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The buckling pressure of an elastic plate floating on water and stressed uniformly along the periphery of an internal hole

LEVEL II

$$w_1(x) + iw_2(x)$$

$$= \int_1^{\infty} \left\{ \left(r^2 + \sqrt{r^4 - 1}\right)^{-\frac{\mu}{2}} + \left(r^2 + \sqrt{r^4 - 1}\right)^{\frac{\mu}{2}} \right\} \frac{e^{\beta x r}}{\sqrt{r^4 - 1}} \, dr$$

Cover: The exact solution of differential equation (1.5) in report enabled us to carry out the analytical study in this report.

CRREL Report 78-14

(12)

The buckling pressure of an elastic plate floating on water and stressed uniformly along the periphery of an internal hole

Shunsuke Takagi

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The analytical solution and the numerical study of the eigenvalue problem for determining the buckling pressure of an infinite elastic plate floating on water and stressed uniformly along the periphery of an internal hole is presented. The boundary conditions considered are the clamped-, simple-, and free-edge conditions. Small buckling pressure occurs only for the free-edge condition. The shape of the deflection for the free-edge condition suggests that buckling is an important mechanism of failure.

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PREFACE

This report was prepared by Dr. Shunsuke Takagi, Research Physical Scientist, Physical Sciences Branch, Research Division, U.S. Army Cold Regions Research and Engineering Laboratory.

This study was funded under DA Project 4A161102AT24, *Research in Snow, Ice and Frozen Ground*; Task A1, *Properties of Cold Regions Materials*; Work Unit 001, *Properties of Snow and Ice.* The mathematical analysis was partly supported by the Army Research Office, Durham, North Carolina, under R&D Project no. 20061102B14C, *Exploratory Research in Mathematics,* in 1972 and 1973. The numerical computation was partly supported in 1976 by the Ice Engineering Program, CRREL.

This report was technically reviewed by Dr. Andrew Assur and Dr. Donald E. Nevel of CRREL.

One of the more difficult theoretical problems in ice engineering is the buckling of ice sheets under horizontal forces. A relevant case may be the end of a pier or an isolated offshore structure. The present report presents mathematical solutions for a somewhat simplified case which eventually, by means of computers, can be used to give guidance for design.

CONTENTS

ILLUSTRATIONS

NOMENCLATURE

a	nondimensional eigenvalue defined in (1.3). See (C.1-1) for the physical meaning.
D	flexural rigidity.
$F(x)$	complex fundamental solution given in the integral form by (4.11), in the series form by (5d.5), and in the asymptotic form by (12.6).
$f_n(x)$	Fuchsian type solutions defined by (3.2), (3.4), (3.5), (3.6).
k	subscript or superscript taking the value 1 or 2. When $k = 1$ or 2, the upper or lower of the double sign $\pm$ (or $\mp$) should be taken.
ℓ_0	the characteristic length defined by $= (D/\gamma)^{1/4}$.
r	radial distance from the center of the hole. Used also as an integration variable.
v	transformed unknown introduced by the contour integral (4.1).
w	deflection.
w_1, w_2	fundamental solutions.
x	$= r/\ell_0$.
x_0	the value of x at the periphery of the internal hole.
x_n	the eigenroots of the determinant equations $D_1, D_2, D_3 = 0$.
β	$= \exp(3\pi i/4) = (-1 + i)/\sqrt{2}$.
γ	density of water. Used also as Euler's constant.
κ	$= \sqrt{a - 1}$.
μ	$= \sqrt{1 - a}$.
ν	Poisson's ratio.

THE BUCKLING PRESSURE OF AN ELASTIC PLATE FLOATING ON WATER AND STRESSED UNIFORMLY ALONG THE PERIPHERY OF AN INTERNAL HOLE

Shunsuke Takagi

INTRODUCTION

The design of shore structures that must withstand oncoming ice sheets is one of the important themes of ice mechanics. During the last decade, field and laboratory measurements have been made on various kinds of structures.[10] However, the failure mechanism of floating ice sheets has not yet been clarified, and the interpretation of the data has not yet been satisfactory.

One of the failure mechanisms of a floating ice plate is buckling. The only analytical study presently available on this subject, however, concerns only the buckling of beams on elastic foundations.[3]

The difficulty of the analytical study of this problem is partly due to the multitudinous boundary geometries of the shore structures and ice plates. For this study we chose an ideally simple case and carried out the analysis of a buckling failure.

1. The problem

Let us suppose we have a thin elastic plate floating on water, extending horizontally to infinity, and stressed with uniform horizontal pressure along the periphery of an internal circular hole. We are interested in formulating buckling pressure and deflection at the point of failure.

The vertical deflection w of an elastic plate that rests on a liquid and is subjected to the middle surface forces N_{xx}, N_{yy}, and N_{xy} is governed by the differential equation

$$D\nabla^4 w + \gamma w = N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2} \qquad (1.1)$$

where D is the flexural rigidity and γ the specific weight of the liquid.[5] Let r be the radial distance from the center of the hole. In our problem, the deformation is cylindrically symmetric around the center of the hole. Then (1.1) becomes (see App. B)

$$l_0^4\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w + w = \frac{1}{\gamma}\left(N_{rr}\frac{d^2 w}{dr^2} + N_{\theta\theta}\frac{1}{r}\frac{dw}{dr}\right) \qquad (1.2)$$

where $l_0 = (D/\gamma)^{1/4}$ is the characteristic length, and N_{rr} and $N_{\theta\theta}$ are the radial and hoop horizontal stresses in the plate.

Following the usual treatment,[5] we assume that the horizontal stress components N_{xx}, N_{xy}, N_{yy} are in equilibrium by themselves. Then they are derived from a biharmonic function ϕ by

$$N_{xx} = \frac{\partial^2 \phi}{\partial y^2}$$

$$N_{yy} = \frac{\partial^2 \phi}{\partial x^2}$$

$$N_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

In the general polar coordinates they become

$$N_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$N_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

$$N_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}\,.$$

In our problem, ϕ is a function of r only and must tend to zero when r becomes infinite. Then $N_{r\theta} = 0$, and N_{rr} and $N_{\theta\theta}$ may be formulated as

$$\begin{aligned} N_{rr} &= -a\gamma\, l_0^4\, r^{-2} \\ N_{\theta\theta} &= a\gamma\, l_0^4\, r^{-2} \end{aligned} \tag{1.3}$$

where a is a nondimensional constant chosen to be positive.

We introduce the nondimensional length x

$$x = r\, l_0^{-1}\,. \tag{1.4}$$

Thus, (1.2) becomes

$$\frac{d^4 w}{dx^4} + \frac{2}{x}\frac{d^3 w}{dx^3} - \frac{1-a}{x^2}\frac{d^2 w}{dx^2} + \frac{1-a}{x^3}\frac{dw}{dx} + w = 0. \tag{1.5}$$

At $x = \infty$, the condition

$$\begin{aligned} w &= 0 \\ \frac{dw}{dx} &= 0 \end{aligned} \tag{1.6}$$

must be satisfied. At $x = x_0$, where x_0 is the value of x at the periphery of the internal hole, we consider three alternative conditions:

(1) the clamped-edge condition:

$$\begin{aligned} w &= 0 \\ \frac{dw}{dx} &= 0 \end{aligned} \tag{1.7}$$

(2) the simple-edge condition:

$$w = 0$$
$$\frac{d^2 w}{dx^2} + \frac{\nu}{x}\frac{dw}{dx} = 0 \tag{1.8}$$

(3) the free-edge condition:

$$\frac{d^2 w}{dx^2} + \frac{\nu}{x}\frac{dw}{dx} = 0$$
$$\frac{d}{dx}\left(\frac{d^2 w}{dx^2} + \frac{1}{x}\frac{dw}{dx}\right) + \frac{a}{x^2}\frac{dw}{dx} = 0 \tag{1.9}$$

where ν is Poisson's ratio.

Eq $(1.8)_2$ and $(1.9)_1$ are found from $M_r = 0$. The second equation of (1.9) is derived from Kirchhoff's condition,[5] $Q_r + (1/r)(\partial M_{r\theta}/\partial\theta) = 0$. The effect of horizontal stress is counted in Q_r, as explained below. In the rectangular coordinates x, y, shears Q_x and Q_y are given by

$$Q_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} + N_{xx}\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}$$
$$Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} + N_{xy}\frac{\partial w}{\partial x} + N_{yy}\frac{\partial w}{\partial y}\,. \tag{1.10}$$

We found these equations by extending Hetényi's[3] one-dimensional treatment to two-dimensional treatment. In the polar coordinates r, θ, shears Q_r and Q_θ are given (see App. B) by

$$Q_r = \frac{\partial M_r}{\partial r} + \frac{1}{r}\left\{M_{rr} + \frac{\partial M_{r\theta}}{\partial\theta} - M_{\theta\theta}\right\} + \frac{\partial w}{\partial r}N_{rr} + \frac{1}{r}\frac{\partial w}{\partial\theta}N_{r\theta}$$
$$Q_\theta = \frac{\partial M_{r\theta}}{\partial r} + \frac{2}{r}M_{r\theta} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial\theta} + \frac{\partial w}{\partial r}N_{r\theta} + \frac{1}{r}\frac{\partial w}{\partial\theta}N_{\theta\theta}\,. \tag{1.11}$$

The constant a is the eigenvalue to be determined to satisfy the boundary conditions at $x = x_0$. The *first step* for the solution of this eigenvalue problem is to discover, given a positive number a, two real functions, $w_1(x)$ and $w_2(x)$, that are the solutions of differential equation (1.5) and meet the boundary conditions at $x = \infty$ in (1.6). The value of x_0 is not yet specified at this stage, and no boundary conditions are imposed in the finite range. We call w_1 and w_2 the *fundamental solutions.* We shall find them later in the following form:

$$w_1 + iw_2 = \int_1^\infty \left\{(r^2 + \sqrt{r^4 - 1})^{-\sqrt{1-a}/2} + (r^2 + \sqrt{r^4 - 1})^{\sqrt{1-a}/2}\right\} e^{\frac{-1+i}{\sqrt{2}}xr} (r^4 - 1)^{-1/2}\, dr \tag{1.12}$$

The *second step* is to express the fundamental solutions as power series of x. Let $f_m(x)$ $(m = 0, 1, 2, 3)$ be the Fuchsian type solutions of (1.5) relative to $x = 0$. We shall find linear combinations

$$w_k(x) = \sum_{m=0}^{3} A_{km} f_m(x) \tag{1.13}$$

where $k = 1$ or 2, by determining constants A_{km} by use of (1.12). The solution $w(x)$ is a linear combination of the fundamental solutions

$$w(x) = A\, w_1(x) + B\, w_2(x). \tag{1.14}$$

The *third step* is to solve the simultaneous equations of A and B that are found by substituting (1.14) into the boundary conditions of (1.7), (1.8), or (1.9). If the algebraic equation found by letting the determinant of the simultaneous equations equal zero has one or more than one non-negative root, a root gives x_0. Our problem is then solved.

2. Abstract of the result

The main feature of the numerical result is as follows:

a. Under the clamped-edge condition, eigenvalue a is in the range $1 < a < \infty$. It does not extend to infinity. Infinitely many eigenvalues exist in the neighborhood of $x_0 = 0$.

b. Under the simple-edge condition, eigenvalue a is in the range $1 < a \leqq \infty$. A single branch extends to infinity. Infinitely many eigenvalues exist in the neighborhood of $x_0 = 0$.

c. Under the free-edge condition, eigenvalue a is in the range $1 - \nu^2 \leqq a \leqq \infty$, where ν is Poisson's ratio of the elastic plate. Two branches extend to infinity. Note that the range of a in this case extends to between 0 and 1. Infinitely many eigenvalues exist in the neighborhood of $x_0 = 0$.

d. Our numerical computation is complete in the range $0 \leqq a \leqq 626$ and in the neighborhood of $a = \infty$, but incomplete in the intermediate region $626 < a < \infty$. We believe that the range presented in this paper covers all the cases of our interest.

e. Buckling deflections take the shapes shown in Figures 10, 11, and 12. In these deflections, curve x_0 under the free-edge condition is observed frequently in laboratory experiments and field tests. Therefore, we may conclude that buckling under the free-edge condition is an important mechanism of deformation and failure.

PART I. FUNDAMENTAL SOLUTIONS

The main objectives of Part I are to find A_{km} in (1.13) and to determine the fundamental solutions in series forms.

3. Fuchsian type solutions

Equation (1.5) has a regular singularity at $x = 0$. The solutions relative to $x = 0$ are the Fuchsian type power series of x. Their indicial numbers are

$$\nu_0 = 0$$

$$\nu_1 = 2$$

$$\nu_2 = 1 + \mu$$

$$\nu_3 = 1 - \mu$$

where

$$\mu = \sqrt{1 - a}\,. \tag{3.1}$$

We express the four solutions with the single formula $f_m(x)$

$$f_m(x) = \sum_{n=0}^{\infty} a_n^{(m)} x^{\nu_m + 4n} \qquad (3.2)$$

where $m = 0, 1, 2, 3$, and

$$a_0^{(m)} = 1$$

and the rest of the coefficients $a_n^{(m)}$ ($n \geqq 1$) are determined by the recurrence formula

$$a_n^{(m)} = -a_{n-1}^{(m)} [(\nu_m + 4n)(\nu_m + 4n - 2)(\nu_m + 4n - 1 - \mu)(\nu_m + 4n - 1 + \mu)]^{-1} \qquad (3.3)$$

The individual forms of the solutions are

$$f_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{3n}(2n)!} \frac{\Gamma\left[\frac{1}{4}(3-\mu)\right]\Gamma\left[\frac{1}{4}(3+\mu)\right]}{\Gamma\left[n+\frac{1}{4}(3-\mu)\right]\Gamma\left[n+\frac{1}{4}(3+\mu)\right]} x^{4n} \qquad (3.4)$$

$$f_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{3n}(2n+1)!} \frac{\Gamma\left[\frac{1}{4}(5-\mu)\right]\Gamma\left[\frac{1}{4}(5+\mu)\right]}{\Gamma\left[n+\frac{1}{4}(5-\mu)\right]\Gamma\left[(n+\frac{1}{4}(5+\mu)\right]} x^{4n+2} \qquad (3.5)$$

$$f_{k+1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{4n} n!} \frac{\Gamma\left[\frac{1}{2}(2\pm\mu)\right]\Gamma\left[\frac{1}{4}(3\pm\mu)\right]\Gamma\left[\frac{1}{4}(5\pm\mu)\right]}{\Gamma\left[n+\frac{1}{2}(2\pm\mu)\right]\Gamma\left[n+\frac{1}{4}(3\pm\mu)\right]\Gamma\left[n+\frac{1}{4}(5\pm\mu)\right]} x^{4n+1\pm\mu} \qquad (3.6)$$

where $k = 1, 2$, and $\Gamma(\;)$ is a gamma function. In (3.6) we have introduced the convention that *the upper or lower of the double sign* $\pm$ (or $\mp$) should be taken according to $k = 1$ or 2, respectively. We use a subscript or superscript k exclusively for this purpose. This convention is observed throughout the paper.

