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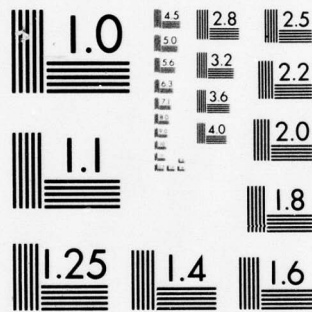
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TRANSIENT ANALYSIS OF STRUCTURAL MEMBERS
BY THE CSDT RICCATI TRANSFER MATRIX METHOD

Technical Report

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A method for direct integration of the dynamic governing partial differential equations of motion for structural members is presented. This technique is called the continuous space discrete time (CSDT) Riccati transfer matrix method. Numerical results for bar and beam example problems indicate that the method is numerically stable and accurate for calculating the dynamic response of linear structural members.			

INTRODUCTION

The most common approach to structural dynamics problems is to discretize the continuous structure in space, using, for example, the finite element model. This leads to a set of ordinary differential equations in time. Integration schemes or modal superposition are employed to solve these equations.

In the present effort the governing partial differential equations for structural members such as rods, plates, and shells are transformed into ordinary differential equations in space by discretizing the time derivatives using a finite difference scheme. A mixed method for structural members, such as the transfer matrix method for beams or numerical integrations for shells, is then applied to solve these ordinary differential equations. These computations in space can be stabilized with the aid of the field method or Riccati transformations.

The method developed in the present paper is called the continuous-space discrete - time (CSDT) Riccati transfer matrix method since only the time variable is treated in discrete form and the Riccati transfer matrix method [1] is employed to eliminate the numerical instabilities so often encountered [2] in spatial calculations. As mentioned above, this method differs from the commonly used direct integration method for structural dynamics, in which the numerical integration is performed on a system of second order differential equations resulting from the usual structural approximations of the spatial geometry of the members [3].

The continuous-space discrete-time method has been used in analog or hybrid computers for the solution of partial differential equations [4, 5, 6, 7, 8]. The central difference formulation was used in discretizing the time derivatives. Stability of the method in the direction of time

has been studied for parabolic heat transfer equations [9] and two methods, i.e. the method of decomposition [10] and the invariant imbedding method [11,12], were developed to eliminate the instability associated with the resulting boundary value problem for the ordinary differential equations in space. Breed [13] used a form of the CSDT approach for the transient analysis of rotating shafts. A central difference time discretization was used in his formulation. However, no effort was made to stabilize the integration for the resulting spatial boundary value problem.

Without such a stabilization, the applications are severely limited. This paper extends the continuous-space discrete-time method to the transient analysis of structural members. The Newmark generalized acceleration formulation is used for time discretization. Advantages of this formulation over the central difference formulation usually adopted for the continuous-space discrete-time method will be demonstrated.

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FORMULATION OF THE CONTINUOUS-SPACE DISCRETE-TIME RICCATI TRANSFER MATRIX METHOD

In general, the partial differential equations of motion for structural members can be expressed as

$$D\dot{W}(x,t) = \sum_{j=1}^2 A_j(x) \frac{\partial^j W(x,t)}{\partial t^j} - F(x,t) \quad (1)$$

where $W(x,t)$ is the N -dimensional column vector of dependent variables, $A_j(x)$ is an N -square spatial matrix, $D(x)$ is an N -square spatial matrix linear differential operator and $F(x,t)$ is the N -dimensional force vector. The terms associated with A_1 and A_2 are usually identified as the damping and inertia terms respectively. If some form of finite difference discretization is used to approximate the inertia and damping terms in equation (1), then the partial differential equations can be transformed into an ordinary differential equation with spatial derivatives only. The linear multistep discretization is used for this purpose due to the fact that most of the commonly used direct integration methods in structural dynamics, for example, the central difference method [14], the Houbolt method [15], the Newmark method [16] and the stiffly stable method [17], can be derived from the linear multistep formulation. This formulation can be written as

$$\sum_{i=0}^p \alpha_i \dot{W}_{n+1-i} = \Delta t \sum_{i=0}^p \beta_i \ddot{W}_{n+1-i} \quad (2)$$

$$\sum_{i=0}^p \gamma_i W_{n+1-i} = \Delta t^2 \sum_{i=0}^p \delta_i \ddot{W}_{n+1-i} \quad (3)$$

where dot notation is used to represent time derivatives, α_i , β_i , γ_i and δ_i are coefficients, Δt is the time step and the subscript $n+1-i$ denotes the time at $(n+1-i)\Delta t$ with $n=0, 1, \dots$. Rearrange equations (2) and (3) in the form

$$\dot{W}_{n+1} = \frac{\alpha_0}{\beta_0 \Delta t} W_{n+1} + P_n \quad (4)$$

$$\ddot{W}_{n+1} = \frac{\gamma_0}{\delta_0 \Delta t^2} W_{n+1} + Q_n \quad (5)$$

$$\text{where } P_n = \frac{1}{\beta_0 \Delta t} \sum_{i=1}^P \alpha_i W_{n+1-i} - \frac{1}{\beta_0} \sum_{i=1}^P \beta_i \dot{W}_{n+1-i} \quad (6)$$

$$Q_n = \frac{1}{\delta_0 \Delta t^2} \sum_{i=1}^P \gamma_i W_{n+1-i} - \frac{1}{\delta_0} \sum_{i=1}^P \delta_i \ddot{W}_{n+1-i} \quad (7)$$

The functions P_n and Q_n involve variables at previous times only and hence can be considered as the historical part in the formulation. Substituting equations (4) and (5) into the governing equations of motion (1) at time $(n+1)\Delta t$, we have

$$DW_{n+1} = \left(\frac{A_1 \alpha_0}{\beta_0 \Delta t} + \frac{A_2 \gamma_0}{\delta_0 \Delta t^2} \right) W_{n+1} + A_1 P_n + A_2 Q_n - F_{n+1} \quad (8)$$

$$\text{or } DW_{n+1} - KW_{n+1} = R_n \quad (9)$$

$$\text{where } K = \frac{A_1 \alpha_0}{\beta_0 \Delta t} + \frac{A_2 \gamma_0}{\delta_0 \Delta t^2} \quad (10)$$

$$R_n = A_1 P_n + A_2 Q_n - F_{n+1} \quad (11)$$

Equation (9) is a differential equation with spatial derivatives only and hence, the dynamic analysis of the structural members can be treated at each time step ($n=0,1,---$) as a static problem by considering R_n to be a generalized external force acting on the member.

In principle, many methods for solving the ordinary differential equations such as the Runge-Kutta and predictor - corrector methods [18] can be applied to solve equation (9) with the prescribed boundary conditions. Here, the mixed method techniques such as the transfer matrix method for beams are used. It would appear reasonable that numerical integration could be employed for shells. Note that the form of equation (9) is similar to the governing equation for a structural member on an elastic foundation with an equivalent elastic foundation modulus K given by equation (10). Also, in the above formulation, Δt must be taken fairly small in order to keep the time discretization error reasonably small. However, a small value of Δt corresponds to a large value of

elastic foundation modulus K. Normally with such a K one would expect to encounter numerical difficulties when, for example, the usual transfer matrix method is applied. Several techniques are available to overcome such numerical difficulties [2]. One of the better methods is to use the Riccati transformation or the Riccati transfer matrix method [1].

Consider the formulation of the Riccati transfer matrix method. Let $[U]$ denote the $N \times N$ general transfer matrix which transfers the state variable $[W]$ from station i to station $i+1$ across segment i , at time $(n+1)\Delta t$. Then

$$[W]_{i+1} = [U]_i [W]_i + [F]_i \quad (12)$$

where F contains the loading terms which are evaluated from the general loading function equation (11). Both W and F are $N \times 1$ matrices. Let the transfer matrix be arranged and partitioned so that

$$\begin{bmatrix} f \\ e \end{bmatrix}_{i+1} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}_i \begin{bmatrix} f \\ e \end{bmatrix}_i + \begin{bmatrix} F_f \\ F_e \end{bmatrix}_i \quad (13)$$

where f contains the $N/2$ state variables corresponding to the homogeneous left hand boundary conditions and e contains the respective complementary $N/2$ state variables.

