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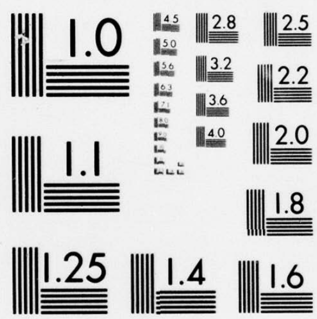


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(6) THE DETERMINATION OF THE DISTRIBUTION OF THE TIME IN THE WAITING LINE AND THE DISTRIBUTION OF THE LENGTH OF A BUSY PERIOD IN GI/G/1 QUEUES.

by

(10) Zeev Barzily

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Zeev Barzily

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# THE DETERMINATION OF THE DISTRIBUTION OF THE TIME IN THE WAITING LINE AND THE DISTRIBUTION OF THE LENGTH OF A BUSY PERIOD IN GI/G/1 QUEUES

by

Zeev Barzily

## 1. Introduction and Summary

We consider a GI/G/1 queueing system in which customer number $n$, $n=1,2,\ldots$, arrives at time $x_n$. We let $x_1 = 0$ and define

$$X_n = x_{n+1} - x_n \, , \quad n=1,2,\ldots \, ,$$

as the interarrival time between the nth and the (n+1)st customers. We assume that $X_1, X_2, \ldots$ are independent identically distributed random variables with a cumulative distribution function $A(\ )$, an expectation $\lambda$, and a variance $\sigma^2$. Let $M_k$ denote the number of the customer who initiates the kth busy period. We assume that the distribution of the time the nth customer stays in service is a function of the number of customers previously served in that same busy period; namely, denoting by $S_n$ the service time of the nth customer, then given that $M_k=\ell<n$, $M_{k+1} > n$, and letting $j = n+1-\ell$, then the nth customer is the jth served in the busy period and $S_n$ is distributed according to a distribution $G_j(\ )$ with an expectation $\mu_j$ and a variance $\nu_j^2 < \infty$.

Let $\rho_k = \mu_k/\lambda$ . Then if there exists a $k_1$ such that for all $k \geq k_1$ $\rho_k < 1$ , the queue is stable and the stochastic process $\{M_n; n=1,2,...\}$ is positive recurrent. We restrict the discussion in the present paper to stable queues.

Denote by $T_n$ the time spent by the nth customer in the waiting line. Provided that the distributions of $X_n$ , n=1,2,... , are nonlattice, then $T_n \Rightarrow T$ , where $\Rightarrow$ denotes convergence of distributions. Let $K_n$ denote the number of customers served in the nth busy period ($K_n = M_{n+1} - M_n$) . Under the assumptions stated above we obtain that $K_1, K_2, ...$ are independent identically distributed random variables. In Section 2 we give a numerical procedure for the determination of the distributions of $T$ and $K_n$ .

Let $B_n$ denote the length of the nth busy period. Clearly,

$$B_n = X_{M_{n+1}} - X_{M_n} .$$

Since the epochs of the starting of busy periods are regeneration points, $B_1, B_2, ...$ are independent identically distributed random variables. In Section 3 we present a numerical procedure for the determination of the distribution function of a busy period.

Several papers have been published recently on numerical analysis of queueing systems (see [1], [3], [4], [5], [6], and [7]). Most of these algorithms rely on the memoryless property of the exponential random variable. In the current algorithm none of the distributions of $S_n$ or $X_n$ has to be of the phase type. Furthermore, both distributions can depend on the number of customers that were served before the nth customer in the same busy period. The disadvantage of the current algorithm is that one must execute several recursive numerical integrations. Hence, when traffic is heavy the computing time can be rather long and the results may contain substantial numerical error. Section 4 includes a more detailed discussion of the results.

## 2. The Distribution of the Time Spent in the Waiting Line

In the previous section we denoted by $T_n$ the time the nth customer spends in the system; we also noted that $T_n \Rightarrow T$ . In this section we determine the distribution $F$ of the random variable $T$ .

