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SOLUTION OF H-EQUATIONS BY ITERATION

C. T. Kelley

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ABSTRACT

A generalization of the Chandrasekhar H-equation is solved by iteration. Such equations are of interest in heat transfer.

SIGNIFICANCE AND EXPLANATION[†]

In certain mathematical models of physical phenomena, there may be several theoretical solutions, but only one of them may make sense in the physical context. When one encounters such a model which cannot be solved in closed form, the question naturally arises as to whether some numerical method will in fact produce the physically significant solution.

In certain heat transfer problems the following integral equation is encountered:

$$(*) \qquad H(x,\omega) = 1 + H(x,\omega)\,\frac{\omega}{2}\int_0^\infty \frac{t}{x+t}\,J_1(t)H(t,\omega)\,dt, \quad x \geq 0 .$$

In (*), $J_1(t)$ is the Bessel function of order 1, ω is a parameter, and h is the function to be determined. Of the several possible solutions to (*), only one is of physical interest.

In this paper, we show that (*) may be solved by iteration and that the solution obtained in this way is the one of physical importance.

AMS(MOS) Subject Classification: 80.45, 45G99, 45L05

Key Words: H-equation, iteration

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[†] The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

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SOLUTION OF H-EQUATIONS BY ITERATION

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I. Introduction.

Consider the integral equation

$$(1.1) \qquad F(x,\omega) = 1 + \omega\, f(x,\omega) \int_0^\infty \frac{t}{x+t}\, \varphi(t)\, f(t,\omega)\,dt.$$

In (1.1), ω is a complex parameter, f is the function to be found, and φ is a measurable function on $(0,\infty)$.

Under certain assumptions on φ it is known that (1.1) has a solution $H(x,\omega)$ that is analytic in ω for $|\omega| < 1$. This solution is of importance in many applications. We call H the physical solution to (1.1). The question considered in this paper is the following: When can H be found by solving (1.1) by iteration?

The question has been answered if $\varphi \geq 0$, $\varphi(x) = 0$ for $0 \leq x \leq 1$, and $\int_1^\infty t^2 \varphi(t)\,dt < \infty$. In this case H is continuous for $|\omega| \leq 1$, $x \geq 1$. These assumptions are natural in neutron transport theory. *Bowden and Zwiefel* [1] have shown that if $\int_1^\infty \varphi(t)\,dt = \frac{1}{2}$ then H may be found for $|\omega| < 1$ by an iterative method. In a more general setting, Mullikin and the author [7] showed that H may be computed for $|\omega| \leq 1$ in this way.

In many cases of physical importance, however, φ may become negative. The methods of Bowden and Zweifel [1] and Rall [9], then give the best known results. If, however, φ is not integrable on $(0,\infty)$, these methods fail as the integral in (1.1) need not even be defined in the Lebesgue sense.

This paper is motivated by work of Crosbie and Sawheny [5], [6]. They consider (1.1) for $\varphi(x) = \frac{1}{2} J_1(x)$, where J_1 is the Bessel function of order one. They obtained numerical results that indicated that H may be found by an iterative method for quite general φ.

We assume φ satisfies the following conditions:

(A1) $\varphi \in L_P(0,0)$ for some P, $1 \leq P < \infty$.

(A2) $\int_0^\infty \left|\frac{\varphi(t)}{t}\right| dt < \infty$.

(A3) Let $k(x) = \lim_{N \to \infty} \int_0^N e^{-|x|t}\, t\, \varphi(t)\,dt$; then $k \geq 0$, and $k \in L_1(-\infty,\infty)$.

(A4) $\int_{-\infty}^\infty k(x)\,dx = 1$.

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(A5) $\lim\limits_{N \to \infty} \int_0^N \varphi(t)dt = 1/2$.