A generalized Riccati transformation at station i is given by

$$[f]_i = [S]_i [e]_i + [P]_i \quad (14)$$

which relates half of the state variables to the remaining state variables at station i . Using equations (13) and (14), it can be shown that a general recurrence relationship could be obtained as [1]

$$[f]_{i+1} = [S]_{i+1} [e]_{i+1} + [P]_{i+1} \quad (15)$$

$$\text{where } [S]_{i+1} = [U_{11}S + U_{12}]_i [U_{21}S + U_{22}]_i^{-1} \quad (16)$$

$$[P]_{i+1} = [U_{11}P + F_f]_i - [S]_{i+1} [U_{21}P + F_e]_i \quad (17)$$

Thus, the matrices $[S]$ and $[P]$ determine the state variables f from e

at any station. Another matrix $[T]$, which transmits the state variables e from station $i+1$ to station i , can also be obtained from equations (13) and (14) as [1]

$$[e]_i = [T]_{i+1}[e]_{i+1} + [Q]_{i+1} \quad (18)$$

$$\text{where } [T]_{i+1} = [U_{21}S + U_{22}]_i^{-1} \quad (19)$$

$$[Q]_{i+1} = -[T]_{i+1}[U_{21}P + F_e]_i \quad (20)$$

To start the calculation, the submatrices of the transfer matrix in equation (13) must be determined. The matrices $[S]$, $[P]$, $[T]$ and $[Q]$ are calculated for each station while moving along the member from left to right with the boundary conditions $[S]_0 = [P]_0 = 0$. When the last station m is reached, equation (14) gives

$$[f]_m = [S]_m[e]_m + [P]_m \quad (21)$$

The $N/2$ known state variables at the right hand boundary are substituted into the above relationship to determine the remaining $N/2$ unknown state variables. Successive applications of equations (18) at each station allows the calculation of $N/2$ state variables e while moving from right to left along the member. At any station, the remaining $N/2$ state variables f are determined from equation (15). This completes the formulation for the continuous - space discrete - time Riccati transfer matrix method.

Due to the generality of the transfer matrix method, this method can be used to solve structural members with variable cross section and arbitrary boundary conditions. In cases when the transfer matrix has to be evaluated numerically, this method can still be used with the same procedures as outlined above. For example, members with continuously varying cross-sectional properties and those with more than four equations of motion, i.e. $N > 4$ in equation (1), numerical integration may be required to evaluate the transfer matrix. The finite difference discretiza-

tion equations (2) and (3) are used to transform the governing partial differential equation into ordinary differential equations, and a numerical method such as the Runge-Kutta or predictor-corrector are applied to provide the transfer matrix $[U]$ in equation (13). Once the transfer matrix is obtained, the Riccati transfer matrix method can be applied.

Concentrated occurrences can be treated the same way as in the Riccati transfer matrix method [1] except for a dashpot or a concentrated mass whose behavior depends on the velocity or acceleration at that point. Consider a segment of beam containing a dashpot (Fig. 1). Let L and R denote the sections just to the left and right of station i where the dashpot is placed. The transfer matrix across the dashpot can be expressed by a point matrix at time $(n+1)\Delta t$ as

$$\begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_L + \begin{bmatrix} 0 \\ 0 \\ 0 \\ C \frac{\partial y_{n+1}}{\partial t} \end{bmatrix} \quad (22)$$

where y , θ , M and V are the displacement, slope, moment and shear force of the beam respectively. When equation (4) is used, equation (22) can be written in the form

$$\begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\alpha_0 C}{\beta_0 \Delta t} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_L + \begin{bmatrix} 0 \\ 0 \\ 0 \\ CP_n \end{bmatrix} \quad (23)$$

The same procedure can be applied to the case where the solution is needed to cross a concentrated mass. The transfer matrix across the mass can be expressed by the point matrix

$$\begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_L + \begin{bmatrix} 0 \\ 0 \\ 0 \\ m \frac{\partial^2 y_{n+1}}{\partial t^2} \end{bmatrix} \quad (24)$$

If the approximation equation (5) is applied, equation (24) will be in the form

$$\begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{mY_0}{\delta_0 \Delta t^2} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ \theta_{n+1} \\ M_{n+1} \\ V_{n+1} \end{bmatrix}_L + \begin{bmatrix} 0 \\ 0 \\ 0 \\ mQ_n \end{bmatrix} \quad (25)$$

The Riccati transfer matrix method using the point matrices (23) and (25) can be used to carry the solution across the dashpot and the concentrated mass. Such in-span indeterminate conditions as moment releases and rigid supports require special attention. The same is true for prescribed time-varying displacements. Consider, for example, the beam in Fig. 2 which has two successive prescribed time dependent displacements δ_{i-1} and δ_i applied at position $x = a_{i-1}$ and a_i . The unknown force $W_{i-1}(t)$ is introduced at $x = a_{i-1}$. The point matrix corresponding to this force is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & W_{i-1}(t) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (26)$$

From the condition $y_i = \delta_i(t_n)$ at $t = n\Delta t$, $x = a_i$ and the predetermined field transfer matrix between a_{i-1} and a_i , the unknown force $W_{i-1}(t)$ can be determined as

$$\delta_i(t_n) = y_{i-1}(t_n) U_{yy} + \theta_{i-1}(t_n) U_{y\theta} + M_{i-1}(t_n) U_{yM} + V_{i-1}(t_n) U_{yV} + W_{i-1}(t_n) U_{yV} + U_{yF} \quad (27)$$

where U_{yy} , $U_{y\theta}$, U_{yM} , U_{yV} and U_{yF} are the first row elements of the field transfer matrix and y_{i-1} , θ_{i-1} , M_{i-1} and V_{i-1} are the state variables at $x = a_{i-1}$. Equation (27) then yields for the unknown force

$$W_{i-1}(t_n) = -y_{i-1}(t_n) \frac{U_{yy}}{U_{yV}} - \theta_{i-1}(t_n) \frac{U_{y\theta}}{U_{yV}} - M_{i-1}(t_n) \frac{U_{yM}}{U_{yV}} - V_{i-1} + \frac{\delta_i(t_n) - U_{yF}}{U_{yV}} \quad (28)$$

The point matrix to transfer across $x = a_{i-1}$ is found from equation (28) as

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -U_{yy}/U_{yV} & -U_{y\theta}/U_{yV} & -U_{yM}/U_{yV} & 0 & (\delta_i(t_n) - U_{yF})/U_{yV} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (29)$$

This point matrix can be used to evaluate the matrices $[S]$, $[P]$, $[T]$ and $[Q]$ for the Riccati transfer matrix method.

The investigation just completed is sufficient to find the unknown force corresponding to an applied displacement using the condition at the next in-span or prescribed time dependent displacement position. This procedure can be applied until the last station is reached. However, the $N/2$ known state variables at the right hand boundary are not enough to determine the remaining $N/2$ unknown state variables because one of the homogeneous boundary conditions was used to determine the point matrix for the preceeding in-span condition or prescribed time dependent displacement. The unused condition at the first (counting along the member

from left to right) prescribed time dependent displacement can replace this used boundary condition by being available for fixing the remaining unknowns. For example, let i denote the station of the first prescribed time dependent displacement and assume the displacement is contained in the state variable $[f]$, then, by equations (15) and (18)

$$[f]_m = [S]_m [e]_m + [P]_m \quad (30)$$

$$[e]_{m-1} = [T]_m [e]_m + [Q]_m \quad (31)$$

$$[e]_{i+1} = [T]_{i+2} [e]_{i+2} + [Q]_{i+2} \quad (32)$$

$$[e]_i = [T]_{i+1} [e]_{i+1} + [Q]_{i+1} \quad (33)$$

$$[f]_i = [S]_i [e]_i + [P]_i \quad (34)$$

By successive substitution of equations (31), (32), (33) and (34), the following relationship between $[f]_i$ and $[e]_m$ is obtained.

$$[f]_i = [S] [e]_m + [P] \quad (35)$$

where $[S] = [S]_i [T]_{i+1} [T]_{i+2} \dots [T]_m$

and $[P] = [S]_i [T]_{i+1} [T]_{i+2} \dots [Q]_m + \dots$
 $+ [S]_i [T]_{i+1} [Q]_{i+2} + [S]_i [Q]_{i+1} + [P]_i$

The prescribed time dependent displacement of equation (35) and the unused homogeneous right hand boundary condition of equation (30) give two necessary equations for the two unknown right hand boundary conditions. Once the boundary conditions are evaluated, the Riccati transfer matrix can be applied.

STABILITY AND ERROR ANALYSIS

Now that the formulation of the CSDT Riccati transfer matrix method using the general multistep method has been presented, it is of interest to ascertain the values for p and the coefficients in equations (2) and

(3) to be used. Usually, it is desirable to make p large so that the truncation error can be reduced [19]. However, in structural dynamics problems wherein oscillatory functions dominate, it is not clear that the truncation error should be used as a measure of total accuracy of the method [20]. Some other factors such as the amplitude decay, period elongation and effects of spurious roots (for $p > 2$) could also be considered as primary accuracy measures. The stability and accuracy analysis of some of the commonly used direct integration schemes in linear structural dynamics based on the finite element model of the equations of motion have been studied by several authors [14,21,22,23,24]. For our formulation, several techniques including the central difference method, the Houbolt method, the Newmark method and the Wilson θ method, have been studied for time discretization and it is found that Newmark's generalized acceleration method with the following relations gives the most satisfactory results

$$y_{n+1} = y_n + \Delta t \dot{y}_n + (\frac{1}{2} - \beta) \Delta t^2 \ddot{y}_n + \beta \Delta t^2 \ddot{y}_{n+1} \quad (36)$$

$$\dot{y}_{n+1} = \dot{y}_n + (1 - \gamma) \Delta t \ddot{y}_n + \gamma \Delta t \ddot{y}_{n+1} \quad (37)$$

The truncation error of this formulation is Δt^2 and no spurious roots are involved. Consider the analysis of a simple bar with the governing partial differential equation

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \quad (38)$$

where $a = \sqrt{EA/m}$, E is the modulus of elasticity, m and A are the density and cross-sectional area of the bar, respectively, and x is in the direction of the bar axis.

