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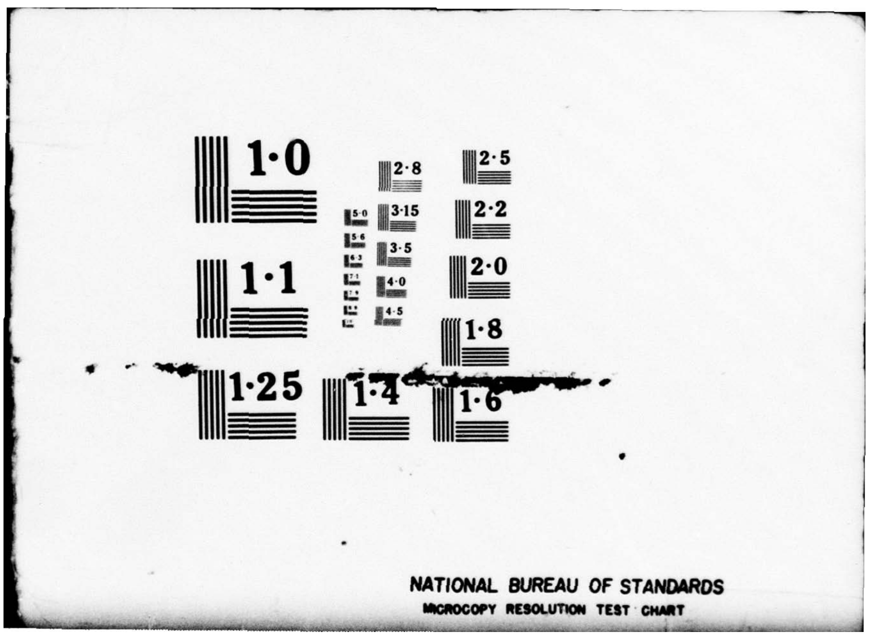
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by

Constantinos Spyros A. Cariniotakis

March, 1978

Thesis Advisor: George J. Thaler

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A method for design of compensation of linear feedback control systems using singular line theory is presented, and application of this design method for self-adaptive control systems is considered.

General design steps and required procedures for using this design method are summarized and examples are presented.

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SINGULAR LINE THEORY AND CONTROL SYSTEMS

by

Constantinos Spyros A. Cariniotakis
Lieutenant Commander, Greek Navy
B.S.E.E., United States Naval Postgraduate School, 1977
M.S.E.E., United States Naval Postgraduate School, 1977

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Author: ______________________

Approved by: ______________________
Thesis Advisor

____
Second Reader

____
Chairman, Department of Electrical Engineering

____
Acting Dean of Science and Engineering

ABSTRACT

The theory of singular lines on the Parameter Plane and in Parameter Space are derived, and applied to control system design.

A method for design of compensation of linear feedback control systems using singular line theory is presented, and application of this design method for self-adaptive control systems is considered.

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TABLE OF CONTENTS

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I. INTRODUCTION

Prior to 1959 analysis and design of control systems was primarily performed by classical methods. These methods are adequate when there is a single variable parameter and a single loop system.

In modern control engineering, where multiloop systems with characteristic equations with two or more variable parameters usually result, the above mentioned design methods become in general very complex. The designer has to complete several designs in a trial and error process and to choose the most suitable solution and to make usually some arbitrary choices based on his experience.

Methods for studying the effect of variations in parameters of a multivariable, multiloop control system have been developed by Mitrovic [1] , Siljak [2] , Thaler [3] , and other investigators. All these methods are based on the same concept and their dependence on a computer is characteristic.

Mitrovic's [1] method which was introduced in 1958, allowed study of the effect of variations of two variable parameters on the location of the characteristic roots of a system. This method consists of specifying as variables the two lowest order coefficients of the characteristic equation. Using then the characteristic equation as a mapping function, constant zeta, omega and sigma curves are transformed from the s-plane into the variable coefficient plane. This method permits the designer to adjust these variable coefficients so that the roots of the characteristic equation may be set at desired locations.

In 1964, Siljak [2] extended Mitrovic's [1] , coefficient plane method into the Parameter Plane method in which the two variable parameters may appear linearly in any or all coefficients of the characteristic equation. The parameter plane method provides the designer with a procedure for factoring a characteristic equation and displaying the results in a parameter plane diagram, and allows him to obtain information about system stability and how it is affected with parameter variations.

The parameter plane method was further extended in 1965 [4] to a general case in which the two variable parameters may appear nonlinearly in the coefficients of the characteristic equation. Another extension of the Parameter Plane method to include more than two variable parameters has been defined as the Parameter Space method [5] .

When in general only two parameters (or coefficients) are involved, for any complex value of s or a pair of (ζ, ω_n) values two simultaneous equations in two unknowns are obtained by setting the real and imaginary part of the characteristic equation equal to zero.

In 1967, Bowie [6] while working with the characteristic equation of an inertially stabilized vehicle on the parameter plane, found that a complete set of roots for the characteristic equation could not be determined in the real root area by using the existing parameter plane theory. He further investigated this problem and found that specific pairs of ζ, ω_n values, make the above mentioned system of equations singular, which thus corresponds to a single straight line in the parameter plane, called a "singular line".

The singular line added a new dimension to the parameter plane because it provides the designer a way to hold a root fixed while varying others, by varying the two parameter values. Another important characteristic of singular lines is also the fact that they can be drawn in a parameter plane diagram by hand, without needing a computer, as in the case of drawing constant ζ or ω_n curves.

Although an analytic study of the two previous mentioned simultaneous equations, which are obtained by setting the real and imaginary part of the characteristic equation equal to zero, makes obvious the existence of singular lines under certain conditions, they were not found earlier due to the standard computer programming practices and limitations.

Singular line theory is based on the special case of linearly dependent parameter plane solution equations.

From the standpoint of the design engineer it was desirable that this theory be investigated in order to establish a standard general set of rules which would provide a method to obtain systems with singular lines.

In this thesis this task is accomplished and possible applications of the singular line theory in control system design are investigated.

II. SINGULAR LINE THEORY

A. SINGULAR CONDITIONS

Consider the characteristic equation:

$$F(s) = \sum_{k=0}^{n} \alpha_k s^k = 0 \qquad (2\text{-}1)$$

where α_k $\{k=0, 1, 2, \ldots, n\}$, is a function of two variable parameters α and β, i.e., $\alpha_k = g_k(\alpha, \beta)$, and s is a point on the s plane which can be expressed in rectangular coordinates as:

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \qquad (2\text{-}2)$$

If for a given pair of ζ_s, ω_{ns} values {where $|\zeta_s| < 1$ and $\omega_{ns} > 0$}, equation (2-1) has an infinite number of α and β real valued pairs which satisfy it, then for this equation singular conditions exist. In this case the point of the s plane which is defined by the pair ζ_s, ω_{ns} is a singular point, and the characteristic equation is singular.

The locus of all points (α, β) on the Parameter Plane which satisfy equation (2-1) in the case of singular conditions is called the singular line. Each such line corresponds to a specific singular point or in other words to a specific pair of ζ_s, ω_{ns} values.

B. SINGULAR LINE ANALYSIS IN THE LINEAR CASE

A simple case, which was analyzed and which appears in many practical problems is the linear case. In this case the two variable parameters α and β appear in the coefficients of the characteristic polynomial in the general form:

$$\alpha_K = \beta_K \alpha + c_K \beta + d_K \tag{2-3}$$

where b_K, c_K and d_K are real constants.

1. Analysis by Using Parameter Plane Equations

Consider the characteristic equation (2-1), where α_K is defined by equation (2-3). By introducing equation (2-2), into equation (2-1), and equating real and imaginary parts to zero, it can be shown [6] that:

$$\begin{aligned} B_1 \alpha + C_1 \beta &= -D_1 \\ B_2 \alpha + C_2 \beta &= -D_2 \end{aligned} \tag{2-4}$$

where

$$\begin{aligned} B_1 &= \sum_{K=0}^{n} (-1)^K b_K \omega_n^K U_{K-1}(\zeta) & B_2 &= \sum_{K=0}^{n} (-1)^K b_K \omega_n^K d_K U_K(\zeta) \\ C_1 &= \sum_{K=0}^{n} (-1)^K c_K \omega_n^K U_{K-1}(\zeta) & C_2 &= \sum_{K=0}^{n} (-1)^K C_K \omega_n^K U_K(\zeta) \\ D_1 &= \sum_{K=0}^{n} (-1)^K d_K \omega_n^K U_{K-1}(\zeta) & D_2 &= \sum_{K=0}^{n} (-1)^K d_K \omega_n^K U_K(\zeta) \end{aligned} \tag{2-5}$$

The functions $U_k(\zeta)$ and $U_{k-1}(\zeta)$ are Chebyshev functions of the second kind for which $U_o(\zeta)=0$, $U_1(\zeta)=1$, and in general:

$$U_k(\zeta) = 2\zeta U_{k-1}(\zeta) - U_{k-2}(\zeta) \tag{2-6}$$

The system (2-4) can be written in the Matrix form as:

$$\begin{bmatrix} B_1 & C_1 \\ B_2 & C_2 \end{bmatrix} \begin{matrix} \alpha \\ \beta \end{matrix} = \begin{bmatrix} -D_1 \\ -D_2 \end{bmatrix}$$

or

$$A \underset{\sim}{x} = B \tag{2-7}$$

When the solution of the system (2-7) is considered there are the following cases:

a. No Solution

When Rank (A) = 1 and Rank (A, B) = 2, which implies that:

$$\frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2} \tag{2-8}$$

then the system has no solution.

b. Unique Solution

When Rank (A) = 2 and Rank (A, B) = 2, which implies that:

$$\frac{B_1}{B_2} \neq \frac{C_1}{C_2} \neq \frac{D_1}{D_2} \tag{2-9}$$

then the system has a unique solution.

c. An Infinite Number of Solutions

When Rank (A) = 1 and Rank (A, B) = 1, which implies that

$$\frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2} \qquad (2\text{-}10)$$

then the system has an infinite number of solutions. Therefore in this case, since there are an infinite number of pairs of α, β values which satisfy the system (2-4) and in consequence the characteristic equation (2-1), it implies that for this equation singular conditions exist.

2. Analysis by Using Complex Variable Theory

The characteristic equation (2-1) is considered again. This equation can be rearranged in the form:

$$f_1(s)\,\alpha + f_2(s)\,\beta + f_3(s) = 0 \qquad (2\text{-}11)$$

where $f_1(s)$, $f_2(s)$ and $f_3(s)$ are polynomials of s. Introducing equation (2-2) into equation (2-11), these polynomials will be in general a complex number with some magnitude and an argument. Thus they can be written as:

$$f_1(s) = r_1\, e^{j\theta_1}$$

$$f_2(s) = r_2\, e^{j\theta_2}$$

$$f_3(s) = r_3\, e^{j\theta_3}$$

Introducing the above equations into equation (2-11) yields:

$$r_1\, e^{j\theta_1}\alpha + r_2\, e^{j\theta_2}\beta + r_3\, e^{j\theta_3} = 0 \qquad (2\text{-}12)$$

Using the Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$, equation (2-12) becomes:

$$r_1\{\cos\theta_1+j\sin\theta_1\}\alpha+r_2\{\cos\theta_2+j\sin\theta_2\}\beta+r_3\{\cos\theta_3+j\sin\theta_3\} = 0$$

or

$$\{r_1\cos\theta_1\alpha+r_2\cos\theta_2\beta+r_3\cos\theta_3\}+j\{r_1\sin\theta_1\alpha+r_2\sin\theta_2\beta+r_3\sin\theta_3\} = 0$$

Equating real and imaginary parts to zero yields the system:

$$\begin{aligned} r_1\sin\theta_1\alpha+r_2\sin\theta_2\,\beta &= -r_3\sin\theta_3 \\ r_1\cos\theta_1\alpha+r_2\cos\theta_2\,\beta &= -r_3\cos\theta_3 \end{aligned} \qquad (2\text{-}13)$$

By applying the same reasoning as in paragraph one, the system (2-13) will have:

a. No Solution

If and only if:

$$\frac{r_1\sin\theta_1}{r_1\cos\theta_1} = \frac{r_2\sin\theta_2}{r_2\cos\theta_2} \neq \frac{r_3\sin\theta_3}{r_3\cos\theta_3}$$

or

$$\tan\theta_1 = \tan\theta_2 \neq \tan\theta_3$$

b. Unique Solution

If and only if:

$$\tan\theta_1 \neq \tan\theta_2 \neq \tan\theta_3$$

c. An Infinite Number of Solutions

If and only if:

$$\tan\theta_1 = \tan\theta_2 = \tan\theta_3 \qquad (2\text{-}14)$$

Equation (2-14) implies that for the characteristic equation (2-1) which has been rearranged in the form (2-11) singular conditions exist, if and only if:

$$\theta_1 = \theta_3 + k_1 \Pi$$

$$\theta_2 = \theta_3 + k_2 \Pi$$

or

$$\begin{aligned} \angle f_1(s) &= \angle f_3(s) + k_1 \Pi \\ \angle f_2(s) &= \angle f_3(s) + k_2 \Pi \end{aligned} \qquad (2\text{-}15)$$

where k_1 and k_2 are integers.

3. Singular Conditions for the Linear Case

Summarizing previously derived results, the conclusion is that, in the linear case where the characteristic equation is written in the general form:

$$F(s) = \sum_{k=0}^{n} a_k \, s^k = 0$$

where $a_k = \beta_k \alpha + c_k \beta + d_k$, singular conditions exist if and only if:

$$\frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$$

When the characteristic equation is considered in the general form:

$$f_1(s)\alpha + f_2(s)\beta + f_3(s) = 0$$

then singular conditions exist, if and only if:

$$\angle f_1(s) = \angle f_3(s) + k_1\Pi$$

$$\angle f_2(s) = \angle f_3(s) + k_2\Pi$$

4. Equation of a Singular Line

When for a given pair of ζ_s, ω_{ns} values singular conditions exist for the characteristic equation:

$$F(s) = \sum_{k=0}^{n} a_k \, s^k = 0$$

or

$$f_1(s)\alpha + f_2(s)\beta + f_3(s) = 0$$

then, according to the previous analysis the two parameters α and β will be related by one of the following equations, respectively:

$$\beta = -\frac{D_1}{C_1} - \frac{B_1}{C_1}\alpha \equiv -\frac{D_2}{C_2} - \frac{B_2}{C_2}\alpha \qquad (2\text{-}16)$$

or

$$\beta = -\frac{r_3\sin\theta_3}{r_2\sin\theta_2} - \frac{r_1\sin\theta_1}{r_2\sin\theta_2}\alpha \equiv -\frac{r_3\cos\theta_3}{r_2\cos\theta_2} - \frac{r_1\cos\theta_1}{r_2\cos\theta_2}\alpha \qquad (2\text{-}17)$$

Each of these equations, is the equation of the singular line corresponding to the singular pair (ζ_s, ω_{ns}) which was considered. This singular line is a straight line on the parameter plane which is tangent to the constant

ζ_s curve at the singular point [6]. It was found in this thesis, that this is also tangent to the constant ω_{ns} curve at the same point.

5. Methods to Test if a Point s is Singular

When the characteristic equation is given and it is desired to determine if a given point (ζ, ω_n), ($|\zeta| < 1$ and $\omega_n > 0$), is singular the following methods can be used.

a. Algebraic Methods

By using the ζ , ω_n values of the point under consideration, the coefficients of equations (2-5) can be evaluated. Then the given point will be singular if and only if equations (2-10) are satisfied.

A second algebraic method consists of the evaluation of the arguments appearing in equation (2-15), after writing the characteristic equation in the form of equation (2-11). Then the given point is singular if and only if equations (2-15) are satisfied.

b. Graphical Method

By this method the characteristic equation must be written in the form of equation (2-11) and the roots of the polynomials $f_1(s)$, $f_2(s)$ and $f_3(s)$ must be evaluated. After these roots have been located on the s-plane, with the use of a spirule the arguments of the previous mentioned polynomials for the point under consideration are determined. Then this point will be singular if and only if, equations (2-15) are satisfied.

c Use of a Computer

There are in general many ways by which a computer can be programmed to answer this problem. One such program is the <<Singular Point Program>> which was developed with this thesis and which appears in Appendix H. In order for this program to be used in the case under consideration, the corresponding value of ζ of the point which is considered is used as an input data. Then at the computer print output the corresponding value of ω_n is checked if it is such that it makes the system (2-4) have an infinite number of solutions. When this is the case, the point (ζ, ω_n) which was considered is a singular point, otherwise it is not singular.

6. Methods to Test if a Characteristic Equation is Singular

In the previous paragraph a characteristic equation was considered to be given and it was desired to determine if a point of the s-plane was a singular point for this equation. In this paragraph the problem is reversed, i.e., a characteristic equation is considered to be given and it is desired to determine if there is at least one point of the s-plane which is singular. For this problem the following methods can be applied:

a. Analytic Method

By this method the characteristic equation is written in the form of equation (2-11). Then, the arguments of $f_1(s)$, $f_2(s)$ and $f_3(s)$ are evaluated as functions of ζ and ω_n, i.e.,

$$\begin{aligned} \underline{/f_1(s)} &= g_1(\zeta,\omega_n) \\ \underline{/f_2(s)} &= g_2(\zeta,\omega_n) \\ \underline{/f_3(s)} &= g_3(\zeta,\omega_n) \end{aligned} \quad (2\text{-}18)$$

Introducing equations (2-18) into the system (2-15) yields:

$$\begin{aligned} g_1(\zeta,\omega_n) &= g_3(\zeta,\omega_n) + k_1\Pi \\ g_2(\zeta,\omega_n) &= g_3(\zeta,\omega_n) + k_2\Pi \end{aligned} \quad (2\text{-}19)$$

When the system (2-19) is solved for the unknowns ζ and ω_n, any acceptable solution, i.e., any solution where the values of ζ and ω_n are real and also $|\zeta| < 1$ and $\omega_n > 0$, will define a pair of ζ, ω_n values, which correspond to a singular point of the characteristic equation. If at least one such point is found then the characteristic equation is singular.

b. Graphical Method

By this method the characteristic equation which is considered must be rearranged again in the form of equation (2-11). Then the roots of the polynomials $f_1(s)$, $f_2(s)$ and $f_3(s)$ are evaluated and they are located on the corresponding points of the s-plane. By the use of a spirule a search on the s-plane must be performed for points which satisfy equations (2-15). This search can be performed either by moving on constant ζ lines or constant ω_n cycles. This method will in general be useful when the polynomials of s which are involved in equation (2-11) are of low order and it is desired to find out if the given characteristic equation is singular for a restricted area of the s-plane which is of interest.

c. Use of a Computer

The Singular Point Program which was mentioned before and which was developed for this purpose corresponds to one among the many ways by which a computer can be programmed to answer this problem.

III. SINGULAR LINE THEORY AND CONTROL SYSTEMS

A. DEFINITIONS

For a control system with two parameters α and β appearing in its characteristic equation the following definitions come as a consequence of the singular line theory which was previously developed.

1. Singular System

A control system is singular, if and only if, its characteristic equation is singular.

2. Singular Characteristic Equation

The characteristic equation of a control system is singular, if and only if, the system of the two simultaneous equations in two unknowns, α and β, which is obtained by setting $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ into the characteristic equation and requiring the summation of reals and the summation of imaginaries to go to zero is singular, for at least one pair of ζ, ω_n values, where $|\zeta| < 1$ and $\omega_n > 0$.

3. Singular Root

A root of the characteristic equation of a control system is singular if it makes the characteristic equation singular.

4. Singular Point

Each singular root corresponds to a specific point of the s plane. Such a point is called a singular point.

B. SINGULAR LINE THEORY IN CONTROL SYSTEM DESIGN

In general the analysis and design of any control system is based on an accepted model for the system under consideration. In the case of a linear control system of any order the no zero, two complex conjugate poles system model is used as a standard one. The accuracy of this approximation is tightly related to the dominant mode concept.

At this point the use of the singular line theory seems to be of utmost importance since, if a system is singular the complex pair which provides the singularity remains fixed and the locations of the other poles may be adjusted to some extent because an infinite number of α

and β pairs is available for their adjustment. In addition some other performance criteria may be satisfied by an appropriate parameter variation.

All these cases are analyzed and further investigated for other probable applications of the singular line theory in control system design.

C. SINGULAR LINE THEORY AND CONTROL SYSTEMS

1. Basic Requirements

From all that has been stated in section II, it seems that in order for the singular line theory to be applied on control systems design and compensation the following basic requirements must be satisfied, i.e., the system under consideration must be singular, a singular root of the system characteristic equation must meet problem specifications when it is considered the dominant root, this root must be made dominant by adjusting the parameter α and β values, and finally the two parameters α and β must be adjustable.

Therefore, the initial problem in general will be to select the structure of the system or the required compensator for a given system, in order for the above basic requirements to be met.

In the following paragraph a preliminary analysis of the form of the characteristic equation of a control system was performed, as the first step in developing the required structure of the control system to obtain the desired singular conditions.

2. The Characteristic Equation of a Control System in the Linear Case

In general the characteristic equation of a control system in the linear case will be in the form:

$$F(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{3-1}$$

where $a_k = b_k \alpha + c_k \beta + d_k$ and b_k, c_k and d_k are real constants. Equation (3-1) can also be rearranged in the following general form:

$$f_1(s)\alpha + f_2(s)\beta + f_3(s) = 0 \tag{3-2}$$

where $f_1(s)$, $f_2(s)$ and $f_3(s)$ are polynomials of s of any order. Assuming that $f_1(s)$ and $f_2(s)$ are not identical to zero, and that $f_3(s) = 0$ or $f_3(s) \neq 0$, yields the following two cases:

a. $f_3(s) \neq 0$

By assuming that $f_3(s) \neq 0$, dividing equation (3-2) by $f_3(s)$ yields:

$$\frac{f_1(s)}{f_3(s)} \alpha + \frac{f_2(s)}{f_3(s)} \beta + 1 = 0 \tag{3-3}$$

or

$$F_1(s)\ \alpha + F_2(s)\ \beta + 1 = 0 \tag{3-4}$$

where $F_1(s)$ and $F_2(s)$ are either a ratio of two polynomials of s or a polynomial of s.

When the characteristic equation of a control system is considered to be in the form of equation (3-4), then the necessary and sufficient conditions for this equation to be singular, according to equations (2-15), are:

$$\begin{aligned} \angle F_1(s) &= k_1\pi \\ \angle F_2(s) &= k_2\pi \end{aligned} \tag{3-5}$$

where k_1 and k_2 are integers.

It must be noticed that according to the process which was followed in the derivation of equations (2-15), they are valid if either $f_1(s)$, $f_2(s)$ and $f_3(s)$ are considered polynomials of s or ratios of two polynomials of s. Based on this remark and the fact that the argument of unity is zero or an even multiple of π, equations (3-5) were derived from (2-15).

Considering now the characteristic equation of a control system in the form of equation (3-4) the following cases may appear:

(1) <u>Case I: $F_1(s) \neq F_2(s)$</u>. In this case a point s of the s-plane is singular for the charactereristic equation (3-4), if and only if, it does not lie on the real axis and satisfies equations (3-5).

In order to determine if a given point in such a case is singular or in order to find the singular points of the characteristic equation (3-4), if any, one of the methods which were described in the previous section can be used. In addition to these methods the following method which is related to the Root locus concept can be applied.

The characteristic equation of the control system is:

$$F_1(s)\ \alpha + F_2(s)\ \beta + 1 = 0 \qquad (3\text{-}6)$$

Suppose that the Root loci of the following equations are drawn on the same drawing:

$A\ F_1(s) = -1$

$B\ F_2(s) = -1$

$C\ F_1(s) = -1$

$D\ F_2(s) = -1$

where A and B are Root locus variable parameters which take values from zero to plus infinity and C and D are Root locus variable parameters which take values from zero to minus infinity.

Based on the angle relation on which each Root locus is drawn, all the points of each of the above

Root loci will satisfy the following equations respectively:

$$\angle F_1(s) = (2n \pm 1)\pi$$

$$\angle F_2(s) = (2n \pm 1)\pi$$

$$\angle F_1(s) = 2n\pi$$

$$\angle F_2(s) = 2n\pi$$

where n is an integer. Therefore, the singular points of the characteristic equation (3-6), i.e., the points of the s-plane which satisfy equations (3-5) simultaneously, will be the points where either the first or the third Root locus intersect with the second or the fourth Root locus.

From the above described method it can be concluded that in the case under consideration either there are no singular points or there is a finite number of such points. Therefore, when in general $F_1(s) \neq F_2(s)$ the characteristic equation of the control system will not always be singular and if it is singular the probability of finding a singular point which meets problem specifications as the operating point of the system will be very low.

(2) <u>Case II: $F_1(s) = F_2(s)$.</u> In this case the characteristic equation (3-4) of the control system becomes:

$$F_1(s)\ (\alpha+\beta) + 1 = 0 \qquad (3\text{-}7)$$

Thus, the singular conditions given by (3-5) become:

$$\angle F_1(s) = k_1\Pi \qquad (3\text{-}8)$$

As it will be shown in the following case, when there is only one angle relation to be satisfied, like that which is given by equation (3-8), then there will be in general an infinite number of singular points. Considering equations (2-5) and (2-16), the equation of the singular line of any singular point will be in the general form:

$$\alpha+\beta = C$$

where C is a real constant. This implies that although in a singular case there will be an infinite number of pairs of α and β values which satisfy equation (3-7), the roots of this equation will be fixed for any of these pairs. For this reason this case can be considered as a trivial one under singular considerations and not useful for applications.

(3) Case III: $F_2(s) = F_1^n(s)$. When $F_2(s) = F_1^n(s)$, where n is an integer, then the characteristic equation (3-4) of the control system becomes:

$$F_1(s)\,\alpha+F_1^n(s)\,\beta+1 = 0 \qquad (3\text{-}9)$$

and the singular conditions given by (3-5) become:

$$\angle F_1(s) = k_1\Pi \qquad (3\text{-}10)$$

i.e., in this case a point s of the s-plane is singular, if and only if it does not lie on the real axis and satisfies equation (3-10).

In order to determine if a point of the s-plane is singular for the characteristic equation (3-9) or in order to find the singular points of this equation one of the methods which were described in the previous section can be used. In addition to these, the Root locus method which was introduced in the first case can be used as follows:

Suppose that the Root loci of the following equations are drawn:

$$A\, F_1(s) = -1$$

$$B\, F_1(s) = -1$$

where A and B are Root locus variable parameters which take values from zero to plus infinity and from zero to minus infinity, respectively.

Based on the angle relation on which each Root locus is drawn, all the points of each Root locus will satisfy the following angle relations respectively:

$$\angle F_1(s) = (2k \pm 1)\pi$$

$$\angle F_1(s) = 2k\pi$$

where k is an integer. Therefore, all the points of these two Root loci will also satisfy equation (3-10). After these remarks it can be concluded that in the case under consideration there will be, in general, an infinite number of singular points. Also the probability of finding a singular point which meets problem specifications as the operating point of the system will be relatively high.

In the simplest case where n = 2, i.e., $F_2(s) = F_1^2(s)$, the characteristic equation (3-9) becomes:

$$F_1(s)\,\alpha + F_1^2(s)\,\beta + 1 = 0 \tag{3-11}$$

and the necessary and sufficient conditions for this equation to be singular will be given again by equation:

$$\angle F_1(s) = k\Pi \tag{3-12}$$

where k is an integer.

(4) <u>Case IV: $F_1(s) = F_2^n(s)$.</u> In this case interchanging the two parameters α and β in the characteristic equation (3-4) yields the previous case.

All the rest of the cases, which may appear can be considered to belong in Case I, and can be analyzed in a similar way.

b. $f_3(s) = 0$

By assuming that $f_3(s) = 0$, equation (3-2) becomes:

$$f_1(s)\,\alpha + f_2(s)\,\beta = 0 \tag{3-13}$$

Since $f_1(s)$ and $f_2(s)$ are not equal to zero, dividing by $f_2(s)\beta$ yields:

$$\frac{f_1(s)}{f_2(s)} \frac{\alpha}{\beta} + 1 = 0 \tag{3-14}$$

or

$$F_1(s) \frac{\alpha}{\beta} + 1 = 0 \tag{3-15}$$

The singular conditions for this case as it can be derived from equations (3-5) are given by the equation:

$$\angle F_1(s) = k\pi \tag{3-16}$$

where k is a real integer or zero. It can be shown by the same procedure as was done in Case III in paragraph 2a(3), that in this case there is in general an infinite number of singular points. Considering equation (2-5) and (2-16) the equation of the singular line of any singular point will be in the form:

$$\frac{\alpha}{\beta} = C \tag{3-17}$$

where C is a real constant. This implies that although in a singular case there will be an infinite number of pairs of α and β values which satisfy equation (3-15), the roots of this equation will be fixed for any of these pairs. For this reason this case also can be considered as a trivial one under singular considerations and not useful for applications.

3. Summary and Conclusions

In the preliminary analysis of the characteristic equation of a control system, which was performed in the previous paragraph, it was found that there is a form of the characteristic equation, which in general has an infinite number of singular points. This is the following:

$$F_1(s)\ \alpha + F_1^{\,n}(s)\ \beta + 1 = 0 \qquad (3\text{-}18)$$

or in the simplest case where n = 2, the above characteristic equation becomes:

$$F_1(s)\ \alpha + F_1^{\,2}(s)\ \beta + 1 = 0 \qquad (3\text{-}19)$$

where $F_1(s) \neq 0$, and it is either a polynomial of s or a ratio of two polynomials of s.

Therefore equation (3-18) corresponds to the most favorable case for the application of singular line theory in control system design and compensation.

The next step is to find the structure of the system (or the required compensator for a given system), which has a characteristic equation corresponding to the form of equation (3-18) or in the simplest case to equation (3-19). This is done in the following section.

IV. ONE INPUT ONE OUTPUT LINEAR FEEDBACK SYSTEMS

A. PRELIMINARY ANALYSIS

In figure (4-1) the block diagram of a unity feedback control system is shown. Suppose that it is desired to compensate the system by making use of singular line theory. Then according to the results of the previous section, the problem is to find the compensation structure or structures which provide a system with a characteristic equation in the general form:

$$1 + \alpha F(s) + \beta F^2(s) = 0 \qquad (4\text{-}1)$$

i.e., a singular system.

Considering a compensation structure then the adjustable parameters are in general all the compensator parameters, i.e., the poles, zeros and gain values, and the gain of the plant. Since the poles and zeros of a compensator can be either real or complex, a preliminary analysis is needed to choose the two parameters α and β from among the gain values.

In the following sections which include a detailed analysis of the singular systems it is shown how one of these parameters, i.e., the parameter α, was introduced in the transfer function of a singular cascade compensator in such a way that its value affects its poles. (See Section VI C and equation (6-18)).

In the preliminary analysis of the problem of finding the compensation structure or structures which provide a system with characteristic equation in the form of equation (4-1), i.e., a singular system, the five most commonly used compensation structures were analyzed as follows:

1. First Compensation Structure

The first compensation structure which was considered is that of figure (4-2).

The characteristic equation of this system is:

$$1 + G(s) + G(s)\,H(s) = 0 \qquad (4\text{-}2)$$

Comparison of equations (4-1) and (4-2) reveals that in order to obtain a singular system the following relations must be satisfied.

$$G(s) = \alpha\, Fg(s)$$
$$H(s) = \frac{\beta}{\alpha}\, Fg(s) \qquad (4\text{-}3)$$

i.e., the feedback compensator must have poles and zeros identical with those of the plant, and the two parameters α and β must be related with the gain values as is shown in equations (4-3). Introducing equations (4-3) into equation (4-2) yields:

$$1 + \alpha\, Fg(s) + \beta\, Fg^2(s) = 0 \qquad (4\text{-}4)$$

which is the characteristic equation of a singular system.

2. Second Compensation Structure

The second compensation structure which was considered is that of figure (4-3).

The characteristic equation of this system is:

$$1+G(s)\,G_c(s) = 0 \qquad (4\text{-}5)$$

Comparison of equations (4-1) and (4-5) reveals that in this case there is no way by which a singular system can be obtained by defining the two parameters α and β between the plant and cascade compensator gain values.

In this case it was also noticed that when G(s) and $G_c(s)$ are defined as:

$$G(s) = kF_g(s) = k\,\frac{(s+z_1)\,(s+z_2)\,\ldots\,(s+z_m)}{s^N(s+p_1)\,(s+p_2)\,\ldots\,(s+p_n)}$$

and

$$G_c(s) = \alpha+\beta\,kF_g(s) = \frac{\alpha\,s^N(s+p_1)\ldots(s+p_n)+\beta k(s+z_1)\ldots(s+z_m)}{s^N(s+p_1)\,(s+p_2)\ldots(s+p_n)}$$

the characteristic equation of the system becomes:

$$1+\alpha kF_g(s)+\beta k^2F_g^{\,2}(s) = 0$$

Since the argument of the second and third term in the above equation are not affected by the appearance of the k and k^2 factors in these terms respectively, the above equation is also singular.

Based on the fact that the cascade compensator which was thus introduced in order for a singular system to be obtained has the same number of poles at the origin as the plant, this singular case (like the singular case, which was analyzed in Section V) has the basic disadvantage that the type number of the plant is not preserved, except in the case where the plant is type zero.

3. Third Compensation Structure

The third compensation structure which was considered is shown in figure (4-4).

Suppose that in general:

$$G(s) = kF_g(s)$$

$$G_c(s) = k_c F_c(s)$$

$$H(s) = k_H F_H(s)$$

The characteristic equation of this system will be:

$$1+G(s)\, G_c(s) + G(s)\, G_c(s)\, H(s) = 0 \qquad (4\text{-}6)$$

Comparison of equations (4-6) and (4-1) reveals that in order to obtain a singular system the following relations must be satisfied:

$$G(s)\, G_c(s) = kk_c\, F_g(s)\, F_c(s) = \alpha F_g(s)\, F_c(s) \qquad (4\text{-}7)$$

and

$$H(s) = k_H F_H(s) = \frac{\beta}{\alpha} F_g(s) F_c(s) \qquad (4\text{-}8)$$

Introducing equations (4-7) and (4-8) into equation (4-6) yields:

$$1+\alpha F_g(s) F_c(s) + \beta F_g^2(s) F_c^2(s) = 0 \qquad (4\text{-}9)$$

which is the characteristic equation of a singular system.

This case can be considered (under singular considerations) identical with the first one of figure (4-2), when $G(s)\ G_c(s)$ is assumed to be one transfer function.

4. Fourth Compensation Structure

The fourth compensation structure which was considered is that of figure (4-5).

The characteristic equation of this system will be in general:

$$1+G(s) H(s) + G(s) G_c(s) = 0 \qquad (4\text{-}10)$$

Comparison of equations (4-1) and (4-10) reveals that for a given plant i.e., given $G(s) = kF_g(s)$, there is an infinite number of ways by which $G_c(s)$ and $H(s)$ can be defined in order for a singular system to be obtained. Based on the fact that certain types of feedback compensation are in common use, e.g., velocity and acceleration feedback, only the following cases were selected to be analyzed as those which have the highest probability of application in practice.

a. $H(s) = \alpha$

When H(s) is a gain adjustment which is defined to be the parameter α, then in order for a singular system to be obtained the following should be the transfer functions of figure (4-5).

$$G(s) = kF_g(s)$$

$$G_c(s) = \beta F_g(s) \qquad (4\text{-}11)$$

$$H(s) = \alpha$$

Introducing equations (4-11) into (4-10) yields:

$$1+\alpha kF_g(s) + \beta kF_g{}^2(s) = 0 \qquad (4\text{-}12)$$

Because the factor k does not affect any argument relation, equation (4-12) will correspond to a singular system.

b. $H(s) = \alpha s$

When H(s) is defined to be a tachometer feedback then in order for a singular system to be obtained the following should be the transfer functions of the blocks in figure (4-5).

$$G(s) = kF_g(s)$$

$$G_c(s) = \beta s^2 F_g(s) \qquad (4\text{-}13)$$

$$H(s) = \alpha s$$

Introducing equations (4-13) into equation (4-10) yields:

$$1+\alpha ks\ F_g(s) + \beta ks^2\ F_g{}^2(s) = 0 \tag{4-14}$$

which is the characteristic equation of a singular system.

It should be noted that a basic steady state performance requirement of the system shown in figure (4-5) is that the cascade compensator $G_c(s)$ must transmit signal at zero frequency, i.e., its transfer function must not have any zero at the origin. But in this case it can happen, if and only if, $F_g(s)$ has at least two poles at the origin, which implies that the plant must be at least type two.

c. $H(s) = \alpha s^2$

When H(s) is defined to be an acceleration feedback, then in order for a singular system to be obtained the following should be the transfer functions of the blocks shown in figure (4-5).

$$G(s) = kF_g(s)$$

$$G_c(s) = \beta s^4\ F_g(s) \tag{4-15}$$

$$H(s) = \alpha s^2$$

Introducing equation (4-15) into equation (4-10) yields:

$$1+\alpha ks^2\ F_g(s) + \beta ks^4\ F_g{}^2(s) = 0 \tag{4-16}$$

which is the characteristic equation of a singular system.

It should be also noted in this case that in order for the cascade compensator $G_c(s)$ to transmit signal at zero frequency its transfer function must not have any zero at the origin, i.e., $F_g(s)$ must have at least four poles at the origin, which implies that the plant must be at least type four. Because such a plant usually cannot be found in practice this case will not be further considered.

5. Fifth Compensation Structure

The last compensation structure which was considered is that of figure (4-6).

By considering the product G(s) $G_{c2}(s)$ as one transfer function then under singular considerations this case turns to be identical with the previous one.

B. SUMMARY

In this section a preliminary analysis was performed on the most commonly used compensation structures in order to find which of these can provide a singular system, and under what conditions, i.e., a system with characteristic equation in the form of equation (4-1).

The results of this analysis have indicated that from a theoretical point of view there are many ways by which a singular system can be obtained by properly selecting the compensation structure and by properly defining the compensators and the two parameters α and β.

The conclusion also from this analysis was that three singular cases must be considered for a further analysis, because they may be useful and applicable in control system design and compensation. These are the following: The first singular case corresponds to the compensation structure of figure (4-2), where the feedback compensator and the two parameters α and β are defined by equation (4-3). The second singular case corresponds to the compensation structure of figure (4-5), where the compensators and the two parameters α and β are defined by equation (4-11). The third singular case corresponds again to the compensation structure of figure (4-5), where the compensators and the two parameters α and β are defined by equation (4-13), where the plant is considered to be at least type two.

In the following three sections which refer to these three singular cases respectively, a detailed analysis is performed.

V. FIRST SINGULAR CASE ANALYSIS

In the previous section the << first singular case >> was defined as that which corresponds to the compensation structure of figure (4-2), where the associated transfer functions and the two parameters α and β are defined as follows:

$$G(s) = \alpha F_g(s) = \alpha \frac{(s+z_1)\ (s+z_2)\ \ldots\ (s+z_m)}{s^N (s+p_1)\ (s+p_2)\ \ldots\ (s+p_n)} \tag{5-1}$$

and

$$H(s) = \frac{\beta}{\alpha}\ Fg(s) = \frac{\beta}{\alpha}\ \frac{(s+z_1)\ (s+z_2)\ \ldots\ (s+z_m)}{s^N (s+p_1)(s+p_2)\ \ldots\ (s+p_n)} \tag{5-2}$$

A. SYSTEM OUTPUT AND ERROR ANALYSIS

The system's transfer function is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{\alpha\ F_g(s)}{1\ +\alpha F_g(s)\ +\beta F_g^2(s)} \tag{5-3}$$

or

$$T(s) = s^N \frac{\alpha \prod_{j=1}^{m} (s+z_j) \prod_{k=1}^{n} (s+p_k)}{s^{2N} \prod_{k=1}^{n} (s+p_k)^2 + \alpha s^N \prod_{k=1}^{n} (s+p_k) \prod_{j=1}^{m} (s+z_j) + \beta \prod_{j=1}^{m} (s+z_j)^2}$$

(5-4)

The output of the system is:

$$C(s) = R(s)\ T(s) \tag{5-5}$$

and the error of the system is:

$$E(s) = R(s)\ \{1 - T(s)\} \tag{5-6}$$

When the final value theorem was applied to equations (5-5) and (5-6) the steady state output and error were evaluated, assuming that all the poles of the transfer function T(s) were on the left half of the s-plane.

The input signals considered are those, which are the most commonly used, i.e., step, ramp and parabolic input.

1. Plant Type Zero

In this case where the plant is type zero, i.e., $N = 0$, the factor s^N which appears in equation (5-4) becomes unity.

a. Step Input

When a step input $R(t) = Au(t)$ or $R(s) = A/s$ was considered then:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \lim_{s\to 0} s\frac{A}{s}T(s) = \frac{A\alpha \prod_{j=1}^{m} z_j \prod_{k=1}^{n} p_k}{\prod_{k=1}^{n} p_k^{\,2} + \alpha \prod_{j=1}^{m} z_j \prod_{k=1}^{n} p_k + \beta \prod_{j=1}^{m} z_j^{\,2}}$$

and

$$e(t) = e_{ss}(t) = \lim_{s \to o} s \frac{A}{s} \{1 - T(s)\} = \frac{A \prod_{k=1}^{n} p_k^{\,2} + \beta \prod_{j=1}^{m} z_j^{\,2}}{\prod_{k=1}^{n} p_k^{\,2} + \alpha \prod_{j=1}^{m} 2_j \prod_{k=1}^{n} p_k + \beta \prod_{j=1}^{m} z_j^{\,2}}$$

with $e(t)$ taken as $t \to \propto$.

b. Ramp Input

When a ramp input $R(t) = Btu(t)$ or $R(s) = B / s^2$ was considered then:

$$C(t)_{t \to \propto} = C_{ss}(t) = \lim_{s \to o} s \frac{B}{s^2} T(s) = \propto$$

and

$$e(t)_{t \to \propto} = e_{ss}(t) = \lim_{s \to o} s \frac{B}{s^2} \{1 - T(s)\} = \propto$$

c. Parabolic Input

When a parabolic input $R(t) = ct^2u(t)$ or $R(s) = 2c/s^3$ was considered then:

$$C(t)_{t \to \propto} = C_{ss}(t) = \lim_{s \to o} s \frac{2c}{s^3} T(s) = \propto$$

and

$$e(t)_{t \to \propto} = e_{ss}(t) = \lim_{s \to o} s \frac{2c}{s^3} \{1 - T(s)\} = \propto$$

2. <u>Plant Type One</u>

In this case where the plant was considered to be type one, i.e., N = 1, the steady error and output of the system were found as follows:

a. Step Input

$$\lim_{t\to\infty} C(t) = C_{ss}(t) = \lim_{s\to 0} s\,\frac{A}{s}\,T(s) = 0$$

and

$$\lim_{t\to\infty} e(t) = e_{ss}(t) = \lim_{s\to 0} s\,\frac{A}{s}\,\{1-T(s)\} = A$$

b. Ramp Input

$$\lim_{t\to\infty} C(t) = C_{ss}(t) = \lim_{s\to 0} s\,\frac{B}{s^2}\,T(s) = \frac{B\alpha \prod_{j=1}^{m} Z_j \prod_{k=1}^{n} p_k}{\beta \prod_{j=1}^{m} Z_j^{\,2}}$$

and

$$\lim_{t\to\infty} e(t) = e_{ss}(t) = \lim_{s\to 0} s\,\frac{B}{s^2}\,\{1-T(s)\} = \infty$$

c. Parabolic Input

$$\lim_{t\to\infty} C(t) = C_{ss}(t) = \lim_{s\to 0} s\,\frac{2c}{s^3}\,T(s) = \infty$$

and

$$\lim_{t\to\infty} e(t) = e_{ss}(t) = \lim_{s\to 0} s\,\frac{2c}{s^3}\,\{1-T(s)\} = \infty$$

3. Plant Type Two

When a type two plant was considered, i.e., N = 2, the steady state error and output of the system were found as follows:

a. Step Input

$$\underset{t \to \infty}{C(t)} = C_{ss}(t) = \lim_{s \to o} s \frac{A}{s} T(s) = 0$$

and

$$\underset{t \to \infty}{e(t)} = e_{ss}(t) = \lim_{s \to o} s \frac{A}{s} \{1-T(s)\} = A$$

b. Ramp Input

$$\underset{t \to \infty}{C(t)} = C_{ss}(t) = \lim_{s \to o} s \frac{B}{s^2} T(s) = 0$$

and

$$\underset{t \to \infty}{e(t)} = e_{ss}(t) = \lim s \frac{B}{s^2} \{1-T(s)\} = \infty$$

c. Parabolic Input

$$\underset{t \to \infty}{C(t)} = C_{ss}(t) = \lim_{s \to o} s \frac{2c}{s^3} T(s) = \frac{2c\alpha \prod_{j=1}^{m} Z_j \prod_{k=1}^{n} p_k}{\beta \prod_{i=1}^{m} Z_j^{\,2}}$$

and

$$\underset{t \to \infty}{e(t)} = e_{ss}(t) = \lim_{s \to o} s \frac{2c}{s^3} \{1-T(s)\} = \infty$$

4. Plant Type Three or Over Three

In all such cases $C_{ss}(t) = 0$, and the error becomes finite or infinity depending on the input signal which is considered.

5. Remarks

From the above results which are summarized in Table I of Appendix E, it can be seen that in general the type of the plant changes due to the form of the feedback path which is introduced in order for a singular system to be obtained. Therefore, compensation of a given system by this method yields a system which in general, does not respond to a given input as might be expected.

Although some of the above cases which were analyzed may have a specific application in control system design and compensation, only the case where both output and error have finite values at steady state seems to be of a more general interest.

The example which follows is referred to that case.

B. EXAMPLE (5-1)

Assume that in figure (4-2) the transfer functions of the plant and the feedback compensator are as follows:

$$G(s) = \alpha F_g(s) = \alpha \frac{s+3}{(s+1)(s+2)(s+4)} \qquad (5\text{-}7)$$

and

$$H(s) = \frac{\beta}{\alpha} F_g(s) = \frac{\beta}{\alpha} \frac{s+3}{(s+1)(s+2)(s+4)} \qquad (5\text{-}8)$$

then the characteristic equation of the compensated system is:

$$1+ \alpha F_g(s) + \beta F_g{}^2(s) = 0 \qquad (5\text{-}9)$$

or after substituting equations (5-7) and (5-8) into (5-9) yields:

$$s^6 +145^5 +(77+\alpha)s^4 +(212 + 10\alpha)s^3 +(308 +35\alpha+\beta)\ s^2 +$$
$$+ (224 + 50\alpha + 6\beta)\ s + (64 + 24\alpha+9\beta) = 0 \qquad (5\text{-}10)$$

By using the method of paragraph 6c, section II, the singular points of the system were found for several values of ζ. They are listed in Appendix A. One of these singular points, i.e., the singular point (ζ_s = 0.60, ω_{ns} = 3.1104) was further considered. Then from equation (2-6) the Chebyshev functions of the second kind were evaluated and from equation (2-5) the coefficients B_1, C_1 and D_1 were found to be:

B_1 = -5.3085

C_1 = 0.6746

D_1 = 41.7741

Then from equation (2-16) the equation of the corresponding singular line was found to be:

$$\beta = - 61.9254 + 7.8693\alpha \qquad (5\text{-}11)$$

With a computer program then, the roots of the characteristic equation (5-10) were found for several different parameter pair values, which satisfy equation (5-11). They are listed in Appendix B. By examining these roots it can be seen that the singular root and a real root are fixed and the other three roots are varying for different pair values.

In figure (5-1) the parameter plane diagram of this singular system is shown for some arbitrarily selected values of ζ and ω_n.

In figure (5-2) the parameter plane diagram of the same singular system is shown for one constant ζ and one constant ω_n curve, which correspond to the singular values of the previously considered singular point, i.e., ζ_s = 0.60 and ω_{ns} = 3.1104. On the same diagram the singular line of this singular point is also shown. It was noticed that these two curves and the singular line are tangent to each other on the same point of the parameter plane.

For a further analysis of this system and of any other system which is referred to the << first singular case >>, the same procedure which is described in the next section and which is referred to the "second singular case", can be followed.

VI. SECOND SINGULAR CASE ANALYSIS

In Section IV, the "second singular case" was defined as that which corresponds to the compensation structure of figure (6-1) where the associated transfer functions and the two parameters α and β are defined as follows:

$$G(s) = kF_g(s) = k\,\frac{(s+Z_1)(s+Z_2)\ \dots\ (s+Z_m)}{s^N(s+p)(s+p_2)\ \dots\ (s+p_n)} \qquad (6\text{-}1)$$

$$G_c(s) = \beta F_g(s) = \beta\,\frac{(s+Z_1)(s+Z_2)\ \dots\ (s+Z_m)}{s^N(s+p_1)(s+p_2)\ \dots\ (s+p_n)} \qquad (6\text{-}2)$$

$$H(s) = \alpha \qquad (6\text{-}3)$$

A. SYSTEM OUTPUT AND ERROR ANALYSIS

The transfer function of the system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)\ G_c(s)}{1 + G(s)\ H(s) + G(s)G_c(s)} \qquad (6\text{-}4)$$

Introducing equations (6-1) through (6-3) into equation (6-4) yields:

$$T(s) = \frac{\beta k \prod_{j=1}^{m} (s+Z_j)^2}{s^{2N} \prod_{k=1}^{n} (s+p_k)^2 + aks^N \prod_{k=1}^{n} (s+p_k) \prod_{j=1}^{m} (s+Z_j) + \beta k \prod_{j=1}^{m} (s+Z_j)^2} \qquad (6\text{-}5)$$

The output of the system is:

$$C(s) = R(s)\,T(s) \tag{6-6}$$

and the error of the system is:

$$E(s) = R(s) - C(s) = R(s)\,\{1-T(s)\} \tag{6-7}$$

When the final value theorem was applied to equations (6-6) and (6-7) the steady state output and error values were evaluated, assuming that all the poles of the transfer function (6-5) lie on the left half plane. This analysis was based again on the three usually considered input signals, i.e., step, ramp and parabolic input.

1. <u>Plant Type Zero</u>

In this case where the plant was type zero, i.e., $N = 0$, the steady state error and output of the system were as follows:

a. Step Input

When a step input $R(t) = Au(t)$ or $R(s) = A/s$ was considered then:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \lim_{s\to 0} s\frac{A}{s}\,T(s) = \frac{A\beta k \prod_{j=1}^{m} z_j^{\,2}}{\prod_{k=1}^{n} p_k^{\,2} + \alpha k \prod_{k=1}^{n} p_k \prod_{j=1}^{m} z_j + \beta k \prod_{j=1}^{m} z_j^{\,2}}$$

and

$$e(t) = e_{ss}(t) = \lim_{s\to o} s\frac{A}{s}\{1-T(s)\} = \frac{A\left\{\prod_{k=1}^{n} p_k^{\,2} + \alpha k \prod_{k=1}^{n} p_k \prod_{j=1}^{m} z_j\right\}}{\prod_{k=1}^{n} p_k^{\,2} + \alpha k \prod_{k=1}^{n} p_k \prod_{j=1}^{m} z_j + \beta k \prod_{j=1}^{m} z_j^{\,2}}$$

(with $t\to\infty$ under $e(t)$)

b. Ramp Input

When a ramp input $R(t) = Btu(t)$ or $R(s) = B/s^2$ was considered then:

$$C(t)_{t\to\infty} = C_{ss}(t) = \lim_{s\to o} s\frac{B}{s^2}T(s) = \infty$$

and

$$e(t)_{t\to\infty} = e_{ss}(t) = \lim_{s\to o} s\frac{B}{s^2}\{1-T(s)\} = \infty$$

c. Parabolic Input

When a parabolic input $R(t) = ct^2u(t)$ or $R(s) = 2c / s^3$ was considered then:

$$C(t)_{t\to\infty} = C_{ss}(t) = \lim_{s\to o} s\frac{2c}{s^3}T(s) = \infty$$

and

$$e(t)_{t\to\infty} = e_{ss}(t) = \lim_{s\to o} s\frac{2c}{s^3}\{1-T(s)\} = \infty$$

2. <u>Plant Type One</u>

In this case where the plant was considered to be type one, i.e., N = 1, the steady state error and output of the system were found to be as follows:

a. Step Input

When a step input was considered then:

$$\lim_{t\to\infty} C(t) = C_{ss}(t) = \lim_{s\to 0} s\,\frac{A}{s}\,T(s) = A$$

and

$$\lim_{t\to\infty} e(t) = e_{ss}(t) = \lim_{s\to 0} s\,\frac{A}{s}\,\{1-T(s)\} = 0$$

b. Ramp Input

When a ramp input was considered then:

$$\lim_{t\to\infty} C(t) = C_{ss}(t) = \lim_{s\to 0} s\,\frac{B}{s^2}\,T(s) = \infty$$

and

$$\lim_{t\to\infty} e(t) = e_{ss}(t) = \lim_{s\to 0} s\,\frac{B}{s^2}\,\{1-T(s)\} = \frac{B\alpha \prod_{k=1}^{n} p_k}{\beta \prod_{j=1}^{m} z_j}$$

c. Parabolic Input

When a parabolic input was considered then:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \lim_{s\to o} s \frac{2c}{s^3} T(s) = \infty$$

and

$$\underset{t\to\infty}{e(t)} = e_{ss}(t) = \lim s \frac{2c}{s^3} \{1-T(s)\} = \infty$$

3. Plant Type Two

In this case where the plant was considered to be type two, i.e., N = 2 the steady state error and output of the system were evaluated as follows:

a. Step Input

When a step input was considered then:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \lim_{s\to o} s \frac{A}{s} T(s) = A$$

and

$$\underset{t\to\infty}{e(t)} = e_{ss}(t) = \lim_{s\to o} s \frac{A}{s} \{1-T(s)\} = 0$$

b. Ramp Input

When a ramp input was considered then:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \lim_{s\to o} s \frac{B}{s^2} T(s) = \infty$$

and

$$\underset{t\to\infty}{e(t)} = e_{ss}(t) = \lim_{s\to o} s \frac{B}{s^2} \{1-T(s)\} = 0$$

c. Parabolic Input

When a parabolic input was considered then:

$$\lim_{t\to\infty} C(t) = e_{ss}(t) = \lim_{s\to 0} s \frac{2c}{s^3} T(s) = \infty$$

and

$$\lim_{t\to\infty} e(t) = e_{ss}(t) = \lim s \frac{2c}{s^3} \{1-T(s)\} = \frac{2c\alpha \prod_{k=1}^{n} p_k}{\beta \prod_{j=1}^{m} z_j}$$

4. <u>Remarks</u>

As a first step toward a further study of the "second singular case", an error and output analysis was performed. From these results which were obtained and which are summarized in Table II of Appendix E, it was noticed that in contradiction with the "first singular case" the type of the plant is preserved in this case, where the compensation structure and the associated compensators of figure (6-1) are introduced in order for a singular system to be obtained.

B. EXAMPLE (6-1)

In this example a type one plant was considered, with transfer function.

$$G(s) = kF_g(s) = 100 \frac{1}{s(s+1)(s+5)} \tag{6-8}$$

Suppose that this plant is compensated by the compensation structure shown in figure (6-1), then according to equations (6-2) and (6-3) the transfer functions of the compensators are:

$$G_c(s) = \beta\, F_g(s) = \beta \frac{1}{s(s+1)(s+5)} \qquad (6\text{-}9)$$

and

$$H(s) = \alpha \qquad (6\text{-}10)$$

The characteristic equation of the compensated system is:

$$1 + G(s)\, H(s) + G(s)\, G_c(s) = 0 \qquad (6\text{-}11)$$

Introducing equations (6-8), (6-9) and (6-10) into equation (6-11) yields:

$$s^6 + 12s^5 + 46s^4 + (60+100\alpha)s^3 + (25+600\alpha)s^2$$
$$+500\,\alpha s + 100\beta = 0 \qquad (6\text{-}12)$$

Using the computer method of paragraph 6c, Section II, the singular points of the system were found for several values of ζ. They are listed in Appendix C.

One of these singular points was further considered, i.e., the singular point ($\zeta_s = 0.40$, $\omega_{ns} = 0.97096$). Then, corresponding to the damping ratio value, Chebyshev functions of the second kind were evaluated from equation (2-6) and from equations (2-5) and (2-16) the equation of the singular line which corresponds to the singular point which was considered, was evaluated to be:

$$\beta = - 4.92497\alpha + 0.25924 \qquad (6\text{-}13)$$

Considering the last two terms of equation (6-12) a necessary condition for stability according to Routh's first criterion is that $\alpha>0$ and $\beta>0$. Since α and β parameters are related by equation (6-13), it implies that a necessary condition for stability is:

$$0< \alpha < 0.05265 \qquad (6\text{-}14)$$

By arbitrarily selecting $\alpha = 0.05$ from the range of values defined by (6-14), the corresponding value of β was found from equation (6-13) to be, $\beta = 0.013$. Then the roots of the characteristic equation (6-12) were evaluated, for this pair of parameter values, by using a computer program with the ZPOLR subroutine. These roots are:

$s_1 = - 0.3884 + j0.8899$ $\qquad$ $s_2 = - 0.3884 - j0.8899$

$s_2 = - 5.2141$ $\qquad$ $s_4 = - 5.0132$

$s_5 = - 0.9270$ $\qquad$ $s_6 = - 0.0701$

i.e., one of the roots is the singular root ($\zeta_s = 0.40, \omega_{ns} = 0.97096$) or $s = - 0.3884 \pm j0.8899$.

In figure (6-2) the parameter plane diagram of this singular system is shown for some arbitrarily selected values of ζ and ω_n.

C. HARDWARE CONSIDERATIONS

In figure (6-3) the block diagram of a unity feedback system (plant) is shown. The open loop transfer function of this system is:

$$G(s) = kF_g(s) = k\,\frac{(s+z_1)(s+z_2)\ \ldots\ (s+z_m)}{s^N(s+p_1)(s+p_2)\ \ldots\ (s+p_n)} \qquad (6\text{-}15)$$

Compensation of this system according to the "second singular case" scheme, yields the compensated singular system of figure (6-4). This compensation structure will be in general complex since it introduces a cascade and a feedback compensator. Therefore, a simplified version may also be desired. This modification is shown in figure (6-5), where the feedback compensator (gain adjustment) has been taken from the plant and has been introduced at the cascade compensator $G_c(s)$. Also the Root locus gain β from $G_c(s)$ has been set as a gain adjustment in front of the minor loop.

The characteristic equation of the systems in figure (6-5) and (6-6) are respectively:

$$1 + \alpha k\, F_g(s) + \beta k\, F_g{}^2(s) = 0 \qquad (6\text{-}16)$$

and

$$1 + \alpha F_g(s) + \beta k F_g^2(s) = 0 \tag{6-17}$$

Comparison of these equations reveals that both are singular and they have the same singular points, but they will be in general different, except when k = 1, in which case they become identical. By this modification the whole compensation structure has been introduced in cascade with the plant, as is shown in figure (6-5).

An equivalent system of figure (6-5) is shown in figure (6-6) where the whole compensation structure of figure (6-5) is replaced by an equivalent cascade compensator.

The following considerations can now be stated concerning the compensation structures of figures (6-4) and (6-5) or (6-6). First both are singular. Second with the compensation structure of figure (6-5) or (6-6), the compensation problem has been simplified and better organized since it will be required to build a filter from distinct components or a filter in an integrated form which will be introduced in cascade with the plant. This is a simpler and in general a more realistic approach to the problem. In the case also where K = 1, i.e., the gain of the plant is equal to unity then these structures are also equivalent. When K ≠ 1 then, although they will have the same singular points (because the singular conditions to be satisfied are identical, in both cases), the corresponding equations of the singular lines will be different. This means that for the same singular point, different points of the parameter plane (α, β), will be associated and also the non-singular characteristic roots, will be in general different.

The cascade compensator $G_{sc}(s)$ which was introduced in figure (6-6) has the transfer function:

$$G_{sc}(s) = \frac{\beta\ (s+z_1)(s+z_2)\ \ldots\ (s+z_m)}{s^N\ (s+p_1)(s+p_2)\ \ldots\ (s+p_n)\ +\alpha(s+z_1)\ \ldots\ (s+z_m)} \tag{6-18}$$

where the subscripts s c stands for << Singular compensator >>. Assuming that a pair of α and β parameter pair values which satisfies problem specifications has been selected, then this compensator and its transfer function can be precisely determined.

This modified singular compensation structure of figure (6-5) or (6-6) was further analyzed as the most applicable due to its simplicity under hardware considerations, in comparison with that of figure (6-4). It must be also noticed that in a specific application, where a gain feedback adjustment at the plant is possible or desired, e.g., when it is desired to eliminate the effect of some nonlinearities that the plant may have, then the initial compensation structure of figure (6-4) may be considered for application.

