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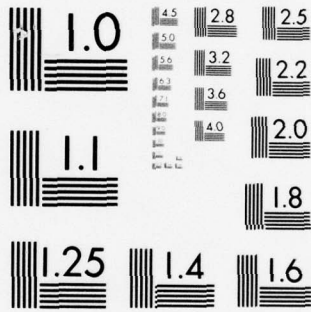
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ABSTRACT

We consider here a model of traffic flow on a road network. For each ordered pair of nodes there is a demand function which expresses travel demand between the two nodes as a function of travel times on the network. Each road (arc) has a delay function which expresses travel time on that arc as a function of total traffic flow. Our objective is to show how an equilibrium of travel times, flows, and demands may be computed under conditions which are simple, general, and plausible.

To solve the network problems we develop techniques for solving the stationary point problem. These techniques for the stationary point problem are the best of which we are aware.

CHAPTER 1

INTRODUCTION

Let $(\mathcal{N}, \mathcal{A})$ be a directed network with nodes i in \mathcal{N} and arcs ij in \mathcal{A} . Our interest is in the traffic flows and travel times on the network.

For each arc ij we are given a delay function f_{ij} which expresses travel time on arc ij as a function of the traffic flows on the arcs of the network. We might expect that travel time along arc ij would depend on the flow on that arc, but it may depend on flows along other arcs as well. In particular a two-way street could be modeled as a pair of opposing arcs, and the flow on one of the arcs might affect the travel time on the other arc.

Another way in which the interactions between different arcs may arise is in the modelling of an intersection. Florian and Nguyen [10] employed two methods of representing an intersection, see Figures 1 and 2. Travel time across the intersection will depend on the traffic in other directions. For instance the time required to make a left turn will be affected by the density of oncoming traffic.

For each pair of distinct nodes i, k we are given a travel demand function $g_{i,k}$ which expresses demand for travel from i to k as a function of travel times between nodes on the network. Demand for travel from i to k would, of course, depend on travel time from

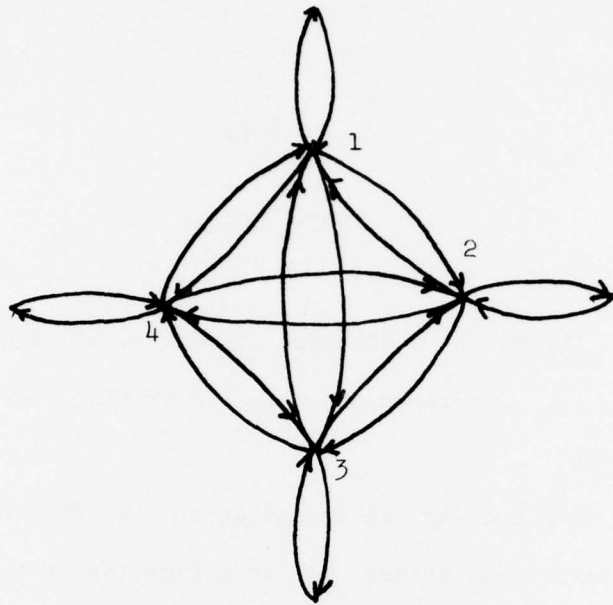


Figure 1

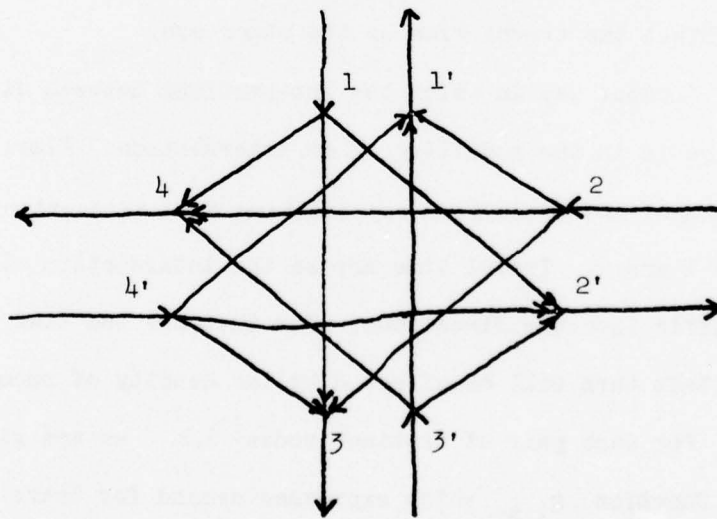


Figure 2

i to k . But it might also depend on travel times between other pairs of nodes, for instance from i to some alternate destination. Later in this chapter we will address some of the issues involved in deriving these demand functions.

The ability to handle these interactions between various arcs and origin-destination pairs is the principal contribution of this paper.

Many models very much like the model presented here have been developed over the last 25 years. The earliest examples of this kind are due to Wardrop [13] and Beckman [2]. These models are based on the notion that the road system has a large community of users, with each user taking the quickest route available, given the actions of other users. In addition, the number of trips made may depend on the time required to make a trip. On the other hand, travel time on a particular road will depend on traffic volume.

Numerous solution procedures have been proposed for these models. The usual approach has been to reformulate the equilibrium problem as a convex programming problem. This has the advantage that convex programming techniques when applied to this type of problem can be very efficient [9]. Rather strong conditions must be satisfied by the functions f and g in order for these procedures to be valid. In particular, f must be integrable and g must have an inverse which is integrable. This will be true if each f_{ij} depends only on the total flow $\sum_k y_{ij,k}$ and each $g_{i,k}$ depends only on $t_{i,k}$ and is monotone decreasing. The equilibrium problem is solved under conditions like these by Beckman [3], Dafermos and Sparrow [5], Florian and Nguyen [9], and others.

Dafermos [4] discusses a model in which travel times on one arc may depend on the flow on another arc in order to model the interaction of traffic going in opposite directions on a two-way street. In order to use the nonlinear programming approach she assumes that the delay functions are continuously differentiable, with symmetric Jacobian matrices. These assumptions are essentially a restatement of the integrability conditions mentioned above. These are reasonable when the street is symmetric and the traffic flow is about the same in each direction. Otherwise, the methods presented here would seem more applicable. This would be particularly true for the modeling of an intersection where the situation would normally be quite unsymmetric.

In practice the demand and delay functions f and g are at best empirical fits and can be endowed with these or any other restrictions which may seem useful. It should be noted, however, that the approach used here does not require such restrictions, and these restrictions can be expected to complicate the handling of interactions between various arcs and origin-destination pairs as described above. Indeed to compute an equilibrium it is only necessary that the delay functions f_{ij} be positive on each arc, that the travel demand functions $g_{i,k}$ be nonnegative and bounded for each pair of nodes, and that the network be complete that is, there is a directed path from every node to every other node. Unfortunately we must pay a price in computational efficiency for the additional generality. For problems which can be solved by convex programming techniques that would be the best approach.

Next we present the mathematical conditions for a user equilibrium. The travel time from node i to node k will be written $t_{i,k}$. The flow on arc ij with destination k will be written $y_{ij,k}$.

We say that the vectors of travel times t and flows x and y are in equilibrium if the following conditions hold.

$$1.a) \quad g_{i,k}(t) = \sum_j y_{ij,k} - \sum_j y_{ji,k} \quad i \neq k, i, k \in \mathcal{N}$$

$$1.b) \quad y \geq 0$$

$$1.c) \quad t_{i,k} \leq f_{ij}(x) + t_{j,k} \quad i \neq k, ij \in \mathcal{A}, k \in \mathcal{N}$$

$$t_{k,k} = 0 \quad k \in \mathcal{N}$$

$$1.d) \quad y_{ij,k} (f_{ij}(x) + t_{j,k} - t_{i,k}) = 0 \quad i \neq k, ij \in \mathcal{A}, k \in \mathcal{N}$$

$$1.e) \quad x_{ij} = \sum_k y_{ij,k} \quad ij \in \mathcal{A}$$

It may be useful to think of this system as a multicommodity network, where all of the traffic destined for a particular node k is a separate commodity, all of which must be shipped to node k via the network. In this way $g_{i,k}(t)$ is the amount of commodity k which must travel from node i to node k . This trip will traverse a path of arcs from i to k .