(A6) If f is a measurable, nonnegative decreasing function on $(0,\infty)$ and

$$0 < \sup_{0 < x \leq 1} |x\, f(x)| < \infty, \text{ then } \lim_{N \to \infty} \int_0^N \varphi(t) f(t)dt > 0.$$

The reader should note that (A4) is merely a normalization condition on φ . As will become clear, (A5) implies (A4). The purpose of the normalization is to make H analytic in the disc $|\omega| < 1$. Our assumptions imply that the integral in (1.1) is defined for nonnegative decreasing f by

$$\int_0^\infty \frac{t}{x+t} \varphi(t) f(t)dt = \lim_{N \to \infty} \int_0^N \frac{t}{x+t} \varphi(t) f(t)dt \ . \tag{1.2}$$

For $\varepsilon \geq 0$, let $C(\varepsilon)$ be the space of complex-valued functions continuous on $[\varepsilon,\infty)$ and having finite limits at infinity. $C(\varepsilon)$ is a Banach space under the sup norm. Moreover Dini's Theorem holds for $C(\varepsilon)$ in the sense that if a sequence of real valued functions f_n in $C(\varepsilon)$ converge monotonically upward to f in $C(\varepsilon)$ pointwise, and $\lim\limits_{x \to \infty} f_n(x) = \lim\limits_{x \to \infty} f(x)$ for all n , then f_n converges to f in the topology of $C(\varepsilon)$.

We will show in section II that H is an analytic $C(0)$ valued function of ω for $|\omega| < 1$ and is a continuous $C(\varepsilon)$ valued function of ω for $|\omega| \leq 1$.

Under assumptions (A1)-(A6) we prove the following iteration results.

Theorem (2.1) *For* $|\omega| \leq 1$, $x \geq 0$ *define*

$$H_0(x,\omega) = 1$$

$$H_{n+1}(x,\omega) = 1 + \omega H_n(x,\omega) \int_0^\infty \frac{t}{x+t} H_n(t,\omega)\varphi(t)dt, \quad n \geq 0.$$

Then for $0 < \varepsilon < 1$

(i) H_n *converges to* H *in* $C(0)$ *uniformly in* ω *for* $|\omega| \leq 1 - \varepsilon$.

(ii) H_n *converges to* H *in* $C(\varepsilon)$ *uniformly in* ω for $|\omega| \leq 1$.

Theorem (2.2) *Let* $0 < \varepsilon < 1$, $|\omega| \leq 1$, *define for* $x \geq 0$

$$K_0(x,\omega) = 1$$

$$K_{n+1}(x,\omega) = [1 - \omega \int_0^\infty \frac{t}{x+t} \varphi(t) K_n(t,\varphi)dt]^{-1}.$$

Then

(i) K_n converges to H in $C(0)$ uniformly in ω for $|\omega| \leq 1 - \varepsilon$.

(ii) K_n converges to H in $C(\varepsilon)$ uniformly in ω for $|\omega| \leq 1$.

Implicit in the statements of these theorems is the fact that all integrals exist in the sense of (1.2). For convenience we write, for g measurable,

$$\lim_{N \to \infty} \int_0^N g(t)dt = \int_* g(t)dt \tag{1.3}$$

when the limit on the left exists.

II <u>Iteration Results</u>

Equation (1.1) is intimately connected to the theory of Wiener-Hopf equations [8]. This connection plays a vital role in the proofs of our results, and we give the relevant details here. For $f \in L_1(-\infty,\infty)$, $\hat{f}$ will denote the Fourier transform of f.

Let $k(x)$ be as in (A3). Assumptions (A3) and (A4) imply that for $|\omega| < 1$, the equation,

$$\gamma(x,\omega) - \omega \int_0^\infty k(x-y)\gamma(y,\omega)dy = \omega k(x),\ x > 0\ , \tag{2.1}$$

has a unique solution $\gamma(x,\omega) \in L_1(0,\infty)$. For $0 \leq \omega < 1$, $\gamma \geq 0$, and γ is an analytic $L_1(0,\infty)$-valued function of ω for $|\omega| < 1$. We write