Let equation (38) be multiplied by $(\frac{1}{2} - \beta)\Delta t^2$ at time $n\Delta t$ and by $\beta\Delta t^2$ at time $(n+1)\Delta t$. The following equation is obtained by adding these two expressions and using equations (36) and (37)

$$\begin{aligned} & \beta a^2 \Delta t^2 \frac{\partial^2 y_{n+1}}{\partial x^2} + (\frac{1}{2} - 2\beta + \gamma) a^2 \Delta t^2 \frac{\partial^2 y_n}{\partial x^2} + (\frac{1}{2} - \gamma + \beta) a^2 \Delta t^2 \frac{\partial^2 y_{n-1}}{\partial x^2} \\ & = y_{n+1} - 2y_n + y_{n-1} \end{aligned} \quad (39)$$

The stability of equation (39) can be investigated using von Neumann's stability criteria [25]. Assume

$$y_n = T^n e^{i\eta x} \quad (40)$$

where T is a function of time only. Substitute equation (40) into equation (39) and solve the resulting relationship for T . Then, two solutions are obtained

$$T_{1,2} = \frac{[2 - (\frac{1}{2} - 2\beta + \gamma)B^2] \pm \sqrt{D}}{2(1 + \beta B^2)} \quad (41)$$

where $D = [(\frac{1}{2} - 2\beta + \gamma)^2 - 4\beta(\frac{1}{2} - \gamma + \beta)]B^4 - 4B^2$
and $B^2 = \eta^2 a^2 \Delta t^2$

For stability, it is necessary to have

$$D < 0 \quad (42)$$

Before drawing conclusions from equation (42), consider the error associated with solution (40) by assuming that condition (42) is satisfied.

Note that

$$y = e^{i\eta(x + at)} \quad (43)$$

is the exact solution of the governing equation of motion (38). The accuracy of the method can be evaluated by comparing equation (40) with equation (43) and rewriting equation (40) in the form

$$y_n = \cos^n \theta e^{i\eta(x + \psi a n \Delta t)} \quad (44)$$

where θ , α and ψ can be evaluated using

$$\cos \theta = \frac{[(2 - XB)^2 + 4(Y + X + \beta)B^2 - (X^2 - 4\beta Y)B^4]^{\frac{1}{2}}}{2(1 + \beta B^2)}$$

$$\alpha = \tan^{-1} \frac{[4(Y+X+\beta)B^2 - (X^2-4\beta Y)B^4]^{\frac{1}{2}}}{2-XB^2}$$

$$\psi = \frac{\alpha}{\eta a \Delta t}$$

$$\text{and } X = \frac{1}{2} - 2\beta + \gamma \quad ; \quad Y = \frac{1}{2} - \gamma + \beta$$

From equations (43) and (44), we can see that these two solutions will be identical to each other if the value of $\cos \theta$ and ψ are equal to 1.

If we set $\cos \theta = 1$, then

$$(\frac{1}{2} - \gamma)(\beta B^2 + 1)B^2 = 0 \quad (45)$$

This equation is satisfied if we set $\gamma = \frac{1}{2}$, i.e. there is no amplitude decay error associated with the method if $\gamma = \frac{1}{2}$. Now, return to equation (42). With $\gamma = \frac{1}{2}$, equation (42) gives

$$D = (1-4\beta)B^4 - 4B^2$$

In order to have $D < 0$ for arbitrary values of B , we must have $\beta \geq \frac{1}{4}$, i.e. the method will be unconditionally stable if we choose $\beta \geq \frac{1}{4}$.

With $\gamma = \frac{1}{2}$ and small Δt , the parameter ψ can be expressed as

$$\psi = \frac{\sqrt{4-(1-4\beta)\eta^4 a^4 \Delta t^4}}{2-(1-2\beta)\eta^2 a^2 \Delta t^2} - \frac{\eta^2 a^2 \Delta t^2}{3} \left[\frac{\sqrt{4-(1-4\beta)\eta^4 a^4 \Delta t^4}}{2-(1-2\beta)\eta^2 a^2 \Delta t^2} \right]^3 + \dots \quad (47)$$

which in no case will be equal to 1. Thus, the method has a period elongation error which is a function of the time step used.

The stability and error analysis for a simple Euler-Bernoulli beam can be carried out in the same fashion since the governing equation of motion for the beam has the same form as equation (38) except that there are fourth order derivatives in x and EA is replaced by EI . As a consequence, all η^2 in the analysis should be changed to η^4 . However, the conclusion is the same, i.e. the method is unconditionally stable with $\beta \geq \frac{1}{4}$ and

has no amplitude decay error if $\gamma = \frac{1}{2}$. Also there is an error in the period elongation which is a function of Δt .

The foregoing stability and error analysis was based on systems with no damping. For the case of non-zero damping, the stability and error analysis would have to include the damping coefficient as an additional variable. Usually, however, small values of the damping coefficient do not change the overall stability characteristics of an integration scheme [3].

It is of interest to note that if we choose the parameters to be $\beta=1$ and $\gamma=3/2$, the Newmark formulation equation (39) is identical to the central difference formulation. We can conclude from the analysis of this section that the central difference formulation used by all of the previous works on continuous-space discrete-time method is unconditionally stable but has amplitude decay and period elongation errors. These errors can be investigated by setting $\beta=1$ and $\gamma=3/2$ in equation (44) which give, for small Δt ,

$$\cos n\theta = e^{-n\eta^2 a^2 \Delta t^2 / 2} + O(\Delta t^4) \quad (48)$$

$$\text{and } \psi = 1 - \eta^2 a^2 \Delta t^2 / 3 + O(\Delta t^4) \quad (49)$$

Thus, we can see that the central difference formulation of the continuous-space discrete-time method has amplitude decay error approximately equal to $e^{-n\eta^2 a^2 \Delta t^2 / 2}$ and the period is increased in the ratio $1/\psi = 1 + \eta^2 a^2 \Delta t^2 / 3$.

NUMERICAL EXAMPLES

In using the direct integration method, the most difficult decision is to select an appropriate time step Δt . On one hand, the time step must be small enough to obtain an accurate solution. On the other hand, the time step must not be smaller than necessary in order to reduce the computa-

tion cost. Usually [3], a time step smaller than $0.01T$ is suggested for the commonly used unconditionally stable direct integration methods where T is the period of the response. Judging from the numerical results of the examples in this section, a larger time step could be used without losing accuracy for the present method. However, in using the Riccati transfer matrix method it is helpful if the length of each section is chosen to be small enough such that all the terms for any element in the transfer matrix are the same order of magnitude [1]. Hence, the section length for the space computations should depend on the time step chosen for the problem in order to satisfy the above criterion.

It is difficult to compare the efficiency of the present method to the other direct integration methods based on finite element models. The present method is especially appropriate for structural members. The cost of the present method in carrying the solution forward one time step could be estimated roughly from Ref. 1 which shows that the use of the Riccati transfer matrix method to transfer from one station to an adjacent station requires $N^{3/2} + 4(N/2)^2$ multiplications, where N is the order of the transfer matrix for the structural member.

Extension bar

Consider the displacement at the tip of a cantilevered elastic bar subjected to a suddenly applied concentrated loading at the free end. The problem parameters are shown in Fig. 3.

The governing equation of motion for this problem is the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2} \quad (50)$$

where $a = \sqrt{EA/m}$ is the wave propagation velocity. Let us assume that the

bar is initially at rest, i.e.

$$y(x, 0) = 0 ; \left. \frac{\partial y(x, t)}{\partial t} \right|_{t=0} = 0 \quad (51)$$

To solve this problem using the CSDT Riccati transfer matrix method, replace the governing equation of motion (50) at time $(n+1)\Delta t$ by the following equation, in which we have employed equation (36)

$$\frac{d^2 y_{n+1}}{dx^2} - \frac{y_{n+1}}{\beta a^2 \Delta t^2} = R(x) ; \quad n=0, 1, 2, \dots \quad (52)$$

$$\text{where } R(x) = - \frac{y_n}{\beta a^2 \Delta t^2} - \frac{\dot{y}_n}{\beta a^2 \Delta t} - \frac{1}{a^2} \left(\frac{1}{2\beta} - 1 \right) \ddot{y}_n$$