Condition (1.a) is the conservation of flow equation. It says that the traffic leaving node i with destination k is the sum of the traffic arriving at node i with destination k and the traffic originating at i with destination k .

Condition (1.b) says that traffic flows cannot be negative.

Conditions (1.c) and (1.d) require that traffic flow by the fastest route available. In (1.c) we require that $t_{i,k}$ not exceed the minimum travel time from i to k given the flows y on the network. Then (1.d) says that traffic may flow only along a route which achieves this minimum travel time. Together (1.c) and (1.d) imply the principle of minimum travel time. This says that if any traffic flows from i to k , that is if $\sum_j y_{ij,k} > 0$, then

$$t_{i,k} = \min_j (f_{ij}(y) + t_{j,k})$$

Equation (1.e) relates the basic flows y to the total arc flows x .

More generally, the functions f and g may be point-to-set maps. In this case we say that travel times t , flows x and y , and arc delays τ are in equilibrium if the following conditions hold.

$$1.a) \quad \sum_j y_{ij,k} - \sum_j y_{ji,k} \in g_{i,k}(t) \quad i \neq k, i, k \in \mathcal{N}$$

$$1.b) \quad y \geq 0$$

$$1.c) \quad \tau \in f(x)$$

$$t_{i,k} \leq \tau_{ij} + t_{j,k} \quad i \neq k, ij \in \mathcal{a}, k \in \mathcal{N}$$

$$t_{kk} = 0 \quad k \in \mathcal{N}$$

$$1.d) \quad y_{ij,k} (\tau_{ij} + t_{j,k} - t_{i,k}) = 0 \quad i \neq k, ij \in \mathcal{a}, k \in \mathcal{N}.$$

$$1.e) \quad x_{ij} = \sum_k y_{ij,k}$$

In the following theorems we require that the delays and demands be expressed in terms of upper semicontinuous convex valued point-to-set maps. In practice these functions are usually continuous and hence satisfy the requirements.

THEOREM 1. Suppose (η, α) is a complete network. Suppose

- i) f , the delay function is positive, convex valued, and upper semicontinuous on $\{x|x \geq 0\}$ and
- ii) g , the demand function is nonnegative, bounded, convex valued, and upper semicontinuous on $\{t|t \geq 0\}$.

Then a solution to the equilibrium problem exists and can be computed by the Eaves-Saigal algorithm. \square

Recently Aashtiani [1] has shown how the equilibrium problem can be formulated as a complementarity problem. Complementarity problems will be defined in Section 1 of Chapter 2. Aashtiani's approach is like the approach used here. He proved an existence theorem which is similar to Theorem 1, but he shows that his solution procedure may not converge under the conditions of his theorem.

The conditions of Theorem 1 would seem to be sufficiently general to allow the modelling of a traffic network. However, many actual systems have been modelled with demand functions which are unbounded when the corresponding travel times are small. Beckmann [3] suggests the following formula for travel demand.

$$g_{i,k}(t) = c \cdot p_i \cdot \frac{p_k}{(t_{i,k})^a}$$

Here p is the vector of populations at various nodes, and a and c are given constants. Utility functions can also be used to derive travel demands. In neither case are the conditions of Theorem 1 satisfied. It will be easy to see that the explicit function given above satisfies the conditions of Theorem 2. Theorem 3 and the discussion preceding it show how a demand function may be derived using utility functions which satisfies the conditions of Theorem 2.

Here $t \geq \epsilon$ means no component of t is less than ϵ .

THEOREM 2. Suppose (\mathcal{N}, a) is a complete network. Let $\epsilon \geq 0$. Suppose

- i) f , the delay function is upper semicontinuous and convex valued,
- ii) Each $f_{ij}(x) > \epsilon$ for all $x \geq 0$, and
- iii) g , the demand function is nonnegative, bounded, convex valued and upper semicontinuous, on $\{t | t \geq \epsilon\}$.

Then a solution to the equilibrium problem exists and can be computed by the Eaves-Saigal algorithm. \square

Theorem 1 is just the special case of Theorem 2 where $\epsilon = 0$. The following paragraphs will show how the assumptions on g in Theorem 2 arise naturally when g is derived as the sum of demands by consumers, each maximizing his own utility function.

We now suppose that the network is used by several consumers j . Each observes the condition of the network as measured by the travel time vector t . Each sets up a new potential demand vector that maximizes his utility. His utility is affected by the amount of travel between the various node pairs on the network, and the total time devoted to travel. This utility is expressed through a function U^j . For simplicity we assume that each consumer has an upper bound \bar{q}^j on the total amount of time he has available for travel. The sum over all j of the demands for travel originating at i and destined for k represents $g_{i,k}(t)$.

Given travel times the vector of t , the j th consumer will find a pair (q^j, r^j) which maximizes

$$U^j(q^j, r^j)$$

subject to

$$\begin{aligned} r^j \cdot t &\leq q^j \\ r^j &\geq 0, \quad q^j \leq \bar{q}^j \end{aligned}$$

where r^j is the vector of travel demands for various node pairs, and q^j is the total amount of time devoted to travel by j . The vector product $r^j \cdot t$ is the sum of the time spent in travel between all node pairs, hence its terms represent an allocation of q^j .

The following theorem shows that the demand function derived as the sum of these individual consumer demands satisfies the requirements of Theorem 2. It tells us that if g is derived as above and $f \geq \epsilon$ for any $\epsilon > 0$ then the hypotheses of Theorem 2 will be satisfied.

THEOREM 3. Suppose that each U^j is convex and continuous. For each $t > 0$ define $g(t) = \sum_j g^j(t)$. Then for each $\epsilon > 0$, g is non-negative, bounded, convex valued, and upper semicontinuous on $\{t | t \geq \epsilon\}$. \square

In Theorems 1 and 2 conditions (1.a) through (1.e) were shown to have a solution and that the Eaves-Saigal algorithm can construct a solution. For the special cases mentioned above where the problem can be solved as a convex program the equilibrium solution can also be shown to be unique. The following theorem extends the uniqueness property to a more general situation.

A point-to-set mapping f is called monotone on $K \subset \mathbb{R}^n$ if for each pair of vectors x^1 and x^2 in K and every $y^1 \in f(x^1)$ and $y^2 \in f(x^2)$, $(x^1 - x^2) \cdot (y^1 - y^2) \geq 0$. The mapping f strictly monotone if for every pair of distinct vectors x^1 and x^2 in K , and every $y^1 \in f(x^1)$ and $y^2 \in f(x^2)$, $(x^1 - x^2) \cdot (y^1 - y^2) > 0$.

The following theorem includes the convex programming uniqueness result as a special case since the gradient of a strictly convex function is strictly monotone.

THEOREM 4. Suppose in addition to the hypotheses of Theorem 2 that f and $-g$ are strictly monotone. Then the set of solutions to the equilibrium problem is convex and the total flows x are unique. \square

The proofs of the theorems of this chapter will be delayed until Chapter 3 since they require the background material of Chapter 2.

CHAPTER 2

THE STATIONARY POINT PROBLEM

The stationary point problem provides a common framework which includes the linear and nonlinear complementarity problems as well as fixed point problems. Applications come from such diverse areas as economics, game theory, mathematical programming and mechanics. For this reason the results of this chapter should be of interest beyond the specific equilibrium problems of this dissertation. However the approach used here is largely dictated by the problems at hand.

In Section 1 of this chapter after giving the necessary basic definitions, we explore some of the more important special cases of the stationary point problem. Section 2 is devoted to existence theorems; in Section 3 we present two theorems which address the question of uniqueness of stationary points.

2.1. Definitions and Examples

Let D be a subset of \mathbb{R}^n . A point-to-set map f from D to subsets of \mathbb{R}^n is called upper semicontinuous if $x^i \rightarrow x$ and $y^i \in f(x^i)$ imply that y^i has a cluster point in $f(x)$. (We must make a slightly stronger definition here than is usual since we do not assume that the image set is compact.)

We say that f is lower semicontinuous if $x^i \rightarrow x$ and $y \in f(x)$ imply that there is a sequence $y^i \rightarrow y$ such that $y^i \in f(x^i)$ for all i .

If f is both upper semicontinuous and lower semicontinuous then f is continuous.