D. STABILITY ANALYSIS

1. Introduction

One of the basic requirements of any control system is stability. The stability concept and related definitions can be found in reference [7]. The first question which arises when stability considerations are made for the singular structure of figure (6-6) is: What effect has the cascade singular compensator which has been introduced, on the stability of the initial uncompensated system? That is, in addition to the fact that this compensator makes the system singular, does it *improve or impair the stability of the* initial system? In either case the question which seeks an answer is: How is it possible from an initially stable or unstable system to obtain a stable singular system? The interpretations of curves and singular lines on a parameter plane diagram and determination of stability is not in general a simple process.

Stability analysis by using the Root locus concept seems to be relatively easy and can give a more complete picture of the behavior of the system. The stability analysis as it can be performed by using the Root locus concept for the case of the compensated singular system is described below.

2. Stability Analysis by the Root Locus Concept

By substituting $F_g(s)$ from equation (6-15) into equation (6-17) and after some manipulations the characteristic equation of the compensated singular system becomes:

$$\alpha\,(s+z_1)(s+z_2)\ldots(s+z_m)s^N(s+p_1((s+p_2)\ldots(s+p_n)+\beta k(s+z_1)^2\ldots(s+z_m)^2+s^{2N}(s+p_1)^2(s+p_2)^2\ldots(s+p_n)^2=0 \tag{6-19}$$

Assuming that the singular points of the singular compensated system have been found and that one of them has been selected as the operating point of the system, then the equation of the corresponding singular line can be evaluated from equations (2-5) and (2-16). This equation will be in general in the form:

$$\beta = A\alpha + B$$

where A and B are real constants. By substituting the parameter β from this equation (6-19) and after some manipulations, the characteristic equation of the singular compensated system becomes:

$$\alpha\frac{Ak(s+z_1)^2(s+z_2)^2\ldots(s+z_m)^2+s^N(s+p_1)(s+p_2)\ldots(s+p_n)(s+z_1)(s+z_2)\ldots(s+z_m)}{Bk(s+z_1)^2(s+z_2)^2\ldots(s+z_m)^2+s^{2N}(s+p_1)^2(s+p_2)^2\ldots(s+p_n)^2}=-1 \tag{6-20}$$

Since A, B and k are known constants the numerator and denominator of the above equation can be factored, which in general yields:

$$\alpha \frac{(s+z_1')(s+z_2') \ldots (s+z_p')}{(s+p_1')(s+p_2') \ldots (s+p_q')} = -1 \qquad (6\text{-}21)$$

where z_j' and p_i' may be real or complex. By treating the parameter α as the Root locus variable parameter the Root locus of the compensated system can be drawn, either by the classical graphical methods or by the use of a computer. In the last case where a computer is used, usually it is possible to proceed directly from equation (6-19) to draw the Root locus of the compensated singular system.

Since it is known that as the Root locus variable parameter α of equation (6-21) varies from zero to infinity the roots of the characteristic equation move on the Root locus segments, starting from the poles and terminating to zeros or at infinity when zeros are not available, and that the singular root and maybe one or more other roots of the characteristic equation (6-21) are fixed, independent of the variations of the parameter α, it implies that some zeros and poles of equation (6-21) have to be identical. Therefore, the numerator and denominator of equation (6-21) must consist of a number of common factors equal to the number of the fixed roots that the characteristic equation of the singular compensated system has.

This number is at least two, corresponding to the system's complex singular root. This fact makes the process of drawing the Root locus of the compensated singular system by the classical graphical methods, much simpler than it initially looks. Thus the Root locus of the singular compensated system consists of two or more fixed points of the s-plane plus the classical Root locus segments.

Considering now system stability all the related questions can be answered from the Root locus study. For example, if any Root locus segment or isolated point lies entirely on the right half of the s-plane, then there is no pair of α and β parameter values for which the compensated system can be stable. If on the other hand all the Root locus segments lie on the left half of the s-plane as well as all the isolated points then all pairs of α and β parameter values will yield a stable system. Finally, if some Root locus segments cross the imaginary axis in one or more points, then the range of the parameter α values for which the compensated singular system is stable can be determined. The corresponding values of the parameter β are then found from the equation of the singular line.

The following example is an application of the above procedure and concepts which were introduced for the case of the singular compensated system.

3. Example (6-2)

In this example the uncompensated system (Plant) of figure (6-3), was considered to have transfer function:

$$G(s) = kF_g(s) = k \frac{1}{s(s+1)(s+5)} \tag{6-22}$$

The characteristic equation of this system is:

$$s^3 + 6s^2 + 5s + k = 0 \tag{6-23}$$

The Root locus of this system is shown in figure (6-7). The gain at the stability limit is k=30. When this system is compensated by a singular cascade compensator the block diagram of the compensated system will be that which is shown in figure (6-5) or the equivalent of figure (6-6). The singular cascade compensator which was thus introduced has a transfer function:

$$G_{sc}(s) = \frac{\beta}{s^3+6s^2+5s+\alpha} \tag{6-24}$$

which was obtained from equation (6-18).

The characteristic equation of the compensated singular system is:

$$1+G(s)\,G_{sc}(s) = 0 \tag{6-25}$$

Introducing equations (6-22) and (6-24) into equation (6-25) yields:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+k\beta = 0 \qquad (6\text{-}26)$$

Two values of the gain k were arbitrarily selected, i.e., k = 20 and k = 35, by which the uncompensated system becomes stable and unstable respectively. These cases were separately analyzed as follows:

a. Uncompensated System Stable, k = 20

For this value of k, the uncompensated system is stable and the characteristic equation (6-26) of the compensated singular system becomes:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+20\beta = 0 \qquad (6\text{-}27)$$

In figure (6-8), the parameter plane diagram of the compensated singular system is shown for arbitrarily selected values of ζ and ω_n. By choosing $\zeta_s = 0.40$ the corresponding value of ω_{ns} was found in the computer by using the <<Singular Point Program>> as $\omega_{ns} = 0.97096$. Therefore, a singular root of the compensated singular system is:

$$s = -\zeta_s\,\omega_{ns} \pm \zeta\omega_{ns}\sqrt{1-\zeta_s^2} = -0.3884 \pm j\,0.8899 \qquad (6\text{-}28)$$

From equations (2-5) and 2-16), the equation of the corresponding singular line was found to be:

$$\beta = 0.24621\alpha - 1.21242 \qquad (6\text{-}29)$$

i.e., if the pair of α and β parameter values satisfies equation (6-29), then the characteristic equation (6-27) of the compensated singular system will have a fixed complex root at the point $s = -0.3884 \pm j\,0.8899$.

Considering now the coefficients of the zero and first power of s terms of the characteristic equation (6-27), a necessary condition for stability is that both these are greater than zero, which implies that $\alpha > 0$ and $\beta > 0$. Based on this observation the roots of the characteristic equation (6-27) were found for a specific range of the parameter α and β values and with each parameter pair satisfying equation (6-29), by using the subroutine ZPOLR in a computer program. These roots are listed in Appendix D. From this list several general remarks can be made about the range of the α and β parameters in the stable region and the variation of the nonsingular roots in this region, i.e., the stable region in terms of the parameter α value is bounded between the values of 5 and 40 approximately, and that within this region the variations of the nonsingular roots is such that the dominance and sensitivity concepts are considerably affected, for different parameter values.

Substituting the value of β from equation (6-29) into equation (6-27) the characteristic equation of the compensated singular system becomes:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+(4.9242\alpha-24.2485) = 0 \qquad (6\text{-}30)$$

Treating then the parameter α as the Root locus variable parameter, equation (6-30) can be written as:

$$\alpha \frac{s^3+6s^2+5s+ 4.9242}{s^6+12s^5+46s^4+60s^3+25s^2 -24.2485} = - 1 \qquad (6\text{-}31)$$

or by factoring numerator and denominator yields:

$$\alpha \frac{(s+5.2230)(s^2+0.7768s + 0.9428)}{(s+5.2230)(s^2+0.7768s+0.9428)(s+4.7195)(s+1.8458)(s-0.5653)} = - 1$$

(6-32)

The numerator and denominator of the left part of equation (6-32) have common factors corresponding to the real root s = -5.2230 and the complex singular root $s = -0.3884 \pm j0.8899$. Therefore, the characteristic equation of the compensated singular system has a fixed real root and a fixed complex root at these points of the s-plane.

The Root locus of this compensated singular system, which was obtained from equation (6-30). By the use of the ROOTLO subroutine in a computer program is shown in figure (6-9).

At this point it must be noticed that, the remarks stated in paragraph two about the fixed roots that a singular system has, the analytical results shown in Appendix D, and the Root locus of figure (6-9) are in agreement. It must also be noticed that the fixed roots of the characteristic equation (6-30) are shown by dark cross marks in a circle, in figure (6-9).

b. Uncompensated System Unstable, $k = 35$

When the gain of the uncompensated system (Plant) has the value of 35, then the uncompensated system is unstable since a pair of complex roots of the characteristic equation (6-23) lie on the right half of the s-plane.

In this case the characteristic equation (6-26) of the compensated singular system shown in figure (6-6) becomes:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+35\beta = 0 \qquad (6\text{-}33)$$

In figure (6-10) the parameter plane diagram of this compensated singular system is shown.

For the same value of ζ_s as in the previous case, i.e., $\zeta_s = 0.40$, the corresponding value of ω_{ns} was found by using the "Singular Point Program" as $\omega_{ns}=0.97096$. Therefore, a singular root of the compensated singular system is $s = -0.3884 \pm j0.8899$. The fact that the singular root remains the same as in the previous case, although the characteristic equation has been changed comes as a consequence of the singular line theory which was developed in section II and III, i.e., if the characteristic equation of the compensated singular system is considered to be in the form of equation (6-17), then in both cases $Fg(s)$ is the same and only the value of k has been changed. Because the singular points according to the singular line theory are the points of the s-plane, which satisfy equation:

$$\angle Fg(s) = n\Pi$$

where n is a real integer or zero, it implies that since $F_g(s)$ is the same in both cases the singular points will remain the same.

From equations (2-5) and (2-16) the equation of the corresponding singular line was found to be:

$$\beta = 0.14069\alpha - 0.69281 \qquad (6\text{-}34)$$

For the same values of the parameter α, as those which were used in the previous case, the corresponding values of the parameter β were found from equation (6-34). For these formed pairs the roots of the characteristic equation (6-33) were evaluated by using the subroutine ZPOLR in a computer program. They are also listed in Appendix D. By comparing the roots of the characteristic equation (6-27) and (6-23), which are listed in Appendix D, it can be seen that they are identical for the same value of the parameter α. The reason for this fact is that when the singular point $s = -0.3884 \pm j0.8899$ or ($\zeta_s = 0.40$, $\omega_{ns} = 0.97096$), is considered, the equation of the corresponding singular line as a function of k, according to equation (2-5) and (2-16) is:

$$\beta = \frac{1}{k}(4.9242\alpha - 24.2485) \qquad (6\text{-}35)$$

When the value of the parameter β from equation (6-35) is substituted in the general form of the characteristic equation of the compensated singular system, i.e., into equation (6-17) yields:

$$1+\alpha\, F_g(s) + (4.9242\alpha - 24.2485)\, F_g^{\,2}(s) = 0$$

or after substituting $F_g(s)$ from equation (6-22) and making the necessary manipulations yields:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+ (4.9242\alpha - 24.2485) = 0 \qquad (6\text{-}36)$$

i.e., the characteristic equation of the compensated singular system is independent of the value of k when the same singular point is considered. Therefore, for the same singular point and for the same value of the parameter α the compensated singular system will have the same characteristic roots independently of the value of k. As a consequence of these remarks not only for k = 20 but for any value of k >0 the Root locus of the compensated singular system will be identical with the Root locus shown in figure (6-9) when the singular point $s = -0.3884 \pm j\,0.8899$ is considered.

4. Summary and Remarks on the Stability of the Compensated Singular System

When the plant to be compensated by a singular cascade compensator has no zeros in its transfer function then the parameter β and the Plant's gain k will appear only in the zero power of s term of the characteristic equation of the compensated singular system as can be seen from equation (6-19). In addition, only the product of β and k will appear in this term, plus a constant when the plant is type zero. When a specific singular operating point of the system is considered (ζ_s, ω_{ns}), then the parameters α and β will be related by the equation of the singular line which corresponds to the above mentioned singular point. According to equation (2-5) and (2-16), the equation of the singular line will be in the form:

$$\beta = \frac{1}{k} (A + B\alpha) \qquad (6\text{-}37)$$

where A and B are constants. From this equation it will be observed on the parameter plane that in the case under consideration when the value of k varies it will change the slope of the singular line which passes from the selected singular point. From this equation also it can be seen that the product βk can be expressed as a function of the parameter α alone. By substituting this product in the characteristic equation (6-19) of the compensated singular system the whole equation will be expressed as a function of the parameter α. Thus the roots of the characteristic equation will depend on the value of α only.

When the plant to be compensated by a singular cascade compensator has one or more zeros in its transfer function then the parameter β and the plant's gain k will appear in three or more power of s terms of the characteristic equation of the compensated singular system as can be seen from equation (6-19). Because they will always appear together in a product form the coefficient C_1 of equation (2-5) can be written in the form:

$$C_1 = k \sum_{j=o}^{n} (-1)^j \, C'_k \, \omega_n{}^j \, U_{j-1}(\zeta) \qquad (6\text{-}38)$$

where C'_k is the constant coefficient of the βk product, in the j power of s term of equation (6-19). When a specific operating singular point of the compensated singular system is considered, i.e., (ζ_s, ω_{ns}), then the parameters α and β will be related by the equation of the corresponding singular line of the above mentioned singular point. Then according to equation (2-5) and (2-16), the equation of the singular line will be:

$$\beta = \frac{1}{k}(A' + B'\alpha) \qquad (6\text{-}39)$$

where A' and B' are constants. By continuing with the same reasoning as in the previous case, the same remarks were found to be applied here.

Therefore it can be stated, that in general the stability of a compensated singular system when a specific singular point is considered, is independent of the value of the plant gain k and depends on the value of the parameter α alone or in other words the stability of the compensated singular system can be interpreted in terms of the parameter α alone. The stable region also (if any) on the parameter plane will be bounded between constant α lines. Thus when the compensated singular system is stable for some range of the parameter α values, it will also be stable for the same range of the parameter a values and for any value of the plant gain k. Therefore, when stability analysis is performed by the Root locus method where the parameter α is used as the Root locus variable parameter then the value of k can be set to any desired value as will be dictated from other design factors.

E. HOW TO OBTAIN A SINGULAR ROOT AT A DESIRED POINT OF THE S-PLANE

1. Introduction

Compensation of a system by using a singular cascade compensator will be in general useful if the compensated system provides a singular root where the specifications of the problem dictate. Therefore, when designing by using the singular line theory the next problem which seeks an answer is: How a singular root can be obtained, where the system's specifications dictate?

The general objective in all cases will be to make use of the advantage that, by varying the parameter values the singular root remains fixed and as other roots are varying the system's performance may be moved toward the system's specifications.

2. Selection of the Singular Point

Assume that a plant with transfer function:

$$G(s) = kF_g(s) = k\frac{(s+z_1)(s+z_2)\ \ldots\ (s+z_m)}{s^N(s+p_1)(s+p_2)\ \ldots\ (s+p_n)} \qquad (6\text{-}40)$$

is going to be compensated by a "singular cascade compensator" whose transfer function according to equation (6-18) is

$$G_{sc}(s) = \beta \frac{(s+z_1)(s+z_2)\ \ldots\ (s+z_m)}{s^N(s+p_1)(s+p_2)\ \ldots(s+p_n) + \alpha(s+z_1)(s+z_2)\ \ldots(s+z_m)} \tag{6-41}$$

then the characteristic equation of the compensated singular system will be:

$$1 + \alpha F_g(s) + \beta k\, F_g^{\,2}(s) = 0 \tag{6-42}$$

According to equation (3-10) the singular points of the system will be the points of the s-plane which satisfy the following equation:

$$\underline{/F_g(s)} = k_1\Pi \tag{6-43}$$

where k, is a real integer or zero. Therefore, the location of the singular points on the s-plane is determined only by the zeros and the poles of the plant, i.e., the singular points are a function of $F_g(s)$ only, and in consequence they can be affected only by changing $F_g(s)$.

In the case where among the singular points there is such a point which satisfies problem specifications as the operating point of the compensated singular system, then there is no need to change $F_g(s)$. Considering now the case where such a singular point is not available then $F_g(s)$ has to be changed in order to obtain a singular point where it is dictated by the problem's specifications. The following examples were selected for illustration of the design procedure in the above case.

a. Example (6-3)

Assume the plant to be compensated has the transfer function:

$$G(s) = kF_g(s) = \frac{1}{s(s+1)(s+5)} \qquad (6\text{-}44)$$

and that problem specifications, i.e., damping ratio, natural undamped frequency, settling time and peak overshoot require a dominant singular root at the point ($\zeta_s = 0.4$, $\omega_{ns} = 0.97096$) or $s_s = -\ 0.3884 \pm j0.8899$.

In figure (6-11) the poles of $F_g(s)$ have been plotted on the s-plane as reference points, in order to measure the angles involved. The desired singular point is also located on the s-plane. By the use of a spirule it was found that:

$$\angle F_g(s) = -\ \{\phi_1 + \phi_2 + \phi_3\} = -\ 180°$$

therefore the point S_s is actually a singular point since for this point equation (6-43) is satisfied for $k_1 = -1$.

This point also was found to be a singular point by the use of the "Singular point program" in example (6-1).

It must be noticed that another way to check if the point ($\zeta_s = 0.4, \omega_{ns} = 0.97096$) is singular for the compensated singular system, is to draw the Root locus of the uncompensated system. Then if the desired operating point lies on the above Root locus (according to the analysis which was developed in paragraph 2a(3), Section III), it is singular. Because this method is more laborious its use in general is not justified, except in the case where the desired operating point is not singular and some trade offs have to be done in order to decide if $F_g(s)$ must be changed (by the appropriate initial compensator) or another operating point which lies on the Root locus of the uncompensated system can be selected as the operating point of the compensated singular system.

b. Example (6-4)

Assume now that the plant to be compensated has transfer function:

$$G(s) = kF_g(s) = k\frac{s+5}{s^2(s+10)(s+20)} \qquad (6\text{-}45)$$

and that problem specifications dictate for an operating singular point ($\zeta_s = 0.6, \omega_{ns} = 1$), or $S_s = -0.6 \pm j\,0.8$. As in the previous example the poles and zeros of $F_g(s)$ were plotted on the s plane as shown in figure (6-12). Then by the use of a spirule the argument of $F_g(s)$ was evaluated and it was found that:

$$\underline{/F_g(s)} = \theta_1 - (2\phi_1 + \phi_2 + \phi_3) = 22.5° - (2x127° + 10.5° + 5°) = -247° \qquad (6\text{-}46)$$

Therefore the desired point is not singular since it does not satisfy equation (6-43). This implies that $F_g(s)$ must be changed somehow in order for the desired point to become singular. From a theoretical point of view it can be achieved as follows:

(1) Introducing an Additional Zero. Suppose that an additional zero is introduced at the point s=-1.82. Then the angle contribution of this zero will be +67° as it can be seen in figure (6-13a). Then, by defining the transfer function of the new obtained system after the above zero was introduced $G_z(s)$, yields:

$$G_z(s) = kF_z(s) = k\{(s+1.82)\ Fg(s)\}$$

and

$$\angle F_z(s) = -180°$$

which implies that the desired operating point ($\zeta_s = 0.6$, $\omega_{ns} = 1$) is singular for the system whose transfer function is:

$$G_z(s) = k\,\frac{(s+5)\ (s+1.82)}{s^2(s+10)\ (s+20)}$$

(2) Introducing an Additional Pole. Suppose that instead of a zero an additional pole is introduced at the point s=-0.58, then the angle contribution of this pole will be -113° as it is shown in figure (6-13b). Then by defining the transfer function of the new system after the pole was introduced, $G_p(s)$ yields:

$$G_p(s) = k\ F_p(s) = k\ \{\frac{1}{s+0.58}\ F_g(s)\}$$

and

$$\angle F_g(s) = -360°$$

which implies that the desired operating point (ζ_s=0.6, ω_{ns}= 1) is singular for the system with transfer function:

$$G_p(s) = k \frac{s+5}{s^2(s+0.58)(s+10)(s+20)}$$

(3) Introducing an Additional Pole and Zero.

Suppose now that an additional pole and a zero are introduced at the transfer function of the initially given plant. This pair of points can be chosen in such a way that their net angle contribution will make the desired singular point (ζ_s= 0.60, ω_{ns}= 1) actually singular. For example if the pole is chosen at the point s = - 16 and the zero at the point s = - 1.66 as it is shown in figure (6-13c), then by defining the transfer function of the new system after this pole and zero have been introduced, $G_{zp}(s)$ yields:

$$G_{zp}(s) = k\, F_{zp}(s) = k \left\{\frac{s+1.66}{s+16} F_g(s)\right\}$$

and

$$\angle F_g(s) = \angle F_g(s) + 73^\circ - 6^\circ = -180^\circ$$

which implies that the desired operating point is singular for the system with transfer function:

$$G_{zp}(s) = k \frac{(s+5)\ (s+1.66)}{s^2(s+10)\ (s+16)\ (s+20}$$

3. Summary and Remarks

Considering in general an uncompensated plant with open loop transfer function G(s) = k $F_g(s)$, where $F_g(s)$ is a ratio of any two polynomials of s, then the argument of $F_g(s)$ evaluated at any point of the s-plane will always lie between the limits:

$$k_1\pi \leqslant \angle F_g(s) \leqslant (k_1+1)\pi$$

where k_1 can be any real integer or zero. Therefore, the maximum angle correction which $F_g(s)$ may require in order to satisfy equation (6-43) for a desired point of the s-plane will be less than 180° , which implies that a single zero or a single pole or a zero and a pole or finally any number of zeros and poles can be properly selected and introduced to $F_g(s)$ in order to make the argument of $F_g(s)$ satisfying the angle requirement for singular conditions, for a specific singular point, which is desired to be the operating point of the compensated singular system.

These poles and zeros in general may be real or complex. In consequence it implies that when such a change of $F_g(s)$ is required an initial cascade compensator must be introduced, before a "singular cascade compensator" is introduced.

Since a compensator with only zeros actually can not be built, the case of introducing one or more zeros alone has to be ignored as unrealistic. Therefore, among the rest of the possibilities available an optimal selection of the initial cascade compensator has to be done. This selection can be based on noise considerations, problem specifications and the "dominance of the singular root" concept which is described in the next part of this section, i.e., except the fact that the Root locus of the uncompensated system will be forced to pass through the point where the desired singular root is located, the shape also of the resulting Root locus must be such that it promises some range of dominancy of the singular root. (See Example (6.6)).

It must also be noticed that as it will be shown in the following analysis when an initial cascade compensator is required, then this requirement will actually affect only the transfer function of the singular cascade compensator and neither the complexity of the rest of the design problem nor the compensation structure (See Example (6-6) and figure (6-28c)). Under these considerations an initial cascade compensator with only zero's may also be considered.

F. DOMINANCE OF THE SINGULAR ROOT

1. Dominant Root Concept

The concept of the dominant root has wide applications in analysis and design of linear systems. Considering a system with a characteristic equation of order n, then there will be n characteristic roots for this system. If one of these roots dominates the transient response, then this root will be called dominant. When the dominant root of a system is a complex root, $s = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2}$, then the overall response of the system for a given input may be closely approximated from the corresponding standard second order system graphs (i.e., the second order system whose the characteristic equation is $s^2+2\zeta\omega_n s+\omega_n^2=0$ and has no zeros).

2. Dominance of the Singular Root

When a singular root has been determined from the problem specifications, then this singular root may be available from the given plant or not. In the last case an initial cascade compensator will be required, in order for the desired singular root to become available. The design process of this initial cascade compensator was previously illustrated.

In either case it will also be desired for the above considered singular root to be dominant for one or more (depends from the specific application) parameter pair values. The design process in all possible cases toward the dominancy of the singular root will be illustrated by the following examples:

3. Example (6-5)

In this example the Plant of example (6-2) was again considered with the Root locus gain k = 20, i.e.:

$$G(s) = k\,F_g(s) = 20\,\frac{1}{s(s+1)(s+5)} \qquad (6\text{-}47)$$

Since the value of the plant's gain does not affect the location of the singular roots on the s-plane, as was shown in example (6-2), the plant under consideration when it is compensated by a "singular cascade compensator" will have the same singular roots with these which are listed in Appendix C for the example (6-1) and (6-2). Some of these roots were considered and their dominancy was investigated.

In figure (6-14a) and (6-14b) the compensated singular system is shown, where in figure (6-14a) the singular compensator is considered to be in integrated form, and in figure (6-14b) the same singular compensator is considered to consist of distinct components.

The singular cascade compensator which thus has been introduced according to equation (6-18) has transfer function:

$$G_{sc}(s) = \frac{\beta}{s(s+1)(s+5)+\alpha} \qquad (6\text{-}48)$$

The characteristic equation of the compensated singular system was found to be:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+20\beta = 0 \qquad (6\text{-}49)$$

a. Singular Root ($\zeta_s = 0.40, \omega_{ns} = 0.97096$)

When the singular root ($\zeta_s = 0.40, \omega_{ns} = 0.97096$) was considered, then the problem which was investigated was: Are there any pairs of parameter α and β values for which this singular root becomes dominant?

From equations (2-5) and (2-16) the equation of the corresponding singular line was found to be:

$$\beta = 0.24631\alpha - 1.21242 \qquad 6\text{-}50)$$

Introducing equation (6-50) into equation (6-49) the characteristic equation of the compensated singular system in terms of the parameter α becomes:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s + (4.9242\alpha - 24.2485) = 0 \quad (6\text{-}51)$$

Treating then the parameter α as the Root locus variable parameter, the Root locus of the compensated singular system for the singular point under consideration was drawn in the computer. It is shown in figure (6-9). The compensated singular system has three fixed roots, i.e., the singular root $s_s=-0.3884 \pm j\,0.8899$ and a real root $s=-5.2230$, which are shown with dark circle marks at the corresponding locations of figure (6-9). The compensated singular system also has three other roots whose location varies on the corresponding Root locus segments according to the value of the parameter α.

Examining the roots of the characteristic equation (6-51) in relation with the Root locus of figure (6-9) it can be seen that the value of the plant gain does not affect the location of the roots of the characteristic equation of the compensated singular system and that the location of the three varying roots is affected only by the value of the parameter α. The dominancy also of the singular root can be affected only by the other two characteristic roots which lie on the same Root locus segments with the singular root. These roots are either real or complex according to the value of the parameter α.

It was found that in the stable region the value of the parameter α is approximately betwen the values of five and fourty. When $\alpha = 9.85$ then the two varying roots in the vicinity of the singular root become identical with the singular root, i.e., for the value of the parameter $\alpha = 9.85$ the singular root is repeated. It was also found that $\alpha=9.85$ not only the singular root but and the real fixed root at $s=-5.2230$ was also repeated.

In an attempt to find the values of the parameter α for which the singular root is dominant (or in other words the system's performance can be well predicted from the corresponding to the singular root second order model) several runs with different parameter pair values each, were performed in the computer using a DSL (Digital Simulation Language) program. In this program the compensated singular system shown in figure (6-14b) was implemented.

When the second order model corresponding to the singular root ($\zeta_s = 0.40, \omega_{ns} = 0.97096$ was considered with transfer function:

$$\frac{C(s)}{R(s)} = \frac{\omega_{ns}^2}{s^2 + 2\zeta_s\omega_{ns}s + \omega_{ns}^2} = \frac{0.9428}{s^2 + 0.77685s + 0.9428} \qquad (6\text{-}52)$$

its transient response characteristic values were calculated. These values together with those which were obtained from the computer for the compensated singular system are listed in Table I of Appendix F.

By examining this table the following remarks can be stated:

(1) There is no pair of parameter values for which all the transient response characteristics of the compensated singular system can be well predicted from the values corresponding to the singular root second order model.

(2) For all the parameter pairs both the singular root and the other root in the vicinity of the singular root contribute to the transient response of the compensated singular system.

(3) Considering that in practice the only transient response characteristic values for which the designer is in general interested, are the settling time and the percent overshoot, then from this point of view, it can be said that the singular root is dominant for $\alpha = 7.8$ and $\beta = 0.708$. Assuming also that there is some tolerance in the system performance then these two parameter pair values can vary within some range according to the tolerance values.

The conclusion from this example was that when dominant mode design is used for compensation of the given plant then the singular line theory provides in general an acceptable solution. In figure (6-15) the time response of the compensated singular system is shown for a unit step input and for the two parameters having the values of $\alpha = 7.8$ and $\beta = 0.71$. In figure (6-16) the time response of the compensated singular system is shown for the parameter pair values $\alpha = 9.85$ and $\beta = 1.21$ for which the singular root and the fixed real root are repeated. This fact as can be seen from this figure causes a very high overshoot value. In figure (6-17) the time response of the compensated singular system is shown for such a pair of values, i.e., $\alpha = 20$ and $\beta = 3.71$, for which the contribution of the singular root and the other root in the vicinity of the singular root can be recognized. In figure (6-18) the time response of the compensated singular system is shown for the case where the characteristic root in the vicinity of the singular root lies on the real axis. In this case because this root is closer to the origin and to the imaginary axis than the singular root, it is dominant and the transient response of the compensated singular system has an exponential form.

b. Singular Root ($\zeta_s = 0.30, \omega_{ns} = 1.15268$)

The second singular root from the table of Appendix C which was examined for dominancy was the singular root ($\zeta_s = 0.30, \omega_{ns} = 1.15268$) or $s = -0.34580 \pm j\,1.09959$.

From equations (2-5) and (2-16) the equation of the corresponding singular line was found to be:

$$\beta = 0.35266\alpha - 2.48732 \qquad (6\text{-}53)$$

Introducing equation (6-53) into equation (6-49) the characteristic equation of the compensated singular system in terms of the parameter α becomes:

$$s^6+12s^5+46s^4(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+7.0532\alpha - 49.7464 = 0 \qquad (6\text{-}54)$$

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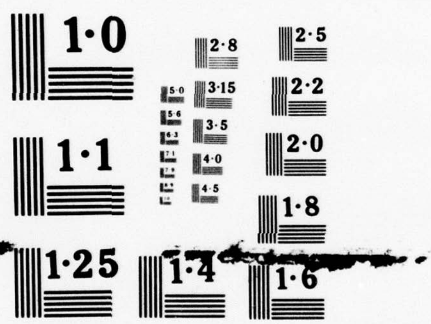

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Treating then the parameter α as the Root locus variable parameter, the Root locus of the compensated singular system for the singular root under consideration was drawn in the computer. It is shown in figure (6-19). This Root locus as can be seen from this figure consists of three points of the s-plane where the three fixed roots of the compensated singular system lie and the classical Root locus segments. The above mentioned three fixed roots are shown with dark circle marks at the corresponding locations, i.e.,

$$s_1 = -\ 5.3084$$

$$s_2 = -\ 0.34580 + j1.09959$$

$$s_3 = -\ 0.34580 - j1.09959$$

In an attempt to find the values of the parameter α for which the singular root is dominant several runs with different parameter pair values each, were performed in the computer using the DSL simulation program. The transient response characteristic values which were obtained from the computer, together with these corresponding to the singular root ($\zeta_s = 0.30, \omega_{ns} = 1.15268$) second order model are listed in Table II of Appendix F.

By examining this table, the same general remarks and conclusions were derived in this case as in the previous one. The basic remark again was that there is a pair of parameter values, i.e., $\alpha = 10.6$, $\beta = 1.25$, for which the singular root is dominant from a practical point of view. In figure (6-20) the time response of the compensated singular system is shown for this pair of parameter values. In figure also (6-21) through (6-23), the time response of the compensated singular system is shown for several characteristic pairs of parameter values.

c. Singular Root ($\zeta_s = 0.75, \omega_{ns} = 0.60667$)

The third singular root which was examined for dominancy was ($\zeta_s = 0.75, \omega_{ns} = 0.60667$) or $s = -0.45500 \pm j0.40127$.

From equations (2-5) and (2-16) the equation of the corresponding singular line was found to be:

$$\beta = 0.09367\alpha - 0.17547 \qquad (6\text{-}55)$$

Introducing equation (6-55) into equation (6-49) the characteristic equation of the compensated singular system in terms of the parameter α becomes:

$$s^6+12s^5+46s^4+(60+\alpha)s^3+(25+6\alpha)s^2+5\alpha s+1.8734\alpha-3.5094 = 0 \qquad (6\text{-}56)$$

Treating then the parameter α as the Root locus variable parameter, the Root locus of the compensated singular system for the singular root under consideration was drawn in the computer. It is shown in figure (6-24). This Root locus (as can be seen from figure (6-24) also consists of three points of the s-plane where the three fixed roots of the compensated singular system lie and the classical Root locus segments. The three points where the three fixed roots are located are shown with dark circle marks at the corresponding locations, i.e.:

$s_1 = -5.0899$

$s_2 = -0.45500 + j0.40127$

$s_3 = -0.45500 - j0.40127$

By the same procedure used in the previous cases the same general remarks and conclusions were derived in this case also. The basic remark again was that there is a pair of parameter values α and β for which the singular root is dominant under a practical point of view. This pair as can be seen from the Table III of Appendix F is: $\alpha = 3.6$, $\beta = 0.162$. In this table the transient response characteristic values which were obtained from the computer together with these corresponding to the singular root ($\zeta_s = 0.75$, $\omega_{ns} = 0.60667$) second order model are listed. In figure (6-25) the time response of the compensated singular system is shown for the previous mentioned parameter pair values.

d. Remarks

It must be noted here that the proceeding design was available for the particular system. It may not be generally available or may not provide an acceptable solution to some problems.

4. <u>Root Loci Correlation</u>

In the previous example if the desired singular root of the compensated singular system could not be made dominant by a pair of parameter values, then an initial cascade compensator would be required, as has been already stated in paragraph two.

Since the initial cascade compensator is applied to the uncompensated system, i.e., the system before the singular cascade compensator has been introduced, in order to affect the compensated singular system, i.e., the system after the singular cascade compensator has been introduced, a prior knowledge of this effect is necessary in order to avoid an undetermined number of trials.

This prior required knowledge or the criterion for the selection of the initial cascade compensator can be based on the correlation between the Root locus of the uncompensated system and the Root locus of the compensated singular system, as can be seen from the following analysis.

When an uncompensated linear control system with unity feedback is considered, its characteristic equation will in general have the form:

$$k \frac{(s+z_1)(s+z_2) \ldots (s+z_m)}{s^N (s+p_1)(s+p_2) \ldots (s+p_n)} = k\, F_g(s) = -1 \qquad (6\text{-}57)$$

where $k\, F_g(s)$ is the open loop transfer function and k is the Root locus gain. The Root locus of this uncompensated system which is drawn with the Root locus gain k as the Root locus variable parameter consists of all the points of the s-plane which satisfy the angle requirement:

$$\angle F_g(s) = (2n\pm1)\Pi$$

Therefore any point S_{RL} of the Root locus of the uncompensated system as it can be seen from equation (6-57) and the previous angle relation satisfies the following two relations:

$$k\, F_g(S_{RL}) = -1$$

and

$$\angle F_g(S_{RL}) = (2n\pm1)\Pi$$

or $$F_g(S_{RL}) < 0 \qquad (6\text{-}58)$$

and $$\angle F_g(S_{RL}) = (2n\pm1)\,\Pi \qquad (6\text{-}59)$$

where n is a real integer.

When the system which was considered is compensated by a singular cascade compensator the characteristic equation of the compensated singular system according to equation (6-17) will be:

$$1+\alpha\, F_g(s) + \beta\, k\, F_g^{\,2}(s) = 0 \qquad (6\text{-}60)$$

Then according to equation (3-12) a point of s-plane is a singular point for the compensated singular system if and only if it does not lie on the real axis and satisfies equation:

$$\underline{/F_g(s)} = nk \qquad (6\text{-}61)$$

where n is a real integer or zero. Therefore all the points of the Root locus of the uncompensated system, which do not lie on the real axis (this restriction is implied from the definition of the singular point in Section II, A) are singular points because they satisfy equation (6-59) and in consequence equation (6-61).

Suppose that one of these points, i.e., the point (ζ_s, ω_{ns}) is selected as the operating point for the compensated singular system. For this point equation (6-58) becomes:

$$F_g(\zeta_s, \omega_{ns}) < 0 \qquad (6\text{-}62)$$

The equation of the corresponding singular line (i.e., the equation which defines the relation between the two parameters α and β for which the selected singular point will always be a root of the characteristic equation) can be found either by using equations (2-5) and (2-16) or by direct substitution of the selected singular point in equation (6-60), which yields:

$$1+\alpha\ F_g(\zeta_s, \omega_{ns}) + \beta\ k\ F_g^2(\zeta_s, \omega_{ns}) = 0 \qquad (6\text{-}63)$$

Solving this equation for βk and substituting it in equation (6-60) yields:

$$1+\alpha \quad F_g(s) - \frac{1+\alpha \; F_g(\zeta_s,\omega_{ns})}{F_g^{\;2}(\zeta_s,\omega_{ns})} F_g^{\;2}(s) = 0 \qquad (6\text{-}64)$$

Equation (6-64) is the characteristic equation of the compensated singular system as a function of the parameter α, when the singular point (ζ_s,ω_{ns}) is considered to be the operating point of the compensated singular system. This equation can be written in the form:

$$\alpha \frac{\{F_g(\zeta_s,\omega_{ns})F_g(s)\} \; \{F_g(\zeta_s,\omega_{ns}) - F_g(s)\}}{\{F_g(\zeta_s,\omega_{ns})+F_g(s)\} \; \{F_g(\zeta_s,\omega_{ns}) - F_g(s)\}} = -1 \qquad (6\text{-}65)$$

or

$$\alpha \frac{F_g(\zeta_s,\omega_{ns})(s+z_1)(s+z_2)\ldots(s+z_m)}{F_g(\zeta_s,\omega_{ns})s^N(s+p_1)(s+p_2)\ldots(s+p_n)+(s+z_1)(s+z_2)\ldots(s+z_n)} = -1 \qquad (6\text{-}66)$$

Considering the Root locus of the uncompensated system, the Root locus of the compensated singular system and the corresponding equations, the following remarks and conclusions can be stated.

a. Number of Fixed Roots

From equation (6-65), the common factor in the numerator and the denominator on the left side of equation, implies that the compensated singular system has a number of fixed roots equal to the order of the characteristic equation of the uncompensated system, e.g., when the order of the characteristic equation of the uncompensated system is three, there will be three fixed roots, two of which have to be the complex pair of the singular roots.

b. Root Loci Correlation

The angle relation which the points of the Root locus of the compensated singular system satisfy is:

$$\angle \frac{F_g(\zeta_s, \omega_{ns}) F_g(s)}{F_g(\zeta_s, \omega_{ns}) + F_g(s)} = (2n \pm 1)\Pi \qquad (6\text{-}67)$$

Then according to equations (6-58) and (6-62) all the points of the Root locus of the uncompensated system satisfy equation (6-67). Therefore all the points of the Root locus of the uncompensated system will be also points of the Root locus of the compensated singular system.

Second, from equation (6-57) and (6-66) it is observed that they have, at the expression on the left side, identical zeros. They also have the same number of poles.

The conclusion from the above remarks is that the Root locus of the compensated singular system will be identical with the Root locus of the uncompensated system with the only difference that the Root locus segments of the compensated singular system may be further extended from the pole side end points, where the corresponding Root locus segments of the uncompensated system terminate, until they reach the poles of the compensated singular system.

Examining the figures (6-7), (6-9), (6-21) and (6-25) they confirm all the above derived remarks and conclusions. Therefore, if an initial cascade compensator is introduced to the uncompensated system this will affect the Root locus of the uncompensated system and the Root locus of the compensated singular system in the same way, i.e., the prior knowledge of the effect of the initial compensator on the compensated singular system can be obtained by studying the effect of the initial compensator on the uncompensated system.

5. Range of Dominancy of the Singular Root

When a singular compensated system is considered and an operating singular point, then the parameters α and β will be related by the equation of the corresponding singular line. In general there will be two ranges of α and β parameter values of primary interest. The first one will be that for which the compensated singular system is stable and the second one will be that within the previous one, in which the singular root is dominant. In general the wider these ranges the more desirable will be the solution. Therefore when an initial cascade compensator will be introduced to the uncompensated system in order to make the singular root dominant, it must be selected in such a way that it will also give a wide range of dominancy of the singular root.

It must be realized that in a self-adaptive singular system where the two parameters α and β are automatically adjusted so that the singular point will remain on the singular line during the operation of the system, the range of dominancy of the singular root will be very important.

The following example is mainly referred to the range of dominancy of the singular root.

6. Example (6-6)

In this example the same plant which is shown in figure (6-26) was again considered. Its transfer function is:

$$G(s) = \frac{20}{s(s+1)(s+5)}$$

For this example, it was assumed that problem specifications require an operating point for the system, i.e., an operating dominant singular root, with damping ratio $\zeta = 0.60$, and natural frequency $\omega_n = 1.2$ rad/sec. A wide range of dominancy was also assumed to be required for the above singular root.

The Root locus of the plant with the Root locus gain used as the Root locus variable parameter is shown in figure (6-29), where the location of the desired singular root is also shown. By studying this figure and taking into account the remarks which were stated in part VI E and VI F, the conclusion was that an initial cascade compensator is needed for the following two reasons: The first is that since the desired singular root does not lie on the Root locus of the plant this Root locus must be reshaped in order to pass through the point where the desired singular root is located, and the second reason is that the shape of the Root locus must be changed in order for this singular root to become dominant with a wide range of dominancy, as it is required from the problem specifications.

After the above remarks and taking into account the noise problem it was decided to compensate the plant initially by a cascade lead filter with transfer function:

$$G_c(s) = \frac{s+2}{s+20} \qquad (6\text{-}68)$$

The compensated system is shown in figure (6-27). The characteristic equation of this system is:

$$s^4+26s^3+125s^2+(100+k)\ s+2k = 0 \qquad (6\text{-}69)$$

where k = 20. Equation (6-69) can be written in the form:

$$k \frac{s+2}{s(s+1)(s+5)(s+20)} = -1 \qquad (6\text{-}70)$$

The Root locus of this system with k used as the Root locus variable parameter is shown in figure (6-30).

At z = 0.6 the corresponding Root locus point is ($\zeta = 0.60$, $\omega_n = 1.232586$). The value of ω_n although it could be measured directly from the Root locus diagram of figure (6-30) was found in the indicated accuracy by the use of the "singular point program". Therefore this initially compensated plant provides the desired singular root (assuming that the small difference between the desired value of ω_n, i.e., $\omega_n = 1.2$ rad/sec and the actual value of $\omega_n = 1.232586$ is within the system's tolerance). The shape also of the Root locus of figure (6-30) is such that promises some range of dominancy of the singular root.

Based on these remarks the initially compensated plant was next compensated by a singular cascade compensator the transfer function of which according to equation (6-18) is:

$$G_{sc}(s) = \frac{\beta\ (s+2)}{s(s+1)(s+5)(s+20)\ +\alpha\ (s+2)} \qquad (6\text{-}71)$$

The compensated singular system is shown in figure (6-28α). The characteristic equation of this system was found to be:

$$s^8+52s^7+926s^6+(6700+\alpha)s^5+(20825+28\alpha)s^4+$$
$$(25000+177\alpha)s^3+(10000+350\alpha+20\beta)s^2+ \qquad (6\text{-}72)$$
$$(200\alpha+80\beta)s+80\beta = 0$$

When the singular root ($\zeta_s = 0.60$, $\omega_{ns} = 1.232586$) was considered, the equation of the corresponding singular line was found from equations (2-5) and (2-16) to be:

$$\beta = 3.31245248\alpha - 219.4468287 \qquad (6\text{-}73)$$

Introducing equation (6-73) into equation (6-72) and after some manipulations yields:

$$\alpha \frac{s^5+28s^4+177s^3+416.249s^2+464.996s+264.996}{s^8+52s^7+926s^6+6700s^5+20825s^4+25000s^3+5611.0635s^2} \cdots$$

$$\cdots \overline{-17555.746s-17555.746} = -1 \qquad (6\text{-}74)$$

or after factoring numerator and denominator yields:

$$\alpha \frac{(s^2+1.479s+1.519)(s+20.204)(s+4.316)(s+2)}{(s^2+1.479s+1.519)(s+20.204)(s+4.316)(s+19.786)} \cdots$$

$$\cdots \overline{(s+5.642)(s+1.4129)(s-0.840)} = -1 \qquad (6\text{-}75)$$

which is in agreement with the conclusions made in paragraph four, i.e., the number of the fixed roots is four which is also the order of the characteristic equation of the system before the *singular cascade compensator* was introduced and that the Root locus of the compensated singular system with the parameter α used as the Root locus variable parameter has identical zeros and the same number of poles as the uncompensated system, i.e., the system before the singular cascade compensator was introduced.

The Root locus of the compensated singular system was drawn by using a computer program in which equation (6-74) was implemented. This is shown in figure (6-31). In this figure the four fixed roots are shown with dark circles at the corresponding locations.

a. Range of Dominancy of the Singular Root

By the use of a DSL simulation program in which the system of figure (6-28b) was implemented, the range of dominancy of the singular root ($\zeta_s = 0.60$, $\omega_{ns} = 1.232586$) was investigated. More than one hundred *properly* selected parameter pair values were used within the system's stable region. The conclusion from this analysis was that the singular root has a very wide range of dominancy. The detailed results which were interpreted on both, the s-plane and the α-β parameter plane, have as follows:

(1) S-plane. The Root locus of the compensated singular system is shown in figure (6-31). This Root locus has been drawn with the parameter α used as the Root locus variable parameter. According to equation (6-72) and (6-75), the compensated singular system has eight characteristic roots, four of which are fixed at the points:

$$s_1 = -4.316 \qquad s_3 = -0.7396+j0.9861$$

$$s_2 = -20.204 \qquad s_4 = -0.7396-j0.9861$$

i.e., two fixed real roots and the fixed complex pair of the singular roots. All these fixed roots are shown in the corresponding locations of figure (6-31) with dark circle marks. The location of each of the other four characteristic roots is varying on the corresponding Root locus segments as a function of the parameter α value. It must be noticed that a specific value of the parameter α defines a unique pair of parameter α and β values since for each value of α corresponds one and only one value of β as can be seen from equation (6-73), which is the equation of the operating singular line.

Interpreting the computer results which were obtained from a DSL simulation and a polynomial root finding program, the following remarks and conclusions can be stated in correlation of the parameter α value and the Root locus of figure (6-31).

(a) $0 < \alpha < 66.25$. Within this range of the parameter α value, the compensated singular system is unstable because there is a real root on the right half of the s-plane. Based on the need for a specific name to be given to this root, since it was found that after the singular root this is the only root which affects the transient response of the compensated singular system and the fact that this root lies on the same Root locus segment with the "singular root", it was called the "non singular root".

(b) $\alpha = 66.25$. For this value of the parameter α the non singular root lies on the origin of the s-plane and the compensated singular system is still unstable or conditionally stable.

(c) $66.25 < \alpha < 81.30$. Within this range of the parameter α value, the non singular root lies on the real axis in the left half of the s-plane. The compensated singular system is stable and the non singular root is dominant. In figure (6-32) the transient response of the compensated singular system is shown for a unit step input and with an arbitrarily selected value of the parameter α within this range, i.e., $\alpha = 75$ for which the corresponding value of β was found from the equation of the operating singular line, (equation (6-73)) to be $\beta = 28.987$.

(d) $\alpha = 81.30$: For this value of the parameter α the non singular root is repeated at the point where the Root locus breaks away from the axis of reals, i.e., at this point the non singular root meets the other real root which was moving to the right on the real axis. This root is again the dominant root. In figure (6-33) the transient response of the compensated singular system is shown for this value of the parameter α. As the value of the parameter α further increases, these two roots form a complex pair of roots. In this case by the name "non singular root" this complex pair will be meant.

(e) $81.30 < \alpha \quad 95$. Within this range of the parameter α value the non singular root is complex and although it has moved away from the real axis it is still closer to the origin and to the imaginary axis than the singular root. Within this range the contribution of the singular root to the transient response of the compensated singular system was negligibly small, and the non singular root was dominant. In figure (6-34) the transient response of the system is shown for $\alpha = 90$.

(f) $95 \leq \alpha < 105$. Within this range of the parameter α value the non singular root has approached closer to the singular root. The contribution of both of these roots to the transient response of the compensated singular system is significant and none of these can be said to be dominant.

When the second order model corresponding to the singular root ($\zeta_s = 0.6, \omega_{ns} = 1.2326$) was considered, with transfer function:

$$\frac{\omega_{ns}^2}{s^2+2\zeta_s\omega_{ns}\,s+\omega_{ns}^2} = \frac{1.5193}{s^2+0.7396s+1.5193} \qquad (6\text{-}76)$$

the corresponding transient response characteristics were calculated and they were found to be as follows:

Time to peak of first overshoot:	$t_p = 3.19$ sec
Settling time :	$t_s = 5.41$ sec
Percent overshoot :	$Mp = 9.48\%$
Transient oscillatory frequence:	$\omega_t = 0.99$ rad/sec

Within this range of parameter α value the transient response characteristics of the compensated singular system were found to be as follows:

Time to peak of first overshoot	:	$t_p \simeq 5$ sec.
Settling time	:	$5 \leq t_s \leq 8$ sec.
Percent overshoot	:	$8 \leq Mp \leq 11\%$
Transient oscillatory frequency	:	$0.6 \leq \omega_t \leq 1$ rad/sec

Because in practical applications the transient response characteristics which are of the basic interest are the settling time and the percent overshoot it seem from the above results that an acceptable solution to the problem can be derived within this range of the parameter α. In figure (6-35) the transient response of the system is shown for α = 100.

(g) $105 \leq \alpha < 131.5$. Within this range of the parameter α value the singular and the non singular root are very close to each other. The contribution of both of these roots to the transient response of the compensated singular system is again significant, and none of these can be called dominant. The transient response was in general characterized by long settling time, which was approximately $t_s \simeq 8.5$ sec and high percent overshoot values, which were found to lie in the range from 11% up to 18.5%. In figure (6-36) the transient response of the compensated singular system is shown for $\alpha = 120$.

(h) $\alpha = 131.5$. For this value of the parameter α the non singular root becomes identical with the singular root, i.e., the singular root is repeated. In figure (6-37) the transient response of the system is shown for this value of the parameter α.

(i) $131.5 < \alpha < 285$. Within this range of the parameter α value the non singular root, *although* it is further away from the origin and the imaginary axis than the singular root, its contribution to the transient response of the compensated singular system is still significant and none of these roots can be called dominant. The transient response characteristics of the compensated singular system were found to be as follows:

Time to peak of first overshoot:	$2.8 \lesssim t_p \lesssim 4$ sec
Settling time :	$7 \lesssim t_s \lesssim 8.8$ sec
Percent overshoot :	$18.96 \lesssim Mp \lesssim 23.6\%$
Transient oscillatory frequency:	$0.98 \lesssim \omega_t \lesssim 1.1$ rad/sec

In figure (6-38) the transient response of the compensated singular system is shown for α = 200.

(j) $285 \lesssim \alpha \lesssim 385$. Within this range of the parameter α value the non singular root has moved further from the imaginary axis and from the origin, and the singular root has become the dominant root but with low degree of dominancy. Within this range, the transient response characteristics of the compensated singular system were found to be as follows:

Time to peak of first overshoot:	$t_p \simeq 2.8$
Settling time :	$6.6 \lesssim t_s \lesssim 7$ sec
Percent overshoot :	$14.22 \lesssim Mp \lesssim 18.96\%$
Transient oscillatory frequency:	$0.99 \lesssim \omega_t \lesssim 1.1$ rad/sec

i.e., the percent overshoot is from 50% up to 100% over the corresponding to the singular root second order model value, and the settling time from 22% up to 30%. In figure (6-39) the transient response of the compensated singular system is shown for α = 330.

(k) $385 \lesssim \alpha \lesssim 1520$. Within this range of the parameter α value the non singular root is so far away from the origin in comparison with the singular root, that its contribution to the transient response of the compensated singular system is either very little or neglegible and the singular root is the dominant root.

Within this range of the parameter α value the transient response characteristics of the compensated singular system were found to be as follows:

Time to peak of first overshoot:	$2.8 \lesssim t_p \lesssim 3.1$ sec
Settling time :	$5.6 \lesssim t_s \lesssim 6.6$ sec
Percent overshoot :	$11.85 \leqslant Mp \leqslant 14.22\%$
Transient oscillatory frequency:	$0.98 \lesssim \omega_t \lesssim 0.99$ rad/sec

i.e., the percent overshoot from 10% up to 50% over the second order model value and the settling time no more than 22% over that corresponding to the second order model value. In figure (6-40) through (6-44) the transient response of the compensated singular system is shown for several values of the parameter α within the range which was considered. It was noticed in this case that although the parameter values vary within a wide range, the transient response of the compensated singular system remains almost unaffected.

(1) $1520 < \alpha < 1922$. Within this range of the parameter α value the non singular root appears again to affect appreciably the transient response of the compensated singular system due to the fact that although it is far away from the origin it is very close to the imaginary axis. The transient response is characterized with long settling time and high frequency of oscillations. In figure (6-45) through (6-47) the transient response of the

compensated singular system is shown for three selected values of the parameter α within the range which was considered.

(m) $\alpha = 1922$. For this value of the parameter α the non singular root lies again on the imaginary axis and the system is unstable or conditionally stable. The corresponding value of ω for the non singular root at that point is $\omega = 8.67$ rad/sec.

(n) $\alpha > 1922$. Within this range of the parameter α value the non singular root lies on the right half of the s-plane and the compensated singular system is unstable. In figure (6-48) the transient response of the system is shown for such a value of α, i.e., $\alpha=2000$.

The conclusion from the above analysis is that there is in general a wide range of the parameter α values for which the singular root is dominant, and which may be restricted according to the tolerance of the transient response characteristic values which are usually determined with problem specifications.

(2) Parameter Plane. In figure (6-49) the parameter plane diagram of the compensated singular system is shown for the singular values of ζ_s and ω_{ns} together with the singular line which is associated with them. As

can be seen in this figure all the above mentioned lines are tangent at the same point of the parameter plane.

In figure (6-50) the same parameter plane diagram is shown for different scale values. On the same diagram the compensated singular system stable region and dominant singular root region have been interpreted in terms of singular line segments.

b. Hardware implementation of the compensated singular system

According to Part C of this Section, the compensated singular system can be implemented as shown in figure (6-28α) or (6-28b) or (6-28c), all of which are equivalent.

Choice of the configuration to be used in practice will be in general determined from the specific application and the type of compensator available, i.e., integrated form or distinct components.

When it is required to maintain the two parameters α and β adjustable or when the compensator of the system is going to be built from distinct components then the configuration of figure (6-28b) seems to be the most representative one for such a case.

Considering also the advance of the technology today in integrated circuits, it must be realized that any required compensator can be built in integrated form. In such a case either figure (6-28α) or figure (6-28c) can be implemented, after the two parameter values have been determined.

c. Remarks

When the singular line theory is considered as a design method for compensation of linear systems then the following remarks can be stated concerning the singular case which was considered in this section.

(1) It can be applied to any type of plant.

(2) The design problem has been simplified and better organized since in general the basic design problem is not to find the required compensator but the appropriate values of the two parameters α and β which have to be used with the singular compensator which was defined by equation (6-18).

(3) The existence of the singular line provides increased flexibility in the design procedure.

It must also be noticed that when the two paramaters α and β are considered as time variant quantities of an adaptive system and its compensator respectively, then by adjusting their initial value and their rate of change in such a way that the operating point moves along the corresponding to the singular root dominant root region, system performance will be essentially unchanged. In this case other characteristic roots of the system will change but since they are not dominant, system performance can not be affected. When the above mentioned rate adjustments are made automatically the compensated singular system becomes self-adaptive. This type of application of the singular line theory in adaptive systems needs to be further investigated.

In the previous example the results which were obtained were related with a mathematical accuracy which in general can not be achieved in practice. In this case a reasonable question will be: How is the performance of the compensated singular system affected when the values of the two parameters α and β are set approximated with respect to their theoretical values? or what is the sensitivity of the singular dominant root with respect to each or both of the two parameter values? This problem is next analyzed.

G. SENSITIVITY ANALYSIS

1. Introduction

The characteristics of a physical component are subject to change for various reasons. Perfect control systems with stable components which can be built or can maintain their design values under any conditions are difficult to obtain in practice. Take for example such a passive component as a resistor. It is almost impossible to obtain a resistor with the exact design resistance value. For this reason a tolerance usually ranging from

1% up to 30% will accompany its specifications. Even if this "ideal" component is obtainable, factors like temperature variations, radiation etc., may eventually cause the resistance value to vary and produce undesirable results on the control system performance. These general rules are also applied when an active or solid state device is considered.

A measure of the effects of parameter variation upon the performance of a control system has in general been demoted as sensitivity. Basically there are two types of parameter variations studied by various investigators. The first one which is the most popular and the most common in practice is the incremental variation case, where the change in parameter values is assumed to be very small, i.e., infinite-simal. The corresponding sensitivity study in this case is called <<Microscopic Sensitivity Analysis >>. The second case is that of relative large variations in parameter values i.e., finite changes. In this case the corresponding sensitivity study is called << Microscopic Sensitivity Analysis >>.

The definition also of the sensitivity coefficient varies among investigators. Generally the sensitivity coefficient S can be defined as the ratio of the amount by which a control system's performance characteristic deviates from its nominal (or original) value when one or more system parameters vary to its nominal value. From this definition the sensitivity coefficient is mathematically given by:

$$S = \frac{\Delta c}{c} \qquad (6\text{-}77)$$

where C denotes the nominal value of the system characteristic which is studied and Δc the deviation from this value when one or more system parameters which are considered vary.

There have been many sensitivity functions defined for the effects of parameter variations on the transfer functions of linear control systems, some of which are the following:

a. The Kokotovic Method

The root sensitivity functions developed by Kokotovic and Siljak provide most general solution to the sensitivity problem for small parameter variations in linear control systems and the most applicable to parameter plane methods.

b. The F.H. Hollister Method

The macroscopic root sensitivity functions defined by F.H. Hollister are applicable for large parameter variations.

c. Bode Method

Bode defines a logarithmic or normalized sensitivity function of a variable, x_i (t, d_j), with respect to a parameter a_j, as:

$$S_{dj}^{xi} \equiv \frac{\partial \ell n \, a_j}{\partial \ell n \, x_i}$$

In general an analytic investigation of the sensitivity problem of a linear control system is neither simple nor easy. By computing sensitivity coefficients limits of individual parameter variations may be deduced and tolerance levels can be formulated.

2. Sensitivity Analysis of a Singular System

The sensitivity analysis of a singular system can be performed by any method applicable to a linear feedback control system. By such a method the sensitivity of any system performance characteristic due to variations of one or more parameter values can be analyzed. Especially when compensation of a control system is designed by the use of the "singular line theory", then the transient response characteristics and the two variable parameters α and β are of the main interest for the singular system which results, at least for the first steps of the design process. Therefore it will be very important to know how transient response characteristics will be affected and how tolerance levels can be established concerning variations of the two parameter α and β values. In terms of sensitivity this could be stated as: How sensitive are the transient response characteristics of a singular system to variations of the two parameter values and how can the corresponding tolerance levels be determined?

The goal of the present analysis is to give an answer to these questions by a relatively simple and accurate way, in which computer use will be optional.

Generally when designing by the use of singular line theory, specifications on transient response characteristics determine the desired operating singular point (see paragraph VI, F, 6). From this singular point or singular root the equation of the corresponding singular line will be evaluated. Some other specifications or physical limitations of the system may also restrict the permissible operating region on the singular line to a finite line segment. Then when the operating point moves or stays fixed on this singular line segment, performance requirements on transient response characteristics are satisfied. The design process up to this point has been described in detail in the previous part of this section.

It will be assumed now, that the operating point moves away from the singular line due to variations of either one or both of α and β parameter values from their design values. This will be in general the case for an actual system. In such a case the acceptable tolerance on the transient response characteristics will determine how far away from the singular line the operating point is permitted to move.

Since there is at least one singular line for each value of ζ for a singular system it will be expected that for a singular system there is an infinite number of singular lines on the parameter plane, each of which corresponds to a different value of ζ. Therefore according to the tolerances, which are given for the transient response characteristics of the compensated singular system, a permissible region can be established on the parameter plane for the operating point, which will be bounded between two such singular lines. This concept not only was applied in the following example but was also further investigated from a theoretical point of view and it was found that it is true and valid, which will be shown later in the example.

Determination of these two singular lines on the parameter plane will not only give the answer to the previous question, but it will give also a lot of other information concerning sensitivity and performance of a singular system for parameter variations. It will also give a precise way for tolerance values establishment for a singular control system.