Differential equation (1.5) has an irregular singularity at $x = \infty$. In other words, the solution relative to $x = \infty$ can be found in the form

$$e^{-\lambda x} \sum_{n=0}^{\infty} p_n x^{-1/2 - n}$$

where λ satisfies $\lambda^4 + 1 = 0$. The series $\sum_{n=0}^{\infty} p_n x^{-n}$ in this equation is asymptotic and divergent. Therefore this equation does not provide any means for determining A_{km} in (1.13).

4. Contour integral solution

In order to find the fundamental solutions, we must transform (1.5) by means of the contour integral

$$w(x) = \int_L v(\zeta) e^{x\zeta} d\zeta \qquad (4.1)$$

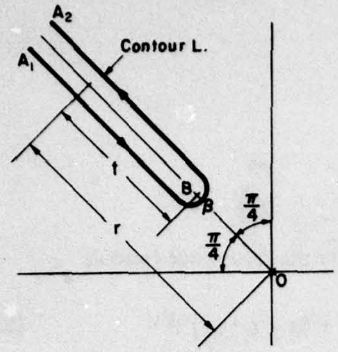

Figure 1. Contour L on the complex plane ζ. Points A_1 and A_2 represent points $\infty\beta$ on the respective branch.

where L is a contour in the complex plane of ζ that shall be determined to let a solution of (1.5) satisfy the boundary condition (1.6) at $x = \infty$. Following the usual procedure (ref. 4, p. 187-188), we arrive at the differential equation of $v(\zeta)$

$$(1 + \zeta^4)\frac{d^3 v}{d\zeta^3} + 10\zeta^3 \frac{d^2 v}{d\zeta^2} + (23 + a)\zeta^2 \frac{dv}{d\zeta} + (9 + 3a)\zeta v = 0. \tag{4.2}$$

The contour L selected for this solution is shown in Figure 1.

To find the solution relative to $\zeta = \infty$, let

$$\zeta = \beta r, \tag{4.3}$$

where

$$\beta = \exp(3\pi i/4). \tag{4.4}$$

Then (4.2) becomes

$$(r^4 - 1)\frac{d^3 v}{dr^3} + 10r^3 \frac{d^2 v}{dr^2} + (23 + a) r^2 \frac{dv}{dr} + (9 + 3a) r v = 0. \tag{4.5}$$

This equation has a regular singular point at $r = \infty$. The indicial numbers λ_m ($m = 0, 1, 2$) at $r = \infty$ are

$$\lambda_0 = 3$$

$$\lambda_1 = 2 + \mu$$

$$\lambda_2 = 2 - \mu$$

where μ is given by (3.1). The solution $v_m(r)$, corresponding to the indicial number λ_m, is

$$v_m(r) = \sum_{n=0}^{\infty} q_n^{(m)} r^{-\lambda_m - 4n} \tag{4.6}$$

where

$$q_n^{(m)} = \prod_{p=0}^{n-1} (4p+\lambda_m)(4p+\lambda_m+2)\left[(4p+\lambda_m+2-\mu)(4p+\lambda_m+2+\mu)\right]^{-1}. \qquad (4.7)$$

The contour L must be such that the point $\zeta = \beta$ is a branch point of $\nu(\zeta)$. This condition is satisfied by the series $\nu_1(r)$ and $\nu_2(r)$, as shown by (4.10) below, but not by $\nu_0(r)$. The series $\nu_1(r)$ and $\nu_2(r)$ can be expressed by means of hypergeometric series $F(\ ,\ ;\ ;\ )$ as

$$\nu_k(r) = r^{-2\mp\mu} F[(2\pm\mu)/4, (4\pm\mu)/4; (2\pm\mu)/2; r^{-4}]. \qquad (4.8)$$

The hypergeometric series are summed up by use of the formula

$$F(a, \tfrac{1}{2}+a; 2a; z) = 2^{2a-1}(1-z)^{-1/2}(1+\sqrt{1-z})^{1-2a} \qquad (4.9)$$

[Ref. 2, p. 556, formula 15.1.14]. Thus $\nu_k(r)$, where $k = 1$ or 2, simplifies to

$$\nu_k(r) = (r^4-1)^{-1/2}\left[(1/2)(r^2+\sqrt{r^4-1})\right]^{\mp\mu/2}. \qquad (4.10)$$

This equation shows that $\zeta = \beta$ is a branch point. Formula (4.9) can be proved by showing that the right-hand side satisfies the hypergeometric differential equation of the left-hand side and that they both satisfy the same initial conditions at $z = 0$.

Suppose that $\nu_k(r)$ on the branch A_1B of L in Figure 1 is given by (4.10). Then, $\nu_k(r)$ on the other branch A_2B of L is given by

$$(r^4-1)^{-1/2}\left[(1/2)(r^2+\sqrt{r^4-1})\right]^{\pm\mu/2}.$$

Thus we find the integral solution

$$F(x) = \int_1^\infty \left\{(r^2+\sqrt{r^4-1})^{-\mu/2} + (r^2+\sqrt{r^4-1})^{\mu/2}\right\} e^{\beta x r}(r^4-1)^{-1/2}\,dr. \qquad (4.11)$$

Then the fundamental solutions $w_1(x)$ and $w_2(x)$ are given by the real and imaginary parts of $F(x)$

$$F(x) = w_1(x) + i\,w_2(x). \qquad (4.12)$$

5. Integration of the integral solution

We shall integrate (4.11) to a linear combination of $f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$. The first step toward this goal is to change the range of integration in (4.11). We use a complex variable $z = \beta r$, where β is given by (4.4), to transform (4.11) to

$$F(x) = \int_\beta^{\infty\beta} \left\{(\beta^{-2}z^2+\sqrt{-z^4-1})^{-\mu/2} + (\beta^{-2}z^2+\sqrt{-z^4-1})^{\mu/2}\right\} e^{zx}\beta^{-1}(-z^4-1)^{-1/2}\,dz. \qquad (5.1)$$

We change the range of integration $\beta \sim \infty\beta$ to the sum of the ranges $\beta \sim 0$ and $0 \sim -\infty$. Thus (5.1) becomes

$$F(x) = \left\{ \int_{\beta}^{0} + \int_{0}^{-\infty} \right\} \left\{ (\beta^{-2} z^2 + i\sqrt{z^4+1}\,)^{-\mu/2} + (\beta^{-2} z^2 + i\sqrt{z^4+1}\,)^{\mu/2} \right\} e^{zx}\, i^{-1}\, \beta^{-1}\, (z^4+1)^{-1/2}\, dz. \tag{5.2}$$

In the above equations, the quantities inside the square roots are made positive in the respective ranges in order to ensure correct transformations. Letting $z = \beta r$ and $z = -r$ in the ranges $\beta \sim 0$ and $0 \sim -\infty$, respectively, (5.2) transforms to a linear combination of the normal range integrals

$$F(x) = ih_1(x) + ih_2(x) - \beta \exp(-\mu\pi i/4)\, g_1(x) - \beta \exp(\mu\pi i/4)\, g_2(x) \tag{5.3}$$

where

$$h_k(x) = \int_0^1 (r^2 + i\sqrt{1-r^4}\,)^{\mp\mu/2}\, e^{\beta x r}\, (1-r^4)^{-1/2}\, dr \tag{5.4}$$

$$g_k(x) = \int_0^{\infty} (r^2 + \sqrt{r^4+1}\,)^{\mp\mu/2}\, e^{-xr}\, (r^4+1)^{-1/2}\, dr. \tag{5.5}$$

Expansion of $\exp(\beta xr)$ transforms (5.4) to a power series of x

$$h_k(x) = h_0^{(k)} + \beta x\, h_1^{(k)} + \frac{1}{2!} (\beta x)^2\, h_2^{(k)} + \dots, \tag{5.6}$$

where

$$h_n^{(k)} = \int_0^1 (r^2 + i\sqrt{1-r^4}\,)^{\mp\mu/2}\, r^n\, (1-r^4)^{-1/2}\, dr. \tag{5.7}$$

Integral (5.5) transforms, as explained in the next section, to an ascending power series of x

$$g_1(x) = g_0^{(1)} + g_1^{(1)} x + g_\mu^{(1)} x^{1+\mu} + g_2^{(1)} x^2 + \dots \tag{5.8}$$

$$g_2(x) = g_0^{(2)} + g_\mu^{(2)} x^{1-\mu} + g_1^{(2)} x + g_2^{(2)} x^2 + \dots \tag{5.9}$$

Integration of (5.7) will be carried out later. Thus we find $F(x)$ in the following form

$$F(x) = B_0 + B_\mu^{(2)} x^{1-\mu} + B_1 x + B_\mu^{(1)} x^{1+\mu} + B_2 x^2 + \dots \tag{5.10}$$

where

$$B_0 = i(h_0^{(1)} + h_0^{(2)}) - \beta(e^{-(\mu\pi i/4)}\, g_0^{(1)} + e^{\mu\pi i/4}\, g_0^{(2)}) \tag{5.11}$$

$$B_1 = i\beta(h_1^{(1)} + h_1^{(2)}) - \beta(e^{-(\mu\pi i/4)}\, g_1^{(1)} + e^{\mu\pi i/4}\, g_1^{(2)}) \tag{5.12}$$

$$B_2 = \frac{1}{2} (h_2^{(1)} + h_2^{(2)}) - \beta(e^{-(\mu\pi i/4)}\, g_2^{(1)} + e^{\mu\pi i/4}\, g_2^{(2)}) \tag{5.13}$$

$$B_\mu^{(k)} = -\beta e^{\mp(\mu\pi i/4)}\, g_\mu^{(k)}. \tag{5.14}$$

In this calculation we have tentatively assumed that $0 < a < 1$. Series are arranged in the ascending order on this assumption. The formulas for the outside of the range $0 < a < 1$ will be derived from the formulas in this range. Note that the entries in (5.10), except $B_1 x$, are the first terms of $f_m(x)$, where $m = 0, 1, 2, 3$. The first-order term, $B_1 x$, is not contained in any of $f_m(x)$; it is proved that $B_1 = 0$. Therefore, the entries in (5.10) are sufficient to express $w_k(x)$ as linear combinations of $f_m(x)$.

If the original series $v_\kappa(r)$ in (4.6) is not summed up to a closed form (4.10), the transformation of integral (5.1) to (5.2) needs the analytical continuation of the integrand. However, the analytical continuation at the singular point $\zeta = \beta$ is not easy. In this particular case, this difficulty is completely overcome by use of the hypergeometric series. Further discussion is presented in the Appendix.

5a. Formulas of $g_n^{(k)}$

To formulate $g_n^{(k)}$ we shall successively develop (5.5) into series of x. Integration of these formulas will be carried out in the next section.