The graph of f is $\{(x,y) | y \in f(x)\}$. A necessary and sufficient condition for f to be upper semicontinuous is that the graph of f is closed and f is bounded on compact subsets.

If $f(x)$ is any single point for each x then the above definition of upper semicontinuity coincides with the usual definition of continuity. The same is true for lower semicontinuity.

If D is a subset of R^n then D^* is the set of all nonempty compact convex subsets of D .

Let K be a convex subset of R^n . Let $f:K \rightarrow R^{n*}$ be upper semicontinuous. A pair (x,y) is a stationary point of (f,K) if $x \in K$, $y \in f(x)$, and $(u-x) \cdot y \geq 0$ for all $u \in K$.

As an example consider the programming problem

$$\begin{array}{l} \min g(x) \\ x \in K \end{array}$$

where g is differentiable.

We now show that for every solution x^* to this minimization problem that $(x^*, \nabla g(x^*))$ is a stationary point of the pair $(\nabla g, K)$. Suppose x^* solves the minimization problem. Then $x^* \in K$. Let $u \in K$. Since g is differentiable we may expand it in a Taylor series for $\lambda \in [0,1]$.

$$\begin{aligned} g(\lambda u + (1-\lambda)x^*) &= g(x^* + \lambda(u-x^*)) \\ &= g(x^*) + \lambda \nabla g(x^*) \cdot (u-x^*) + \lambda \alpha(\lambda) \end{aligned}$$

Here $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ is a function which satisfies $\lim_{\lambda \rightarrow 0} \alpha(\lambda) = 0$. Since $g(x^*) \leq g(x^* + \lambda(u-x^*))$ for all $\lambda \in [0,1]$, $\nabla g(x^*) \cdot (u-x^*) + \alpha(\lambda) \geq 0$ for all $\lambda \in [0,1]$. Since $\lim_{\lambda \rightarrow 0} \alpha(\lambda) = 0$ we have $\nabla g(x^*) \cdot (u-x^*) \geq 0$. Since this is true for all $u \in K$, $(x^*, \nabla g(x^*))$ is a stationary point of the pair $(\nabla g, K)$.

Given a convex cone K in \mathbb{R}^n , its polar K^+ is the set of all vectors y in \mathbb{R}^n such that $x \cdot y \geq 0$ for all $x \in K$. If $f: K \rightarrow \mathbb{R}^{n^*}$ then the complementarity problem (f, K) is to find (x, y) satisfying

$$x \in K, \quad y \in f(x) \cap K^+, \quad \text{and} \quad x \cdot y = 0$$

Much effort has been expended in studying the complementarity problem. Moreover, a wide range of applied problems have been reduced to it. However, in many instances the stationary point problem is the more convenient reduction. The next result from Saigal [12] relates the two problems.

Proposition 1. Suppose K is a convex cone and $f: K \rightarrow \mathbb{R}^{n^*}$. Then (x, y) solves the complementarity problem if and only if it solves the stationary point problem.

Proof. Suppose (x, y) solves the stationary point problem. Then $y \in f(x)$ and $x \cdot y \leq u \cdot y$ for all $u \in K$. Since $0 \in K$, $x \cdot y \leq 0$. Also $x \in K$ implies $\rho x \in K$ for $\rho \geq 0$. So for $\rho \geq 0$, $(\rho-1)x \cdot y \geq 0$. Since $\rho-1$ changes sign, we must have $x \cdot y = 0$, and thus $u \cdot y \geq 0$ for all $u \in K$ which says that $y \in K^+$, so (x, y) solves the complementarity problem.

Next let (x, y) solve the complementarity problem. Hence for all $u \in K$, $u \cdot y \geq 0 = x \cdot y$ or $(u-x) \cdot y \geq 0$. Now I need only to note that $x \in K$ and $y \in f(x)$. \square

The following proposition concerns another important special case of the stationary point problem. It also illustrates the relationship between the stationary point problem and stationary points of mathematical programming problems.

Proposition 2. Suppose $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and $K = \{x \in \mathbb{R}^n | Ax \leq b\}$. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^{n^*}$. Then (x, y) solves the stationary point problem (f, K) if and only if $x \in K$, $y \in f(x)$, and there is a $\pi \geq 0$ for which $\pi \cdot (Ax - b) = 0$ and $A^T \pi + y = 0$.

Proof. Let (x, y) solve the stationary point problem. Then x solves this linear program.

$$\begin{aligned} \min y \cdot x \\ Ax \leq b \end{aligned}$$

Let π be the solution to the dual

$$\begin{aligned} \max -b \cdot \pi \\ A^T \pi + y = 0 \\ \pi \geq 0 \end{aligned}$$

Now $\pi \cdot (Ax - b) = 0$ since $A^T \pi + y = 0$ and $y \cdot x = -b \cdot \pi$.

Conversely if $Ax \leq b$, $\pi \geq 0$, $\pi \cdot (Ax - b) = 0$, and $A^T \pi + y = 0$ then $-b \cdot \pi = -\pi \cdot Ax = y \cdot x$ so x solves the primal linear program. If in addition $y \in f(x)$ then (x, y) solves the stationary point problem. \square

2.2. Existence Theorems

This section has two main theorems which deal with the stationary point problem. Theorem 3 is a very general but non-constructive result. The proof is adapted from Saigal [12], but the theorem is more general in that it deals with the stationary point problem rather than the complementarity problem. If A and B are sets then we write $A \setminus B$ for the intersection of A and the complement of B .

Theorem 3. Let $K \subset \mathbb{R}^n$ be closed and convex, $C \subset K$ be compact, and let $f: K \rightarrow \mathbb{R}^{n*}$ be upper semicontinuous. Suppose that for all $x \in K \setminus C$, $y \in f(x)$ there is a $u \in C$ for which $(x-u) \cdot y \geq 0$. Then the stationary point problem (f, K) has a solution.

Proof. Define D to be any compact convex set $C \subset D \subset K$ such that for all $x \in C$, $y \in K \setminus D$ there is a $\lambda > 0$ such that $(1-\lambda)x + \lambda y \in D$. Let E be any compact convex set containing $f(D)$.

For each $y \in \mathbb{R}^n$ define $\pi(y)$ to be the set of all $x \in D$ which minimize $x \cdot y$ on D . Then π is upper semicontinuous and for each $y \in \mathbb{R}^n$, $\pi(y)$ is nonempty, convex, and compact.

Define $\Gamma: D \times E \rightarrow D^* \times E^*$ by $(x, y) \rightarrow (\pi(y), f(x))$. By the Eilenberg-Montgomery fixed point theorem [8], Γ has a fixed point. Call it (\hat{x}, \hat{y}) .

I now need only show that $\hat{x} \cdot \hat{y} \leq x \cdot \hat{y}$ for all $x \in K$. Suppose to the contrary that there is some $x \in K$ with $x \cdot \hat{y} < \hat{x} \cdot \hat{y}$. Then $x \in K \setminus D$. Also $\hat{x} \notin C$ since if $\hat{x} \in C$ then there is a $\lambda > 0$ such that $(1-\lambda)\hat{x} + \lambda x \in D$, but this implies $[(1-\lambda)\hat{x} + \lambda x] \cdot \hat{y} < \hat{x} \cdot \hat{y}$ which contradicts the definition of \hat{x} . So $\hat{x} \notin C$.

Then there is a $u \in C$ with $(\hat{x}-u) \cdot \hat{y} \geq 0$ or $\hat{x} \cdot \hat{y} \geq u \cdot \hat{y}$. There is a $\lambda > 0$ with $(1-\lambda)u + \lambda x \in D$, so

$$[(1-\lambda)u + \lambda x] \cdot \hat{y} = (1-\lambda)u \cdot \hat{y} + \lambda x \cdot \hat{y} < \hat{x} \cdot \hat{y}$$

which contradicts the definition of \hat{x} . So for each $x \in K$, $\hat{x} \cdot \hat{y} \leq x \cdot \hat{y}$ and (\hat{x}, \hat{y}) is a stationary point of (f, K) . \square

The basis for the constructive results of this section is the algorithm of Eaves and Saigal [7] for systems of nonlinear equations. The next lemma gives a general condition on a point-to-set map $f: \mathbb{R}^n \rightarrow \mathbb{R}^{n^*}$ under which an $x \in \mathbb{R}^n$ may be approximated which satisfies $0 \in f(x)$. The algorithm solves the system of equations using piecewise linear approximations. The solution is approached by solving finer and finer approximations.