Equation (52) has the same form as a bar with an elastic foundation whose transfer matrix is [26]

$$\begin{bmatrix} y \\ P \end{bmatrix}_{i+1} = \begin{bmatrix} \cosh \alpha l & \sinh \alpha l / EA \alpha \\ EA \alpha \sinh \alpha l & \cosh \alpha l \end{bmatrix} \begin{bmatrix} y \\ P \end{bmatrix}_i \quad (53)$$

where $\alpha^2 = 1/\beta a^2 \Delta t^2$ and l is the section length between station i and $i+1$. With a fixed left end boundary condition, the submatrices necessary to perform the Riccati transfer matrix method are

$$\begin{aligned} [f] &= y ; & [e] &= P \\ [U_{11}] &= \cosh \alpha l ; & [U_{12}] &= \sinh \alpha l / EA \alpha \\ [U_{21}] &= EA \alpha \sinh \alpha l ; & [U_{22}] &= \cosh \alpha l \\ [F_f] &= - \int_0^l R(x) \sinh \alpha (l-x) / \alpha dx ; & [F_e] &= - \int_0^l R(x) EA \cosh \alpha (l-x) dx \end{aligned} \quad (54)$$

As mentioned earlier, $\beta \geq \frac{1}{4}$ should be used in the calculation. In order to choose a proper value for β and to see more clearly the error involved in the present formulation, suppose the bar is one section long. The displacement at the tip of the bar for the first time step would be

$$y_1 = \frac{\beta P_0 a \Delta t \sinh \alpha L}{EA \cosh \alpha L} \quad (55)$$

The exact solution for the problem is [27]

$$y = \frac{P_0 a t}{EA} \quad \text{for } t < \frac{2L}{a} \quad (56)$$

By comparing equations (55) and (56), we can see that the best choice for the value of β is 1 and in this case the error term due to the discrete-time process is $\sinh \alpha L / \cosh \alpha L$. This term will approach unity as $\alpha \rightarrow \infty$, i.e. $\Delta t \rightarrow 0$.

The exact and computed results (with $\beta=1$) for the displacement at the tip of the bar are shown in Fig. 4. Several values of β were tried and the solution does not show much sensitivity. Although the present method was proved to be unconditionally stable, the selection of Δt used in performing the integration determines the accuracy of the method. Here, the accuracy calculations were made for different values of Δt and are displayed in Fig. 5. These solutions indicate that $\Delta t = 0.025T$ gives satisfactory results.

This same problem is also solved with damping coefficient $C = 10 \text{ lb-sec/in}$ in order to see the effect of viscous damping in the application of the CSDT Riccati transfer matrix method. Figure 6 shows the results of the present method and the exact solution found using integral transforms.

Euler-Bernoulli beam with simple supports

A uniform Euler-Bernoulli beam hinged at both ends with a bending moment applied at the end $x=L$ as shown in Fig. 7 is analyzed by the present method. Numerical and exact [27] results for the displacement at the middle of the beam are shown in Fig. 8 where zero initial conditions are assumed. The numerical results match very well with the exact solution even when the time step 0.025T is used.

The accuracy of the present method as a function of Δt for this problem is illustrated numerically in Fig. 9.

In order to see the effect of amplitude decay error in using the central difference formulation for the present method, this example problem was also solved by setting $\beta=1$ and $\gamma=3/2$. These results are shown in Fig. 8. Hartree [4] proposed the use of the h^2 extrapolation process to improve the accuracy of the central difference formulation of the CSDT method. This process was applied and the improvement to the central difference formulation is illustrated in Fig. 8.

Cantilevered Euler-Bernoulli beam

This example is taken from Ref. 28 where the Newmark method was applied. The problem considered was an elastic cantilever beam subjected to a suddenly applied concentrated loading at the free end. The parameters and the finite element idealization of the beam used in Ref. 28 are shown in Fig. 10. The numerical results for the displacement at the tip of the beam by the present method and the results by the Newmark method [28] are shown in Fig. 11.

An integration time step of 1×10^{-6} seconds was used in Ref. 26. The solution became unstable if the time step increased to 1×10^{-5} seconds. In the present method, the beam was divided into ten sections and a time step of 1×10^{-4} seconds could be satisfactorily used. This is about 100 times larger than the value employed in Ref. 28.

The same type of problem has also been solved in Ref. 29 using the Wilson θ method with the following parameters: $E = 3 \times 10^7$ psi; $m = 0.000733$ lb_m/in; $L = 120$ in, $I = 3$ in⁴. Figure 12 shows the normalized displacement at the tip of the beam.

Timoshenko beam with concentrated loading

Davids [30] solved a cantilever Timoshenko beam by using the so-called direct analysis method. The parameters for the beam given in Ref. 30 are: $E = 3 \times 10^7$ psi; $\nu = 0.3$; $\gamma = \rho g = 0.3$ lb/in³; $A = 1$ in²; $A_s = 0.833$ in²; and $I_T/I = 1$, where I is the transverse moment of inertia and I_T is the rotary moment of inertia. Note that for the Timoshenko beam, the time derivative appears in both the mass inertia term and the rotary inertia term and hence, the finite difference formulation equations (36) and (37), have to be applied to both terms. The resulting equations have the same form as for the static response of a beam with equivalent extension and rotary foundations subjected to generalized external moments and forces. The general forms of the transfer matrix for such a beam can be obtained from Ref. 26.

The numerical results from Ref. 30 and the present method with ramped shear and moment acting on the free end of the beam are shown in Figs. 13 and 14 respectively. Only the ratio of the transverse moment of inertia to the rotary moment of inertia was given in Ref. 30. No values were indicated. A one inch square beam was selected for our calculations and hence $I = I_T = 0.0833$ in⁴. The time step of $\Delta t = 10^{-6}$ seconds was chosen. In the direct analysis method, the time step is determined from $\Delta t = \Delta x/C$ where $C = \sqrt{EI/\rho I_T}$ is the dilatational wave velocity for the problem under consideration. The section length Δx is arbitrary and is varied until any reduction in this quantity will not yield any significant change in the resulting solution of the problem. For the parameters given in this example problem $C = 1.968 \times 10^5$ in/sec. and hence if 30 sections are used as is the case with the present method, the time step will be 2.54×10^{-7} seconds which is much smaller than the time step used by the present method.

Beams with distributed loading

The dynamic response of a cantilevered rectangular beam subjected to a uniform distributed ramp pressure over its length was analyzed by the general purpose finite element program MARC [31]. The pressure load is ramped in the first increment to -655.65 psi and then brought down with constant slope to zero at time .01 seconds. The model with essential parameters is shown in Fig. 15. The beam was modeled with three two-dimensional, rectangular section beam column elements in MARC. Three different methods of analysis were employed. They are the Newmark method, the Houbolt method and the method of modal superposition. The displacement at the free end of the beam is shown in Fig. 16. This problem was also solved by MARC using the Timoshenko beam element which allows transverse shear as well as axial straining. The results from Newmark and modal superposition methods are shown in Fig. 17.

This problem is solved by the present method with a lumped mass model. The mass of the beam is lumped at points 2,3 and 4. The point matrix for a concentrated mass with the effect of rotary inertia is

$$[U]_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \rho I_T \ddot{\theta}_{n+1} \\ 0 & 0 & 0 & 1 & \rho A y_{n+1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (57)$$

When Newmark's formulation equation (36) is used, equation (57) can be written as

$$[U]_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \rho I_T / \beta \Delta t^2 & 1 & 0 & F_{M1} \\ \rho A / \beta \Delta t^2 & 0 & 0 & 1 & F_{V1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (58)$$

where $F_{M1} = -\rho I_T (\ddot{\theta}_n + \Delta t \dot{\ddot{\theta}}_n + (\frac{1}{2} - \beta) \Delta t^2 \ddot{\ddot{\theta}}_n) / \beta \Delta t^2$

and $F_{V1} = -\rho A (\ddot{y}_n + \Delta t \dot{\ddot{y}}_n + (\frac{1}{2} - \beta) \Delta t^2 \ddot{\ddot{y}}_n) / \beta \Delta t^2$

The field matrix for the massless beam with the effect of shear deformation is

$$[U]_2 = \begin{bmatrix} 1 & -\ell & -\ell^2/2EI & -\ell^3/6EI + \ell/GA_s & F_y \\ 0 & 1 & \ell/EI & \ell^2/2EI & F_\theta \\ 0 & 0 & 1 & \ell & F_M \\ 0 & 0 & 0 & 1 & F_V \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (59)$$

where the loading function F_y , F_θ , F_M and F_V are

$$F_y = p(t)(\ell^4/24EI - \ell^2/2GA_s), F_\theta = -p(t)\ell^3/6EI, F_M = -p(t)\ell^2/2, F_V = -p(t)\ell$$

The solution is carried out first for the Euler-Bernoulli beam with transfer matrices (58) and (59) by setting $I_T = 1/GA_s = 0$. The result is shown in Fig. 16. The solution for the Timoshenko beam is also calculated with the results shown in Fig. 17.

Elastic-plastic material

Consider a cantilevered bar with an elastic-plastic material and a constant step velocity of 100 in/sec suddenly imposed on the free end. This problem has been solved in Ref. 28 using the Newmark method with a finite element model of the bar. The parameters used in Ref. 28 and the elastic-plastic model for the materials together with the finite element model are shown in Fig. 18.