Let

$$\overline{F}(t) = \lim_{n\to\infty} P[T_n > t] \ , \tag{2.1}$$

and denote by $L_n$ the customer who initiated the busy period in which the nth customer is served; clearly,

$$L_n = \text{Max } \{M_j : M_j \leq n\} \ .$$

To calculate $P[T_n > t]$ we use the fact that the epochs of the arrival of customers to an empty system $(x_{M_1}, x_{M_2}, \ldots)$ are regeneration points, hence it is convenient to apply the following conditioning,

$$P[T_n>t] = \sum_{j=1}^{n} P[T_n>t \mid L_n = n-j+1] \, P[L_n = n-j+1] \ . \tag{2.2}$$

We first notice that for $j=1$ ,

$$P[T_n>t \mid L_n=n] = 0 \ , \quad \text{for all } t \geq 0 \ . \tag{2.3}$$

When $j > 1$ ,

$$\begin{aligned} P[T_n>t \mid L_n=n-j+1] &= P\left[\sum_{i=1}^{j-1}(S_{n-j+i}-X_{n-j+i})>t \mid L_{n-j+1}=n-j+1, \right. \\ &\qquad \left. (S_{n-j+1}-X_{n-j+1})>0, \ \ldots, \ \sum_{i=1}^{j-1}(S_{n-j+i}-X_{n-j+i})>0\right] \\ &= P\left[\sum_{i=1}^{j-1}(S_i-X_i)>t \mid (S_1-X_1)>0, \ \ldots, \ \sum_{i=1}^{j-1}(S_i-X_i)>0\right] \ . \end{aligned} \tag{2.4}$$

To simplify our notation we let $H_2(t) = P[S_1 - X_1 > t]$ and

$$H_j(t) = P\left[(S_1-X_1)>0,\ \sum_{i=1}^{2}(S_i-X_i)>0,\ \ldots,\ \sum_{i=1}^{j-2}(S_i-X_i)>0,\ \sum_{i=1}^{j-1}(S_i-X_i)>t\right],$$

$$j=3,\ldots,\ t>0,$$

so that expression (2.4) yields

$$P[T_n>t \mid L_n=n-j+1] = H_j(t)/H_j(0). \qquad (2.5)$$

Note that the RHS of (2.5) is independent on $n$. As for the calculation of the probability $P[L_n=n-j+1]$, we make use of the fact that $(n-L_n)$ is the backwards recurrent time of the discrete process $M_j$, $j=1,2,\ldots$ at epoch $n$. Hence, denoting

$$E[K_n] = \beta, \quad n=1,2,\ldots,$$

and

$$P[K_n \geq k] = \overline{P}(k),$$

we obtain

$$\lim_{n\to\infty} P[L_n=n-j+1] = \overline{P}(j)/\beta, \quad j=1,2,\ldots, \qquad (2.6)$$

where $K_n$ (as defined in the introduction) is the number of customers served in the nth busy period. We now realize that the numerator of (2.6) equals the denominator of (2.5); thus, combining (2.1), (2.2), (2.5), and (2.6), we get

$$\overline{F}(t) = \frac{1}{\beta}\sum_{j=2}^{\infty} H_j(t), \quad t \geq 0, \qquad (2.7)$$

where $\beta$ can be obtained by using

$$\beta = 1 + \sum_{j=2}^{\infty} H_j(0). \qquad (2.8)$$

We will now obtain the recursive formulae for the evaluation of $H_j(t)$. Let

$$U_j(x) = \int_{\max\{-x,0\}}^{\infty} G_j(u+x)\,dF(u) , \qquad -\infty < x < \infty , \tag{2.9}$$

then the calculation of $H_2(t)$ is straightforward:

$$H_2(t) = P[S_1 - A_1 > t] = 1 - U_1(t) , \qquad t > 0 . \tag{2.10}$$

The calculation of $H_j(t)$ , j=3,4,... , is carried out recursively as follows:

$$H_{j+1}(t) = \int_{-\infty}^{\infty} P\left[(S_1-X_1)>0,\ \sum_{i=1}^{2}(S_i-X_i)>0,\ \ldots,\ \sum_{i=1}^{j-1}(S_i-X_i)>0,\ \sum_{i=1}^{j}(S_i-X_i)>t \;\middle|\; S_j-X_j=u\right] dP[S_j-X_j \leq u]$$

$$= \int_{-\infty}^{\infty} P\left[(S_1-X_1)>0,\ \ldots,\ \sum_{i=1}^{j-1}(S_i-X_i)>0,\ \sum_{i=1}^{j-1}(S_i-X_i)>t-u\right] dU_j(u)$$

$$= \int_{-\infty}^{t} H_j(t-u)\,dU_j(u) + H_j(0)(1-U_j(t)) , \qquad j=3,4,\ldots . \tag{2.11}$$