Letting $x = 0$ in (5.5), we find $g_0^{(k)}$:

$$g_0^{(k)} = \int_0^\infty (r^2 + \sqrt{r^4+1}\,)^{\mp(\mu/2)} (r^4+1)^{-1/2}\, dr. \tag{5a.1}$$

To find $g_\mu^{(2)}$, subtract $g_0^{(2)}$ in (5a.1) from $g_2(x)$ in (5.5),

$$g_2(x) - g_0^{(2)} = \int_0^\infty (r^2 + \sqrt{r^4+1}\,)^{\mu/2} (e^{-rx} - 1)(r^4+1)^{-1/2}\, dr.$$

Letting $\xi = rx$, the right-hand side of this equation becomes

$$= x^{1-\mu} \int_0^\infty (\xi^2 + \sqrt{\xi^4 + x^4}\,)^{\mu/2} (e^{-\xi} - 1)(\xi^4 + x^4)^{-1/2}\, d\xi. \tag{5a.2}$$

Letting $x = 0$ inside the last integral, we find

$$g_\mu^{(2)} = 2^{\mu/2} \int_0^\infty \xi^{\mu-2} (e^{-\xi} - 1)\, d\xi. \tag{5a.3}$$

To find $g_1^{(2)}$, we subtract the product of $x^{1-\mu}$ and (5a.3) from (5a.2)

$$g_2(x) - g_0^{(2)} - g_\mu^{(2)} x^{1-\mu}$$

$$= x^{1-\mu} \int_0^\infty \left\{ (\xi^4 + x^4)^{-1/2} (\xi^2 + \xi^4 + x^4)^{\frac{1}{2}\mu} - 2^{u/2} \xi^{\mu-2} \right\} (e^{-\xi} - 1)\, d\xi.$$

Letting $\xi = rx$, the right-hand side of this equation transforms to

$$= x \int_0^\infty \phi_1(r) r \frac{e^{-rx} - 1}{rx}\, dr \tag{5a.4}$$

where

$$\phi_1(r) = (r^2 + \sqrt{r^4+1})^{½\mu} (r^4+1)^{-½} - 2^{\mu/2} r^{\mu-2}. \qquad (5a.5)$$

Because of the inequality $1 \geqq (1 - e^{-u})/u \geqq 1 - u/2$, the integrand of (5a.4) is uniformly bounded. We can, therefore, let $x \to 0$ inside the integral. Thus

$$g_1^{(2)} = -\int_0^\infty \phi_1(r)\, r\, dr. \qquad (5a.6)$$

To find $g_2^{(2)}$, we substract the product of x and (5a.6) from (5a.4)

$$g_2(x) - g_0^{(2)} - g_\mu^{(2)} x^{1-\mu} - g_1^{(2)} x = x^2 \int_1^\infty \phi_1(r)\, r^2 \frac{e^{-xr} - 1 + rx}{r^2 x^2}\, dr.$$

Because of the inequality $0 \geqq (1 - u - e^{-u})/u^2 \geqq -½$, the integrand of this integral is uniformly bounded. We can, therefore, let $x \to 0$ inside the integral. Thus

$$g_2^{(2)} = \frac{1}{2} \int_0^\infty \phi_1(r)\, r^2\, dr. \qquad (5a.7)$$

To find $g_1^{(1)}$, we simply differentiate $g_1(x)$ in (5.5) with regard to x and let $x = 0$ in the result. Thus we find

$$g_1^{(1)} = -\int_0^\infty (r^2 + \sqrt{r^4+1})^{-\mu/2} [r/(r^4+1)^{-½}]\, dr. \qquad (5a.8)$$

To find $g_\mu^{(1)}$, we use (5a.1) and (5a.8) to derive the formula

$$g_1(x) - g_0^{(1)} - g_1^{(1)} x = \int_0^\infty (r^2 + \sqrt{r^4+1})^{-\mu/2} (e^{-rx} - 1 + rx)(r^4+1)^{-½}\, dr.$$

Letting $\xi = rx$, the right-hand side of this equation becomes

$$= x^{1+\mu} \int_0^\infty (\xi^4 + x^4)^{-½} (\xi^2 + \sqrt{\xi^4 + x^4})^{-\mu/2} (e^{-\xi} - 1 + \xi)\, d\xi. \qquad (5a.9)$$

Letting $x = 0$ inside the integral, we find

$$g_\mu^{(1)} = 2^{-\mu/2} \int_0^\infty \xi^{-\mu-2} (e^{-\xi} - 1 + \xi)\, d\xi. \qquad (5a.10)$$

To find $g_2^{(1)}$, we use (5a.9) and (5a.10) to derive the formula

$g_1(x) - g_0^{(1)} - g_1^{(1)} x - g_\mu^{(1)} x^{1+\mu}$

$$= x^{1+\mu} \int_0^\infty \left\{ (\xi^4 + x^4)^{-1/2} \left(\xi^2 + \sqrt{\xi^4 + x^4}\right)^{-\mu/2} - 2^{-\mu/2} \xi^{-\mu-2} \right\} (e^{-\xi} - 1 + \xi)\, d\xi.$$

Letting $\xi = rx$, this transforms to

$$= x^2 \int_0^\infty \phi_2(r)\, r^2 \frac{e^{-rx} - 1 + rx}{r^2 x^2}\, dr \tag{5a.11}$$

where

$$\phi_2(r) = (r^4 + 1)^{-1/2} \left(r^2 + \sqrt{r^4 + 1}\right)^{-\mu/2} - 2^{-\mu/2} r^{-\mu-2}.$$

Because the integrand of (5a.11) is uniformly bounded, we can let $x \to 0$ inside the integral. Thus we find

$$g_2^{(1)} = \frac{1}{2} \int_0^\infty \phi_2(r)\, r^2\, dr. \tag{5a.12}$$

5b. Evaluation of $g_n^{(k)}$

The independent variable η defined by

$$r^2 + \sqrt{r^4 + 1} = \eta^{-1/2} \tag{5b.1}$$

is useful for the following integrations. This equation yields

$$r^2 = (1 - \eta)/(2\eta^{1/2})$$

$$\sqrt{1 + r^4} = (1 + \eta)/(2\eta^{1/2})$$

$$r\, dr = -\left[(1 + \eta)/(4\eta^{3/2})\right] d\eta.$$

Substituting these in (5a.1), we find

$$g_0^{(k)} = (2\sqrt{2})^{-1} B\left(\frac{1}{2}, \frac{1 \pm \mu}{4}\right) = (2\sqrt{2\pi})^{-1} \Gamma\left(\frac{1+\mu}{4}\right) \Gamma\left(\frac{1-\mu}{4}\right) \sin \frac{\pi(1 \mp \mu)}{4}. \tag{5b.2}$$

Use of η enables us to integrate (5a.8) simply to

$$g_1^{(1)} = -\mu^{-1}. \tag{5b.3}$$

Use of η transforms (5a.6) to

$$g_1^{(2)} = -\frac{1}{4} \int_0^1 \left\{ 1 - (1 + \eta)(1 - \eta)^{(\mu/2)-1} \right\} \eta^{-(\mu/4)-1}\, d\eta.$$

By letting $1+\eta = 2-(1-\eta)$, the last integral divides into two parts,

$$= -\frac{1}{4} J_1 - \frac{1}{2} J_2$$

where

$$J_1 = \int_0^1 [1-(1-\eta)^{\mu/2}]\, \eta^{-(\mu/4)-1}\, d\eta$$

which integrates to

$$J_1 = -4/\mu + 2B(\mu/2, 1-\mu/4)$$

by the partial integration of $\eta^{-(4/\mu)-1}$. The remaining integral J_2

$$J_2 = \int_0^1 [(1-\eta)^{\mu/2} - (1-\eta)^{(\mu/2)-1}]\, \eta^{-(\mu/4)-1}\, d\eta$$

integrates to

$$J_2 = -B(\mu/2, 1-\mu/4).$$

Thus we find

$$g_1^{(2)} = \mu^{-1}. \tag{5b.4}$$

We show the results of (5b.3) and (5b.4) in a single formula

$$g_1^{(k)} = \mp \mu^{-1}. \tag{5b.5}$$

Applying the partial integration of $\zeta^{\mu-2}$, (5a.3) integrates; applying the two times of partial integration of $\zeta^{-\mu-2}$, (5a.10) integrates. We show the results in a single formula

$$g_\mu^{(k)} = \pm 2^{\mp\mu/2} [\mu(1\pm\mu)]^{-1}\, \Gamma(1\mp\mu). \tag{5b.6}$$

We express $g_2^{(2)}$ in (5a.7) and $g_2^{(1)}$ in (5a.12) in a single formula

$$g_2^{(k)} = \frac{1}{2} \int_0^\infty \left\{ (r^4+1)^{-1/2} (r^2+\sqrt{r^4+1})^{\mp\mu/2} - 2^{\mp\mu/2} r^{-2\mp\mu} \right\} r^2\, dr.$$

The use of η transforms this to

$$g_2^{(k)} = \frac{1}{8\sqrt{2}} \int_0^1 \left\{ 1-(1+\eta)(1-\eta)^{-1\mp(\mu/2)} \right\} \eta^{-\frac{5}{4}\pm\frac{\mu}{4}} (1-\eta)^{1/2}\, d\eta.$$

Letting $1+\eta = 2-(1-\eta)$, this divides into two integrals

$$g_2^{(k)} = (8\sqrt{2})^{-1} K_1 + (4\sqrt{2})^{-1} K_2,$$

where

$$K_1 = \int_0^1 \left\{ 1 - [(1-\eta)^{\mp\mu/2}] \right\} \eta^{-\frac{5}{4}\mp\frac{\mu}{4}} (1-\eta)^{1/2}\, d\eta$$

which integrates to

$$K_1 = \frac{2}{-1 \pm \mu} \left\{ B\left(\frac{1}{2}, \frac{3 \pm \mu}{4}\right) - (1 \pm \mu)\, B\left(\frac{1 \pm \mu}{2}, \frac{3 \pm \mu}{4}\right) \right\}$$

by the partial integration of $\eta^{-\frac{5}{4}\pm\frac{\mu}{4}}$. The remaining integral K_2

$$K_2 = \int_0^1 [(1-\xi)^{\mp\mu/2} - (1-\xi)^{\mp(\mu/2)-1}]\, \xi^{\pm(\mu-5)/4} (1-\xi)^{1/2}\, d\xi$$

integrates to

$$K_2 = -B\left(\frac{1 \mp \mu}{2}, \frac{3 \pm \mu}{4}\right).$$

Thus we find

$$g_2^{(k)} = [4\sqrt{2}\,(-1 \pm \mu)]^{-1}\, B\left(\frac{1}{2}, \frac{3 \pm \mu}{4}\right)$$

$$= -[\sqrt{2\pi}\,(1-\mu^2)]^{-1}\, \Gamma\left(\frac{3+\mu}{4}\right) \Gamma\left(\frac{3-\mu}{4}\right) \sin\frac{\pi(1 \pm \mu)}{4}. \qquad (5b.7)$$

5c. Integration of $h_n^{(k)}$

Let

$$r^2 + i\sqrt{1-r^4} = \zeta. \qquad (5c.1)$$

Then

$$r^2 = (1+\zeta^2)/(2\zeta)$$

$$\sqrt{1-r^4} = i(1-\zeta)^2/(2\zeta)$$

$$r\,dr = [(\zeta^2-1)/(4\zeta^2)]\, d\zeta.$$

Substituting these in (5.7), we get

$$h_n^{(k)} = i2^{-(n+1)/2} \int_i^1 \zeta^{-\frac{n+1}{2}\mp\frac{\mu}{2}} (1+\zeta^2)^{(n-1)/2}\, d\zeta. \qquad (5c.2)$$

For $n = 1$, this integrates to

$$h_1^{(k)} = \mp (i/\mu)\,[1 - \exp(\mp \mu\pi i/4)]\,. \tag{5c.3}$$

For $\eta = 0$ and $\eta = 2$, we divide the contour of (5c.2) into two parts

$$h_n^{(k)} = i2^{-(n+1)/2}\left(I_n^{(k)} - J_n^{(k)}\right)$$

where

$$I_n^{(k)} = \int_0^1 \zeta^{\mp\frac{\mu}{2} - \frac{n+1}{2}}\,(1 + \zeta^2)^{(n-1)/2}\,d\zeta$$

$$J_n^{(k)} = \int_0^i \zeta^{\mp\frac{\mu}{2} - \frac{n+1}{2}}\,(1 + \zeta^2)^{(n-1)/2}\,d\zeta\,.$$

Letting $\zeta = i\sqrt{x}$, the latter integrates to

$$J_n^{(k)} = \frac{1}{2}\exp\left[\frac{\pi i}{4}(\mp\pi + 1 - n)\right]\Gamma\left(\frac{\mp\mu + 1 - n}{4}\right)\Gamma\left(\frac{n+1}{4}\right)\Gamma\left(\frac{\mp\mu + 3 + n}{4}\right).$$

Transforming the Gamma-functions, we get

$$J_0^{(k)} = (2\sqrt{\pi})^{-1}\,\Gamma\left(\frac{1+\mu}{4}\right)\Gamma\left(\frac{1-\mu}{4}\right)\exp\left[\frac{\pi i(1 \mp \mu)}{4}\right]\sin\frac{\pi(1 \pm \mu)}{4}$$

$$J_2^{(k)} = -4[(1-\mu^2)\sqrt{\pi}\,]^{-1}\,\Gamma\left(\frac{3+\mu}{4}\right)\Gamma\left(\frac{3-\mu}{4}\right)\exp\left[\frac{\pi i(-1 \mp \mu)}{4}\right]\sin\frac{\pi(1 \mp \mu)}{4}\,.$$

Thus we find

$$h_0^{(1)} + h_0^{(2)} = i2^{-1/2}\left(I_0^{(1)} + I_0^{(2)}\right) + (2\sqrt{2\pi}\,)^{-1}\,\Gamma\left(\frac{1+\mu}{4}\right)\Gamma\left(\frac{1-\mu}{4}\right)\left(\cos\frac{\mu\pi}{2} - i\right) \tag{5c.4}$$

$$h_2^{(1)} + h_2^{(2)} = i2^{-3/2}\left(I_2^{(1)} + I_2^{(2)}\right) + \sqrt{2}\,[\sqrt{\pi}(1-\mu^2)]^{-1}\,\Gamma\left(\frac{3+\mu}{4}\right)\Gamma\left(\frac{3-\mu}{4}\right)\left(\cos\frac{\mu\pi}{2} + i\right). \tag{5c.5}$$

In fact, $h_n^{(1)} + h_n^{(2)}$ is real. To show this, let $r^2 = \cos\theta$ in (5.7) and get

$$h_n^{(1)} + h_n^{(2)} = \int_2^{\pi/2} (\cos\theta)^{(n-1)/2}\cos\mu\theta\,d\theta$$

which is real. Thus by taking the real parts of (5c.4) and (5c.5), we find

$$h_0^{(1)} + h_0^{(2)} = (2\sqrt{2\pi}\,)^{-1}\,\Gamma\left(\frac{1+\mu}{4}\right)\Gamma\left(\frac{1-\mu}{4}\right)\cos\frac{\mu\pi}{2} \tag{5c.6}$$

and

$$h_2^{(1)} + h_2^{(2)} = (1-\mu^2)^{-1}\sqrt{\frac{2}{\pi}}\ \Gamma\left(\frac{3+\mu}{4}\right)\Gamma\left(\frac{3-\mu}{4}\right)\cos\frac{\mu\pi}{2} \tag{5c.7}$$

because $l_0^{(k)}$ and $l_2^{(k)}$ are real.

