\gamma} \to \infty \;\forall\; \gamma > 0$ but $\frac{\ell-k}{k} \to c$ for $0 \leq c \leq \infty$ then

$$\underset{k\to\infty}{\ell t} \; \frac{\ell}{k} \; \frac{L(\ell) - L(\ell - k)}{L(\ell - k)} = 0.$$

Then for $\alpha = \delta + 1$,

$$\underset{n\to\infty}{\ell t} \quad P(\mu_n \leq \beta_n + \frac{x}{\chi_n}) = \exp(-e^{-x}) \tag{2.3}$$

for all $-\infty < x < \infty$. We write

$$\chi_n = (2\ell n_2 n)^{\frac{1}{2}} \; ; \; \beta_n = \chi_n + \frac{(1/\alpha - 1/2)\ell n_3 n + \ell n(H_\alpha/\sqrt{4\pi})}{\chi_n}$$

and

$$H_\alpha = \underset{T\to\infty}{\ell t} \; \frac{1}{T} \int_0^\infty e^u P(\max_{0\leq t\leq T} \zeta(t) > u)du$$

where $\zeta(t)$ is a separable nonstationary Gaussian process with $E\,\zeta(t) = -|t|^{\alpha}$ and $\mathrm{Cov}(\zeta(s), \zeta(t)) = |s|^{\alpha} + |t|^{\alpha} - |s-t|^{\alpha}$.

Before we proceed to prove (2.3), we will state and prove two lemmas about $E\, Y_k Y_\ell$ which will simplify the proof of the Theorem.

Lemma 2.1: For $[\exp(\varepsilon m)] \leq k < \ell < [\exp(\varepsilon(m+1))]$, $\varepsilon > 0$ and m sufficiently large.

$$(2.4) \qquad 1 - (\tfrac{1}{2}+\varepsilon')\frac{v(\ell-k)}{v(k)} \leq E\, Y_k Y_\ell \leq 1 - (\tfrac{1}{2}-\varepsilon')\frac{v(\ell-k)}{v(k)} \quad .$$

By ε' we denote a small positive number depending on ε, not necessarily the same at all places. Also $[\cdot]$ denotes integral part.

Proof of Lemma 2.1: Since $v(k) = \sum_{j=0}^{k-1} \sum_{i=-j}^{j} r_i$, (2.1) and Theorem 1 in ([4], pg. 273) gives that $v(k)$ is regularly varying with exponent $\alpha = \delta + 1$ and

$$(2.5) \qquad \frac{1/\alpha\, k^{\alpha} L(k)}{v(k-1)} \to 1 \quad \text{as } k \to \infty.$$

Define $\omega_{k\ell} = \sum_{j=1}^{-k} \sum_{i=j}^{j+k} r_i$. Then

$$E\, Y_k Y_\ell = \frac{v(k) + \omega_{k\ell}}{(v(k)v(\ell))^{\frac{1}{2}}}$$

$$(2.6) \qquad = 1 - \frac{(v(k)v(\ell))^{\frac{1}{2}} - (v(k) + \omega_{k\ell})}{(v(k)v(\ell))^{\frac{1}{2}}}$$

$$= 1 - \frac{v(k)v(\ell) - (v(k) + \omega_{k\ell})}{v(k)v(\ell)\{1 + (v(k) + \omega_{k\ell})(v(k)v(\ell))^{-\frac{1}{2}}\}} .$$

Notice that $v(\ell) = v(k) + v(\ell-k) + 2\,\omega_{k\ell}$ and define $E_{k\ell} = \dfrac{v(\ell-k) + 2\,\omega_{k\ell}}{v(k)}$.
Then the R.H.S. above is equal to

$$(2.7) \qquad 1 - \frac{v(\ell-k) - \omega_{k\ell}^2/v(k)}{v(\ell)\{1 + (v(k) + \omega_{k\ell})(v(k)v(\ell))^{-\frac{1}{2}}\}}$$

$$= 1 - \frac{v(\ell-k)}{v(k)}\{1 - \frac{\omega_{\ell k}^2}{v(k)v(\ell-k)}\}(1 + E_{k\ell})^{-\frac{1}{2}}(1 + (1 + E_{k\ell})^{-\frac{1}{2}} + \frac{\omega_{k\ell}}{(v(k)v(\ell))^{\frac{1}{2}}}).$$

We need to show that the product of the last three terms in (2.7) could be bounded above and below by $\frac{1}{2} \pm \varepsilon'$ as $k \to \infty$. At (a), (b) and (c) below we find bounds for various quantites involved. First we look at $v(\ell-k)/v(k)$. The argument used here is repeated several times during the rest of the text. We have $0 \leq \ell-k \leq \varepsilon' k$. If $(\ell-k) \leq k^{\theta}$ for some $0 < \theta < 1$ then we can write $v(\ell-k) \leq L(\ell-k)\, k^{\alpha\theta}$ for large k and

$$\frac{v(\ell-k)}{v(k)} \leq \frac{L(\ell-k)}{L(k)k^{\alpha(1-\theta)}} \quad .$$

Since $L(t)$ is slowly varying function, we know that $L(t)\, t^{-\gamma} \to 0$ and $L(t)\, t^{\gamma} \to \infty$ as $t \to \infty$ for all $\gamma > 0$. If $k^{\theta} < \ell-k \leq \varepsilon' k$ then by (2.2(i)), $\frac{L(\ell-k)}{L(k)}$ is bounded. Thus

(a) $$v(\ell-k)|v(k) \leq \varepsilon'$$

for all k sufficiently large. Next, we will look at $\omega_{k\ell} = \sum_{j=1}^{\ell-k} \sum_{i=j}^{j+k} r_i$. For large k, by (2.1) we have

$$|\omega_{k\ell}| \leq (\text{const.})(\ell-k)\ell^{\delta}L(\ell) \; .$$

Hence

$$\frac{|\omega_{k\ell}|^2}{v(k)v(\ell-k)} \leq (\text{const.})\left(\frac{\ell-k}{k}\right)^{1-\delta} \frac{L^2(\ell)}{L(k)L(\ell-k)} \; .$$