Although different specifications or different systems may give rise to different boundaries for the operating point on the parameter plane or different criteria in order to determine these boundaries, the procedure which must be followed will be in general the same. This will be illustrated by the following example.

3. Example (6-7)

Consider again the plant of example (6-6) with transfer function:

$$G(s) = \frac{20}{s(s+1)(s+5)}$$

which is compensated by an initial cascade compensator with transfer function:

$$G_c(s) = \frac{s+2}{s+20}$$

The Root locus of the resulting system which is drawn with the gain of the plant used, as the Root locus variable parameter, is shown in figure (6-51).

It is already known (see paragraph VI, F, 6) that any point of the above Root locus, which does not lie on the real axis is a singular point for the compensated singular system. It will be assumed that the specifications on the transient response characteristics for the compensated singular system require the operating point to be the Root locus point which is located at ($\zeta_s = 0.60$, $\omega_{ns} = 1.23$) and that according to the given tolerance on the transient response characteristics which are interpreted in terms of the second order model, this singular point is permitted to move within the finite Root locus segment which is bounded between the Root locus points:

$$(\zeta_s' = 0.65, \ \omega_{ns}' = 1.03)$$

and

$$(\zeta_s'' = 0.55, \ \omega_{ns}'' = 1.58)$$

These points can be determined by the intersection of the Root locus with the corresponding constant ζ lines (see figure (6-51)) or can be evaluated by using the "Singular point program" in the computer. In the second case, although more precise values can be obtained for ω_{ns}' and ω_{ns}'' it will be seen later in this example that such an accuracy is not actually needed.

Since the above defined Root locus segment has been established by interpreting problem specifications in terms of second order model, will be actually the permissible region for the operating singular root, if and only if, this singular root has a high degree of dominancy anywhere within this region.

By, either studying the Root locus of figure (6-51) or taking into account the results obtained from example (6-6), a basic remark concerning the dominancy of the singular root in the case which is considered is that as ζ_s decreases, the degree of dominancy of the singular root also decreases because it moves away from the origin and from the imaginary axis, and the contribution of the non-singular root (see example (6-6)) becomes more significant. Based on this remark and the need for the given specifications on the transient response characteristics to be met, it was decided the permissible region for the operating singular root locus to be restricted between the points:

$$(\zeta_s = 0.60,\ \omega_{ns} = 1.23)$$

and

$$(\zeta'_s = 0.65,\ \omega'_{ns} = 1.03)$$

which are shown in figure (6-51).

At this point it must be noticed that such a decision is based absolutely on personal judgments the correctness of which can be checked by simulation of the compensated singular system at the limiting acceptable values for the two parameters α and β, which were obtained from this analysis, as it will be shown later in this example.

By introducing a singular cascade compensator, the characteristic equation of the compensated singular system becomes:

$$\begin{aligned}&s^8+52s^7+926s^6+(6700+\alpha)s^5+(20825+28\alpha)s^4+\\&(25000+177\alpha)s^3+(10000+350\alpha+20\beta)s^2+\\&(200\alpha+80\beta)s + 80\beta = 0\end{aligned} \qquad (6\text{-}78)$$

When the singular point ($\zeta_s = 0.60, \omega_{ns} = 1.23$) was considered, the equation of the corresponding singular line was found from equation (2-5) and (2-16) to be:

$$\beta = 3.304\alpha - 218.355 \qquad (6\text{-}79)$$

Comparison of equations (6-79) and (6-73) reveals that the use of the computer for evaluation of the polar coordinates of a singular point and from these the equation of the corresponding singular line to be calculated, does not give any significant improvement on the accuracy of the equation of the singular line, for analysis and design purposes.

When the singular root ($\zeta_s' = 0.65$, $\omega_{ns}' = 1.03$) was considered the equation of the corresponding singular line was found from equation (2-5) and (2-16) to be:

$$\beta = 2.444\alpha - 138.097 \qquad (6\text{-}80)$$

The above two singular lines were drawn on the parameter plane diagram which is shown in figure (6-52). After taking into account the results of example (6-6), the characteristic equation of the compensated singular system, and equations (6-79) and (6-80) of the boundary singular lines, the stable region A B C A B'C'A, and the region D E F G in which the varying singular root has a high degree of dominancy were approximately established on the parameter plane diagram of figure (6-52).

When the operating point lies at the point A where the two singular lines intersect, then the compensated singular system has both the corresponding singular roots as characteristic roots. When the operating point lies within the region D E F G where the varying singular root is dominant with a high degree of dominancy, then the specifications on the transient response characteristics for the compensated singular system are met, i.e:

Peak overshoot	M_p:	From 6.8% up to 12.6%
Settling time	t_s:	From 4.6 sec up to 5.9 sec
Trans. oscil. freq.	ω_t:	From 0.78 rad/sec up to 1.32rad/sec

This was actually verified in the computer by a DSL simulation program of the compensated singular system, in which several pairs of α and β parameter values which define points in the region D E F G of the parameter plane diagram of figure (6-52) were considered. In figure (6-53) through (6-56) the time response of the system for some of these points is shown.

a. Sensitivity Study

Each operating point within the region D E F G of figure (6-52) lies on a singular line (it will be shown later in this example), which corresponds to a singular root (ζ_s, ω_{ns}). These polar coordinates of this singular root can be approximated for any such point from the corresponding values of the boundary singular lines. For example the point M_1 and any other point which lies half way between the boundary singular lines will correspond to a singular root ($\zeta_s = 0.625$, $\omega_{ns} = 1.13$). When such a point, i.e., the point M_1 ($\alpha = 900$, $\beta = 2400$) is considered the corresponding transient response characteristics can be evaluated from the second order model which corresponds to the singular root ($\zeta_s = 0.625$, $\omega_{ns} = 1.13$).

Considering now that due to variations of either one or both of the parameter α and β values the operating point moves to another location within the region D E F G of the parameter plane of figure (6-52), then the corresponding new ζ_s and ω_{ns} values can be approximated and from these the corresponding transient response characteristics. Therefore the sensitivity coefficient which relates the transient response characteristics and the variations of the two parameter α and β values can be evaluated and from these sensitivity studies can be made as it is shown below:

(1) <u>Sensitivity of the Peak Overshoot due to Variations of the Parameter α</u>. Let the operating point move from M_1($\alpha = 900$, $\beta = 2400$) to M_2 ($\alpha = 990$, $\beta = 2400$) due to a +10 % change of the parameter α value. The corresponding singular points were approximated as ($\zeta_s = 0.625$, $\omega_{ns} = 1.13$) and ($\zeta_s' = 0.64$, $\omega_{ns}' = 1.07$)

respectively. The corresponding percent peak overshoots as they can be found from the second order model are $Mp = 8\%$ and $Mp' = 7.3\%$. Therefore according to the definition of the sensitivity coefficient which was given in paragraph one:

$$S_{\alpha}^{Mp} = \frac{Mp' - Mp}{Mp} = \frac{7.3-8}{8} = -0.0875$$

(2) <u>Sensitivity of the Peak Overshoot due to Variations of the Parameter β</u>. Let now the operating point more from $M_1(\alpha = 900, \beta = 2400)$ to $M_3(\alpha = 900, \beta = 2640)$ due to a +10% change of the parameter β value. The corresponding singular points were approximated as ($\zeta_s = 0.625$, $\omega_{ns} = 1.13$) and ($\zeta_s'' = 0.61$, $\omega_{ns}'' = 1.19$) respectively. The corresponding percent overshoot was evaluated from the second order model to be $Mp = 8\%$ and $Mp'' = 8.9\%$. Therefore the sensitivity coefficient which relates the peak overshoot and the +10% variation of the parameter β value from its value at the initial point M_1 will be:

$$S_{\beta}^{Mp} = \frac{Mp'' - Mp}{Mp} = \frac{8.9-8}{8} = 0.1125$$

Comparison of the above evaluated two sensitivity coefficients reveals that when the operating point M_1 is considered, then the peak overshoot of the compensated singular system is more sensitive to variations of the parameter β than it is to variations of the parameter α (almost 1.28 times more sensitive), and that the effect of the variations of these two parameters on the Peak overshoot of the compensated singular system is opposite.

It must be noticed that although the values obtained for the sensitivity coefficients are approximate, the conclusions which can be derived from a relative comparison must be considered as very accurate for analysis and design purposes.

In the same way any sensitivity problem which relates the transient response characteristics of the compensated singular system and variations of either one or both of the parameter α and β values can be studied.

b. Selection of the Operating Point

In general the operating point M of a compensated singular system must be selected where the singular root has the highest degree of dominancy (which results in the best possible approximation of the transient response characteristics of the compensated singular system by the corresponding second order model) and on a point halfway between the boundary singular lines (which will give the maximum tolerance for variations of the two parameters α and β, when it is considered that both can vary simultaneously). Under these considerations the best location for the operating point in the case which was studied in this example seems to be at the point M(α = 1310 , β = 3500), which is shown in figure (6-52). It must be noticed that some other specifications of system's performance, or other limitations may require another location for the operating point M to be determined within the region D E F G.

c. Tolerance Levels Establishment on Parameter α and β Variations

After the operating point M has been chosen, then tolerance levels on the two parameters α and β variations must be established. These tolerance levels must be determined in such a way that the operating point will be restricted to lie within a circle which has center the point M itself and is tangent to the two boundary singular lines and to any other boundary line in the neighborhood of this point. For the case of the example which was studied the required tolerance levels for both the α and β parameters are ±7.5% from the corresponding nominal values of α = 1310 and β = 3500, as it can be evaluated from figure (6-52), i.e., for no more than 7.5% variations towards both directions of each parameter α and β value, the operating point will lie within the circle shown in figure (6-52) whose center is at the point M_1 and the performance of the compensated singular system will be such that the specifications will be met.

4. Comments

In establishing tolerance levels in practice some trade offs can be made by considering each parameter separately since the nature of each is different, i.e., the parameter β is always associated with a gain value while the parameter α is associated either with another gain value or with pole locations (see figure (6-28a,b,c)). If for example, in the previous case, which was studied above, the poles of the singular compensator can be set very accurately, due to the nature of the singular compensator which is going to be used, then its gain β will not be required to be set with a higher than ±15% accuracy, in order for the operating point to be restricted within the desired circle which was mentioned above, as can be seen from figure (6-52).

Another general remark which was derived from figure (6-52) is that as far away from the intersection of the two boundary singular lines the operating point M is located, the less sensitive the compensated singular system becomes to the parameter α and β variations.

The above comments together with the results obtained up to this point from the sensitivity analysis reveal that design of compensation by using singular line theory leads to very accurate setting of tolerance levels for the two parameter α and β values and very easy and accurate way of determining the optimal operating point for the compensated singular system either on the s-plane or on the α-β plane.

Although all the required information in order to make a conclusion about the sensitivity of a compensated singular system transient response characteristics, due to variations of the parameter α and β values, have not been derived, based on the results of the analysis which was performed up to this point, it is believed that in general compensation of a system by using singular line theory may lead to control systems which are not sensitive or at least very sensitive to parameter values variations.

In the previous example it was shown that when the operating point moves within the region D E F G of figure (6-52) then the transient response characteristics of the compensated singular system can be well approximated by the corresponding second order model. Actually there is no question about it, when the operating point lies on a singular line, like the two boundary singular lines D E and G F of the previous mentioned operating region. In the case where the operating point lies between these boundaries, although the results from example (6-7) reveal that the method is still valid, some reasonable questions which may rise, in this case from a theoretical point of view are: Is the compensated system still singular? Do singular conditions exist? Does the operating point still lie on a singular line? or have all the characteristics of the singular conditions been lost? These questions were answered as follows:

When a linear control system with negative unity feedback, whose open loop transfer function is $G(s) = kF_g(s)$ is compensated by a singular cascade compensator, the characteristic equation of the compensated singular system will be:

$$1 + \alpha\, F_g(s) + \beta\, kF_g^{\,2}(s) = 0$$

It has been shown (see paragraph III, C, 3) that such a system has in general an infinite number of singular points, i.e., points of the s-plane which satisfy equation:

$$\angle F_g(s) = n\Pi$$

where n is an real integer or zero. When such a singular point has been specified, the equation of the corresponding singular line can be determined either by direct substitution of the value of the singular point s in the characteristic equation of the compensated singular system or by using equations (2-5) and (2-16). From the equation of this singular line then, an infinite number of α and β pair values corresponding to the singular point under consideration, can be determined. In the opposite case where a pair of α and β values has been specified the equation of the corresponding singular line and corresponding to this equation the singular point can also be determined.

This will be illustrated by considering the following two cases from example (6-7) where the operating point moves within the region D E F G. In the first case the operating point arbitrarily was assumed to lie at the point $M_1(\alpha=900,\ \beta=2400)$ of figure (6-52). Then the characteristic equation (6-78) of the compensated singular system becomes:

$$s^8+52s^7+926s^6+7600s^5+46025s^4+184300s^3+373000s^2$$
$$+372000s+192000 = 0$$

By the use of a computer program with the ZPOLR subroutine the roots of the above equation were found to be:

$s_1 = -\ 0.7044558+j0.8843676$ $\qquad s_2 = -\ 0.7044558-j0.8843676$
$s_3 = -\ 0.8651241+j5.8923703$ $\qquad s_4 = -\ 0.8651241-j5.8923703$
$s_5 = -\ 22.1206845$ $\qquad s_6 = -\ 20.1762447$
$s_7 = -\ 4.4148436$ $\qquad s_8 = -\ 2.1490673$

Each of the above complex roots was checked and it was found that each lies on the Root locus of figure (6-51). Therefore each of these complex roots is also a singular root, (see statement after equation (6-61)). A more general theoretical proof of this statement which is always true, can be derived from paragraph VI, F, 4.

Because the operating region of the parameter plane, where the operating point M_1 lies has been designed as dominant root region only the first singular root, i.e., the singular root $s_{1,2}$, which also was dominant, was further considered. This root is:

$$S_{1,2} = -\ 0.7044558 \pm j\ 0.8843676$$

or

$$(\zeta_s = 0.623, \quad \omega_{ns} = 1.131)$$

When this singular root was considered, from equation (2-5) and (2-16), the equation of the corresponding singular line was found to be:

$$\beta = 2.8472\alpha - 162.1855 \qquad (6\text{-}81)$$

Since the coordinates of M_1, i.e., $\alpha = 900$ and $\beta = 2400$ satisfy the above equation it implies that this point lies on a singular line of a dominant singular root whose coordinates are ($\zeta_s = 0.623$, $\omega_{ns} = 1.131$).

Two points are interesting here. First, the above coordinates of the dominant singular root were approximated from the corresponding values of the two boundary singular lines of figure (6-52), during the sensitivity study of example (6-7), (see paragraph G3 a (1) of this section), and second, when the other singular root was considered, then by the same method, it was found that the operating point M_1 lies also on the singular line corresponding to this root, i.e., finally the operating point M_1 lies on the intersection of two singular lines. Of course under dominant root considerations this last singular line is of no interest.

It was next assumed that the operating point was moved to a new location $M_5(\alpha = 1430, \beta = 4280)$ of the dominant root region D E F G of figure (6-52). Then the characteristic equation of the compensated singular system was evaluated to be:

$$s^8+52s^7+926s^6+8130s^5+60825s^4+278110s^3+596100s^2$$
$$+628400\ s + 342400 = 0$$

By the same method as in the previous case the roots of this equation were found to be:

$s_1 = -\ 0.7257602+j0.9470602$ $\qquad$ $s_2 = -\ 0.7257602-j0.9470602$

$s_3 = -\ 0.3981063+j7.5192493$ $\qquad$ $s_4 = -\ 0.3981063-j7.5192493$

$s_5 = -\ 23.1172585$ $\qquad$ $s_6 = -\ 4.3550711$

$s_7 = -\ 20.1934085$ $\qquad$ $s_8 = -\ 2.0865288$

Each of the above complex roots was also checked and it was found that it lies on the Root locus of figure (6-51). Therefore they are also singular roots. In this case the dominant singular root is:

$$s_{1,2} = -0.7257602 \pm j0.9470602$$

or

$$(\zeta_s = 0.608,\ \omega_{ns} = 1.193)$$

When this singular root was considered, from equation (2-5) and (2-16), the equation of the corresponding singular line was found to be:

$$\beta = 3.1307\alpha - 196.1499 \qquad (6\text{-}82)$$

Since the coordinates of the operating point M_5 which was considered, i.e., $\alpha = 1430$ and $\beta = 4280$, satisfy the above equation, it implies that the operating point M_5 lies on the singular line which is defined by equation (6-82). It must also be noticed that equivalent remarks like these which were stated in the previous case about the other complex root $s_{3,4}$ also were found to apply and in this case.

By the same way the singular lines and singular points corresponding to any point within the region D E F G or to any other point of the parameter plane, can be evaluated.

The conclusion from these results is that independently of where the operating point lies on the α-β plane, singular conditions always exist.

The results from the example which was studied are summarized in Table I of Appendix G.

H. STEADY STATE ACCURACY OF A SINGULAR SYSTEM

1. Introduction

A measure of control system accuracy is usually the steady state error. Related concepts and definitions can be found in reference [2].

When steady state error considerations are made for a singular compensated system shown in figure (6-5), the question which arises is: What is the effect of the singular cascade compensator on the error constant K_x (which is also called the DC gain or zero-frequency gain), of the system which is compensated by a singular cascade compensator?

The goal of the present study is for an answer to be given to this question and for the required design procedure to be established.

2. Steady State Error Analysis of a Singular System

It is known that if a finite error exists in steady state operation of a control system its magnitude is determined by the reciprocal of the error constant K_x.

When a unity feedback control system is considered with open loop transfer function:

$$G(s) = k\,F_g(s) = k\,\frac{(s+z_1)(s+z_2)\,\ldots\,(s+z_m)}{s^N(s+p_1)(s+p_2)\,\ldots\,(s+p_n)}.$$

then the error constant K_x is:

$$K_x = \lim_{s\to o} s^N\,G(s) \tag{6-83}$$

or

$$K_x = \frac{k\,z_1 z_2 \ldots z_m}{p_1 p_2 \ldots p_n} \tag{6-84}$$

When the above considered system is compensated by a singular cascade compensator whose transfer function $G_{sc}(s)$ is given by equation (6-18), then the error constant of the compensated singular system is:

$$K_{sx} = \lim_{s\to o} s^N\,G_{sc}(s)\,G(s) \tag{6-85}$$

or

$$K_{sx} = \frac{k\beta\,z_1 z_2 \ldots z_m}{\alpha\,p_1 p_2 \ldots p_n} = \frac{\beta}{\alpha}\,K_x \tag{6-86}$$

i.e., the error constant of the compensated singular system differs from the error constant of the system before the singular cascade compensator has been introduced by a factor equal to the ratio β/α of the two parameter values. Since an operating singular line does not in general pass through the origin of the α-β plane, the ratio β/α will not in general remain constant when the operating point is moved along a singular line.

When an operating region on the α-β plane is considered, which actually will be the case, i.e., the parameter values will be fixed, then the error constant of the compensated singular system will have also a fixed value.

Since the ratio β/α corresponds to the slope of a straight line through the origin of the α-β plane, it implies that the locus of all the points of the α-β plane with a fixed value of the ratio β/α is a straight line through the origin, and that the locus of all the points of the α-β plane with a specified range of values of the ratio β/α is a region of the α-β plane which is bounded between two such straight lines through the origin.

Therefore interpretation of any steady state accuracy specifications in terms of the β/α ratio value leads to "steady state error boundary lines" establishment on the α-β plane as is shown in the following example.

3. <u>Example (6-8)</u>

In this example it was assumed that steady state accuracy specifications were established for the compensated singular system of the previous example (6-7), and that interpretation of these in terms of the error constant K_{sx} set the following restriction on the β/α ratio value, i.e.:

$$\frac{\beta}{\alpha} > 1 \qquad (6\text{-}87)$$

The corresponding "steady state error boundary line" for this case is the straight line through the origin with slope equals to one. This was drawn on the α-β plane shown in figure (6-52). Then all the points of the α-β plane above this line satisfy steady state accuracy specifications of the problem. Therefore any point of the dominant singular root region D E F G shown in figure (6-52) will also give an acceptable steady state error constant for the compensated singular system.

In the same way, when another system or different specifications are given concerning the steady state accuracy establishment of the corresponding "steady state error boundary lines" or line, on the parameter plane the required information for analysis and design purpose can be derived.

VII. THIRD SINGULAR CASE ANALYSIS

In this section the "third singular case" which was defined in Section IV, was considered for analysis. This singular case corresponds to the compensation structure of figure (4-5) where:

$$G(s) = kF_g(s)$$
$$G_C(s) = \beta s^2 F_g(s) \qquad (7\text{-}1)$$
$$H(s) = \alpha s$$

and the plant $G(s) = kF_g(s)$ has at least two poles at the origin, i.e., the cascade compensator has no zero's at the origin.

The characteristic equation of the compensated singular system is:

$$1+\alpha ks\ F_g(s)+\beta k\ (sF_g(s)^2) = 0$$

A. SYSTEM OUTPUT AND ERROR ANALYSIS

The transfer function of the compensated singular system is:

$$T(s)= \frac{C(s)}{R(s)} = \frac{\beta ks^2 F_g{}^2(s)}{1+\alpha ks\ F_g(s)+\beta ks^2\ F_g{}^2(s)} \qquad (7\text{-}2)$$

and the output and error are respectively;

$$C(s) = R(s)\,T(s) \tag{7-3}$$

and

$$E(s) = R(s)\,\{1 - T(s)\} \tag{7-4}$$

1. <u>Plant Type Two</u>

When the plant is type two, i.e.:

$$G(s) = kF_g(s) = k\frac{(s+z_1)\,(s+z_2)\ldots(s+z_m)}{s^2(s+p_1)\,(s+p_2)\ldots(s+p_n)} \tag{7-5}$$

equations (7-3) and (7-4) become respectively:

$$C(s) = \frac{\beta k \prod_{j=1}^{m}(s+z_j)^2}{s^2\prod_{k=1}^{n}(s+p_k)^2 + \alpha ks\prod_{j=1}^{m}(s+z_j)\prod_{k=1}^{n}(s+p_k) + \beta k\prod_{j=1}^{m}(s+z_j)^2}\,R(s) \tag{7-6}$$

and

$$E(s) = \frac{s^2\prod_{k=1}^{n}(s+p_k)^2 + \alpha ks\prod_{j=1}^{m}(s+z_j)\prod_{k=1}^{n}(s+p_k)}{s^2\prod_{k=1}^{n}(s+p_k)^2 + \alpha ks\prod_{j=1}^{m}(s+z_j)\prod_{k=1}^{n}(s+p_k) + \beta k\prod_{j=1}^{m}(s+z_j)^2}\,R(s) \tag{7-7}$$

When the final value theorem was applied to the above equations the steady state output and error were evaluated assuming that all the poles of the transfer function T(s) lie on the left half of the s-plane.

a. Step Input

When a step input $R(t) = Au(t)$ or $R(s) = A/s$ was considered it was found that:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = A$$

and

$$\underset{t\to\infty}{E(t)} = E_{ss}(t) = 0$$

b. Ramp Input

When a ramp input $R(t) = Btu(t)$ or $R(s) = B/s^2$ was considered it was found that:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \infty$$

and

$$\underset{t\to\infty}{E(t)} = E_{ss}(t) = \frac{B\alpha \prod_{k=1}^{n} p_k}{\beta \prod_{k=1}^{m} z_j}$$

c. Parabolic Input

When a parabolic input $R(t) = Ct^2u(t)$ or $R(s) = 2C/s^3$ was considered it was found that:

$$\underset{t\to\infty}{C(t)} = C_{ss}(t) = \infty$$

and

$$\underset{t\to\infty}{E(t)} = E_{ss}(t) = \infty$$

2. <u>Plant Type Three</u>

In the same way, the steady state output and error value for a type three plant were found to be as follows:

a. Step Input

$$\lim_{t \to \infty} C(t) = C_{ss}(t) = A$$

and

$$\lim_{t \to \infty} E(t) = E_{ss}(t) = 0$$

b. Ramp Input

$$\lim_{t \to \infty} C(t) = C_{ss}(t) = \infty$$

and

$$\lim_{t \to \infty} E(t) = E_{ss}(t) = \frac{B\alpha \prod_{k=1}^{n} p_k}{\beta \prod_{j=1}^{m} z_j}$$

c. Parabolic Input

$$\lim_{t \to \infty} C(t) = C_{ss}(t) = \infty$$

and

$$\lim_{t \to \infty} E(t) = E_{ss}(t) = \infty$$

3. Remarks

The general form of the characteristic equation of the compensated singular system in the case which was studied in this section is:

$$1+\alpha ks\, F_g(s) + \beta k\{s\, F_g(s)\}^2 = 0 \tag{7-8}$$

where the transfer function of the plant is:

$$G(s) = kF_g(s) \tag{7-9}$$

By defining:

$$F(s) = sF_g(s)$$

the characteristic equation (7-8) becomes:

$$1+\alpha kF(s) + \beta k\, F^2(s) = 0 \tag{7-10}$$

which is the characteristic equation of a singular system.

Comparison of equations (7-10) and (6-16) reveals that the design process which has to be followed when compensation is performed by the compensation scheme, which corresponds to the "third singular case" will be in general the same with that corresponding to the "second singular case". The only difference will be that instead of using the Root locus of the uncompensated system as the basic design tool, the Root locus of the uncompensated system with one pole less than the actual number of poles at the origin has to be used. This will be illustrated in the following example.

B. EXAMPLE (7-1)

In this example the plant shown in figure (7-1) was considered with open loop transfer function:

$$G(s) = kF_g(s) = 100 \frac{1}{s^2(s+2)} \quad (7\text{-}11)$$

This plant is unstable and it was assumed that problem specifications dictate for a stable system with a dominant root with polar coordinates:

$\zeta = 0.40$

and

$2<\omega_n<3$

The compensated singular system is shown in figure (7-2). As can be seen the whole design problem consists of finding the appropriate pair of α and β parameter values by which the given specifications can be met. This was performed in three steps which follow:

1. Step One

The first step was to determine the required operating singular root for the compensated singular system.

The Root locus of the uncompensated system with a pole less than the actual number of poles at the origin is shown in figure (7-3), i.e., this is the Root locus of equation:

$$k\frac{1}{s(s+2)} = -1 \qquad (7\text{-}12)$$

By the same reasoning as in the case of the "second singular case" it can be shown that any point of the above Root locus, which does not lie on the real axis is a singular point of the compensated singular system. (See Section VI, F, 4, b). Considering next the specifications the desired singular point was located on the Root locus at the point (ζ = 0.40, ω_n = 2.5).

2. Step Two

The second step was to find the equation of the singular line, corresponding to the selected singular point.

From equation (7-9) the characteristic equation of the compensated singular system was found to be:

$$s^4+4s^3+(4+100\alpha)s^2+200\alpha s+100\beta = 0 \qquad (7\text{-}13)$$

When the singular root (ζ_s= 0.40, ω_{ns} = 2.5) of the above characteristic equation was considered, the equation of the corresponding singular line was found from equation (2-5) and (2-16) to be:

$$\beta = 6.25\alpha-0.390625 \qquad (7\text{-}14)$$

3. Step Three

The third step which was the last one was to find the required operating point of the singular line which was defined by equation (7-14).

Because it was also desired that the design process for this problem be completed without using a computer this step was performed as follows.

Introducing equation (7-14) into (7-13) the characteristic equation of the compensated singular system becomes:

$$s^4+4s^3+(4+100\alpha)s^2+200\alpha s+625\alpha-39.0625 = 0 \qquad (7\text{-}15)$$

Although according to the results which were stated in Section VI, F, 4, the Root locus of the compensated singular system, i.e., the Root locus of equation (7-15), with the parameter α used as the Root locus variable parameter has the shape of the Root locus of equation (7-12) shown in figure (7-3), and therefore the drawing of the Root locus of equation (7-15) is not actually required, it was done in this example for a more detailed illustration.

Equation (7-15) can be written as:

$$\alpha\frac{100s^2+200s+625}{s^4+4s^3+4s^2-39.0625} = -1$$

or after factoring numerator and denominator:

$$\alpha \frac{\{s+(1+j2.291)\}\ \{s+(1-j2.291)\}}{\{s+(1+j2.291)\}\ \{s+(1-j2.291)\}\{s+3.693\}\ \{s-1.693\}} = -1$$

The Root locus of the above equation is shown in figure (7-4). This root locus which was drawn by hand, consists of a fixed point there where the singular root (ζ_s=0.40, ω_{ns}=2.5) or $s=-1\pm j2.291$ is located and the regular Root locus segments.

Comparison of figures (7-3) and (7-4) reveals that the two Root loci have actually the same shape.

*Because the singular root and the complex non-*singular root (See Section VI, F,4,a,(1),(c)), have always equal real parts, in this case, in order for the singular root to be dominant its residue must be much greater than that corresponding to the non-singular root, i.e., the singular root must lie much closer to the origin of the s-plane than the non-singular root. Therefore the non-singular root must lie far away above the singular root which is shown on the Root locus either of figure (7-3) or (7-4).

By arbitrarily selecting the distance of the non-singular root from the origin to be relative very large, in comparison with that which corresponds to the singular root, dominance of the singular root in general shall be expected, e.g., let the non-singular root be:

$$s = -1 \pm j50$$

Then the left part of equation (7-15) must be equal to the product of the singular root and non-singular root. This yields $\alpha = 25.07$. From equation (7-14) the corresponding value of the parameter β, was found to be $\beta = 156.31$.

Therefore the required parameter pair values or the desired operating point of the α-β plane is (α=25.07, β=156.31).

4. Remarks

The compensated singular system shown in figure (7-2) was simulated in the computer by using a DSL program in which the two parameters α and β were set at the above indicated values.

From this program it was found that the time response characteristics of the compensated singular system were exactly the same as these corresponding to the singular root second order model. It is shown in figure (7-5).

It must also be noticed that the desired performance of the compensated singular system can be achieved by an infinite number of α and β parameter pair values, which correspond to a different selection in the location of the non-singular root on the s-plane.

Criterion for the selection of the appropriate location for the non-singular root can be based either on sensitivity or and steady state accuracy considerations as follows:

a. Sensitivity Considerations

As the location of the non-singular root moves away from the axis of reals on the Root locus of figure (7-4) the value of the Root locus variable parameter α increases.

Considering next the parameter α-β plane and the singular line (equation (7-14)), which corresponds to the operating singular root ($\zeta_s = 0.40$, $\omega_{ns} = 2.5$), as the value of the parameter α increases the operating point (α,β) moves away from the origin on the singular line, but according to the results obtained from the sensitivity study in the previous section it yields a less sensitive compensated singular system with respect to variation on the parameter α and β values. These remarks can be used accordingly in each case.

b. Steady State Accuracy Considerations

When the error constant of the compensated singular system was considered it was found to be:

$$K_{sx} = \frac{\beta}{2\alpha}$$

When problem specifications concerning the error constant K_{sx} are also established, then an additional criterion for the selection of the location of the non-singular root will be the desired value of the parameter α for which the ratio β/α is such that the optimal error constant coefficient can be achieved.

In general the ratio β/α can be effected either by selecting a different location for the non-singular root or by selecting a different operating singular point, i.e., operating on a different singular line or by introducing additional gain adjustments at the forward path of the minor loop of figure (7-2).

It must be noticed also that any other specifications can be taken under consideration in each design step of the previous illustrated example.

VIII. EXTENSIONS OF THE SECOND SINGULAR CASE COMPENSATION STRUCTURE

A. INTRODUCTION

Among the three singular cases, which were previously analyzed, the most interesting for applications in design of compensation of a linear feedback system by the dominant mode method, appears to be the "second singular case" (See Section VI).

The basic advantages of this case are that the plant type number is preserved, it can be used for any type of plant and for any plant structure.

On the other hand the basic limitation of the "first singular case" (See Section V) is that in general the type number of the plant is not preserved, and for the "third singular case" (See Section VII) is that the plant has to be at least type two.

For the above reasons, the "second singular case" was further considered and some extensions of its compensation structure were studied.

B. SECOND SINGULAR CASE COMPENSATION STRUCTURES

1. The Basic Structure

The basic compensation structure of the "second singular case" is shown in figure (6-4) where:

$$G(s) = kF_g(s) = k\frac{(s+z_1)\ (s+z_2)\ldots(s+z_m)}{s^N(s+p_1)\ (s+p_2)\ldots(s+p_n)} \tag{8-1}$$

represents the transfer function of the plant to be compensated.

$$H(s) = \alpha \tag{8-2}$$

represents a gain adjustment, introduced as a negative feedback from the output to the input of the plant, and

$$G_c(s) = \beta F_g(s) = \beta\frac{(s+z_1)\ (s+z_2)\ldots(s+z_m)}{s^N(s+p_1)\ (s+p_2)\ldots(s+p_n)} \tag{8-3}$$

represents the transfer function of a cascade compensator whose poles and zeros are identical with these of the plant and its Root locus gain is the second adjustment of the compensation structure.

This compensation structure defined above, corresponds to a singular system which was studied in Section VI as the "second singular case".

Its transfer function has the general form:

$$T_B(s) = \frac{C(s)}{R(s)} = \frac{\beta k \prod_{j=1}^{m} (s+z_j)^2}{s^{2N} \prod_{k=1}^{n} (s+p_K)^2 + \alpha k s^N \prod_{k=1}^{n} (s+p_K) \prod_{j=1}^{m} (s+z_j) + \beta k \prod_{j=1}^{m} (s+z_j)^2}$$

(8-4)

The error constant K_{sx-B} was evaluated and it was found to be:

$$K_{sx-B} = \frac{\beta}{\alpha} \frac{\prod_{j=1}^{m} z_j}{\prod_{k=1}^{n} p_K} \qquad (8\text{-}5)$$

The characteristic equation also of this basic singular compensation structure was found to be in the general form:

$$1+\alpha k F_g(s)+\beta k F_g^{\,2}(s) = 0 \qquad (8\text{-}6)$$

From this basic structure six modified singular compensation structures were derived. They were analyzed as follows:

2. Modified Structures

a. First Modified Structure

The first modified singular compensation structure is shown in figure (8-1). It was derived from the previous basic structure and has been analyzed in Section VI.

In figure (8-1) G(s) denotes the transfer function of the plant to be compensated and $G_{sc}(s)$ the transfer function of the singular cascade compensator, which according to equation (6-18) its transfer function is:

$$G_{sc}(s) = \frac{\beta \prod_{j=1}^{m} (s+z_j)}{s^N \prod_{k=1}^{n} (s+p_\kappa) + \alpha \prod_{j=1}^{m} (s+z_j)} \tag{8-7}$$

where s^N, p_k and z_j, represents the poles at the origin, the other poles and the zeros of the plant respectively.

The transfer function of this compensation structure was found to be:

$$T_1(s) = \frac{C(s)}{R(s)} = \frac{\beta k \prod_{j=1}^{m} (s+z_j)^2}{s^{2N} \prod_{k=1}^{n} (s+p_\kappa)^2 + \alpha s^N \prod_{k=1}^{n} (s+p_\kappa) \prod_{j=1}^{m} (s+z_j) + \beta k \prod_{j=1}^{m} (s+z_j)^2} \tag{8-8}$$

The error constant K_{sx-1} was also found (See equation (6-85)) to be:

$$K_{sx-1} = \frac{\beta}{\alpha} \frac{k \prod_{j=1}^{m} z_j}{\prod_{k=1}^{n} p_\kappa} = \frac{\beta}{\alpha} K_\chi \tag{8-9}$$

where K_x denotes the error constant of the plant, before it was compensated by the singular cascade compensator.

The characteristic equation of this compensation structure will be in the general form:

$$1+\alpha F_g(s)+\beta k F_g^2(s) = 0 \tag{8-10}$$

The basic advantages of this modified singular compensation structure with respect to the previous basic one, are the following two:

(1) Simpler Structure. The compensation structure is simpler because only a single cascade filter is required. Historically it is realized that series compensation has played a prominent role in system design.

(2) Greater Error Constant. Comparison of equations (8-5) and (8-9), reveals that the error constant of this structure is greater than the error constant of the previous basic one by a factor equal to the Root locus gain k of the plant.

b. Second Modified Structure

The second modified singular compensation structure is shown in figure (8-2). It was derived from the previous one by changing the location of the singular compensator from the forward path at the feedback path. Of course this modified singular structure is no longer a unity feedback control system.

For this compensation structure the transfer function was evaluated and it was found to be:

$$T_2(s) = \frac{C(s)}{R(s)} = \frac{ks^N \prod_{k=1}^{n}(s+p_K) \prod_{j=1}^{m}(s+z_j) + \alpha k \prod_{j=1}^{m}(s+z_j)^2}{s^{2N} \prod_{k=1}^{n}(s+p_K)^2 + \alpha s^N \prod_{k=1}^{n}(s+p_K) \prod_{j=1}^{m}(s+z_j) + \beta k \prod_{j=1}^{m}(s+z_j)^2} \tag{8-11}$$

The characteristic equation also of this compensation singular structure was found to be:

$$1+G(s)\, G'(s) = 0$$

or

$$1+\alpha F_g(s)+\beta k\, F_g^2(s) = 0 \tag{8-12}$$

In order for steady state accuracy to be investigated the following considerations were made:

The error of the system is:

$$E=R-C=(1-T_2)R \tag{8-13}$$

Introducing equation (8-11) into equation (8-13) yields:

$$E(s) = \frac{s^{2N}\prod_{k=1}^{n}(s+p_{\kappa})^{2} + (\alpha-k)s^{N}\prod_{k=1}^{n}(s+p_{\kappa})\prod_{j=1}^{m}(s+z_{j}) + (\beta k-\alpha k)\prod_{j=1}^{m}(s+z_{j})^{2}}{s^{2N}\prod_{k=1}^{n}(s+p_{\kappa})^{2} + \alpha s^{N}\prod_{k=1}^{n}(s+p_{\kappa})\prod_{j=1}^{m}(s+z_{j}) + \beta k\prod_{j=1}^{m}(s+z_{j})^{2}}\,R(s) \qquad (8\text{-}14)$$

When a type one plant was considered and a ramp input, i.e., $N=1$ and $R(s)=B/s^{2}$, and when the final value theorem was applied to equation (8-14), assuming a stable system, the steady state error was evaluated as:

$$\underset{t\to\propto}{E(t)} = E_{ss}(t) = \propto$$

For a physical interpretation of the above result the following considerations were also made on the compensation structure which is shown in figure (8-2).

For a type one plant, and a step input, at steady state $\acute{E}$ has to be zero, i.e., $\acute{E}_{ss} = 0$. Because $\acute{E} = R - \acute{C}$, it implies that $R_{ss} = \acute{C}_{ss}$. Since C_{ss} is a constant and the system is not a unity feedback system $C_{ss} \neq \acute{C}_{ss}$. Therefore $C_{ss} \neq R$.

When a type one plant was again considered with a ramp input, at steady state $\acute{E}$ has to be constant, i.e., E'_{ss}= Constant. Because $E' = R - C'$ this implies that $\dot{R} = \dot{C}'_{ss}$. Since C_{ss} is a ramp and the system is not a unity feedback system, $\dot{C}_{ss} \neq \dot{C}'_{ss}$. Therefore $\dot{C}_{ss} = \dot{R}$, which implies that an infinite error E_{ss} must be expected. This result was also derived previously by analytical study of equation (8-14).

Based on these remarks and results, the conclusion is that such a system does not reproduce the input signal at the output or in other words at steady state the output does not follow the input, which may be an advantage or a disadvantage depending on the specific application, and which can be eliminated as will be seen by the following modified singular compensation structure.

When a type one plant was considered again and a step input applied, the steady state error was found to be:

$$E_{ss} = \left(1 - \frac{\alpha}{\beta}\right) A$$

where A is the value of the step input, i.e., at steady state the output differs from the input by the value of E_{ss}.

In order for all the above mentioned problems to be eliminated when this is desired, the following considerations were made on this modified singular structure.

When R was considered as the input signal which the output of the system was desired to reproduce, i.e., to follow, then the error will be given by equation:

$$E = R - C \tag{8-15}$$

By considering also an input which is not followed by the output to be R', then according to the notations used:

$$C = T_2 R' \tag{8-16}$$

Introducing equation (8-16) into equation (8-15) yields:

$$E = R - T_2 R' \tag{8-17}$$

By assuming that the actual input to the system R' is correlated to the desired input R by a function Y, i.e:

$$R' = YR \tag{8-18}$$

then equation (8-17) becomes:

$$E = R - T_2\, YR = (1 - T_2 Y)\, R \tag{8-19}$$

Introducing equation (8-11) into equation (8-19) yields:

$$E(s) = \frac{s^{2N}\prod_{k=1}^{n}(s+p_k)^2 + (\alpha - kY)s^N \prod_{k=1}^{n}(s+p_k)\prod_{j=1}^{m}(s+z_j) + (\beta k - \alpha k Y)\prod_{j=1}^{m}(s+z_j)^2}{s^{2N}\prod_{k=1}^{n}(s+p_k)^2 + \alpha s^N \prod_{k=1}^{n}(s+p_k)\prod_{j=1}^{m}(s+z_j) + \beta k \prod_{j=1}^{m}(s+z_j)^2}\, R(s)$$

(8-20)

In order for the steady state error to be finite for a ramp input to a type one plant or zero for a step input, Y has to be defined as:

$$Y \equiv \frac{\beta}{\alpha} \tag{8-21}$$

Introducing equation (8-21) into equation (8-18) yields:

$$R' = \frac{\beta}{\alpha} R \tag{8-22}$$

After the above result was obtained a third modified structure was derived and is analyzed as follows:

c. Third Modified Structure

The third modified singular compensation structure is shown in figure (8-3), where R is the input to the system which is to be followed at steady state by the output of the system.

The transfer function of this modified structure was found to be:

$$T_3(s) = \frac{C(s)}{R(s)} = \frac{\beta}{\alpha} \frac{k s^N \prod_{k=1}^{n}(s+p_K) \prod_{j=1}^{m}(s+z_j) + \alpha k \prod_{j=1}^{m}(s+z_j)^2}{s^{2N} \prod_{k=1}^{n}(s+p_K)^2 + \alpha s^N \prod_{k=1}^{n}(s+p_K) \prod_{j=1}^{m}(s+z_j) + \beta k \prod_{j=1}^{m}(s+z_j)^2}$$

(8-23)

The characteristic equation was also found to be in the general form:

$$1+\alpha F_g(s)+\beta k F_g^2(s) = 0 \qquad (8\text{-}24)$$

By the same procedure as in the case of the first modified structure, the error constant K_{sx-3} for this case was found to be:

$$K_{sx-3} = \frac{\alpha\beta}{\alpha^2-\beta k} k \frac{\prod_{j=1}^{m} z_j}{\prod_{k=1}^{n} p_k} = \frac{\alpha\beta}{\alpha^2-\beta k} K_x \qquad (8\text{-}25)$$

where K_x denotes the error constant of the plant before it was compensated. For the examples which were worked out in Section VI, the ratio $\alpha\beta/\{\alpha^2-\beta k\}$ turns to be greater than unity. Therefore, when such is the case the error constant of the uncompensated plant will be improved when the plant is compensated.

When this compensation structure was compared with the first modified structure it was found that they have the same characteristic equation but different error constant and transfer function. Therefore the stable region on the α-β plane will be the same in both cases. Also for the same operating point (α,β), although they will have the same characteristic roots, the residues of these roots will be different in each case due to the different numerator of their transfer functions. Since the dominancy of a characteristic root is a function of its residue, in addition to the distance of the location of this root on the s-plane from the imaginary axis, a different degree of dominancy for the operating singular root must be expected in general in each case.

For a specific plant and a specific operating point on the α-β plane a further investigation of the dominancy of the operating singular root can be performed either by a computer simulation of each compensation structure and comparing their time response characteristics with those corresponding to the singular root second order model or by analytical evaluation of the residues corresponding to the characteristic roots of each case.

d. Fourth Modified Structure

The fourth modified singular structure is shown in figure (8-4). It was derived from the first one by properly defining the feedback compensator in order for these two compensation structures to have the same open loop transfer function, i.e.:

$$G(s)\,G_{sc}(s) = \frac{G(s)}{1+G(s)\,G_{sc}'(s)} \qquad (8\text{-}26)$$

from which the transfer function of the filter $G'_{sc}(s)$ was evaluated to be:

$$G'_{sc}(s) = \frac{s^N \prod_{k=1}^{n}(s+p_k)\{s^N \prod_{k=1}^{n}(s+p_k)+\alpha \prod_{j=1}^{m}(s+z_j)-\beta \prod_{j=1}^{m}(s+z_j)\}}{\beta k \prod_{j=1}^{m}(s+z_j)^2} \tag{8-27}$$

Because G(s) defined by equation (8-1) denotes the transfer function of a physical control system, it implies that $N+n>m$, i.e., the above defined filter has more zeros than poles. In practice a structure of this type is realized by introducing additional poles as needed, but placing them well outside the bandwidth of the system. Whether this can be done without affecting the singularity characteristics has not been investigated. Based on the argument requirement for singular conditions (See equation (3-12 and (10-11)), it is believed that if these additional poles are placed on the s-plane in such locations that their net contribution to the argument of $G'_{sc}(s)$ is negligibly small, then it can be assumed that the singularity characteristics of the system remain unaffected.

The transfer function of this structure was found to be:

$$T_4(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)+G(s)\ G'_{sc}(s)} \tag{8-28}$$

or after some manipulations:

$$T_4(s) = \frac{C(s)}{R(s)} = \frac{\beta k \prod_{j=1}^{m} (s+z_j)^2}{s^{2N} \prod_{k=1}^{n} (s+p_k)^2 + \alpha s^N \prod_{k=1}^{n} (s+p_k) \prod_{j=1}^{m} (s+z_j) + \beta k \prod_{j=1}^{m} (s+z_j)^2} \tag{8-29}$$

The characteristic equation of this compensation structure was found to be in the general form:

$$1+\alpha F_g(s)+\beta k\ F_g^2(s) = 0 \tag{8-30}$$

By the same procedure also, as in the case of the first modified structure, the error constant K_{sx-4} for this case was found to be:

$$K_{sx-4} = \frac{\beta}{\alpha} K_x \tag{8-31}$$

Comparison of the results which were obtained for the first modified compensation structure with these obtained above, reveals that the performance of the system in both cases is the same.

e. Fifth Modified Structure

The fifth modified singular compensation structure is shown in figure (8-5). It was derived from the second one shown in figure (8-2) by splitting the feedback path into two parallel paths as shown, such that their sum is the compensator transfer function:

$$G_{sc}(s) \equiv 1+G''_{sc}(s) \tag{8-32}$$

from which the transfer function of the minor loop feedback filter was found to be:

$$G''_{sc}(s) = \frac{(\beta-\alpha)\prod_{j=1}^{m}(s+z_j)-s^N\prod_{k=1}^{n}(s+p_k)}{s^N\prod_{k=1}^{n}(s+p_k)+\alpha\prod_{j=1}^{m}(s+z_j)} \tag{8-33}$$

The transfer function of this compensation structure is:

$$T_s(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)+G(s)\,G''_{sc}(s)}$$

or after some manipulations it becomes:

$$T_s(s) = \frac{C(s)}{R(s)} = \frac{ks^N \prod_{k=1}^{n}(s+p_k) \prod_{j=1}^{m}(s+z_j) + \alpha k \prod_{j=1}^{m}(s+z_j)^2}{s^{2N} \prod_{k=1}^{n}(s+p_k)^2 + \alpha s^N \prod_{k=1}^{n}(s+p_k) \prod_{j=1}^{m}(s+z_j) + \beta k \prod_{j=1}^{m}(s+z_j)^2} \tag{8-34}$$

The characteristic equation of this compensation structure was found to be in the general form:

$$1+\alpha F_g(s) + \beta k F_g^2(s) = 0 \tag{8-35}$$

Also when steady state accuracy considerations were made for this compensation structure the results were the same as these for the second modified structure. Comparison of the results corresponding to these two cases reveals that they are equivalent under performance considerations.

f. Sixth Modified Structure

This is shown in figure (8-6), and it was derived from the third modified structure with the same reasoning by which the fifth modified structure was derived from the second one.

It was also found that the third and sixth modified structures have the same transfer function, characteristic equation, error constant and thus under system performance considerations they are equivalent.

3. Remarks

Although the "second singular case" can be used for compensation of any control system, i.e., it may have general application, its "basic structure" has certain disadvantages, which have been stated in the previous analysis and which may not be acceptable in some cases.

When the characteristic equation of the basic structure is considered, it will have the general form:

$$1+\alpha kF_g(s)+\beta kF_g{}^2(s) = 0$$

On the other hand the six modified structures, which were analyzed above, have certain advantages over the "basic structure" and a common characteristic, i.e., they have the same characteristic equation, which has the general form:

$$1+\alpha F_g(s)+\beta kF_g{}^2(s) = 0$$

Also their performance characteristics can be mathematically correlated.

This fact aids the designer in making his decision on the problem of the structure without additional design effort and protects from arbitrary choices.

In the following example all these modified singular compensation structures were used to compensate the same plant in order that the derived analytical results could be compared. The compensated system performance was also studied in each of these cases.

4. Example (8-1)

In this example the plant of example (6-7) was considered for which the transfer function G(s) of the system before any singular filter has been introduced is:

$$G(s) = \frac{20(s+2)}{s(s+1)\ (s+5)\ (s+20)} \tag{8-36}$$

Also as operating point from the dominant root region of the α-β plane corresponding to the singular root (ζ_s=0.60, ω_{ns}=1.23), the point (α=920, β=2500) was arbitrarily selected. Then each of the six modified singular compensation structures was considered separately and the performance of each for a step input, R=1 Volt and a ramp input R=0.1 rad/sec, was studied in the computer as follows:

a. First Modified Structure

This is shown in figure (8-1). According to equations (8-7) and (8-36) the transfer function of the singular cascade compensator $G_{sc}(s)$ was found to be:

$$G_{sc}(s) = \frac{2500\ (s+2)}{s(s+1)\ (s+5)\ (s+20)+920\ (s+2)}$$

By simulation of this system in the computer with the use of a DSL program the time response characteristics were obtained for a step and a ramp input. They are shown in figure (8-1A) and (8-1B) respectively.

The characteristics of the transient response corresponding to the singular root (ζ_s=0.60, ω_{ns}=1.23) were evaluated and they were found to be:

Time to peak of first overshoot: t_p = 3.19 sec
Settling time : t_s = 5.41 sec
Percent overshoot : Mp = 9.48%
Transient oscillatory frequency: ω_t = 0.99 rad/sec

When the step input was considered then it was found from the computer print output that:

Steady state output value : C_{ss} = R = 1 Volt
Time to peak of first overshoot: t_p = 3.2 sec
Settling time : t_s = 5.4 sec
Percent overshoot : Mp = 11%
Transient oscillatory frequency: ω_t = 0.98 rad/sec

When the ramp input was considered then it was found also from the computer print output that:

Steady state output value : $C_{ss} = R - 0.002$

Steady state output rate : $\dot{C}_{ss} = \dot{R} = 0.1$

Settling time : ts = 5.4 sec

These results can also be derived approximately from the corresponding figures of the time response characteristics, figure (8-1A,B).

b. Second Modified Structure

When the singular compensator $G_{sc}(s)$ was located at the feedback path as shown in figure (8-2), the transient response characteristics which were obtained are shown in figure (8-2A) and (8-2B) for a step and a ramp input respectively.

The exact transient and other characteristic values which were read from the computer print output have as follows:

(1) <u>Step Input</u>.

Steady state output value : $C_{ss} = 0.368 = \frac{\alpha}{\beta} R$

Time to peak of first overshoot : $t_p = 3.2$ sec

Settling time : $t_s = 5.4$ sec

Percent overshoot from C_{ss} : $Mp = 10.2\%$

Transient oscillatory frequency : $\omega_t = 0.92$ rad/sec

(2) Ramp Input.

Steady state output value : C_{ss} followed a ramp different than the input ramp.

Steady state output rate : $\dot{C}_{ss} = 0.036 = \frac{\alpha}{\beta} \dot{R}$

Settling time : $t_s = 5.4$ sec

c. Third Modified Structure

When a gain adjustment equals with 2.717, which corresponds to the ratio β/α value, i.e:

$$\frac{\beta}{\alpha} = \frac{2500}{920} = 2.717$$

was introduced at the indicated in figure (8-3) location the third modified singular structure was obtained.

The transient response characteristics which were obtained in this case for the same step and ramp input as in the previous cases, are shown in figure (8-3A) and (8-3B) respectively.

The exact transient and other characteristic values which were read from the computer print output have as follows:

(1) Step Input.

Steady state output value : $C_{ss} = 1 = R$

Time to peak of first overshoot : $t_p = 3.2$ sec

Settling time : $t_s = 5.4$ sec

Percent overshoot : $M_p = 10.2\%$

Transient oscillatory frequency : $\omega_t = 0.92$ rad/sec

(2) Ramp Input.

Steady state output value : $C_{ss} = R - 0.002$

Steady state output rate : $\dot{C}_{ss} = 0.1 = \dot{R}$

Settling time : $t_s = 5.4$ sec

d. Fourth Modified Structure

The fourth modified structure which is shown in figure (8-4) requires a feedback filter $G'_{sc}(s)$. After the transfer function $G'_{sc}(s)$ was evaluated from equations (8-27) and (8-36) the system was simulated in the computer. The transient response characteristics for a step R=1 Volt and a ramp R=0.1 rad/sec input, which were obtained in this case were the same with these corresponding to the first modified structure. They are shown in figure (8-4A) and (8-4B) respectively.

e. Fifth Modified Structure

This modified structure involves a different feedback singular filter $G''_{sc}(s)$ whose transfer function was evaluated from equations (8-33) and (8-36). The transient response characteristics for a step R=1 Volt and a ramp R=0.1 rad/sec input, which were obtained when this structure was simulated in the computer are shown in figure (8-5A) and (8-5B) respectively, which are the same with these corresponding to the second modified structure.

f. Sixth Modified Structure

This is shown in figure (8-6) and was implemented from the previous one by introducing a gain adjustment as it is indicated in figure.

Simulation of this system in the computer for a step R=1 Volt and a ramp R=0.1 rad/sec input, gave the same transient and steady state characteristics as in the case corresponding to the third modified structure. They are also shown in figures (8-6A) and (8-6B) respectively for each input which was considered.

5. Comment

The above presented results which were obtained from the computer not only confirm the corresponding analytical results but also make clear the flexibility which the designer has when compensation is performed by using singular line theory.

IX. DESIGN OF COMPENSATION BY USING A SINGULAR CASCADE COMPENSATOR

A. INTRODUCTION

Consider the plant which is shown in figure (4-1), and that this plant is going to be compensated by a singular cascade compensator. The open loop transfer function of this plant is:

$$G(s) = kF_g(s) = k\frac{(s+z_1)\ (s+z_2)\ldots(s+z_m)}{s^N(s+p_1)\ (s+p_2)\ldots(s+p_n)} \tag{9-1}$$

It was assumed that the above transfer function is known and that the problem's specifications have been interpreted in terms of the desired dominant characteristic root (ζ, ω_n) for the compensated singular system. It was also assumed that tolerance on the polar coordinates ζ and ω_n values for this root have been established from the problem's specifications.

The compensated singular system is shown in figure (6-6) where $G_{sc}(s)$ denotes the transfer function of the required singular cascade compensator, whose standard form is:

$$G_{sc}(s) = \frac{\beta(s+z_1)\ (s+z_2)\ldots(s+z_m)}{s^N(s+p_1)\ (s+p_2)\ldots(s+p_n)+\alpha(s+z_1)\ (s+z_2\ldots(s+z_m)} \tag{9-2}$$

as was derived in Section VI C.

Therefore the design problem in general will be limited to the problem of determining the required parameter pair values. For different types of applications this problem can be stated in terms of the parameter plane as: What must be the operating point, or the operating line segment, or the operating region of the parameter plane in order that the performance of the compensated singular system be as desired?

The required design procedure in all these cases has been developed in Section VI. In this section the design process in the case where an operating point is desired to be defined on the parameter plane and from this the precise form of the transfer function (9-2), was formulated in a sequence of basic design steps which must be followed in a case.

B. DESIGN PROCESS

The required design process in general when an acceptable operating point of the parameter plane is desired to be found, is relatively simple and can usually be completed with satisfactory accuracy for most applications with graphical and mathematical techniques, by the following four basic design steps.

1. First Step

The first step consists of drawing the Root locus of equation:

$$kF_g(s) = -1 \tag{9-3}$$

2. Second Step

The second step consists of insertion of an initial cascade compensator, when the shape of the above Root locus is such that either an acceptable operating singular point cannot be located on this Root locus or although it can be located it is not acceptable under dominancy considerations (See VI, F,2).

When an initial cascade compensator is required then this requirement will actually affect only the transfer function of the singular cascade compensator and neither the complexity of the rest of the design process nor the compensation structure (See example (6-6) and figure (6-28)).

In the simple case where such an initial compensator will not be required then this step will consist of determining an acceptable operating singular point on the Root locus of equation (9-3).

3. Third Step

The third step consists of determining the equation of the singular line corresponding to the selected operating singular point (See VI D3a equation (6-29)).

4. Fourth Step

This step which is the last one consists of determining the required operating point of the parameter plane, i.e., the required pair of α and β parameter values (See examples (6-6), (6-7), (6-8)).

X. EXTENSION OF THE SINGULAR LINE THEORY TO PARAMETER SPACE

A. INTRODUCTION

The parameter space is an extension of the parameter plane when there are three or more parameters.

Considering the characteristic equation:

$$F(s) = \sum_{k=0}^{n} a_k s^k = 0 \qquad (10\text{-}1)$$

where n is a real integer and a_k is a real valued coefficient which may be a function of three or more parameters, i.e:

$$a_k = g(\alpha, \beta, \gamma, \delta, \ldots n^{th}) \qquad (10\text{-}2)$$

where n^{th} denotes the n^{th} parameter.

The point s can be defined in polar coordinates as:

$$s = -\zeta_{\omega n} \pm j\, \omega_n \sqrt{1-\zeta^2} \qquad (10\text{-}3)$$

B. SINGULAR CONDITIONS

If for a given pair ζ_s, ω_{ns} values, where $|\zeta_s|<1$ and $\omega_{ns}>0$, equation (10-1) has an infinite number of real valued sets of $\alpha, \beta, \gamma, \delta$, ... n^{th} parameter values which satisfy it, then for this equation singular conditions exist. In this case the point (ζ_s, ω_{ns}) is a singular point. This equation also for which singular conditions exist is called singular.

C. SINGULAR CONDITIONS ANALYSIS IN THE LINEAR CASE

A simple case which was analyzed is the linear case, in which the variable parameters appear in the coefficients of the characteristic polynomial in the general form:

$$a_k = b_k\,\alpha + c_k\,\beta + d_k\,\gamma + e_k\,\delta + \dots + \dots$$

$$+\{\text{Coefficient of the } n^{th} \text{ parameters}\}_k\{n^{th} \text{ parameter}\} +$$

$$+\{\text{Constant term}\}_k$$

where b_k, c_k, ... are real constants.