$$\gamma(x,\omega) = \sum_{n=1}^{\infty} \omega^n \gamma_n(x)\ . \tag{2.2}$$

In (2.2), $\gamma_n(x) \geq 0$ for $x > 0$, $\gamma_n \in L_1(0,\infty)$. Then the unique solution to (1.1) which is analytic in ω for $|\omega| < 1$ and $C[0,\infty]$ valued is

$$H(x,\omega) = 1 + \hat{\gamma}(ix,\omega)\ . \tag{2.3}$$

For $|\omega| < 1$, we have the factorization, valid for real λ,

$$(1 - \omega\hat{k}(\lambda))(1 + \hat{\gamma}(\lambda,\omega))(1 + \hat{\gamma}(-\lambda,\omega)) = 1\ . \tag{2.4}$$

From (2.3) and (2.4) we conclude

$$\frac{\partial}{\partial x} H(x,\omega) < 0 \quad \text{for} \quad x > 0\ ,\ 0 \leq \omega < 1\ , \tag{2.5}$$

and

$$\frac{\partial}{\partial \omega} H(x,\omega) > 0 \quad \text{for} \quad x \geq 0\ ,\ 0 \leq \omega < 1\ . \tag{2.6}$$

$$H(x,\omega) = 1 + \sum_{n=1}^{\infty} \omega^n \hat{\gamma}_n(ix)\ ,\ |\omega| < 1. \tag{2.7}$$

$$\hat{\gamma}_n(ix) > 0 \quad \text{for} \quad n \geq 1\ ,\ x \geq 0\ . \tag{2.8}$$

$$\hat{\gamma}_n(i\cdot) \in C[0,\infty] \quad \text{for} \quad n \geq 1 \tag{2.9}$$

Let Σ denote the class of nonnegative decreasing continuous functions f on $(0,\infty)$ which satisfy

$$\sup_{0 < x \leq 1} |xf(x)| < \infty\ . \tag{2.10}$$

For $f \in \Sigma$ define

(2.11) $$L\,f(x) = \int_* \frac{t}{x+t}\,\varphi(t)f(t)dt$$

(2.12) $$M\,f(x) = \int_0^\infty \frac{x}{x+t}\,\varphi(t)f(t)dt.$$

The integral defining M is a Lebesque integral for each fixed $x > 0$ by assumption (A1). Assumptions (A1) and (A6) imply that Mf and Lf are in $C(0)$ for every $f \in \Sigma$.

We note that for $0 \le \omega < 1$, H is in Σ. We may rewrite (1.1) as

(2.13) $$H(\omega) = 1 + \omega\, L(H(\omega))H(\omega)\ .$$

We now prove the main lemma.

<u>Lemma</u> (2.1). Let $\varepsilon > 0$, then

$$\lim_{\omega \to 1^-} H(\omega) = H(1)$$

exists in $C(\varepsilon)$. Moreover $H(1) \in \Sigma$, and for $x > 0$, we have

(2.14) $$H(1) = 1 + L(H(1))H(1)\ .$$

<u>Proof.</u> By (2.4) we have that

$$H(0,\omega) = (1-\omega)^{-1/2} = 1 + \omega \int_* \varphi(t)H(t,\omega)dt\ H(0,\omega).$$

Hence

(2.15) $$H^{-1}(\omega) = (1-\omega)^{1/2} + \omega\, M(H(\omega))\ .$$

Consider the function F given for $x \ge 0$ by,

(2.16) $$F(x) = \int_* \frac{t}{x+t}\,\varphi(t)dt.$$

Assumptions (A2) and (A6) imply that $F \in C(0)$ and $F \ge 0$. For $\varepsilon > 0$, $x \ge 0$, we have

$$F(x) - F(x+\varepsilon) = \int_0^\infty \frac{\varepsilon}{(x+t)(x+\varepsilon+t)}\,\varphi(t)dt > 0\ .$$