The fact that we need $(\ell-k)$ sufficiently large to make the approximation $v(\ell-k) \sim (\ell-k)^{1+\delta} L(\ell-k)$ is inconsequential here, since, if $(\ell-k)$ is too small then the L.H.S. above is obviously small for k large. If $(\ell-k) \leq k^{\theta}$ then the

R.H.S. above tends to zero as $k \to \infty$ by same argument as at (a) and if $(\ell-k) > k^{\theta}$ then $L(\ell)/L(\ell-k)$ is bounded by (2.2(i)) and $L(\ell)/L(k)$ is always bounded for the range of ℓ and k under consideration. Thus

(b) $$\frac{|\omega_{k\ell}|^2}{v(k)v(\ell-k)} \leq \varepsilon'$$

for all k sufficiently large. Obviously $\frac{|\omega_{k\ell}|}{(v(k)v(\ell))^{\frac{1}{2}}} \leq \varepsilon'$ as well. Finally,

(c) $$\frac{|\omega_{k\ell}|}{v(k)} \leq \frac{\ell-k}{k}\left(\frac{\ell}{k}\right)^{\delta}\frac{L(\ell)}{L(k)} \leq \varepsilon$$

for all k sufficiently large. Substituting in (2.7) we get the result.

Lemma 2.2: If $\ell \geq k \to \infty$ such that $\ell-k \geq \ell^{\theta}$ for some $\theta > 0$ then

(2.8) $$\lim_{k\to\infty} \frac{|\omega_{k\ell}|}{(v(k)v(\ell))^{\frac{1}{2}}} \cdot \left(\frac{\ell}{k}\right)^{\gamma} = 0$$

for some $\gamma > 0$.

Proof of Lemma 2.2: We can write

$$\begin{aligned}
|\omega_{k\ell}| &= \left|\sum_{u=1}^{k} \sum_{i=-(\ell-j)}^{\ell-j} r_i - v(k)\right| \\
&\lesssim \sum_{j=1}^{k} (\ell-j)^{\delta} L(\ell-j) \\
&\simeq 1/\alpha \left(\ell^{\alpha} L(\ell) - (\ell-k)^{\alpha} L(\ell-k)\right) \\
&\lesssim 1/\alpha \left(\ell^{\alpha}(L(\ell) - L(\ell-k)) + \alpha k \ell^{\delta} L(\ell-k)\right) \\
&\simeq k\ell^{\delta} L(\ell-k) \left\{1 + \frac{\ell}{\alpha k} \frac{L(\ell) - L(\ell-k)}{L(\ell-k)}\right\}.
\end{aligned}$$

If ℓ/k is bounded then we can bound the terms in the bracket above by a constant since $\frac{L(\ell) - L(\ell-k)}{L(\ell-k)}$ is always bounded for $\ell-k \geq \ell^\theta$ by (2.2(i)). If $\ell/k \to \infty$ however, then $\frac{\ell}{k} \frac{L(\ell) - L(\ell-k)}{L(\ell-k)} \to 0$ by (2.2(ii)). Thus

$$\frac{|\omega_{k\ell}|}{(v(k)v(\ell))^{\frac{1}{2}}} \leq (\text{const.}) \frac{k\ell^{\delta} L(\ell-k)}{(k\ell)^{\alpha/2} (L(k)L(\ell))^{\frac{1}{2}}}$$

$$= (\text{const.}) \left(\frac{k}{\ell}\right)^{\frac{1-\delta}{2}} \frac{L(\ell-k)}{(L(k)L(\ell))^{\frac{1}{2}}} .$$

Choose $0 < \gamma < \frac{1-\delta}{2}$. Then the R.H.S. above is $o((k/\ell)^\gamma)$ as $k \to \infty$ because here both $L(\ell-k)/L(k)$ and $L(\ell-k)/L(\ell)$ are bounded due to (2.2(i)) (Notice that the case $\delta = 1$ is excluded).

We now turn to the proof of the Theorem.

<u>Proof of Theorem 2.1</u>: The proof *is split into two parts. First establishing*

$$\text{(2.9)} \qquad \liminf_{n\to\infty} P\left(\mu_n \leq \beta_n + \frac{x}{\chi_n}\right) \geq \exp(-e^{-x})$$

for $-\infty < x < \infty$. We recall that $\mu_n = \max_{0<k<n} Y_k$ where $Y_k = (v(k))^{-\frac{1}{2}} \sum_{i=1}^{k} \chi_i$. The constants β_n and χ_n are defined after (2.3).

We will split $\{Y_i, 1 \leq i \leq n\}$ into blocks $t_m \leq i < t_{m+1}$, $m = 0,1,\ldots,m_0$ where $t_m = [\exp(\varepsilon m)]$ for some $\varepsilon > 0$ and m_0 is such that $t_{m_0} \simeq n$. We will find a lower bound for the probability in L.H.S. of (2.9) by treating Y_i in different blocks as independent since $E\, Y_k Y_\ell$ will be shown to be nonnegative. Now, suppose $E\, Y_k Y_\ell$ was, in fact equal to $1 - \frac{1}{2}\left(\frac{\ell-k}{k}\right)^\alpha$. Let $\xi(t)$ be a standard stationary Gaussian process with covariance function $1 - (\frac{1}{2} + \varepsilon)|t|^\alpha$ for $0 \leq t \leq \varepsilon$. We can sample $\{\xi(t), 0 \leq t \leq \varepsilon\}$ at points s_j where $s_1 = 1/t_m$ and $s_k = s_{k-1} + 1/s_{k-1}$. It would be easy to see that $\max_k \xi(s_k)$ provides a stochastic upper bound for

$\max_{t_m \le k \le t_m+1} Y_k$. However, by using Lemma 2.1, we can get explicit bounds for $E\, Y_k Y_\ell$ only when $(\ell-k)$ is large. Thus it is necessary to delete all but a subsequence $\{Y_{t_{m,h}}\}$ of variables from the m^{th} block. The proof will follow this general outline. To start off, we also need to exclude $Y_1, \dots, Y_{(\ell nn)^{1/2}}$ from consideration, since all the bounds gotten in the Lemmas work only for large k.