In this case equation (10-1) can be rearranged in the form:

$$f_1(s)\alpha + f_2(s)\,\beta + f_3(s)\,\gamma + \dots + f_n(s)\{n^{th} \text{ parameter}\} +$$

$$+ f_{n+1}(s) = 0 \qquad (10\text{-}4)$$

where $f_1(s)$, $f_2(s)$, ... are polynomials of s. By analogy as in the case of the parameter plane (See Section IIB2) equation (10-4) can be written as:

$$r_1 e^{j\theta_1}\,\alpha + r_2 e^{j\theta_2}\,\beta + r_3 e^{j\theta_3}\,\gamma + \dots + r_n e^{j\theta_n}\{n^{th} \text{ parameter}\} + r_{n+1} e^{j\theta_{n+1}} = 0$$

or

$$r_1(\cos\theta_1+j\sin\theta_1)\alpha+r_2(\cos\theta_2+j\sin\theta_2)\beta+\ldots+\ldots$$

$$+r_n(\cos\theta_n+j\sin\theta_n)\{n^{th}\text{ parameter}\}+r_{n+1}(\cos\theta_{n+1}+j\sin\theta_{n+1})=0$$

Equating then real and imaginary parts to zero, yields:

$$r_1\sin\theta_1\ \alpha+r_2\ \sin\theta_2\ \beta+\ldots+r_n\sin\theta_n\{n^{th}\}+r_{n+1}\sin\theta_{n+1}=0$$

and

$$r_1\cos\theta_1\ \alpha+r_2\cos\theta_2\ \beta+\ldots+r_n\cos\theta_n\{n^{th}\}+r_{n+1}\cos\theta_{n+1}=0$$

According to [1] the above system has an infinite number of solutions if and only if:

$$\frac{r_1\sin\theta_1}{r_1\cos\theta_1}=\frac{r_2\sin\theta_2}{r_2\cos\theta_2}=\frac{r_3\sin\theta_3}{r_3\cos\theta_3}=\ldots=\frac{r_n\sin\theta_n}{r_n\cos\theta_n}=\frac{r_{n+1}\sin\theta_{n+1}}{r_{n+1}\cos\theta_{n+1}}$$

or

$$\tan\theta_1=\tan\theta_2=\tan\theta_3=\ldots=\tan\theta_n=\tan\theta_{n+1} \qquad (10\text{-}5)$$

Equation (10-5) implies that for the characteristic equation (10-1) or (10-4) singular conditions exist if and only if:

$$\angle f_1(s) = \angle f_{n+1}(s) + K_1 \pi$$

$$\angle f_2(s) = \angle f_{n+1}(s) + K_2 \pi$$

$$\angle f_3(s) = \angle f_{n+1}(s) + K_3 \pi \qquad (10\text{-}6)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\angle f_n(s) = \angle f_{n+1}(s) + K_n \pi$$

Assuming that $f_{n+1}(s) \neq 0$ and dividing equation (10-4) by $f_{n+1}(s)$ yields:

$$\frac{f_1(s)}{f_{n+1}(s)} \alpha + \frac{f_2(s)}{f_{n+1}(s)} \beta + \frac{f_3(s)}{f_{n+1}(s)} \gamma + \cdots + \frac{f_n(s)}{f_{n+1}(s)} n^{th} + 1 = 0$$

(10-7)

or

$$F_1(s)\ \alpha + F_2(s)\ \beta + F_3(s)\ \gamma + \ldots + F_n(s)\ n^{th} + 1 = 0$$

(10-8)

When the characteristic equation is considered in the form of equation (10-8) then the necessary and sufficient conditions for singular conditions given by equation (10-6) become:

$$\angle F_1(s) = K_1\pi$$
$$\angle F_2(s) = K_2\pi \qquad (10\text{-}9)$$
$$\angle F_3(s) = K_3\pi$$
$$\cdots\cdots\cdots\cdots$$
$$\angle F_n(s) = K_n\pi$$

where any of the K_1, K_2,...K_n is real integers.

D. SINGULAR THEORY AND CONTROL SYSTEMS

From equations (10-8) and (10-9) and by the same procedure which was used in Section III for the case of the two parameters, it was found that in this case of the parameter space where there are n parameters in the characteristic equation of a control system the general form of a singular characteristic equation is:

$$\alpha C_1 F^{m_1}(s) + \beta C_2 F^{m_2}(s) + \gamma\, C_3 F^{m_3}(s) + \ldots + \{n^{th}\} C_n F_n^{\,m_n}(s) + C_{n+1} = 0 \qquad (10\text{-}10)$$

where C_1, C_2, C_3,..., C_n, C_{n+1} are real constants and m_1, m_2, m_3,..., m_n are n different real integers. (Otherwise when there are two or more of these identical then the number of the actual parameters can be reduced by combining terms of equal powers of F(s)).

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SINGULAR LINE THEORY AND CONTROL SYSTEMS,(U)
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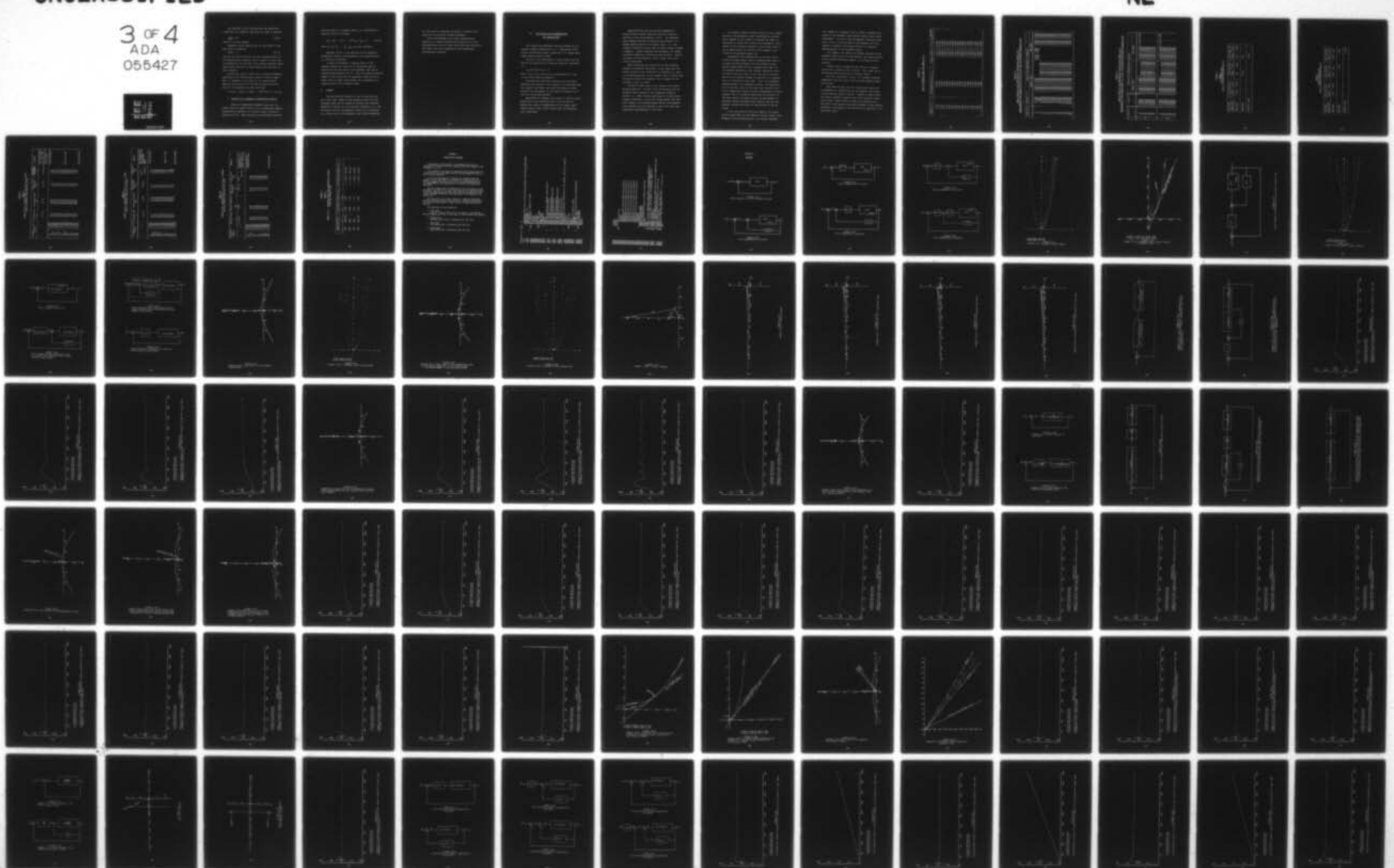

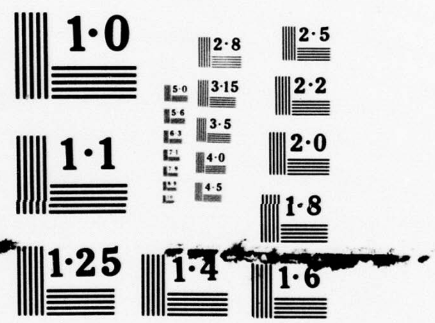
1·0
2·8
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1·4
1·6

For equation (10-10) the necessary and sufficient conditions, for singular conditions are given by equation:

$$\angle F(s) = K\pi \qquad (10\text{-}11)$$

where K is a real integer.

Equation (10-11) implies that all the points of the Root locus of equation:

$$KF(s) = -1 \qquad (10\text{-}12)$$

which do not lie on the real axis are singular points for the characteristic equation (10-10), where K in this case is considered to be the Root locus variable parameter which takes values from zero to plus infinity and from zero to minus infinity.

Finally any control system with n variable parameters appearing in its characteristic equation linearly dependent, is singular, if and only if, its characteristic equation can be written in the form of equation (10-10) which in the simplest case takes the form:

$$1 + \alpha F(s) + \beta F^2(s) + \gamma F^3(s) + \ldots n^{th} F^n(s) = 0 \qquad (10\text{-}13)$$

1. <u>Equation of a Singular n Dimensional Surface</u>

When an n dimensional parameter space was considered then the general form of the corresponding singular characteristic equation of a control system is given by equation (10-10). Then according to the previous analysis when the value of a singular point s_s is substituted in equation (10-10) it becomes:

$$\alpha C_1' + \beta C_2' + \gamma C_3' + \ldots + n^{th} C_n' + C_{n+1}' = 0 \qquad (10\text{-}14)$$

where C_1', C_2', C_3', ..., C_n', C_{n+1}' are real constants.

Equation (10-14) is the equation of the singular n dimensional surface, corresponding to the singular point s_s, which was considered.

In order to determine a singular point of this singular surface the values of n-1 parameters must be defined and then the value of the parameter left can be evaluated from equation (10-14). Thus one among an infinite number of real valued sets of parameter values can be defined, which in other words defines the coordinates of a singular point in the parameter space.

E. COMMENT

From the previous analysis it can be seen that any non-linear case, either in the parameter plane on in the parameter space can be studied by defining any different parameter's product as a new variable parameter, e.g., the non-linear case in the parameter plane can be considered as a linear case in the parameter space (three dimensions for this case) by defining the product $\alpha\beta$ equals with γ which will be the third variable parameter.

After transforming a non-linear characteristic equation in a linear characteristic equation as it was described above then all these which have been stated for the linear case can be applied for the transformed characteristic equation.

XI. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

This thesis has developed practical methods for designing singular compensators, i.e., compensators which produce the desired singular conditions in a single input-single output control system.

The key to the development of these methods was the form of the characteristic equation which was considered, i.e:

$$\alpha f_1(s)+\beta f_2(s)+f_3(s) = 0$$

where: $f_1(s)$, $f_2(s)$ and $f_3(s)$ are polynomials of s and α,β are real variable parameters.

Necessary and sufficient conditions for the above characteristic equation to be singular were determined and the singular line theory was then investigated to make possible the development of the required standard set of rules for the design engineer.

Application of singular line theory in control system design led to the conclusion that it can be used in general for design of compensation of any single input single output linear control system with the following basic advantages.

Required filters for any desired compensation structure have transfer functions which are known as a function of the two parameters α and β. The parameter plane diagram required for analysis and synthesis of a design problem consists of straight lines, i.e., the singular lines which can be drawn by hand, without needing a computer. Design techniques and procedures are characterized by simplicity, flexibility and accuracy. Finally, tolerances on the parameter α and β values can be precisely established.

When considering the sensitivity of the transient response to parameter variations, it was found that the control system is less sensitive to variations in α and β if the operating point on the singular line is remote from the point at which the singular line is tangent to the constant ζ_s and ω_{ns} curves.

The basic design tools of the singular line theory design method are: The Root locus of the plant and the parameter plane diagram of the compensated singular system, i.e., under certain conditions or in other words singular conditions, a classical design method (The Root locus method), and a modern design method (the parameter plane method) can work together in such a way that each compensates disadvantages of the other.

For example stability analysis which is not a simple process in the parameter plane is performed on the Root locus diagram in the s-plane or studying the effect of variations in either one or both of the parameter α and β values on the transient response of the system, which is not a simple process on the Root locus diagram is performed in the parameter plane, etc.

Conceptually the compatibility of the Root locus classical design method, which is adequate when there is a single variable parameter with the parameter plane modern design method, which is adequate when there are two variable parameters, when singular conditions exist, is based on the fact that in this case one of the parameters can be defined through the equation of a singular line, as a function of the other. Thus a two parameter problem becomes a one parameter problem. The similarity also of the Root locus of the plant with the Root locus of the compensated singular system, independently of which singular line is considered generalized the validity of the above concept and made possible the development of graphical design techniques which simplify the work and markedly increase the power of visualization of the designer.

It was also realized that the singular line theory could be applicable in self-adaptive control system, since movement of the operating point (α,β) along a dominant line segment of a singular line or within a dominant root region in the parameter plane implies almost fixed system performance. In such a case the adaptive controller need only to drive the system back to the dominant root line segment or dominant root region and not to a specific operating point in the parameter plane.

The singular line theory was further extended in the parameter space and in the most general case where products of the variable parameters appear in the characteristic equation.

Further study is needed for the case of a singular system with a negative error constant. This study can be initialized from an analysis of equation (8-25).

Also the correlation, if any, of a singular feedback compensator with the linear state variable feedback maybe interesting.

Some other related areas for future work, which are recommended are: Applications of the singular line theory in self-adaptive control systems. Singular surface theory and control systems, singular line theory and multi-input, multi-output linear control systems, and finally any of the above in the general case where products of the variable parameters also appear in the characteristic equation, i.e., non-linear cases.

APPENDIX A

TABLE OF SINGULAR POINTS FOR EXAMPLE (5-1)

Value of ζ_s	Corresponding Value of ω_{ns}	Singular Points: $s = -\zeta_s \omega_{ns} \pm \omega_{ns}\sqrt{1-\zeta_s^2}$
0.10	19.9747	$-1.9975 \pm j\ 19.8746$
0.15	13.2949	$-1.9942 \pm j\ 13.1445$
0.20	9.9479	$-1.9896 \pm j\ 9.7469$
0.25	7.9333	$-1.9833 \pm j\ 7.6814$
0.30	6.5844	$-1.9753 \pm j\ 6.2811$
0.35	5.6149	$-1.9652 \pm j\ 5.2598$
0.40	4.8825	$-1.9530 \pm j\ 4.4749$
0.45	4.3047	$-1.9371 \pm j\ 3.8442$
0.50	3.8375	$-1.9188 \pm j\ 3.3234$
0.55	3.4550	$-1.9002 \pm j\ 2.8855$
0.60	3.1104	$-1.8662 \pm j\ 2.4883$
0.65	2.8175	$-1.8314 \pm j\ 2.1411$
0.70	2.5599	$-1.7919 \pm j\ 1.8281$
0.75	2.3280	$-1.7460 \pm j\ 1.5398$
0.80	2.1225	$-1.6980 \pm j\ 1.2735$
0.85	1.9425	$-1.6511 \pm j\ 1.0233$
0.90	1.7851	$-1.6066 \pm j\ 0.7781$
0.95	1.6505	$-1.5680 \pm j\ 0.5154$

APPENDIX B

TABLE OF ROOTS OF THE CHARACTERISTIC EQUATION FOR EXAMPLE (5-1)

CORRESPONDING TO THE SINGULAR POINT (ζ_s=0.60, ω_{ns}=3.1104) FOR DIFFERENT PARAMETER PAIRS

Value of α	Value of β	ROOTS OF THE CHARACTERISTIC EQUATION					
		COMPLEX ROOTS		REAL ROOTS			
6	-14.7096	-1.866 ± j 2.488		-0.226	-2.454	-3.268	-4.320
7	- 6.8403	-1.866 ± j 2.488		-0.570	-2.283	-3.268	-4.148
8	1.029	-1.866 ± j 2.488		-1.095	-1.927	-3.267	-3.978
9	8.8983	-1.866 ± j 2.488	-1.590 ± j 0.674			-3.267	-3.819
10	16.7676	-1.866 ± j 2.488	-1.661 ± j 1.074			-3.267	-3.678
11	24.6369	-1.866 ± j 2.488	-1.720 ± j 1.388			-3.268	-3.561
12	32.5062	-1.866 ± j 2.488	-1.767 ± j 1.662			-3.267	-3.468
13	40.3755	-1.866 ± j 2.488	-1.732 ± j 1.459			-3.267	-3.535
14	48.2448	-1.866 ± j 2.488	-1.831 ± j 2.134			-3.267	-3.338
15	56.1141	-1.866 ± j 2.488	-1.853 ± j 2.343			-3.268	-3.266
16	63.9834	-1.866 ± j 2.488	-1.870 ± j 2.538			-3.268	-3.268
17	71.8527	-1.866 ± j 2.488	-1.884 ± j 2.721			-3.269	-3.232
18	79.7220	-1.866 ± j 2.488	-1.896 ± j 2.893			-3.267	-3.208
19	87.5913	-1.866 ± j 2.488	-1.905 ± j 3.057			-3.267	-3.190
20	95.4606	-1.866 ± j 2.488	-1.913 ± j 3.214			-3.267	-3.175
25	134.8071	-1.866 ± j 2.488	-1.939 ± j 3.907			-3.267	-3.122
30	174.1536	-1.866 ± j 2.488	-1.953 ± j 4.498			-3.267	-3.094
35	213.5001	-1.866 ± j 2.488	-1.962 ± j 5.021			-3.267	-3.076
50	331.5400	-1.866 ± j 2.488	-1.976 ± j 6.339			-3.267	-3.049
75	528.2721	-1.866 ± j 2.488	-1.985 ± j 8.072			-3.267	-3.030
100	725.0046	-1.866 ± j 2.488	-1.989 ± j 9.495			-3.268	-3.022
200	1511.9346	-1.866 - j 2.488	-1.995 - j13.790			-3.267	-3.010

APPENDIX D

TABLE OF ROOTS OF THE CHARACTERISTIC EQUATION FOR EXAMPLE (6-2) CORRESPONDING TO THE SINGULAR POINT (ζ_s = 0.40, ω_{ns}=0.97096) FOR DIFFERENT PARAMETER PAIRS

K	Value of α	Value of β	ROOTS OF THE CHARACTERISTIC EQUATION: COMPLEX ROOTS		REAL ROOTS			
20	5	0.01863	-0.3884 ± j 0.8899		-5.2230	-5.0040	-0.0154	-0.9808
	10	1.24968	-0.3884 ± j 0.8899	-0.3852 ± j 0.9072	-5.2230	-5.2290		
	15	2.48073	-0.3884 ± j 0.8899	-0.2897 ± j 1.3320	-5.2230	-5.4200		
	20	3.71178	-0.3884 ± j 0.8899	-0.2060 ± j 1.6300	-5.2230	-5.5880		
	25	4.94283	-0.3884 ± j 0.8899	-0.1308 ± j 1.8660	-5.2230	-5.7380		
	30	6.17388	-0.3884 ± j 0.8899	-0.0623 ± j 2.0650	-5.2230	-5.8750		
	35	7.40493	-0.3884 ± j 0.8899	-0.0009 ± j 2.2390	-5.2230	-6.0020		
	40	8.63598	-0.3884 ± j 0.8899	-0.0598 ± j 2.3930	-5.2230	-6.1200		
	45	9.86703	-0.3884 ± j 0.8899	-0.1150 ± j 2.5340	-5.2230	-6.2300		
	50	11.09808	-0.3884 - j 0.8899	-0.1671 - j 2.6620	-5.2230	-6.3310		
35	5	0.01064	-0.3884 ± j 0.8899		-5.2230	-5.0040	-0.0154	-0.9808
	10	0.71409	-0.3884 ± j 0.8899	-0.3852 ± j 0.9072	-5.2230	-5.2290		
	15	1.41754	-0.3884 ± j 0.8899	-0.2897 ± j 1.3320	-5.2230	-5.4200		
	20	2.12099	-0.3884 ± j 0.8899	-0.2060 ± j 1.6300	-5.2230	-5.5880		
	25	2.82444	-0.3884 ± j 0.8899	-0.1308 ± j 1.8660	-5.2230	-5.7380		
	30	3.52789	-0.3884 ± j 0.8899	-0.0623 ± j 2.0650	-5.2230	-5.8750		
	35	4.23134	-0.3884 ± j 0.8899	-0.0009 ± j 2.2390	-5.2230	-6.0020		
	40	4.93479	-0.3884 ± j 0.8899	-0.0598 ± j 2.3930	-5.2230	-6.1200		
	45	5.63824	-0.3884 ± j 0.8899	-0.1150 ± j 2.5340	-5.2230	-6.2300		
	50	6.34169	-0.3884 - j 0.8899	-0.1671 - j 2.6620	-5.2230	-6.3310		

APPENDIX E

<u>TABLE I</u>
FIRST SINGULAR CASE
RESULTS OF OUTPUT AND ERROR ANALYSIS

Plant Type	Plant Type Zero		Plant Type One		Plant Type Two	
Steady State	Output	Error	Output	Error	Output	Error
Step Input	Finite	Finite	Zero	Finite	Zero	Finite
Ramp Input	$\propto$	$\propto$	Finite	$\propto$	Zero	$\propto$
Parabolic Input	$\propto$	$\propto$	$\propto$	$\propto$	Finite	$\propto$

TABLE II

SECOND SINGULAR CASE
RESULTS OF OUTPUT AND ERROR ANALYSIS

Plant Type	Plant Type Zero		Plant Type One		Plant Type Two	
Steady State	Output	Error	Output	Error	Output	Error
Step Input	Finite	Finite	Finite	Zero	Finite	Zero
Ramp Input	$\propto$	$\propto$	$\propto$	Finite	$\propto$	Zero
Parabolic Input	$\propto$	$\propto$	$\propto$	$\propto$	$\propto$	Finite

APPENDIX F

TABLE I

EXAMPLE (6-5): SINGULAR POINT (ζ_s=0.40, ω_{ns}=0.97096)
Transient Response Characteristic Values

Parameter pair values		Settling Time t_s	Percent Overshoot Mp_t	Time of First Overshoot t_p	Transient Oscillatory Frequency ω_t	Remarks
α	β	10.3 sec	25.38%	3.53 sec	0.89Rad/sec	Corresponding to the singular root second order model, values.
α =4	β =-0.966					System Unstable
=7	= 0.511	12.8	12.08	7.2	0.65	
=7.5	= 0.634	12.0	20.75	6.6	0.71	
=7.6	= 0.659	12.0	22.30	6.6	0.71	
=7.7	= 0.683	12.0	23.72	6.4	0.71	
=7.8	= 0.708	12.0	25.18	6.4	0.75	
=7.9	= 0.733	12.0	26.51	6.4	0.75	
=8	= 0.757	12.0	27.79	6.2	0.75	
=9	= 1.003	11.4	38.28	5.8	0.83	
=9.85	= 1.213	11.0	44.47	5.4	0.88	Singul.rt rptd.
=10	= 1.250	10.6	45.40	5.4	0.89	
=11	= 1.496	10.2	50.13	5.2	0.92	
=12	= 1.742	9.8	53.32	4.8	0.92	
=13	= 1.988	13.5	55.60	4.8	0.92	
=14	= 2.235	14.0	57.15	4.6	1.05	
=20	= 3.712	14.2	56.80	4.0	1.43	
=30	= 6.173	50.0	46.81	3.4	2.24	
=40	= 8.636					System Unstable

TABLE II

EXAMPLE (6-5): SINGULAR POINT (ζ_s=0.3, ω_{ns}= 1.15268)
Transient Response Characteristic Values

Parameter pair values		Settling Time t_s	Percent Overshoot Mp_t	Time of First Overshoot t_p	Trans.Oscil. Freq. ω_t	Remarks
α	β	11.6 sec	37.23%	2.86 sec	ω_t=1.01 Rad/sec	Corresponding to the singular root second order model
=7	=-0.019					System Unstable
=10	= 1.039	10.4	27.77	5.6	0.87	
=10.5	= 1.216	10.6	35.88	5.4	0.87	
=10.6	= 1.251	11.6	37.31	5.4	0.87	
=10.7	= 1.286	12.4	38.65	5.4	0.92	
=10.8	= 1.321	12.6	40.03	5.2	0.92	
=10.9	= 1.357	13.0	41.44	5.2	0.92	
=11	= 1.392	13.8	42.76	5.2	0.92	
=12	= 1.745	15.0	53.49	4.8	0.92	
=14.1	= 2.485	16.4	69.10	4.6	1.01	Singul.rt.rptd.
=15	= 2.803	16.6	72.65	4.4	1.12	
=20	= 4.566	16.8	83.81	3.8	1.31	
=25	= 6.329	20.4	84.43	3.4	1.96	
=30	= 8.092	31.2	82.05	3.2	1.96	
=35	= 9.856	99.8	77.76	3.0	2.24	
=40	=11.619					System Unstable

TABLE III

EXAMPLE (6-5): SINGULAR POINT ($\zeta_s = 0.75$, $\omega_{ns} = 0.060667$)
Transient Response Characteristic Values

Parameter pair values		Settling Time t_s	Percent Overshoot M_{pt}	Time of First Overshoot t_p	Trans. Osci. Freq. ω_t	Remarks
α	β	8.8 sec	2.84%	7.82 sec	0.40 Rad/sec	Correspond. to the singular root, second order model
=1	=-0.082					System Unstable Exponential Trans.
=2	= 0.012					
=3.5	= 0.152	13.0	2.36	12.8	0.30	
=3.6	= 0.162	13.0	2.78	12.2	0.30	
=4	= 0.199	12.8	5.11	11.0	1.31	
=6	= 0.387	10.4	8.46	8.0	0.32	
=7	= 0.480	9.6	7.86	7.4	0.36	
=8	= 0.574	8.8	6.66	7.0	0.40	
=10	= 0.761	11.2	3.46	6.4	0.42	
=15	= 1.230	9.8	5.47	8.4	0.51	
=20	= 1.698	11.2	6.27	7.4	0.60	
=25	= 2.166	13.2	5.91	6.6	1.96	
=30	= 2.635	55.2	4.80	6.2	1.99	
=35	= 3.103					System Unstable

APPENDIX G

TABLE I

EXAMPLE (6-7): OPERATING POINTS AND CORRESPONDING SINGULAR POINTS AND SINGULAR LINES.

Operating Point		Corresponding Singular Point		Corresponding Singular Line
α	β	ζ_s	ω_{ns}	
900	2755.245	0.600	1.230	$\beta = 3.304\,\alpha - 218.355$
1430	4280	0.608	1.193	$\beta = 3.131\,\alpha - 196.150$
900	2400	0.623	1.131	$\beta = 2.847\,\alpha - 162.185$
900	2061.503	0.650	1.030	$\beta = 2.444\,\alpha - 138.097$

APPENDIX H

SINGULAR POINT PROGRAM

The purpose of this program is to determine the roots of a characteristic equation which are singular points with respect to the parameters α and β.

This program is applicable to polynomials whose coefficients are of the form ($b\alpha+c\beta+d$), where α and β are variable parameters and b, c, and d are real constants.

For a given characteristic equation this program solves the system (2-4), by the Cramer's rule method for a desired value of ζ and for a range of values for ω_n, which has been determined within the program and can be changed when it is desired, by changing the card 0035.

The four columns which are generated at the print output are the following: The first is a list of the values for ω_n, which were used for the calculations and the other three are the corresponding numerators and common denominator from the Cramer's rule solution of the system (2-4).

Any value of ω_n in the first column, for which the associated values of the other three columns are zero's, together with the value of ζ which was used defines a singular point for the characteristic equation.

The input data for the program are:

a. First card:
Value of ζ and N, format (F4.2, I2) where ζ is the desired value of damping ratio and N is the order of the characteristic equation.

b. Second card:
Constant coefficients in ascending order (8E 10.5)

c. Third card:
Alfa coefficients in ascending order (8E 10.5)

d. Fourth card:
Beta coefficients in ascending order (8E 10.5)

```
FORTRAN IV G LEVEL  21

0001              DIMENSION B(20),C(20),D(20),UZ(20),WN(1000),ANUM(1000),BNUM(1000),
                 1CDEN(1000)
0002              READ (5,36) Z,N
0003              WRITE(6,37)
            C     INITIALIZE ALL ARRAYS TO ZERO
0004          400 DO 4 J=1,20
0005              B(J)=0.0
0006              C(J)=0.0
0007              D(J)=0.0
0008              UZ(J)=0.0
0009            4 CONTINUE
0010          500 DO 5 J=1,1000
0011              ANUM(J)=0.0
0012              BNUM(J)=0.0
0013              CDEN(J)=0.0
0014            5 CONTINUE
0015              N1=N+1
            C     READ AND WRITE THE CONSTANTS COEFFICIENTS
0016              READ (5,32) (D(K),K=1,N1).
0017              WRITE (6,29)
0018              WRITE (6,33) (D(K),K=1,N1)
            C     READ AND WRITE THE ALFA COEFFICIENTS
0019              READ (5,32) (B(K),K=1,N1)
0020              WRITE (6,30)
0021              WRITE (6,33) (B(K),K=1,N1)
            C     READ AND WRITE THE BETA COEFFICIENTS
0022              READ (5,32) (C(K),K=1,N1)
0023              WRITE (6,31)
0024              WRITE (6,33) (C(K),K=1,N1)
0025              WRITE(6,38) Z
            C     GENERATE THE CHEBYSHEV FUNCTIONS OF THE SECOND TYPE FOR A GIVEN Z
0026              UZ(1)=-1.0
0027              UZ(2)=0.0
0028              N2=N+2
0029          100 DO 1 M=3,N2
0030              M1=M-1
0031              M2=M-2
0032              UZ(M)=2.0*Z*UZ(M1)-UZ(M2)
0033            1 CONTINUE
            C     GENERATE THE B1,B2,C1,C2,D1,D2 COEFFICIENTS
0034              WRITE (6,34)
0035          200 DO 2 II=1,1000
0036              WN(II)=II/20.0
0037              B1=0.C
0038              B2=0.0
```

```
      C1=0.0
      C2=0.0
      D1=0.0
      D2=0.0
300      DO 3 I2=1,N1
         L=I2-1
         LL=I2+1
         B11=((-1.0)**L)*B(I2)*(WN(I1)**L)*UZ(I2)
         B1=B1+B11
         B22=((-1.0)**L)*B(I2)*(WN(I1)**L)*UZ(LL)
         B2=B2+B22
         C11=((-1.0)**L)*C(I2)*(WN(I1)**L)*UZ(I2)
         C1=C1+C11
         C22=((-1.0)**L)*C(I2)*(WN(I1)**L)*UZ(LL)
         C2=C2+C22
         D11=((-1.0)**L)*D(I2)*(WN(I1)**L)*UZ(I2)
         D1=D1+D11
         D22=((-1.0)**L)*D(I2)*(WN(I1)**L)*UZ(LL)
         D2=D2+D22
  3      CONTINUE
      ANUM(I1)=C1*D2-C2*D1
      BNUM(I1)=B2*D1-B1*D2
      CDEN(I1)=B1*C2-B2*C1
      WRITE (6,35)WN(I1),ANUM(I1),BNUM(I1),CDEN(I1)
    2 CONTINUE
      STOP
   29 FORMAT (//,4X,'CONSTANT COEFFICIENTS IN ASCENDING ORDER',//)
   30 FORMAT (//,4X,'ALFA COEFFICIENTS IN ASCENDING ORDER',//)
   31 FORMAT (//,4X,'BETA COEFFICIENTS IN ASCENDING ORDER',//)
   32 FORMAT (8F10.3)
   33 FORMAT (4X,8E14.7)
   34 FORMAT ('1','VALUE OF WN',14X,'ANUM=C1D2-C2D1',11X,'BNUM=B2D1-B1D2
     1',11X,'COMMON DEN=B1C2-B2C1',//)
   35 FORMAT (5X,F11.6,16X,E13.5,13X,E13.5,13X,E13.5)
   36 FORMAT (F4.2,I2)
   37 FORMAT(' ',5X,'THE INPUT DATA IS',//)
   38 FORMAT(////,10X,'THE VALUE OF Z IS :   Z=',F4.2,//)
      END
```

APPENDIX I

FIGURES

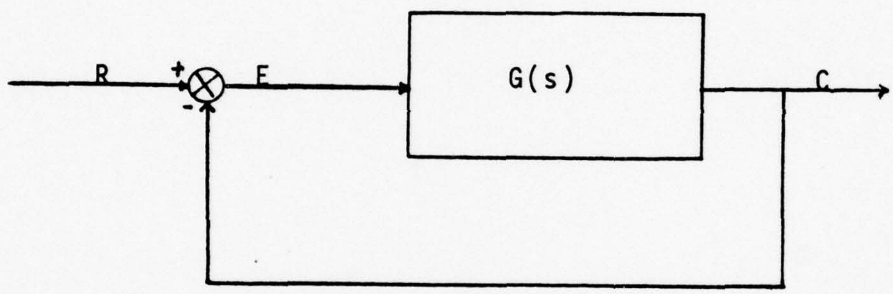


FIGURE (4-1)
Block diagram of a unity feedback system

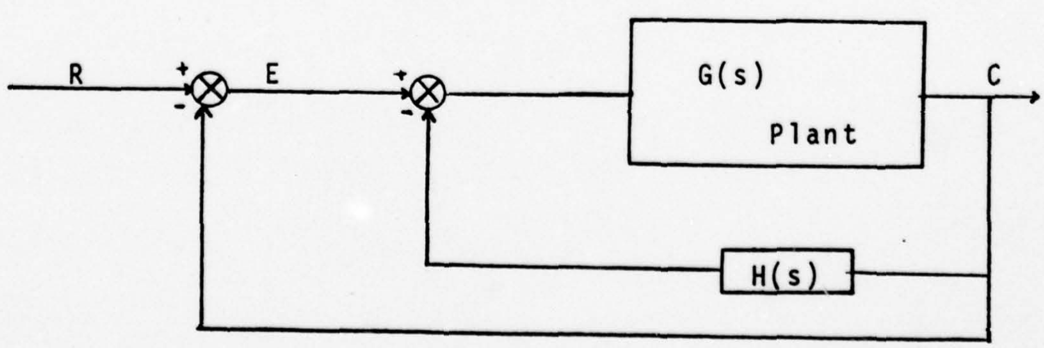


FIGURE (4-2)
First compensation structure

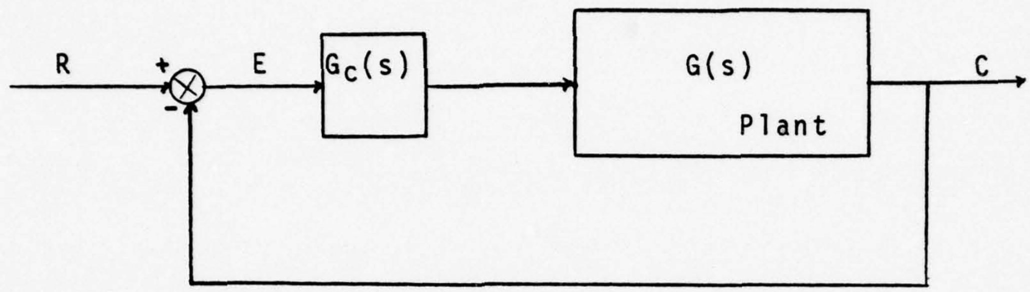


FIGURE (4-3)
Second compensation structure

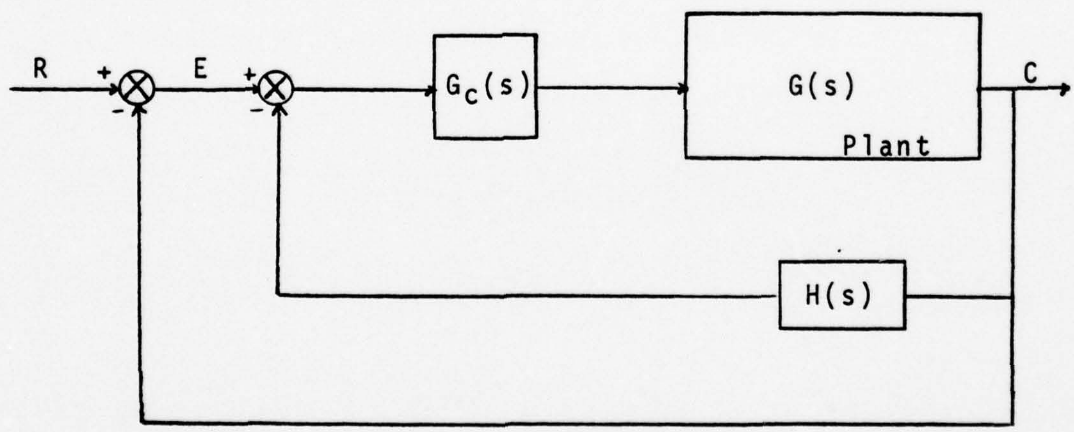


FIGURE (4-4)
Third compensation structure

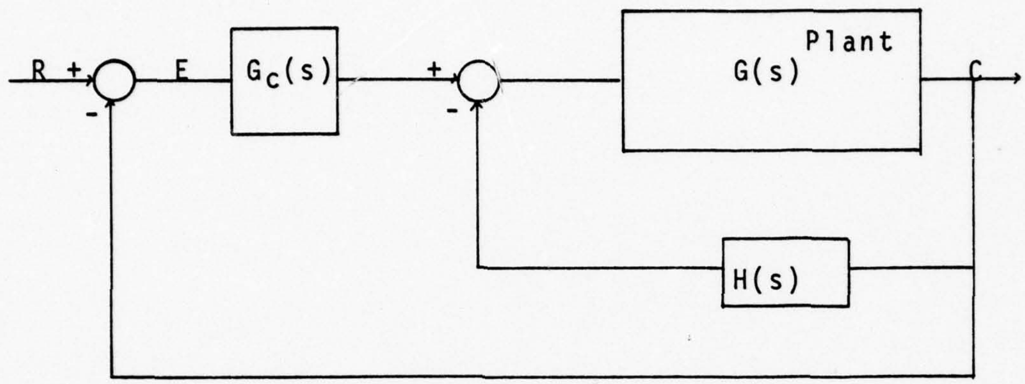


FIGURE (4-5)
Fourth compensation structure.

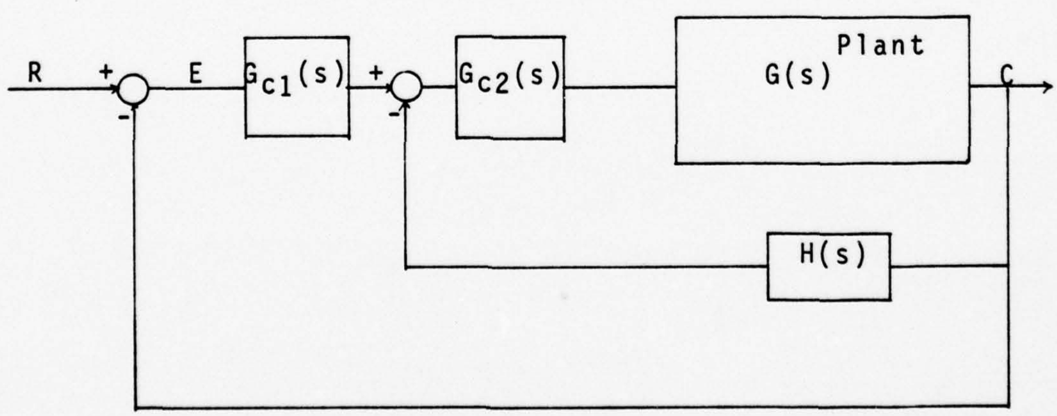
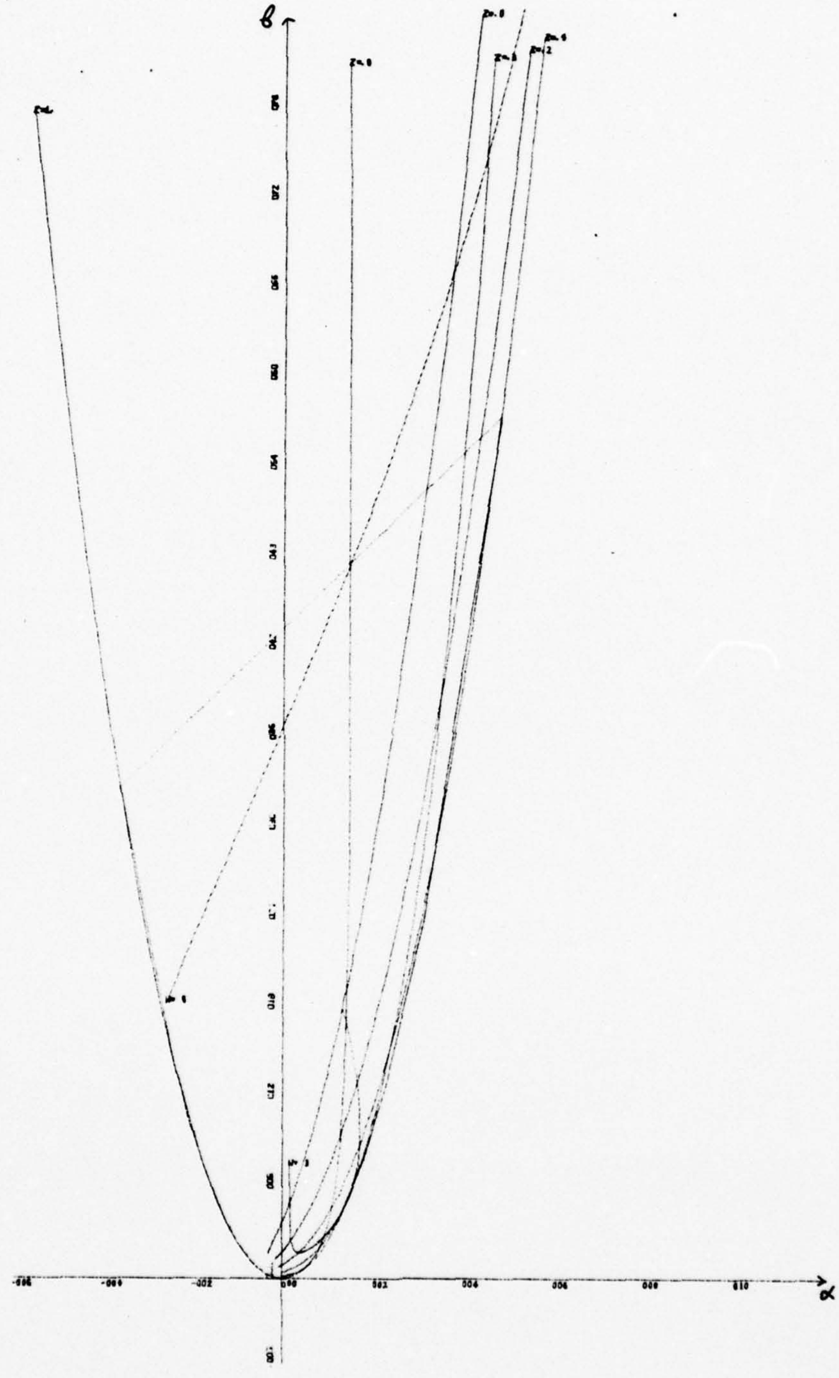
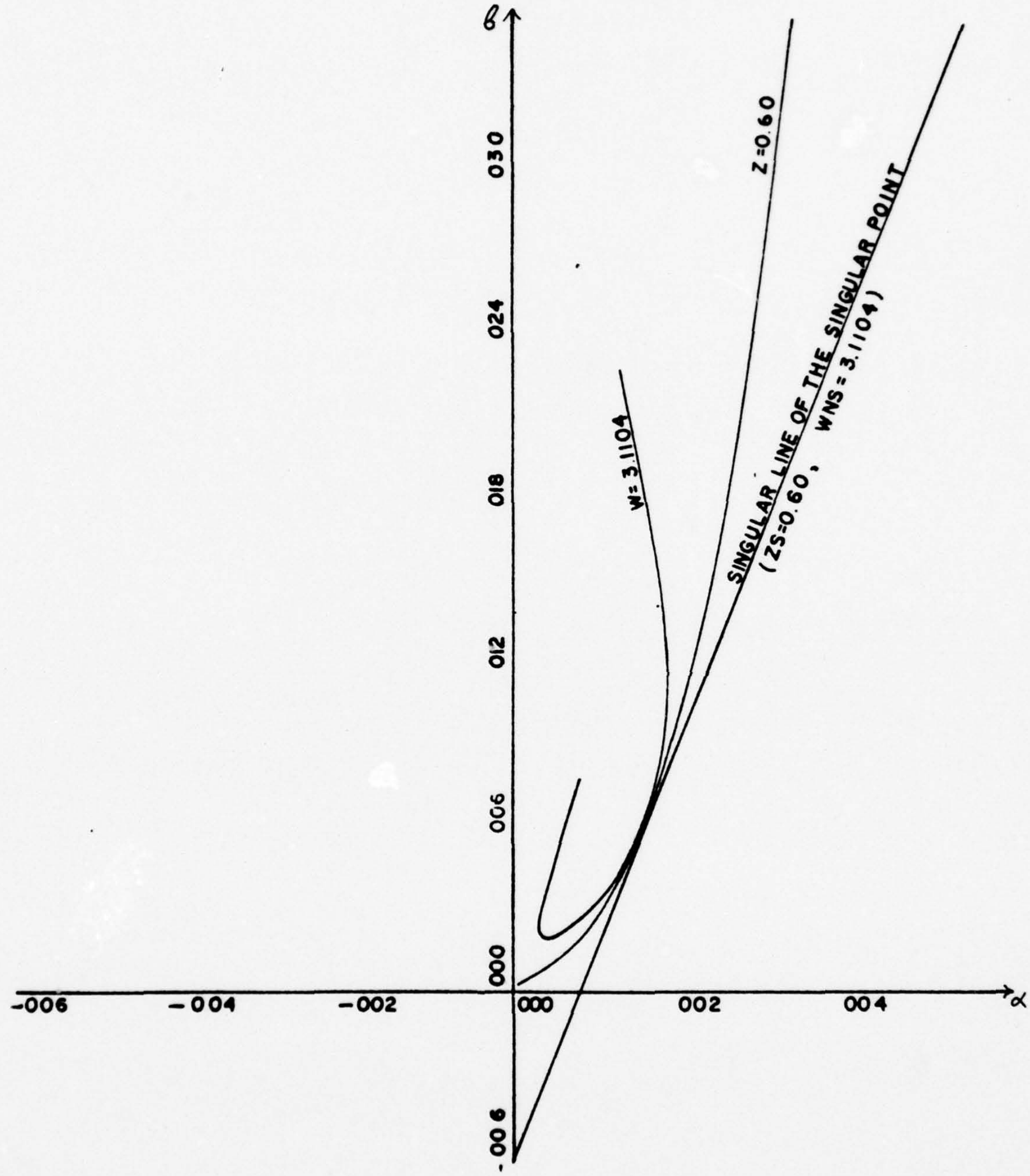


FIGURE (4-6)
Fifth compensation structure.



X-SCALE=2.00E+01 UNITS INCH.
Y-SCALE=6.00E+01 UNITS INCH.

FIGURE (5-1)
Example (5-1):Parameter plane diagram.



X-SCALE =2.00 E+01 UNITS INCH.
Y-SCALE =6.00 E+01 UNITS INCH.

FIGURE (5-2)

Example (5-1):Parameter plane diagram showing a singular line.

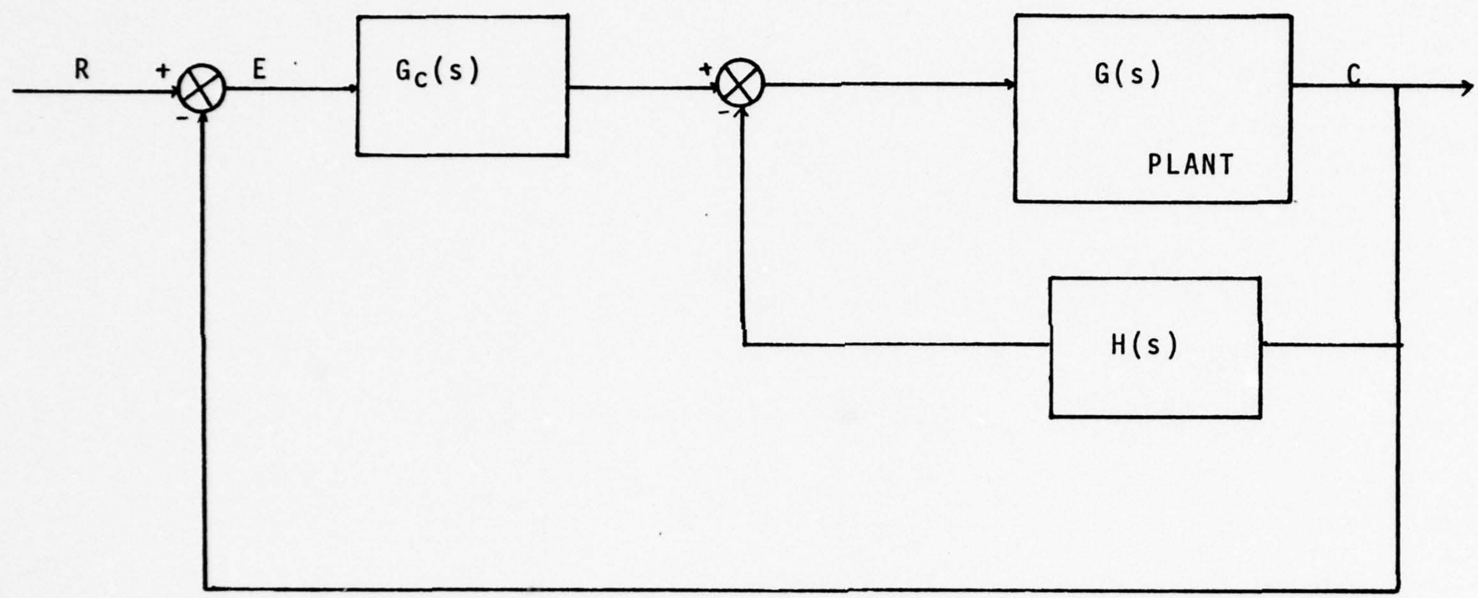
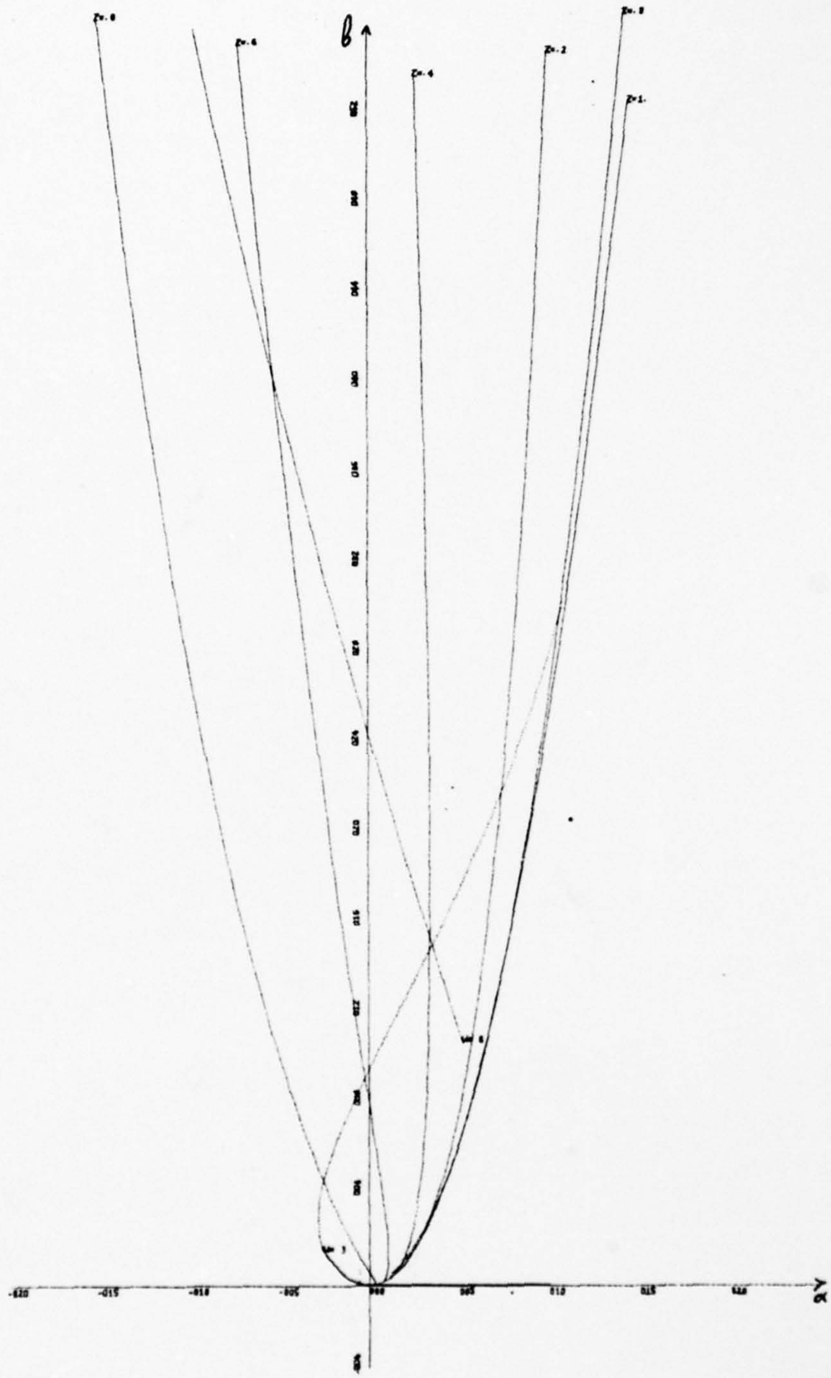


FIGURE (6-1)
Block diagram corresponding to the second singular case.



X-SCALE = 5.00E-01 UNITS INCH.
Y-SCALE = 4.00E+00 UNITS INCH.

FIGURE (6-2)
Example (6-1): Parameter plane diagram.

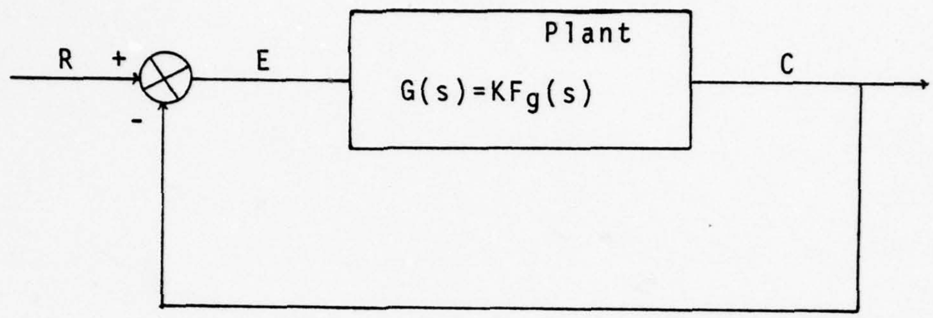


FIGURE (6-3)
Block diagram of the plant.

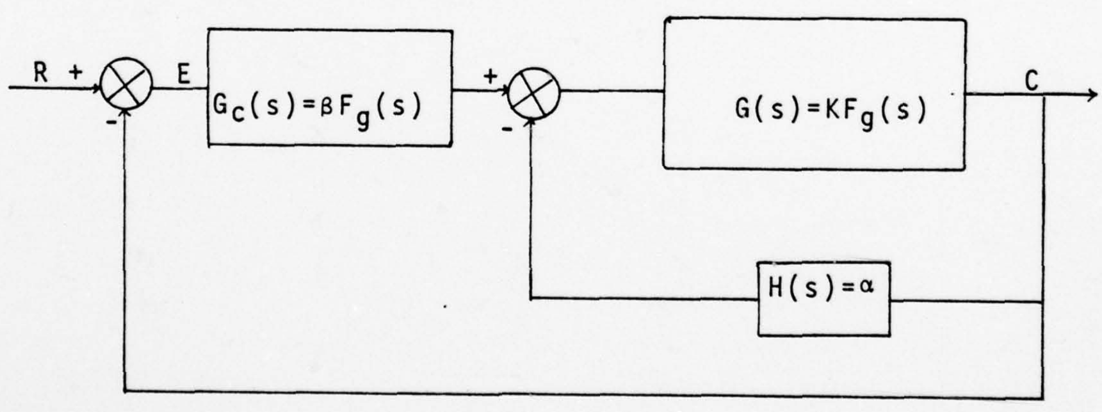


FIGURE (6-4)
Block diagram of the compensated singular system corresponding to the "second singular case",scheme.

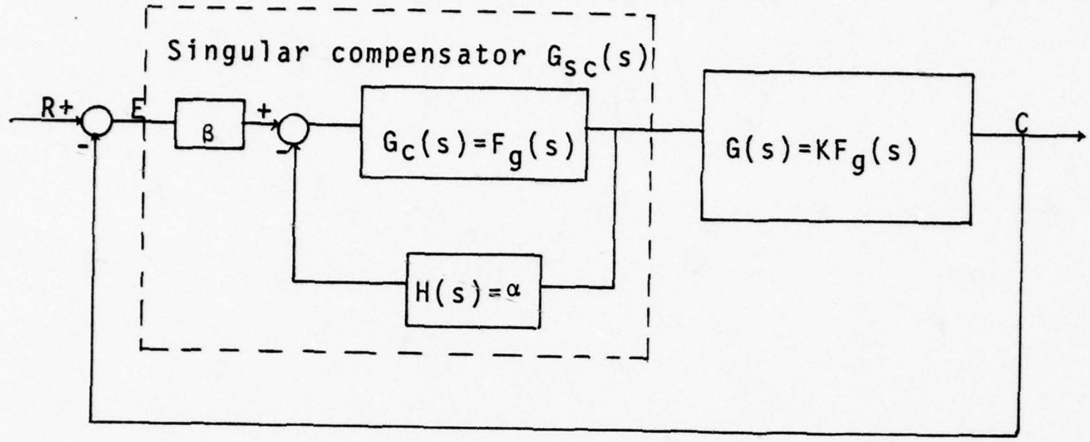


FIGURE (6-5)
Block diagram of the compensated singular system corresponding to the modified second singular case,scheme.

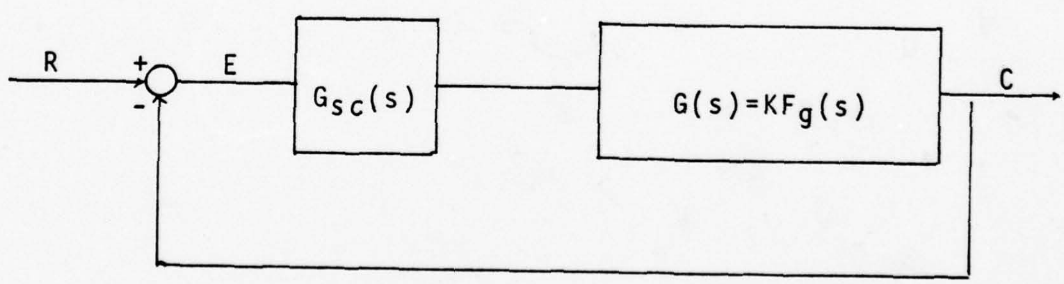


FIGURE (6-6)
Simplified block diagram of the modified second singular case,scheme.