An examination of expression (2.11) reveals that it is highly unlikely that tractable analytic expressions for $H_j(t)$ , j=2,3,... , can be obtained. We are compelled, therefore, to approximate $\overline{F}(t)$ by

$$\overline{F}_0(t) = \sum_{j=L}^{N} H_j(t) \Big/ \left(1 + \sum_{j=2}^{N} H_j(0)\right) , \tag{2.12}$$

while the $H_j(t)$ , j=2,3,... are calculated numerically.

To assess the magnitude of the numerical error that results from summing up only N elements in (2.12), one should first assess the value of $\sum_{j=N+1}^{\infty} H_j(t)$ . This can often be achieved by using the fact that

$$H_j(t) \leq P\left[\sum_{i=1}^{j-1} (S_i-X_i) > t\right] . \tag{2.13}$$

The RHS of (2.13) can sometimes be approximated using the central limit theorem. If the c.d.f.'s $G_k$ , $k=1,2,\ldots$ , satisfy $G_k \leq G_0$ , where $G_0$ is a c.d.f. having an expectation $\mu_0$ $(\mu_0<\lambda)$ and a variance $\nu_0^2$ , then for $j$ sufficiently large we may write

$$P\left[\sum_{i=1}^{j-1}(S_i-X_i)>t\right] \cong 1-\phi\left(\frac{t-(j-1)(\mu_0-\lambda)}{\sqrt{(j-1)\left(\nu_0^2+\sigma^2\right)}}\right),$$

where $\phi$ denotes the standard normal c.d.f. If $\left(t-(j-1)(\mu_0-\lambda)\right)/\sqrt{(j-1)\left(\nu_0^2+\sigma^2\right)} \geq 1$ , then after some algebraic manipulations we obtain

$$1-\phi\left(\frac{t-(j-1)(\mu_0-\lambda)}{\sqrt{(j-1)\left(\nu_0^2+\sigma^2\right)}}\right) \leq \frac{1}{\sqrt{2\pi}}\exp\left(-t\,\frac{\lambda-\mu_0}{\nu_0^2+\sigma^2}\right)\exp\left(-\frac{(j-1)(\lambda-\mu_0)^2}{2\left(\nu_0^2+\sigma^2\right)}\right).$$

Now we apply the facts that $H_j(0) \geq H_{j+1}(0)$ and $H_j(0) \geq H_j(t)$ , and denoting

$$N_t = \min\left\{j:\ H_N(0) \geq \frac{1}{\sqrt{2\pi}}\exp\left(-t\,\frac{\lambda-\mu_0}{\nu_0^2+\sigma^2}\right)\exp\left(-\frac{(j-1)(\lambda-\mu_0)^2}{2\left(\nu_0^2+\sigma^2\right)}\right),\ j=N+1,N+2,\ldots\right\},$$

we get

$$\sum_{j=N+1}^{\infty} H_j(t) \leq C_N(t)\ , \qquad (2.14)$$

where

$$C_N(t) = (N_t-N)H_N(0) + \frac{\exp\left(-(\lambda-\mu_0)\left(t+(\lambda-\mu_0)N_t/2\right)/\left(\nu_0^2+\sigma^2\right)\right)}{\sqrt{2\pi}\left(1-\exp\left(-(\lambda-\mu_0)^2/2\left(\nu_0^2+\sigma^2\right)\right)\right)}.$$

Combining (2.7), (2.8), (2.12), and (2.14) yields

$$\frac{\sum_{j=2}^{N} H_j(t)}{\sum_{j=2}^{N} H_j(t) + C_N(t)} \leq \frac{\overline{F}_0(t)}{\overline{F}(t)} \leq 1 + \frac{C_N(0)}{1 + \sum_{j=2}^{N} H_j(0)} . \qquad (2.16)$$

The cases in which use of the central limit theorem cannot be justified should be handled on an individual basis. Here it may sometimes be very difficult to obtain the required bounds.