For the first part of the proof, define $u_n = \beta_n + x/\chi_n$. Then for $\mu_n' = \max_{(\ell nn)^{1/2} \le k \le n} Y_k$

$$|P(\mu_n \le u_n) - P(\mu_n' \le u_n)| \le P(\max_{0 \le k \le (\ell nn)^{1/2}} Y_k > u_n)$$

$$\le (\ell nn)^{1/2}\, \phi(u_n)/u_n$$

where $\phi(u) = (2\Pi)^{-1/2} \exp(-u^2/2)$. The R.H.S. above tends to zero. Thus we can replace μ_n by μ_n' in (2.3).

We will prove (2.9) with μ_n replaced by μ_n'. For k and ℓ large $E\, Y_k Y_\ell \ge 0$ since by (2.1), $v(k) + 2\,\omega_{k\ell} > 0 \Rightarrow \omega_{k\ell} > -v(k)/2$ and $v(k) + \omega_{k\ell} > v(k)/2 > 0$. This, in view of (2.6) shows that $E\, Y_k Y_\ell \ge 0$ for k and ℓ large. By Slepian's Lemma [11],

$$P(\mu_n' \le u_n) \ge \prod_{m=m_1}^{m_0} P(\max_{t_m \le \ell < t_{m+1}} Y_\ell \le u_n) \tag{2.10}$$

where m_1 is such that $t_{m_1} \sim (\ell nn)^{1/2}$.

The next step is to select a proper subsequence $t_{m,k}$ from the m^{th} block. Define $t_{m,h} = [\exp(\varepsilon(m^2+h)^{1/2})]$ for $h = 0,1,\dots,(2m+1)$ i.e., $t_{m,0} = t_m$ and $t_{m,2m+1} = t_{m+1}$. Define $Z_\ell = Y_\ell - Y_{t_{m,h}}$ for $t_{m,h} < \ell \le t_{m,h+1}$, $h = 0,1,\dots,2m$.

Then

$$(2.11)\qquad P(\max_{t_m\le \ell<t_{m+1}} Y_\ell \le u_n) \ge P(\max_{0\le h\le 2m} Y_{t_{m,h}} \le u_n - \frac{\ln m}{m^{\alpha/2}})$$

$$- P(\max_{t_m<\ell<t_{m+1}} Z_\ell > \frac{\ln m}{m^{\alpha/2}})$$

where $\alpha' < \alpha$. We will get an upperbound for the R.H.S. above by using the following result.

<u>Theorem</u>: (Marcus and Shepp[6]): Let $\{\xi_i, i \ge 1\}$ be a sequence of jointly Gaussian random variables with $P\{\sup_{i\ge 1} |Y_i| < \infty\} = 1$. Then, letting $\sigma^2 = \sup_{i\ge 1} \mathrm{var}(Y_i)$, for $\rho>0$ and all sufficiently large t, we have

$$(2.12)\qquad P(\sup_{i\ge 1} |Y_i| > t) < \exp\{-(1 - \rho)t^2/(2\sigma^2)\} .$$

In order to apply this to Z_ℓ, we need to verify that $P(\max_{t_m\le \ell<t_{m+1}} Z_\ell < \infty) = 1$. It is sufficient to verify that $\max_{t_m\le \ell<t_{m+1}} Y_\ell$ is finite with probability one. By (2.4), for $t_m \le k < \ell < t_{m+1}$ and m large,

$$(2.13)\qquad E\, Y_k Y_\ell \ge 1 - (\tfrac{1}{2} + \varepsilon')\left(\frac{\ell-k}{k}\right)^\alpha \frac{L(\ell-k)}{L(k)} \ge 1 - \varepsilon' \left(\frac{\ell-k}{k}\right)^{\alpha'} .$$

We write the first inequality above for the sake of convenience. For $(\ell-k)$ too small, we may not be able to write $v(\ell-k) \sim (\ell-\kappa)^\alpha L(\ell-k)$. However, the second inequality is correct for large k since we have chosen $\alpha' < \alpha$. The argument for other values of $(\ell-k)$ is similar to the one used before. Now, for values of k and ℓ under consideration $0 \le \frac{\ell-k}{k} < \varepsilon$. By Slepian's Lemma [11], $\max_{t_m\le \ell<t_{m+1}} Y_\ell$ is stochastically smaller than the maximum of an α'-process (i.e.,

correlation near zero is $1 - (\text{const.})\,|t|^{\alpha'}$) on the set $[0,\varepsilon]$. The later is obviously finite with probability one. Thus from (2.12) we have

$$\text{(2.14)} \qquad P(\max_{t_m \le \ell < t_{m+1}} Z_\ell > \frac{\ell n\, m}{m^{\alpha'/2}}) \le \exp\{-(\frac{1-\rho}{2})\,(\frac{\ell n\, m}{m^{\alpha'/2}})^2 \cdot \frac{1}{\sigma_m^2}\}$$

where $\sigma_m^2 = \max_{t_m \le \ell < t_{m+1}} \operatorname{Var}(Z_\ell)$. But for $t_{m,h} \le \ell < t_{m,h+1}$,

$$\begin{aligned}
\operatorname{Var}(Z_\ell) &= V(Y_\ell - Y_{t_{m,h}}) \\
&= 2(1 - E\,Y_\ell Y_{t_{m,h}}) \\
&\le 2\varepsilon' \left(\frac{\ell - t_{m,h}}{t_{m,h}}\right)^{\alpha'} \quad \text{using (2.13)} \\
&\le \varepsilon' \left(\frac{t_{m,h+1} - t_{m,h}}{t_{m,h}}\right)^{\alpha'} \\
&\le \varepsilon' m^{-\alpha'} \quad \text{for all } h = 0,1,\ldots,2m.
\end{aligned}$$