We say U separates $v \notin U$ from infinity if every connected set containing v and not meeting U is bounded.

Let $C \subset \mathbb{R}^n$. We define $\text{conv}(C)$ to be the intersection of all convex sets containing C . We write $\text{dia}(C)$ for

$$\sup_{x, y \in C} \|x - y\| .$$

In proving the following theorem we list some of the most important properties of the Eaves-Saigal algorithm. A complete description of the algorithm may be found in Eaves and Saigal [7].

Lemma 4. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^{n*}$ be upper semicontinuous. Let $v \in \mathbb{R}^n$. Let $U \subset \mathbb{R}^n$ be a closed set which separates v from infinity. Let D be the union of all connected sets containing v but not meeting U . Define $G: \mathbb{R}^n \rightarrow \mathbb{R}^{n*}$ by $G(x) \triangleq \text{conv}(f(x) \cup \{x-v\})$. Suppose that $0 \notin G(U)$. Then there is a $\delta > 0$ such that if the initial grid size is less than δ then the Eaves-Saigal algorithm will not evaluate f outside D . Therefore it will converge to a connected set $C_\infty \neq \emptyset$ with $0 \in f(x)$ for all $x \in C_\infty$.

Proof. Given a starting point v and an initial grid size δ the Eaves-Saigal algorithm generates a sequence of sets C_i , $i = 0, 1, 2, \dots$ with the following properties.

$$(2.1) \quad v \in \text{conv}(C_0)$$

$$(2.2) \quad \text{dia}(C_i) < \delta$$

$$(2.3) \quad 0 \in \text{conv}(f(C_i) \cup (C_i - v))$$

$$(2.4) \quad C_{i+1} = C_i \cup \{s_i\} \setminus \{t_i\}$$

$$(2.5) \quad \text{if } C_{i_j} \rightarrow x \text{ on a subsequence then } 0 \in \text{conv}(f(C_{i_j})) \text{ for all large } j.$$

$$(2.6) \quad \text{If all } C_i \text{ lie in a bounded set, then } \text{dia}(C_i) \rightarrow 0.$$

Let $C_\infty = \overline{\lim} C_i$. If $x \in C_\infty$ then $0 \in f(x)$. If all the C_i lie in a bounded set then C_∞ is a nonempty, closed, connected set.

D is a connected set containing v , and not meeting U , so D is bounded since U separates v from infinity.

We now use an indirect argument to verify the theorem. If the conclusion does not hold then there is a sequence $\delta^k \rightarrow 0$ such that for each k the algorithm produces a sequence C_i^k , $i = 0, 1, 2, \dots$ which is not contained in D . For each k let x^k be the first point in the sequence C_i^k which is not in D . By (2.4) $C_i^k - \{x^k\} \subset D$ so $d(x^k, D) < \delta^k$.

Let \bar{x}^k be any point in D with $d(x^k, \bar{x}^k) < \delta^k$. Since $x^k \notin D$ there is a point $u^k \in \text{conv}(x^k, \bar{x}^k) \cap U$. Therefore $d(x^k, U) < \delta^k$.

Let x^* be any cluster point of the x^k . Then $d(x^*, U) = 0$, so $x^* \in U$ since U is closed. But since for all i and all k , $0 \in \text{conv}(G(C_i^k))$, it follows that $0 \in G(x^*)$. This contradiction proves the theorem. \square

In practice neither δ nor the set U will be known to the user of the algorithm. However, if it is known that the function f and starting point v satisfy the hypotheses of the lemma for some U then the following process may be employed to find a solution.

- 1.) Choose δ^k and M^k , $k = 0, 1, 2, \dots$ such that $\delta^k \rightarrow 0$ and $M^k \rightarrow \infty$. Set $k = 0$.
- 2.) Apply the algorithm to f using starting point v and initial grid size δ^k . If no points $x \in R^n$ are generated with $\|x-v\| > M^k$ then the algorithm must converge to a set of solutions. Otherwise go to 3.

3.) Set $k = k+1$. Go back to 2.

Eventually, a k will be found for which δ^k is small enough and $\|x-v\| < M^k$ for all $x \in D$. Lemma 4 tells us that the algorithm will converge to a set of solutions for this k .

In practice grid size does not appear to be critical. If the algorithm seems to be straying too far from the starting point, it would probably be better to look for errors in the formulation than to try again with a smaller grid size.

Lemma 5. If p, q , and r are in R^n and $\|p-q\| < \|p-r\|$ then $(p-r)(q-r) > 0$.

Proof.

$$\|p-q\|^2 < \|p-r\|^2$$

$$\|(p-r) - (q-r)\|^2 < \|p-r\|^2$$

$$\|p-r\|^2 - 2(p-r)(q-r) + \|q-r\|^2 < \|p-r\|^2$$

$$-2(p-r)(q-r) + \|q-r\|^2 < 0$$

$$(p-r)(q-r) > 0 . \quad \square$$

The next theorem is a constructive version of Theorem 3. In order to be able to compute a solution we need only slightly stronger conditions than in Theorem 3.

A function $g:R^n \rightarrow R$ is convex by definition if for all $x, y \in R^n$, $\lambda \in [0,1]$, $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$. If g is convex then we define the subgradient map of g by

$$\partial g(x) = \{y \in \mathbb{R}^n \mid y \cdot (x-u) \geq g(x) - g(u) \text{ for all } u \in \mathbb{R}^n\} .$$

If g is convex then $\partial g(x)$ is nonempty, compact and convex for each x and ∂g is upper semicontinuous.

Theorem 6. Let $g_i: \mathbb{R}^n \rightarrow \mathbb{R}$ $i = 1, \dots, m$ be convex. Let $K = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i = 1, \dots, m\}$. Let $C \subset K$ be compact. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^{n^*}$ be upper semicontinuous. Suppose that there is a $u^0 \in C$ such that $g_i(u^0) < 0, i = 1, \dots, m$. Suppose that for all $x \in K \setminus C, y \in f(x)$ there is a $u \in C$ such that $(x-u) \cdot y > 0$. Then the stationary point problem (f, K) has a solution which can be computed by the Eaves-Saigal algorithm using any starting point.

Proof. For each $x \in \mathbb{R}^n$ define $F(x)$ to be the set of all $\sum_{i=0}^m \lambda_i y^i$ which satisfy these conditions.

$$\begin{aligned} \sum_{i=0}^m \lambda_i &= 1 & \lambda_i &\geq 0 \\ y^0 &\in f(x) & \lambda_0 &= 0 \text{ if for any } i \neq 0, g_i(x) > 0 \end{aligned}$$

(2.7) For $i = 1, \dots, m$

$$y^i \in \partial g_i(x) \quad \lambda_i = 0 \text{ if } g_i(x) < 0.$$

First we need to show that F is upper semicontinuous. Suppose that $x^j \rightarrow x$, and $w^j \in F(x^j)$ for each j . For each j there exist

λ^j and $y^{i,j}$, $i = 0, \dots, m$ which satisfy (2.7) and such that $w^j = \sum_{i=1}^m \lambda_i^j y^{i,j}$. Since the x^j converge, they lie in some compact set. The $y^{i,j}$ are in the (compact) images of this compact set under $f, \partial g_1, \dots, \partial g_m$. Thus there is a subsequence on which the $y^{i,j}$ and λ^j converge, say to y^i and λ . In the following we will deal only with this subsequence.

Since the g_i are continuous, for all j sufficiently large and for all $i = 1, \dots, m$, $g_i(x^j) > 0$ if $g_i(x) > 0$, and $g_i(x^j) < 0$ if $g_i(x) < 0$. If for any $i = 1, \dots, m$, $g_i(x) > 0$ then $\lambda_0^j = 0$ for large j , so $\lambda_0 = 0$. If $i = 1, \dots, m$ and $g_i(x) < 0$ then $\lambda_i^j = 0$ for large j , so $\lambda_j = 0$. Since f and the ∂g_i are upper semi-continuous, $y^0 \in f(x)$ and $y^i \in \partial g_i(x)$. Therefore λ, x , and y^i , $i = 0, \dots, m$ satisfy (2.7). Let $w = \sum_{i=0}^m \lambda^i y^i$. Then $w \in F(x)$ and on the subsequence $w^j \rightarrow w$ so F is upper semicontinuous.

