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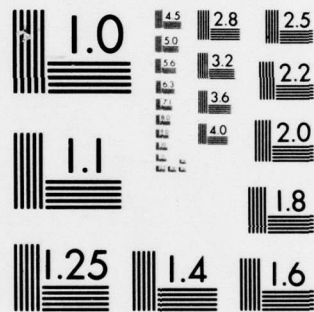


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POSITIVE KERNELS AND STOCHASTIC INTEGRALS

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POSITIVE KERNELS AND STOCHASTIC INTEGRALS

Marc A. Berger

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ABSTRACT

This report is intended to expand the applicability of positive kernel theory to probabilistic settings and stochastic integrals. The main result states that if $a(t)$ is a positive kernel, and $\{\beta(t) : t \geq 0\}$ a Brownian motion, then

$$(*) \quad \int_0^T \xi(t) \int_0^t a(t-\tau) \xi(\tau) d\beta(\tau) d\beta(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 dt \geq 0, \text{ a.s.}$$

for every stochastic process $\{\xi(t) : t \geq 0\}$ which has a stochastic differential $d\xi(t)$ with respect to $\beta(t)$, and for every $T \geq 0$. The implication of (*) concerning energy estimates for certain Ito-Volterra equations is discussed, and examples are provided.

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SIGNIFICANCE AND EXPLANATION

A concept useful in establishing the stability of a solution to certain linear and non-linear integral equations is that of a positive kernel. This is a function $a(t)$ such that

$$(*) \quad \int_0^T x(t) \int_0^t a(t-\tau)x(\tau)d\tau dt \geq 0$$

for every continuous function $x(t)$ and every $T \geq 0$. For example, if $\lambda \geq 0$ then $e^{-\lambda t}$ is a positive kernel. We show that this concept

generalizes to a probabilistic setting, where stochastic integrals are used. But the analogous estimate (*) no longer holds. There is a significant correction term that must be added to the left side of (*).

We provide this term, and consider its implication concerning energy estimates for certain stochastic integral equations. The one-dimensional linear homogeneous stochastic differential equation serves as an illustrative example.

POSITIVE KERNELS AND STOCHASTIC INTEGRALS

Marc A. Berger

INTRODUCTION

In the theory of Volterra equations of the form

$$(V) \quad x(t) + \int_0^t a(t-\tau)g(x(\tau))d\tau = f(t); \quad t \geq 0$$

a concept useful for studying stability is that of a positive kernel $a(t)$. This is one for which

$$\int_0^T x(t) \int_0^t a(t-\tau)x(\tau)d\tau dt \geq 0$$

for every $x(t) \in C[0, \infty)$, and for every $T \geq 0$. A discussion of such kernels, and their implications concerning the stability of (V), appear in a large number of places, including MacCamy and Wong [4], and Nohel and Shea [5].

Let $a(t)$ be a positive kernel. Then if $F(t) \in C^1[0, \infty)$

$$\int_0^T x(t) \int_0^t a(t-\tau)x(\tau)dF(\tau)dF(t) \geq 0$$

for every $x(t) \in C[0, \infty)$, and for every $T \geq 0$. To extend the setting a bit, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. If $\{G(t) : t \geq 0\}$ is an a.s. differentiable process, then

$$(*) \quad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)dG(\tau)dG(t) \geq 0, \quad \text{a.s.}$$

for every a.s. continuous stochastic process $\{\xi(t) : t \geq 0\}$, and for every $T \geq 0$.

Suppose now that there is a Brownian motion $\{\beta(t) : t \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the stochastic integral

$$I_a(T; \xi) = \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t); \quad T \geq 0$$

where $\{\xi(t) : t \geq 0\}$ is a nonanticipating and a.s. continuous stochastic process.

If it was possible to interpret the stochastic integral as a classical Stieltjes integral, then (*) would be valid with $G(t) = \beta(t)$. But of course, this is not so.

In fact the estimate (*) no longer holds. For if $a(t) \equiv 1$ then

$$I_a(T; \xi) = \frac{1}{2} \left| \int_0^T \xi(t) d\beta(t) \right|^2 - \frac{1}{2} \int_0^T |\xi(t)|^2 dt, \quad \text{a.s.; } T \geq 0.$$

What is true, however, is that whenever $a(t)$ is a positive kernel

$$I_a(T; \xi) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 dt \geq 0, \quad \text{a.s.}$$

for every $T \geq 0$. This is the content of the theorem in §1. The precise technical hypotheses on $a(t)$ and $\xi(t)$ are also provided in §1. In §2 we present an estimate concerning solutions of

$$\xi(t) + \int_0^t a(t-\tau) g(\xi(\tau)) d\beta(\tau) = f(t); \quad t \geq 0$$

when they exist.