Therefore F is decreasing. By (2.13), we have

(2.17) $$H^{-1}(x,\omega) \ge \omega\, x\, F(x) \ge \omega\, x\, F(0)\ .$$

Hence, for $x \ge \varepsilon$, $0 \le \omega < 1$, $H(x,\omega) \le \frac{1}{\omega\varepsilon} F^{-1}(0)$. Hence, for $x > 0$, $\lim_{\omega \to 1^-} H(x,\omega) = H(x,1)$ exists by (2.6). Moreover $H(x,1)$ is decreasing in x and $\sup_{0<x\le 1} |x\, H(x,1)| \le F^{-1}(0) < \infty$. Hence $M\,H(1) \in C(0)$ by assumption (A1). By the dominated convergence theorem and assumption (A1) we have, for $x > 0$,

(2.18) $$M\,H(1)(x) = \lim_{\omega \to 1^-} M\,H(\omega)(x) = H^{-1}(x,\omega)\ .$$

Hence $H(1) \in \Sigma$ and $H(1)$ satisfies (2.14). Dini's theorem then implies that the limit in (2.18) exists in $C(\varepsilon)$ for $\varepsilon > 0$. This completes the proof.

Now for $x > 0$, $|H(x,\omega)| \leq \sum_{n=1}^{\infty} |\omega|^n \hat{\gamma}_n(ix) + 1 = H(x,|\omega|) \leq H(x,1)$. Hence for $\varepsilon \geq 0$, H is an analytic $\mathcal{C}(\varepsilon)$-valued function of ω for $|\omega| < 1$. For $\varepsilon > 0$, H is a continuous $\mathcal{C}(\varepsilon)$ valued function of ω for $|\omega| \leq 1$.

We now prove the main results.

__Theorem__ (2.1) Assume φ satisfies (A1)-(A6). For $|\omega| \leq 1$, $x \geq 0$ define

$$H_0(\omega) = 1$$

$$H_{n+1}(\omega) = 1 + \omega H_n(\omega)\, L(H_n(\omega)) \qquad n \geq 0 .$$

Then, for $0 < \varepsilon < 1$,

(i) $H_n(\omega)$ converges to $H(\omega)$ in $\mathcal{C}(0)$ uniformly in ω for $|\omega| \leq 1 - \varepsilon$.

(ii) $H_n(\omega)$ converges to $H(\omega)$ in $\mathcal{C}(\varepsilon)$ uniformly in ω for $|\omega| \leq 1$.

__Proof.__ Let $0 < \varepsilon < 1$. Observe that if $g(x) = 1 + \int_0^{\infty} e^{-xt} G(t)dt$, and $G \in L_1$ is non-negative, then $1 + Lg(x)$ has the same form as

$$Lg(x) = \int_0^{\infty} e^{-xt}[k(t) + \int_0^{\infty} k(x+t)G(s)ds]dt . \tag{2.19}$$

The definition of H_n implies, therefore that for $m \geq 1$ there are functions $P_{n,m} \in L_1(0,\infty)$, $P_{n,m} \geq 0$, such that for $x \geq 0$, $|\omega| \leq 1$,

$$H_n(x,\omega) = \sum_{m=0}^{2^{n-1}} \omega^m \int_0^{\infty} e^{-xt} P_{n,m}(x)dx . \tag{2.20}$$

It is easy to see that

$$P_{n,m}(x) = \gamma_m(x) , \qquad 1 \leq m \leq n . \tag{2.21}$$

We claim that $P_{n,m}(x) \leq \gamma_m(x)$ for all m and n. We prove this by induction on n. For $n = 0$ the result is clear. Assume (2.21) for $n = N$; from the definition of H_n and (2.19) we have,

$$\begin{aligned}
P_{N+1,m}(t) = {} & \sum_{\substack{j+\ell=m-1 \\ j,\ell \geq 1}} \int_0^t P_{N,\ell}(t-s) \int_0^{\infty} k(s+r)P_{N,j}(r)drds \\
& + \int_0^t P_{N,m-1}(t-s)k(s)ds + \int_0^{\infty} k(t+s)P_{N,m-1}(s)ds \\
\leq {} & \sum_{\substack{j+\ell=m-1 \\ j,\ell \geq 1}} \int_0^t \gamma_\ell(t-s) \int_0^{\infty} k(s+r)\gamma_j(r)drdx \\
& + \int_0^t \gamma_{m-1}(t-s)k(s)ds + \int_0^{\infty} k(t+s)\gamma_{m-1}(s)ds \\
= {} & \gamma_m(x) .
\end{aligned} \tag{2.22}$$