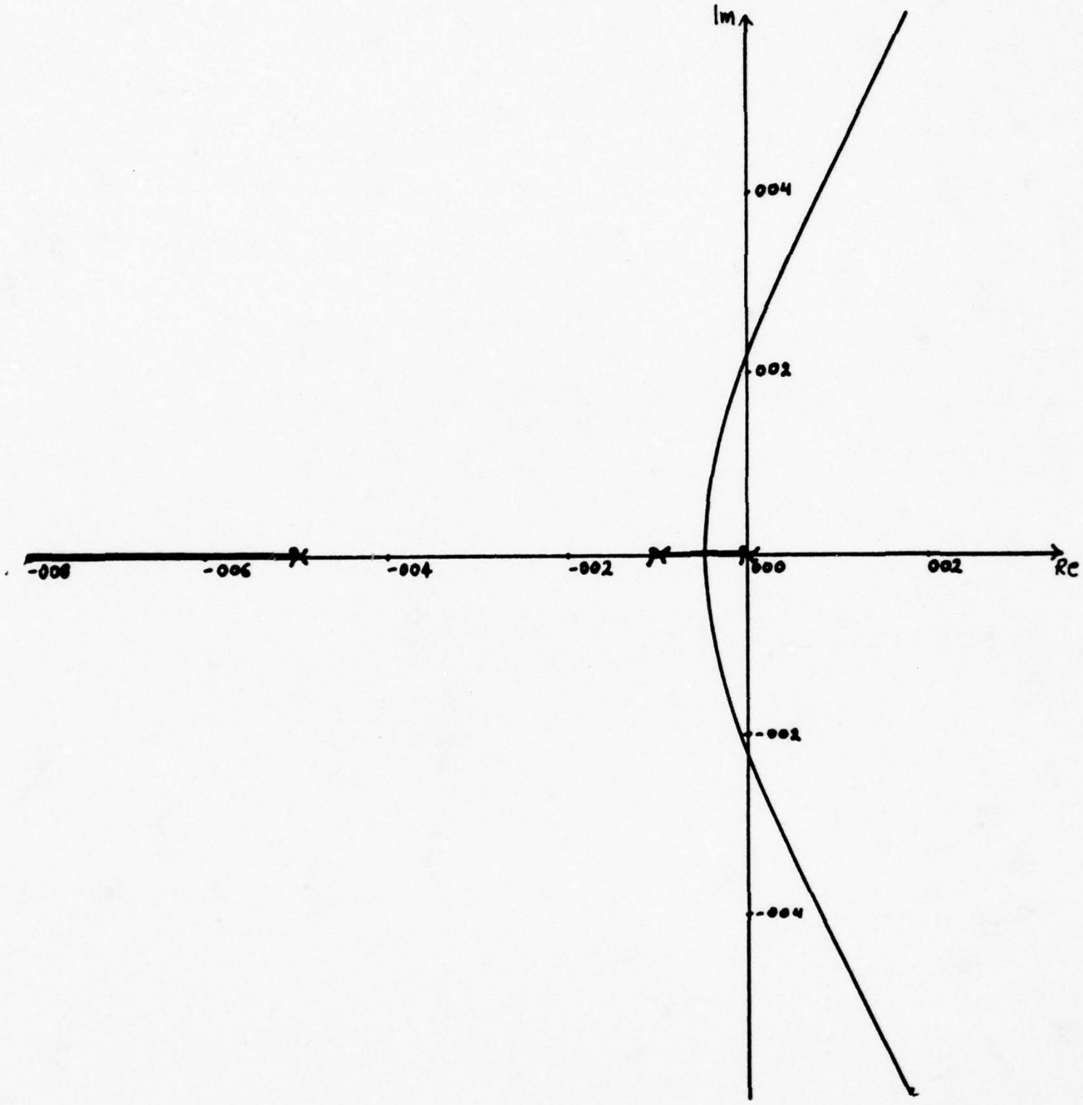
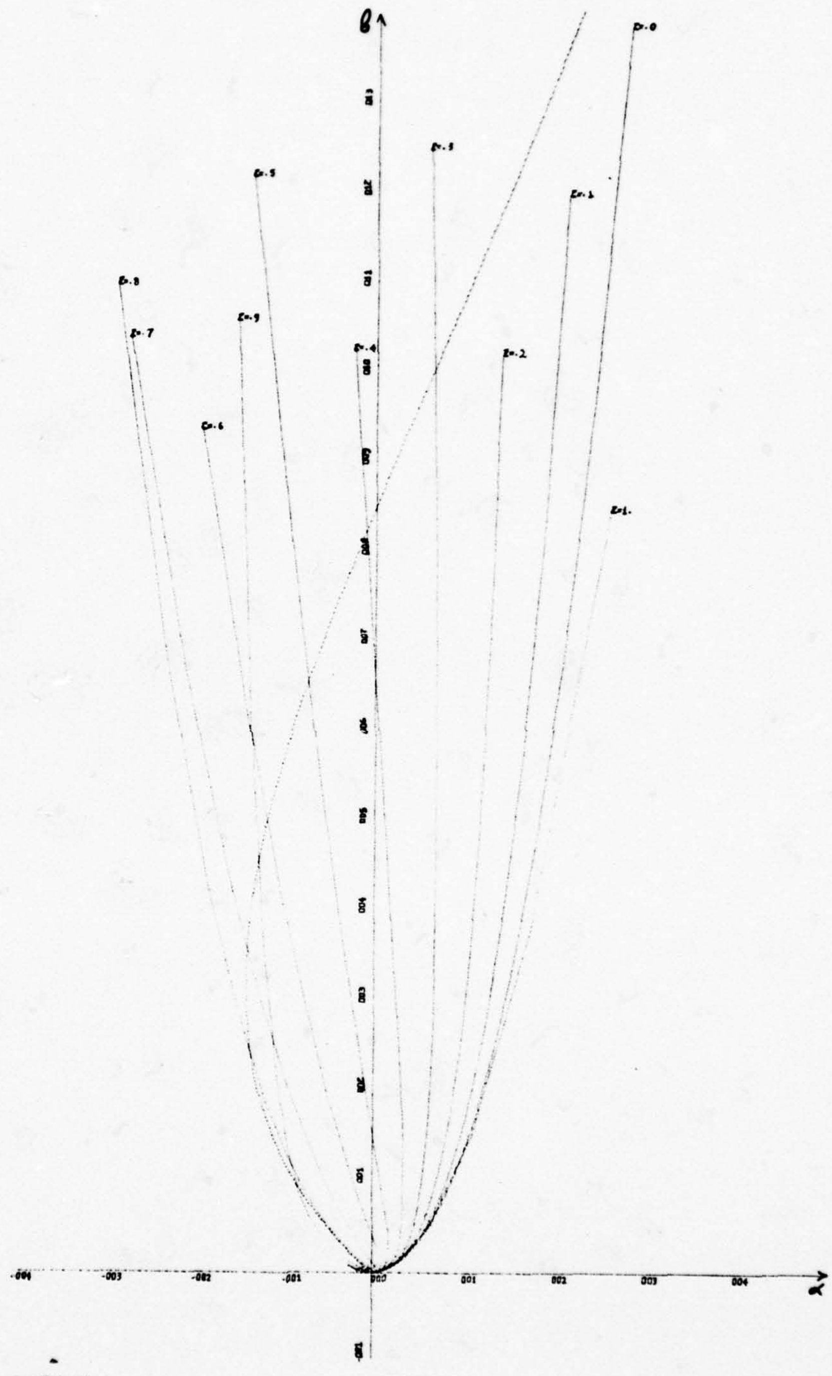


FIGURE (6-7)
Example (6-2): Root locus of the uncompensated system.



X-SCALE= 1.00E+02 UNITS INCH.
Y-SCALE= 1.00E+02 UNITS INCH.

FIGURE (6-8)
Example (6-2): Parameter plane diagram,K=20.

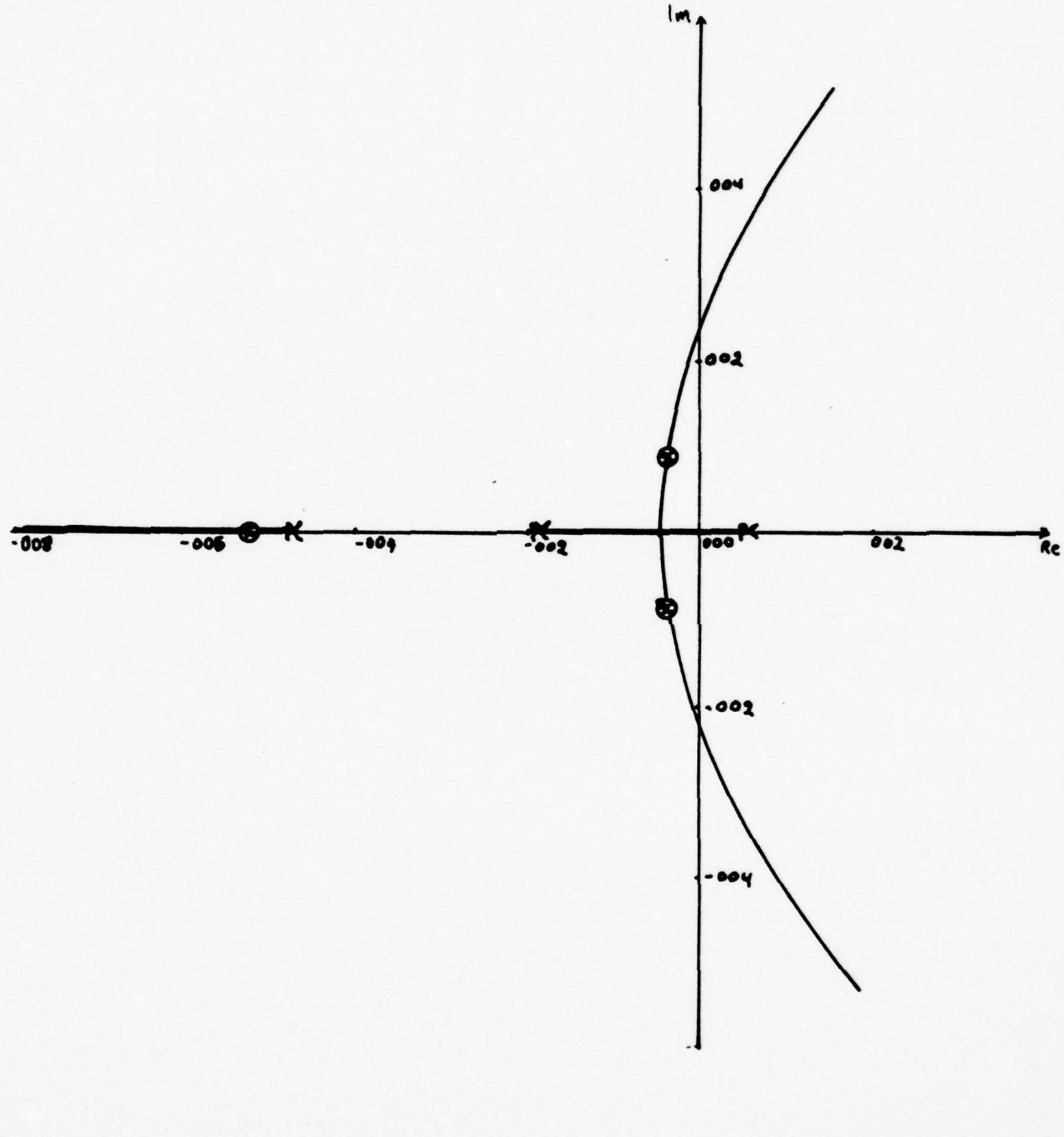
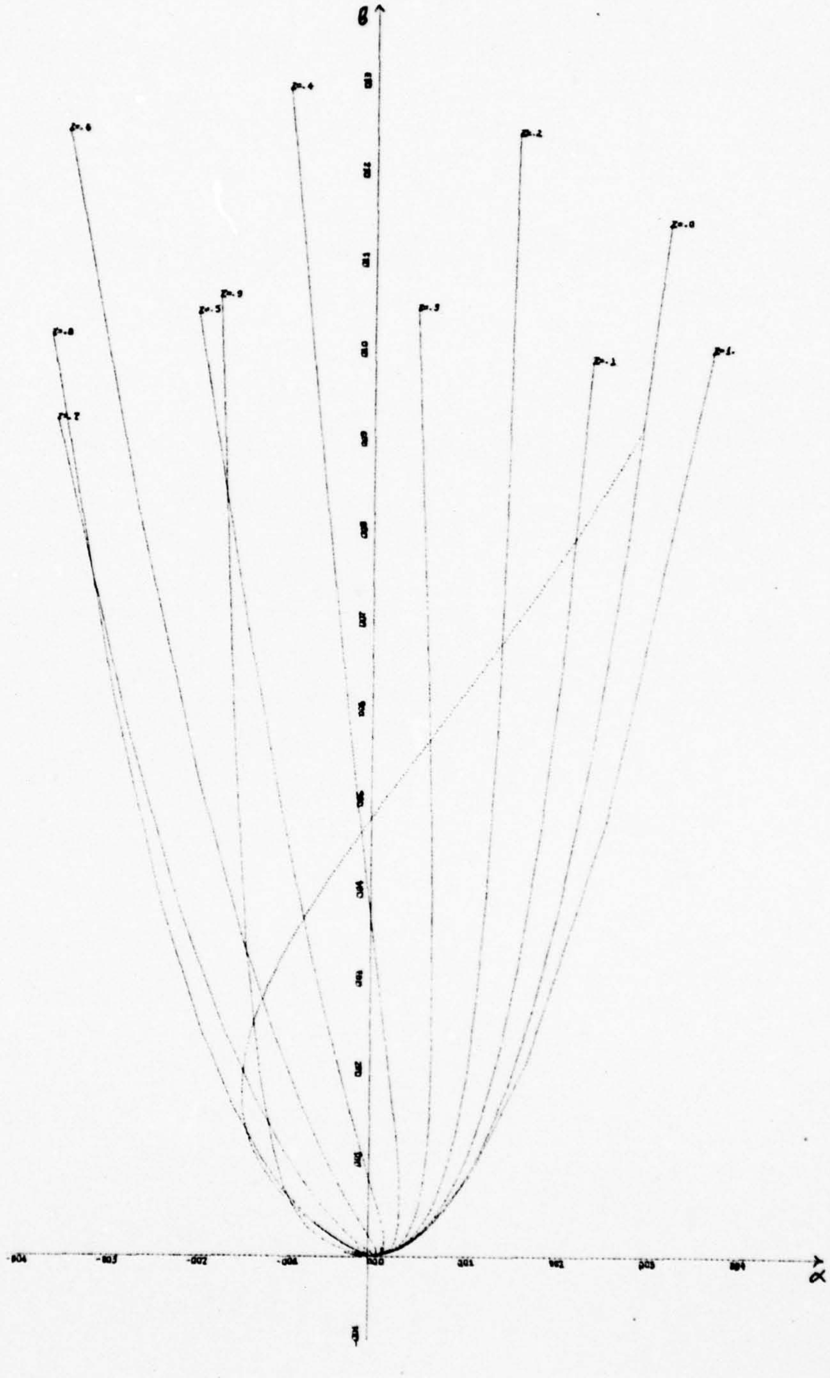


FIGURE (6-9)
Example (6-2): Root locus of the compensated singular system corresponding to the singular root: $s=-0.3884\pm j0.8899$ or ($\zeta_s=0.40, \omega_{ns}=0.97096$)



X-SCALE = 1.00 E+02 UNITS INCH.
Y-SCALE = 1.00 E+02 UNITS INCH.

FIGURE (6-10)
Example (6-2): Parameter plane diagram, K=35.

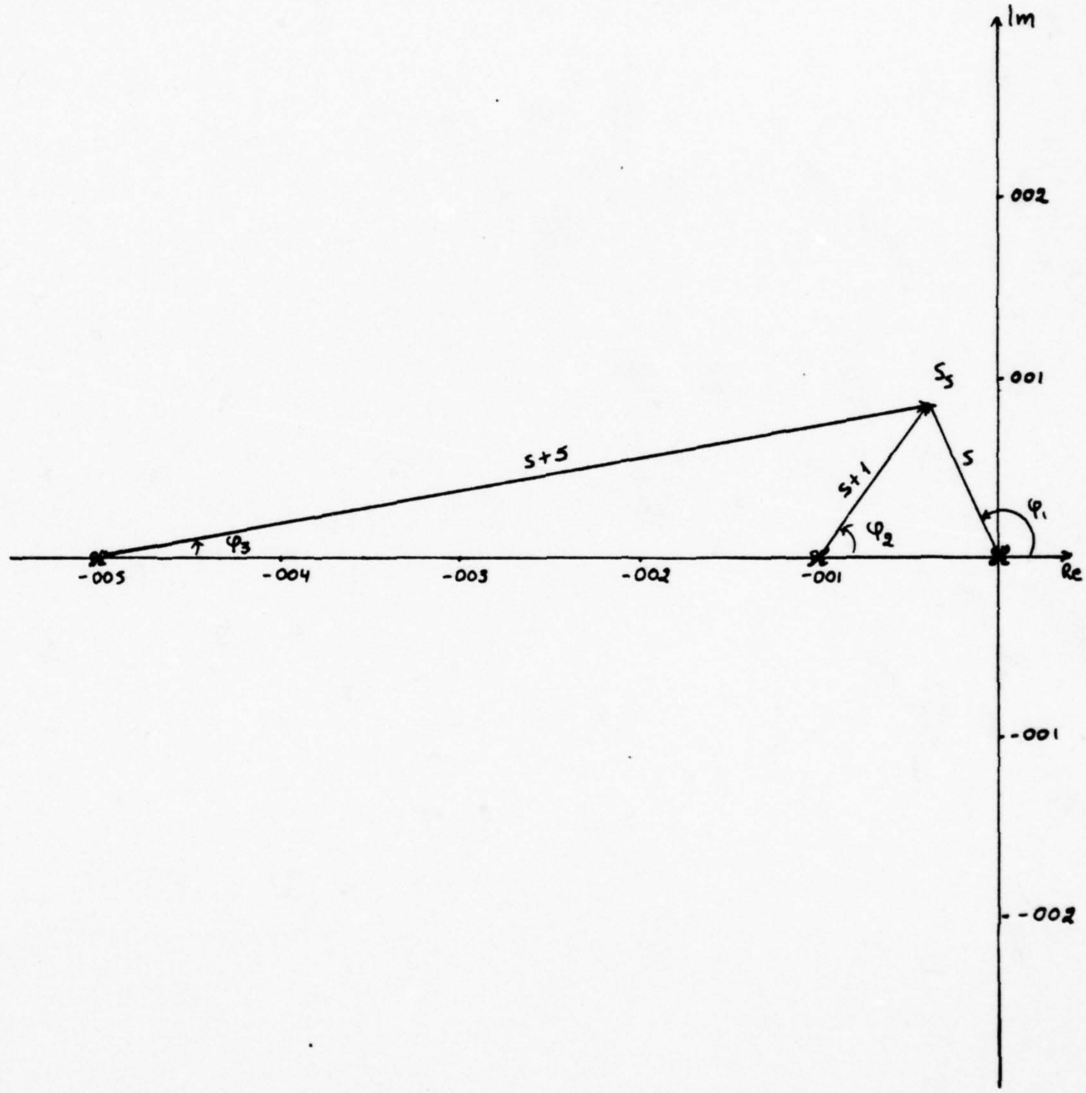


FIGURE (6-11)
Example (6-3): S-plane diagram.

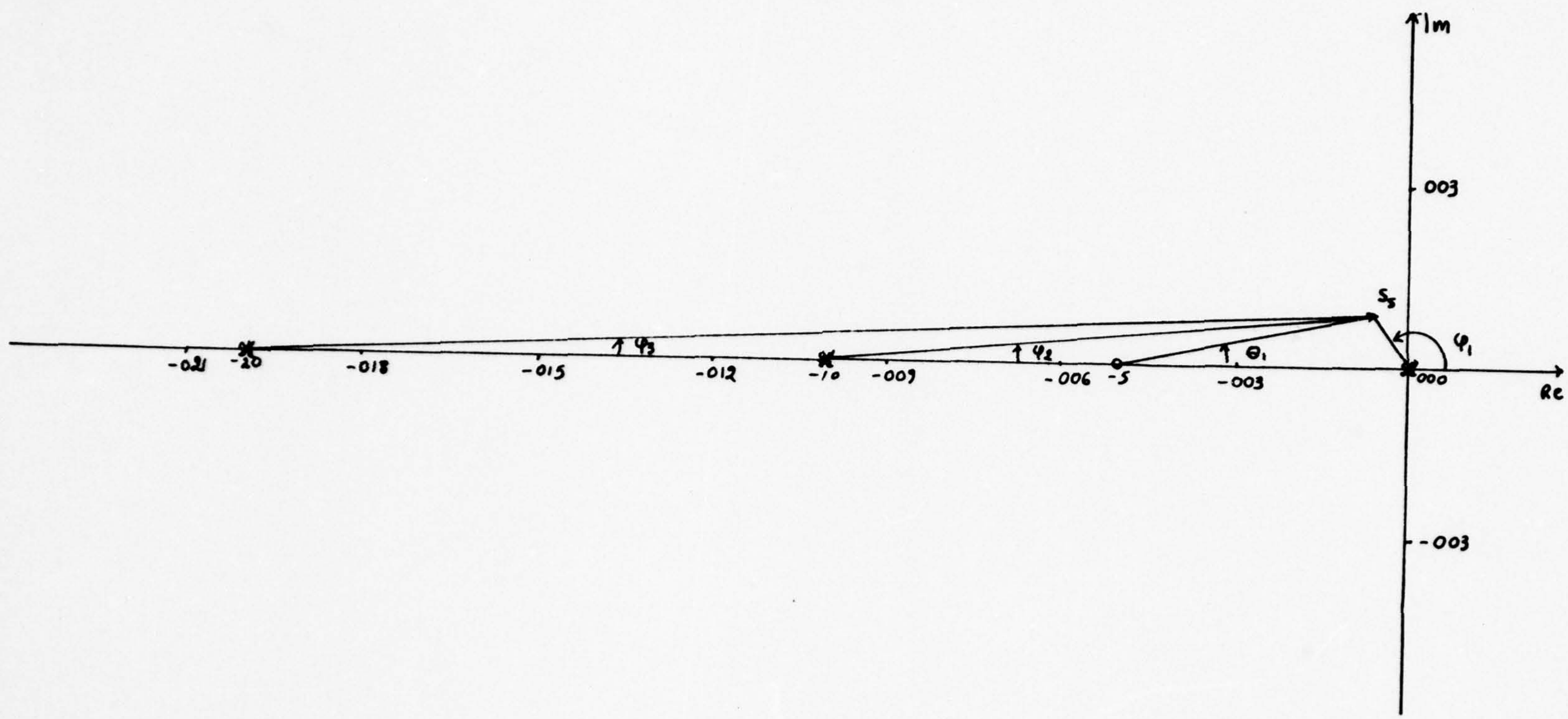


FIGURE (6-12)
Example (6-4): S-plane diagram.

FIGURE (6-13a)
Example (6-4): Introduction of an additional zero.

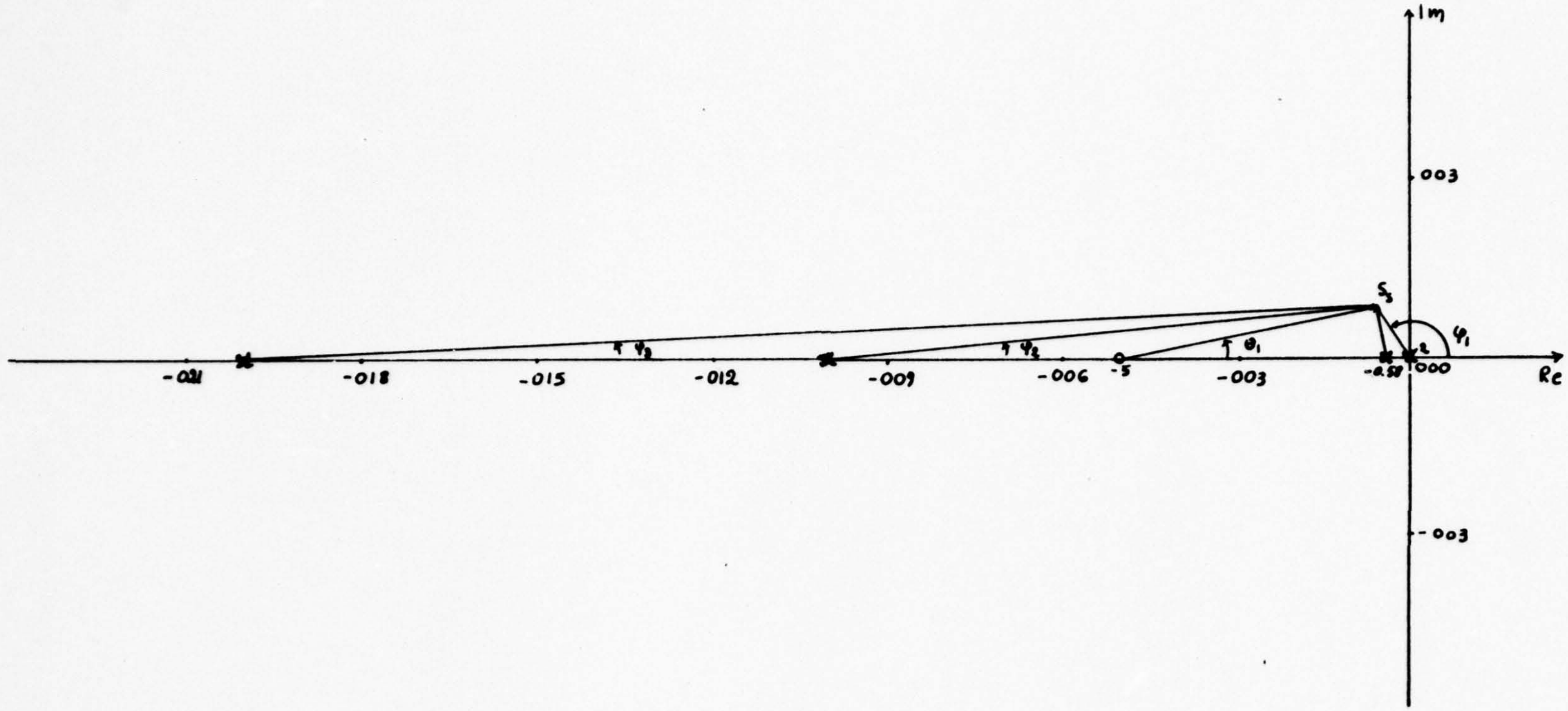


FIGURE (6-13b)
Example (6-4): Introduction of an additional pole.

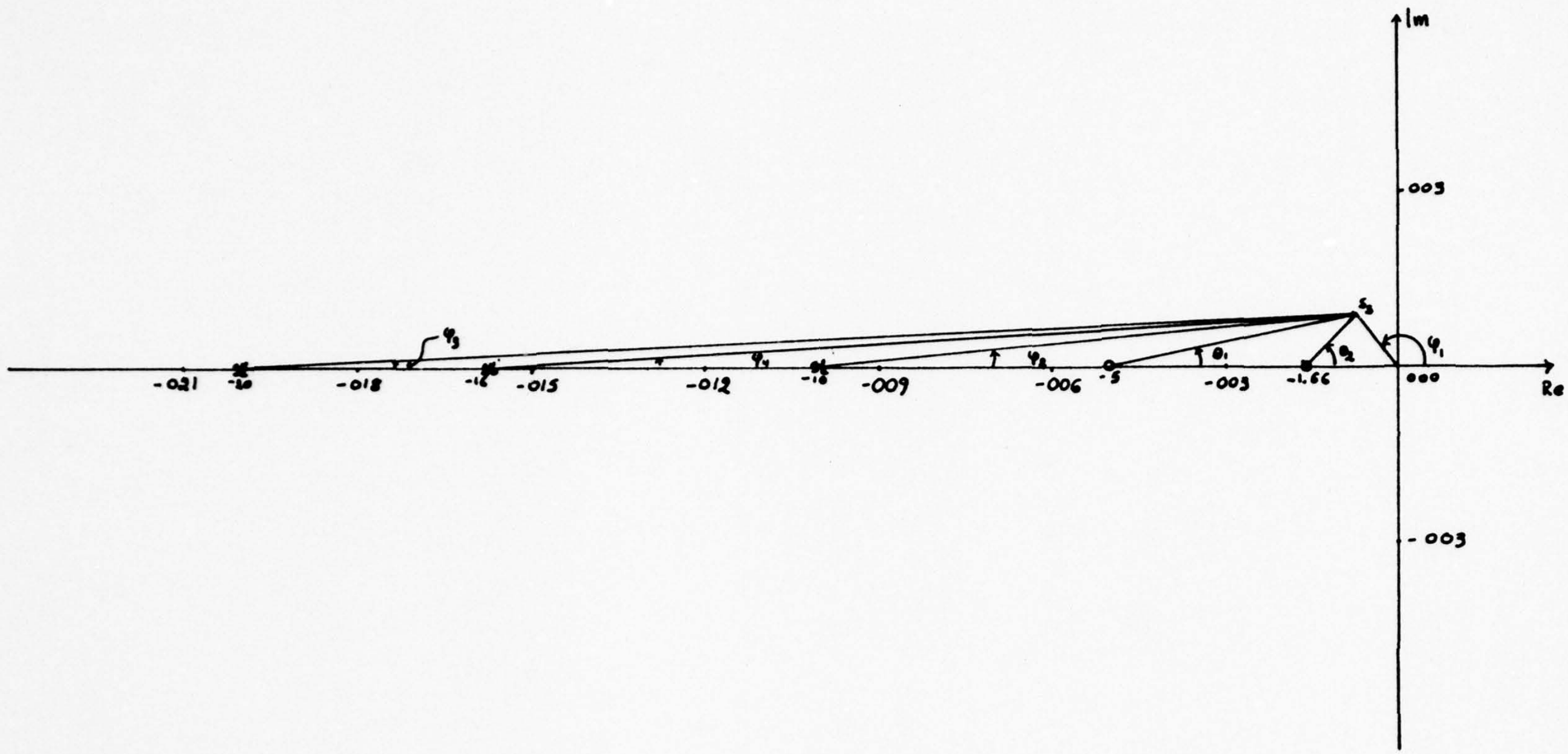


FIGURE (6-13c)
Example (6-4): Introduction of an additional pole and zero.

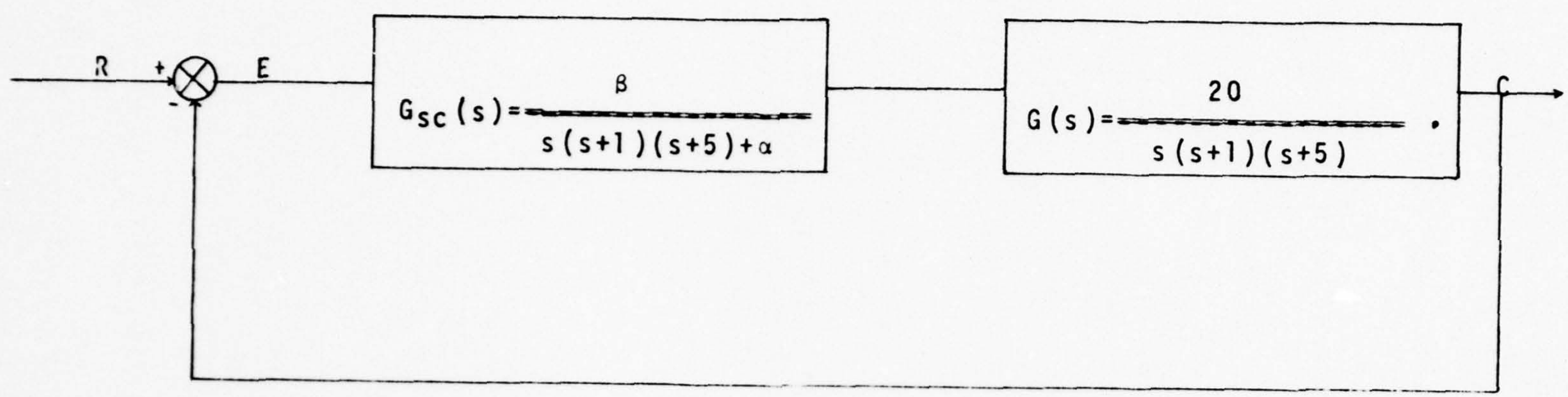


FIGURE (6-14a)
Example (6-5): Block diagram of the compensated singular system with the singular compensator considered in integrated form.

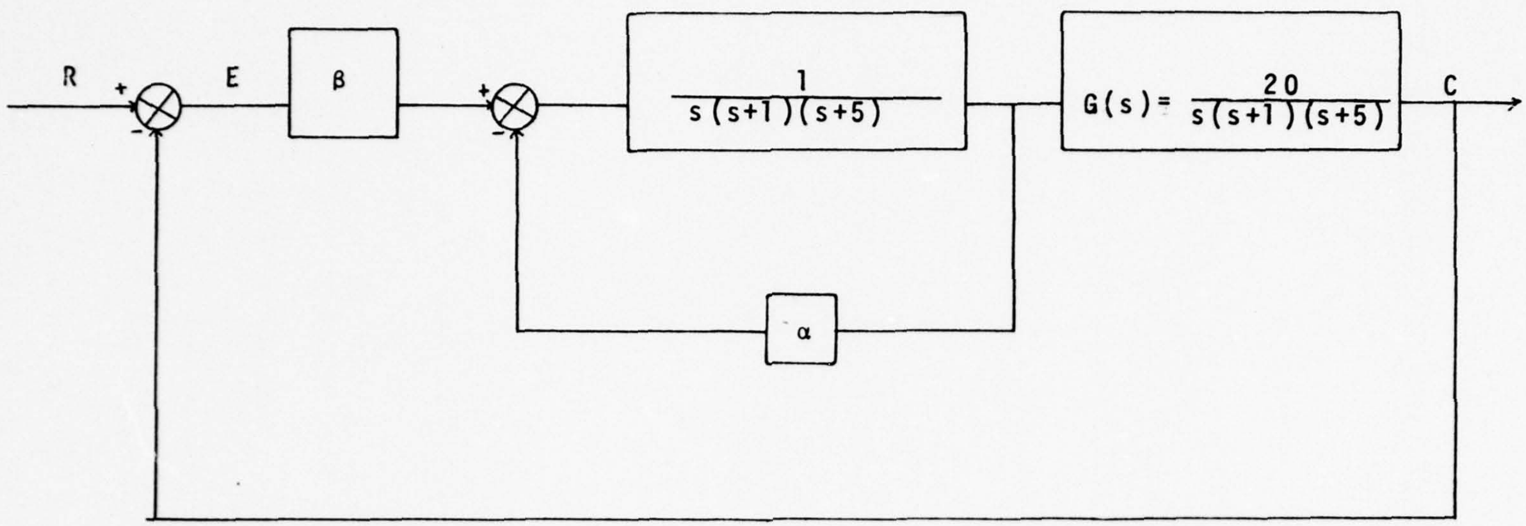
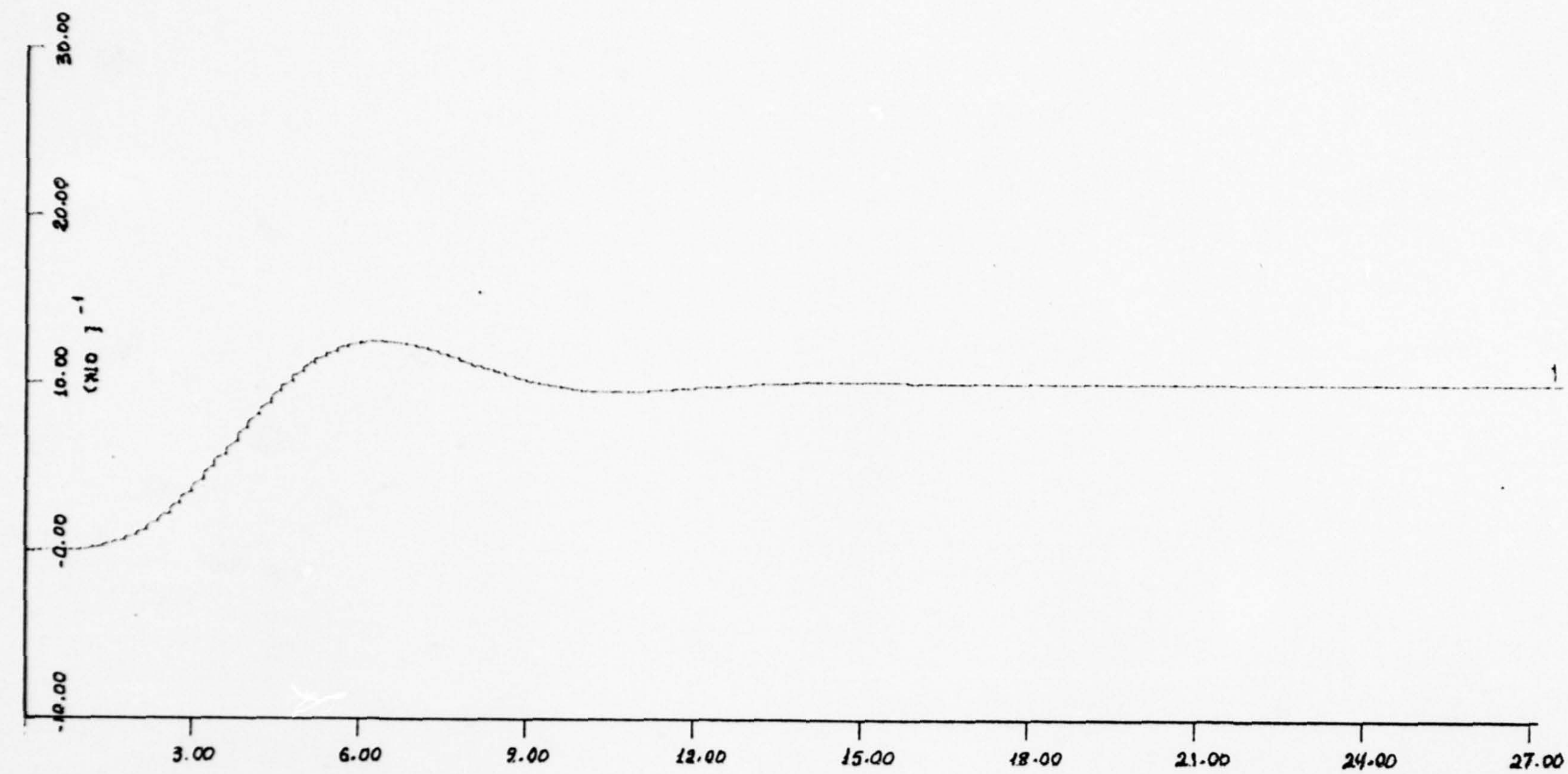
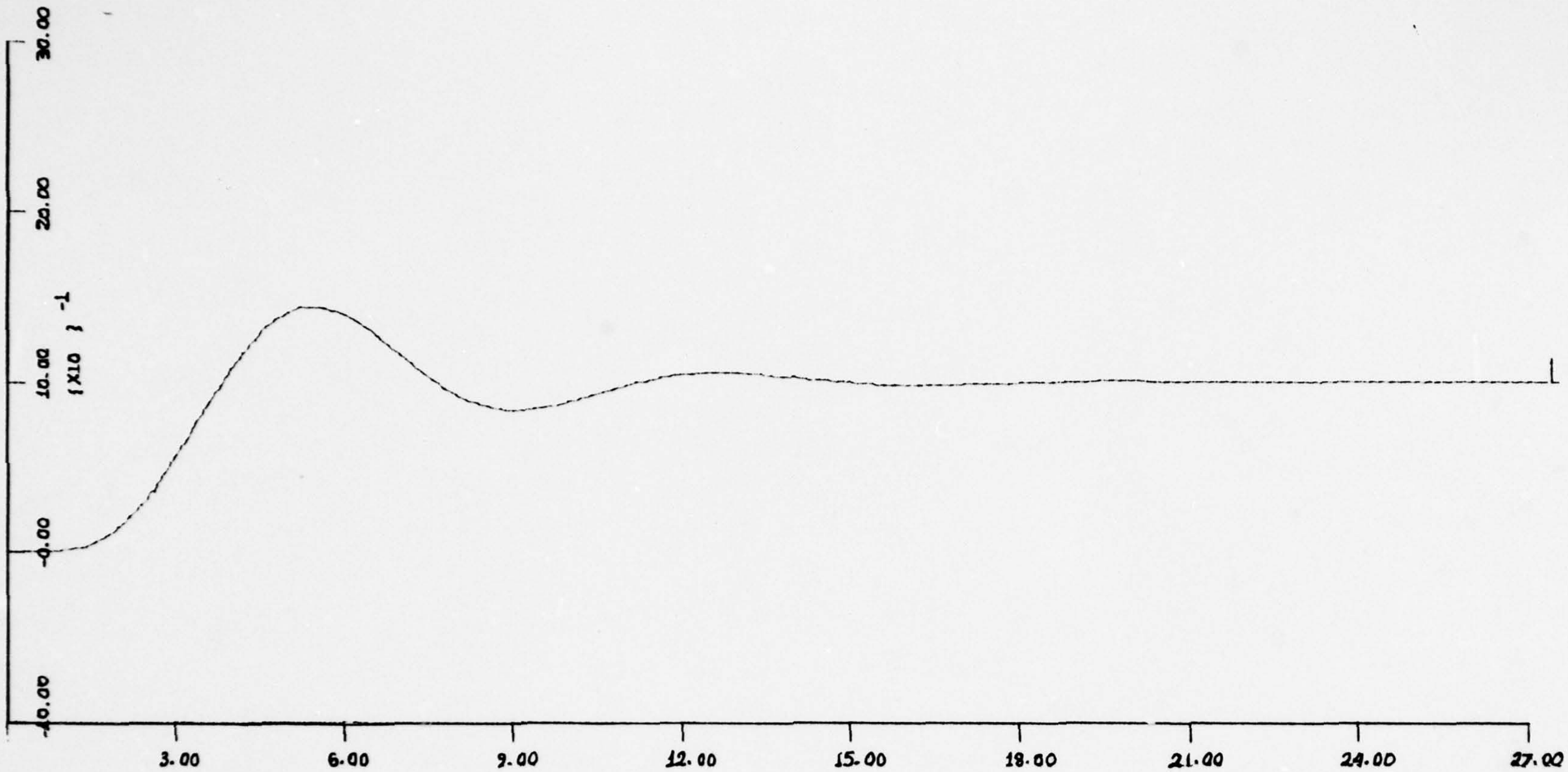


FIGURE (6-14b)
Example (6-5): Block diagram of the compensated singular system with the singular compensator considered to consist of distinct components.



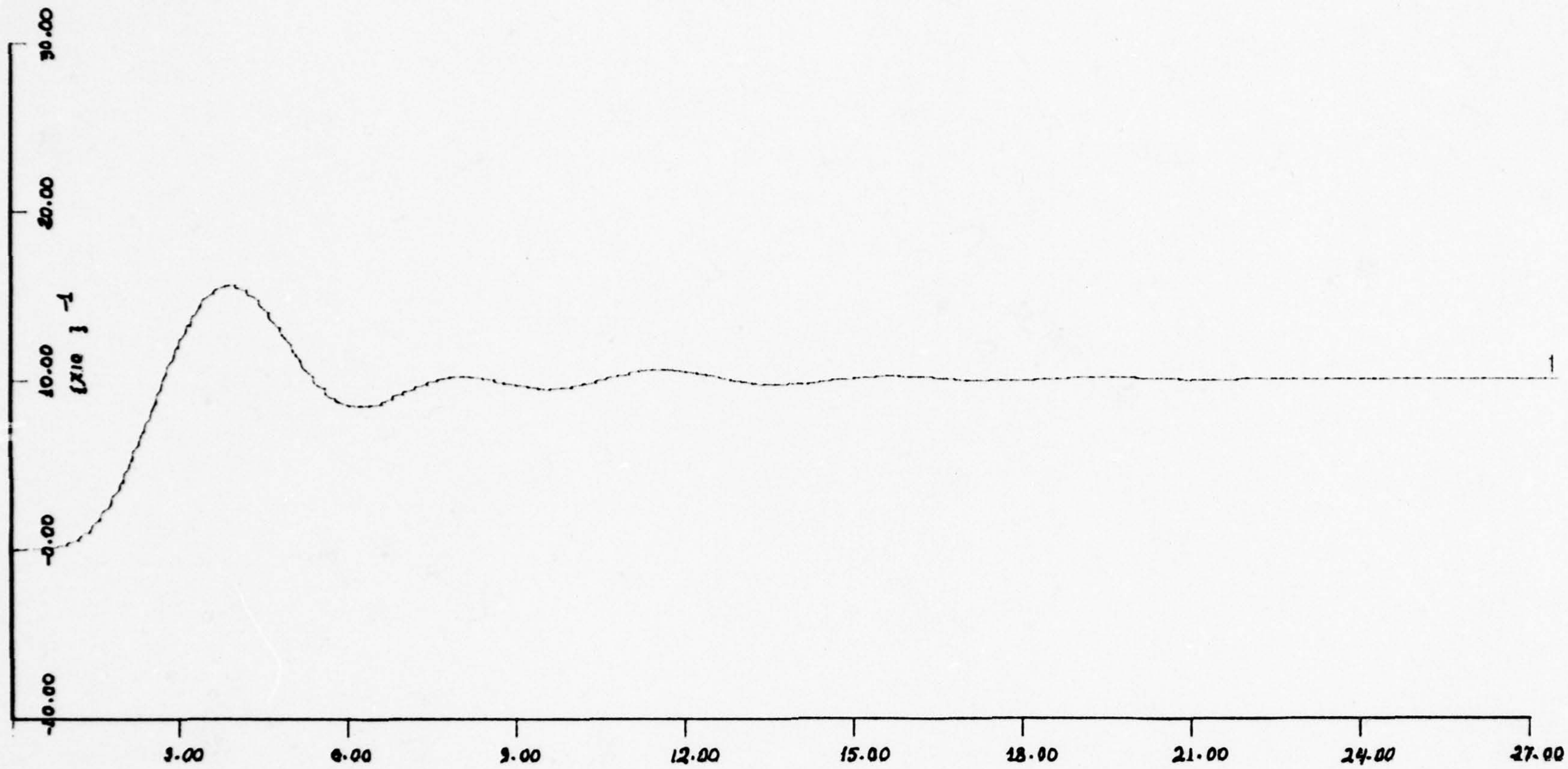
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-15)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values: A=7.8,B=0.71)



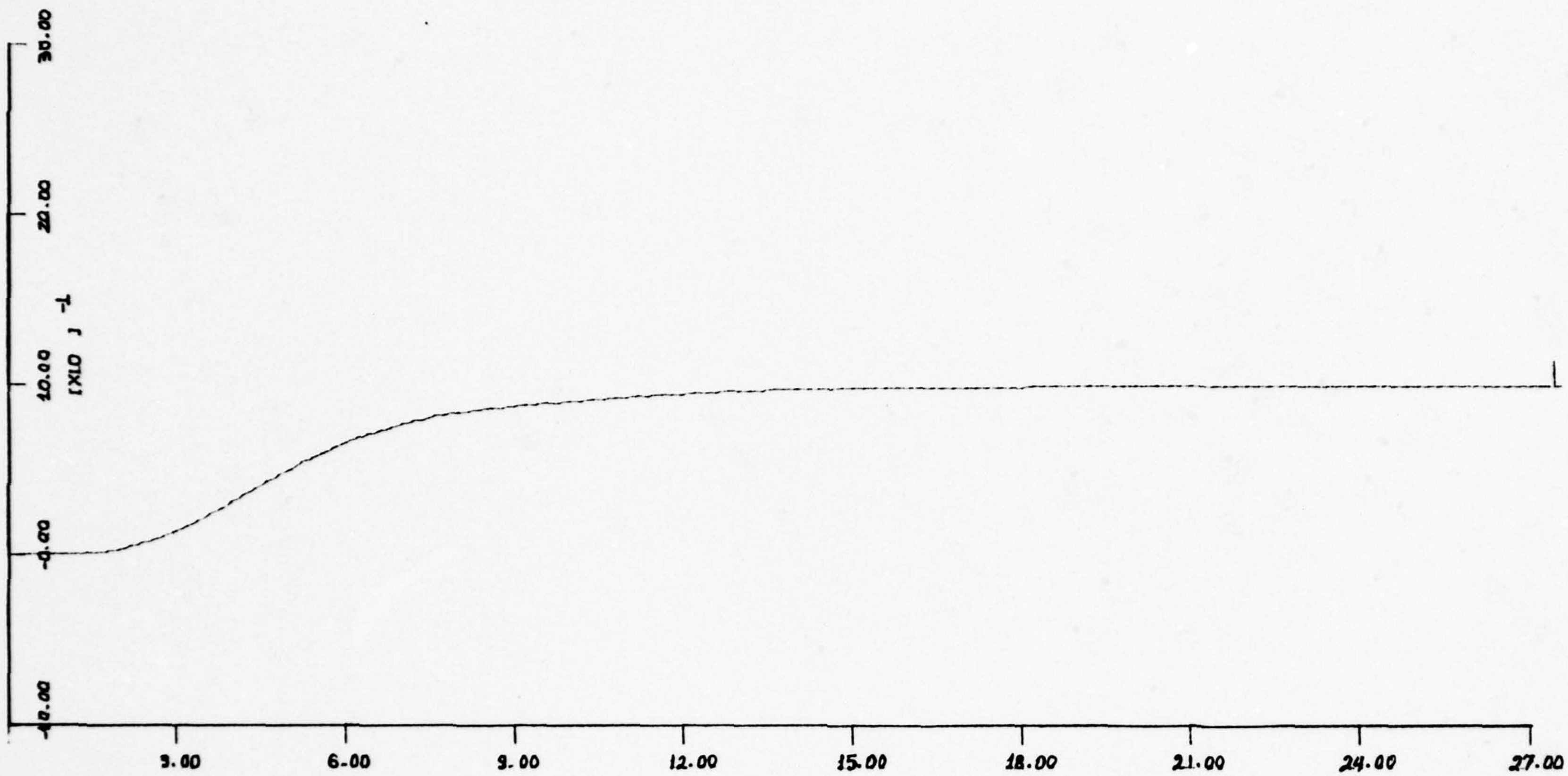
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-16)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values:A=9.85,B=1.123)



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-17)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values: A=20,B=3.712)



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-18)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values:A=6,B=0.265)

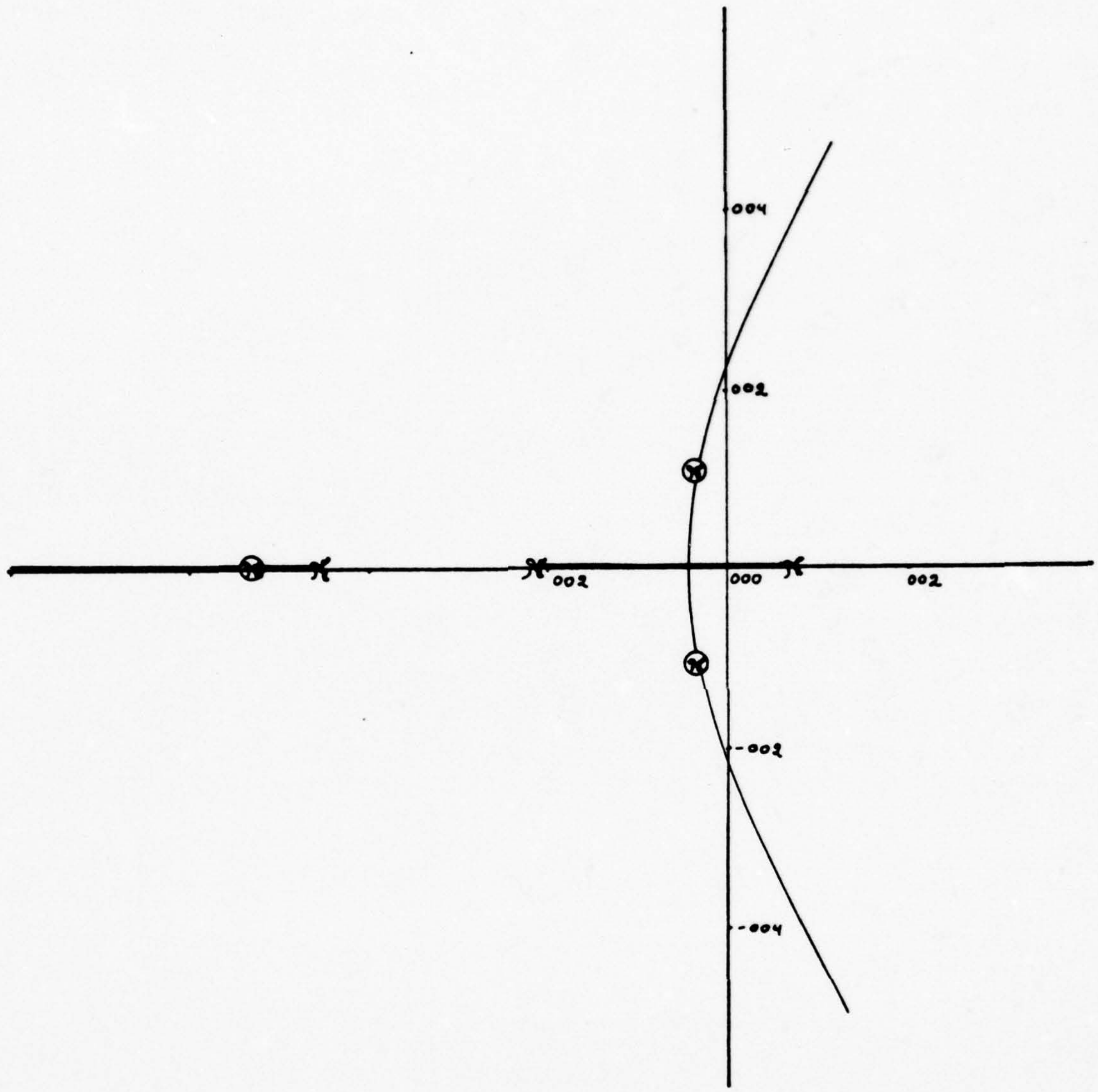
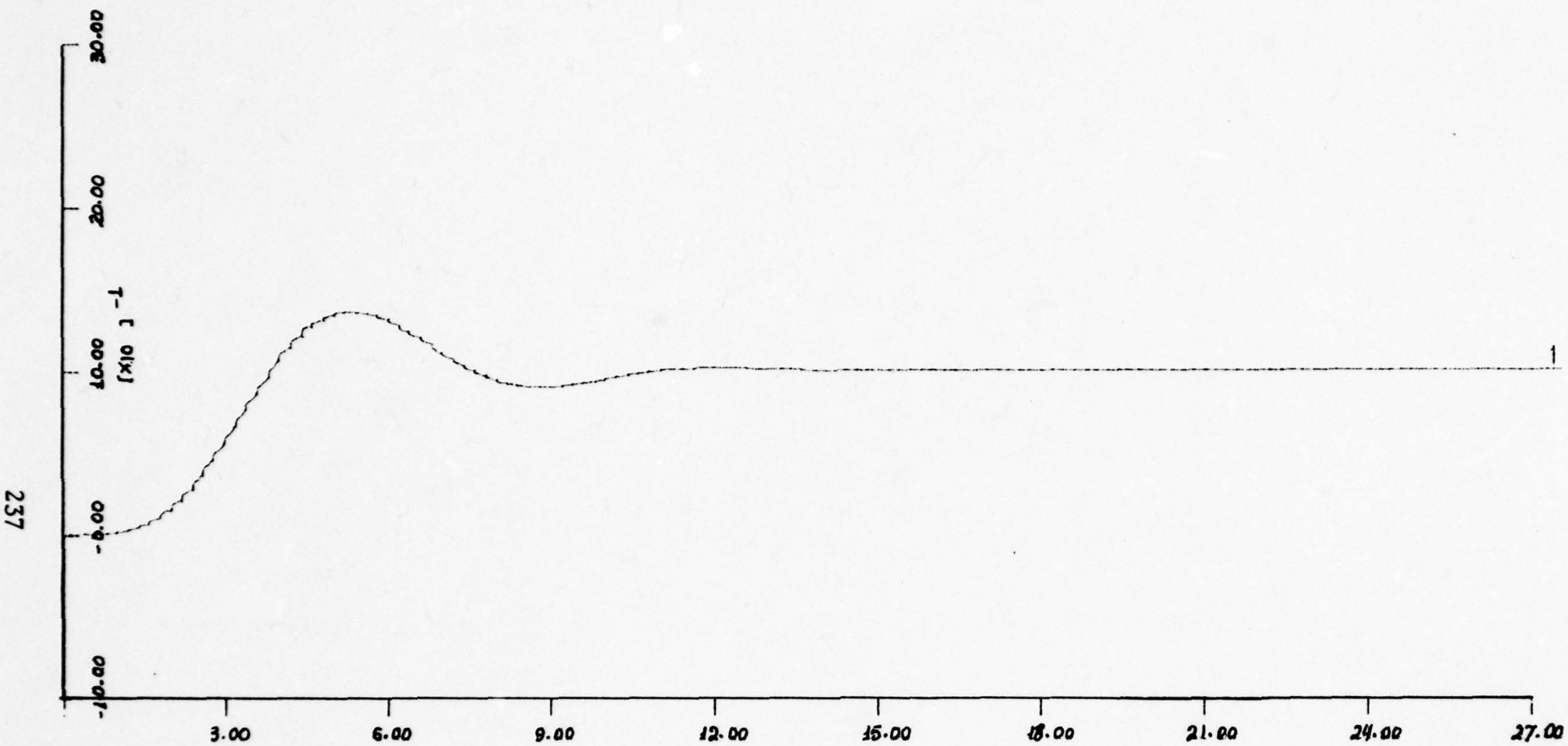
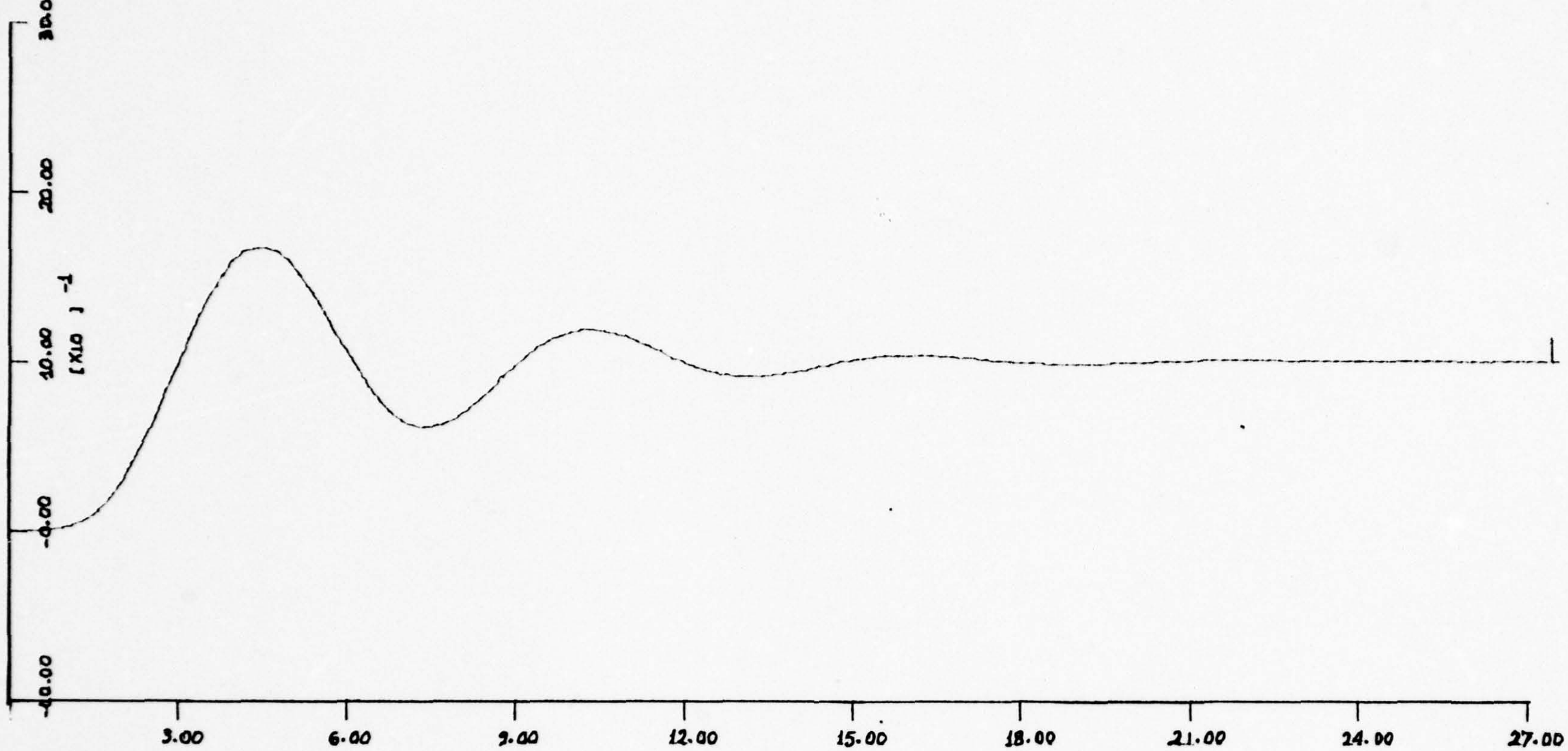


FIGURE (6-19)
Example(6-5): Root locus of the compensated singular system corresponding to the singular root (ζ_s=0.30, ω_{ns}=1.15268).