§1. Basic Estimate

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a Brownian motion $\{\beta(t) : t \geq 0\}$.

Let S be the space of real-valued functions $a(t) \in C^1[0, \infty)$ such that $e^{-\lambda t} a(t) \in L^1(0, \infty)$ for all $\lambda > 0$. A function $a(t) \in S$ is said to be of positive type if

$$(1.1) \quad \int_0^T x(t) \int_0^t a(t-\tau)x(\tau)d\tau dt \geq 0$$

for every real-valued function $x(t) \in C[0, \infty)$, and for every $T \geq 0$. For a discussion and characterization of functions of positive type, and some illustrative examples, the reader is referred to MacCamy and Wong [4], and Nohel and Shea [5].

Theorem:

Let $a(t) \in S$ be of positive type. Then

$$(1.2) \quad \int_0^T \xi(t) \int_0^t a(t-\tau)\xi(\tau)d\beta(\tau)d\beta(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 dt \geq 0, \text{ a.s.}$$

for every real-valued stochastic process $\{\xi(t) : t \geq 0\}$ which has a stochastic differential $d\xi(t)$ with respect to $\beta(t)$, and for every $T \geq 0$.

The proof of this theorem relies on the following three results.

Lemma 1 (Correction Formula):

Let $\{\phi(\tau, t) : 0 \leq \tau \leq t \leq T\}$ be a real-valued t -nonanticipating stochastic process which has a stochastic differential $\partial_\tau \phi(\tau, t)$ with respect to $\beta(\tau)$, and satisfies

$$(1.3) \quad \int_0^T \int_0^t |\phi(\tau, t)|^2 d\tau dt < \infty, \text{ a.s.}; \quad \int_0^T |\phi(t, t)| dt < \infty, \text{ a.s.}$$

Then

$$(1.4) \quad \int_0^T \int_\tau^T \phi(\tau, t) d\beta(t) d\beta(\tau) = \int_0^T \int_0^t \phi(\tau, t) d\beta(\tau) d\beta(t) + \int_0^T \phi(t, t) dt, \text{ a.s.}$$

Lemma 2:

Let $\Pi = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$. If $a(t) \in S$ is of positive type, then $\hat{a}(z) \geq 0$ for $z \in \Pi$, where

$$(1.5) \quad \hat{a}(z) = \operatorname{Re} \int_0^\infty e^{-zt} a(t) dt.$$

Lemma 3:

If $a(t) \in L^1(-\infty, \infty)$ is an even function, then for a.e. $t \in (-\infty, \infty)$

$$(1.6) \quad a(t) = \frac{1}{\pi} \int_{-\infty}^\infty e^{i\tau t} \hat{a}(i\tau) d\tau.$$

Lemma 1 can be found in Berger [1], or Berger and Mizel [3]. The definitions of t -nonanticipating, $\partial_t \phi(\tau, t)$ and the two double integrals in (1.4) are also provided there. Lemma 2 can be found in Nohel and Shea [5]. Lemma 3 can be proved directly from the Fourier Inversion Formula.

Proof of Theorem:

Following Nohel and Shea [5] extend $a(t)$ evenly to $(-\infty, \infty)$; that is, $a(t) = a(-t)$, $t < 0$. And define $a_\lambda(t) \in L^1(-\infty, \infty)$, for $\lambda > 0$, by $a_\lambda(t) = e^{-\lambda|t|} a(t)$, $t \in (-\infty, \infty)$. Let

$$(1.7) \quad \xi_T(t) = \int_0^T e^{-i\tau t} \xi(\tau) d\beta(\tau); \quad t \in (-\infty, \infty).$$

By Lemma 1, for $T \geq 0$

$$(1.8) \quad \int_0^T \xi(t) \int_0^t a(t-\tau) \xi(\tau) d\beta(\tau) d\beta(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 dt = Q_a(T; \xi), \quad \text{a.s.}$$

where

$$(1.9) \quad Q_a(T; \xi) \equiv \frac{1}{2} \int_0^T \xi(t) \int_0^T a(t-\tau) \xi(\tau) d\beta(\tau) d\beta(t); \quad T \geq 0.$$