Numerical examples

A computer program was written for the calculation of $\overline{F}_0(t)$ . The recursive integrations (2.11) are carried out via the discretization of $X$ and $S$ . The ranges for which the densities are substantial are divided into intervals of length $\Delta$ . It is assumed that the discretized random variables take the value of the middle of an interval with a probability equal to the concentration of their densities in the interval.

Example 1: Interarrival times gamma distributed.

$$\frac{dA(x)}{dx} = \frac{7.0^{10.5}}{\Gamma(10.5)} x^{9.5} e^{-7.0x} , \quad 0 < x .$$

The service times are normally distributed $G_k \sim N\left(\mu_k = 0.95\left(1 + \frac{0.15}{k}\right), \sigma=0.15\right)$ . The function $\overline{F}_0(t)$ for $N=7$ and $\Delta=0.025$ (computing time 40 seconds on the IBM 370/148) is given in Figure 1. The RHS and LHS of Equation (2.16) were also calculated taking $\mu_0 = 1.0$ and $\nu_0 = 0.15$ ; the results are given in Table 1.

TABLE 1

THE RHS AND LHS OF (2.16) FOR THE DATA OF EXAMPLE 1

| t | 0.0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 |
|---|---|---|---|---|---|---|---|---|---|
| RHS | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.01 | 1.01 | 1.01 | 1.01 |
| LHS | 0.98 | 0.98 | 0.97 | 0.96 | 0.93 | 0.90 | 0.83 | 0.74 | 0.62 |