5d. Complex fundamental solution

We get

$$B_0 = \frac{1}{2\sqrt{2\pi}}\ \Gamma\left(\frac{1+\mu}{4}\right)\ \Gamma\left(\frac{1-\mu}{4}\right) \tag{5d.1}$$

by substituting (5b.2) and (5c.4) into (5.11). We get

$$B_1 = 0 \tag{5d.2}$$

by substituting (5b.5) and (5c.3) into (5.12). We get

$$B_2 = \frac{i}{\sqrt{2\pi a}}\ \Gamma\left(\frac{3+\mu}{4}\right)\Gamma\left(\frac{3-\mu}{4}\right) \tag{5d.3}$$

by substituting (5b.7) and (5c.5) into (5.13). We get

$$B_\mu^{(k)} = \mp\frac{\Gamma(2\mp\mu)}{\mu a}\ 2^{\mp\mu/2}\exp\frac{(3\mp\mu)\pi i}{4} \tag{5d.4}$$

by substituting (5b.6) into (5.14). Thus we find the complex fundamental solution

$$F(x) = B_0 f_0(x) + B_2 f_1(x) + B_\mu^{(1)} f_2(x) + B_\mu^{(2)} f_3(x). \tag{5d.5}$$

5e. Fundamental solutions for $0 < a < 1$

When $0 < a < 1$, functions $f_m(x)$ ($m = 0, 1, 2, 3$) are real. Therefore, fundamental solutions $w_1(x)$ and $w_2(x)$ are found by separating the coefficients of (5d.5) into real and imaginary parts. Thus

$$w_1(x) = p_0 f_0(x) + p_1 f_2(x) + p_2 f_3(x) \tag{5e.1}$$

$$w_2(x) = q_0 f_2(x) + q_1 f_2(x) + q_2 f_3(x) \tag{5e.2}$$

where

$$p_0 = \frac{1}{2\sqrt{2\pi}}\ \Gamma\left(\frac{1+\mu}{4}\right)\ \Gamma\left(\frac{1-\mu}{4}\right) \tag{5e.3}$$

$$q_0 = \frac{1}{\sqrt{2\pi a}}\ \Gamma\left(\frac{3+\mu}{4}\right)\ \Gamma\left(\frac{3-\mu}{4}\right) \tag{5e.4}$$

$$p_k = \mp\frac{\Gamma(2\mp\mu)}{\mu a}\ 2^{\mp\mu/2}\cos\frac{(3\mp\mu)\pi}{4} \tag{5e.5}$$

$$q_k = \mp\frac{\Gamma(2\mp\mu)}{\mu a}\ 2^{\mp\mu/2}\sin\frac{(3\mp\mu)}{4}\ . \tag{5e.6}$$

The convention with regard to k is observed in (5e.5) and (5e.6).

6. Fundamental solutions for $a = 1$

When $a = 1$, (1.5) reduces to

$$\frac{d^4 w}{dx^4} + \frac{2}{x}\frac{d^3 w}{dx^3} + w = 0. \tag{6.1}$$

Nevel[6] gave the Fuchsian type solutions of this equation with the notations

$$\mathrm{nev}_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{3n}(2n)!}\left[\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(n+\frac{3}{4}\right)}\right]^2 x^{4n}$$

$$\mathrm{nev}_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{4n}(n!)^2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(n+\frac{3}{4}\right)\Gamma\left(n+\frac{5}{4}\right)}\, x^{4n+1}$$

$$\mathrm{nev}_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{3n}(2n+1)!}\left[\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(n+\frac{5}{4}\right)}\right]^2 x^{4n+2}$$

$$\mathrm{nel}_1(x) = \mathrm{nev}_1(x)\log x -$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{4^{4n}(n!)^2}\,\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(n+\frac{3}{4}\right)\Gamma\left(n+\frac{5}{4}\right)}\sum_{k=1}^{n}\left(\frac{1}{4k-1}+\frac{1}{2k}+\frac{1}{4k+1}\right)x^{4n+1}.$$

To find the fundamental solutions, the unknown function $w(x)$ must be transformed to the unknown function $v(\zeta)$ defined by the contour integral (4.1). The transformed differential equation is

$$(1+\zeta^4)\frac{dv}{d\zeta} + 2\zeta^3 v = 0$$

and we find

$$v(\zeta) = (1+\zeta^4)^{-1/2}$$

Therefore the complex solution $w(x)$ for $a = 1$ is given in the integral form

$$w(x) = \int_L \frac{e^{\zeta x}}{\sqrt{1+\zeta^4}}\, d\zeta. \tag{6.2}$$

The contour L is the one shown in Figure 1. To find the fundamental solutions in the form of the linear combination of the nev functions, J. Dieudonné,[1] as explained in Nevel,[7] expanded (6.2) into power series in the neighborhood of $x = 0$, and determined the first few coefficients. The fundamental solutions thus found are denoted here by w_1^N and w_2^N

$$w_1^N(x) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{2\pi}}\,\mathrm{nev}_0(x) - \frac{\pi}{2\sqrt{2}}\,\mathrm{nev}_1(x) + \frac{\Gamma^2\left(\frac{3}{4}\right)}{2\sqrt{2\pi}}\,\mathrm{nev}_2(x) \tag{6.3}$$

$$w_2^N(x) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}} \, \text{nev}_0(x) - \frac{\Gamma^2\left(\frac{3}{4}\right)}{4\sqrt{\pi}} \, \text{nev}_2(s) + \text{nev}_1(x) - (1 - \gamma + \log\sqrt{2}) \, \text{nel}_1(x) \tag{6.4}$$

where γ is Euler's constant 0.5772156.

We shall show in the following that $w_1(x)$ and $w_2(x)$ in (5e.1) and (5e.2), respectively, give

$$\lim_{a \to 1} w_1(x) = w_1^N(x) + \sqrt{2} \, w_2^N(x)$$

$$\lim_{a \to 1} w_2(x) = w_1^N(x) - \sqrt{2} \, w_2^N(x) .$$

To show this, note that

$$\lim_{a \to 1} p_0 f_0(x) = (2\sqrt{2\pi})^{-1} \, \Gamma^2\left(\frac{1}{4}\right) \text{nev}_0(x)$$

$$\lim_{a \to 1} q_0 f_2(x) = (2\sqrt{2\pi})^{-1} \, \Gamma^2\left(\frac{3}{4}\right) \text{nev}_2(x)$$

$$\lim_{a \to 1} f_{k+1}(x) = \text{nev}_1(x).$$

We shall prove, therefore, that

$$\lim_{a \to 1} [p_1 f_2(x) + p_2 f_3(x)] = -\sqrt{2}\left(1 - \gamma + \log\sqrt{2} + \frac{\pi}{4}\right) \text{nev}_1(x) + \sqrt{2} \, \text{nel}_1(x) \tag{6.5}$$

$$\lim_{a \to 1} [q_1 f_2(x) + q_2 f_3(x)] = \sqrt{2}\left(1 - \gamma + \log\sqrt{2} - \frac{\pi}{4}\right) \text{nev}_1(x) - \sqrt{2} \, \text{nel}_1(x). \tag{6.6}$$

The formula

$$p_1 f_2(x) + p_2 f_3(x)$$

in the left-hand side of (6.5) transforms to

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{4n} \, n!} \, x^{4n+1} \lim_{\mu \to 0} \frac{1}{\mu} \left[- C_1(n) + C_2(n) \right]$$

by use of (5e.5) and (3.6), where

$$\underset{k}{C(n)} = 2^{\mp\mu/2} \, \Gamma(2 \mp \mu) \, \frac{\Gamma\left[\frac{1}{2}(2 \pm \mu)\right] \Gamma\left[\frac{1}{4}(3 \pm \mu)\right] \Gamma\left[\frac{1}{4}(5 \pm \mu)\right]}{\Gamma\left[n + \frac{1}{2}(2 \pm \mu)\right] \Gamma\left[n + \frac{1}{4}(3 \pm \mu)\right] \Gamma\left[n + \frac{1}{4}(5 \pm \mu)\right]} \, x^{\pm\mu} \cos \frac{(3 \mp \mu)\pi}{4} .$$

Taking the limit, we find that

$$\lim_{\mu \to 0} \frac{1}{\mu} [-C_1(n) + C_2(n)] =$$

$$-\sqrt{2}\left[n!\left(\frac{5}{4}\right)_n\left(\frac{3}{4}\right)_n\right]^{-1}\left\{1-\gamma-\log\frac{x}{\sqrt{2}}+\frac{\pi}{4}+\sum_{p=1}^{n}\left(\frac{1}{4p+1}+\frac{1}{2p}+\frac{1}{4p-1}\right)\right\}$$

where we have introduced the convention that the summation $\sum_{p=1}^{n}$ disappears when $n = 0$.
$(Z)_n = Z(Z+1)...(Z+n-1)$ is Pochhammer's symbol (ref. 2, p. 246). This equation proves (6.5).

The formula

$$q_1 f_2(x) + q_2 f_3(x)$$

in the left-hand side of (6.6) transforms to

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{4n} n!} x^{4n+1} \lim_{\mu \to 0} \frac{1}{\mu} [-S_1(n) + S_2(n)]$$

by use of (5e.6) and (3.6), where $S_k(n)$ is given by replacing $\cos[(3 \mp \mu)\pi/4]$ in $C_k(n)$ with $\sin[(3 \mp \mu)\pi/4]$. Taking the limit, we find that

$$\lim_{\mu \to 0} \frac{1}{\mu} [-S_1(n) + S_2(n)] =$$

$$\sqrt{2}\left[n!\left(\frac{5}{4}\right)_n\left(\frac{3}{4}\right)_n\right]^{-1}\left\{1-\gamma-\log\frac{x}{\sqrt{2}}-\frac{\pi}{4}+\sum_{p=1}^{n}\left(\frac{1}{4p+1}+\frac{1}{2p}+\frac{1}{4p-1}\right)\right\}$$

where, by convention, the summation $\sum_{p=1}^{n}$ disappears when $n = 0$. This proves (6.6).

7. Fundamental solutions for $a = 0$

When $a = 0$, (1.5) reduces to

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx}\right)^2 w + w = 0$$

as may be derived by putting $N_{rr} = 0$ and $N_{\theta\theta} = 0$ in (1.2). We decompose this equation into the following two equations

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \mp i\right) w = 0, \tag{7.1}$$

shown with the double sign. The solutions of the above two equations, satisfying the boundary conditions (1.6) at $x = \infty$, are

$$w = \operatorname{ker} x \pm i \operatorname{kei} x \tag{7.2}$$

giving the fundamental solutions $\operatorname{ker} x$ and $\operatorname{kei} x$. Thus, for $a = 0$, we have

$$w(x) = A \ker x + B \operatorname{kei} x . \tag{7.3}$$

We shall prove that $w_1(x)$ in (5e.1) and $w_2(x)$ in (5e.2) satisfy

$$\lim_{a \to 0} w_1(x) = \sqrt{2} \ker x \tag{7.4}$$

$$\lim_{a \to 0} w_2(x) = \sqrt{2} \operatorname{kei} x . \tag{7.5}$$

First we note that

$$\lim_{\mu \to 1} f_0(x) = \lim_{\mu \to 1} f_3(x) = \operatorname{ber} x$$

$$\lim_{\mu \to 1} f_1(x) = \lim_{\mu \to 1} f_2(x) = 4 \operatorname{bei} x .$$

Denoting

$$1 - \mu = \lambda$$

we rewrite (5e.1) to

$$w_1 = + \frac{1}{\mu(1+\mu)} 2^{-\mu/2} \Gamma(2-\mu) \cdot \frac{1}{\lambda} \sin \frac{\lambda\pi}{4} \cdot f_2 +$$

$$+ \frac{1}{\lambda} \left\{ \frac{2}{\pi} \Gamma\left(\frac{2-\lambda}{4}\right) \Gamma\left(\frac{4+\lambda}{4}\right) \cdot f_0 + \frac{1}{(1-\lambda)(2-\lambda)} 2^{(1-\lambda)/2} \Gamma(3-\lambda) \cos \frac{\lambda\pi}{4} \cdot f_3 \right\} .$$

Then, letting $\lambda \to 0$

$$\lim_{\lambda \to 0} w_1 = \frac{\pi}{2\sqrt{2}} \operatorname{bei} x + \sqrt{2} (\log 2 - \alpha) \operatorname{ber} x + \sqrt{2} \lim_{\lambda \to 0} \left(\frac{\partial f_0}{\partial \lambda} - \frac{\partial f_3}{\partial \lambda} \right) .$$

By use of f_0 in (3.4) and f_3 in (3.6), we find that

$$\lim_{\lambda \to 0} \left(\frac{\partial f_0}{\partial \lambda} - \frac{\partial f_3}{\partial \lambda} \right) = - \log x \operatorname{ber} x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{4^{2n} (2n)!^2} \sum_{p=1}^{2n} \frac{1}{p}$$

which proves (7.4).

We rewrite (5e.2) to

$$w_2 = \frac{1}{\mu(1+\mu)} 2^{\mu/2} \Gamma(2+\mu) \cdot \frac{1}{\lambda} \sin \frac{\lambda\pi}{4} \cdot f_3 +$$

$$+ \frac{1}{\lambda} \left\{ \frac{1}{\sqrt{2\pi}(2-\lambda)} \Gamma\left(\frac{2+\lambda}{4}\right) \Gamma\left(\frac{4-\lambda}{4}\right) \cdot f_1 - \frac{1}{(1-\lambda)(2-\lambda)} 2^{-\frac{1-\lambda}{2}} \Gamma(1+\lambda) \cos \frac{\lambda\pi}{4} \cdot f_2 \right\}$$

Letting $\lambda \to 0$

$$\lim_{\lambda \to 0} w_2 = \frac{\pi}{2\sqrt{2}} \operatorname{ber} x - \sqrt{2}\,(1 - \gamma + \log\sqrt{2}) \operatorname{bei} x + \frac{1}{2\sqrt{2}} \lim_{\lambda \to 0} \left(\frac{\partial f_1}{\partial \lambda} - \frac{\partial f_2}{\partial \lambda} \right).$$

By use of f_1 in (3.5) and f_2 in (3.6), we find that

$$\lim_{\lambda \to 0} \left(\frac{\partial f_1}{\partial \lambda} - \frac{\partial f_2}{\partial \lambda} \right) = 4 \log x \operatorname{bei} x - 4 \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n+2}}{4^{2n+2}\,[(2n+1)!]^2} \sum_{p=2}^{2n+1} \frac{1}{p}$$

which proves (7.5).