Using the fact that (see Berger and Mizel [3])

$$(1.10) \quad \lim_{\lambda \rightarrow 0} \sup_{0 \leq t \leq T} [|a_\lambda(t) - a(t)| + |a'_\lambda(t) - a'(t)|] = 0; \quad T \geq 0$$

and Lemma 3, it follows that

$$\begin{aligned}
 Q_a(T; \xi) &= \frac{1}{2} \lim_{\lambda \rightarrow 0} \int_0^T \xi(t) \int_0^T a_\lambda(t-\tau) \xi(\tau) d\beta(\tau) d\beta(t), \text{ a.s.} \\
 (1.11) \quad &= \frac{1}{2\pi} \lim_{\lambda \rightarrow 0} \int_0^T \xi(t) \int_0^T \left[\int_{-\infty}^{\infty} e^{is(t-\tau)} \hat{a}(\lambda + is) ds \right] \xi(\tau) d\beta(\tau) d\beta(t), \text{ a.s.} \\
 &= \frac{1}{2\pi} \lim_{\lambda \rightarrow 0} \int_{-\infty}^{\infty} |\tilde{\xi}_T(s)|^2 \hat{a}(\lambda + is) ds, \text{ a.s.}
 \end{aligned}$$

Thus, by Lemma 2, $Q_a(T; \xi) \geq 0$, a.s. for every $T \geq 0$. And the result is apparent now from (1.8). ■

We note that from the more general version of the Correction Formula which appears in Berger and Mizel [3], one obtains a generalization of (1.2) to continuous martingales $\{y(t) : t \geq 0\}$. Namely,

$$(1.12) \quad \int_0^T \xi(t) \int_0^t a(t-\tau) \xi(\tau) dy(\tau) dy(t) + \frac{1}{2} a(0) \int_0^T |\xi(t)|^2 d\langle y, y \rangle(t) \geq 0, \text{ a.s.}$$

52. Ito-Volterra Equations

We consider next the Ito-Volterra equation

$$(I-V) \quad \xi(t) + \int_0^t \sigma(t-\tau)g(\xi(\tau))d\beta(\tau) + \int_0^t b(t-\tau)\xi(\tau)d\tau = \varphi(t); \quad t \geq 0.$$

In Berger [1] existence and uniqueness of a solution to (I-V) is established under the hypotheses

$$(a_1) \quad \sup_{0 \leq t \leq T} |\sigma(t)| = \|\sigma\|_T < \infty, \quad \sup_{0 \leq t \leq T} |b(t)| = \|b\|_T < \infty; \quad T \geq 0$$

(a₂) $g(x)$ is Lipschitz continuous on $\mathbb{R}$, and there exists a constant $K > 0$ such that for all $x \in \mathbb{R}$

$$(2.1) \quad |g(x)|^2 \leq K(1 + |x|^2)$$

$$(a_3) \quad \sup_{0 \leq t \leq T} \mathbb{E} |\varphi(t)|^2 < \infty; \quad T \geq 0$$

Let $b^*(t)$ be the resolvent kernel for $b(t)$. That is, $b^*(t)$ satisfies

$$(2.2) \quad b^*(t) + \int_0^t b(t-\tau)b^*(\tau)d\tau = b(t); \quad t \geq 0.$$

Then (I-V) can be put in the form

$$(2.3) \quad \xi(t) + \int_0^t a(t-\tau)g(\xi(\tau))d\beta(\tau) = f(t); \quad t \geq 0$$

where

$$a(t) = \sigma(t) - \int_0^t b^*(t-\tau)\sigma(\tau)d\tau; \quad t \geq 0$$

$$(2.4) \quad f(t) = \varphi(t) - \int_0^t b^*(t-\tau)\varphi(\tau)d\tau; \quad t \geq 0.$$

For the linear case, $g(x) = x$, the solution $\xi(t)$ can be written in terms of a resolvent kernel, as is done in Berger [1], and Berger and Mizel [2]. The result is

$$(2.5) \quad \xi(t) = f(t) + \int_0^t k(\tau, t)f(\tau)d\alpha(\tau), \quad \text{a.s.}; \quad t \geq 0$$

where

$$\alpha(t) = \beta(t) + a(0)t; \quad t \geq 0$$

$$(2.6) \quad k(\tau, t) = \sum_{n=1}^{\infty} (-1)^n k_n(\tau, t); \quad 0 \leq \tau \leq t$$

$$k_1(\tau, t) = a(t-\tau), \quad k_{n+1}(\tau, t) = \int_{\tau}^t a(t-s) k_n(s, \tau) d\beta(\tau), \quad n = 1, 2, \dots; \quad 0 \leq \tau \leq t.$$

The definition of the integral on the right of (2.5) can be found in Berger [1]; or Berger and Mizel [2] or [3].