The problem is solved first for the elastic wave propagation. The results for the velocity and stress of the bar obtained from Ref. 28 and the present method are shown in Fig. 19. The plastic wave propagation

problem is then solved using the elastic-plastic model shown in Fig. 18. The results from Ref. 28 and the present method with $\beta=1$ are shown in Fig. 20.

A possible formulation for the application of the CSDT Riccati transfer matrix method to non-linear structural members is proposed in Ref. 32. Both material and geometric nonlinearities are considered and usually an iteration process would be needed. However, for this problem, advantage can be taken of the fact that the general stress-strain curve in the elastic-plastic model is composed of two straight lines and, hence, the solution can be carried out by using the elastic modulus E to form the necessary transfer matrix if the calculated stress on a particular section of the bar is less than the yield stress σ_y . When the calculated stress exceeds σ_y , the transfer matrix is re-computed based on the plastic modulus E_p .

CONCLUSIONS

The usual approach to transient structural dynamics is based on continuous time, discrete space ordinary differential equations. Here we propose a discrete-time, continuous-space approach, with the Riccati transformation used to stabilize the spatial computations for structural members.

The generalized Newmark acceleration formulation has been chosen to approximate the time derivatives in the governing equations of motion. The resulting method is unconditionally stable and has no amplitude decay error if the two parameters in the formulation are chosen as $\gamma = \frac{1}{2}$ and $\beta > \frac{1}{4}$. The method, however, has period elongation error proportional to the time step used in the integration. Selection of the proper integration time step is always a problem in the use of a direct integration method. Although most of the commonly used methods are unconditionally stable, there are amplitude decay and period elongation errors associated with the use of a large Δt . Usually, a time step smaller than $0.01T$ is suggested for most of the methods where T is the period of the response. Although the same difficulty is experienced in using the present method for bar and beam vibration problems it appears as though a somewhat larger time step can be used with sufficient accuracy. This conclusion, however, should be scrutinized after this method is applied to other types of structures.

Numerical examples included bar and beam vibration problems. Although the geometries and loading conditions were simple, the proposed method applies to more complicated situations. Due to the generality of the transfer matrix method, the method of this paper can be applied to structural members with arbitrary boundary conditions, in-span supports, and geometric and material nonuniformities.

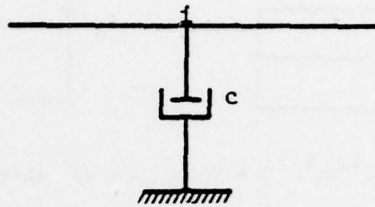


Fig. 1 Dashpot concentrated occurrence

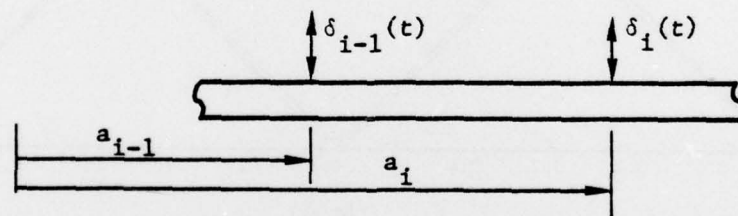
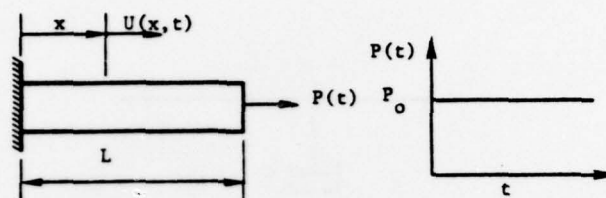


Fig. 2 Beam with prescribed time dependent displacements



$$m = 0.002 \text{ lb-sec}^2/\text{in}^2; L = 10\text{in}; E = 10^7 \text{ lb/in}^2; A = 1 \text{ in}^2; P_0 = 10^6 \text{ lb}$$

Fig. 3 Cantilever bar with a concentrated loading at the free end

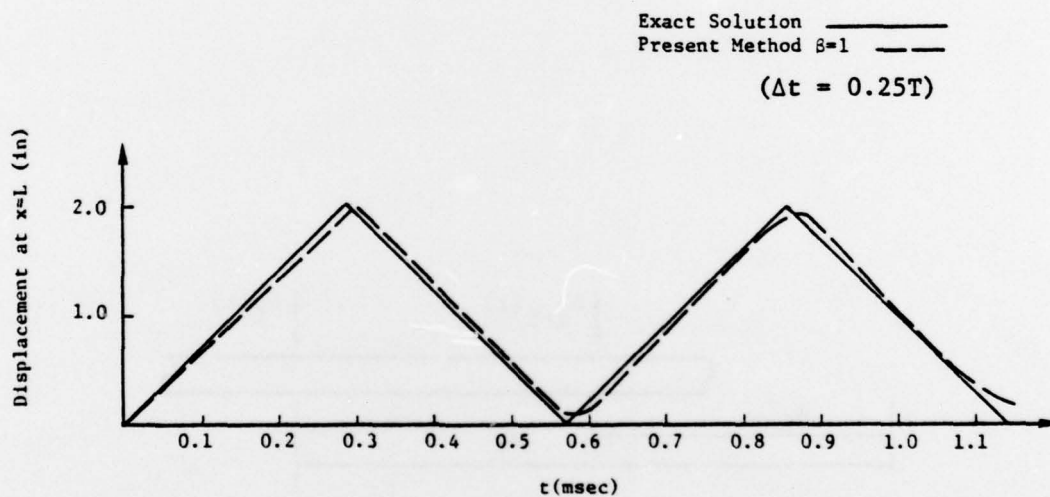


Fig. 4 Results for the vibration of the cantilever bar of Fig. 3

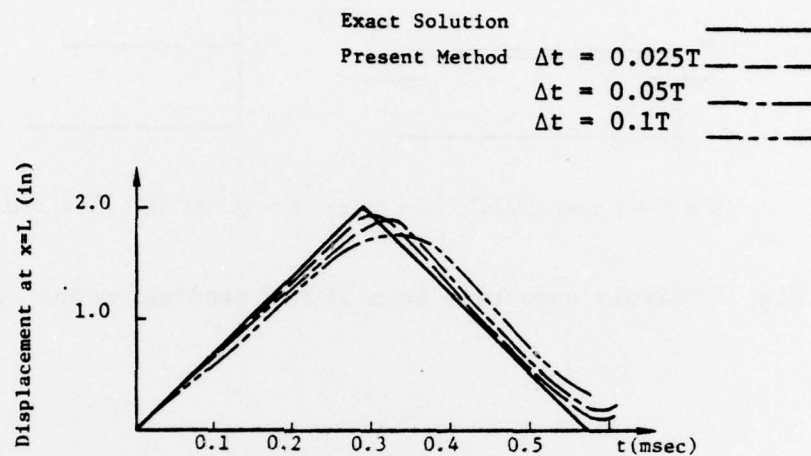


Fig. 5 Effects of the size of Δt for the vibration of the cantilever bar of Fig. 3

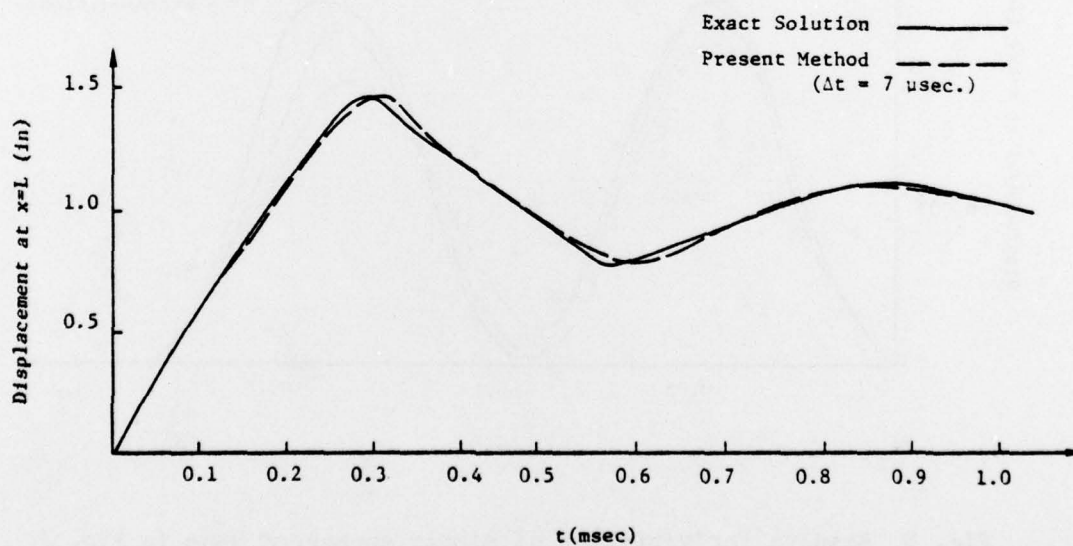
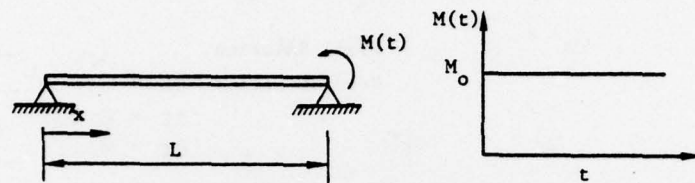