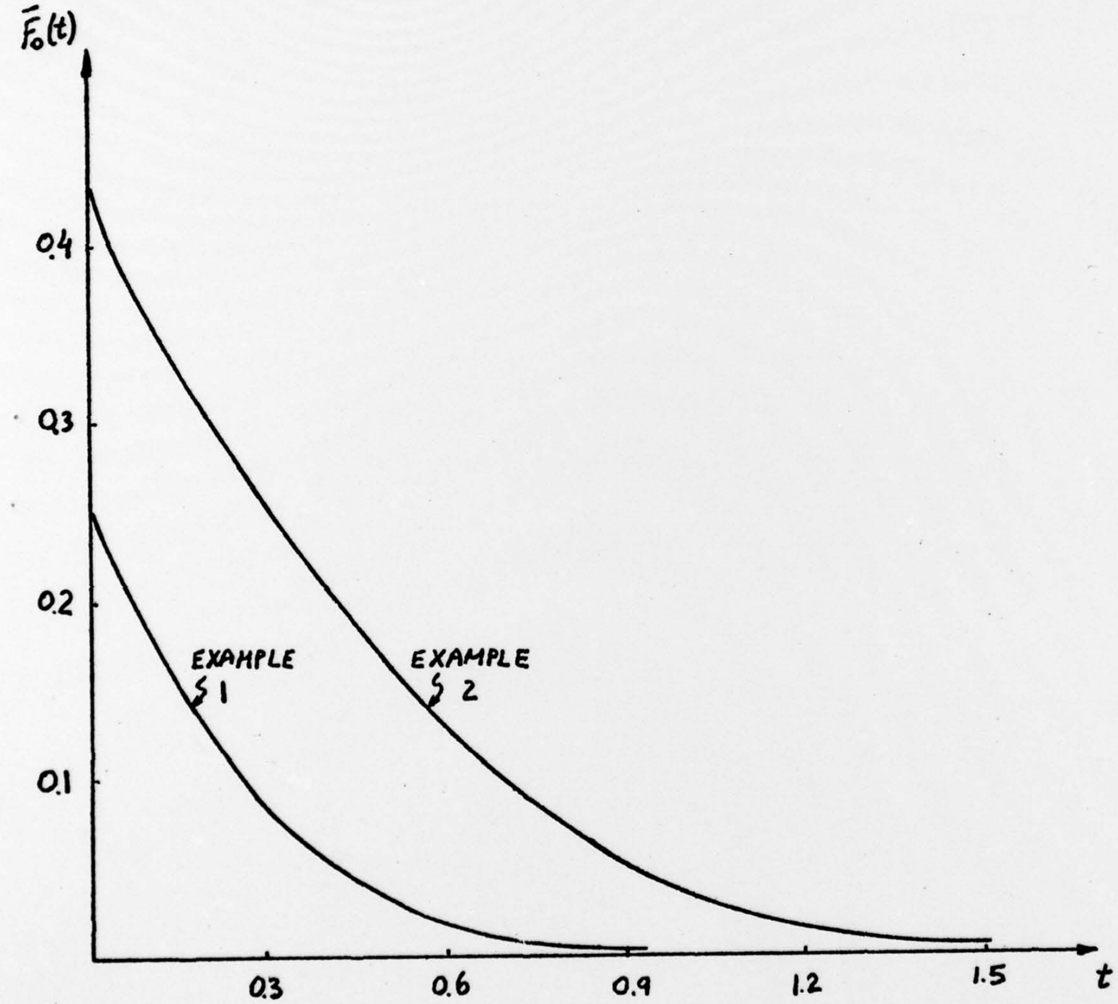


Figure 1. The function $\overline{F}_0(t)$ .

Example 2: Interarrival times uniformly distributed on (0,1]. Service times have the following distribution:

$$dG_k(x) = \begin{cases} \frac{1}{2} & , \quad x=0 \\ \frac{1}{2} \frac{(2k-1)!}{(k-1)!^2} x^{k-1} (1-x)^{k-1} dx , & 0<x<1 \end{cases} \qquad k=1,2,3,4 \; ;$$

for $k>4$ we have $G_k = G_4$. The function $\overline{F}_0(t)$ for $N=15$ and $\Delta=0.025$ (computing time 52 seconds on the IBM 370/148) is given in Figure 1. The bounds (2.16) were calculated here taking $G_0 = G_1$, and the results are presented in Table 2.

TABLE 2

THE RHS AND LHS OF (2.16) FOR THE DATA OF EXAMPLE 2

| t | 0.0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 |
|---|---|---|---|---|---|---|---|---|
| RHS | 1.03 | 1.03 | 1.03 | 1.04 | 1.04 | 1.04 | 1.05 | 1.06 |
| LHS | 0.93 | 0.91 | 0.88 | 0.83 | 0.74 | 0.63 | 0.43 | 0.32 |

## 3. The Distribution of the Length of a Busy Period

In the introduction we denoted by $B_n$ the length of the nth busy period and noticed that $B_1, B_2, \ldots$ are independent identically distributed random variables. In this section we calculate the cumulative distribution function of $B_n$. Define

$$\overline{R}(t) = P[B_n > t] , \quad t \geq 0 .$$

We calculate $\overline{R}(t)$ using the following formula:

$$\overline{R}(t) = \sum_{k=1}^{\infty} P[B_n > t, K_n = k] . \qquad (3.1)$$

Denote

$$Q_1(t,x) \;=\; P[S_1>t,\; S_1-A_1>x]$$

and

$$Q_n(t,x) \;=\; P\left[\sum_{i=1}^{n} S_i>t,\; S_1-A_1>0,\; \ldots,\; \sum_{i=1}^{n-1}(S_i-A_i)>0,\; \sum_{i=1}^{n}(S_i-A_i)>x\right] ,$$

$$n=2,3,\ldots \; .$$

The elements in (3.1) are calculated as follows. For $k=1$ we obtain

$$P\left[B_n>t,\; K_n=1\right] \;=\; \int_t^{\infty} \left(1-F_1(s)\right) dG_1(s) \;=\; 1 - G_1(t) - Q_1(t,0) \; ; \quad (3.2)$$

for $k>1$ we get

$$\begin{aligned}
P\left[B_n>t,\; K_n=k\right] \;&=\; P\left[\sum_{i=1}^{k} S_i>t,\; S_1-A_1>0,\; \ldots,\; \sum_{i=1}^{k-1}(S_i-A_i)>0,\; \sum_{i=1}^{k}(S_i-A_i)\leqq 0\right] \\
&=\; P\left[\sum_{i=1}^{k} S_i>t,\; S_1-A_1>0,\; \ldots,\; \sum_{i=1}^{k-1}(S_i-A_i)>0\right] \\
&-\; P\left[\sum_{i=1}^{k} S_i>t,\; S_1-A_1>0,\; \ldots,\; \sum_{i=1}^{k}(S_i-A_i)>0\right] \qquad (3.3) \\
&=\; \int_{s=0}^{\infty} P\left[\sum_{i=1}^{k} S_i>t,\; S_1-A_1>0,\; \ldots,\; \sum_{i=1}^{k-1}(S_i-A_i)>0 \;\middle|\; S_k=s\right] dG_k(s) \\
&-\; Q_k(t,0) \\
&=\; \int_{s=0}^{t} Q_{k-1}(t-s,\, 0)\,dG_k(s) + Q_{k-1}(0,0)\left(1-G_k(t)\right) - Q_k(t,0) \; .
\end{aligned}$$