Corollary:

Let $a(t)$, given by (2.4), belong to S , and let $\xi(t)$ be a solution to (I-V). If $a(t)$ is of positive type, then

$$(2.7) \quad \int_0^T g(\xi(t)) [\xi(t) - f(t)] d\beta(t) \leq \frac{1}{2} a(0) \int_0^T |g(\xi(t))|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$, where $f(t)$ is given by (2.4). Similarly, if $-a(t)$ is of positive type, then

$$(2.8) \quad \int_0^T g(\xi(t)) [\xi(t) - f(t)] d\beta(t) \geq \frac{1}{2} a(0) \int_0^T |g(\xi(t))|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$.

Proof: Both (2.7) and (2.8) follow directly from (2.3) and (1.2). ■

As an example, consider the stochastic differential equation

$$d\xi(t) = -\mu\xi(t)d\beta(t) - \lambda\xi(t)dt; \quad t \geq 0$$

$$(2.9) \quad \xi(0) = c.$$

This equation can be written in the form (2.3) with

$$(2.10) \quad a(t) = \mu e^{-\lambda t}, \quad f(t) = c e^{-\lambda t}; \quad t \geq 0.$$

If $\mu \geq 0$ and $\lambda \geq 0$ then $a(t)$ is of positive type, and the corollary provides the estimate

$$(2.11) \quad \int_0^T \xi(t) [\xi(t) - f(t)] d\beta(t) \leq \frac{1}{2} \mu \int_0^T |\xi(t)|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$.

As another example, consider the pair of stochastic differential equations

$$\begin{aligned}d\xi(t) &= -\mu\xi(t)d\beta(t) - \lambda\tilde{\xi}(t)dt; \quad t \geq 0 \\d\tilde{\xi}(t) &= -\tilde{\mu}\tilde{\xi}(t)d\beta(t) - \tilde{\lambda}\tilde{\xi}(t)dt + \alpha\xi(t)dt; \quad t \geq 0 \\ \xi(0) &= c, \quad \tilde{\xi}(0) = \tilde{c}.\end{aligned}$$

These equations can be combined and written in the form (2.3) where $a(t)$ and $f(t)$ are both solutions of the differential equation

$$(2.13) \quad x''(t) + \tilde{\lambda}x'(t) + \alpha\lambda x(t) = 0; \quad t \geq 0$$

with initial conditions

$$(2.14) \quad a(0) = \mu, \quad a'(0) = -\lambda\tilde{\mu}; \quad f(0) = c, \quad f'(0) = -\lambda\tilde{c}.$$

If $\mu \geq 0$, $\tilde{\lambda} \geq 0$, $\tilde{\lambda}^2 \leq 4\alpha\lambda$, $\tilde{\lambda}\mu = 2\lambda\tilde{\mu}$ then $a(t)$ is of positive type, and the corollary provides the estimate (2.11) for every $T \geq 0$.

As a final example, consider the integro-differential equation (see Berger [1])

$$(2.15) \quad d\xi(t) = \left[\int_0^t \sigma(t-\tau)g(\xi(\tau))d\beta(\tau) \right] d\beta(t) + \left[\int_0^t b(t-\tau)g(\xi(\tau))d\tau \right] dt; \quad t \geq 0 \\ \xi(0) = c$$

If $\sigma(t)$ and $b(t)$ are both of positive type, then it follows from (1.1) and (1.2) that

$$(2.16) \quad G(\xi(T)) \geq G(c) - \sigma(0) \int_0^T |g(\xi(t))|^2 dt + \int_0^T g'(\xi(t)) \left| \int_0^t \sigma(t-\tau)\xi(\tau)d\beta(\tau) \right|^2 dt, \quad \text{a.s.}$$

for every $T \geq 0$, where $G(x) = 2 \int_0^x g(y)dy$, $x \in \mathbb{R}$.

If $g'(x) \geq 0$, $x \in \mathbb{R}$, then the last term on the right is positive, and can be dropped from the inequality. In fact, if $g(x) = x$ the Gronwall Inequality can be used to obtain the estimate

$$(2.17) \quad |\xi(T)| \geq |c| e^{-\frac{1}{2}\sigma(0)T}, \quad \text{a.s.}$$

for every $T \geq 0$. The reader can check that for the case $\sigma(t) \equiv \sigma \geq 0$, $b(t) \equiv 0$ the solution of (2.15) is a.s.

$$(2.18) \quad \xi(t) = ce^{-\frac{1}{2}\sigma t} \cosh \sqrt{\sigma}\beta(t), \quad t \geq 0.$$

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