Hence the R.H.S. of (2.14) is at most

$$\exp\{-(\text{const.})(\ell n\, m)^2\}$$

and the R.H.S. of (2.10) is at least

$$\prod_{m=m_1}^{m_0} \{P(\max_{0 \le h \le 2m} Y_{t_{m,h}} \le u_n - \frac{\ell n\, m}{m^{\alpha'/2}}) - \exp(-(\text{const.})(\ell n\, m)^2)\}$$

$$\text{(2.15)} \qquad = E_{m_1} \prod_{m=m_1}^{m_0} P(\max_{0 \le h \le 2m} Y_{t_{m,h}} \le u_n - \frac{\ell n\, m}{m^{\alpha'/2}})$$

where $E_{m_1} = \prod_{m=m_1}^{m_0} \{1 - \exp(-(\text{const.})(\ell n\ m)^2)/ P(\max_{0\leq h\leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ell n\ m}{m^{\alpha'/2}}\}$. We have established before that $\max_{0\leq h\leq 2m} Y_{t_{m,h}}$ is finite with probability 1 for all m. Hence $P(\max_{0\leq h\leq 2m} Y_{t_{m,h}} \leq u_n - \frac{\ell n\ m}{m^{\alpha'/2}}) \to 1$ as $n \to \infty$ and

$$E_{m_1} \geq \exp\{-(\text{const.}) \sum_{m=m_1}^{\infty} e^{-(\ell n\ m)^{3/2}}\}.$$

The R.H.S. above tends to one as $n \to \infty$. It remains to show that the product in R.H.S. in (2.15) is bounded below by the required limit. Now, since $t_{m,h}$ and $t_{m,j}$ are sufficiently far apart for $h \neq j$, we can write

$$E\, Y_k Y_\ell \geq 1 - (\tfrac{1}{2} + \varepsilon')\, \frac{L(\ell-k)}{L(k)}\, \left(\frac{\ell-k}{k}\right)^{\alpha}$$

for values of k and ℓ in the set $\tau_m = \{t_{m,h},\ h = 0,1,\ldots,2m\}$. Notice that for such values of k and ℓ, $(\ell-k)/k$ cannot tend to zero very rapidly i.e., $(\ell-k)/k^{1-\gamma} \to \infty\ \forall\ \gamma > 0$. Hence applying (2.2(ii)) we have $L(\ell-k)/L(\ell) \to 1$. Also, since $\ell \leq (1+\varepsilon)k$, $L(\ell)/L(k) \to 1$ as well. By choosing an appropriate value of ε', we can write for large k that

$$E\, Y_k Y_\ell \geq 1 - (\tfrac{1}{2} + \varepsilon')\left(\frac{\ell-k}{k}\right)^{\alpha}$$

for $k,\ell \in \tau_m$. Let $\{\xi(t),\ t \geq 0\}$ be standard stationary Gaussian process with covariance function $1 - (\tfrac{1}{2} + \varepsilon')\ |t|^{\alpha}$ for $0 \leq t \leq \varepsilon$. We see that the covariance matrix of $\{Y_\ell,\ \ell \in \tau_m\}$ is bounded below by that of $\{\xi_{\frac{\ell - t_m}{t_m}},\ \ell \in \tau_m\}$. Thus $\max_{0\leq h\leq 2m} Y_{t_{m,h}}$ is stochastically bounded above by $\max_{0\leq h\leq 2m} \xi_{\frac{t_{m,h} - t_m}{t_m}}$ which is at most

$\max_{0\le t\le\varepsilon} \xi(t)$. Thus the product in the R.H.S. of (2.15) is asymptotically at least

$$\text{(2.16)} \qquad \exp\{-\sum_{m=m_1}^{m_0} P(\max_{0\le t\le\varepsilon} \xi(t) > u_n - \frac{\ln m}{m^{\alpha'/2}})\}$$

$$\sim \exp\{-\sum_{m=m_1}^{m_0} \varepsilon(u_n - \frac{\ln m}{m^{\alpha'/2}})^{2/\alpha-1} \phi(u_n - \frac{\ln m}{m^{\alpha'/2}})(\tfrac{1}{2} + \varepsilon')^{1/\alpha} H_\alpha\}$$

by Lemma (3.8) of Pickands [9]. H_α is defined in the statement of Theorem 2.1. Substituting the value for u_n, we have the R.H.S. of (2.16) asymptotically equal to

$$\text{(2.17)} \quad \exp\{-\frac{\varepsilon}{\ln m}(1 + 2\varepsilon')^{1/\alpha} \exp(-x + o(1)) \sum_{m=m_1}^{m_0} \exp(u_n \ln m/m^{\alpha'/2})\} .$$

Spliting the sum into two parts say $m_1 \le m \le (m_0)^{\frac{1}{2}}$ and $m_0^{\frac{1}{2}} < m \le m_0$, we see that for large n

$$\sum_{m=m_1}^{m_0} \exp(u_n \ln m/m^{\alpha'/2}) \le m_0^{\frac{1}{2}} \exp(u_n) + m_0 \exp(1/(\ln m)^{\alpha'/2}) = m_0(1 + o(1)).$$

Substituting in (2.17) we get that

$$\liminf_{n\to\infty} P(\mu_n \le \beta_n + x/\chi_n) \ge \exp(-(1 + 2\varepsilon')^{1/\alpha} e^{-x})$$

for all $\varepsilon' > 0$. This concludes the proof of (2.9).

It is much easier to show the otherside viz,

$$\text{(2.18)} \qquad \limsup_{n\to\infty} P(\mu_n \le \beta_n + x/\chi_n) \le \exp(-e^{-x}) .$$

We just exclude the appropriate number of Y_k and then compare them with variables selected from an appropriate α-process by Berman's Lemma [1]. Let u_n, m_0 and m_1

be the same as defined before. We will define new subsequences $t_{m,h}$ and the sets τ_m. To avoid clumsy notation, we will use the same symbols. Thus, it should be noted that $t_{m,h}$ and τ_m are defined two different ways in this section and two different ways in Section 3. All efforts to try to define a subsequence that will work for both parts of the proof in each section have failed so far. Care should be taken in noting an appropriate definition of $t_{m,h}$ and τ_m in reading different parts of the proof. Let

$$t_{m,h} = [\exp\{\varepsilon(m + \frac{h}{(\ln m)^{2/\alpha}})\}]$$

and

$$\tau_m = \{t_{m,h};\ 0 \le h \le [(1-\varepsilon)(\ln m)^{2/\alpha}] = m_h \quad \text{say}\} .$$