The functions $Q_k(t,x)$ , $k \geqq 1$ , can be calculated recursively. First,

$$Q_1(t,x) \;=\; \int_{\max\{t,x\}}^{\infty} F_1(s-x)\,dG_1(s) \; , \qquad (3.4)$$

then for $k > 1$ we have

$$Q_k(t,x) = \int_{y=0}^{\infty} \int_{s=0}^{\infty} P\left[\sum_{i=1}^{k} S_i \geq t,\ S_1-A_1>0,\ \ldots,\ \sum_{i=1}^{k-1}(S_i-A_i)>0,\right.$$

$$\left.\sum_{i=1}^{k}(S_i-A_i)>x \mid S_k=s,\ A_k=y\right] dG_k(s)dF(y)$$

$$= \int_{y=0}^{\infty} \int_{s=0}^{\infty} P\left[\sum_{i=1}^{k-1} S_i \geq t-s,\ S_1-A_1>0,\ \ldots,\ \sum_{i=1}^{k-1}(S_i-A_i)>\max\{0,\ x-s+y\}\right]$$

$$\cdot\ dG_k(s)dF(y)\ .$$

After some manipulations we obtain for $x \leq t$ ,

$$Q_k(t,x) = \int_{y=0}^{t-x} \int_{s=0}^{x+y} Q_{k-1}(t-s,\ x-s+y)dG_k(s)dF(y)$$

$$+ \int_{y=t+x}^{\infty} \int_{s=0}^{x+y} Q_{k-1}(0,\ x-s+y)dG_k(s)dF(y)$$

$$+ \int_{y=0}^{t-x} \int_{s=x+y}^{t} Q_{k-1}(t-s,\ 0)dG_k(s)dF_k(y) \tag{3.5}$$

$$+ Q_{k-1}(0,0)\left[\left(1 - G_k(t)F(t-x)\right) + \int_{y=t-x}^{\infty}\left(1 - G_k(x+y)\right)\right]dF_k(y)\ ,$$

and if $x > t$ then

$$Q_k(t,x) = Q_k(x,x)\ . \tag{3.6}$$

As in the previous section, we approximate $\overline{R}(t)$ by

$$\overline{R}_0(t) = \sum_{k=1}^{N} P[B_n \geq t,\ K_n=k]\ . \tag{3.7}$$

To assess the sum of the remaining elements we can use the fact that for $k > 1$ ,

$$\sum_{k=N+1}^{\infty} P[B_n \geq t,\ K_n=k] \leq \sum_{k=N+1}^{\infty} P[K_n=k] = H_{N+1}(0)\ .$$

In Figure 2 we present the function $\overline{R}_0(t)$ for the two examples specified in Section 2. The numerical integration is carried out here, as in the examples of Section 2, via the discretization of the random variables. This program requires much longer computing periods because two-dimensional integrations have to be carried out in many points. The computation of $\overline{R}_0(t)$ on the IBM 370/148 for the data of Example 1, taking $N=4$ and $\Delta=0.15$ , required six minutes $(H_4(0) = 0.011)$ , and the computations for Example 2's data $(N=5 , \Delta=0.1)$ also required six minutes $(H_5(0) = 0.046)$ .

## 4. Conclusions

This paper presents numerical procedures for the determination of the distributions of the time a customer spends in the waiting line and the distribution of a busy period. The algorithm can use standard distribution functions as well as empirical distribution functions, and it can efficiently handle service times and interarrival times that are discrete random variables. Although our computer programs are quite inefficient and we believe that the computing times can be substantially reduced, it is expected that the required CPU times will generally be long, especially in heavy traffic. When the distributions involved are continuous, the smaller $\Delta$ is, the more accurate are the results, and the longer are their computing times. The "bounds" given in (2.16) are only approximate bounds since they make use of the central limit theorem. It is expected, however, that they give a good indication of the error resulting from replacing $\overline{F}(t)$ and $\overline{R}(t)$ by $\overline{F}_0(t)$ and $\overline{R}_0(t)$ , respectively.