8. Eigenvalues for $a = 0$

When $a = 0$, no horizontal pressure works on the plate, and buckling should not take place under any boundary conditions. We shall prove below that this is true under the boundary conditions (1.7), (1.8), and (1.9).

The following formulas are needed for the proof. Substituting (7.2) into (7.1), we find the relations

$$\operatorname{ker}'' x + x^{-1} \operatorname{ker}' x + \operatorname{kei} x = 0 \tag{8.1}$$

and

$$\operatorname{kei}'' x + x^{-1} \operatorname{kei}' x - \operatorname{ker} x = 0. \tag{8.2}$$

We shall prove that *no positive number* x_0 *can satisfy the clamped-edge condition* (1.7). The determinant of (1.7) is given by

$$D_1 = \begin{vmatrix} \operatorname{ker} x & \operatorname{ker}' x \\ \operatorname{kei} x & \operatorname{kei}' x \end{vmatrix}$$

when (7.3) is used. Differentiating D_1, we find the differential equation

$$\frac{dD_1}{dx} + x^{-1} D_1 = \operatorname{ker}^2 x + \operatorname{kei}^2 x.$$

Solving this under the boundary condition that $D_1 = 0$ at $x = \infty$, we find

$$D_1 = -\frac{1}{x} \int_x^{\infty} \xi(\operatorname{ker}^2 \xi + \operatorname{ker}^2 \xi)\, d\xi$$

which is negative for any positive x, proving our contention.

We shall prove that *no positive number* x_0 *can satisfy the simple-edge condition* (1.8). The determinant of (1.8) transforms to

$$D_2 = \begin{vmatrix} \operatorname{ker} x & (1-\nu)x^{-1} \operatorname{ker}' x + \operatorname{kei} x \\ \operatorname{kei} x & (1-\nu)x^{-1} \operatorname{kei}' x - \operatorname{ker} x \end{vmatrix}.$$

This becomes

$$D_2 = -\frac{1-\nu}{x}\int_x^\infty \xi(\mathrm{ker}^2\,\xi + \mathrm{kei}^2\,\xi)\,d\xi - (\mathrm{ker}^2 x + \mathrm{kei}^2 x)$$

which is negative for any positive x, proving our contention.

Finally we shall prove that *no positive number x_0 can satisfy the free-edge condition* (1.9). The determinant of (1.9) transforms to

$$D_3 = \begin{vmatrix} (1-\nu)x^{-1}\,\mathrm{ker}'x + \mathrm{kei}\,x & -\mathrm{kei}'x \\ (1-\nu)x^{-1}\,\mathrm{kei}'x - \mathrm{ker}\,x & \mathrm{ker}'x \end{vmatrix} .$$

This equation becomes

$$D_3 = \frac{1-\nu}{x}(\mathrm{ker}'^2 x + \mathrm{kei}'^2 x) + \frac{1}{x}\int_x^\infty \xi(\mathrm{ker}^2\,\xi + \mathrm{kei}^2\,\xi)\,d\xi$$

which is positive for any positive x, proving our contention.

9. Fundamental solutions for $a > 1$

For $a > 1$, μ defined in (3.1) becomes imaginary and must be replaced with $\mu = i\kappa$, where

$$\kappa = \sqrt{a-1}\ . \tag{9.1}$$

To compute $\Gamma(x+iy)$ for small y, we used the formulas

$$|\Gamma(x+iy)/\Gamma(x)|^2 = \prod_{n=0}^{\infty} [1 + y^2/(x+n)^2]^{-1} \tag{9.2}$$

and

$$\mathrm{Arg}\,\Gamma(x+iy) = y\psi(x) + \sum_{n=0}^{\infty}\Big[y/(x+n) - \tan^{-1}[y/(x+n)]\Big] \tag{9.3}$$

(ref. 2, p. 256). These formulas can be proved by use of Euler's formula for the Gamma function (ref. 8, p. 237).

Using these formulas for small κ, coefficients of $F(x)$ in (5d.5) become

$$B_0 = \frac{1}{2}(2\pi)^{-1/2}\,\Gamma^2\left(\frac{1}{4}\right)\prod_{p=0}^{\infty}[1 + \kappa^2(4p+1)^{-2}]^{-1} \tag{9.4}$$

$$B_2 = \frac{i}{a}(2\pi)^{-1/2}\,\Gamma^2\left(\frac{3}{4}\right)\prod_{p=0}^{\infty}[1 + \kappa^2(4p+3)^{-2}]^{-1} \tag{9.5}$$

$$B_\mu^{(k)} = \mp(a\kappa)^{-1} R\exp\left(\pm\frac{\kappa\pi}{4} + \frac{\pi i}{4} \mp i\kappa\log\sqrt{2} \mp i\Theta\right) \tag{9.6}$$

where R and Θ are defined by

$$\Gamma(2 \mp i\kappa) = R \exp(\mp i\Theta).$$

They are given by

$$R = [\pi\kappa a/\sinh(\pi\kappa)]^{1/2}$$

$$\Theta = \kappa(1-\gamma) + \sum_{n=0}^{\infty} [\kappa/n - \tan^{-1}(\kappa/n)] . \qquad (9.7)$$

There is a relation

$$B_0 B_2 = i\pi[2a \cosh(\kappa\pi/2)]^{-1}$$

Equation (9.4) for B_0 and (9.5) for B_2 were used for $\kappa \leqq 11$ in our computation. Equation (9.7) for Θ was used for $\kappa \leqq 5$. Beyond these respective ranges, the asymptotic expansion of the Gamma-function was used to re-evaluate B_0, B_2 and Θ.

Functions $f_0(x)$ and $f_1(x)$ are real. To decompose the complex function $f_{k+1}(x)$ into the real and imaginary parts, we must transform the denominator of (3.3) to

$$(4n + 1 \pm \mu)(4n - 1 \pm \mu)(4n)(4n \pm 2\mu)$$

which reduces to

$$= 8n[2n(16n^2 + 4 - 5a) \pm i\kappa(32n^2 - a)] .$$

Thus we find

$$f_{k+1}(x) = \sum_{n=0}^{\infty} (-1)^n P_n^{(0)} \exp(\mp i\rho_n)\, x^{4n+1\pm i\kappa}$$

where

$$P_0^{(0)} = 1$$

$$\rho_0 = 0$$

$$P_n^{(0)} = 8^{-2n}(n!)^{-1} \left\{ \prod_{p=1}^{n} \left[p^2\left(4p^2 + 1 - \frac{5}{4}a\right)^2 + (a-1)\left(4p^2 - \frac{a}{8}\right)^2 \right] \right\}^{-1/2}$$

$$\rho_n = \sum_{p=1}^{n} \zeta_p$$

for $n \geqq 1$, in which

$$\tan\zeta_p = \kappa(4p^2 - 8)/\left[p\left(4p^2 + 1 - \frac{5a}{4}\right)\right]$$

where

$$-\pi < \zeta_p \leqq \pi .$$

Fundamental solutions $w_1(x)$, $w_2(x)$, and their derivatives are the real and imaginary parts of $f(x)$ and its derivatives, respectively. We formulated them up to the third derivatives. The following formulas apply when $m = 0, 1, 2,$ and 3:

$$\frac{d^m w_1}{dx^m} = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{2\pi}} \prod_{p=0}^{\infty} \left(1 + \frac{a-1}{(4p+1)^2}\right)^{-1} \frac{d^m f_0}{dx^m} -$$

$$- \frac{R\exp(\kappa\pi/4)}{a\kappa} \sum_{n=0}^{\infty} (-1)^n P_n^{(m)} x^{4n+1-m} \cos(\theta_1 - \rho_n + \phi_n^{(m)}) +$$

$$+ \frac{R\exp(-\kappa\pi/4)}{a\kappa} \sum_{n=0}^{\infty} (-1)^n P_n^{(m)} x^{4n+1-m} \cos(\theta_2 + \rho_n - \phi_n^{(m)}) \tag{9.8}$$

$$\frac{d^m w_2}{dx^m} = \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{2\pi a}} \sum_{p=0}^{\infty} \left(1 + \frac{a-1}{(4p+3)^2}\right)^{-1} \frac{d^m f_1}{dx^m} -$$

$$- \frac{R\exp(\kappa\pi/4)}{a\kappa} \sum_{n=0}^{\infty} (-1)^n P_n^{(m)} x^{4n+1-m} \sin(\theta_1 - \rho_n + \phi_n^{(m)})$$

$$+ \frac{R\exp(-\kappa\pi/4)}{a\kappa} \sum_{n=0}^{\infty} (-1)^n P_n^{(m)} x^{4n+1-m} \sin(\theta_2 + \rho_n - \phi_n^{(m)}), \tag{9.9}$$

where

$$\theta_k = \frac{\pi}{4} \pm \kappa \log \frac{x}{\sqrt{2}} \mp (1-\gamma)\kappa \mp \sum_{p=0}^{\infty} \left(\frac{\kappa}{p+2} - \tan^{-1} \frac{\kappa}{p+2}\right)$$

$$P_n^{(1)} = P_n^{(0)} \left[(4n+1)^2 + \kappa^2\right]^{1/2}$$

$$P_n^{(2)} = P_n^{(1)} \left[(4n)^2 + \kappa^2\right]^{1/2}$$

$$P_n^{(3)} = P_n^{(2)} \left[(4n-1)^2 + \kappa^2\right]^{1/2}$$

$$\phi_n^{(0)} = 0$$

$$\phi_n^{(1)} = \tan^{-1} \left[\kappa/(4n+1)\right]$$

$$\left.\begin{aligned} \phi_0^{(2)} &= \phi_0^{(1)} + \frac{\pi}{2} \\ \phi_n^{(2)} &= \phi_n^{(1)} + \tan^{-1}\left[\kappa/(4n)\right] \end{aligned}\right\} \quad \text{for } n \geq 1$$

$$\left.\begin{aligned}\phi_0^{(3)} &= \phi_0^{(2)} + \pi - \tan^{-1}\kappa\\ \phi_n^{(3)} &= \phi_n^{(2)} + \tan^{-1}[\kappa/(4n-1)].\end{aligned}\right\} \quad \text{for } n \geq 1.$$

In our numerical computation these formulas were effective only in the range $0 \leqq x \leqq 10$.

PART II. ASYMPTOTIC EXPANSIONS

Values of a series solution developed in Part I must overlap on a certain range of x with the values of an asymptotic expansion determined corresponding to the respective series solution. The series must be used for any x larger than the overlapping range.

10. Asymptotic expansion for $0 < a < 1$

We can define the analytical contin ..ion of $\nu_k(r)$ in (4.8) by the analytical continuation of the hypergeometric function $F(\ ,\ ;\ ;\ )$. Thus we can transform $\nu_k(r)$ originally defined in the range $1 < r < \infty$ to the one defined in the neighborhood of $r = 1$

$$\nu_k(r) = \nu_1^{(k)} F\left(\frac{1}{4}(2+\mu), \frac{1}{4}(2-\mu); \frac{3}{2}; 1-r^4\right) +$$

$$+ \nu_2^{(k)} r^2 (r^4-1)^{-1/2} F\left(\frac{1}{4}(2+\mu), \frac{1}{4}(2-\mu); \frac{1}{2}; 1-r^4\right), \tag{10.1}$$

where

$$\nu_1^{(k)} = -2\sqrt{\pi}\ \Gamma\left(\frac{1}{2}(2\pm\mu)\right)\left\{\Gamma\left(\frac{1}{4}(2\pm\mu)\right)\ \Gamma\left(\pm\frac{1}{4}\mu\right)\right\}^{-1} \tag{10.2}$$

$$\nu_2^{(k)} = \sqrt{\pi}\ \Gamma\left(\frac{1}{2}(2\pm\mu)\right)\left\{\Gamma\left(\frac{1}{4}(2\pm\mu)\right)\ \Gamma\left(\frac{1}{4}(4\pm\mu)\right)\right\}^{-1}. \tag{10.3}$$

Double signs may not appear in the hypergeometric functions on the right-hand side of (10.1) because of their symmetric properties with regard to the first and second parameters.

Letting $r = 1 + t$ and developing the hypergeometric functions on the right-hand side of (10.1) into a power series, we can find a complex-form asymptotic expansion for $0 \leqq a \leqq 1$.

$$F(x) \sim (\nu_1^{(1)} + \nu_1^{(2)})\, e^{-x/\sqrt{2}} \left\{ \frac{\pi}{\sqrt{x}} \exp\left(\frac{ix}{\sqrt{2}} + \frac{\pi i}{8}\right) + \frac{P_1\sqrt{\pi}}{2\sqrt{x^3}} \exp\left(\frac{ix}{\sqrt{2}} + \frac{3\pi i}{8}\right) + \right.$$

$$\left. + \frac{3P_2\sqrt{\pi}}{4\sqrt{x^5}} \exp\left(\frac{ix}{\sqrt{2}} + \frac{5\pi i}{8}\right) + \ldots\right\}$$

$$+ (\nu_2^{(1)} + \nu_2^{(2)})\, e^{-x/\sqrt{2}} \left\{ \frac{1}{x} \exp\left(\frac{ix}{\sqrt{2}} + \frac{\pi i}{4}\right) + \frac{B_1}{x^2} \exp\left(\frac{ix}{\sqrt{2}} + \frac{\pi i}{2}\right) + \right.$$

$$\left. + \frac{2B_2}{x^3} \exp\left(\frac{ix}{\sqrt{2}} + \frac{3\pi i}{4}\right) + \ldots\right\} \tag{10.4}$$

where

$$P_1 = -(1+2a)/4$$
$$P_2 = (9+20a+4a^2)/96$$
$$B_1 = -(3+a)/6$$
$$B_2 = -(3+a)(5+a)/120.$$

Asymptotic expansions of $w_1(x)$ and $w_2(x)$ are given by the real and imaginary parts of (10.4), respectively.