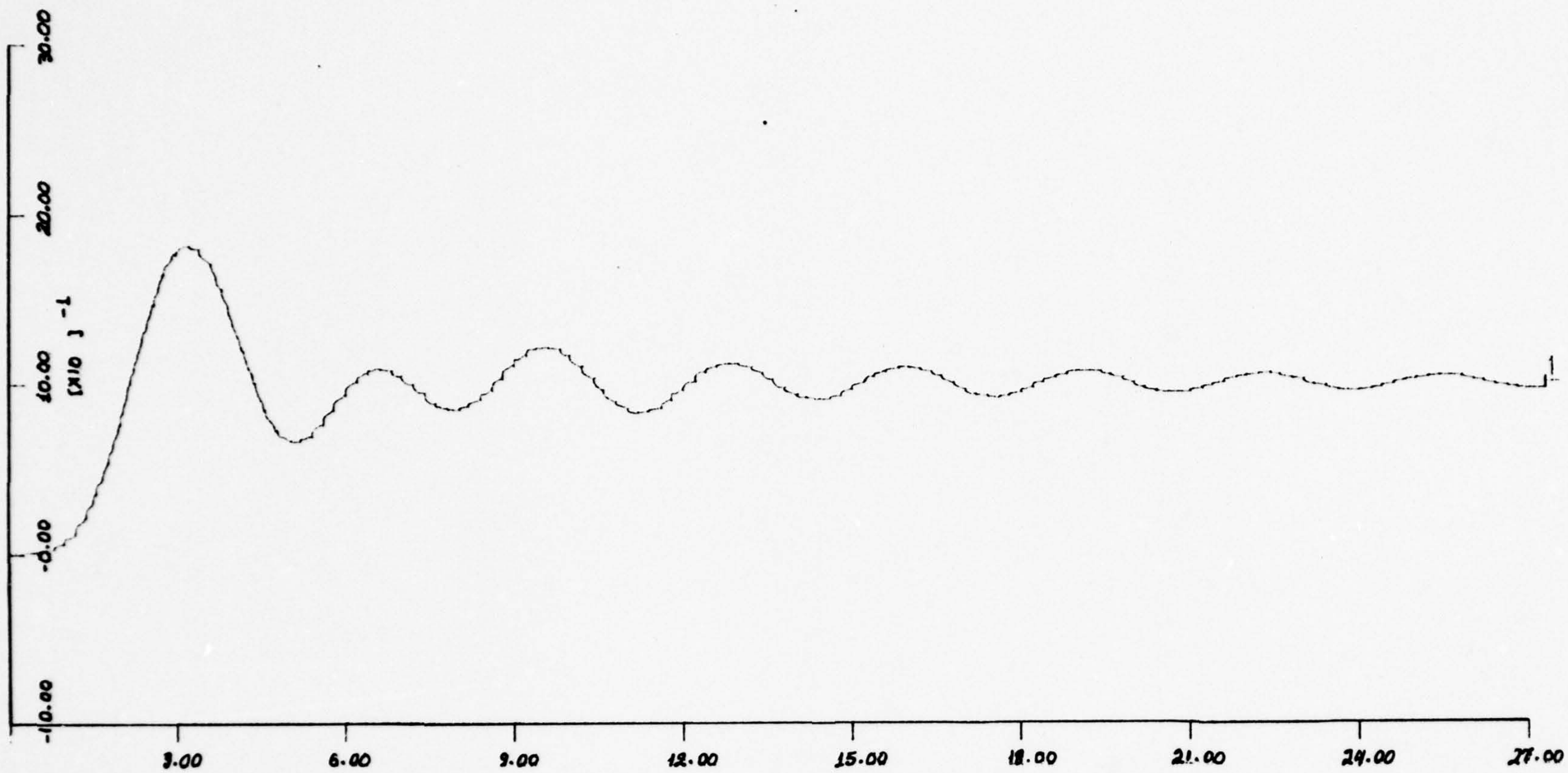
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-20)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values:A=10.6,B=1.251)



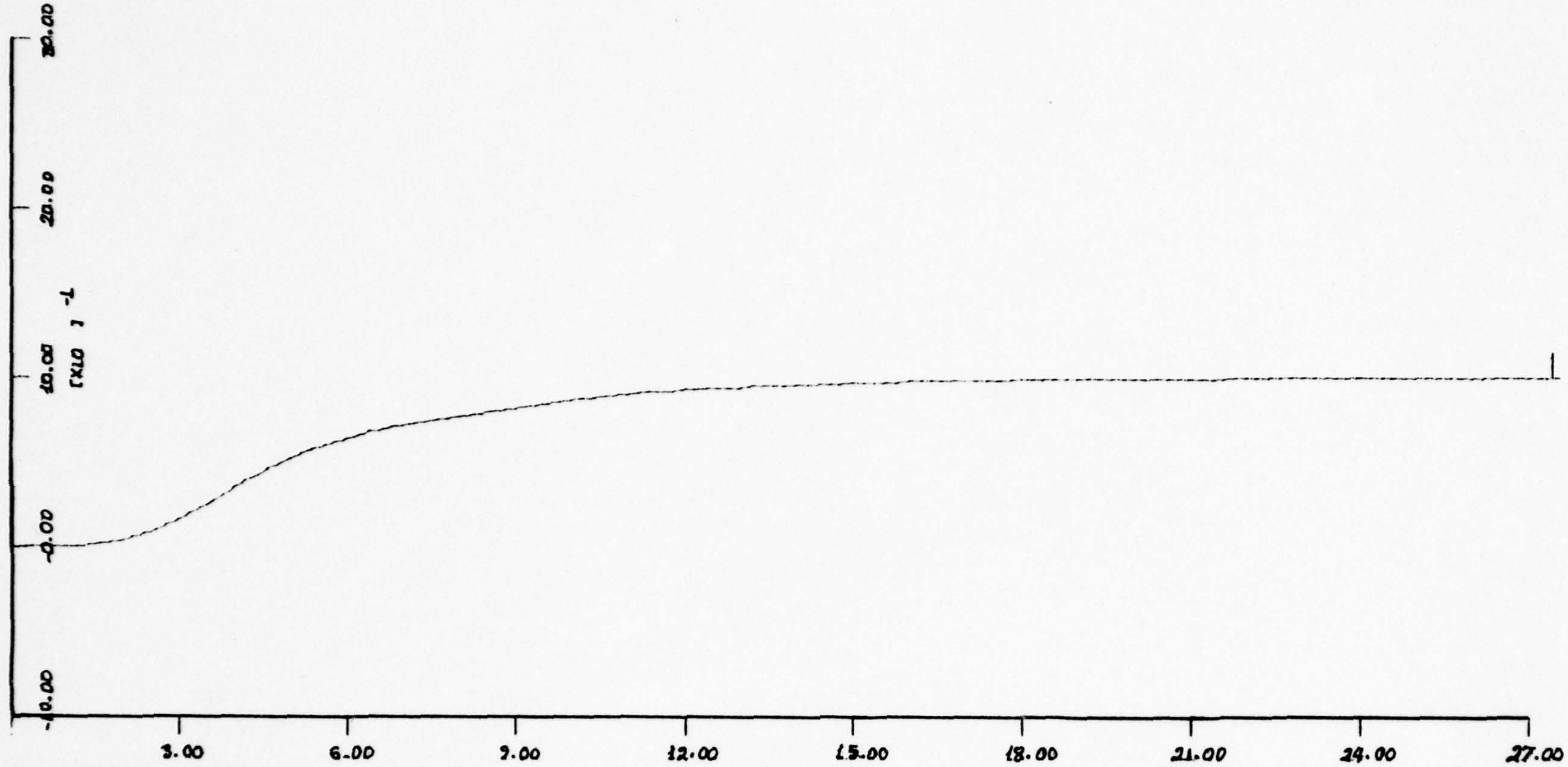
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-21)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values:A=14.1,B=2.485)



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-22)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values: A=30, B=8.092)



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-23)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values: A=8, B=0.334)

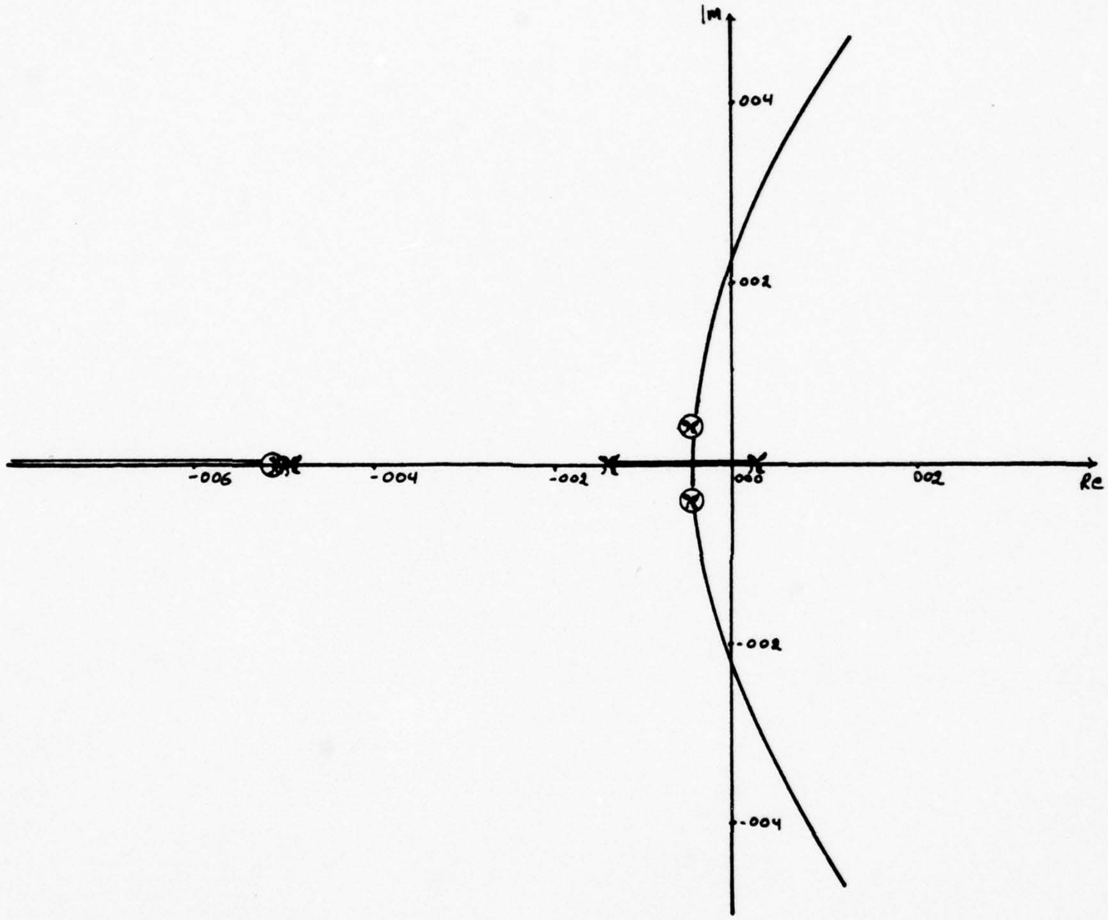
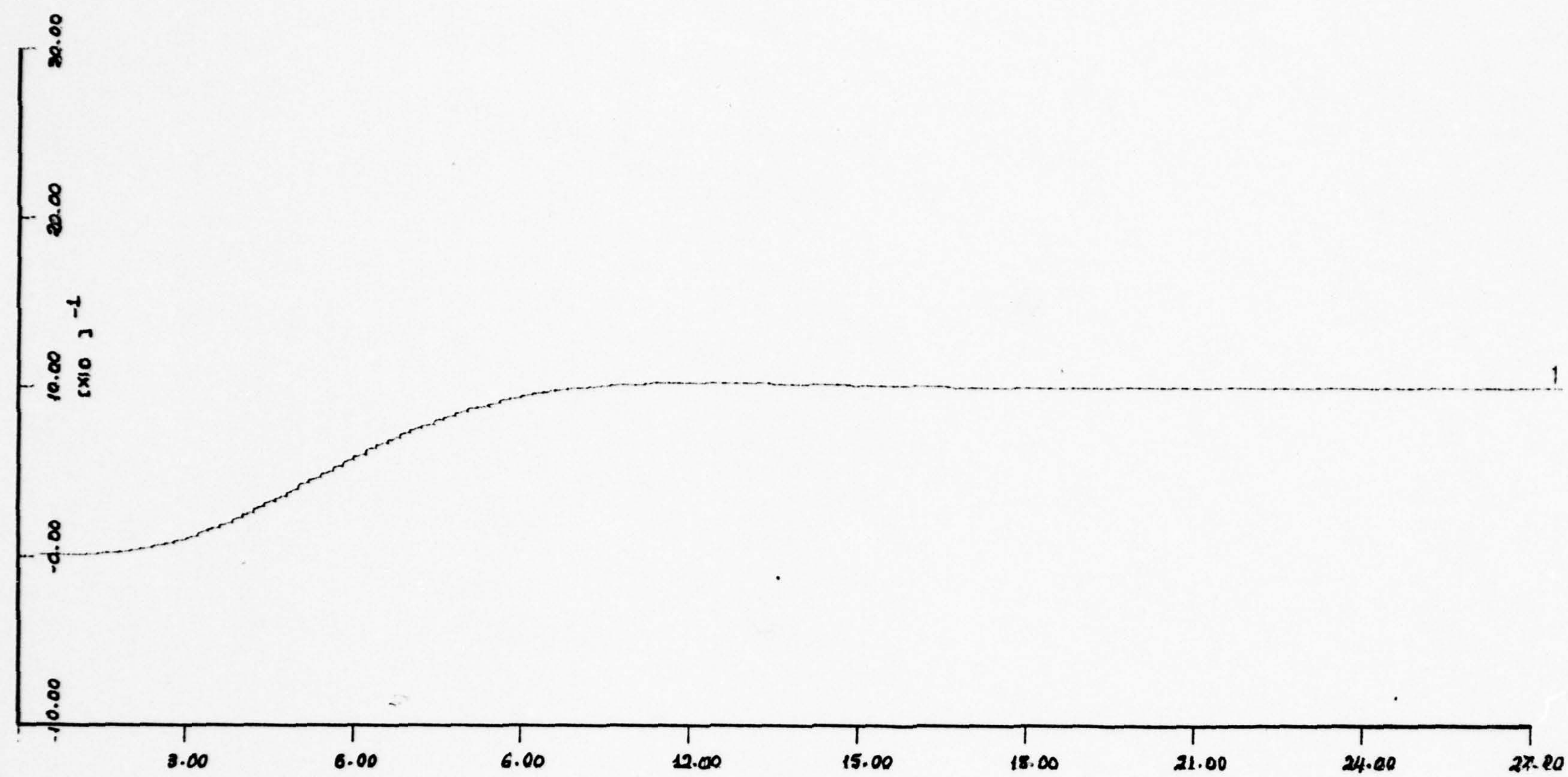


FIGURE (6-24)
Example (6-5): Root locus of the compensated singular system corresponding to the singular root (ζ_s=0.75, ω_{n_s}=0.60667).



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-25)
Example (6-5):Time response of the compensated singular system for a step input.
(Parameter values: A=3.6, B=0.162)

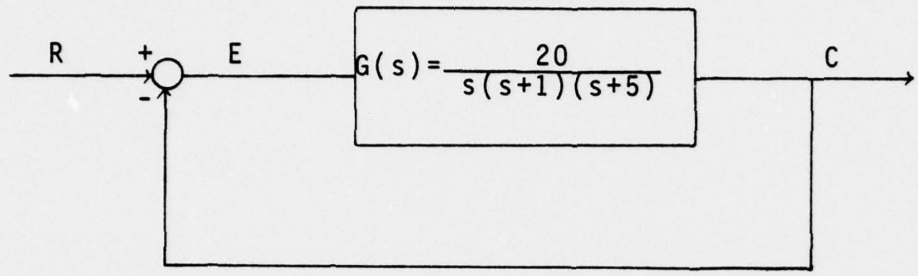


FIGURE (6-26)
Example (6-6): Block diagram of the plant.

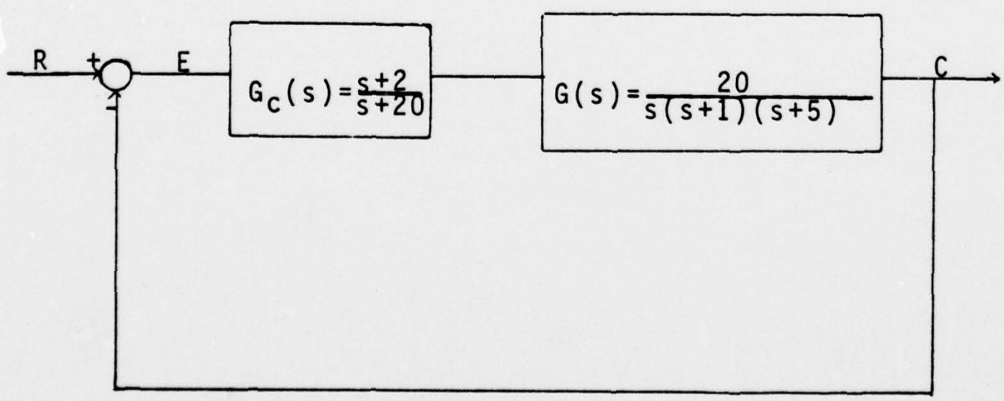


FIGURE (6-27)
Example (6-6): Block diagram of the initially compensated plant.

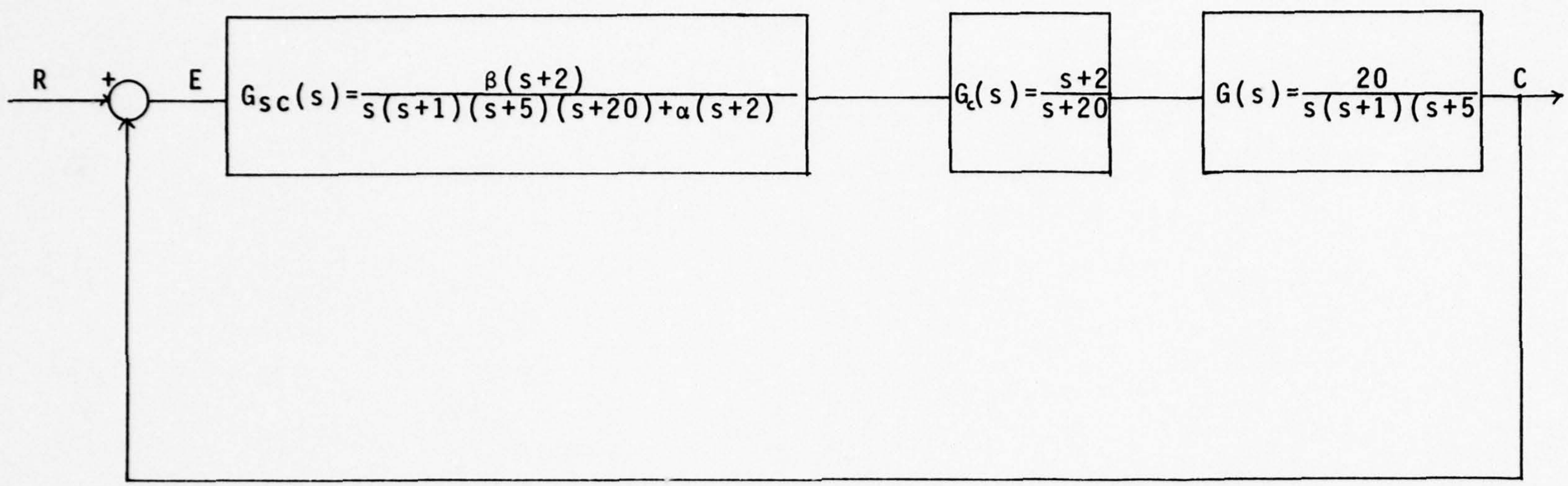


FIGURE (6-28a)
Example (6-6): Block diagram of the compensated singular system.

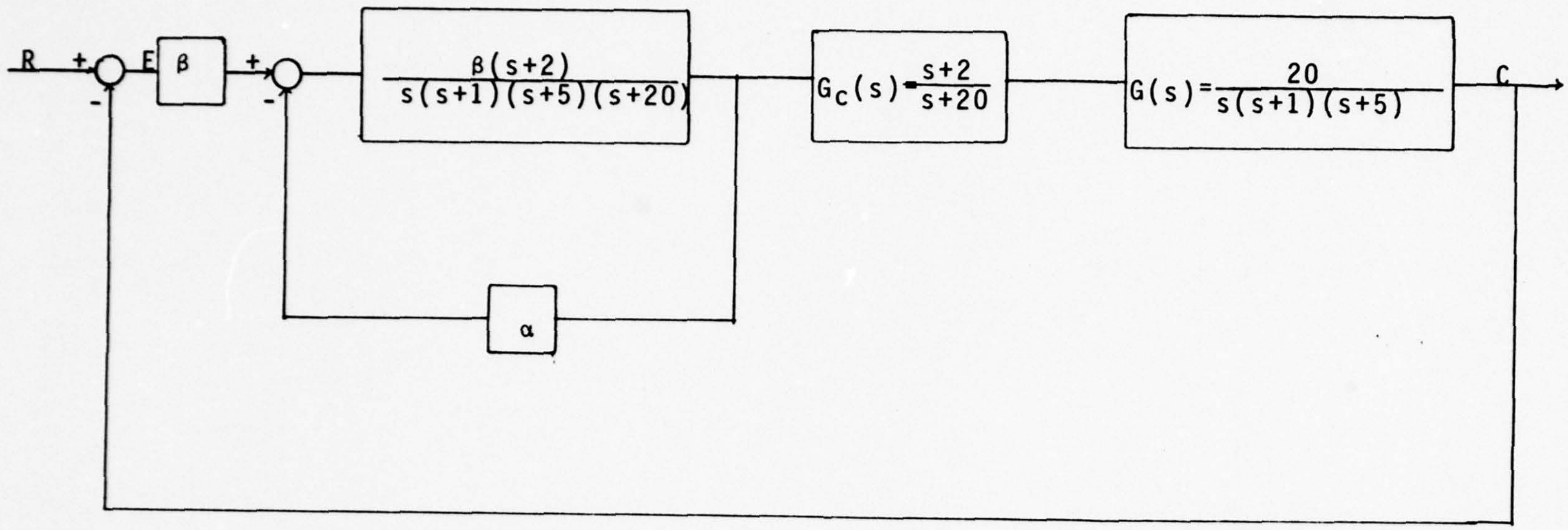


FIGURE (6-28b)
Example (6-6): Block diagram of the compensated singular system.
(Equivalent block diagram with that shown in figure (6-28a))

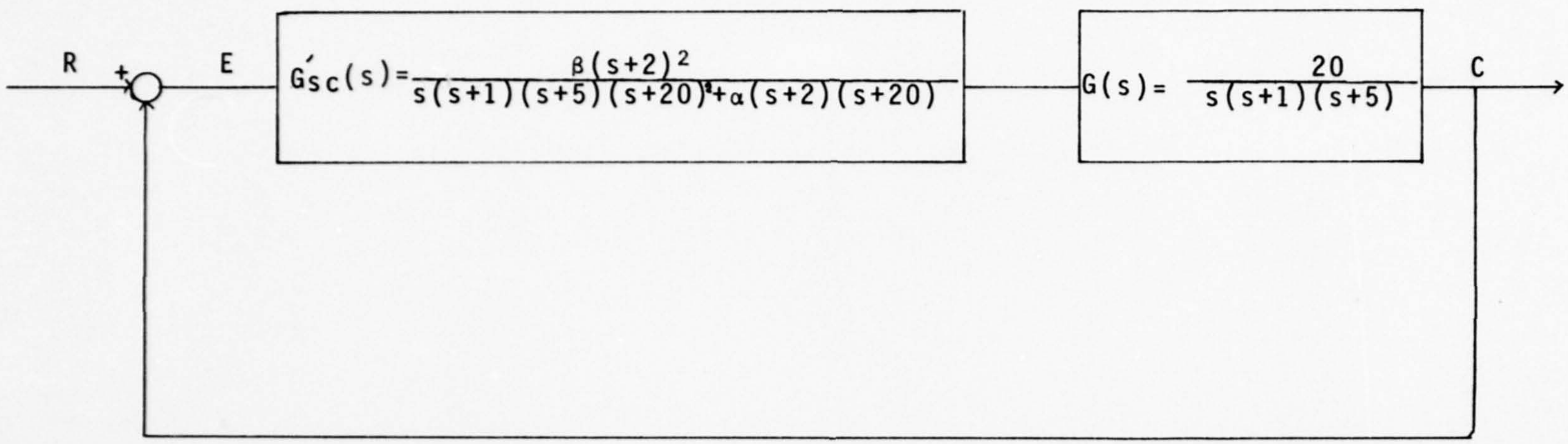


FIGURE (6-28c)
Example (6-6): Equivalent block diagram of the compensated singular system where $G_{sc}(s)$ represents the overall compensator which is formed by combining the singular and the initial compensators.

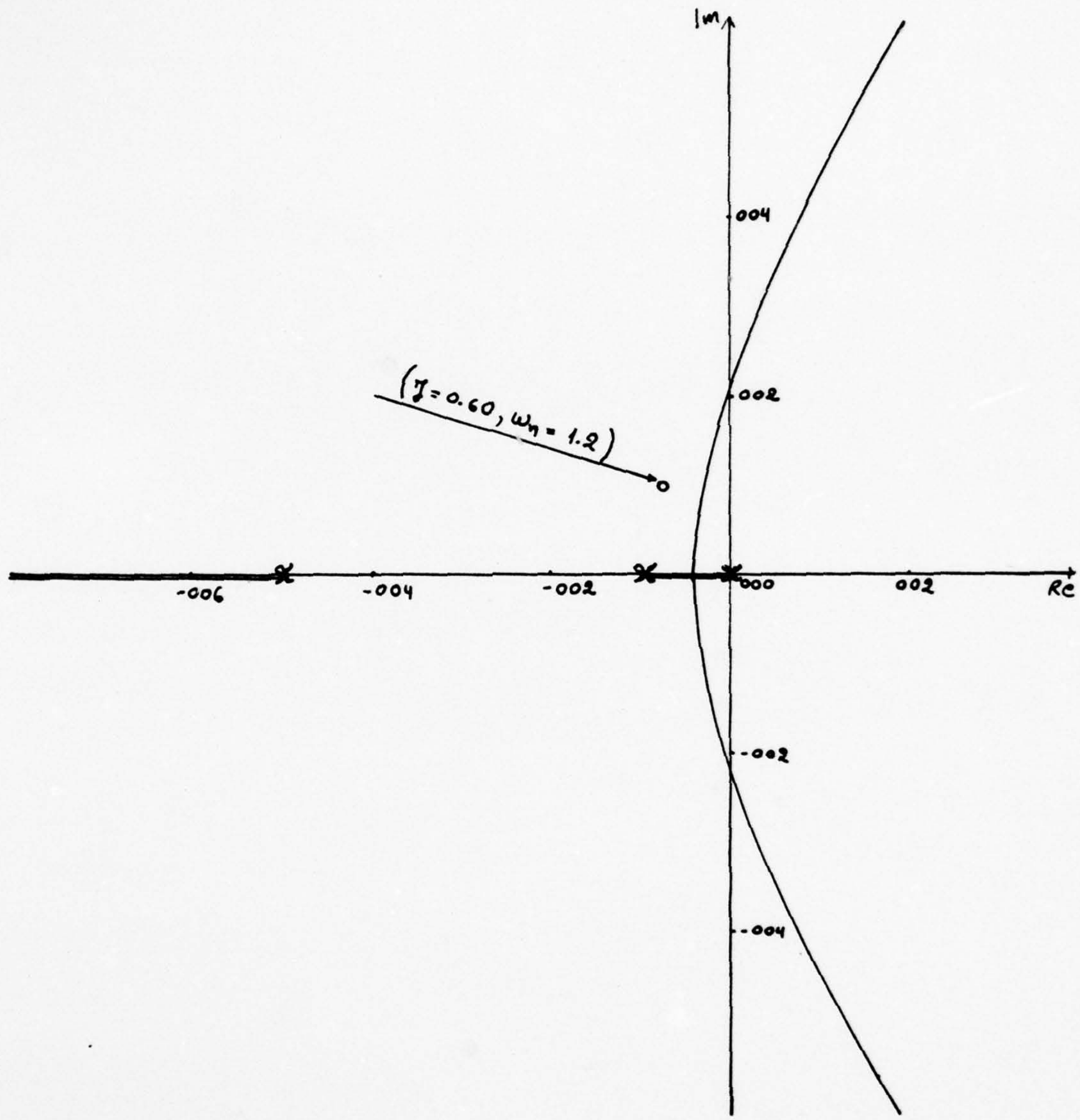


FIGURE (6-29)
Example (6-6): Root locus of the uncompensated system.

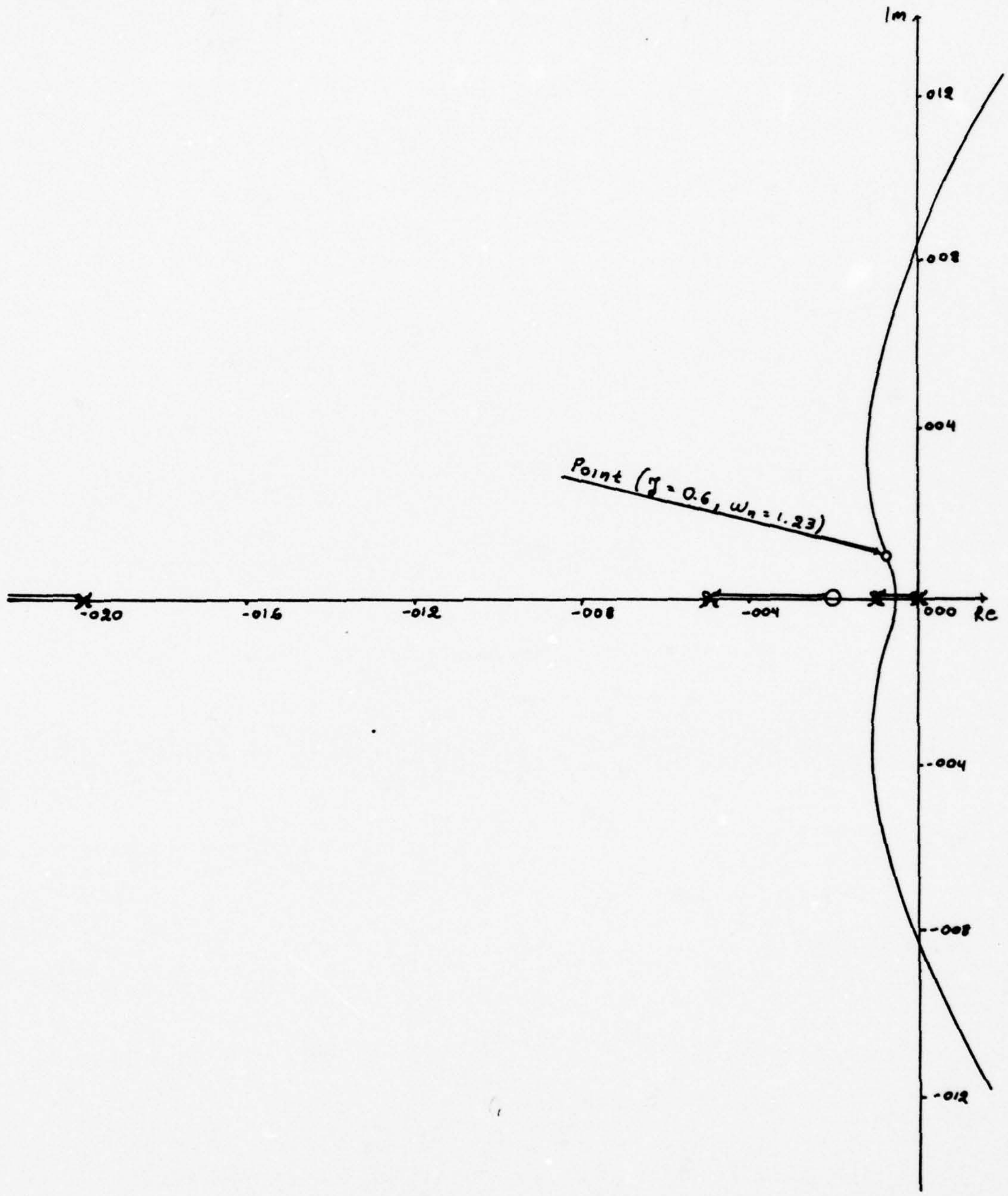


FIGURE (6-30)
Example (6-6): Root locus of the plant after it was compensated by an initial cascade compensated with transfer function(s+2)/(s+20).

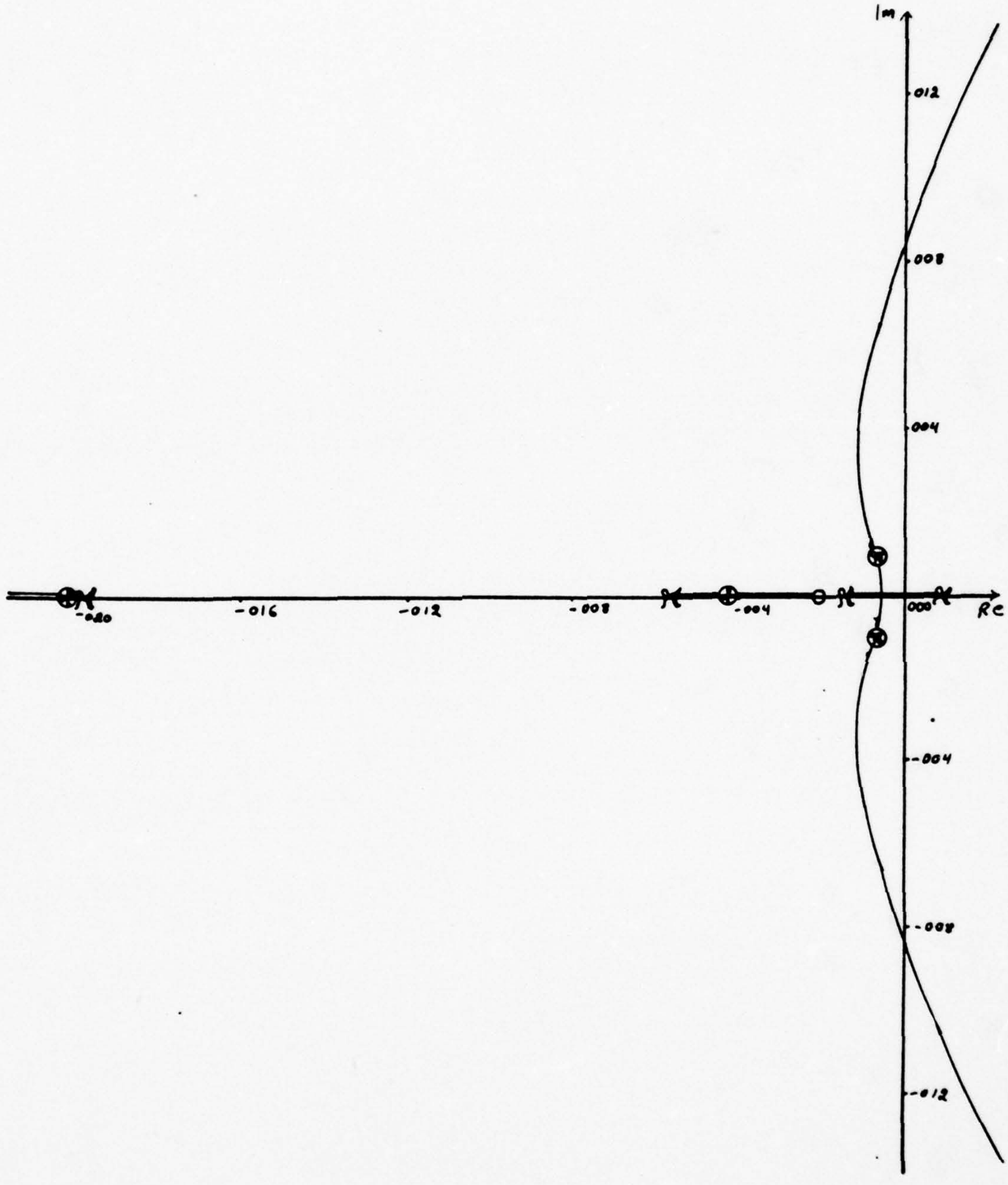
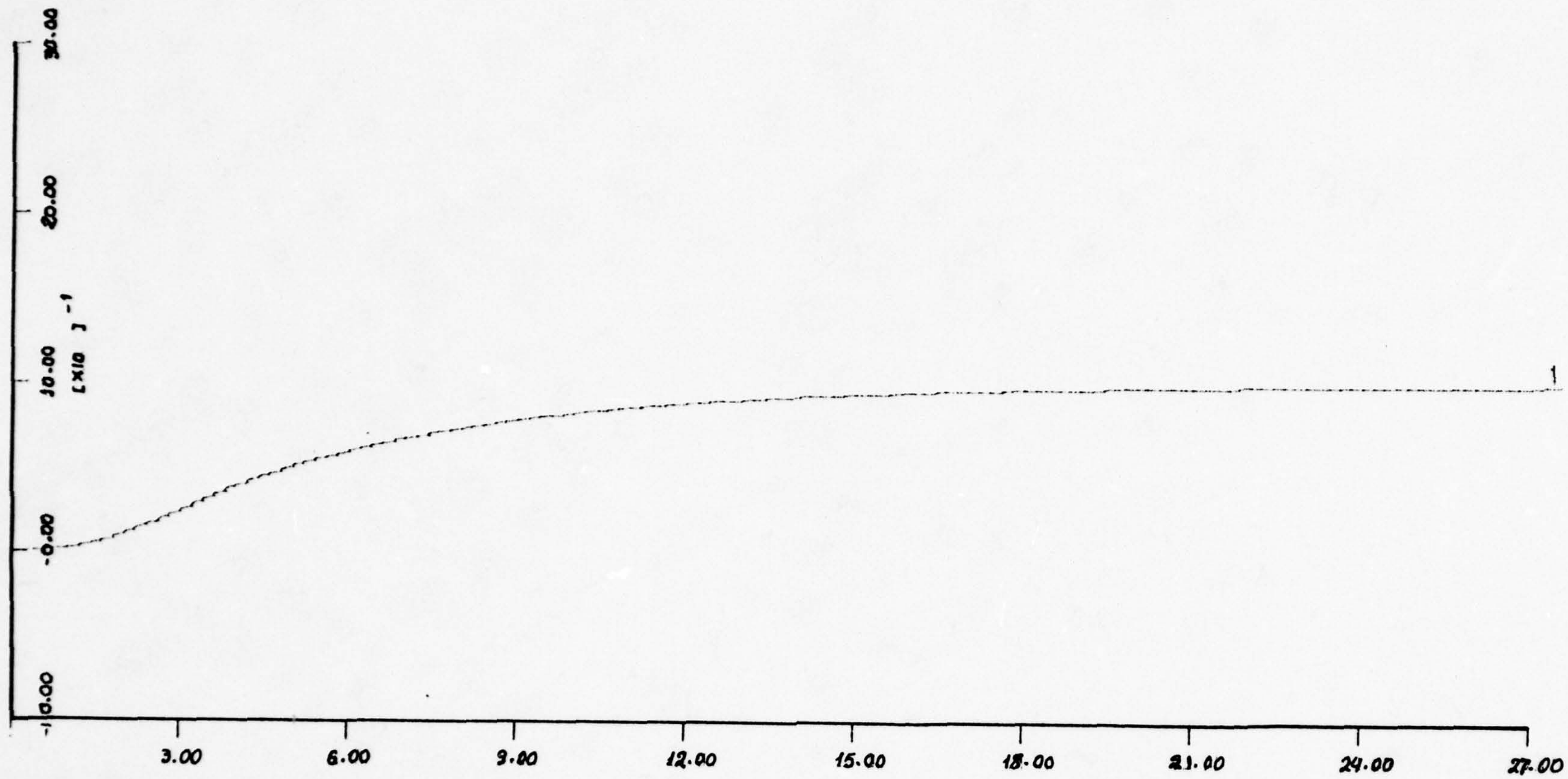
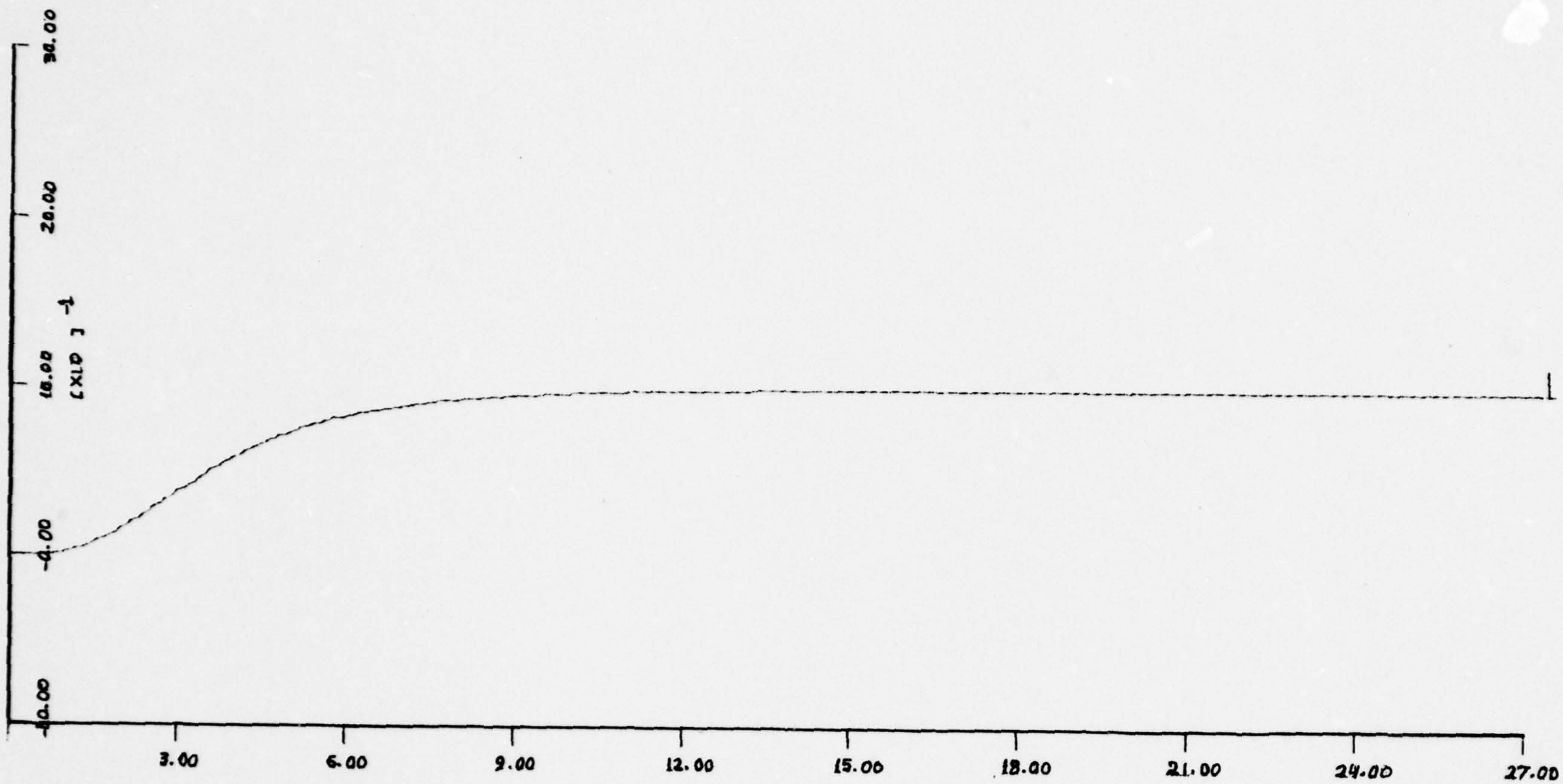


FIGURE (6-31)
Example (6-6):Root locus of the singular compensated system corresponding to the singular root (ζ_s=0.6, ω_{ns}=1.2326) or (s= -0.7395±j0.9861)^s



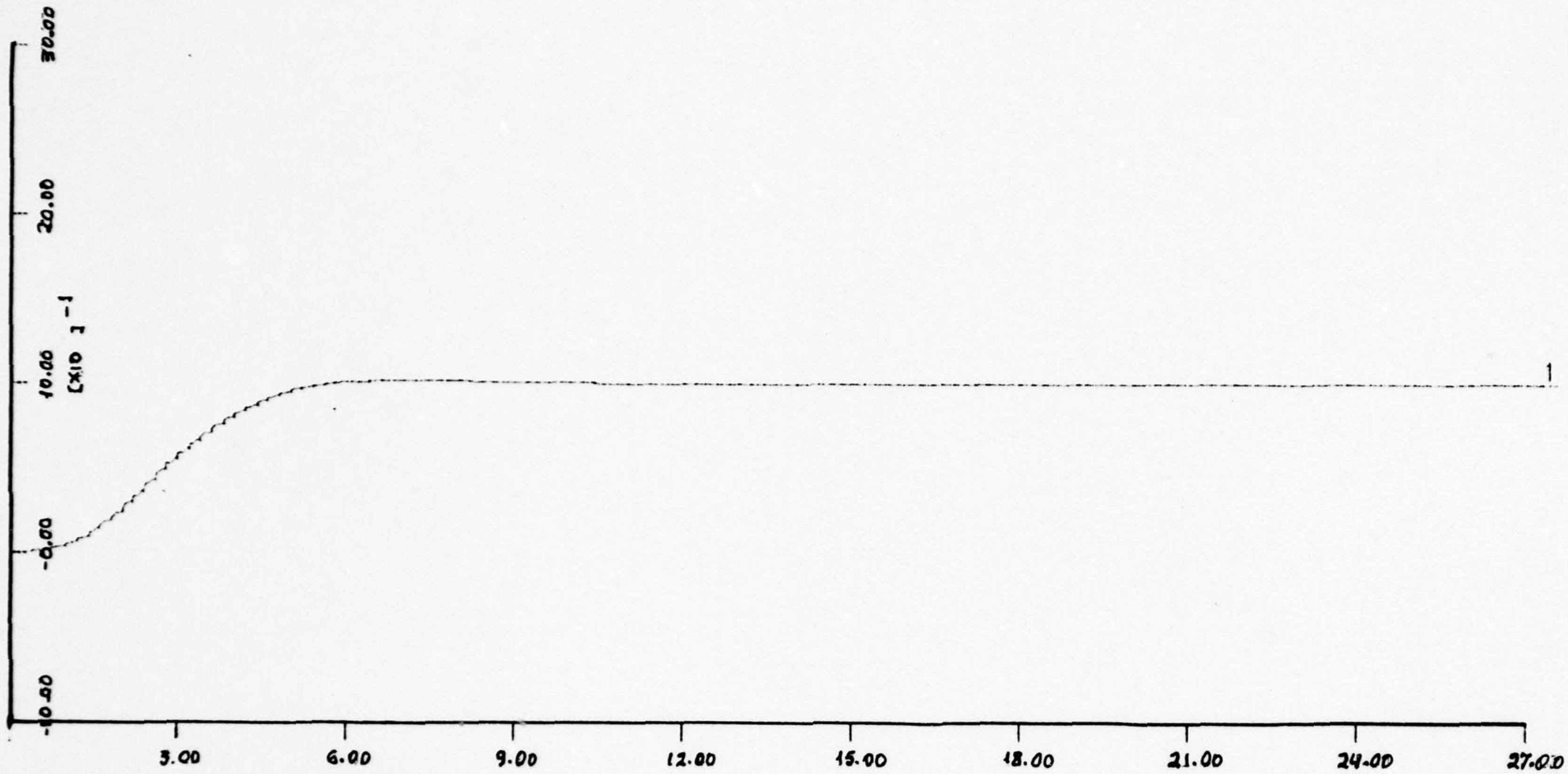
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-32)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=75, B=28.987)



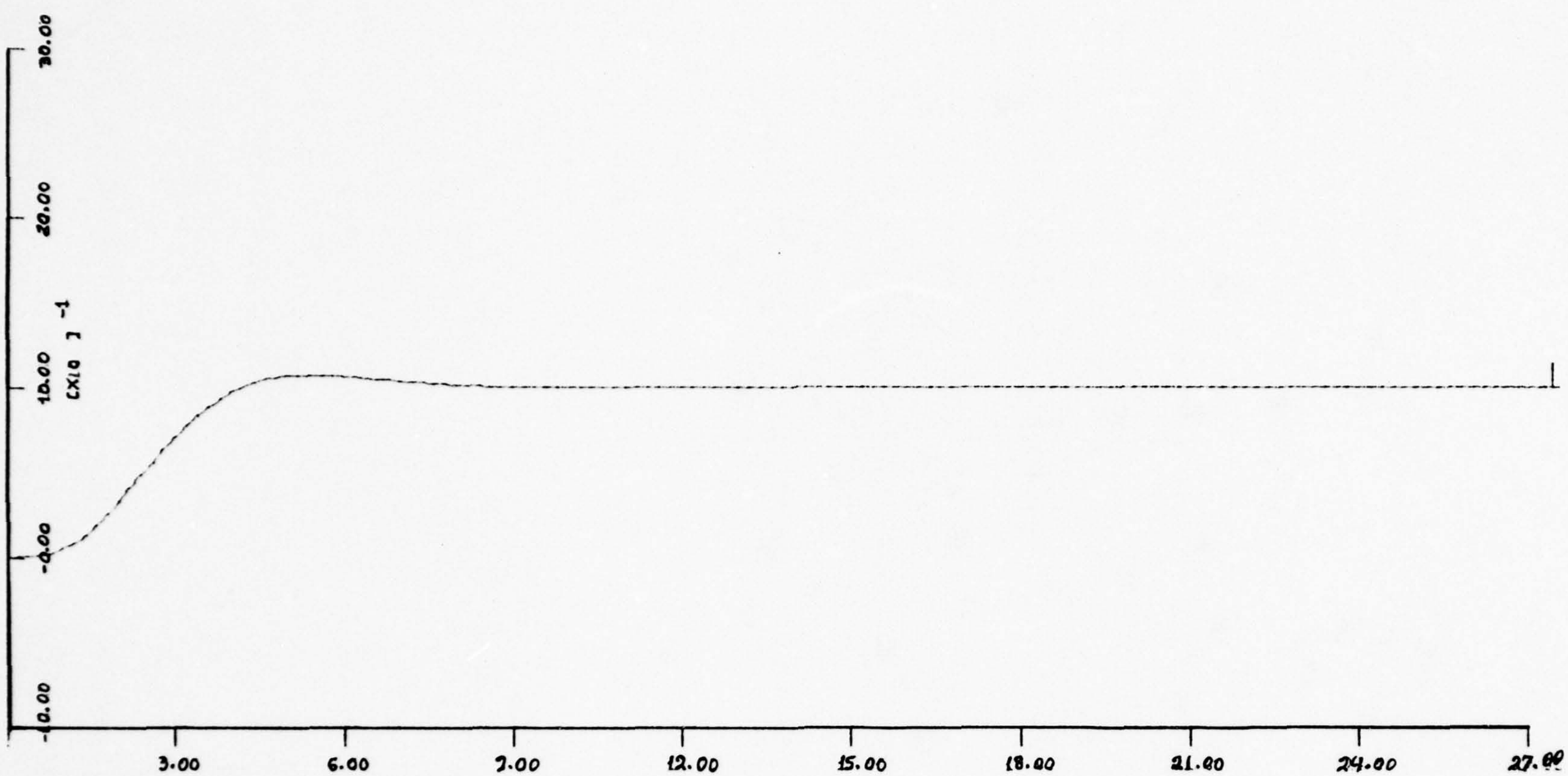
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-33)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=81.3, B=49.856)



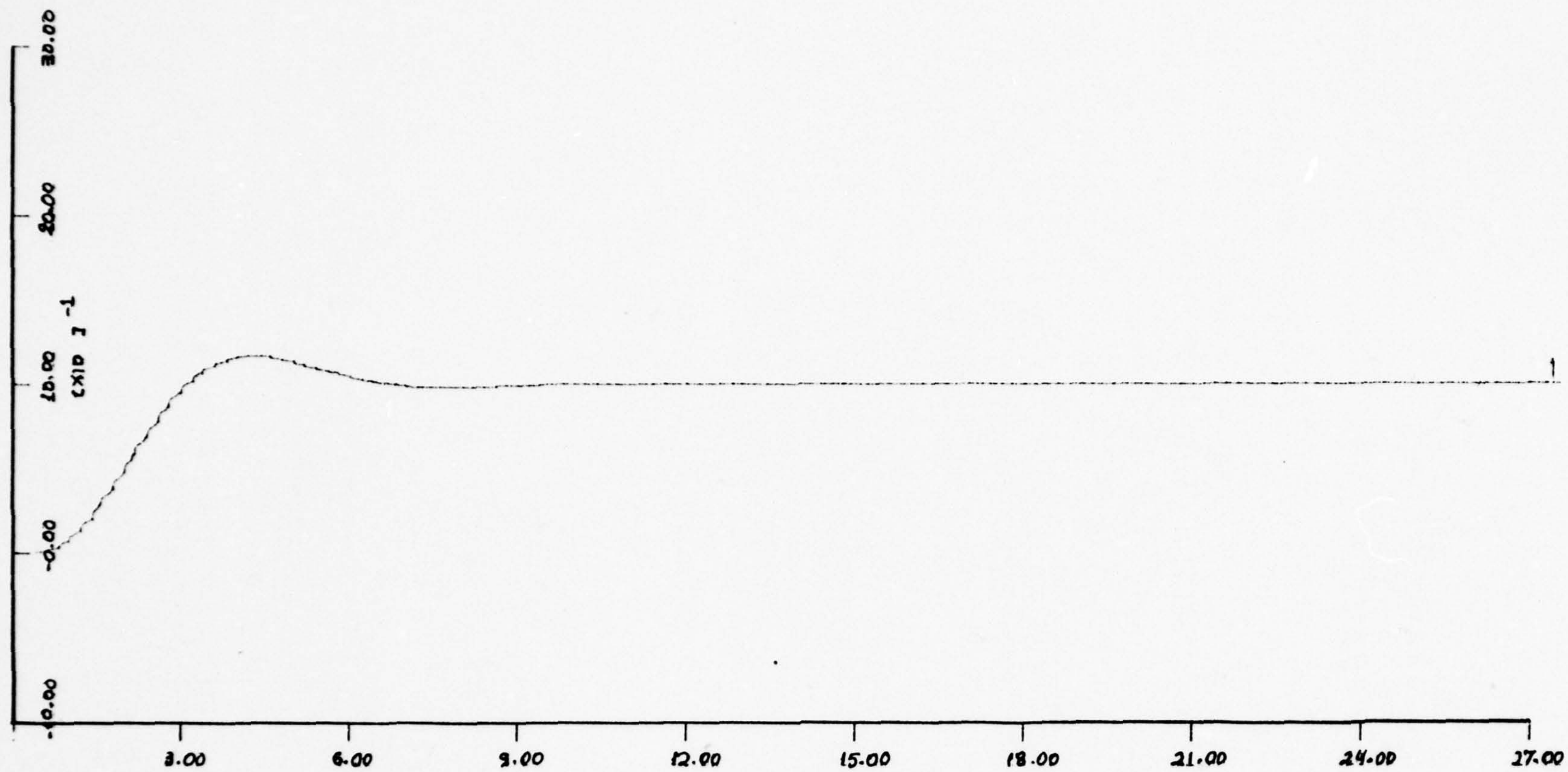
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-34)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=90, B=78.674)



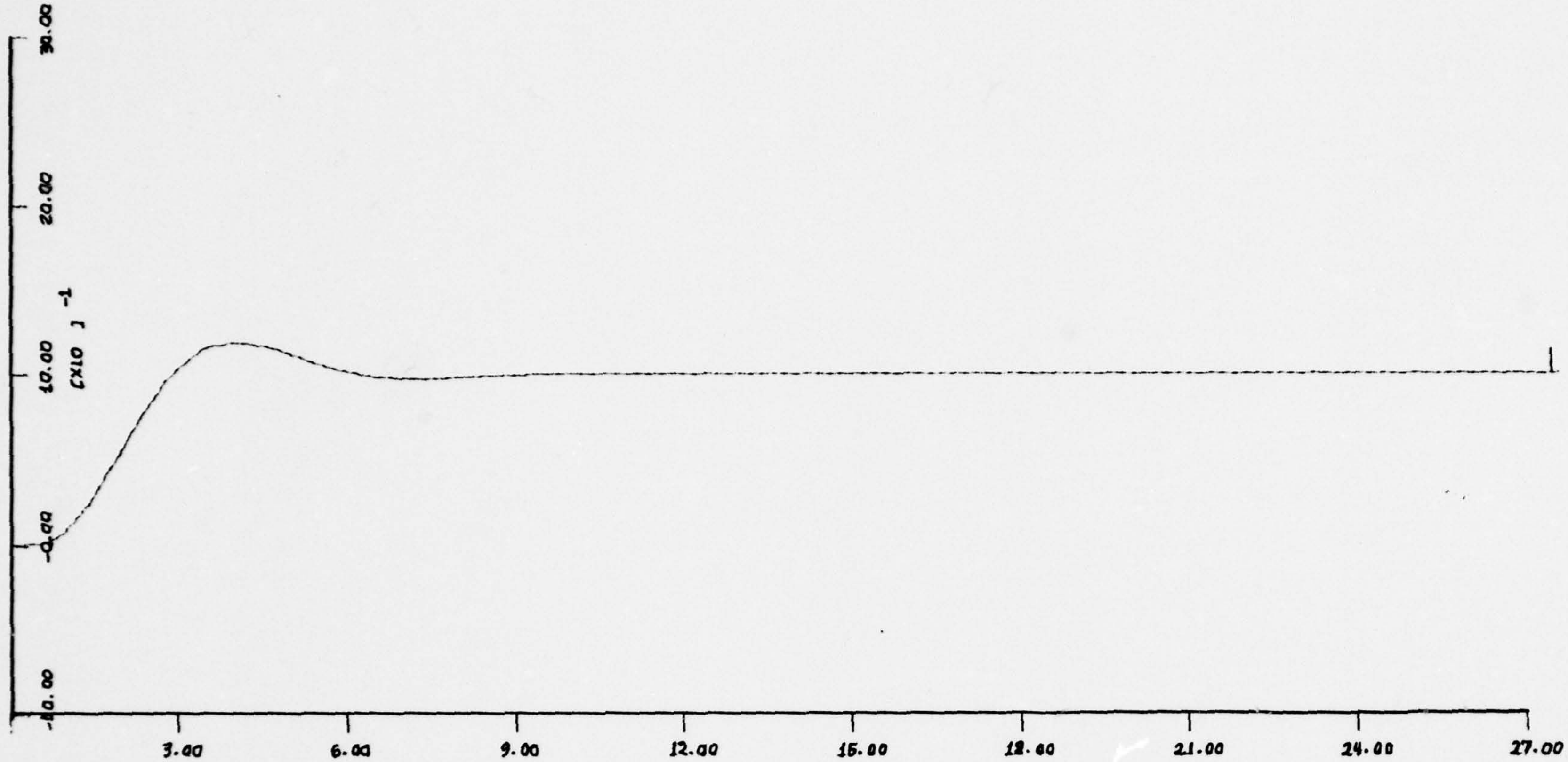
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-35)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=100, B=111.798)



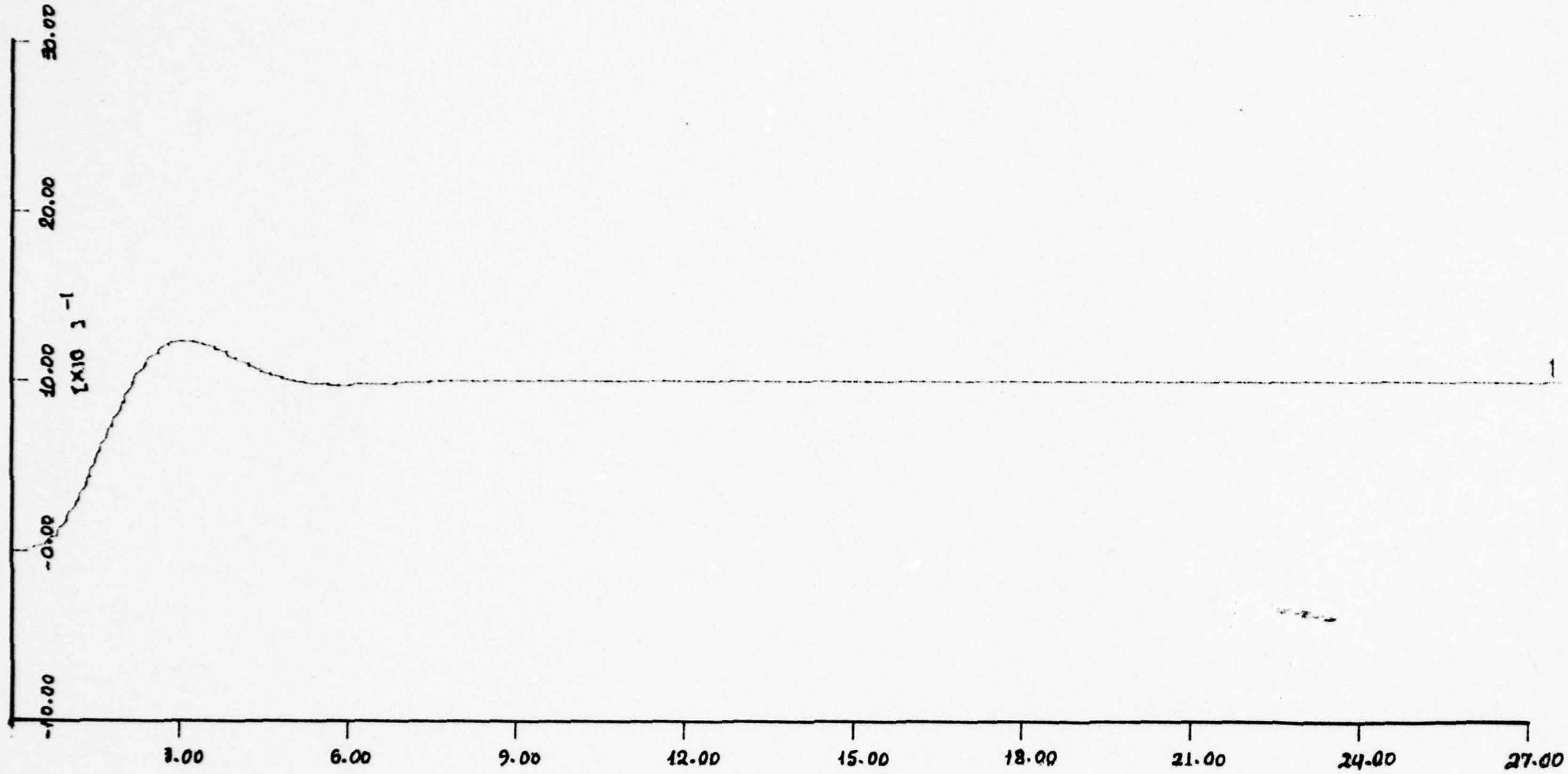
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-36)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=120, B=178.047)