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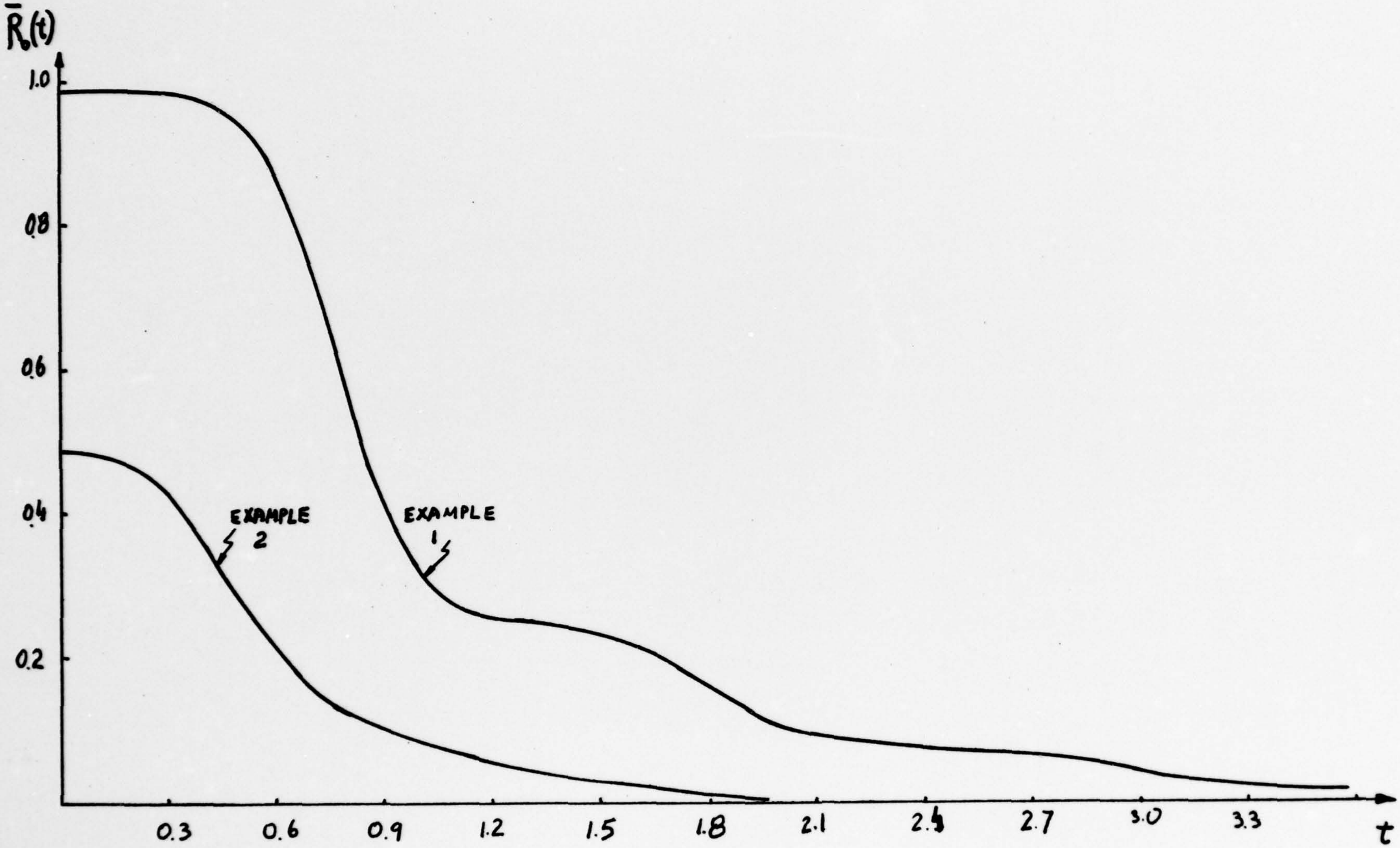


Figure 2. The function of $\bar{R}_0(t)$ .

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