11. Asymptotic expansion for $1 \leqq a \leqq 2$

A form of asymptotic expansion for $a \geqq 1$ is found by letting $\mu = i\kappa$ in the coefficients of $\nu_k^{(1)} + \nu_k^{(2)}$ $(k = 1, 2)$. In this case, formulas (9.2) and (9.3) need to be modified to include the case $x = 0$. The modified formulas are

$$|\Gamma(iy)|^2 = y^{-2} \prod_{n=1}^{\infty} [1 + (y/n)^2]^{-1} \tag{11.1}$$

$$\operatorname{Arg}\Gamma(iy) = -\frac{\pi}{2}\operatorname{sign}(y) - y\gamma + \sum_{n=1}^{\infty}\left(\frac{y}{n} - \tan^{-1}\frac{y}{n}\right) \tag{11.2}$$

where

$$\operatorname{sign}(y) = 1 \quad \text{for } y > 0$$
$$= -1 \quad \text{for } y < 0.$$

The result of the transformation becomes extremely simple:

$$\nu_1^{(1)} + \nu_1^{(2)} = \cos(\kappa \log 2) \tag{11.3}$$

and

$$\nu_0^{(1)} + \nu_0^{(2)} = \kappa \sin\left[\kappa\left(\frac{1}{4}\gamma + \log\sqrt{2}\right)\right]. \tag{11.4}$$

Substituting these into (10.4) the complex form asymptotic expansion for $1 \leqq a$ can be found. We found, however, that this formula was not effective for large values of a. Therefore, we used this formula only for $1 \leqq a \leqq 2$.

12. Asymptotic expansion for $2 \leqq a \leqq \infty$

Letting $\mu = i\kappa$, the integral solution (4.11) transform to

$$\frac{1}{2}F(x) = \int_1^{\infty} e^{\beta x r} \cos\left[\frac{\kappa}{2}\log\left(r^2 + \sqrt{r^4 - 1}\right)\right](r^4 - 1)^{-1/2}\,dr. \tag{12.1}$$

Expanding the integrand in the neighborhood of $r = 1$ by letting $r = 1 + t$, and using the approximations

$$\log\left(r^2 + \sqrt{r^4 - 1}\right) = 2\sqrt{t} + O(t^{3/2})$$

and

$$\sqrt{r^4 - 1} = 2\sqrt{t} + 0(t)$$

we find the integral asymptotic solution

$$F(x) \sim \int_0^\infty e^{\beta x(1+t)} \cos(\kappa\sqrt{t}) \frac{dt}{\sqrt{t}} . \tag{12.2}$$

To evaluate this integral, define the function

$$G_k(x) = \frac{1}{2} \int_0^\infty e^{\beta xt \pm i\kappa\sqrt{t}} \frac{dt}{\sqrt{t}} . \tag{12.3}$$

Then (12.2) becomes

$$F(x) \sim e^{\beta x} [G_1(x) + G_2(x)] . \tag{12.4}$$

Letting $t = \xi^2$, (12.3) transforms to

$$G_k(x) = \int_0^\infty \exp\left(\beta x \left(\xi \mp \frac{\beta\kappa}{2x}\right)^2 + \frac{\kappa^2}{4\beta x}\right) d\xi .$$

Define z by

$$\beta x \left(\xi \mp \frac{\beta\kappa}{2x}\right)^2 = -z^2 .$$

The root z of this equation, satisfying the condition that the real part of z must approach positive infinity as $\xi \to \infty$, is

$$z = \exp\left(-\frac{\pi i}{8}\right) x^{1/2} \xi \mp \exp\left(\frac{5\pi i}{8}\right) \frac{1}{2} \kappa x^{-1/2} .$$

Use of z thus defined transforms (12.3) to

$$G_k(x) = x^{-1/2} \exp\left(\frac{\kappa^2}{4\beta x} + \frac{\pi i}{8}\right) \int_{\alpha_k}^{\infty \exp(5\pi i/8)} \exp(-z^2)\, dz$$

where we have introduced

$$\alpha_k = \mp \exp\left(\frac{5}{8}\pi i\right) \frac{1}{2} \kappa x^{-1/2} .$$

Transforming the contour of integration to the sum of the two contours, $\alpha_k \sim 0$ and $0 \sim +\infty$, we find

$$G_1(x) + G_2(x) = \sqrt{\pi/x}\, \exp[\kappa^2/(4\beta x) + \pi i/8] . \tag{12.5}$$

Thus we get

$$F(x) \sim \sqrt{\frac{\pi}{x}} \exp\left(\beta x + \frac{\kappa^2}{4\beta x} + \frac{\pi i}{8}\right). \tag{12.6}$$

This equation gives a little worse approximation in the neighborhood of $a = 1$ than the one derived in section 11.

PART III. EIGENVALUES

13. Range of eigenvalues

The clamped-edge condition yields eigenvalues in the range $1 < a < \infty$. To show this, we rewrite (1.5) to

$$\frac{d^2}{dx^2}\left(x\frac{d^2 w}{dx^2}\right) + xw = -(1-a)\frac{d}{dx}\left(x\frac{dw}{dx}\right). \tag{13.1}$$

We integrate the product of w and this equation between x_0 and ∞; use of the boundary condition (1.7) simplifies the successive partial integrations. Thus we get

$$a - 1 = \int_{x_0}^{\infty} x\left[\left(\frac{d^2 w}{dx^2}\right)^2 + w^2\right]dx \Bigg/ \int_{x_0}^{\infty} \frac{1}{x}\left(\frac{dw}{dx}\right)^2 dx \tag{13.2}$$

to prove our contention.

For the simple-edge condition, a similar treatment yields

$$a - 1 = \left\{-\nu\left[\left(\frac{dw}{dx}\right)^2\right]_{x_0}^{\infty} + \int_{x_0}^{\infty} x\left[\left(\frac{d^2 w}{dx}\right)^2 + w^2\right]dx\right\} \Bigg/ \int_{x_0}^{\infty} \frac{1}{x}\left(\frac{dw}{dx}\right)^2 dx. \tag{13.3}$$

The right-hand side of this equation does not seem to be nonnegative; however, the following numerical computation shows that it is nonnegative and that $1 \leqq a \leqq \infty$.

In the case of the free-edge condition, the boundary condition does not simplify the partial integrations. The following numerical computation shows that $1 - \nu^2 \leqq a \leqq \infty$. Eigenvalue a can be less than 1 only in this case.

14. Eigenvalues for the free-edge condition

Define operators

$$L = \frac{d^2}{dx^2} + \frac{\nu}{x}\frac{d}{dx} \tag{14.1}$$

$$M = \frac{d^3}{dx^3} + \frac{1}{x}\frac{d^2}{dx^2} - \frac{1-a}{x^2}\frac{d}{dx}. \tag{14.2}$$

Then the determinant D_3 for the determination of A and B in (1.14) is given by

$$D_3 = \begin{vmatrix} L(w_1) & M(w_1) \\ L(w_2) & M(w_2) \end{vmatrix}. \tag{14.3}$$

Root x_0 of D_3 in the range $1 - \nu^2 \leqq a \leqq 2$ are shown in Figures 2 and 3.

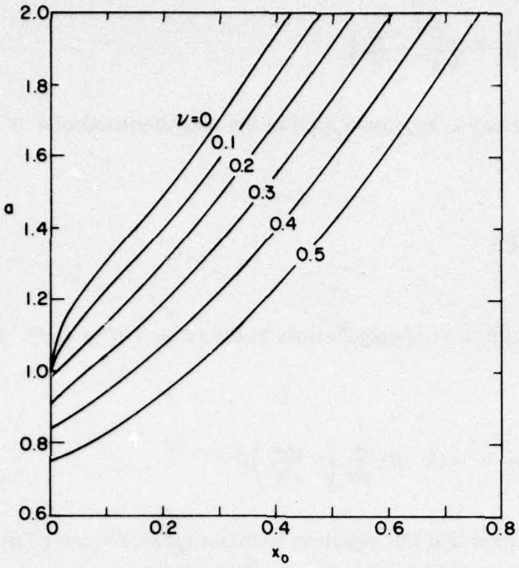

Figure 2. Free-edge condition roots x_0 in the range $1 - \nu^2 \leqq a \leqq 2$.

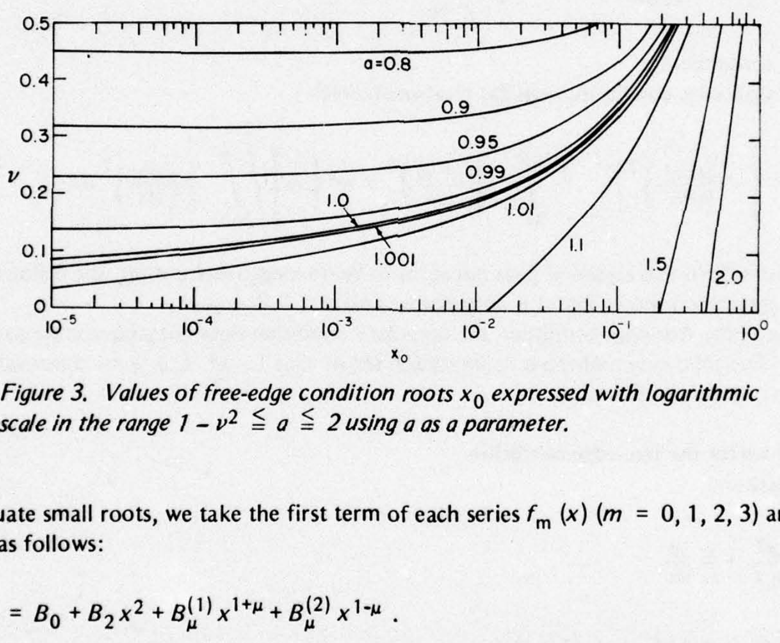

Figure 3. Values of free-edge condition roots x_0 expressed with logarithmic scale in the range $1 - \nu^2 \leqq a \leqq 2$ using a as a parameter.

To evaluate small roots, we take the first term of each series $f_m(x)$ $(m = 0, 1, 2, 3)$ and approximate $F(x)$ as follows:

$$F(x) = B_0 + B_2 x^2 + B_\mu^{(1)} x^{1+\mu} + B_\mu^{(2)} x^{1-\mu} . \tag{14.4}$$

With this approximation, the roots of $D_3 = 0$ are given by letting the real part of $L(F)$ equal zero. To prove this, rewrite (14.3) to the complex form

$$D_3 = \frac{1}{-2i} \begin{vmatrix} L(F) & L(\bar{F}) \\ M(F) & M(\bar{F}) \end{vmatrix} \tag{14.5}$$

Note that the use of (14.4) yields

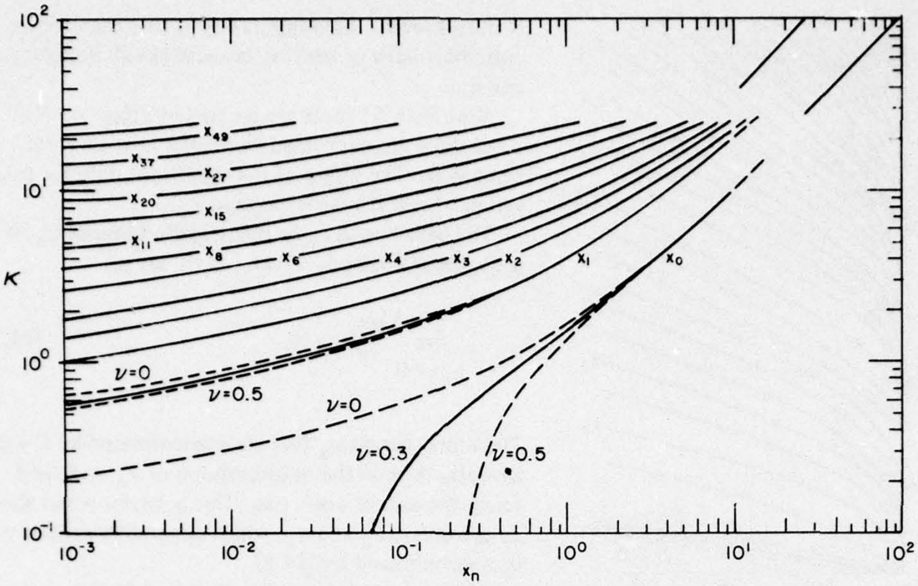

Figure 4. Free-edge condition roots x_n ($n = 0, 1, 2 \ldots$) as functions of $\kappa = \sqrt{a-1}$ in the range $0.01 < a < \infty$.

$$M(F) = 2a\,B_2/x.$$

Use of this equation in (14.5) proves our contention, because B_2 is purely imaginary for any values of a as shown by (5d.3).

When $0 < a < 1$, we can find in this way that the small roots x_0 are given by

$$\left(\frac{1}{\sqrt{2}}\,x_0\right)^{2\mu} = \frac{(\nu-\mu)\,\Gamma(1+\mu)\,\cos[(3+\mu)\pi/4]}{(\nu+\mu)\,\Gamma(1-\mu)\,\cos[(3-\mu)\pi/4]}\,. \tag{14.6}$$

Because x_0 must be nonnegative, this equation shows that the condition $\nu \geqq \mu$, i.e. $a \geqq 1 - \nu^2$, must be met. When $\nu = \mu$, x_0 becomes zero. Therefore, each curve in Figure 2 terminates on the axis of ordinates at $a = 1 - \nu^2$.