Fig. 6 Results for the vibration of the cantilever bar of Fig. 3 with viscous damping included



$$m = 0.002 \text{ lb-sec}^2/\text{in}^2; L = 10\text{in}; E = 10^7 \text{ lb/in}^2; I = 100\text{in}^4; M_0 = 10^6 \text{ lb-in}$$

Fig. 7 Simply supported beam with a bending moment applied at the end

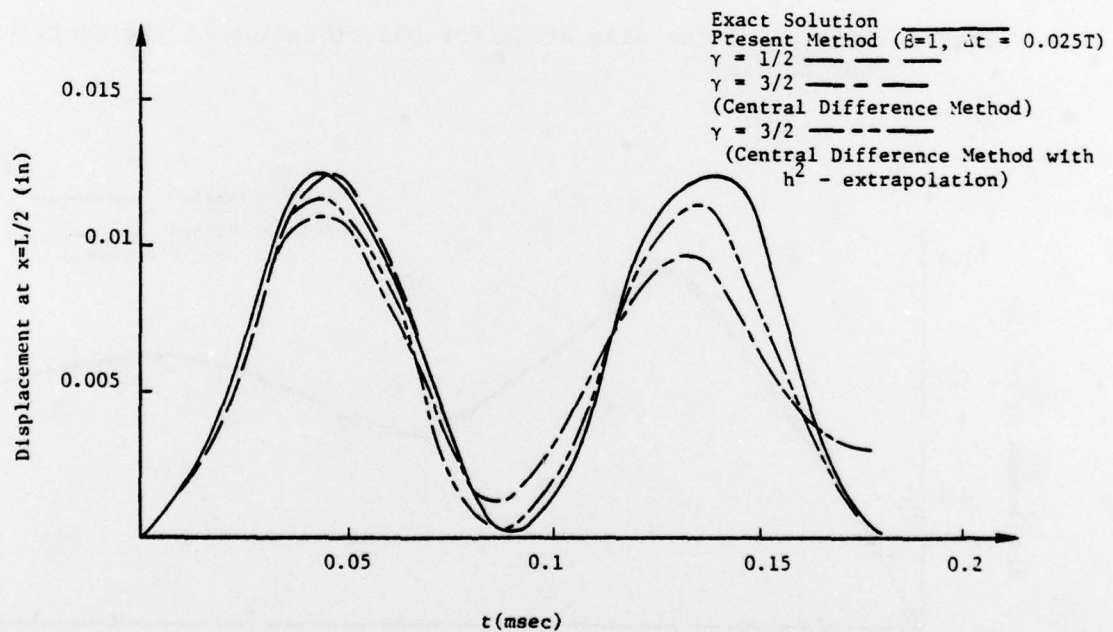


Fig. 8 Results for vibration of simply supported beam in Fig. 7

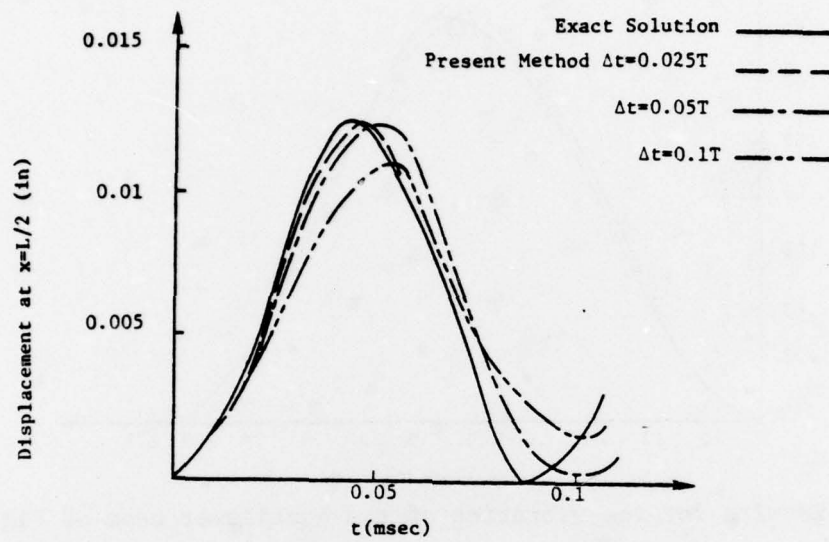


Fig. 9 Effects of the size of Δt for the vibration of the simply supported beam in Fig. 7