(i.e., we have clipped a small portion from the right hand side for each of the sets τ_m). Now

$$(2.19) \qquad P(\mu_n \le u_n) \le P\{\bigcap_{m=m_1}^{m_0} (\max_{t_{m,h}\in\tau_m} Y_{t_{m,k}} \le u_n)\}$$

$$\le \prod_{m=m_1}^{m_0} P(\max_{t_{m,h}\in\tau_m} Y_{t_{m,h}} \le u_n) + C \sum_{p>q=m_1}^{m_0}\sum^{m_0} \sum_{\ell=1}^{P_\ell} \sum_{h=1}^{q_h} \rho_{P\ell qh} \exp(-\frac{u_n^2}{1+\rho_{P\ell qh}}) .$$

The last inequality follows from Berman's Lemma. C is some positive constant and

$$\rho_{P\ell qh} = E(Y_{t_{P,\ell}} Y_{t_{q,h}}) \quad \text{if } P \ne q$$

$$= 0 \qquad\qquad \text{if } P = q .$$

Notice also that the largest value of $(1-\rho^2_{P\ell qh})^{-\frac{1}{2}}$ depends only on ε and is absorbed in C. By Lemma 2.1,

$$E\, Y_{t_{P,\ell}} Y_{t_{q,h}} \leq 1 - (\tfrac{1}{2} - \varepsilon') \frac{v(t_{P,\ell} - t_{q,h})}{v(t_{q,h})} .$$

$$\sim 1 - (\tfrac{1}{2} - \varepsilon') \frac{L(t_{P,\ell} - t_{q,h})}{L(t_{q,h})} \left(\frac{t_{P,\ell} - t_{q,h}}{t_{q,h}} \right)^{\alpha} .$$

Because of the above definitions, when $P \neq q$, $t_{P,\ell} - t_{q,h} \gtrsim t_{q,h}(e^{\varepsilon^2} - 1) \sim \varepsilon^2 t_{q,h}$. By the same reasoning as before and condition (2.2), we have $L(t_{P,\ell} - t_{q,h})/L(t_{q,h}) \sim 1$. Thus

$$\text{(2.20)} \qquad E\, Y_{t_{P,\ell}} Y_{t_{q,h}} \leq 1-\varepsilon'$$

for all $P \neq q$ and q large. For all m, $m_h \leq (\ln m_0)^{2/\alpha}$. Substituting, an upperbound for the sum in the R.H.S. of (2.19) is

$$\text{(2.21)} \quad c \sum_{q=m_1}^{m_0} \{m_0^{\eta} (\ln m_0)^{4/\alpha} \exp(-\frac{u_n^2}{2-\varepsilon'}) + \sum_{p=q+(m_0)^{\eta}}^{m_0} (\ln m_0)^{4/\alpha} \rho_{P\ell qh} \exp(-\frac{u_n^2}{1+\rho_{P\ell qh}})\}$$

for all $0 < \eta < 1$. $u_n^2 \sim 2 \ln_2 n$ and $m_0 \sim \frac{(\ln n)}{\varepsilon}$. We choose $\eta < \varepsilon'/(2-\varepsilon')$ for the value of ε' that works in (2.20). This makes the sum of the first term in (2.21) to be o(1). For the second part of the sum, we know by Lemma 2.2 that

$$E\, Y_{t_{P,\ell}} Y_{t_{q,h}} \leq \left(\frac{v(t_{q,h})}{v(t_{P,\ell})}\right)^{\frac{1}{2}} + \left(\frac{t_{q,h}}{t_{P,\ell}}\right)^{\gamma}$$

for some $\gamma > 0$. But $v(t_{q,h})/v(t_{P,\ell}) \sim (t_{q,h}/t_{P,\ell})^{\alpha}$ for large q. Thus for some $\gamma' > 0$

$$\rho_{p\ell qh} \leq (t_{q,h}/t_{p,\ell})^{\gamma'}$$

$$\leq \exp(-\gamma'\varepsilon\, m_0^{\eta})$$

for all $p \geq q + (m_0)^{\eta}$. Thus the sum in (2.21) is o(1). To bound the product in the R.H.S. of (2.19), we define a standard stationary Gaussian process $\{\xi(t),\ 0 \leq t \leq \varepsilon\}$ (different from the one in the first part of the proof) with covariance function $1-(\frac{1}{2}-\varepsilon')|t|^{\alpha}$ near origin. We bound

$$P(\max_{t_{m,h}\in\tau_m} Y_{t_{m,h}} \leq u_n) \leq P(\max_{1\leq h\leq m_h} \xi_{\frac{t_{m,h}-t_m}{t_m}} \leq u_n) .$$

By Lemma (3.8) of [9], the R.H.S. above is asymptotically equal to

$$\varepsilon(1-\varepsilon)(1-\varepsilon')^{1/\alpha} H_{\alpha}\, u_n^{2/\alpha-1}\, \phi(u_n).$$

Substituting in the product we get the desired result (2.18). The details of the last argument are very much similar to those in the first part of the proof and hence are not repeated. This finishes the proof of Theorem 2.1.

3. STRONGLY DEPENDENT CASE:

In last section we considered the cases when $v(n) \sim n^{\alpha}L(n)$ for $0 < \alpha < 2$. In this section we will have $\alpha = 2$. The sequence $\{X_i,\ i \geq 0\}$ and μ_n are as described in Section 2.

