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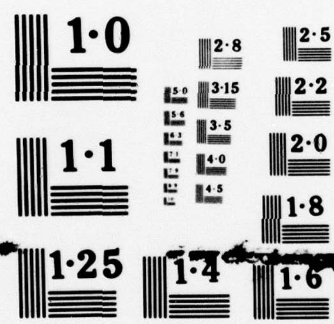
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This paper considers stochastic differential equations depending on a small parameter ε, which enters as a coefficient in the noise term of the equation. The Ventcel-Freidlin estimates give an asymptotic formula as $\varepsilon \to 0$ for the probability that a solution trajectory hits a given compact set B during a given time interval. A stochastic control method to obtain such estimates is outlined. Detailed proofs are given elsewhere.

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ACCOMPANYING STATEMENT

LARGE DEVIATIONS FOR DIFFUSIONS DEPENDING
ON SMALL PARAMETERS: A STOCHASTIC CONTROL METHOD

by

Wendell H. Fleming

This paper summarizes a stochastic control method to derive estimates of Ventcel-Freidlin type that a solution trajectory of a stochastic differential equation hits a given set B during a given time interval. A more detailed treatment of the method is given in the author's paper, "Exit probabilities and optimal stochastic control" to appear in Applied Mathematics and Optimization.

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LARGE DEVIATIONS FOR DIFFUSIONS DEPENDING
ON SMALL PARAMETERS: A STOCHASTIC CONTROL METHOD

Wendell H. Fleming+

ABSTRACT

This paper considers stochastic differential equations depending on a small parameter ε, which enters as a coefficient in the noise term of the equation. The Ventcel-Freidlin estimates give an asymptotic formula as $\varepsilon \to 0$ for the probability that a solution trajectory hits a given compact set B during a given time interval. A stochastic control method to obtain such estimates is outlined. Detailed proofs are given elsewhere.

1. The problem. Consider a Markov diffusion process ξ^ε on n-dimensional R^n which obeys the stochastic differential equations

$$d\xi^\varepsilon = b[t,\xi^\varepsilon(t)]dt + \sqrt{\varepsilon}\,\sigma[t,\xi^\varepsilon(t)]dw, \qquad s \le t, \tag{1.1}$$

with initial data $\xi^\varepsilon(s) = x$. Here w is an n-dimensional brownian motion and ε a positive parameter. Let B be a compact set, $B \subset R^n$, and let τ_B^ε denote the first time t such that $\xi^\varepsilon(t) \in B$. If $\xi^\varepsilon(t) \in R^n - B$ for all $t \ge s$ we set $\tau_B^\varepsilon = +\infty$. Given $T > s$, let

$$q_B^\varepsilon = P(\tau_B^\varepsilon \le T). \tag{1.2}$$

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Let us also consider the unperturbed system

$$d\xi^0 = b[t,\xi^0(t)]dt, \quad s \le t, \tag{1.3}$$

with the same initial data $\xi^0(s) = x$. If $\tau_B^0 > T$, then $q_B^\varepsilon \to 0$ as $\varepsilon \to 0$. The Ventcel-Freidlin estimates [3], [5] give a more precise statement about the rate of convergence to 0 of q_B^ε, under certain assumptions about the coefficients b,σ in (1.1) and the set B. A different method, based on stochastic control ideas, was used in [1], [2] to obtain estimates of Ventcel-Freidlin type. In this note we outline this stochastic control method, as it applies to the problem of estimating q_B^ε.

2. Upper and lower estimates. Let us assume that b is Lipschitz on R^{n+1}, and that σ together with its inverse σ^{-1} are bounded and Lipschitz. Some results in which σ degenerates in a particular way appear in [4]. Let $a = \sigma\sigma'$, and consider the variational integrand

$$L(t,\phi,\dot\phi) = \frac{1}{2}(b(t,\phi) - \dot\phi)'a^{-1}(t,\phi)(b(t,\phi) - \dot\phi). \tag{2.1}$$

Let

$$\Gamma_B = \{\phi \in C([s,T];R^n): \phi(s) = x, \exists\, \theta \in [s,T] \ni \phi(\theta) \in B\},$$

$$I_B = \min_{\Gamma_B} \int_s^T L[t,\phi(t),\dot\phi(t)]dt. \tag{2.2}$$

(The integral on the right side is defined to be $+\infty$ if ϕ is not absolutely continuous.) Let

$$I_B^\varepsilon = -\varepsilon \log q_B^\varepsilon. \tag{2.3}$$

Theorem 1. $I_B \le \liminf_{\varepsilon\to 0} I_B^\varepsilon$.

This theorem gives an upper estimate for q_B^ε. We shall outline a stochastic control proof in §3, and refer to [2] for details.

In order to state a lower estimate, we make the following additional assumption: There exists a relatively open set $\Gamma_B^0 \subset \Gamma_B$ such that

(2.4) $$\inf_{\Gamma_B^0} \int_s^T L[t,\phi(t),\dot{\phi}(t)]dt = I_B.$$

By relatively open we mean that given $\phi \in \Gamma_B^0$ there exists $\delta > 0$ such that $\psi \in C([s,T];R^n)$, $\psi(s) = x$ and $||\psi-\phi|| < \delta$ imply $\psi \in \Gamma_B^0$.

Theorem 2. If (2.4) holds, then $I_B \geq \limsup_{\varepsilon\to 0} I_B^\varepsilon$.