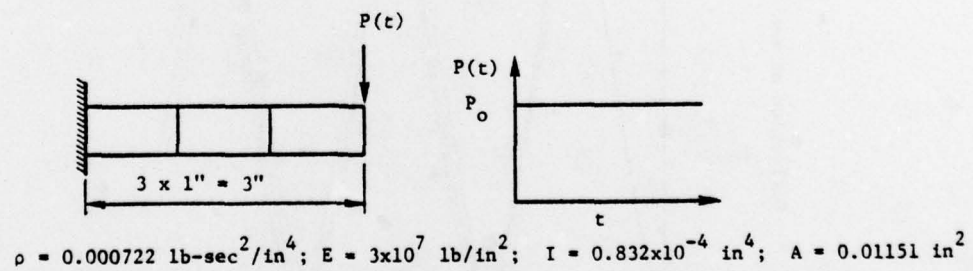


Fig. 10 Cantilever beam with a concentrated loading at the free end

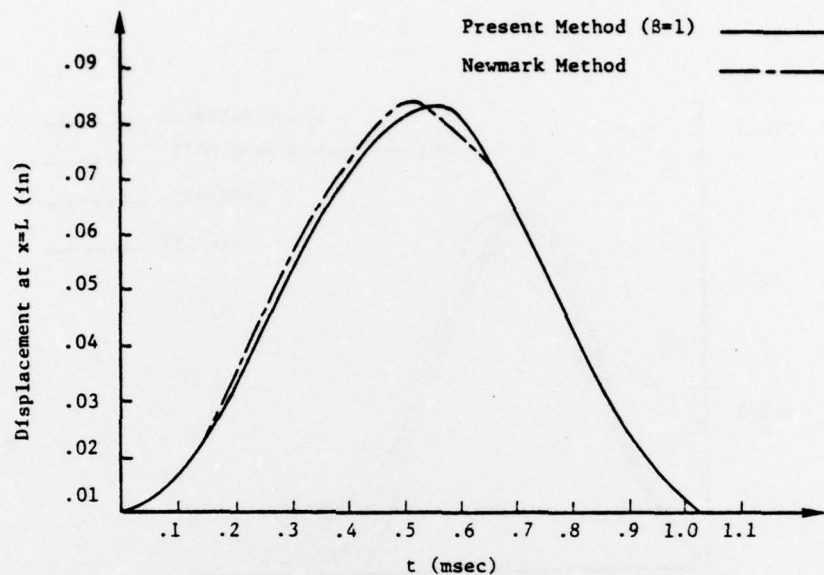


Fig. 11 Results for the vibration of the cantilever beam of Fig. 10

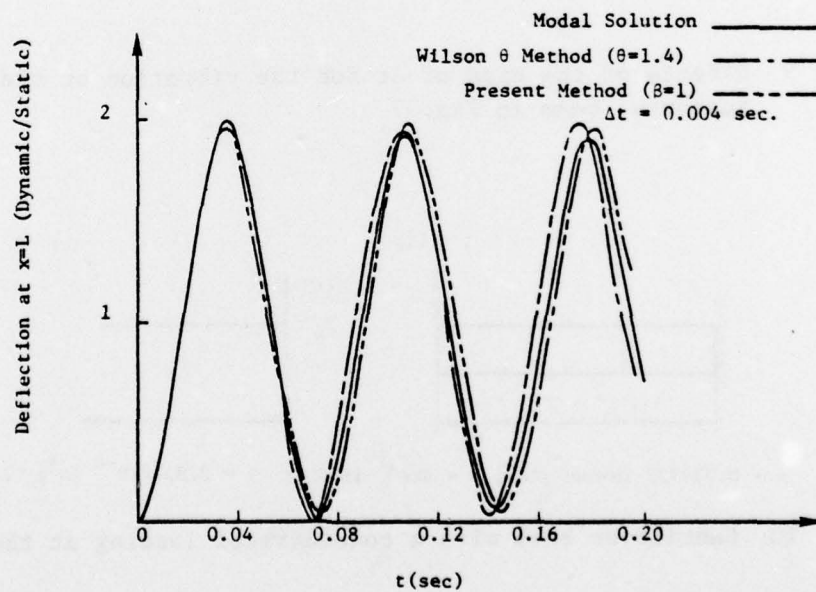


Fig. 12 Results for the vibration of the cantilever beam in Ref. 29

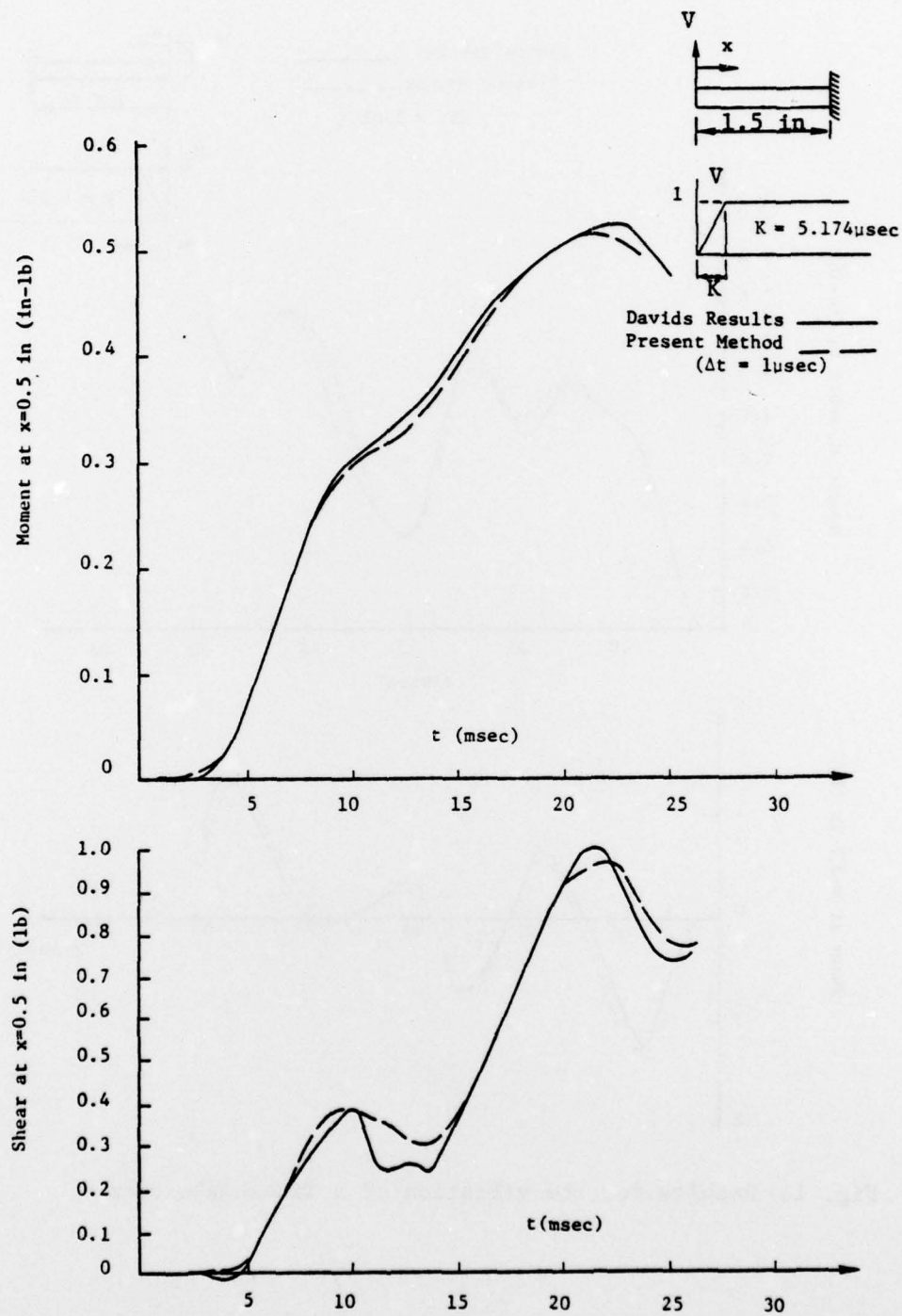


Fig. 13 Results for the vibration of a Timoshenko beam

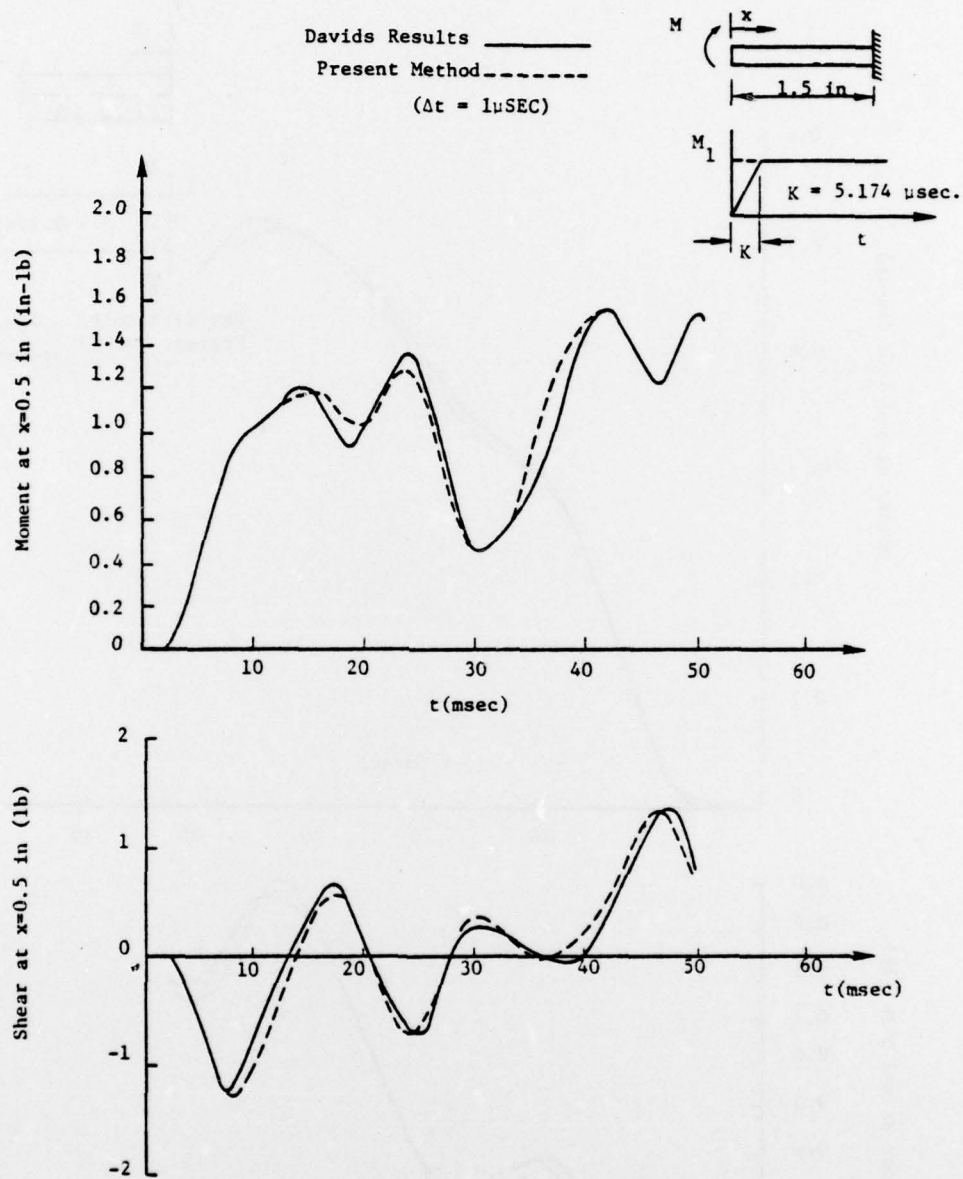
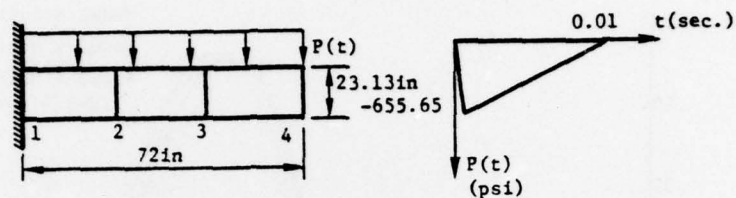


Fig. 14 Results for the vibration of a Timoshenko beam



$$\rho = 7.675 \times 10^{-4} \text{ lb-sec}^2/\text{in}^4 ; E = 3 \times 10^7 \text{ psi} ; A = 14.72 \text{ in}^2 ; \nu = 0.3$$