Theorem 3.1: Let f be a probability density function on the real line and set

$$A_k = \{(x,y)/-\infty < x < \infty;\ 0 \leq y < \infty \text{ and } f(x+k) > y\} .$$

Assume that the correlation function $r_k = EX_0X_k$ satisfies

$$(3.1)\quad \begin{aligned}&\text{(i)}\quad r_k = \int_{-\infty}^{\infty} f(x) \wedge f(x+k)dx\\ &\text{(ii)}\quad \text{for } k \geq k_0,\ A_k \cap A_0 \subseteq A_\ell \cap A_0 \text{ for } 0 \leq k \leq \ell.\end{aligned}$$

Let $r_k \to 0$ be a slowly varying function such that for large k

$$(3.2)\qquad \sum_{i=-k}^{k} r_i = (2 + E_k)k\, r_k$$

with $E_k = o\left(\frac{1}{(\ell n\ k)^\theta}\right)$ nonincreasing and $\theta > 0$. Then

$$(3.3)\qquad \underset{n\to\infty}{\ell t}\ P(\mu_n \leq \beta_n + x/\chi_n) = \exp(-e^{-x})$$

for all $-\infty < x < \infty$. Here

$$\chi_n = (2\ \ell n_2\ 1/rn)^{\frac{1}{2}}$$

and

$$\beta_n = \chi_n + \tfrac{1}{2}\,\frac{\ell n_2\ 1/r_n - \ell n\ 4\Pi}{\chi_n}.$$

Condition (3.1) is used and discussed in Mittal and Ylvisaker [7]. From discussion there, we have

$$X_i = (1-r_k)^{\frac{1}{2}} W_i^{\,k} + I_k \qquad 0 \leq i \leq k$$

where $W_i^{\,k}$ and I_k are independent normal with mean zero, $\mathrm{var}(W_i^{\,k}) = 1$ and $E\, I_k I_\ell = r_k \quad \forall\ \ell \geq k$. Thus

$$Y_k = \frac{\sum_{i=1}^{k} X_i}{(v(k))^{\frac{1}{2}}} = (1-r_k)\ \frac{\sum_{i=1}^{k} W_i^{\,k}}{(v(k))^{\frac{1}{2}}} + \frac{k\, I_k}{(v(k))^{\frac{1}{2}}}.$$

The general idea of the proof is to show that the variables $\sum_{i=1}^{k} W_i^{\,k}/(v(k))^{\frac{1}{2}}$ are too small and hence we can replace the process $\{Y_k\}$ by $\{kI_k|(v(k))^{\frac{1}{2}}\}$. (In fact, we replace it by $\eta_k = I_k/(r_k)^{\frac{1}{2}}$. We deal with the new process in very much similar ways as the last section. The following Lemma does the first part of the proof.

<u>Lemma 3.1</u>: Under the conditions of Theorem 3.1,

$$\lim_{k\to\infty} (\ell n\ k)^{\delta}\ \zeta_k = 0 \quad \text{a.s.} \tag{3.4}$$

for all $\delta > 0$ sufficiently small, where $\zeta_k = Y_k - \eta_k$.

<u>Proof of Lemma 3.1</u>: Let us define $t_k = [\exp(k^{\gamma})]$ for $0 < \gamma < 1$ and

$$A_k = A_k(\varepsilon) = \{ \max_{t_k \le j < t_{j+1}} \zeta_j > \varepsilon\}$$

for $\varepsilon > 0$, $k = 0,1,\ldots$. The result follows by symmetry and the use of Borel-Cantelli Lemma if we show that $\sum_{k=1}^{\infty} P(k) < \infty$. To compute $P(A_k)$, we find lower bound on the correlations of ζ_j, $t_k \le j \le t_{k+1}$. We first note that as a result of the assumed conditions, $r_k \ge 0$ and for sufficiently large k, $r_k \le r_i$ $\forall\ i \le k$. Now for $j \le \ell$,

$$E\ \zeta_\ell \zeta_j = E\ Y_\ell Y_j - \frac{jr_j + \sum_{j+1}^{\ell} r_i}{(r_j\ v(\ell))^{\frac{1}{2}}} - \frac{j(r_\ell)^{\frac{1}{2}}}{(v(j))^{\frac{1}{2}}} + (\frac{r_\ell}{r_j})^{\frac{1}{2}} . \tag{3.5}$$

But

$$r_\ell^{\frac{1}{2}} \{ \frac{1}{r_j^{\frac{1}{2}}} - \frac{j}{(v(j))^{\frac{1}{2}}} \} = r_\ell^{\frac{1}{2}} \{\frac{(v(j))^{\frac{1}{2}} - jr_j^{\frac{1}{2}}}{(r_j v(j))^{\frac{1}{2}}} \} \sim (\frac{r_\ell}{r_j})^{\frac{1}{2}} E_j \ge 0 \text{ for large j}$$

and

$$E\, Y_\ell Y_j - \frac{jr_j + \sum_{j+1}^{\ell} r_i}{(r_j\, v(\ell))^{\frac{1}{2}}} > \left(\frac{v(j)}{v(\ell)}\right)^{\frac{1}{2}} - \frac{\ell\, r_j^{\frac{1}{2}}}{(v(\ell))^{\frac{1}{2}}}$$

$$= \frac{(v(j))^{\frac{1}{2}} - \ell\, r_j}{(v(\ell))^{\frac{1}{2}}}$$

$$\sim \frac{jr_j^{\frac{1}{2}}}{\ell r_\ell^{\frac{1}{2}}}\left(E_j - \frac{\ell-j}{j}\right) .$$

For $t_k \le \ell, j < t_{k+1}$, $\frac{\ell-j}{j} \le (\text{const.})\, k^{-(1-\gamma)}$. Thus for large j, since $r_j \ge r_\ell$,

$$E\, \zeta_\ell \zeta_j \ge E_j \cdot \frac{j}{\ell} \ge E_j\left(1 - \frac{c_1}{k^{1-\gamma}}\right)$$

where $c > 0$ is a constant. We know that $v(Z_j) = 2\left(1 - \frac{jr_j^{\frac{1}{2}}}{(v(j))^{\frac{1}{2}}}\right) \sim E_j$ and E_j is nonincreasing. Hence

$$E\, \frac{\zeta_j \zeta_\ell}{(v(Z_j)v(Z_\ell))^{\frac{1}{2}}} \ge 1 - c_1\, k^{-(1-\gamma)} \quad \text{for } t_k \le j, \ell < t_{k+1}$$