It is clear that $F(x)$ is nonempty, convex and compact for each x .

Let $v \in \mathbb{R}^n$. Let $M = \sup_{u \in C} \|u-v\|$. To show that F satisfies the hypotheses of Theorem 4 it only remains to show that for all $x \in \mathbb{R}^n$ with $\|x-v\| > M$ that $0 \notin \text{conv}(F(x) \cup \{x-v\})$.

So suppose $x \in \mathbb{R}^n$ and $\|x-v\| > M$. Let $w \in F(x)$. There exist λ, y^0, \dots, y^m which satisfy (2.7) such that $w = \sum_{i=0}^m \lambda_i y^i$. There is a $u^0 \in C$ such that $g_i(u^0) < 0$, $i = 1, \dots, m$. We distinguish two cases according to whether $\lambda_0 = 0$.

If $\lambda_0 > 0$ then $x \in K \setminus C$ so there is a $u \in C$ such that $(x-u) \cdot y^0 > 0$. Choose α sufficiently small that $(x - (\alpha u^0 + (1-\alpha)u)) \cdot y^0 > 0$. Let

$u' = \alpha u^0 + (1-\alpha)u$. Then $g_i(u') < 0$, $i = 1, \dots, m$. For each $i = 1, \dots, m$ if $\lambda_i > 0$ then $g_i(x) \geq 0$, so by the gradient inequality $(x-u') \cdot y^i \geq g_i(x) - g_i(u') > 0$. Now $(x-u') \cdot w = \sum_{i=0}^m \lambda_i (x-u') \cdot y^i$. This sum is positive since no term is negative and some term is positive. Since $\|x-v\| > \|v-u'\|$ we have $(x-u') \cdot (x-v) > 0$ by Lemma 5. Therefore $0 \notin \text{conv}(\{w\} \cup \{x-v\})$.

But if $\lambda_0 = 0$ then for each $i = 1, \dots, m$ if $\lambda_i > 0$ then $g_i(x) \geq 0$ so by the gradient inequality $(x-u^0) \cdot y^i \geq g_i(x) - g_i(u^0) > 0$. Now $(x-u^0) \cdot w = \sum_{i=1}^m \lambda_i (x-u^0) \cdot y^i$. This sum is positive since some term is positive and no term is negative. Since $\|x-v\| > \|v-u^0\|$ we have $(x-u^0) \cdot (x-v) > 0$ by Lemma 5. Therefore $0 \notin \text{conv}(\{x\} \cup \{x-v\})$.

Since $0 \notin \text{conv}(\{w\} \cup \{x-v\})$ for all $w \in F(x)$, $0 \notin \text{conv}(F(x) \cup \{x-v\})$.

By Theorem 4 we may use the Eaves-Saigal algorithm to approximate an $x \in R^n$ with $0 \in F(x)$. For such an x there exist λ and y^i , $i = 0, \dots, m$ which satisfy (2.7) and $\sum_{i=0}^m \lambda_i y^i = 0$. For each $i = 1, \dots, m$ if $\lambda_i > 0$ then $g_i(x) \geq 0$ so by the gradient inequality $(x-u^0) \cdot y^i \geq g_i(x) - g_i(u^0) > 0$. If $\lambda_0 < 1$ then

$$\lambda_0 (x-u^0) \cdot y^0 = - \sum_{i=1}^m \lambda_i (x-u^0) \cdot y^i \leq \sum_{i=1}^m \lambda_i (g_i(u^0) - g_i(x)) < 0 .$$

Thus $\lambda_0 > 0$. Let $\bar{u} \in K$. Then

$$\lambda_0 (x-\bar{u}) \cdot y^0 = - \sum_{i=1}^m \lambda_i (x-\bar{u}) \cdot y^i \leq \sum_{i=1}^m \lambda_i (g_i(\bar{u}) - g_i(x)) \leq 0 .$$

So $(x-\bar{u}) \cdot y^0 \leq 0$ and (x, y^0) is a stationary point of (f, K) . \square

2.3. Uniqueness Theorems

In this section we explore the question of uniqueness of stationary points.

Let $f:K \rightarrow \mathbb{R}^{n^*}$ where $K \subset \mathbb{R}^n$ is closed and convex. The mapping f is called monotone on K if for every pair of vectors x^1 and x^2 in K and every $y^1 \in f(x^1)$ and $y^2 \in f(x^2)$, $(x^1-x^2) \cdot (y^1-y^2) \geq 0$. The mapping f is strictly monotone if for every pair of distinct vectors x^1 and x^2 in K , and every $y^1 \in f(x^1)$ and $y^2 \in f(x^2)$, $(x^1-x^2) \cdot (y^1-y^2) > 0$. The following is a useful uniqueness result for the stationary point problem.

THEOREM 7. Let $f:K \rightarrow \mathbb{R}^{n^*}$ be strictly monotone. If there is a solution to the stationary point problem (f,K) then it is unique.

Proof. Suppose (x^1, y^1) and (x^2, y^2) solve (f,K) . Then $y^1 \in f(x^1)$, $y^2 \in f(x^2)$ and for all $u \in K$, $(u-x^1) \cdot y^1 \geq 0$ and $(u-x^2) \cdot y^2 \geq 0$. In particular $(x^2-x^1) \cdot y^1 \geq 0$ and $(x^1-x^2) \cdot y^2 \geq 0$. Summing these we have $(x^1-x^2) \cdot (y^1-y^2) \leq 0$. But this is impossible since f is strictly monotone. \square

The following theorem is based on the same idea, but is slightly more general. By using linearity and relaxing the strict monotonicity we obtain not a unique solution, but a convex set of solutions. However, in many cases we may be able to determine the entire set of solutions.

THEOREM 8. Let $K \subset \mathbb{R}^n$ and $B \in \mathbb{R}^{m \times n}$. Let $L = \{Bx | x \in K\}$. Let $f: L \rightarrow \mathbb{R}^{m^*}$ be strictly monotone. Let $A \in \mathbb{R}^{n \times n}$ be positive semi-definite. Define $g: K \rightarrow \mathbb{R}^{n^*}$ by $g(x) = B^T f(Bx) + Ax$. Then the set of solutions (x, y) to the stationary point problem (g, K) is convex and Bx has the same value for all of these solutions.

Proof. Suppose (x^1, y^1) and (x^2, y^2) solve (g, K) . Then there exist w^1 and w^2 such that for $i = 1, 2$, $y^i = B^T w^i + Ax^i$, $w^i \in f(Bx^i)$ and for all $u \in K$, $(u - x^i) \cdot (B^T w^i + Ax^i) \geq 0$. In particular $(x^2 - x^1) \cdot (B^T w^1 + Ax^1) \geq 0$ and $(x^1 - x^2) \cdot (B^T w^2 + Ax^2) \geq 0$. Summing these $(x^1 - x^2) \cdot (B^T(w^1 - w^2) + A(x^1 - x^2)) \leq 0$ or $(x^1 - x^2) \cdot A(x^1 - x^2) + B(x^1 - x^2) \cdot (w^1 - w^2) \leq 0$. But since A is positive semi-definite $B(x^1 - x^2) \cdot (w^1 - w^2) \leq 0$. Since f is strictly monotone this implies that $Bx^1 = Bx^2$.

