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EXPERIMENTS WITH LINEAR FRACTIONAL PROBLEMS.(U)

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EXPERIMENTS WITH LINEAR FRACTIONAL PROBLEMS

by

GABRIEL R. BITRAN

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FOREWORD

The Operations Research Center at the Massachusetts Institute of Technology is an interdepartmental activity devoted to graduate education and research in the field of operations research. The work of the Center is supported, in part, by government contracts and grants. The work reported herein was supported by the Office of Naval Research under Contract N00014-75-C-0556.

Richard C. Larson
Jeremy F. Shapiro

Co-Directors

ABSTRACT

In this paper ~~we~~ presents the results of a limited number of experiments with linear fractional problems. Six solution procedures were tested and the results are expressed in number of simplex-like pivots required to solve a sample of twenty problems randomly generated.

Two main approaches emerge from the literature to solve the linear fractional problem:

$$v = \max \{ f(x)=n(x)/d(x) \ : \ x \in F \} \qquad (P)$$

where $n(x)=c_o+cx$, $d(x)=d_o+dx$, $F=\{ x \in R^n : Ax=b \ x \geq 0 \}$, c_o and d_o are real numbers, c and d are real n-vectors, A is an mxn real matrix and b is a real m-vector. We assume in this note that F is compact and that $\min \{ d(x) : x \in F \} > 0$.

Charnes and Cooper [4] transform problem (P) into the linear program:

$$v = \max \{ c_o t+cy \ : \ Ay-bt=0, \ d_o t+dy=1, \text{ and } t,y \geq 0 \}.$$

This approach has been extended to the nonlinear version of (P) by Bradley and Frey [3] and Schaible [7]. The second approach solves a sequence of linear problems or at least one pivot step of each linear program over the original feasible set by updating the objective function. Algorithms in this category are related to ideas first presented by Isbell and Marlow [5] and Martos [6]. Similar algorithms have been proposed by several other authors. The interested reader is referred to the excellent bibliography collected by I.M. Stancu-Minasian [8]. Methods in the second approach propose to solve (P) through a sequence of linear programs:

$$r(x^k) = \max \{ r(x^k,x)=n(x)-f(x^k)d(x) \ : \ x \in F \} \quad k=0,1,2... \qquad (LP_k)$$

where x^o is a given feasible point and x^k for $k > 1$ is defined in Isbell and Marlow's procedure as being the optimal solution to (LP_{k-1}) and as the first feasible basis in (LP_{k-1}) for which $r(x^{k-1},x) > 0$ in Martos' procedure. Both algorithms terminate at iteration k_o for which $r(x^{k_o})=0$. In this case $x^{k_o}=x_{optimal}$. It is worth noting that Wagner and Yuan [9] related the two main approaches by showing that Martos' algorithm is equivalent to Charnes and Cooper's method in the sense that both algorithms lead to an identical sequence of pivoting operations. Bitran and Magnanti [1] have extended the connection between these approaches by relating them to generalized programming. No theoretical or empirical evidence has been given, in the past, indicating which of the several existing algorithms is to be preferred.

In this note we present the results, in number of simplex-like pivots, of twenty problems of type (P), randomly generated, solved by the following six algorithms (each problem when solved by each of the six procedures was started with the same basic feasible solution):

A) Maximize $n(x)$ over the feasible set obtaining the optimal solution x^*. Next, apply Isbell and Marlow's algorithm with $x^o=x^*$.

B) Minimize $d(x)$ over the feasible set obtaining the optimal solution x^*. Next, apply Isbell and Marlow's algorithm with $x^o=x^*$.

C) Maximize $g(x)=[c-(cd/dd)d]x$ over F obtaining the optimal solution x^*. Next, apply Isbell and Marlow's algorithm with $x^o=x^*$ (Bitran and Novaes [2] suggested the objective function $g(x)$).

D) Isbell and Marlow's algorithm.

E) Martos' algorithm.

F) The author considered relevant to compare these algorithms with the number of pivots necessary to solve the linear programs:

$$\max \{ n(x)-vd(x) : x \in F\} \qquad \text{(LP)}$$

where for each of the twenty problems (P), v is chosen as its optimal value. The optimal value of (LP) is zero and any solution to (LP) is optimal in the fractional program (P) ([1]). Note that (LP) corresponds to (LP_k) with $x^k=x_{\text{optimal}}$.