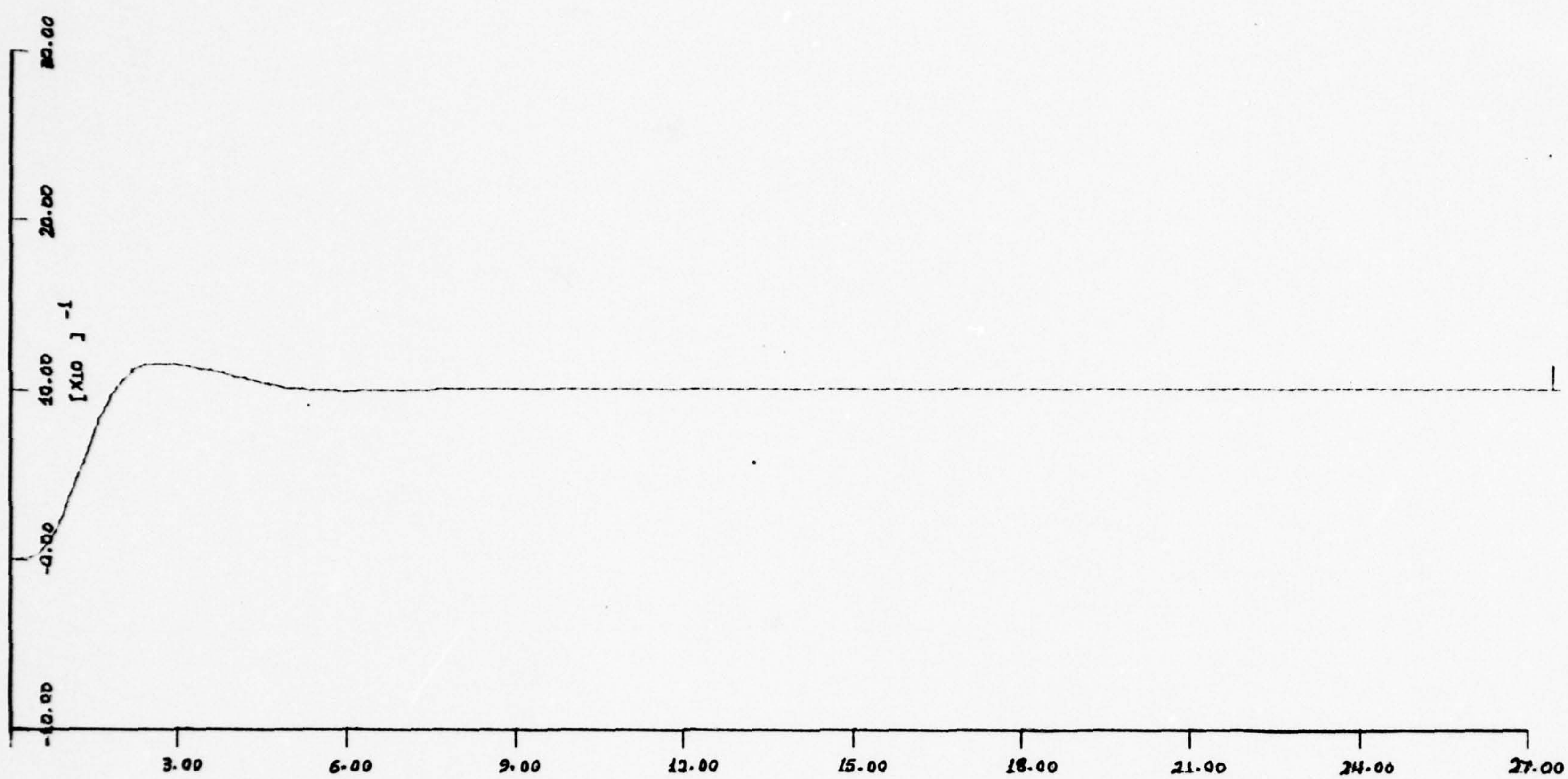
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-37)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=131.5, B=216.141)



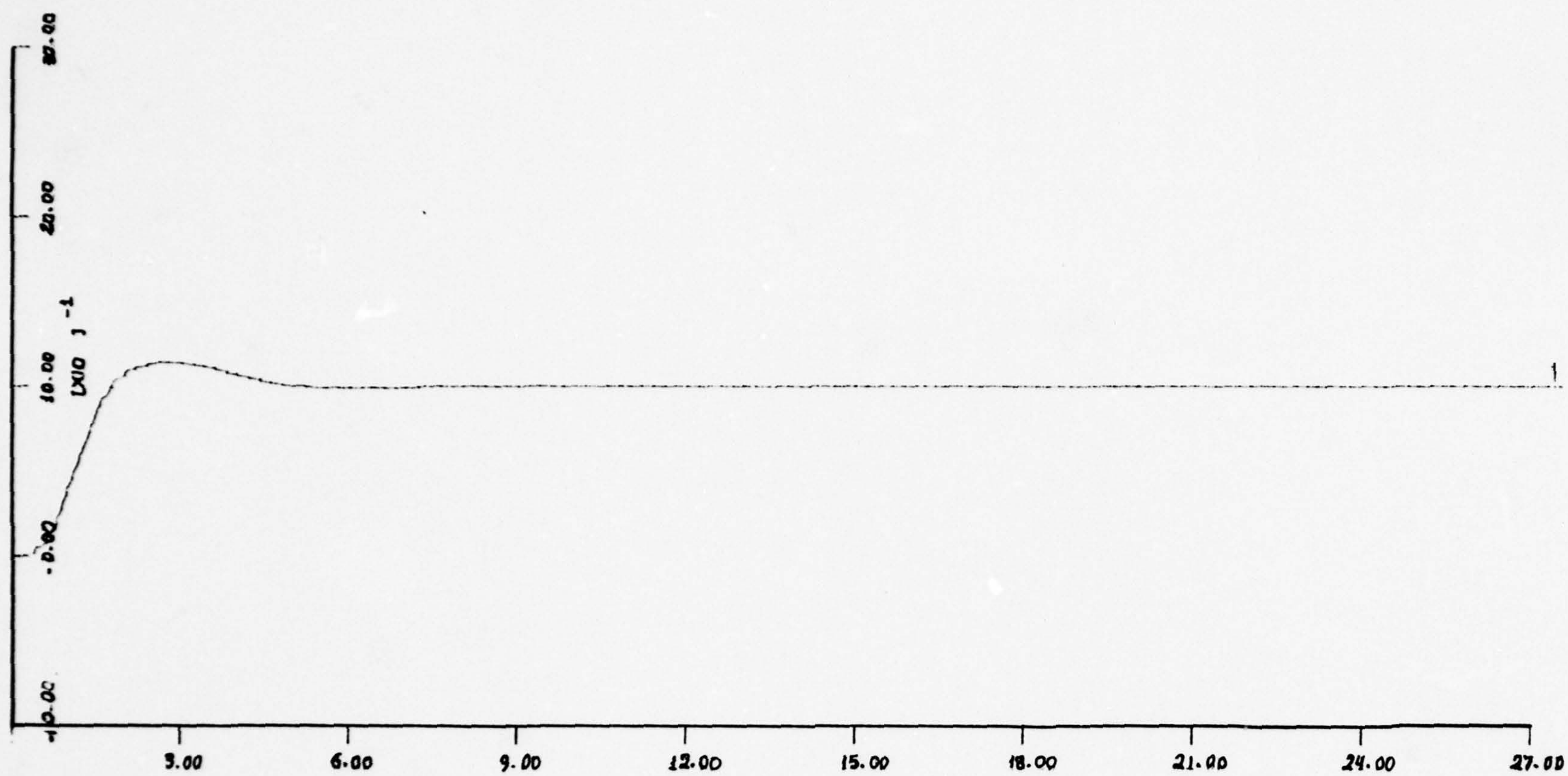
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-38)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=200, B=443.044)



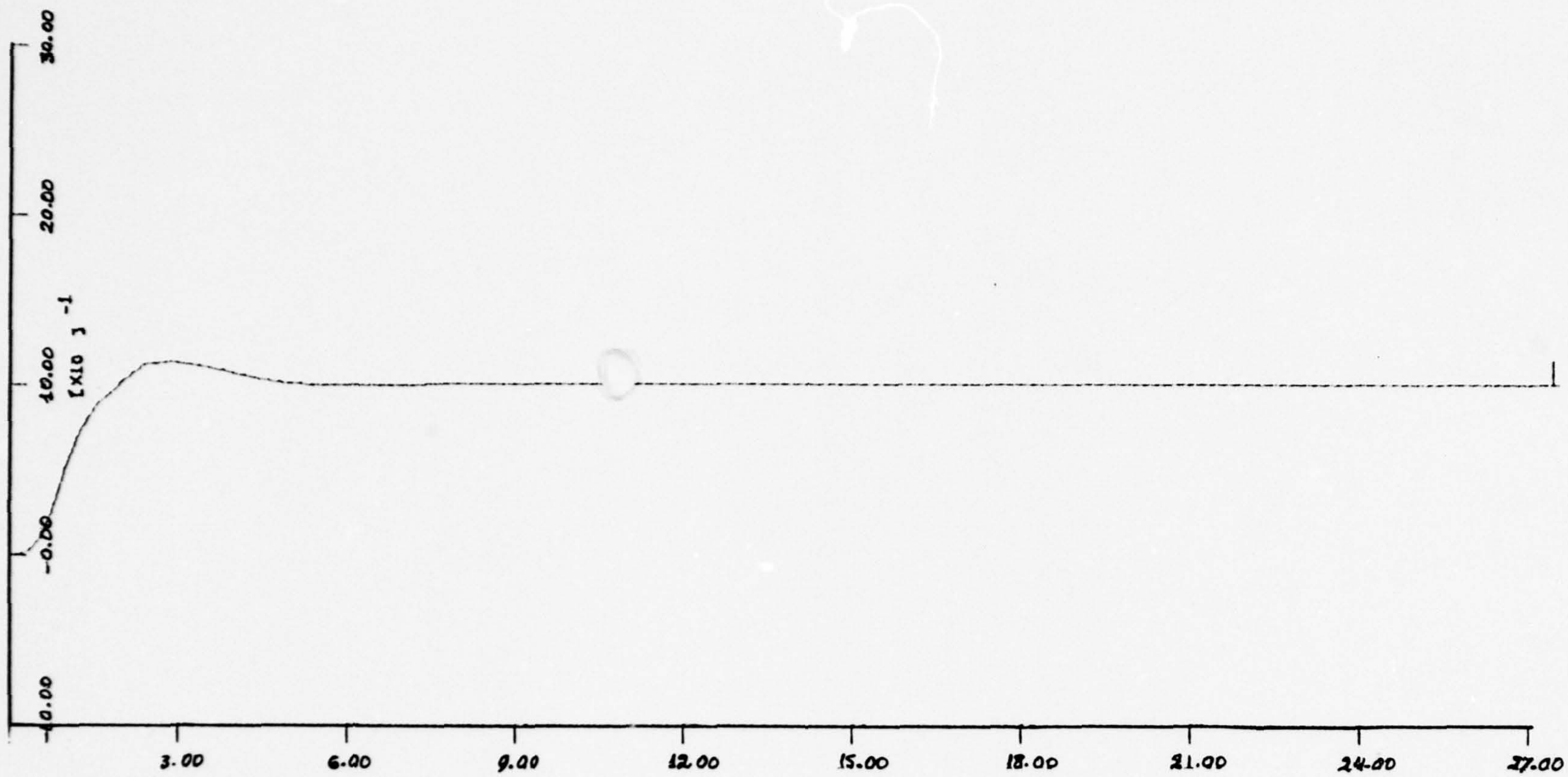
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-39)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=330, B=873.662)



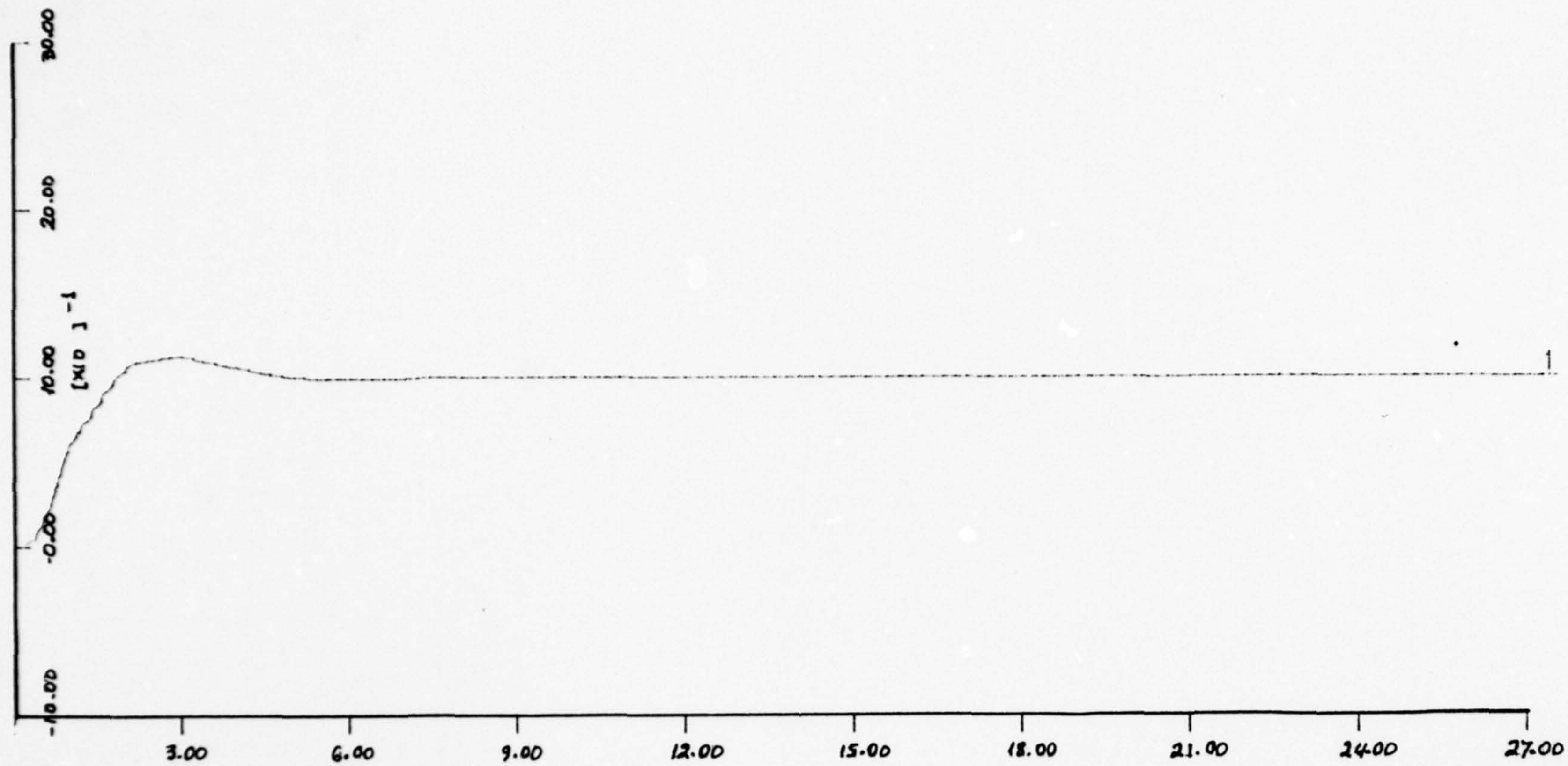
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-40)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=400, B=1105.534)



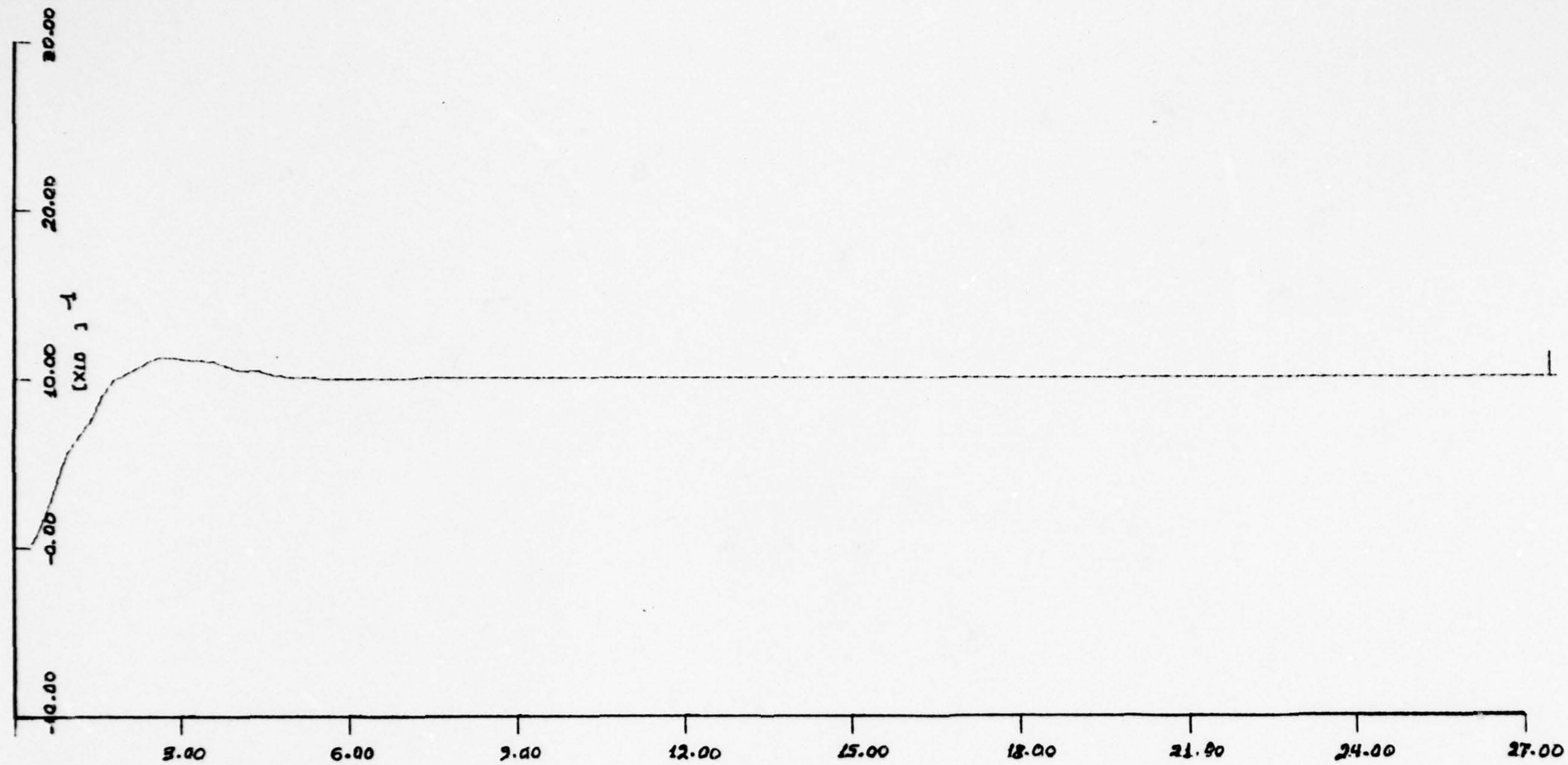
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-41)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=700, B=2099.270)



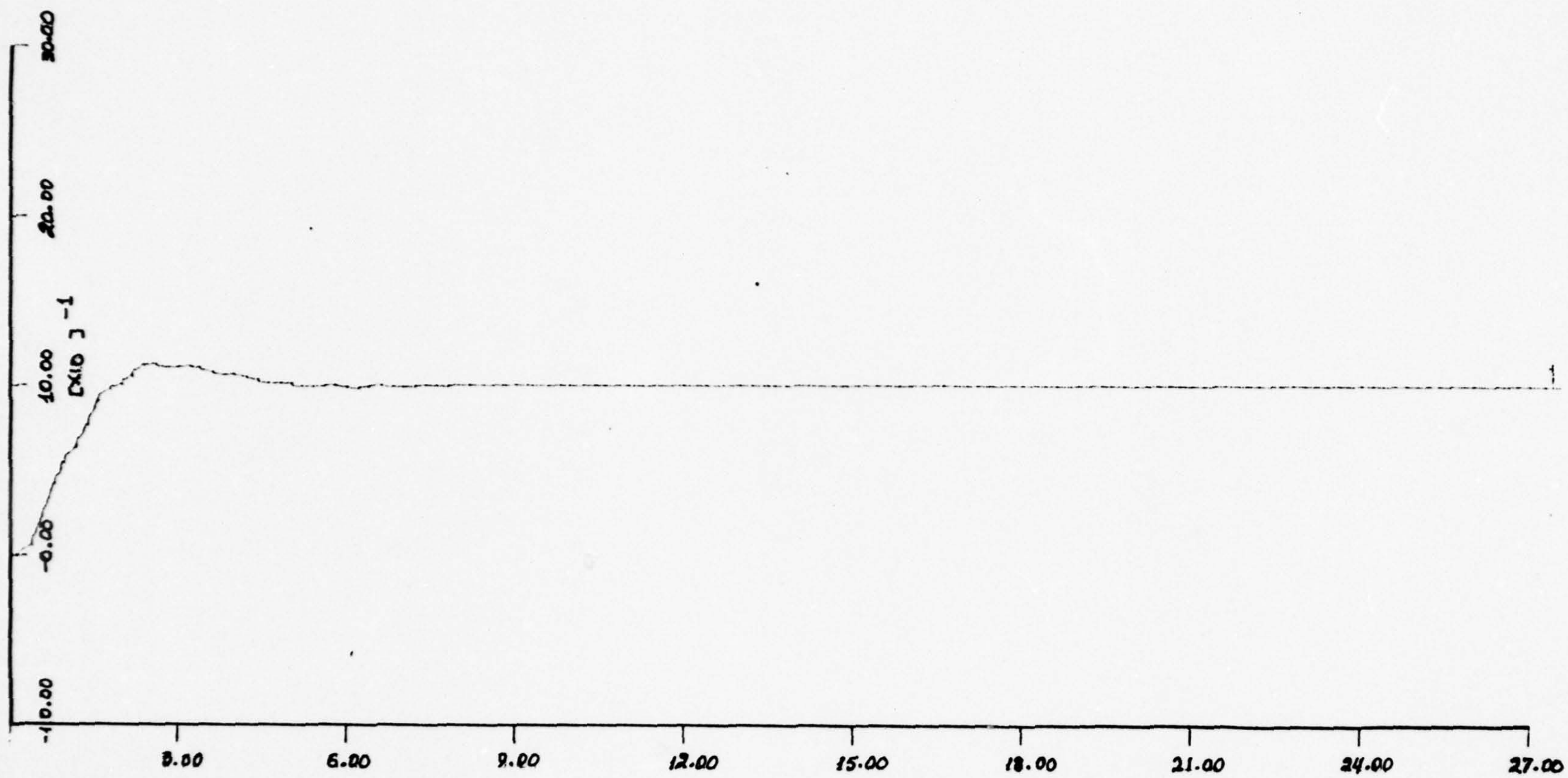
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-42)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=1000, B=3093.006)



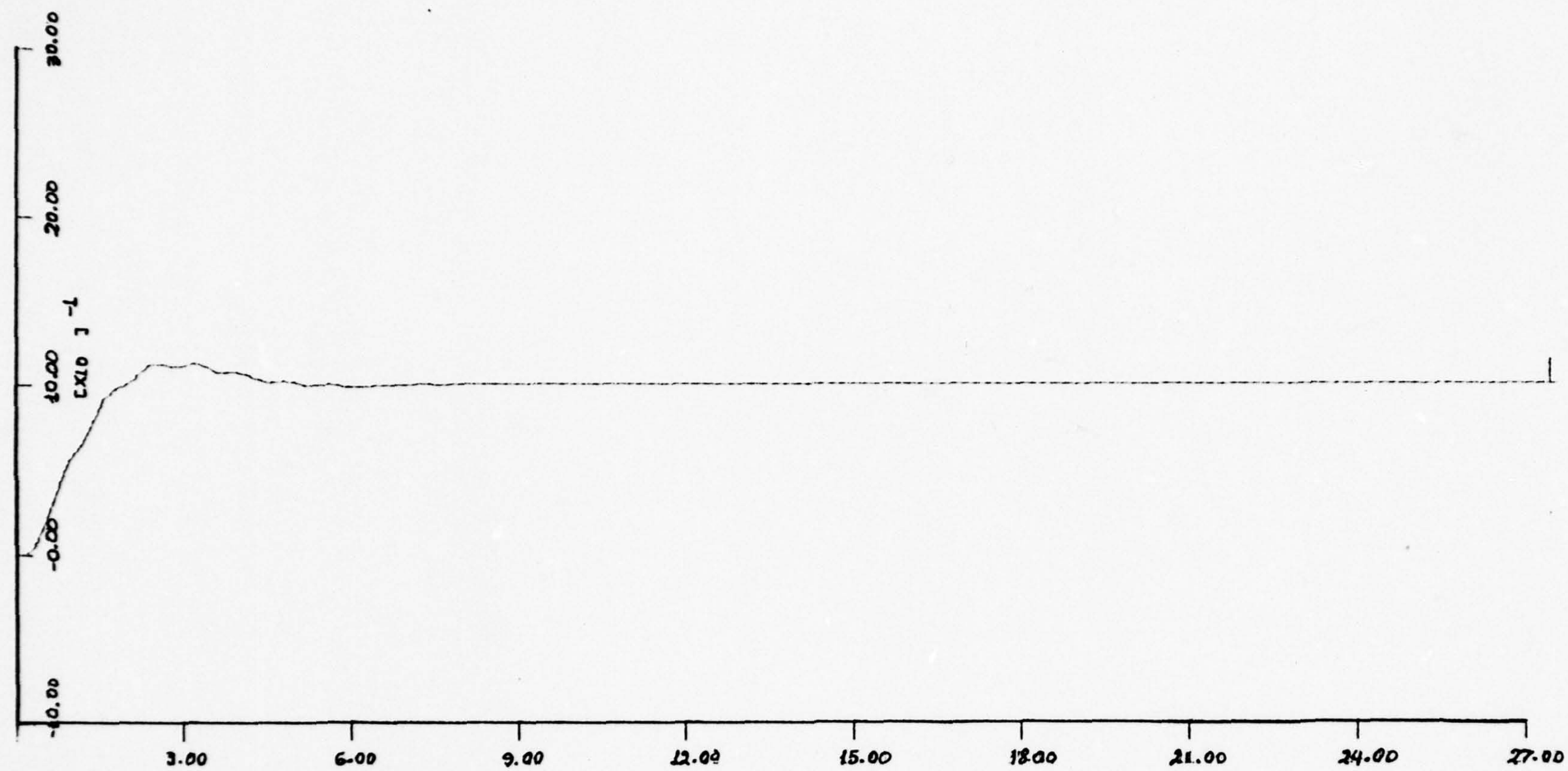
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-43)
Example(6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=1300, B=4086.74)



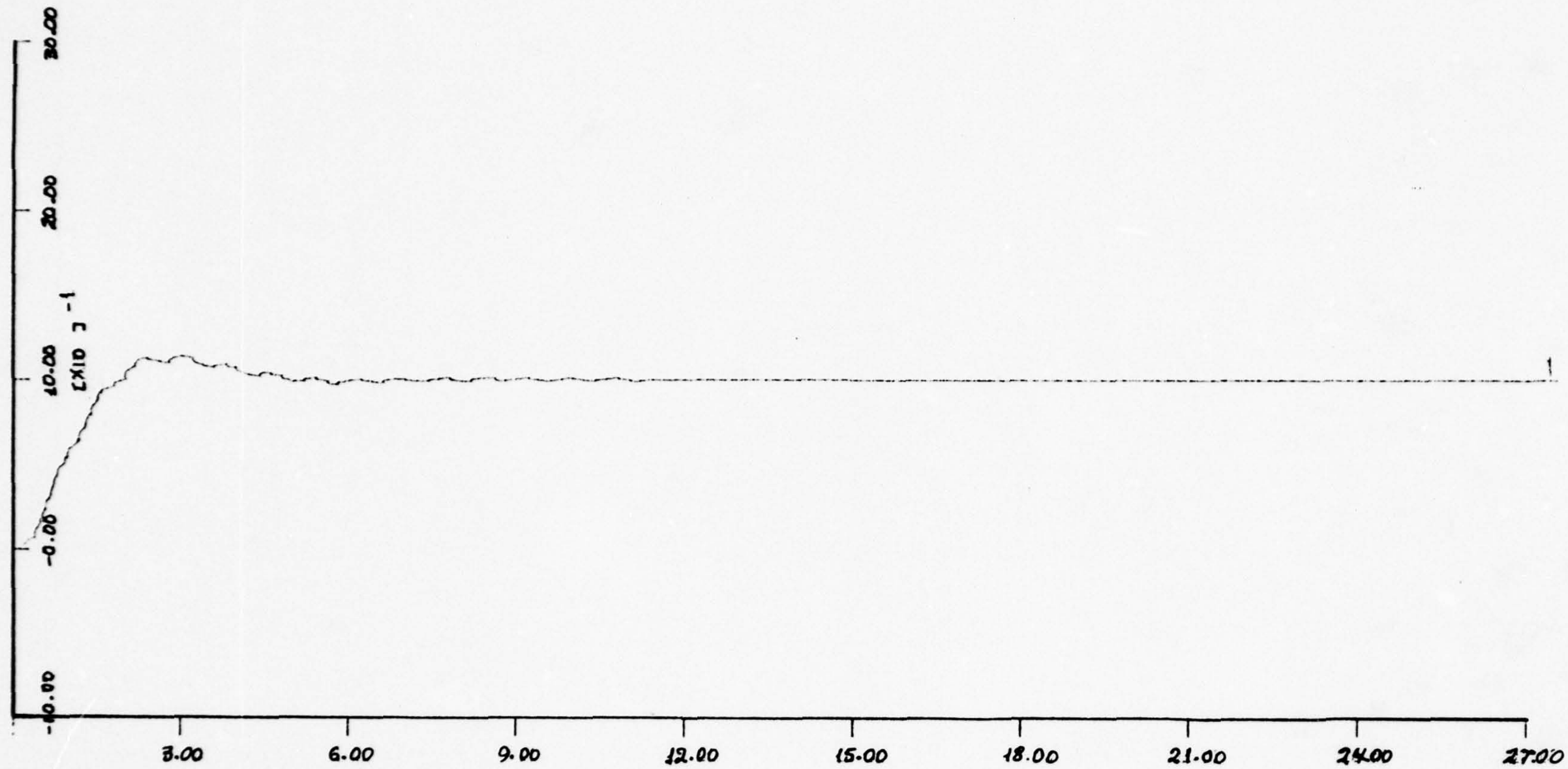
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/inch.

FIGURE (6-44)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=1500, B=4749.23)



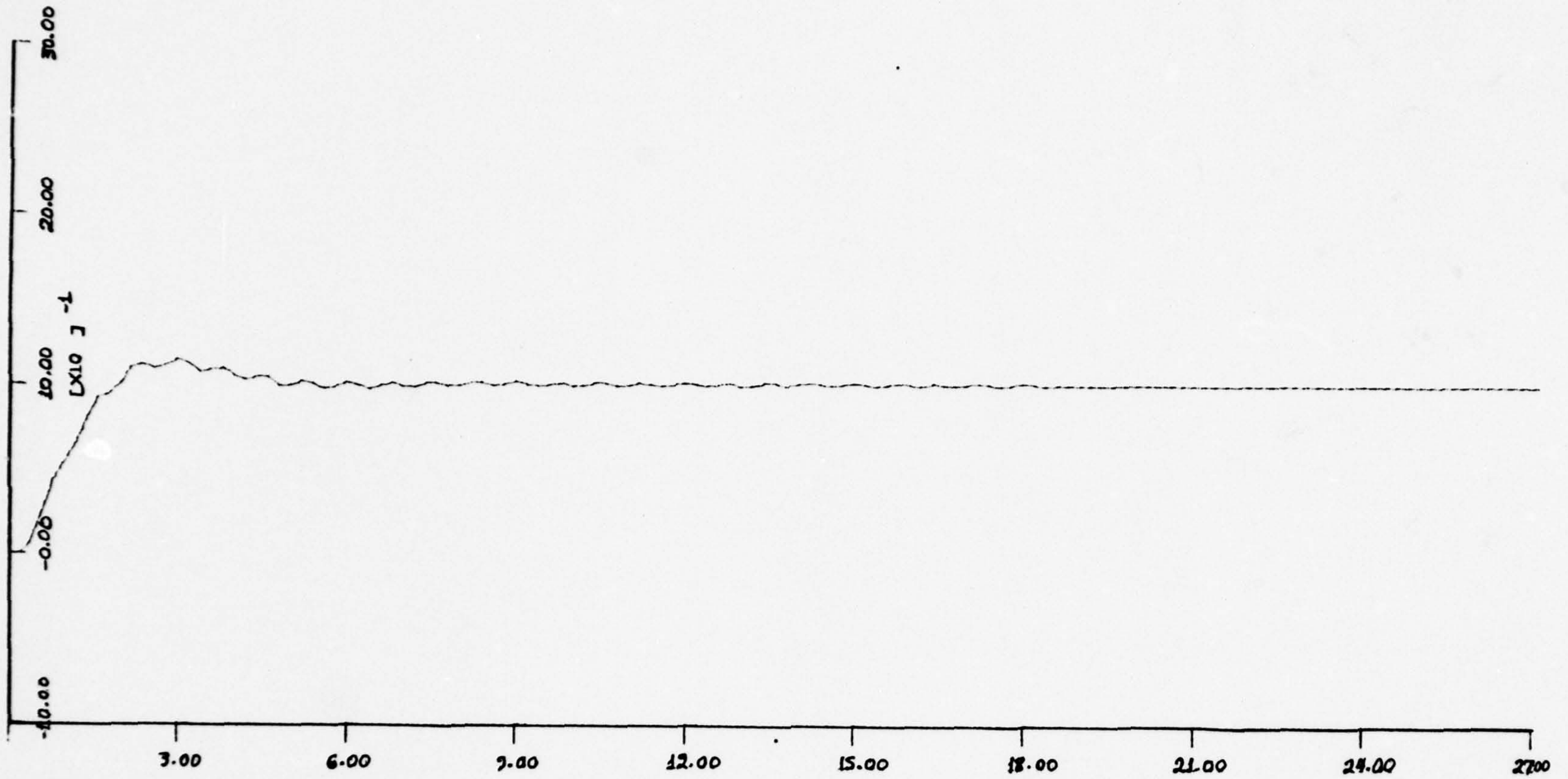
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-45)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=1550, B=-914.855)



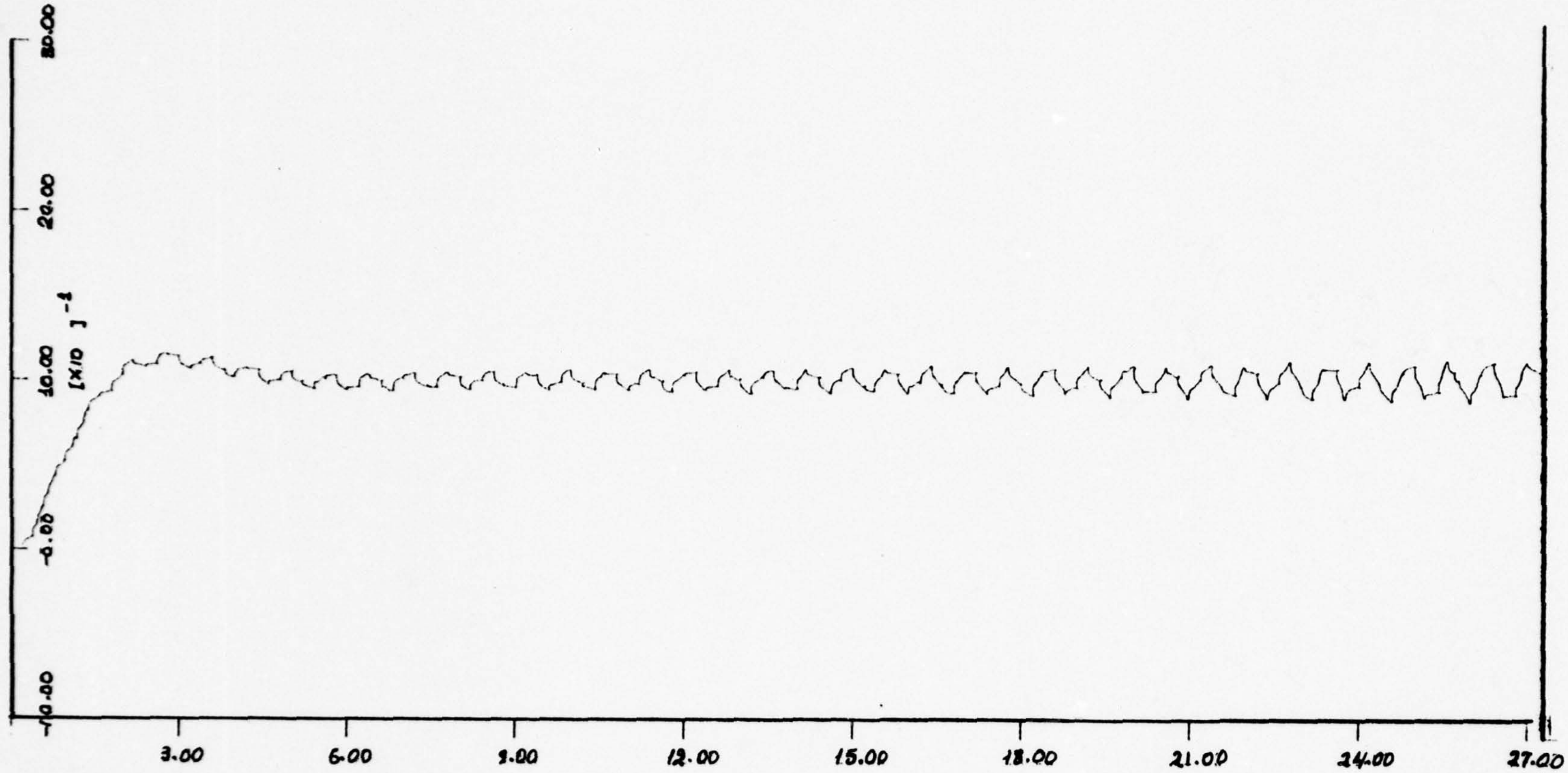
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-46)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=1700, B=5411.722)



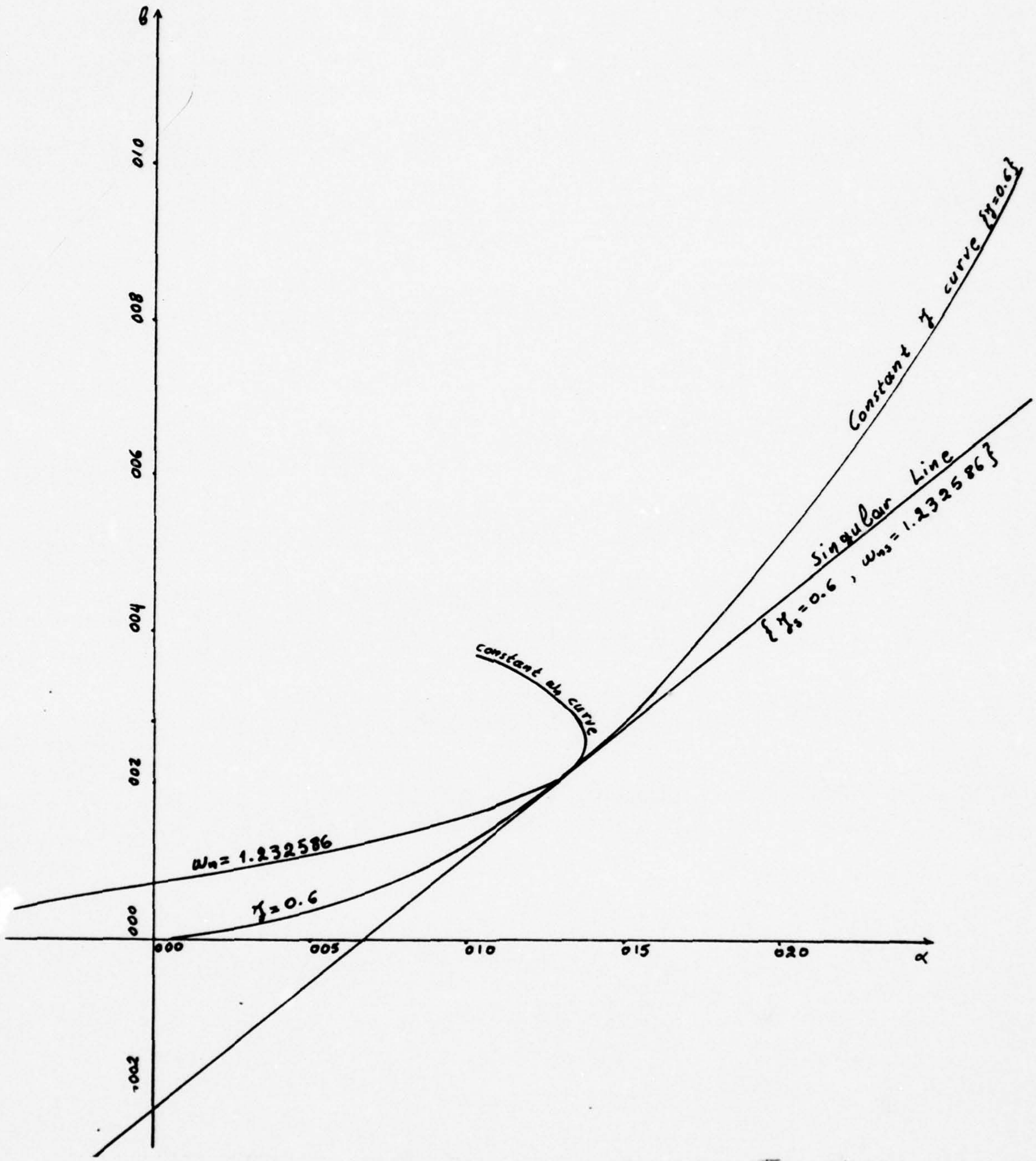
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 units/Inch.

FIGURE (6-47)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=1800, B=5742.968)



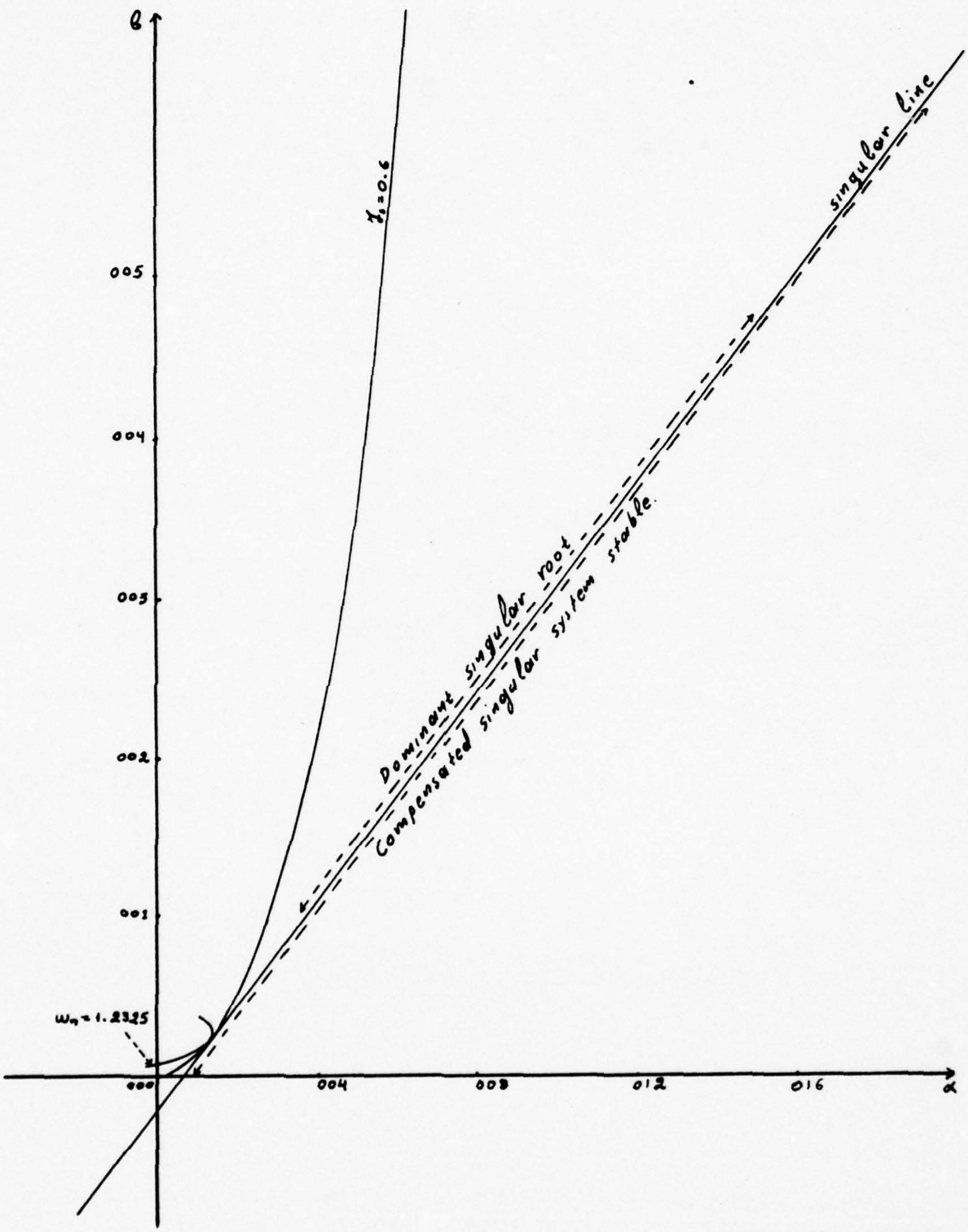
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-48)
Example (6-6):Time response of the compensated singular system for a step input.
(Parameter values: A=2000, B=6405.45)



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+02 UNITS INCH.

FIGURE (6-49)
Example (6-6): Parameter plane diagram of the compensated singular system. Singular point (ζ_s=0.6, ω_{ns}=1.232586)



X-SCALE=4.00E+02 UNITS INCH.
Y-SCALE=1.00E+03 UNITS INCH.

FIGURE (6-50)
Example (6-6): Parameter plane diagram of the compensated singular system. Singular point (ζ_s=0.6, ω_{ns}=1.2325)

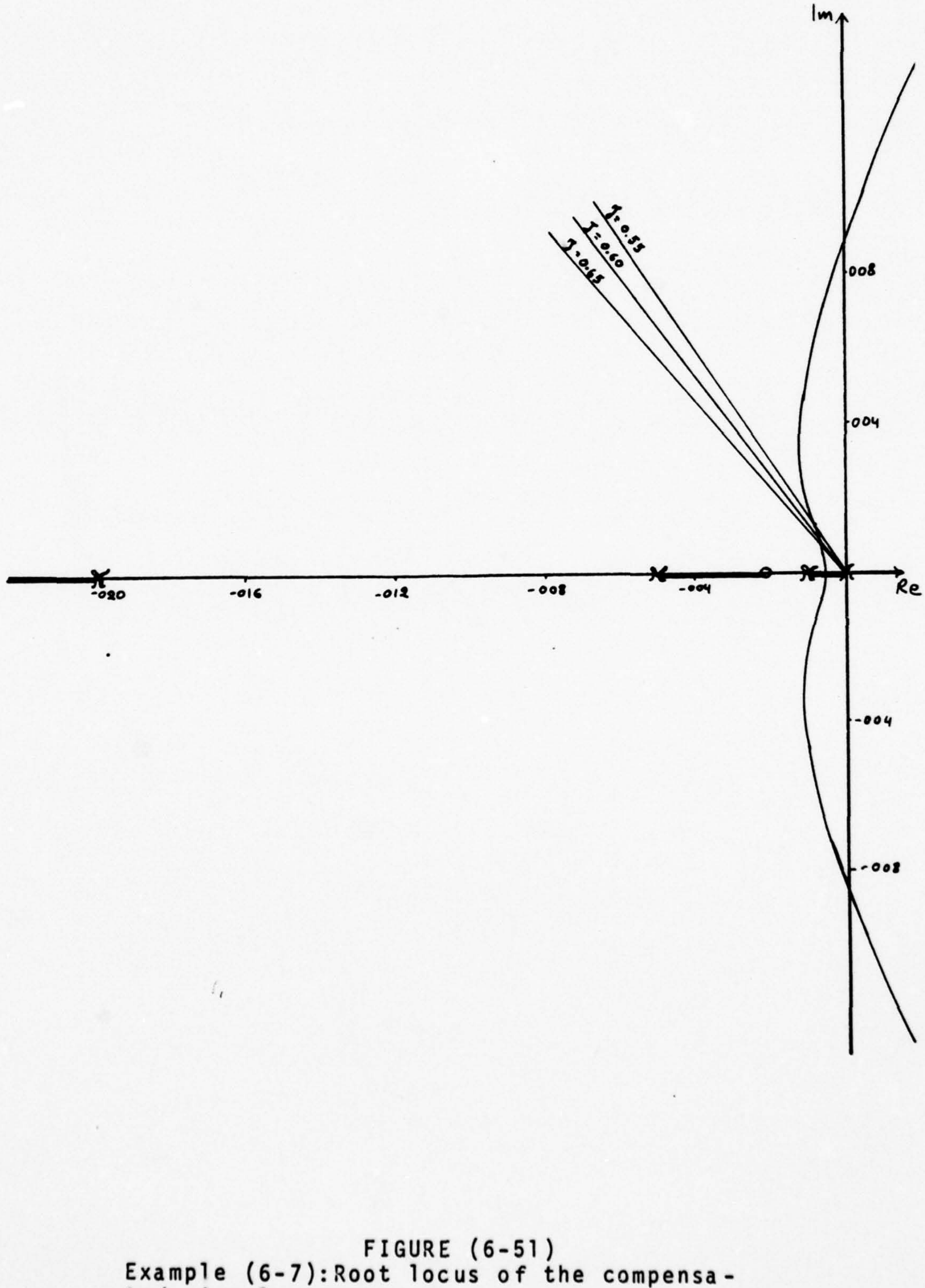


FIGURE (6-51)
Example (6-7): Root locus of the compensated singular system.

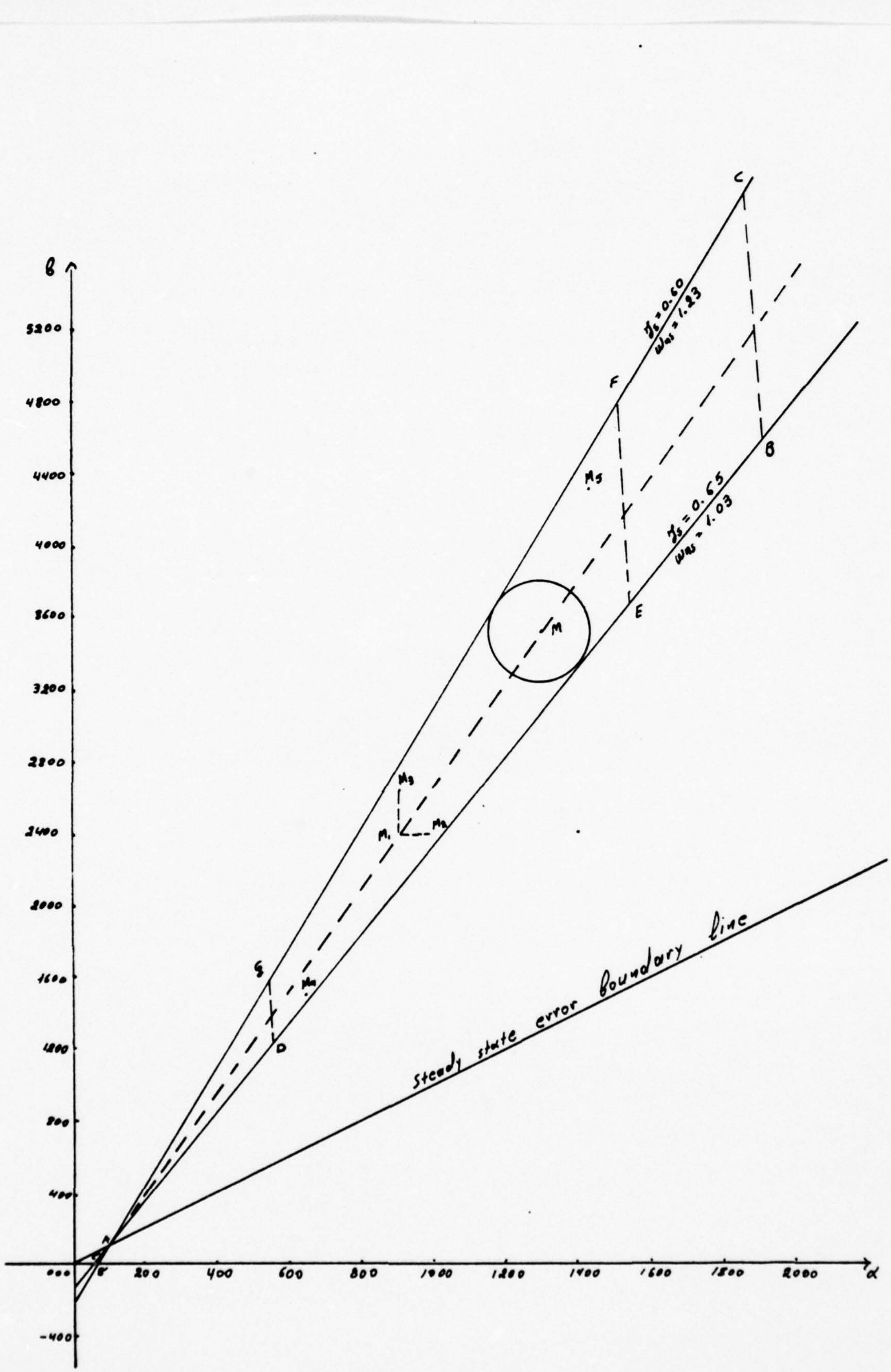
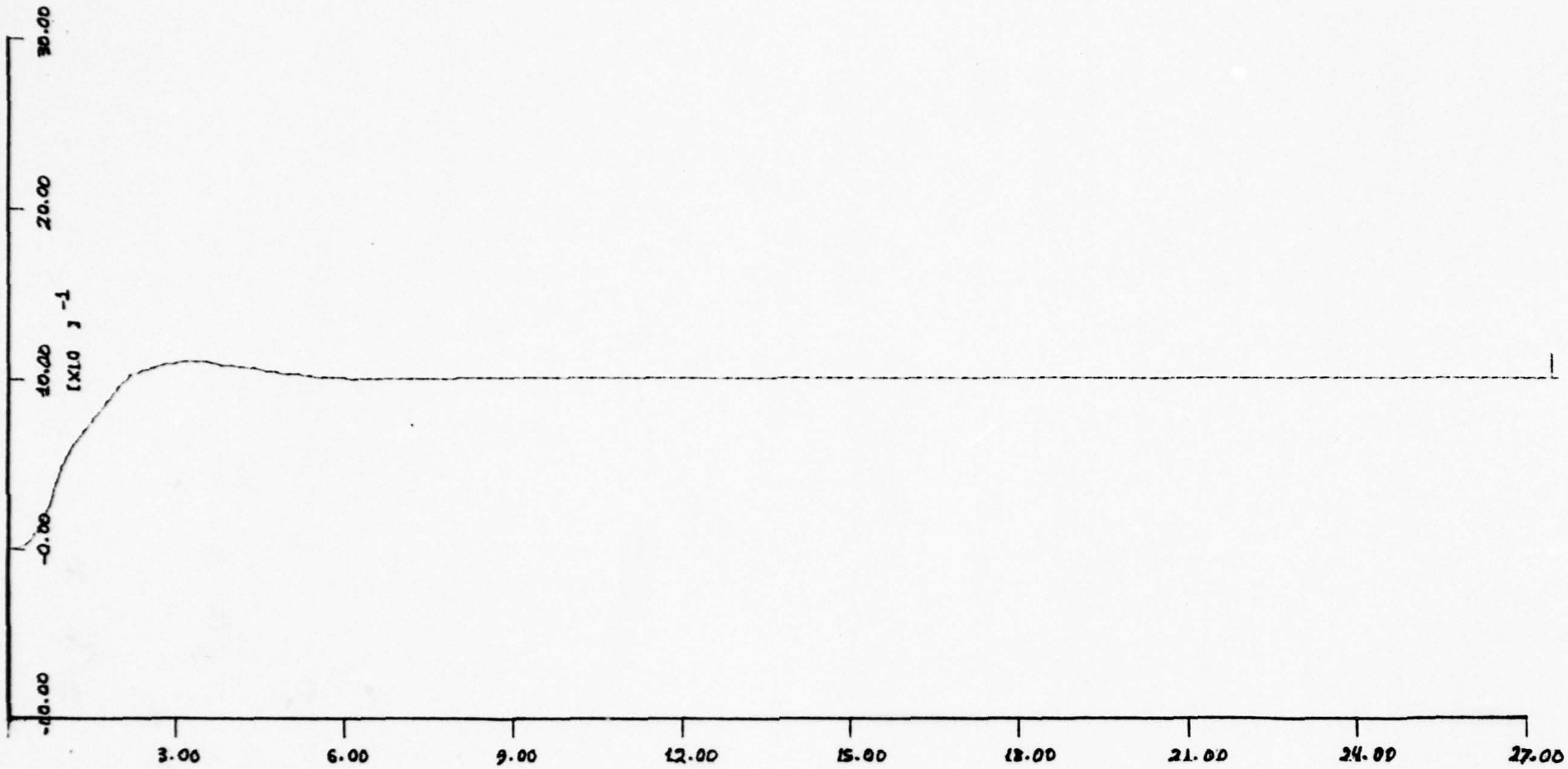
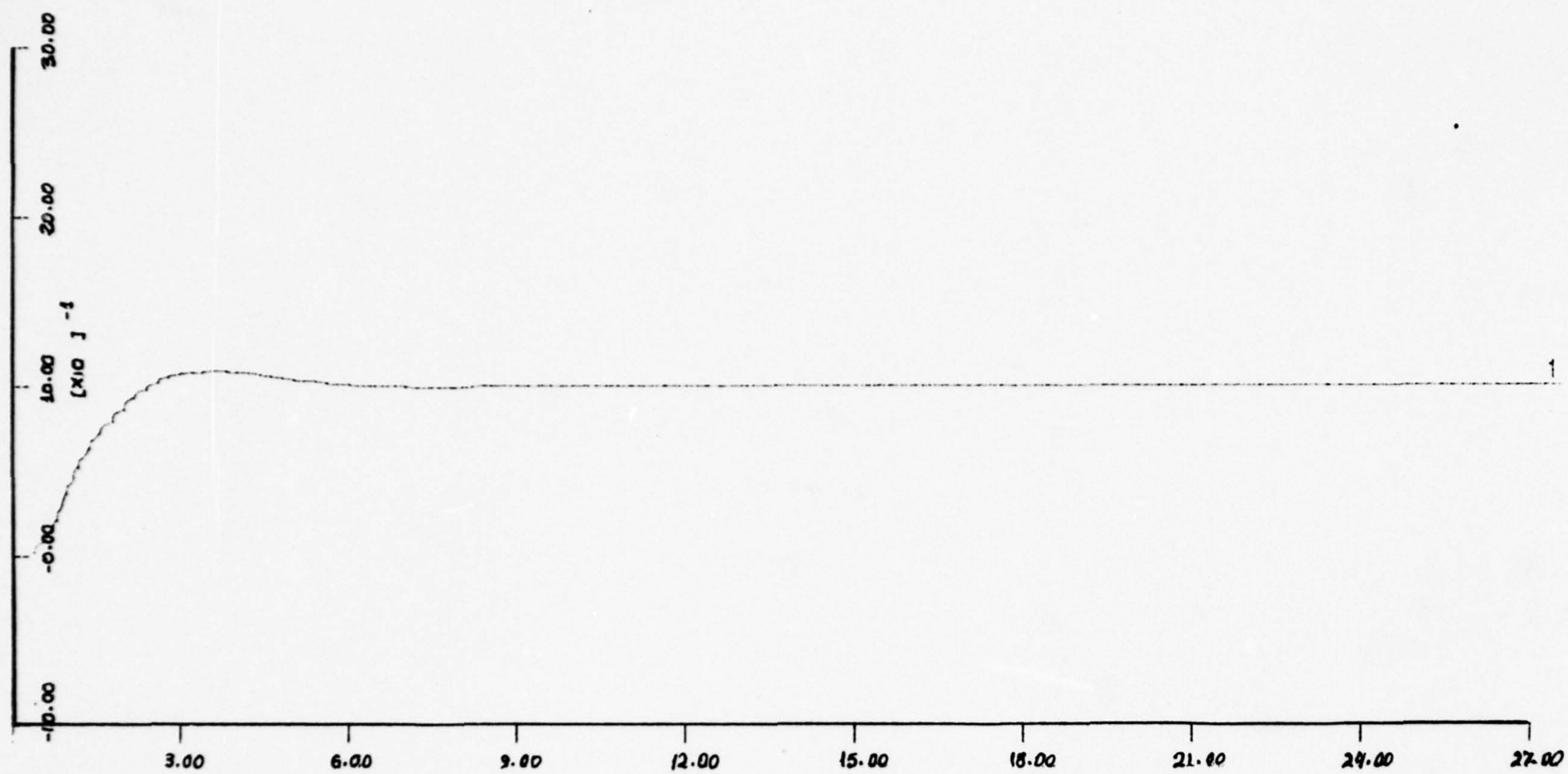


FIGURE (6-52)
Example(6-7): Parameter plane diagram and boundary lines.