Let $\lambda \in [0, 1]$, $x = \lambda x^1 + (1 - \lambda)x^2$, $y = \lambda y^1 + (1 - \lambda)y^2$, and $w = \lambda w^1 + (1 - \lambda)w^2$. Then $Bx = Bx^1 = Bx^2$ and $y = B^T w + Ax$. Since w^1 and w^2 are in $f(Bx)$ and $f(Bx)$ is convex, $w \in f(Bx)$. So $y \in g(x)$. Let $u \in K$. Since A is positive semi-definite

$$\begin{aligned} (u-x)Ax &\geq \lambda(u-x^1)Ax^1 + (1-\lambda)(u-x^2)Ax^2 \\ (u-x)B^T w &= \lambda(u-x)B^T w^1 + (1-\lambda)(u-x)B^T w^2 \\ &= \lambda(u-x^1)B^T w^1 + (1-\lambda)(u-x^2)B^T w^2 \\ (u-x)(B^T w + Ax) &\geq \lambda(u-x^1)(B^T w^1 + Ax^1) + (1-\lambda)(u-x^2)(B^T w^2 + Ax^2) \geq 0 \end{aligned}$$

Therefore (x, y) solves (g, K) and the theorem follows. \square

CHAPTER 3

PROOFS OF THE MAIN THEOREMS

This chapter is devoted to the proofs of the theorems of Chapter 1. Theorem 1.1 will not be given a separate proof since it is clearly a special case of Theorem 1.2. Here $t \geq \epsilon$ means no component of t is less than ϵ .

Proof of Theorem 1.2.

Throughout this proof $t_{k,k}$ will represent the constant zero. Thus $t \in R^m$ where $m = |\mathcal{N}| \cdot (|\mathcal{N}| - 1)$. For computational efficiency we will make the additional requirement that $y_{ij,i} = 0$ for all arcs ij . In the following we will show that a solution to the equilibrium problem can be computed which satisfies this restriction. For this reason we will regard y as a vector in R^n where $n = q \cdot (|\mathcal{N}| - 1)$ and $q = |\mathcal{A}|$ is the total number of arcs. Define $B \in R^{q \times n}$ by

$$(By)_{ij} = \sum_k y_{ij,k} \quad ij \in \mathcal{A}.$$

Then we may write

$$(B^T x)_{ij,k} = x_{ij} \quad i \neq k, ij \in \mathcal{A}, k \in \mathcal{N}.$$

Define a matrix $A \in R^{m \times n}$ by

$$(Ay)_{i,k} = \sum_j y_{ij,k} - \sum_j y_{ji,k} \quad i \neq k, i,k \in \mathcal{N}.$$

Then we may write

$$(A^T t)_{ij,k} = t_{i,k} - t_{j,k} \quad i \neq k, ij \in \mathcal{Q}, k \in \mathcal{N}.$$

In these formulas $y_{ij,i}$ and $t_{k,k}$ are to be read as zero where they appear.

For each $l \neq k, l, k \in \mathcal{N}$ choose a path from l to k . Let $d^{\ell,k}$ be a vector in R^n such that $d_{ij,k}^{\ell,k} = 1$ if ij is on the path and all other components of $d^{\ell,k}$ are zero. Thus $Ad^{\ell,k}$ is a vector in R^m such that $(Ad^{\ell,k})_{\ell,k} = 1$ and all other components are zero.

Choose $b \in R^m$ so that $b \geq g(t)$ for all $t \geq \epsilon$. Choose $h \in R^n$ so that for all $ij \in \mathcal{Q}, k \in \mathcal{N}, i \neq k$

$$h_{ij,k} > \sum_{\ell} (b_{\ell,k} + 1)$$

Let $K = \{(y,t) \in R^{m+n} | t \geq \epsilon, 0 \leq y \leq h\}$. Define $G: R^{m+n} \rightarrow R^{m+n*}$ by

$$G(y,t) \triangleq \begin{pmatrix} B^T f(By) - A^T t \\ Ay - g(t) \end{pmatrix}$$

We proceed to show that the stationary point problem (G,K) may be solved as described in Chapter 2, and that this solution will also solve the equilibrium problem.

Let $y^0 = \sum_{i,k} (b_{i,k} + 1) d^{i,k}$. Then for each $ij \in \mathcal{Q}, i \neq k \in \mathcal{N}$ we have the following.

$$y_{ij,k}^0 = \sum_{\ell} (b_{\ell,k} + 1) d_{ij,k}^{\ell,k} \leq \sum_{\ell} (b_{\ell,k} + 1) < h_{ij,k}$$

$$y^0 < h$$

$$Ay^0 = \sum_{i,k} (b_{i,k} + 1) Ad^{i,k} = b + e$$

Here $e \in R^m$ is a vector of ones. Let

$$M = -\inf\{(y-y^0) \cdot B^T \tau \mid 0 \leq y \leq h, \tau \in f(By)\}.$$

Then $M \geq 0$ since $0 \leq y^0 \leq h$. Let

$$C = \{(y,t) \in R^{m+n} \mid (y,t) \in K, e \cdot t \leq M\}$$

If $(y,t) \in K \setminus C$ then for each $\gamma \in g(t)$, $\tau \in f(By)$

$$\begin{pmatrix} y - y^0 \\ t \end{pmatrix} \begin{pmatrix} B^T \tau - A^T t \\ Ay - \gamma \end{pmatrix} = (y-y^0) \cdot B^T \tau + t \cdot (Ay - \gamma)$$

$$\geq -M + e \cdot t > 0.$$

So by Theorem 2.6 the stationary point problem (G,K) can be solved.

Suppose $((y,t), (B^T \tau, \gamma))$ solves the stationary point problem (G,K) . By Proposition 2.2 we may write this in the following way.

$$t \geq \epsilon, \quad 0 \leq y \leq h, \quad \gamma \in g(t), \quad \tau \in f(By)$$

$$\left. \begin{aligned} r_{i,k} &\leq \sum_j y_{ij,k} - \sum_j y_{ji,k} \\ (t_{i,k} - \epsilon) \cdot (r_{i,k} - \sum_j y_{ij,k} + \sum_j y_{ji,k}) &= 0 \end{aligned} \right\} \quad i \neq k, \quad i, k \in \mathcal{N}$$

$$\tau_{ij} + t_{j,k} - t_{i,k} \begin{cases} \geq 0 & \text{if } y_{ij,k} < h_{ij,k} \\ \leq 0 & \text{if } y_{ij,k} > 0 \end{cases} \quad ij \in \alpha, k \in \eta, i \neq k$$

If for some i, k , $r_{i,k} < \sum_j y_{ij,k} - \sum_j y_{ji,k}$, then $t_{i,k} = \epsilon$ and for all arcs $i_1 i_2$ along some path from i to k , $y_{i_1 i_2, k} > 0$. Therefore $t_{i_1, k} > t_{i_2, k} + \epsilon$ since $\tau_{i_1 i_2} > \epsilon$. But since $t_{k,k} = 0$, $t_{i,k} > 0$ by transitivity along the path. So $r_{i,k} = \sum_j y_{ij,k} - \sum_j y_{ji,k}$ after all.

If for some $ij \in \alpha$, $k \in \eta$, $i \neq k$, $y_{ij,k} = h_{ij,k}$ then $y_{ij,k} > \sum_l b_{l,k} \geq \sum_l r_{l,k}$. That is, the traffic on arc ij with destination k is greater than the traffic from all origins with destination k . This is only possible if there is a loop which includes ij such that for all $i_1 i_2$ in this loop $y_{i_1 i_2, k} > 0$. This implies that $t_{i_1, k} > t_{i_2, k}$ since $\tau_{i_1 i_2} > 0$. But this is impossible since it implies by transitivity along the loop that $t_{i,k}$ is less than itself. Therefore $y < h$.

Since $t \geq \epsilon$ it follows that $t \geq 0$. Since $\tau > 0$, $t \geq 0$ and $t_{k,k} = 0$ for all k , $\tau_{ij} + t_{j,i} - t_{i,i} > 0$.

Taking the last three paragraphs into account we may write the solution conditions for the stationary point problem in this way:

$$t \geq 0, \quad y \geq 0, \quad r \in g(t), \quad \tau \in f(By)$$

$$r_{i,k} = \sum_j y_{ij,k} - \sum_j y_{ji,k} \quad i \neq k, i, k \in \eta$$

$$\left. \begin{array}{l} r_{ij} + t_{j,k} - t_{i,k} \geq 0 \\ (r_{ij} + t_{j,k} - t_{i,k}) \cdot y_{ij,k} = 0 \end{array} \right\} ij \in \mathcal{Q}, k \in \mathcal{N}$$

But these are precisely the desired equilibrium conditions if we let $x = By$. \square

Before proving Theorem 1.3 we will need the following lemma on the minimum of a continuous function. If $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and $D \subset \mathbb{R}^n$ then we define

$$\arg \min_{x \in D} f(x) \triangleq \{x \in D \mid f(x) \leq f(y) \text{ for all } y \in D\}$$

Lemma. Let $K: \mathbb{R}^m \rightarrow D^*$ be continuous, where D is a compact subset of \mathbb{R}^n . Let $f: \mathbb{R}^n \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a continuous function. For each $z \in \mathbb{R}^m$, $w \in \mathbb{R}^k$ let $J(z, w) \triangleq \arg \min_{x \in K(z)} f(x, w)$. Then J is upper semi continuous.