For small μ, expansion in terms of μ transforms (14.6) to

$$\ln\left(\frac{1}{\sqrt{2}}\,x_0\right) = \frac{\pi}{4} - \gamma - \frac{1}{\nu} + \frac{1}{3}\,\mu^2\left(-\zeta(3) - \nu^{-3} + \frac{1}{32}\,\pi^3\right) \tag{14.7}$$

where $\zeta(3)$ is Riemann's zeta-function. Our numerical computation showed that (14.7) gives a close approximation over the entire lengths of the curves in the neighborhood of $a = 1 - 0$ in Figure 3.

The roots in the range $1.01 \leqq a < 626$ are shown in Figure 4. Equations (9.8) and (9.9) were used to calculate w_0, w_2, and their derivatives in (14.3).

The small roots for $a > 1$ are found by letting $\mu = i\kappa$ in $L(F)$ and taking the real part. There are infinitely many roots x_n that can be determined by solving the equation

$$\tan\left(\kappa \log \frac{x}{\sqrt{2}}\,n + \frac{\pi}{4} - \Theta + \tan^{-1}\kappa + \tan^{-1}\frac{\kappa}{\nu}\right) = \exp(\kappa\pi/2). \tag{14.8}$$

If x_n is a root of the equation, x_{n+1} given by

$$x_{n+1} = x_n \exp(-\pi/\kappa) \tag{14.9}$$

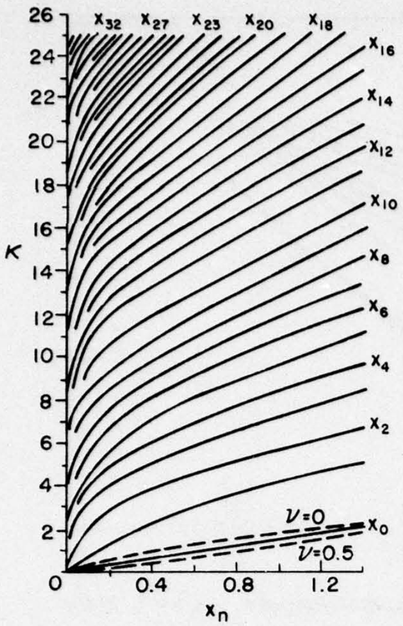

Figure 5. Behavior of the free-edge condition roots x_n ($n = 0, 1, 2, ...$) in the neighborhood of x_n for $\kappa \geqq 0$.

is another root. Equation (14.9) is true even for an extremely large or small κ, because (14.8) holds for any κ.

More than 57 roots are plotted in Figure 4. Not all of them are connected with solid lines to avoid confusion. The effect of ν is significant only for small values of κ in the curves x_0 and x_1.

The behavior of x_n in the neighborhood of $x_n = 0$ is shown in Figure 5. From (14.8), we get

$$\lim_{x_n \to 0} \frac{dx_n}{d\kappa} = 0. \tag{14.10}$$

Therefore, curves x_n that are approximated by (14.8) abruptly drop in the neighborhood of $x_n = 0$, and touch the axis of ordinates. This behavior is not shown by the curves x_0 and x_1, which apparently cannot yet be approximated by (14.8).

When $a = 1$, we simplify (6.3) and (6.4) by taking the first terms of nev functions, substitute these approximations in (14.3), and find that the small roots are given by

$$x_0 \doteqdot \sqrt{2} \exp[-(1/\nu) - \gamma + \pi/(2\sqrt{2})]. \tag{14.11}$$

This calculation is fairly easy because $M(x) = 0$ and $M(x \log x) = 0$. Equation (14.11) shows that x_0 tends to zero, when ν approaches zero as the limit.

The asymptotic behavior of the large roots can be found by using the asymptotic complex fundamental solution $F(x)$ in (12.6) to compute $L(F)$ and $M(F)$. Assuming that ζ defined by

$$\zeta = \kappa^2/(4x^2) \tag{14.12}$$

is of the ordinary magnitude for large x, we find

$$L(F) \sim F(x)\,(\beta^2 - 2\zeta + \beta^{-2}\zeta^2)$$

$$M(F) \sim F(x)\,(1 + \zeta^2)\left[\exp\left(\frac{\pi i}{4}\right) - \zeta \exp\left(-\frac{\pi i}{4}\right)\right].$$

Thus, we discover that there are two asymptotic roots

$$x_h = \kappa/\sqrt{4\zeta_h} \tag{14.13}$$

where

$$\zeta_h = 2 \mp \sqrt{3}\,. \tag{14.14}$$

In the above equations, subscript h is defined by

$$h = k - 1 \tag{14.15}$$

where the old convention for suffix k is still observed. The two lines expressing the two equations in (14.13) are shown in the upper right corner of Figure 4.

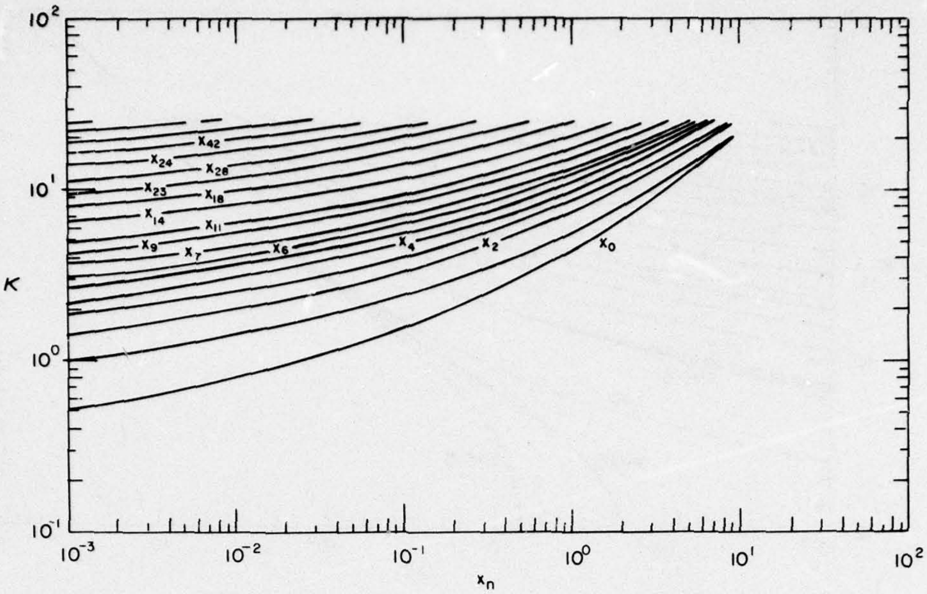

Figure 6. Clamped-edge condition roots x_n (n = 0, 1, 2 ...) as functions of $\kappa = \sqrt{a-1}$ in the range $0.1 \leqq \kappa \leqq 25$.

Therefore, as a increases indefinitely, the infinitely many roots x_n in Figure 4 must converge with either one of the two asymptotic roots given by (14.13). We increased κ up to 25 (i.e. a = 626). As shown in Figure 4, curve x_0 and a few curves immediately to the left of curve x_0 appear to converge to the right-hand side asymptotic branch. However, we did not pursue the mode of convergence any further.

If κ becomes larger than 25, some curves extend to the right of $x = 10$, which was the limit of the use of the series (9.8) and (9.9). When roots x_n are larger than 10, we should analytically continue series (9.8) and (9.9) to a new range of x. However, we did not attempt this, since our engineering interest was originally in small values of a. We presently consider that the mode of convergence in the range $626 < a < \infty$ is not important to us.

15. Eigenvalues for the clamped-edge and simple-edge conditions

The determinants D_1 and D_2 to be used in the case of clamped-edge and simple-edge conditions, respectively, for the determination of A and B in (1.14) are

$$D_1 = \begin{vmatrix} w_1 & w_1' \\ w_1 & w_2' \end{vmatrix} \tag{15.1}$$

and

$$D_2 = \begin{vmatrix} w_1 & L(w_1) \\ w_2 & L(w_2) \end{vmatrix} . \tag{15.2}$$

The roots for the clamped- and simple-edge conditions in the range $0.1 \leqq \kappa \leqq 25$ are shown in Figures 6 and 7, respectively. Infinitely many roots accumulate, as shown in Figures 8 and 9, in the neighborhood of $x = 0$, when a is in the range $1 < a \leqq \infty$ (i.e. excluding the case $a = 1$). The clamped-edge roots are independent of ν, because ν does not appear in the boundary conditions. As

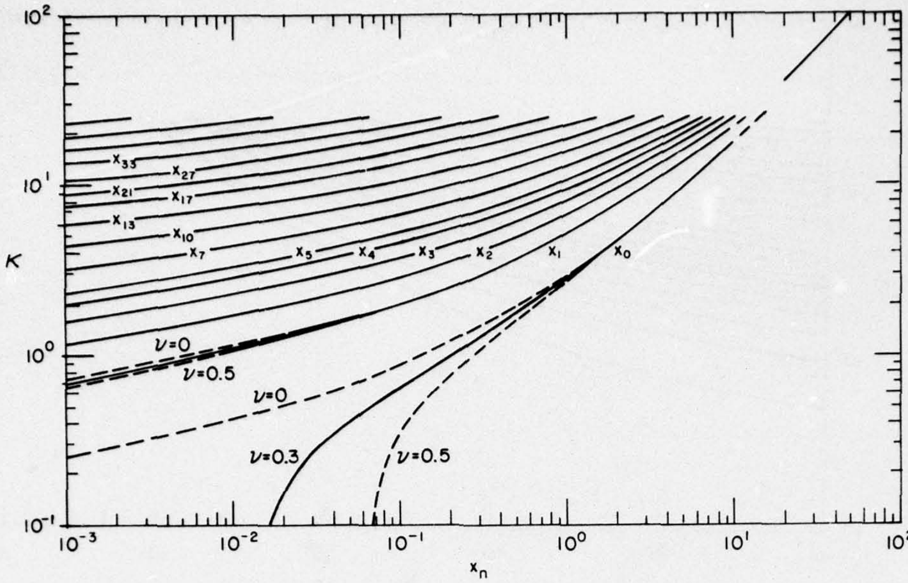

Figure 7. Simple-edge condition roots x_n (n = 0, 1, 2, ...) as functions of $\kappa = \sqrt{a - 1}$ in the range of $0.1 \leqq \kappa \leqq 25$.

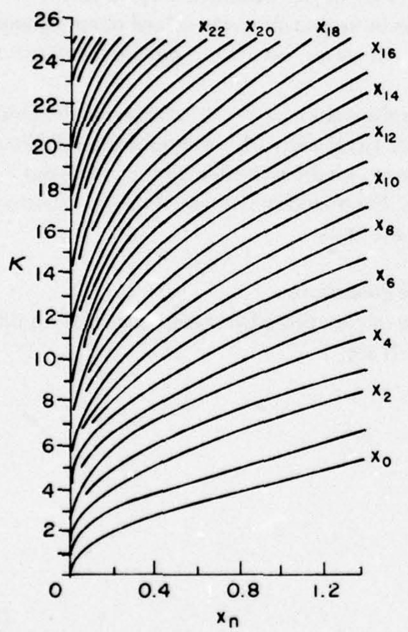

Figure 8. Behavior of the clamped-edge condition roots x_n (n = 0, 1, 2, ...) in the neighborhood of x_n = 0 for $\kappa \geqq 0$.

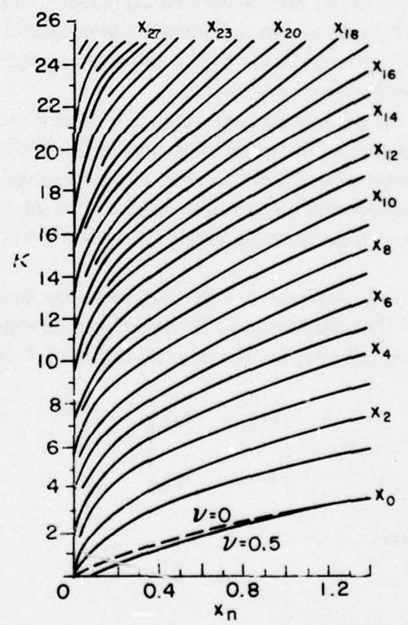

Figure 9. Behavior of the simple-edge condition roots x_0 (n = 0, 1, 2, ...) in the neighborhood of x_n = 0 for $\kappa \geqq 0$.

shown in Figure 9, the effect of ν on the simple-edge condition roots are significant only for small x_0 and small x_1.

To discuss the small roots, we express D_1 and D_2 in complex forms. We find that D_1 and D_2 are equal to the imaginary parts of $-FF'$ and $-FL(\bar{F})$, respectively, where F is given by (14.4). Letting $\mu = ik$, we find that the small roots x_n of D_1 and D_2 are given by

$$\tan[\kappa \log(x_n/\sqrt{2}) - \Theta + (\pi/4) + \tan^{-1}\kappa] = -\exp(-\kappa\pi/2)$$

and

$$\tan[\kappa \log(x_n/\sqrt{2}) - \Theta + (\pi/4) + \tau] = \exp(-\kappa\pi/2),$$

respectively, where we have introduced τ defined by

$$\tan\tau = \kappa(1+\nu)/(1-a+\nu), \quad 0 \leqq \tau < \pi.$$

In both cases, therefore, if x_n is a root, x_{n+1} given by

$$x_{n+1} = x_n \exp(-\pi/\kappa) \tag{15.3}$$

is another root. Because (14.10) still holds in these cases, curves x_n in Figures 8 and 9 are tangent to the axis of ordinates.

The roots for the case $a = 1$ are small and are found by taking the first terms of nev functions in (6.3) and (6.4). Using D_1 in (15.1) and D_2 in (15.2), we find that there is no clamped-edge condition root and that there are small simple-edge condition roots

$$x_0 \doteqdot \sqrt{2}\exp[-(1/\nu) - \gamma - (\pi/4)]. \tag{15.4}$$

Therefore, x_0 found under the simple-edge condition tends to zero when ν approaches zero as the limit.