Fig. 15 Cantilever beam with distributed loading

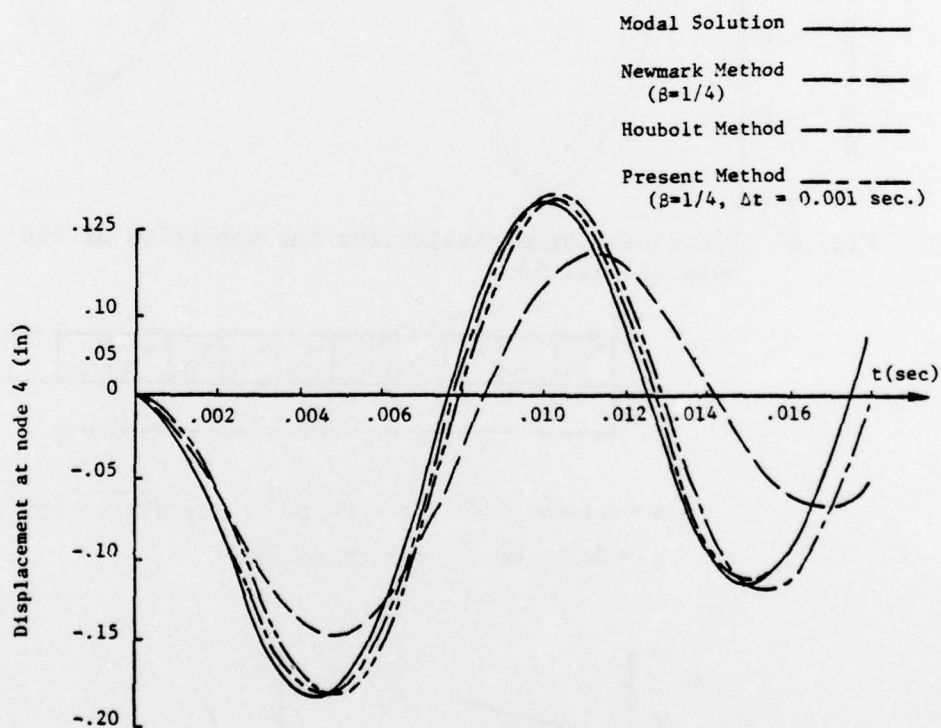


Fig. 16 Euler-Bernoulli beam results for the vibration of the cantilever beam of Fig. 15

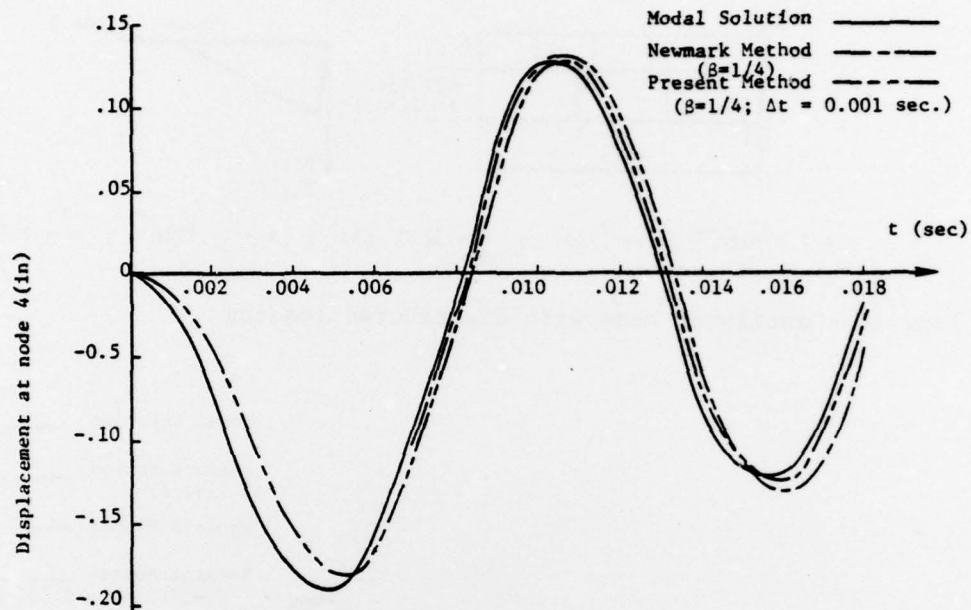
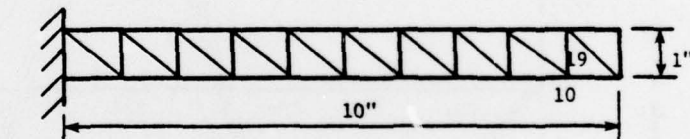


Fig. 17 Timoshenko beam results for the vibration of the cantilever beam of Fig. 15



$$m = 1 \text{ lb-sec}^2/\text{in}^2 ; E = 10^7 \text{ psi} ; C = \sqrt{EA/m} = 1000 \text{ in/sec:}$$

$$\sigma_y = 50,000 \text{ psi} ; E_p = 250,000 \text{ psi}$$

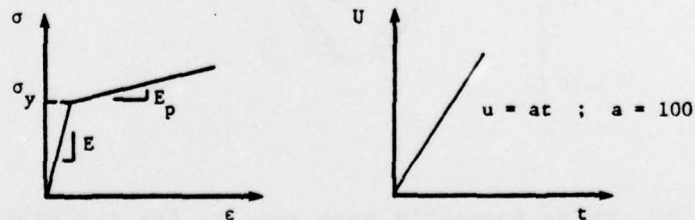


Fig. 18 Cantilever bar with elastic-plastic material

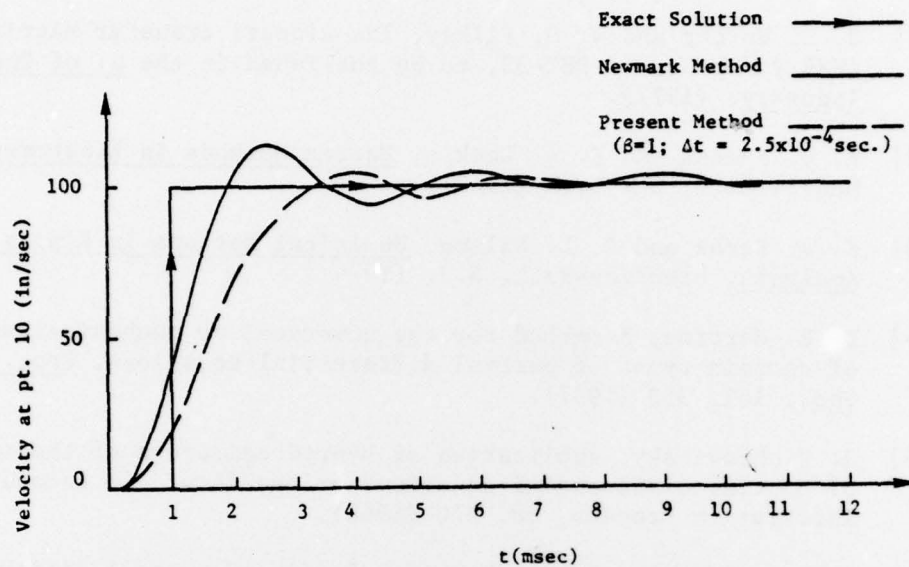


Fig. 19 Elastic wave solution for the vibration of the elastic-plastic material bar of Fig. 18

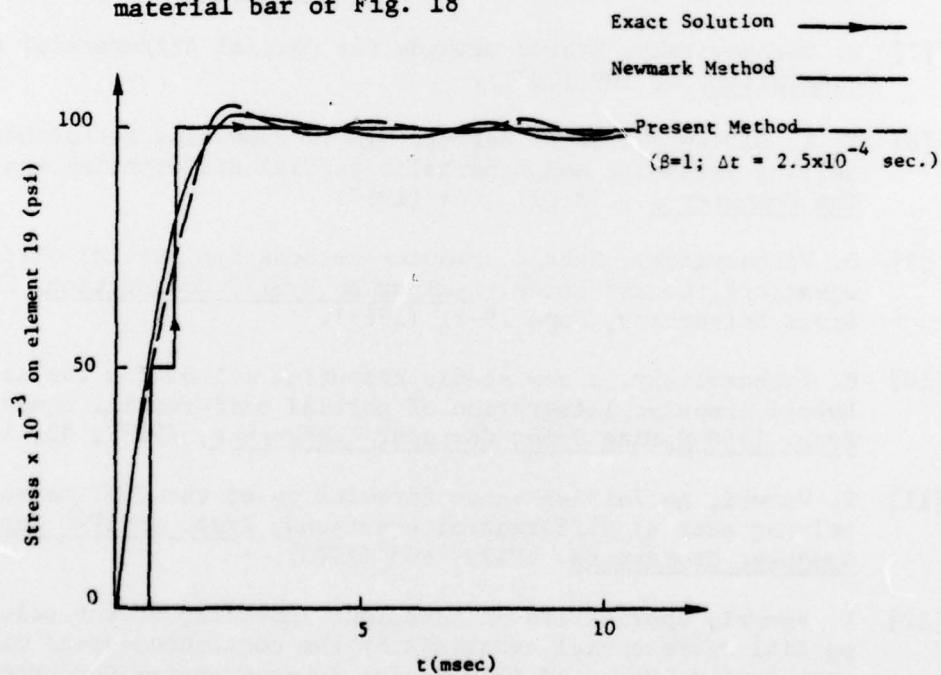


Fig. 20 Plastic wave solution for the vibration of the elastic-plastic material bar of Fig. 18

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