Proof. Suppose $z^i \rightarrow z$, $w^i \rightarrow w$, and $x^i \in J(z^i, w^i)$ for all i . Since each $x^i \in D$ there is a subsequence on which the x^i converge, say to x . Since K is upper semicontinuous, $x \in K(z)$. Hereafter we will work only on this subsequence. If $\hat{x} \in K(z)$ then since K is lower semicontinuous there is a sequence $\hat{x}^i \rightarrow \hat{x}$ such that $\hat{x}^i \in K(z^i)$ for all i . Therefore

$$f(x, w) = \lim_i f(x^i, w^i) \leq \lim_i f(\hat{x}^i, w^i) = f(\hat{x}, w).$$

So $x \in J(z, w)$ which implies that J is upper semicontinuous. \square

Proof of Theorem 1.3. For each $j = 1, \dots, p$ and each $t \geq \epsilon$ let

$$K^j(t) = \{(q^j, r^j) \mid r^j \geq 0, q^j \leq \bar{q}^j, r^j \leq q^j\}.$$

Then for each $(q^j, r^j) \in K^j(t)$ we have $0 \leq q^j \leq \bar{q}^j$ and

$$e \cdot r^j \leq \frac{1}{\epsilon} t \cdot r^j \leq \frac{1}{\epsilon} q^j \leq \frac{1}{\epsilon} \bar{q}^j.$$

Therefore $\bigcup_{t \geq \epsilon} K^j(t)$ is bounded. Define

$$h^j(t) \triangleq \arg \min_{(q^j, r^j) \in K^j(t)} U(q^j, r^j)$$

Then n^j is bounded on $\{t \mid t \geq \epsilon\}$.

For each $t \geq \epsilon$ there exists $(q^j, r^j) \in K^j(t)$ such that $r^j > 0$, $q^j < \bar{q}^j$ and $r^j \cdot t < q^j$. So by [6, Corollary II.3.2] K^j is continuous. By the lemma h^j is upper semicontinuous. But g^j can be obtained by deleting the first component in the range of h^j . Therefore g^j is nonnegative, bounded, convex valued, and upper semicontinuous on $\{t \mid t \geq \epsilon\}$, and so is g since $g = \sum_j g^j$. \square

Proof. of Theorem 1.4. In the proof of Theorem 1.2 we introduced a stationary point problem (G, K) and showed that solutions to (G, K) are also solutions to the network equilibrium problem. Here we show the opposite, namely that solutions to the equilibrium problem solve the stationary point problem. The theorem will follow when we have shown that (G, K) satisfies the hypotheses of Theorem 2.8.

Since g is nonnegative and $-g$ is strictly monotone, g is positive. Suppose y , t , τ , and γ solve the equilibrium problem. We may state this as follows.

$$t \geq 0, \quad y \geq 0, \quad \gamma \in g(t), \quad \tau \in f(By)$$

$$r_{i,k} = \sum_j y_{ij,k} - \sum_j y_{ji,k} \quad i \neq k, i, k \in \mathcal{N}$$

$$\left. \begin{array}{l} \tau_{ij} + t_{j,k} - t_{i,k} \geq 0 \\ (\tau_{ij} + t_{j,k} - t_{i,k}) \cdot y_{ij,k} = 0 \end{array} \right\} ij \in \mathcal{A}, k \in \mathcal{N}$$

For any pair of nodes i, k since $r_{i,k} > 0$ then for all arcs $i_1 i_2$ along some path from i to k , $y_{i_1 i_2, k} > 0$. Therefore $t_{i_1, k} > t_{i_2, k} + \epsilon$ since $\tau_{i_1 i_2} > \epsilon$. But since $t_{k, k} = 0$, $t_{i, k} > \epsilon$ by transitivity along the path. So $t > \epsilon$.

If for some $ij \in \mathcal{A}$, $k \in \mathcal{N}$, $i \neq k$, $y_{ij,k} \geq h_{ij,k}$ then $y_{ij,k} > \sum_{\ell} b_{\ell, k} \geq \sum_{\ell} r_{\ell, k}$. That is, the traffic on arc ij with destination k is greater than the traffic from all origins with destination k . This is only possible if there is a loop which includes ij such that for all $i_1 i_2$ in this loop $y_{i_1 i_2, k} > 0$. This implies that $t_{i_1, k} > t_{i_2, k}$ since $\tau_{i_1 i_2} > 0$. But this is impossible since it implies by transitivity along the loop that $t_{i, k}$ is less than itself. Therefore $y < h$.

Since $t > \epsilon$ and $y < h$, we have in fact a solution to (G, K) .

To show that the hypotheses of Theorem 2.8 are satisfied we make the following definitions.

$$\bar{A} = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} \quad \bar{B} = \begin{pmatrix} B & 0 \\ 0 & I_{m \times m} \end{pmatrix}$$

$$\bar{f}(x, t) = (f(x), -g(t)) .$$

Clearly $G(y, t) = \bar{B}^T \bar{f}(\bar{B}(y, t)) + \bar{A}(y, t)$. \bar{f} is strictly monotone because f and $-g$ are strictly monotone. It is easy to see that \bar{A} is positive semidefinite. Therefore the set of solutions to the stationary point problem (and the equilibrium problem) is convex, and $\bar{B}(y, t)$ has the same value for all such solutions. In particular t is unique and $B y$ is unique but the flows y may not be unique. \square

CHAPTER 4

COMPUTATION

The purpose of this chapter is to demonstrate the computational viability of the procedures described in Chapters 2 and 3.

4.1. Formulation and Approach

The method used to solve the network problems is outlined in the proofs of Theorems 1.2, 2.4 and 2.6. The proof of Theorem 1.2 (Chapter 3) shows how the network problem can be formulated as a stationary point problem which satisfies the conditions of Theorem 2.6. The proof of Theorem 2.6 shows how such a stationary point problem can be formulated as a problem of finding a zero of a point-to-set map.

H. Z. Aashtiani [1] has had some success with an approach to solving traffic network problems which uses Lemke's algorithm as a subroutine. However, no convergence proof is offered for this process.

Considerable latitude is available in solving a stationary problem (f, K) using the Eaves-Saigal algorithm. The remainder of this section is devoted to a trick which uses this latitude to reduce the time required to solve certain stationary point problems.

In the proof of Theorem 2.6 we defined a map F by setting $F(x)$ equal to the set of all $\sum_{i=0}^m \lambda_i y^i$ such that

$$\sum_{i=0}^m \lambda_i = 1 \qquad \lambda_i \geq 0$$

$$y^0 \in f(x) \qquad \lambda_0 = 0 \text{ is for all } i \neq 0, g_i(x) > 0$$

For $i = 1, \dots, m$

$$y^i \in \partial g_i(x) \qquad \lambda_i = 0 \text{ if } g_i(x) < 0.$$

In solving a point-to-set map F using the Eaves-Saigal algorithm a unique member of $F(x)$ must be chosen for each point $x \in R^n$. It is often the case that f and ∂g_i are continuous, so that the y^i are unique. However we still must choose the weights λ_i .

In addition we may choose the triangulation or grid on which the Eaves-Saigal algorithm operates. In the case where the functions g_i are affine it is possible to choose a triangulation in which every simplex which meets the interior of K is contained in K . Using such a triangulation with a choice of λ where $\lambda_0 = 1$ if $x \in K$, we obtain greatly superior convergence. This is due to the fact that a solution to the piecewise-linear approximation at large grid sizes is a good approximation to the actual solution. This is not true for most choices of weights λ and of the triangulation. Finding a suitable triangulation is particularly easy in the case where the constraints on K are only upper and lower bounds, as they are for the network problems of Chapter 1.

In the next section we will demonstrate empirically the usefulness of this technique in the case of the traffic flow problems.