Theorem 2 is an immediate consequence of the first Ventcel-Freidlin estimate [3, p. 332]. That estimate is rather easily obtained from the Girsanov formula. On the other hand, the usual proof of Theorem 1 is based on a somewhat more technically complicated second Ventcel-Freidlin estimate [3, p. 334]. Our method furnishes an alternative proof, as well as a different intuition for arriving at such results.

From Theorems 1 and 2 one has $I_B^\varepsilon \to I_B$ as $\varepsilon \to 0$, which is the desired result about the probability of large deviations.

3. Stochastic control method. Let us fix $T > 0$ and consider $0 < s < T$. Let $Q = (0,T) \times (R^n-B)$, and write $q_B^\varepsilon = q_B^\varepsilon(s,x)$. Then $q_B^\varepsilon \in C^{1,2}(Q)$ and satisfies in Q the backward partial differential equation

(3.1) $$\frac{\partial q_B^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\sum_{i,j=1}^n a_{ij}(s,x)\frac{\partial^2 q_B^\varepsilon}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(s,x)\frac{\partial q_B^\varepsilon}{\partial x_i} = 0.$$

The function $I_B^\varepsilon = -\varepsilon \log q_B^\varepsilon$ satisfies the nonlinear equation

(3.2) $$\frac{\partial I_B^\varepsilon}{\partial s} + \frac{\varepsilon}{2}\sum_{i,j=1}^n a_{ij}(s,x)\frac{\partial^2 \phi_B^\varepsilon}{\partial x_i \partial x_j} + H(s,x,\nabla\phi_B^\varepsilon) = 0,$$

where for each row vector p

(3.3) $$H(s,x,p) = -\frac{1}{2}pa(s,x)p' + p\cdot b(s,x).$$

The function $H(s,x,\cdot)$ is dual to $L(s,x,\cdot)$, in the sense of duality for concave and convex functions. If one puts formally $\varepsilon = 0$ in (3.2), one gets the Hamilton-Jacobi equation associated with the variational integrand L. The fact that Γ_B includes only curves ϕ which reach B by time T corresponds formally to the condition $q_B^\varepsilon(T,x) = 0$ for $x \notin B$, and hence $I_B^\varepsilon(T,x) = +\infty$. The connection with the Hamilton-Jacobi equation indicates, but of course does not prove, results of the kind stated in §2.

<u>Lemma</u>. <u>Let</u> $B = \partial D$, <u>where</u> $D \subset R^n$ <u>is open, bounded with</u> C^2 <u>boundary</u> ∂D. <u>Then</u> $I_B \leq \liminf_{\varepsilon \to 0} I_B^\varepsilon$, <u>for any</u> $(s,x) \in (0,T) \times D$.

This result is proved in [2, §7], by an argument which we sketch below. From the Lemma, Theorem 1 is obtained as follows. There exist $D_1 \subset D_2 \subset \ldots$, with union $R^n - B$, such that $\partial D_n = \tilde{B}_n \cup \{|x| = n\}$, with $\tilde{B}_n$ of class C^2 and $\tilde{B}_n$ contained in the n^{-1}-neighborhood of B. We take $B_n = \partial D_n$ and note that $q_B^\varepsilon \leq q_{B_n}^\varepsilon$,

$$\liminf_{\varepsilon \to 0} I_B^\varepsilon \geq \liminf_{\varepsilon \to 0} I_{B_n}^\varepsilon \geq I_{B_n} .$$

Moreover, $I_{B_n} \to I_B$ as $n \to \infty$.

The proof in [2, §7] of the Lemma is based on a "penalty" function method. Let $\Phi(x)$ be Lipschitz, with $\Phi(x) > 0$ for $x \notin \partial D$, $\Phi(x) = 0$ for $x \in \partial D$. Let $J^\varepsilon(s,x)$ be the solution of (3.2) in $(0,T) \times D$, with $J^\varepsilon(s,x) = 0$ for $0 < s < T$, $x \in \partial D$, $J^\varepsilon(T,x) = \Phi(x)$. Then $J^\varepsilon(s,x)$ satisfies the dynamic programming equation for an optimal stochastic control problem in which the drift term $b[t,\xi^\varepsilon(t)]$ in (1.1) is replaced by an arbitrary, bounded, nonanticipative control process $v(t)$. Instead of (1.1), one now has the stochastic differential equation

$$d\eta = v(t)dt + \sqrt{\varepsilon}\, \sigma[t,\eta(t)]dw.$$

The quantity to be minimized is

$$E\{\int_s^\theta L(t,\eta(t),v(t))dt + \Phi[\theta,\eta(\theta)]\},$$

where $\theta = \min(T$, exit time from D of $\eta(t))$. The minimum of this expression is $J^\varepsilon(s,x)$. It is shown in [2] that $\liminf_{\varepsilon \to 0} J^\varepsilon \geq J$, where $J(s,x)$ is the minimum of $\int_s^\theta L[t,\phi(t),\dot{\phi}(t)]dt + \Phi[\theta,\phi(\theta)]$ taken among all (ϕ,θ) with $\phi(s) = x$ and either $\theta = T$ or $\theta < T$, $\phi(\theta) \in \partial D$. For $M = 1,2,\ldots$ we now take $\Phi = \Phi_M = M\Psi$ for fixed Ψ, and write $J^\varepsilon = J^{\varepsilon M}$, $J = J^M$. It is easy to show that $I \leq \liminf_{M \to \infty} J^M$. Moreover, $J^{\varepsilon M} \leq I^\varepsilon$. Then $J^M \leq \liminf_{\varepsilon \to 0} I_M^\varepsilon$

for each M, from which the Lemma follows.

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