The last equality is a consequence of the fact that

(2.23) $$\hat{\gamma}_m(ix) = \sum_{\substack{j+\ell=m-1 \\ j,\ell \geq 1}} \hat{\gamma}_j(ix) L(\hat{\gamma}_\ell(i\cdot))(x) + \hat{\gamma}_{m-1}(ix) + L(\hat{\gamma}(i\cdot))(x) ,$$

which is an easy consequence of (1.1).

A similar argument shows that

(2.24) $$P_{n,m} \leq P_{n+1,m} .$$

Hence we have, for $n,k \geq 0$, $x \geq 0$, $|\omega| \leq 1$,

(2.25) $$|H_{n+k}(x,\omega) - H_n(x,\omega)| \leq H(x,|\omega|) - H_n(x,|\omega|) \leq \sum_{m=n+1}^{\infty} \hat{\gamma}_m(ix)|\omega|^m .$$

For $x \geq \varepsilon$, $|\omega| \leq 1$ we have

(2.26) $$\sum_{m=n+1}^{\infty} \hat{\gamma}_m(ix)|\omega|^m \leq H(x,1) - \sum_{m=1}^{n} \hat{\gamma}_m(ix) .$$

The right side of (2.26) goes to zero as n becomes large uniformly for $x \geq \varepsilon$, $|\omega| \leq 1$. This proves (ii).

For $x \geq 0$, $|\omega| \leq 1 - \varepsilon$ we have

(2.27) $$\sum_{m=n+1}^{\infty} \hat{\gamma}(ix)|\omega|^m \leq H(x,1-\varepsilon) - \sum_{m=1}^{n} \hat{\gamma}(ix)|1-\varepsilon|^m .$$

The right side of (2.27) converges to zero as n becomes large uniformly for $x \geq 0$, $|\omega| \leq 1-\varepsilon$. This proves (i) and completes the proof.

Theorem (2.2) Let $0 < \varepsilon < 1$, define, for $|\omega| \leq 1$, $x \geq 0$,

$$K_0(x,\omega) = 1$$

$$K_{n+1}(x,\omega) = [1 - \omega\, L(K_n(\cdot,\omega))(x)]^{-1} , \quad n \geq 0 .$$

Then

(i) K_n converges to H in $C(0)$ uniformly in ω for $|\omega| \leq 1-\varepsilon$.

(ii) K_n converges to H in $C(\varepsilon)$ uniformly in ω for $|\omega| \leq 1$

Proof. Note that $H_0(x,\omega) = K_0(x,\omega) \leq H(x,\omega)$. An induction argument similar to that used in the proof of Theorem (2.1) will show that, for $x > 0$, $0 \leq \omega \leq 1$.

(2.28) $$K_n(x,\omega) \leq H(x,\omega) ,$$

and

(2.29) $$K_n(x,\omega)[1-\omega\, L(H_{n-1}(\cdot,\omega)(x)]^{-1} \geq H_n(x,\omega) .$$

As in the proof of Theorem (2.1) we also have for $|\omega| \leq 1$, $x > 0$, $n \geq 0$, $m \geq 0$,

(2.30) $$|K_{n+m}(x,\omega) - K_n(x,\omega)| \leq K_{n+m}(x,|\omega|) - K_n(x,|\omega|).$$

This completes the proof.

Systems of nonlinear equations similar to (1.1) are of interest in the kinetic theory of gases [2], [4]. It is possible that the methods of this paper will generalize to that setting.

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