4.2. Examples.

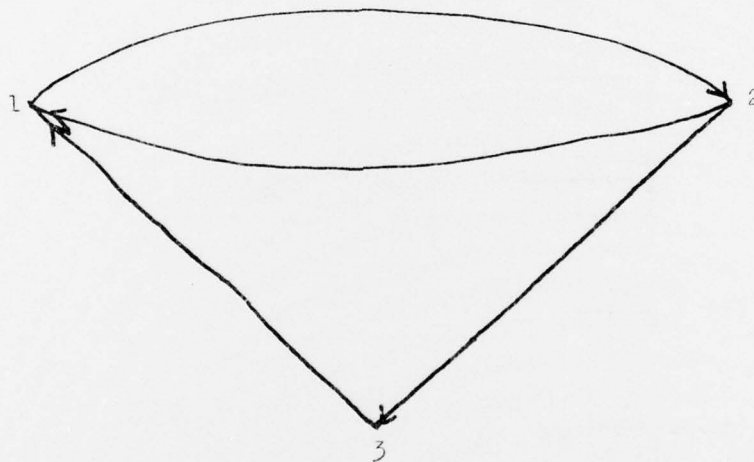
In this section two examples are given of the network formulation of Chapter 1. In order to solve these problems we prepared a computer code which provides an interface between the network problem and the

Eaves-Saigal algorithm. We used a program by Romesh Saigal to execute the Eaves-Saigal algorithm.

All of the computer runs were made on an IBM 370/168 using FORTRAN H with the highest level of optimization.

If $D \subset \mathbb{R}^n$ then the diameter of D is $\sup_{x,y \in D} \|x-y\|$. The initial grid sizes reported below refer to the diameter of the simplices on which the function is first evaluated. The final grid sizes refer to the diameter of the final simplex.

The first example is based on the following network.



Here $\eta = \{1,2,3\}$ and $\alpha = \{12,21,23,31\}$. The delay functions are

$$f_{12}(y) = 10 + e^{(y_{12}-10)} + 1.25 \log(y_{21} + 1.0)$$

$$f_{21}(y) = 10 + e^{(y_{21}-10)} + 1.25 \log(y_{12} + 1.0)$$

$$f_{23}(y) = 4 + e^{(y_{23}-12)}$$

$$f_{31}(y) = 4 + e^{(y_{31}-20)}$$

where

$$y_{ij} = \sum_k y_{ij,k}$$

The travel demand functions are as follows.

$$g_{1,2}(t) = \frac{80}{t_{1,2} + 1}$$

$$g_{1,3}(t) = \frac{120}{t_{1,3} + 1}$$

$$g_{2,1}(t) = \begin{cases} \frac{40}{t_{2,1} + 1} & \text{if } t_{2,1} \geq t_{2,3} \\ \frac{100}{t_{2,1} + 1} & \text{if } t_{2,1} \leq t_{2,3} \end{cases}$$

$$g_{2,3}(t) = \begin{cases} \frac{80}{t_{2,3} + 1} & \text{if } t_{2,1} \geq t_{2,3} \\ \frac{20}{t_{2,3} + 1} & \text{if } t_{2,1} \leq t_{2,3} \end{cases}$$

$$g_{3,1}(t) = \frac{60}{t_{3,1} + 1}$$

$$g_{3,2}(t) = \frac{100}{t_{3,2} + 1}$$

Where more than one function value is given at a particular point (e.g. $t_{2,1} = t_{2,3}$) the value of g is the convex hull of the two values. In this case some of the travelers from node 2 will go to either 1 or 3 depending on which is closest. If the travel times are equal then those travelers who want to go to either 1 or 3 will be divided between the two destinations.

Arcs 12 and 21 represent a two-way street. Note that although the function is symmetric with respect to the two arcs, the Jacobian matrix will not be symmetric unless the flows in the two directions are equal. Therefore, the method proposed by Dafermos [4] would not be applicable here. Furthermore the demand functions are neither invertible nor continuous. Therefore the full power of Theorem 1.1 is required to solve this problem.

Here are the equilibrium values.

i,k	$t_{i,k}$	$g_{i,k}(t)$
1,2	19.30	3.94
1,3	28.43	4.08
2,1	13.22	2.81
2,3	9.13	7.90
3,1	4.09	11.79
3,2	23.38	4.10

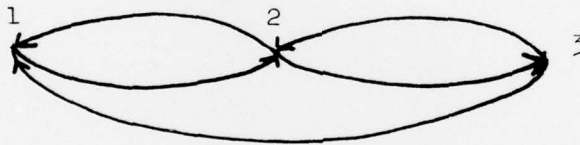
ij,k	$y_{ij,k}$	f_{ij}
12,2	8.04	19.30
12,3	4.08	19.30
21,1	1.15	13.22
21,3	0.00	13.22
23,1	1.66	9.13
23,3	11.97	9.13
31,1	13.46	4.09
31,2	4.10	4.09

This problem was run with and without the special alignment discussed in Section 1. In each run the initial grid size was 18.7 and the final grid size was 2.8×10^{-7} . The same answer was found in each case.

In the first run the triangulation was aligned with K. In this case the algorithm required 886 function evaluations, 979 pivots, and 1.70 seconds CPU time. The starting vector had 1.25 in each component.

In the second run no alignment was used. This time the algorithm required 1598 function evaluations, 1690 pivots and 2.83 seconds CPU time. The starting vector had 1.00 in each component.

The second example is based on the following network.



Now $\eta = \{1, 2, 3\}$ and $\alpha = \{12, 21, 23, 32, 31\}$. The delay functions are

$$f_{12} = 10 + e^{(y_{12}-10)} + 1.25 \cdot \log(y_{21} + 1)$$

$$f_{21} = 10 + e^{(y_{21}-10)} + 1.25 \cdot \log(y_{12} + 1)$$

$$f_{23} = 5 + 0.5 \cdot e^{(y_{23}-10)} + 0.625 \cdot \log(y_{32} + 1)$$

$$f_{32} = 5 + 0.5 \cdot e^{(y_{32}-10)} + 0.625 \cdot \log(y_{23} + 1)$$

$$f_{31} = 15.0 + 1.5e^{(y_{31}-5.0)}$$

where

$$y_{ij} = \sum_k y_{ij,k}$$

The travel demand functions are identical to the earlier example.

Here are the equilibrium values.

i,k	$t_{i,k}$	$g_{i,k}(t)$
1,2	13.30	5.59
1,3	23.55	4.89
2,1	13.05	2.85
2,3	10.25	7.11
3,1	15.39	3.66
3,2	8.04	11.06

ij,k	$y_{ij,k}$	f_{ij}
12,2	5.59	13.30
12,3	4.89	13.30
21,1	2.85	13.05
21,3	0.00	13.05
23,1	0.00	10.25
23,3	12.00	10.25
32,1	0.00	8.04
32,3	11.06	8.04
31,1	3.66	15.39
31,2	0.00	15.39

This problem was also run with and without the special alignment discussed in Section 1. In each run the initial grid size was 20.0 and the final grid size was 2.9×10^{-7} . The same answer was found in each case.

In the first run the triangulation was aligned with K. In this case the algorithm required 1107 function evaluations, 1204 pivots and 2.39 seconds CPU time. Without alignment the algorithm required 1921 function evaluations, 2016 pivots and 3.92 seconds CPU time. The first starting vector had 1.25 in each component, the second 1.00 in each component.

From these runs we may conclude that the procedure developed here is a viable approach to the traffic equilibrium but that the alignment technique significantly reduces the computation time.

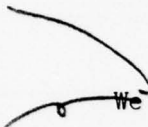
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TRAFFIC NETWORK EQUILIBRIA by Richard Louis Asmuth

 We consider here a model of traffic flow on a road network. For each ordered pair of nodes there is a demand function which expresses travel demand between the two nodes as a function of travel times on the network. Each road (arc) has a delay function which expresses travel time on that arc as a function of total traffic flow. Our objective is to show how an equilibrium of travel times, flows, and demands may be computed under conditions which are simple, general, and plausible.

To solve the network problems ~~we develop~~ ^{are developed} techniques for solving the stationary point problem. These techniques for the stationary point problem are the best of which we are aware.



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