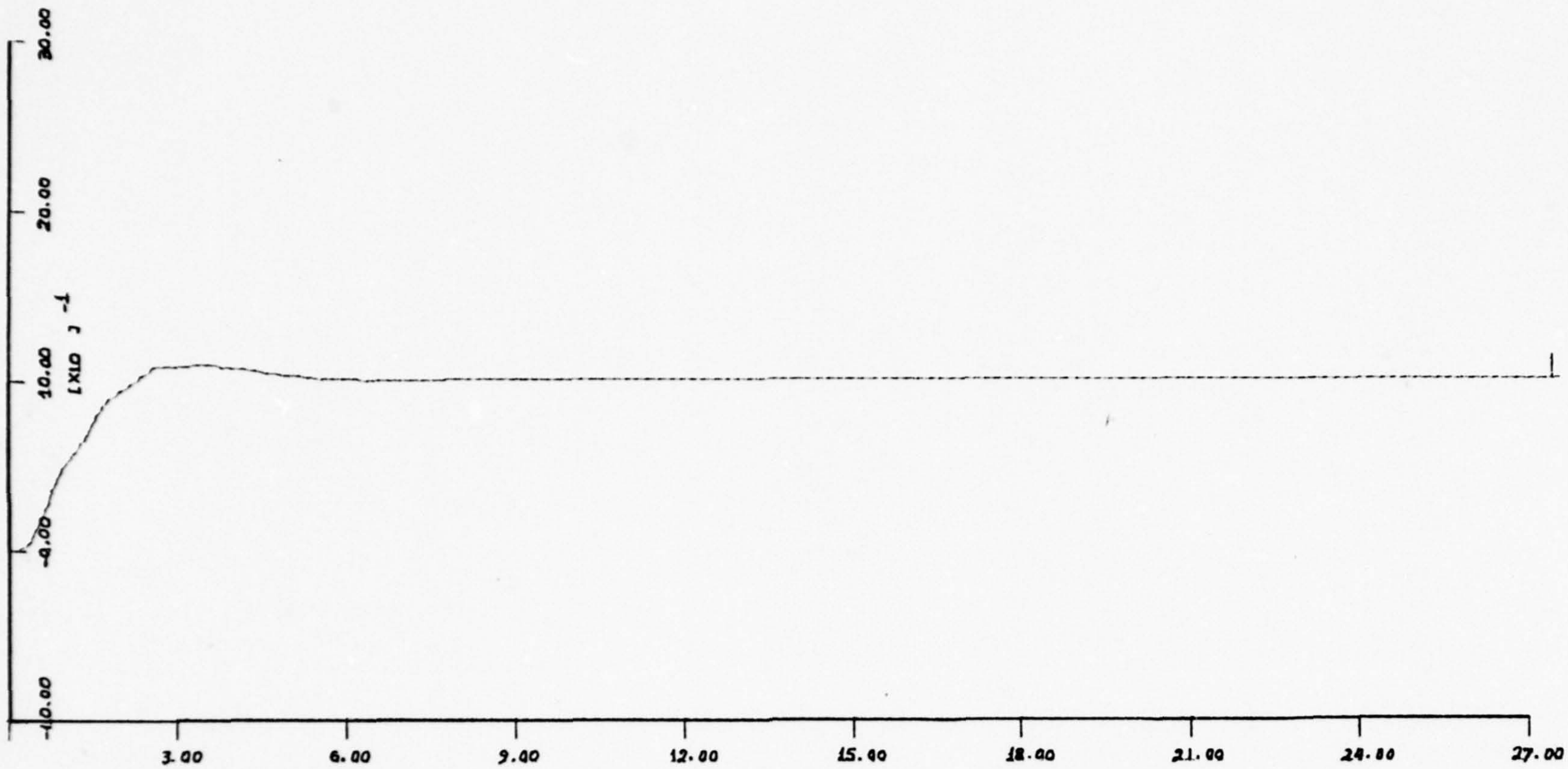
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-53)
Example (6-7):Time response of the compensated singular system for a step input.
(Parameter values: A=900, B=2400)



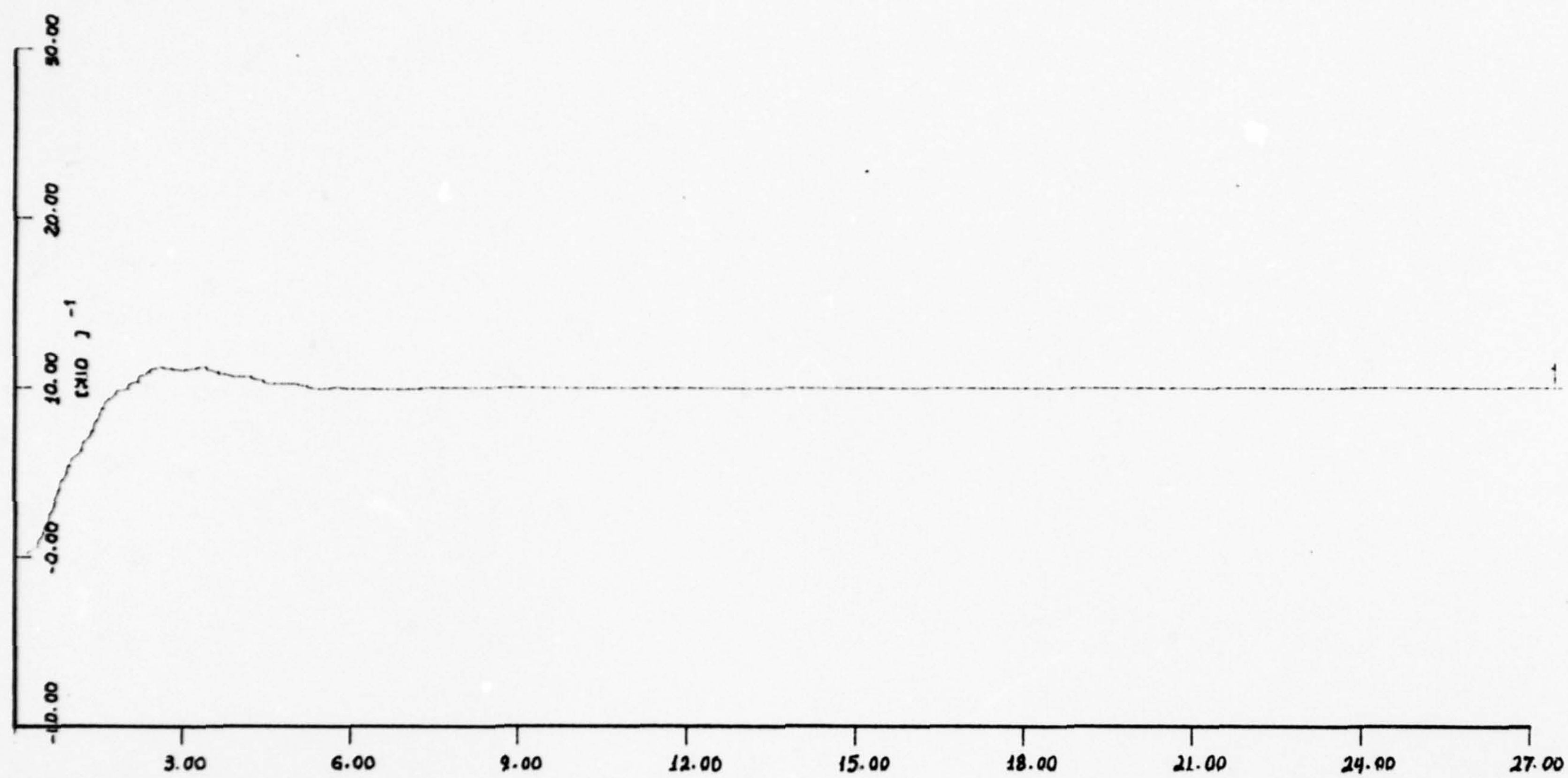
X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-54)
Example (6-7):Time response of the compensated singular system for a step input.
(Operating point on the parameter plane M_4(A=650,B=1480)).



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-55)
Example (6-7):Time response of the compensated singular system for a step input. (Operating point on the parameter plane M(A=1310,B=3500)).



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (6-56)
Example (6-7):Time response of the compensated singular system for a step input.
(Operating point on the parameter plane M_5(A=1430,B=4280)).

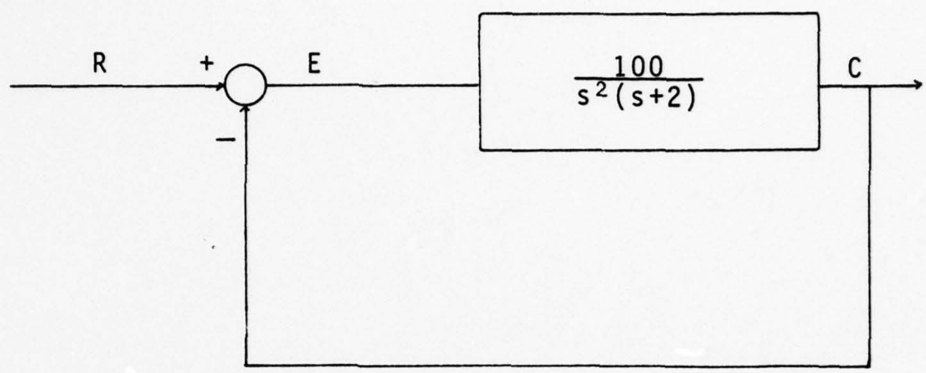


FIGURE (7-1)
Example (7-1): Block diagram of the initial system (Plant).

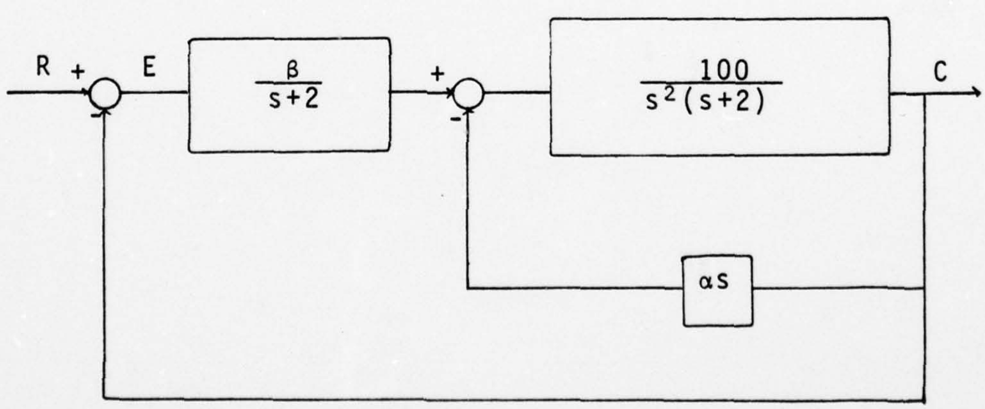


FIGURE (7-2)
Example (7-2): Block diagram of the compensated singular system.

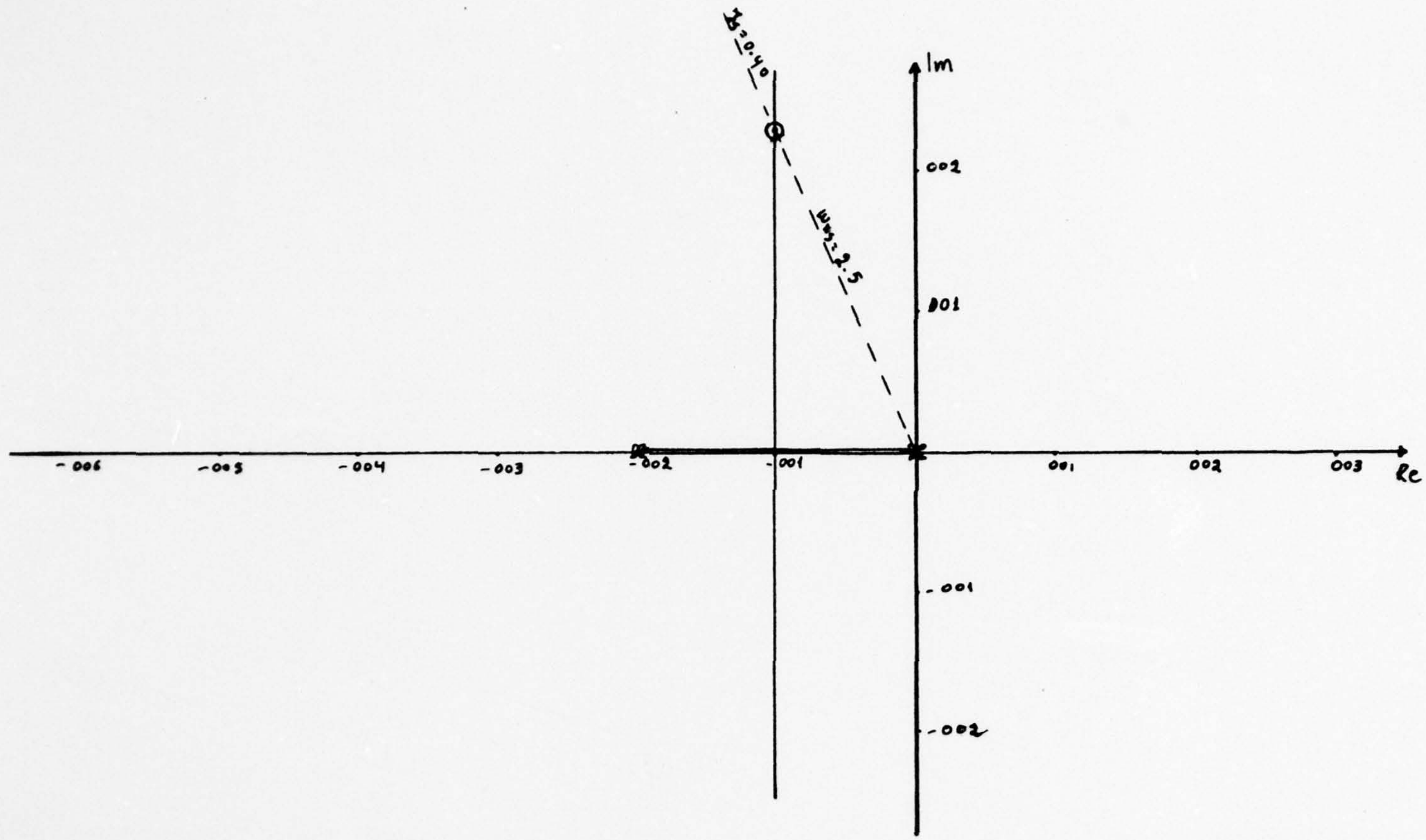


FIGURE (7-3)
Root locus diagram.

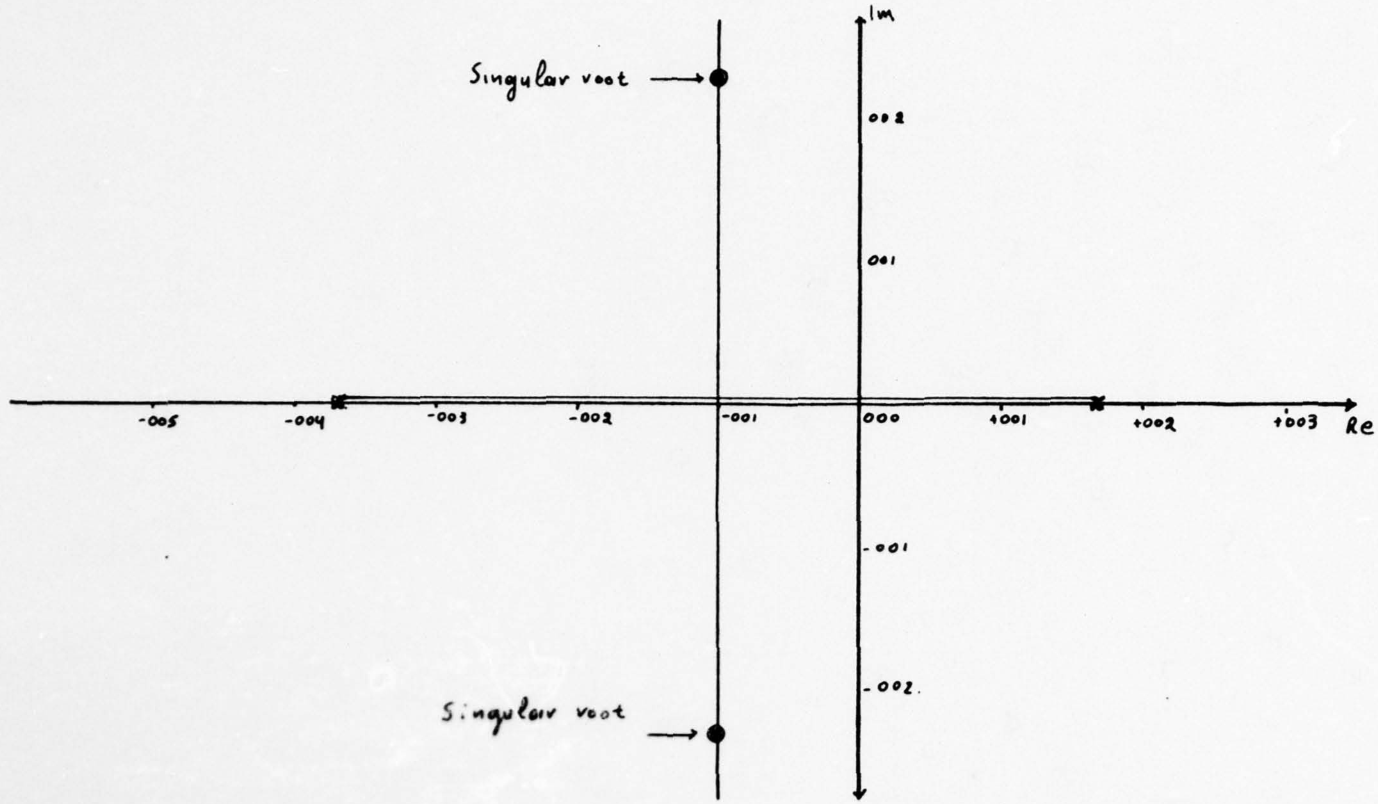
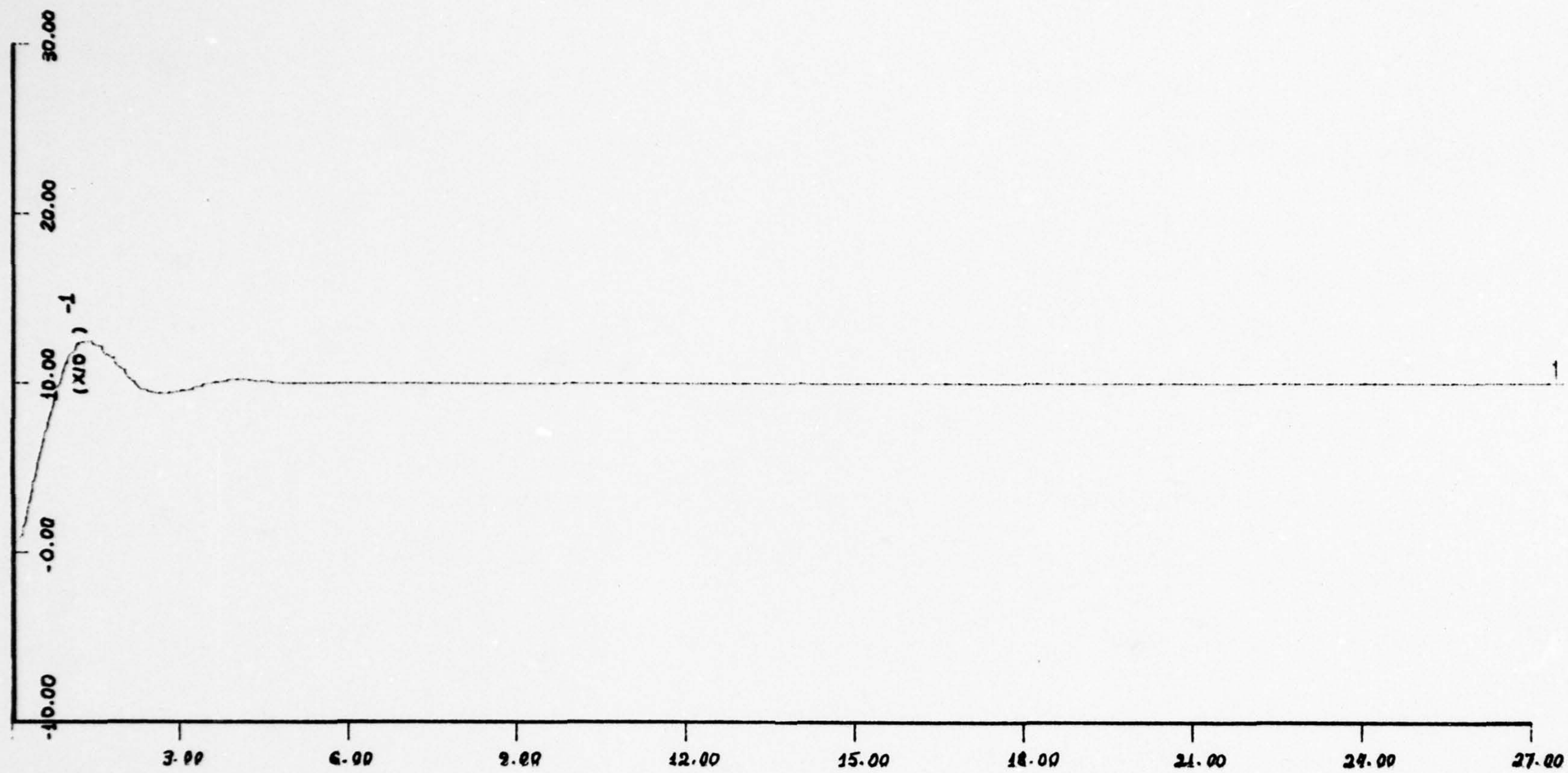


FIGURE (7-4)
Root locus of the compensated singular system.



X-Scale=3.00 Units/Inch.
Y-Scale=1.00 Units/Inch.

FIGURE (7-5)
Example (7-1):Time response of the compensated singular system for a step input.
(Parameter values: A=25.07,B=156.31)

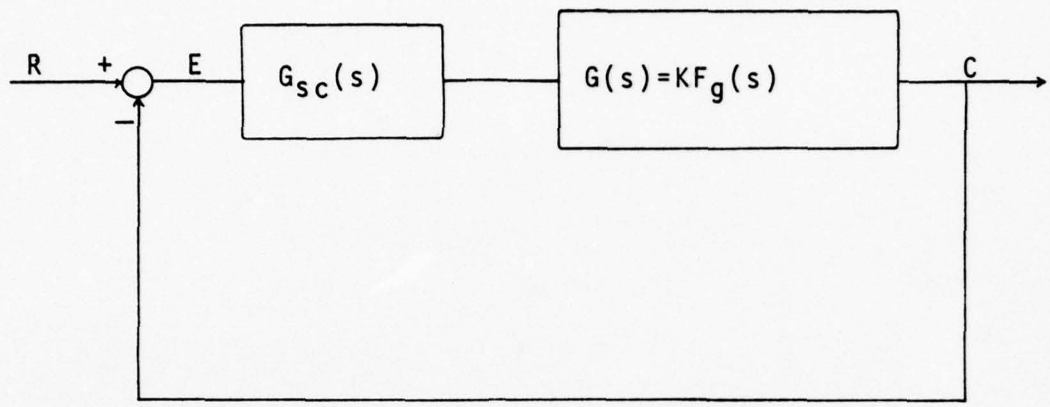


FIGURE (8-1)
First modified singular compensation structure.

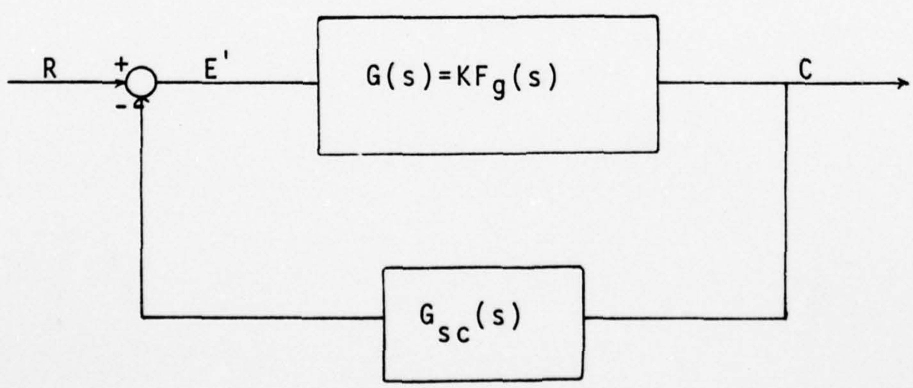


FIGURE (8-2)
Second modified singular compensation structure.

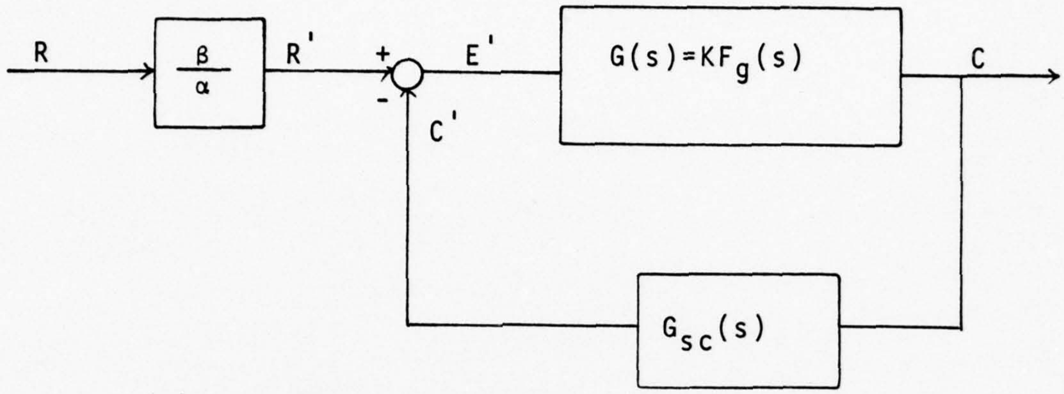


FIGURE (8-3)
Third modified singular compensation structure.

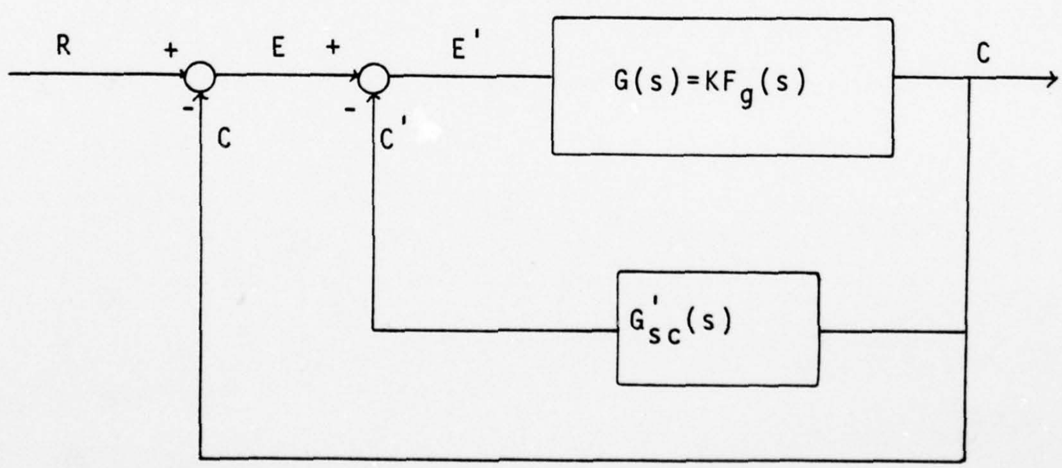


FIGURE (8-4)
Fourth modified singular compensation structure.

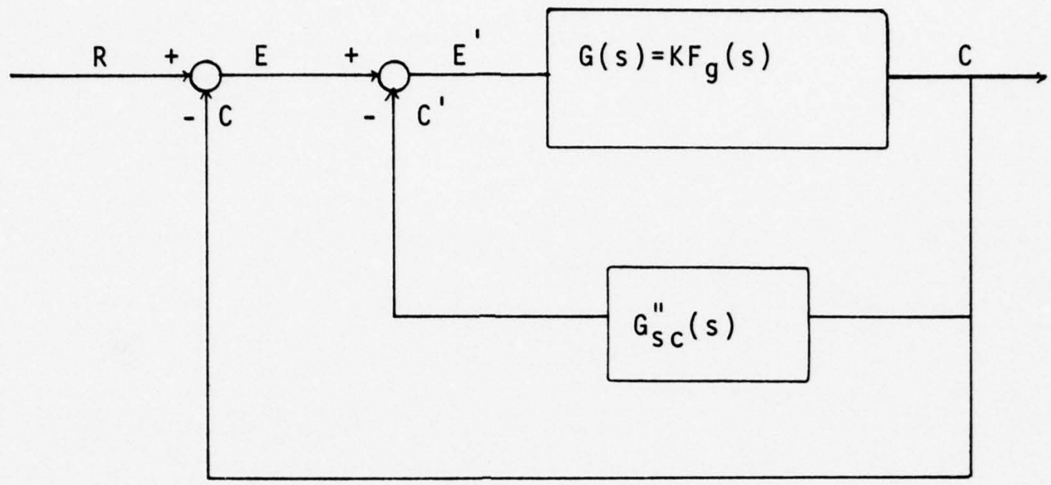


FIGURE (8-5)
Fifth modified singular compensation structure.

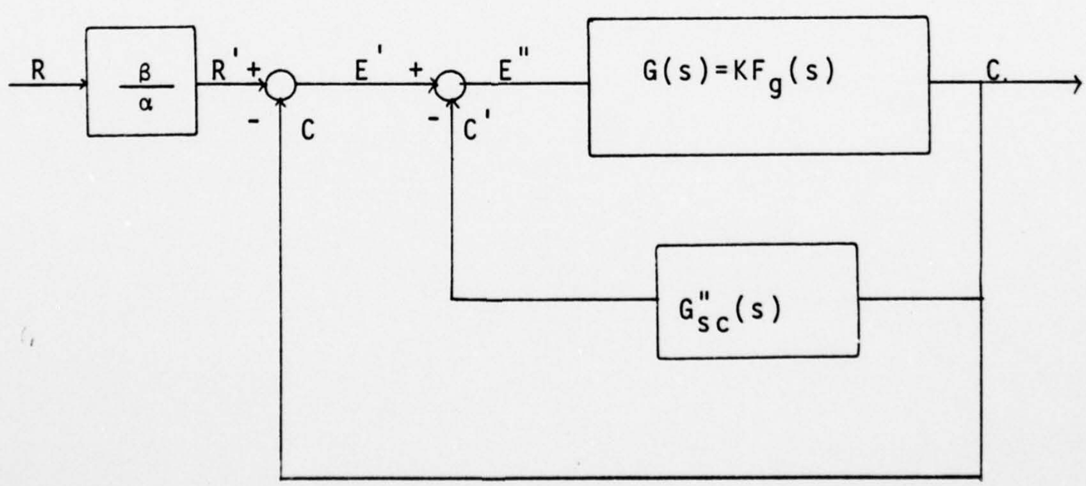
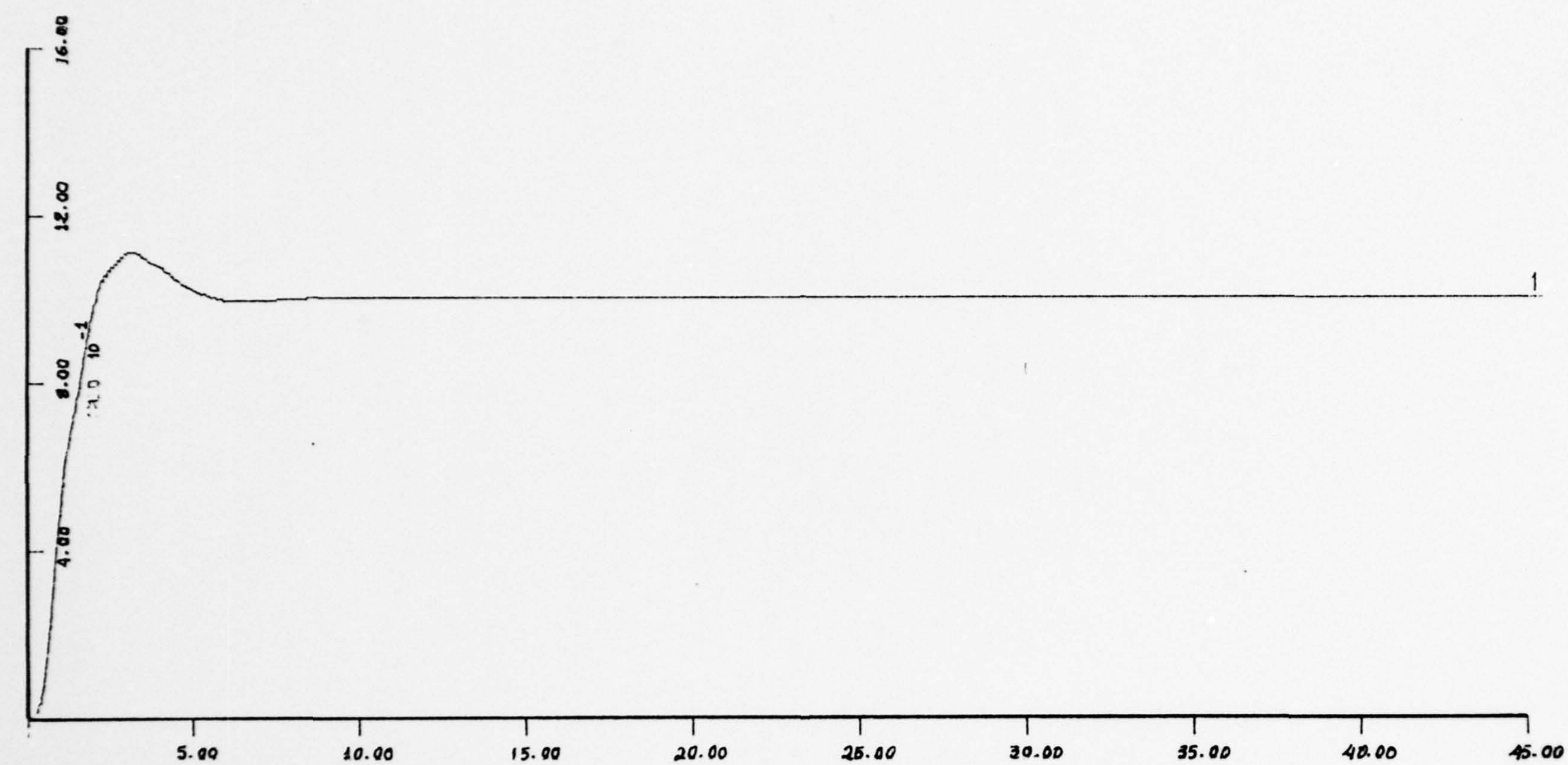
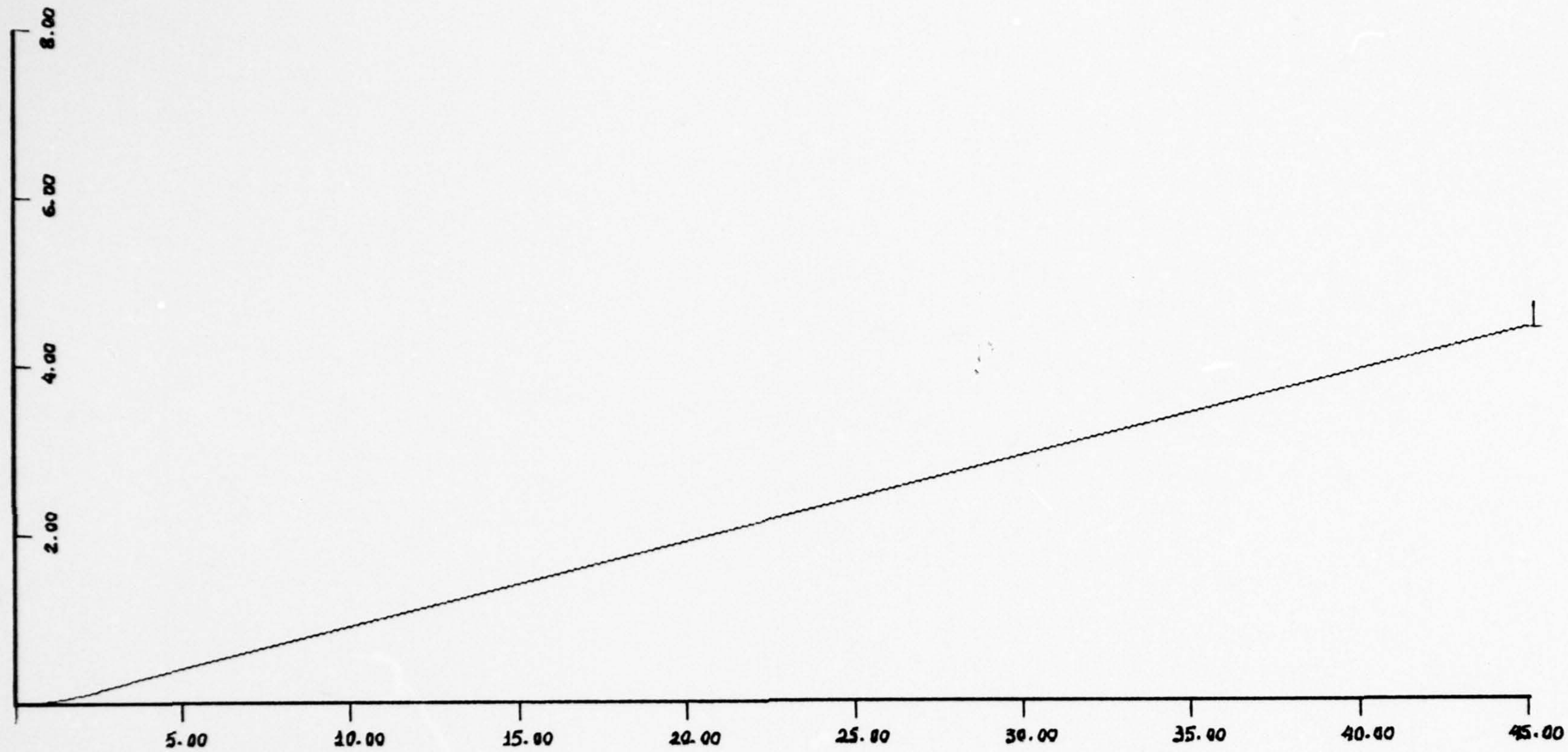


FIGURE (8-6)
Sixth modified singular compensation structure.



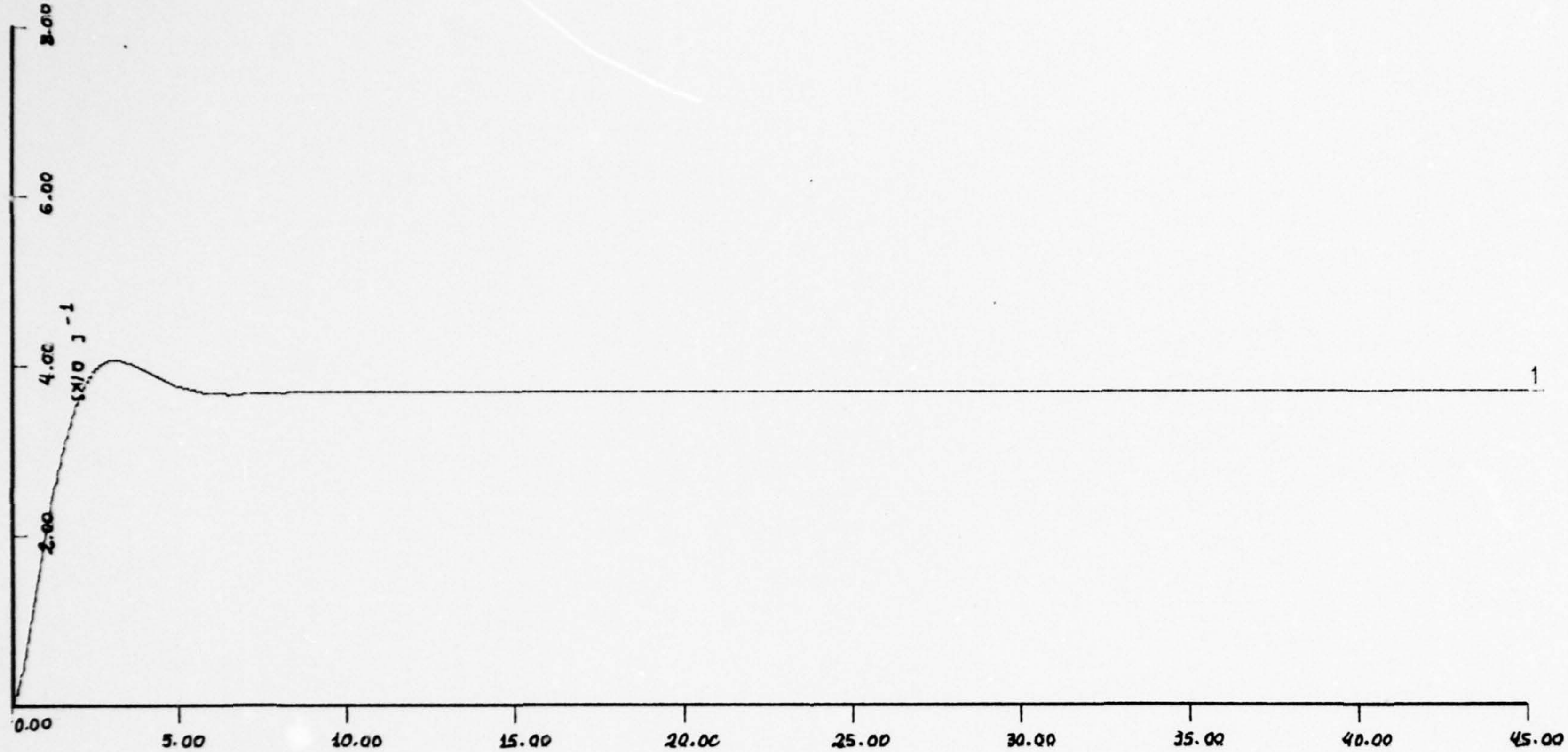
X-Scale=5.00 Units/inch
Y-Scale=0.40 Units/inch

FIGURE (8-1A)
First modified singular structure.Time response of the system for a step input.



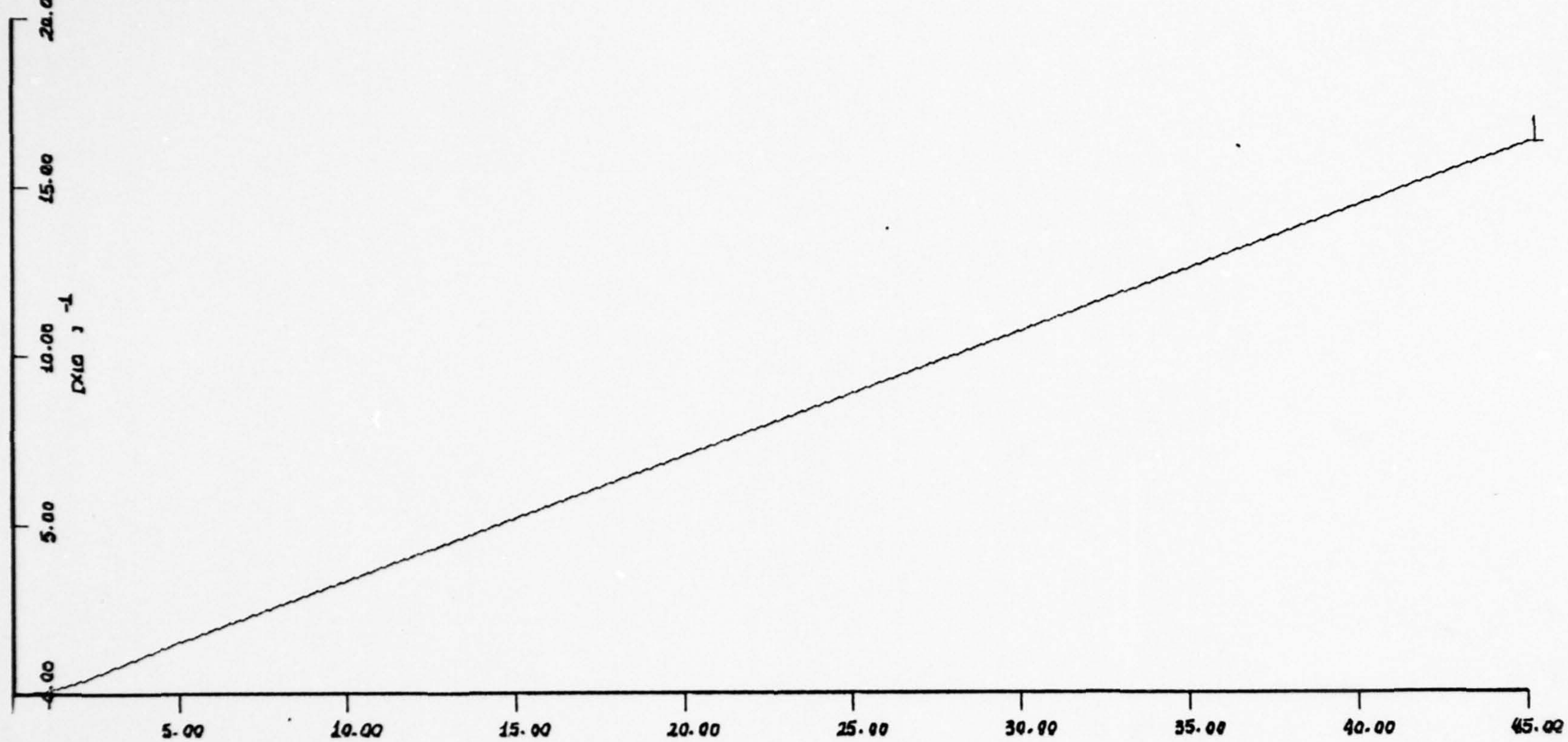
X-Scale=5.00 Units/Inch
Y-Scale=2.00 Units/Inch

FIGURE (8-1B)
First modified singular structure.Time response of the system for a ramp input.



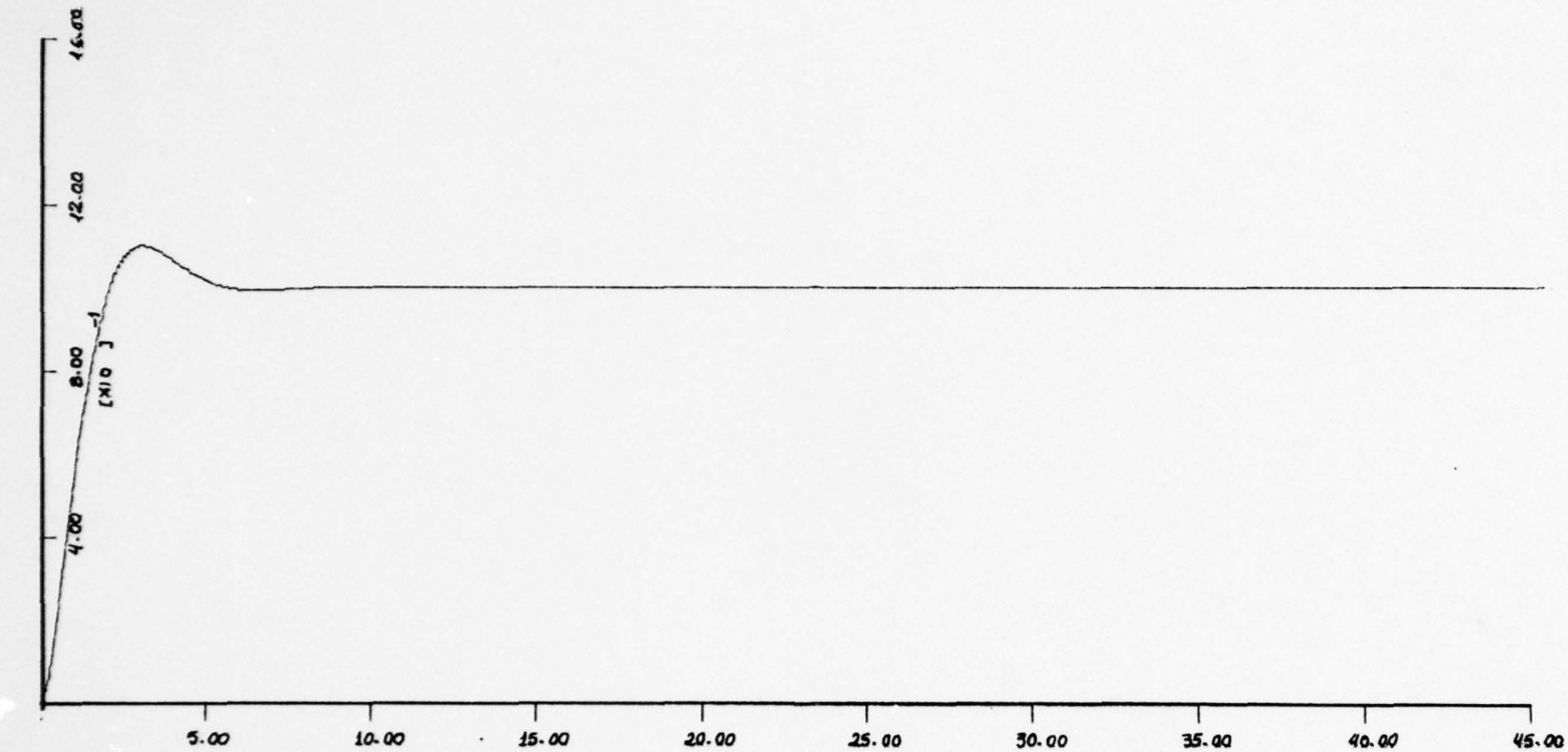
X-Scale=5.00 Units/Inch
Y-Scale=0.20 Units/Inch

FIGURE (8-2A)
Second modified singular structure. Time response of the system for a step input.



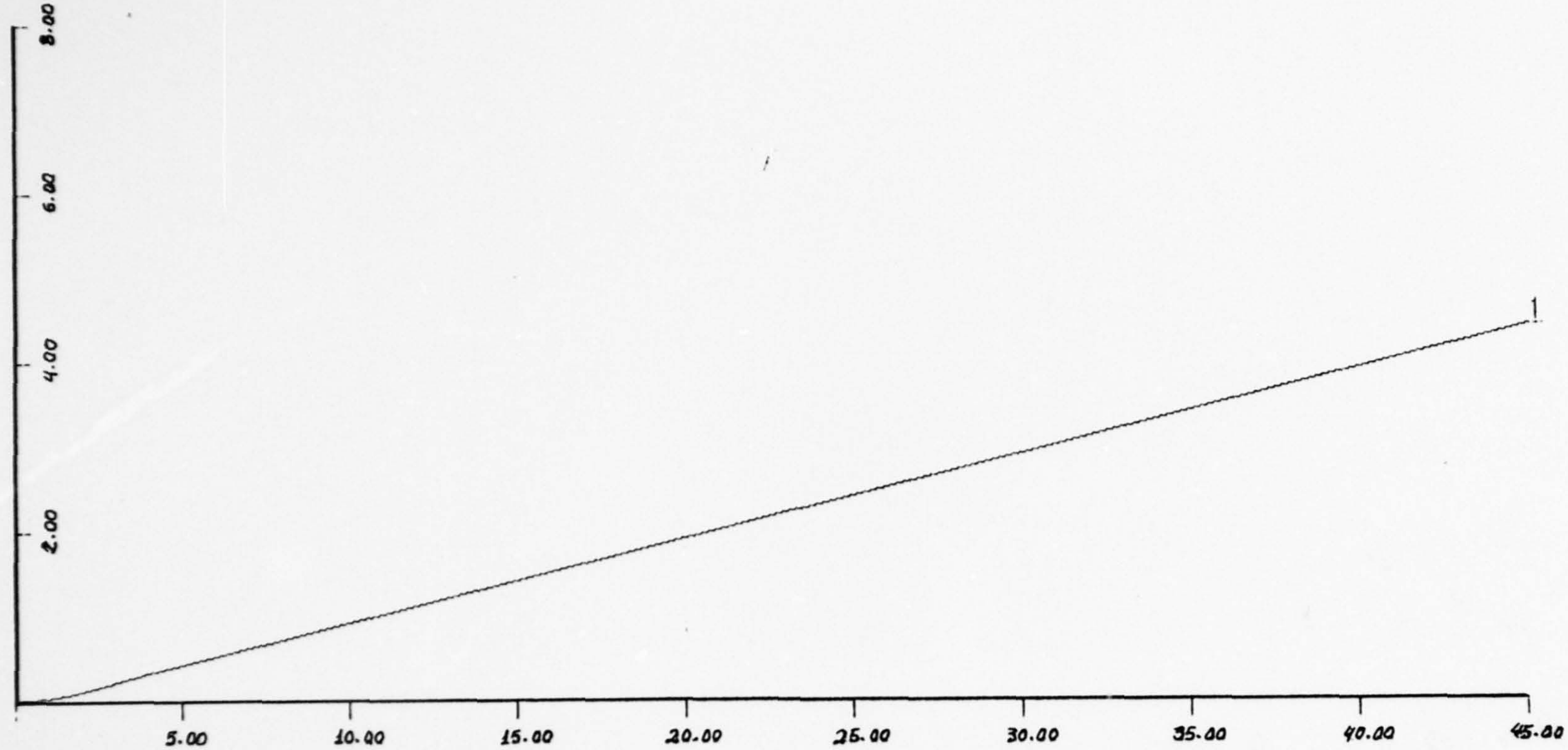
X-Scale=5.00 Units/inch
Y-Scale=0.50 Units/inch

FIGURE (8-2B)
Second modified singular structure.Time response of the system for a ramp input.



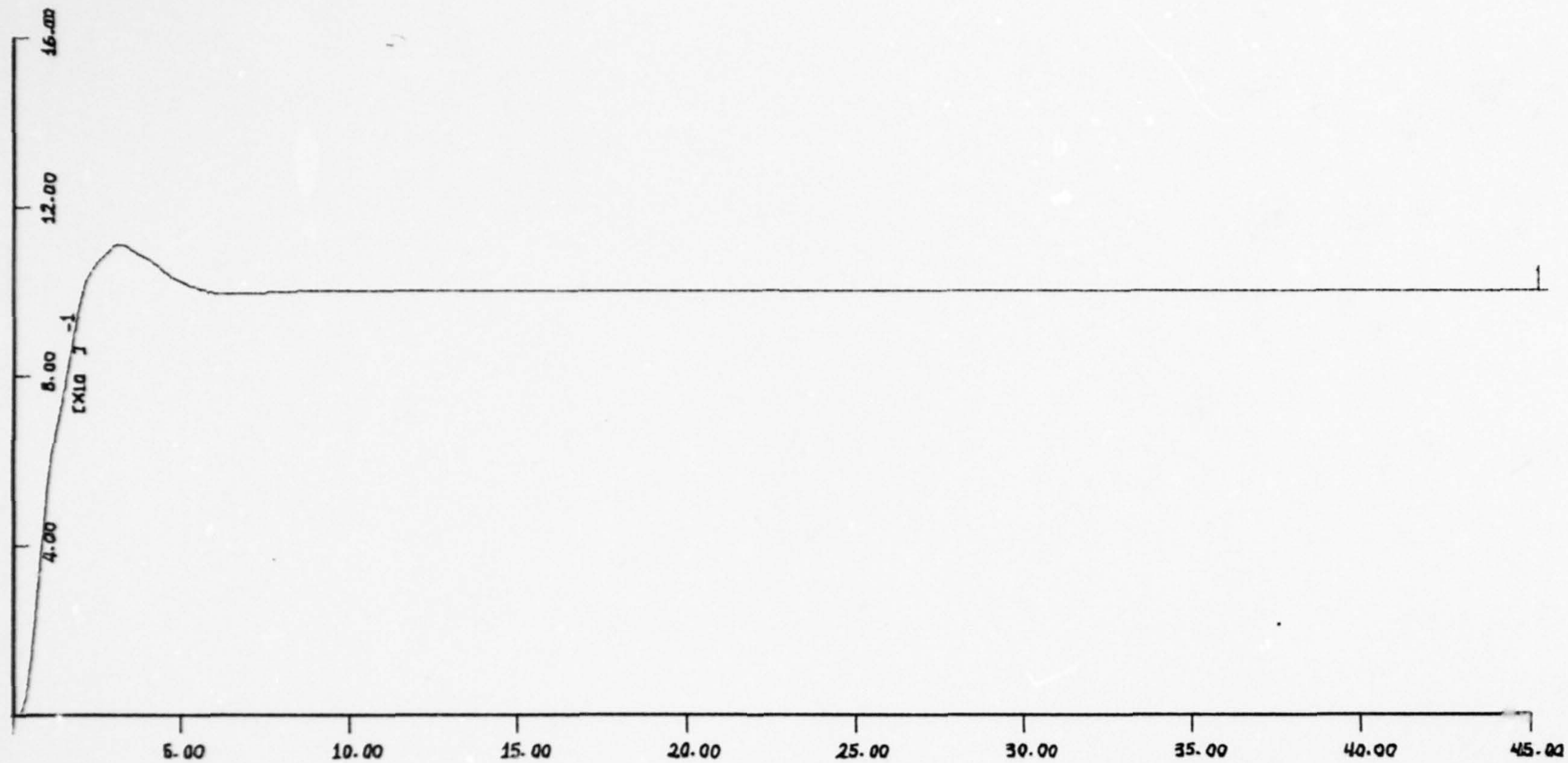
X-Scale=5.00 Units/inch
Y-Scale=0.40 Units/inch

FIGURE (8-3A)
Third modified singular structure.Time response of the system for a step input.



X-Scale=5.00 Units/inch
Y-Scale=2.00 Units/inch

FIGURE (8-3B)
Third modified singular structure.Time response of the system for a ramp input.



X-Scale=5.00 Units/inch
Y-Scale=0.40 Units/inch

FIGURE (8-4A)
Fourth modified singular structure.Time response of the system for a step input.