Now,

$$P(A_k) = P\left(\max_{t_k \le j < t_{k+1}} \zeta_j > \varepsilon\right)$$

(3.6)
$$\le P\left(\max_{t_k \le j < t_{k+1}} \frac{\zeta_j}{(v(Z_j))^{\frac{1}{2}}} > \frac{\varepsilon}{(E_{t_k})^{\frac{1}{2}}}\right)$$

$$\le P\left\{\left(\frac{c_1}{k^{1-\gamma}}\right)^{\frac{1}{2}} M^*_{t_{k+1}-t_k} + \left(1 - \frac{c_1}{k^{1-\gamma}}\right)^{\frac{1}{2}} U > \varepsilon\, k^{\frac{\gamma\theta}{2}}\right\}$$

where M^*_n is maximum of n independent standard normal variables and U also standard normal, independent of M^*_n. An upperbound for the R.H.S. of (3.6) is

$$P(M^*_{t_{k+1}-t_k} > \frac{\varepsilon}{2\,c_1^{\frac{1}{2}}}\,k^{\frac{1-\gamma}{2}+\frac{\gamma\theta}{2}}) + 1 - \Phi(\frac{\varepsilon}{2}\,k^{\frac{\gamma\theta}{2}})$$

where $\Phi(x) = \int_{-\infty}^{x} \phi(u)du$. Noticing that $t_{k+1}-t_k \sim \exp(k^\gamma)$, we see that the above is summable over k. This completes the proof of the Lemma.

Next, we can replace μ_n in (3.3) by $\nu_n = \max_{0\leq k\leq n} I_k/r_k^{\frac{1}{2}} = \max_{0\leq k\leq n} \eta_k$ because

$$\chi_n|\mu_n - \nu_n| \leq \chi_n \max_{0\leq k\leq n} \zeta_k$$

and

$$\chi_n/(\ell n\ n)^\delta \to 0 \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \delta > 0.$$

For the proof of (3.3) with ν_n in place of μ_n, we follow the same procedure as in Section 2. We compare $\{\eta_k\}$ with a sequence sampled from an appropriately chosen (α=1) process. Unlike Section 2, the frequency of sampling here depends on the covariance function $\{r_k\}$. Some of the details of Section 2 become easier here in view of the observation that $I_k = B(r_k)$ where B denotes a standardized Brownian motion. Sketch of the proof is given below avoiding repetition of arguments in Section 2.

Define (different from all previous definitions)

$$t_k = [r^{-1}(e^{-\varepsilon k})] \ ; \quad t_{m,k} = [r^{-1}(e^{-\varepsilon(m^2+k)^{\frac{1}{2}}})]$$

and

$$\tau_m = \{t_{m,k} \ ; \quad 0 \leq k \leq 2m\} .$$

Here $r^{-1}(\delta) = \min_t r(t) = \delta$. Since r is monotone, t_k is nondecreasing. Also $r(r^{-1}(\delta)) = \delta$. By similar arguments that lead to (2.10) we have

$$P(\nu_n' \leq u_n) \geq \prod_{m=m_1}^{m_0} P(\max_{t_{m,k}\in\tau_m} \eta_{t_{m,k}} \leq u_n) \tag{3.7}$$

where $u_n = \beta_n + x/\chi_n$; m_0 and m_1 are such that $t_{m_1} \sim (\ell n\ n)^{\frac{1}{2}}$ and $m_0 = \min\{m | t_m \geq n\}$. Also $\nu_n' = \max_{(\ell n\ n)^{\frac{1}{2}} \leq k \leq n} \eta_k$. Define $Z_\ell = \eta_\ell - \eta_{t_{m,k+1}}$ for $t_{m,k} \leq \ell < t_{m,k+1}$. Then

$$P(\max_{t_m \leq \ell < t_{m+1}} \eta_\ell \leq u_n) \geq P(\max_{0 \leq k \leq 2m} \eta_{t_{m,k}} \leq u_n - \frac{\ell n\ m}{m^{\frac{1}{2}}})$$

$$- P(\max_{t_m \leq \ell < t_{m+1}} Z_\ell > \frac{\ell n\ m}{m^{\frac{1}{2}}}).$$

But

$$P(\max_{t_m \leq \ell < t_{m+1}} Z_\ell > \frac{\ell n\ m}{m^{\frac{1}{2}}}) \leq (2m+1)\, P(\max_{t_{m,k} \leq \ell < t_{m,k+1}} \frac{I_\ell}{r_\ell^{\frac{1}{2}}} - \frac{I_{t_{m,k+1}}}{r_{t_{m,k+1}}^{\frac{1}{2}}} > \frac{\ell n\ m}{m^{\frac{1}{2}}})$$

$$\leq (2m+1)\{P(\max_{t_{m,k} \leq \ell < t_{m,k+1}} \frac{I_\ell - I_{t_{m,k+1}}}{r_\ell^{\frac{1}{2}}} > \frac{\ell n\ m}{m^{\frac{1}{2}}})$$

$$+ P(\max_{t_{m,k} \leq \ell < t_{m,k+1}} I_{t_{m,k+1}} (r_\ell^{-\frac{1}{2}} - r_{t_{m,k+1}}^{-\frac{1}{2}}) > \frac{\ell n\ m}{m^{\frac{1}{2}}})\}$$

$$\leq (2m+1)\{P(\max_{t_{m,k} \leq \ell < t_{m,k+1}} (I_\ell - I_{t_{m,k+1}}) > \frac{\ell n\ m}{m^{\frac{1}{2}}} r_{t_{m,k+1}}^{\frac{1}{2}}) \tag{3.8}$$

$$+ P(I_{t_{m,k+1}} (r_{t_{m,k+1}}^{-\frac{1}{2}} - r_{t_{m,k}}^{-\frac{1}{2}}) > \frac{\ell n\ m}{m^{\frac{1}{2}}})\}.$$

Since $I_\ell = B(r_\ell)$, we can write $I_\ell - I_{t_{m,k+1}}$ as $B(r_\ell - r_{t_{m,k+1}})$. Using proposition 12.20 of Brieman [2], we get the first probability in the R.H.S. of (3.8) to be atmost

$$(3.9) \qquad 2P(I_{t_{m,k}} - I_{t_{m,k+1}} > \frac{\ln m}{m^{\frac{1}{2}}} r^{\frac{1}{2}}_{t_{m,k+1}}).$$