The characteristics of the data of the twenty randomly generated problems are the following:

$n=40$, $m=20$, the absolute value of each a_{ij}, the (i,j)th element of each matrix A was randomly generated in the interval $(0,10]$. The density of negative elements being 20%. Each component b_i $i=1,2.,,,m$ of each right hand side b was defined as $\sum_{j=1}^{n} a_{ij}/2$. The objective function coefficients c_o, c_j, d_o $d_j j=1,2..n$ were generated in the intervals $[-1000 \leq c_o,\ c_j \leq 1000;\ 0 < d_o,\ d_j \leq 1]$, $[1 \leq c_o,\ c_j \leq 1000;\ 1 \leq d_o,\ d_j \leq 2]$ or $[-1000 \leq c_o,\ c_j \leq -1;\ 1 \leq d_o,\ d_j \leq 2]$. The reason for choosing such intervals was to obtain five problems with an angle θ between the gradients of the numerator and denominator, i.e., $\cos\ \theta = \frac{cd}{\|c\|\|d\|}$, in each of the four intervals $[0, \frac{\pi}{4})$, $[\frac{\pi}{4}, \frac{\pi}{2})$, $[\frac{\pi}{2}, \frac{3\pi}{4})$, $[\frac{3\pi}{4}, 2\pi)$ in an attempt to identify a correlation between the algorithms tested and the geometry of linear fractional programs. The geometric properties of problem (P) are consequences of the following facts.

i) The hyperplanes $n(x)-Ld(x)=0$ contain for each L both the sets $\{x \in R^n$: $f(x)=L\}$ and $CE=\{x \in R^n: n(x)=0 \text{ and } d(x)=0\}$. The set CE is called the center of the problem because as L varies the hyperplanes rotate about it giving a "star" centered at CE ([2]).

ii) The objective function $f(x)$ is pseudo-concave and quasiconvex on the set $\{x \in R^n: d(x) > 0\}$, i.e., $f(y) > f(x)$ if and only if $\nabla f(x)(y-x) > 0$.

In R^2 the geometry of (P) ([2]) suggests that procedure (C) would perform better than (A) and (B) for high and low values of θ ($\theta \in [0,\pi]$). Table 1 shows the results obtained. For the first and last five problems a total of 178 pivots was necessary with procedure (C) while 233 and 363 pivots were required with procedures (A) and (B) respectively. The corresponding standard deviations being 3.70, 6.01 and 7.90. For the twenty problems selected Martos' algorithm performed better than the preceding four and in some cases required fewer pivots than procedure (F). Algorithms (C) and (D) were practically equivalent and were followed by (A), while (B) performed poorly. The computer code used to solve the twenty problems by the six algorithms was an adaptation of Burroughs' commercial code TEMPO.

PROBLEM NUMBER	A	B	C	D	E	F	COS θ
1	24	21	13	11	12	12	.873
2	21	34	18	18	15	15	.858
3	23	34	12	12	10	10	.819
4	19	39	21	21	18	17	.770
5	32	46	22	23	21	19	.730
MEAN	23.8	34.8	17.2	17.0	15.2	14.6	
STANDARD DEVIATION	4.44	8.18	4.07	4.77	3.97	3.26	
6	22	32	20	21	15	15	.569
7	25	57	28	23	18	18	.500
8	19	16	16	16	15	15	.370
9	22	51	18	20	11	11	.132
10	19	39	18	26	20	16	.076
MEAN	21.4	39.0	20.0	21.2	15.8	15.0	
STANDARD DEVIATION	2.24	14.46	4.19	3.31	3.06	2.28	
11	12	38	18	18	11	10	-.103
12	21	47	21	21	19	20	-.289
13	18	33	20	22	20	21	-.424
14	18	36	20	20	19	22	-.485
15	33	50	31	25	26	19	-.613
MEAN	20.4	40.8	22.0	21.2	19.0	18.4	
STANDARD DEVIATION	6.94	6.55	4.60	2.31	4.77	4.32	
16	19	51	17	17	15	15	-.720
17	16	30	13	12	13	16	-.747
18	16	39	20	21	13	15	-.820
19	33	33	22	23	21	24	-.840
20	30	36	20	24	18	19	-.874
MEAN	22.8	37.8	18.4	19.4	16.0	17.8	
STANDARD DEVIATION	7.25	7.25	3.14	4.41	3.10	3.43	
TOTAL # OF ITERATIONS	442	762	388	394	330	329	
MEAN	22.1	38.1	19.4	19.7	16.5	16.4	
STANDARD DEVIATION	5.75	9.88	4.42	4.20	4.07	3.79	

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