But $r_{t_{m,k}} \sim \exp(-\varepsilon(m^2+k)^{\frac{1}{2}})$. Thus

$$\begin{aligned} \operatorname{var}(I_{t_{m,k}} - I_{t_{m,k+1}}) &= r_{t_{m,k}} - r_{t_{m,k+1}} \\ &\sim \exp(-\varepsilon(m^2 + k + 1)^{\frac{1}{2}})[e^{\frac{-\varepsilon}{2m}} - 1] \\ &\sim \frac{\varepsilon}{2m} r_{t_{m,k+1}} \end{aligned}$$

and (3.9) is at most $\dfrac{\varepsilon\,\phi(2\,\frac{\ln m}{\varepsilon})}{\ln m}$. Substituting values for $r_{t_{m,k}}$, we get approximately the same bound for the second probability in the R.H.S. of (3.8). Hence

$$(3.10) \qquad P(\max_{t_m \le \ell < t_{m+1}} Z_\ell > \frac{\ln m}{m^{\frac{1}{2}}}) \lesssim \frac{2(2m+1)\phi(\frac{2\ln m}{\varepsilon})}{\ln m}.$$

The R.H.S. of (3.10) is summable over m. Following the same line as in Section 2, it only remains to show that

$$P(\max_{0 \le k \le 2m} \eta_{t_{m,k}} \le u_n) \sim \varepsilon \; u_n \phi(u_n)(\tfrac{1}{2} + \varepsilon').$$

(Notice that the constant H_α for α=1 is explicitly given in Theorem 4.4 of [8]). Let $\{\xi(t),\ t \ge 0\}$ be a standard stationary Gaussian process with covariance function $1 - (\frac{1}{2} + \varepsilon')\ |t|$ near origin. Then

$$E(\eta_{t_{m,k+\ell}}\,\eta_{t_{m,k}}) = (r_{t_{m,k+\ell}}/r_{t_{m,k}})^{\frac{1}{2}}$$

$$\sim \exp\{\tfrac{\varepsilon}{2}((m^2+k+\ell)^{\frac{1}{2}} - (m^2+k)^{\frac{1}{2}})$$

$$\gtrsim 1 - \frac{(1+\varepsilon)\varepsilon\,\ell}{4(m^2+k)^{\frac{1}{2}}}$$

$$\gtrsim 1 - (\tfrac{1}{2}+\varepsilon)(\tfrac{\varepsilon\ell}{2m})$$

$$= E(\xi_{\frac{t_{m,\ell}-t_m}{t_m}} \cdot \xi_0)\ .$$

It follows in the manner similar to Section 2 that

$$\lim_{n\to\infty} P(\mu_n \leq \beta_n + x/\chi_n) \geq \exp(-e^{-x}). \tag{3.11}$$

(It should be noted that $r_{t_{m_0}} \sim \exp(-\varepsilon m_0) \gtrsim e^{-\varepsilon} r_n$. Strictly speaking, the normalized constants for the ξ process should be with r_n replaced by $r_{t_{m_0}}$ but it is easy to see that this does not make any difference in the limit.)

The procedure for the rest of the proof is now apparent. We define a new

$$t_{m,k} = [r^{-1}\{\exp(-\varepsilon(m + \frac{k}{(\ell n\ m)^2}))\}]\ .$$

The fact that $\rho_{P\ell qh} = E\,\eta_{tP,\ell}\,\eta_{tq,h}$ decreases rapidly to zero for $p-q \geq m_0$ is quite obvious. The rest of the arguments are very similar and hence not repeated.

4. EXAMPLES AND DISCUSSION: Examples of covariance functions $\{r_k\}$ satisfying (2.1) and (2.2) are given by convex $\{r_k\}$, $r_k = k^{-\gamma}(\ell n\ k)^{\ell}$ for $k \geq k_0$, $0 < \gamma < 1$ and $\ell \in \mathbb{R}$. (Defined appropriately on $[0,k_0]$ to make it convex). For $\{r_k\}$

satisfying (3.1) and (3.2), we give again convex $\{r_k\}$, $r_k = (\ell n\ k)^{-\ell}$ for $k \geq k_0$ and $\ell > 0$.

The proof of the result $\ell t_{n\to\infty} P(\nu_n \leq \beta_n + x/\chi_n) = \exp(-e^{-x})$ can be greatly reduced by the observation $I_k/r_k^{\frac{1}{2}} = B(r_k)/r_k^{\frac{1}{2}}$ and the result of Darling and Erdös [3] as given in (3.1) of Robbins and Siegmund [10]. Darling and Erdös do not state their result in the form (3.1) of [10]. Also their proof of Theorem 1 in [3] is quite complicated. Theorem 1 of [3] is a particular case of Theorem 2.1 and the last part of Theorem 3.1 offers an alternative proof for (3.1) of [10].

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 6
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER: 9
4. TITLE (and Subtitle): Extreme Value Distribution for Normalized Sums from Stationary Gaussian Sequences.
5. TYPE OF REPORT & PERIOD COVERED: TECHNICAL rept.,
6. PERFORMING ORG. REPORT NUMBER: 14 Mimeo Series No. 1160
7. AUTHOR(s): 10 Yashaswini Mittal
8. CONTRACT OR GRANT NUMBER(s): 15 N00014-75C-0809
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Department of Statistics, University of North Carolina, Chapel Hill, North Carolina 27514
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: Statistics & Probability Program, Office of Naval Research, Arlington, VA
12. REPORT DATE: 11 March 1978
13. NUMBER OF PAGES: 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 30p.
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report): Approved for Public Release: Distribution Unlimited
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number): Normalized sums, extreme value distribution, stationary Gaussian sequences, Brownian motion.
20. ABSTRACT (Continue on reverse side if necessary and identify by block number):

Limiting distribution of the maximum of normalized sums of stationary Gaussian processes is studied.

DD FORM 1473, 1 JAN 73 — EDITION OF 1 NOV 65 IS OBSOLETE

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