For large κ, use of the asymptotic complex fundamental solution (12.6) yields that there is no clamped-edge asymptotic root, and that there is one branch of simple-edge asymptotic roots, given by

$$x = \kappa/2 \tag{15.5}$$

which is shown on the upper-right corner of Figure 7.

16. Deflection

We calculated the deflections corresponding to roots x_0, x_1, and x_2 by applying the normalization

$$w(x_n) = 1 \tag{16.1}$$

for the free-edge condition, and

$$\max|w| = 1 \tag{16.2}$$

for the clamped- and simple-edge conditions. The cases for $a = 17$ are shown in Figures 10, 11, and 12. All the curves we calculated — but the cases for $a = 17$ most conspicuously — showed that the suffix n of x_n is the number of bumps the curve x_n must pass through as x increases from $x = x_n$ before curve x_n assumes the same shape as curve x_0.

Even when a is in the range $1 - \nu^2 \leqq a < 1$, curve x_0 in the case of the free-edge condition assumes a shape similar to curve x_0 in Figure 10. This shape of deflection has often been observed

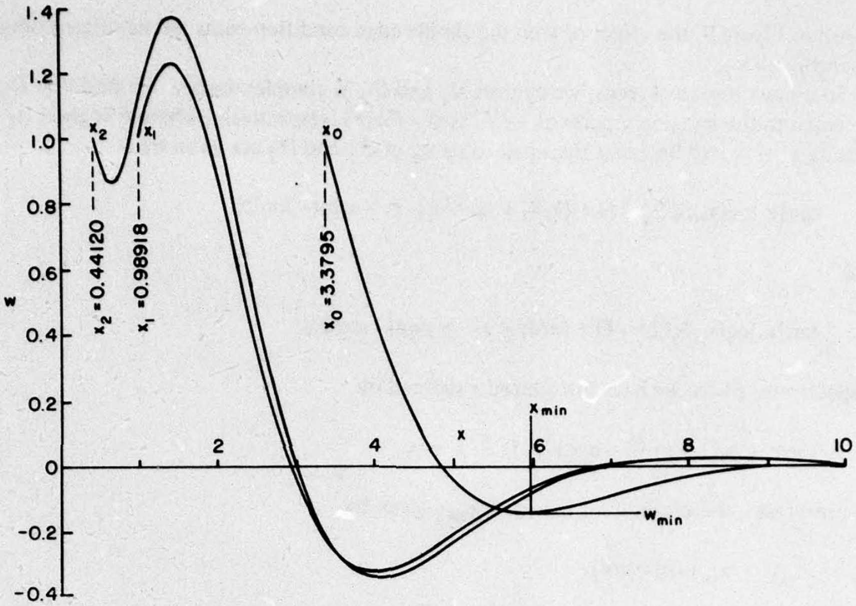

Figure 10. Normalized deflection curves x_0, x_1, and x_2 for the free-edge condition, when a = 17.

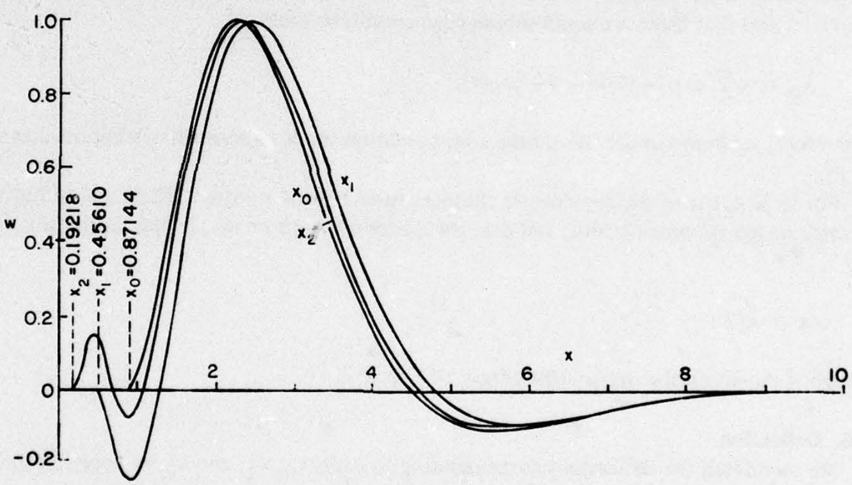

Figure 11. Normalized deflection curves x_0, x_1, and x_2 for the clamped-edge condition when a = 17.

in both laboratory and field when floating ice plates are compressed. We are now convinced that buckling under the free-edge condition frequently takes place.

However, the shapes shown in these figures may be different from the actual ones. First, the maximum deflection is not necessarily upward; in Figures 10 and 12, the normalization compels the maximum to go upward. Second, the maximum point in the real deformation may either go above the water and not touch it, or go downward and be underwater. The assumption introduced for the derivation of the differential equation (1.5) is violated in these cases. The deflection shown in these figures, therefore, may not be actual.

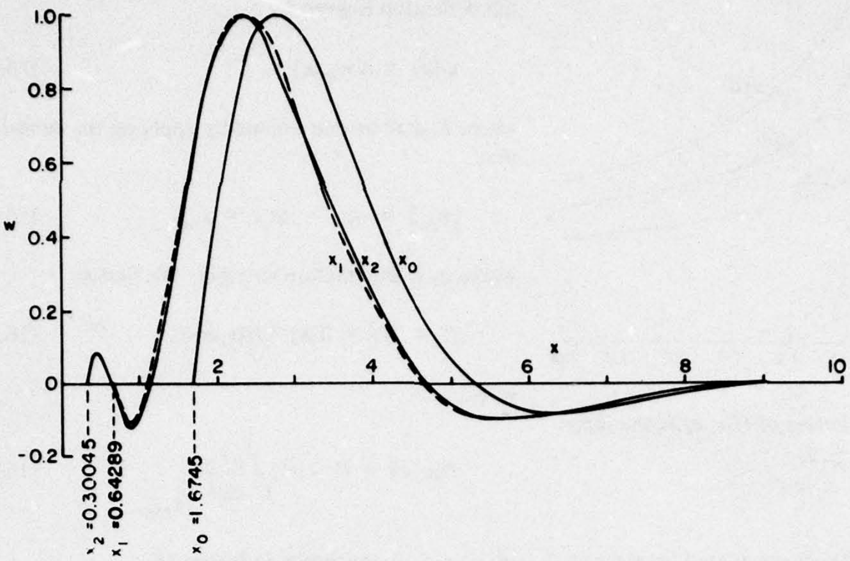

Figure 12. Normalized deflection curves x_0, x_1, and x_2 for the simple-edge condition, when $a = 17$.

To estimate the size of the fracture deflection, we will discuss the curve x_0 under the free-edge condition. The other edge conditions can be similarly discussed. The fracture takes place when the stress at x_{min} (see Fig. 10 for the definition) reaches the fracture stress σ_f. In the general polar coordinates, stress components σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$ are given (see App. B) by

$$\sigma_{rr} = -\frac{Eh}{2(1-\nu)}\left(\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{r}\frac{\partial w}{\partial r} + \frac{\nu}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)$$

$$\sigma_{\theta\theta} = -\frac{Eh}{2(1-\nu)}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right) \qquad (16.3)$$

$$\sigma_{r\theta} = -\frac{Eh}{2(1+\nu)}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)$$

where h is the thickness of the plate, E, Young's modulus. In our case of axisymmetry, introducing the nondimensional length x defined by (1.4), the above formulas become

$$\sigma_{rr} = -\frac{Eh}{2(1-\nu)\ell_0^2}\frac{d^2 w}{dx^2} + \frac{\nu}{x}\frac{dw}{dx}$$

$$\sigma_{\theta\theta} = -\frac{Eh}{2(1-\nu)\ell_0^2}\left(\frac{1}{x}\frac{dw}{dx} + \nu\frac{d^2 w}{dx^2}\right) \qquad (16.4)$$

$$\sigma_{r\theta} = 0.$$

Therefore, at point x_{min}, where $dw/dx = 0$, we have

$$|\sigma_{rr}| > |\sigma_{\theta\theta}|.$$

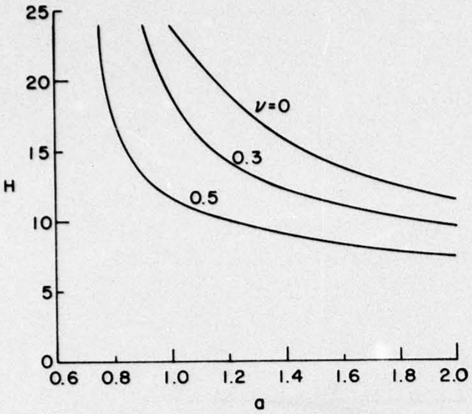

Figure 13. Values of $H(\nu, a)$ in the range $1 - \nu^2 \leq a \leq 2$.

Let $w_n(x)$ be the $w(x)$ normalized at $x = x_0$. Then the deflection is given by

$$w(x) = K w_n(x) \tag{16.5}$$

where K shall be determined by applying the condition that

$$|\sigma_{rr}| = \sigma_f \qquad \text{at } x = x_{min} \tag{16.6}$$

where σ_f is the fracture strength. We find K:

$$K = 2\ell_0^2 \, \sigma_f \, (Eh)^{-1} H(\nu, a) \tag{16.7}$$

where

$$H(\nu, a) = (1 - \nu^2) \Big/ \left. \frac{d^2 w}{dx^2} \right|_{x_{min}} \tag{16.8}$$

Values of $H(\nu, a)$, calculated for the case $1 - \nu^2 \leq a \leq 2$, are shown in Figure 13.

ACKNOWLEDGMENT

During his visit at Northwestern University in 1958, Professor Jean Dieudonné (Maître de Conférence à la Faculté des Sciences de Nice) solved the boundary value problem of (6.1) at the request of the U.S. Army Snow, Ice and Permafrost Research Establishment, then at Evanston, Illinois, as recorded in references (6) and (8). His method inspired me to develop the method of transforming the integral solution (1.12) to the linear combination (1.13).

The analytical derivation of the range of eigenvalues in section 11 was shown to me by Dr. Ben Noble, Director of the Mathematical Research Center, University of Wisconsin at Madison. Dr. C.W. Cryer, MRC, studied the eigenvalue problem numerically. Dr. J.B. McLeod, University of Oxford, England, then visiting at MRC, studied this problem theoretically. These people showed me the misconception I had held at that time with regard to the clamped- and simple-edge eigenvalues.

The comparison of the buckling pressures in Appendix C was brought to my attention by Dr. D.E. Nevel, CRREL.

The numerical computation was carried out with the help of two students at Dartmouth College: D.M. Fitchet in 1975 and J.A. Bagger in 1976.

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APPENDIX A. ANALYTICAL CONTINUATION AT THE SINGULAR POINT

We analytically continued $\nu_k(r)$ in (4.8) across the singular point $z = \beta$ in Figure 1 by use of the hypergeometric function $F(\ , \ ; \ ; \ )$, as shown in (10.1). Let us now suppose that $\nu_k(r)$ is a series whose analytical continuation is not readily available. If the following proposition is applicable, we can define the analytical continuation:

Proposition: Let $\phi_0(t)$, $\phi_1(t)$, and $\phi_2(t)$, where

$$t = r - 1 \tag{A.1}$$

be the Fuchsian-type series solutions of the differential equation (4.5) in the neighborhood of $r = 1$, i.e., $x = \beta$ in Figure A1. If a linear relationship

$$\nu_k(r) = A_k\phi_0(t) + B_k\phi_1(t) + C_k\phi_2(t) \tag{A.2}$$

with constants A_k, B_k, C_k holds true at least at one point in the hatched crescent shape region in Figure A1, the analytical continuation of $\nu_k(r)$ is defined in the circle that has center B and passes through B_1 and B_3.

Proof: The hatched crescent shape region is the common region of convergence of the two types of series $\nu_k(r)$ and $\phi_m(t)$ $(m = 0, 1, 2)$. In fact, the region of convergence of $\nu_k(r)$ is outside the circle that has a center at 0 and passes through B, B_1, B_2, and B_3; the region of convergence of $\phi_m(t)$ $(m = 0, 1, 2)$ is inside the circle that has a center at B and passes through B_1 and B_3. If coefficients A_k, B_k, and C_k are non-zero at one point in the hatched regions, these coefficients must be constant at this point, because all the determinants of the simultaneous equations

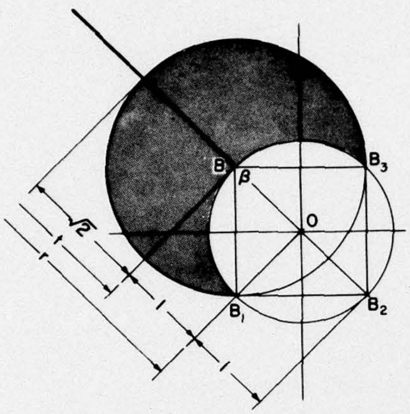

Figure A1. The common region of convergence of $\nu_k(r)$ and $\phi_m(t)$ $(m = 0, 1, 2)$ on the complex plane ζ, where $\beta = (-1 + i)/\sqrt{2}$.

$$\nu_k = A_k\phi_0 + B_k\phi_1 + C_k\phi_2$$

$$\nu_k' = A_k\phi_0' + B_k\phi_1' + C_k\phi_2'$$

$$\nu_k'' = A_k\phi_0'' + B_k\phi_1'' + C_k\phi_2''$$

are Wronskians. If (A.2) holds true at one point in the common region of convergence, (A.2) must be true at any point in the common region of convergence, because both sides of (A.2) satisfy the same differential equation (4.5) and the same initial conditions. The proposition is thus proved.

However, the analytical continuation defined by use of the hypergeometric function in (4.8) is not the type of (A.2), as shown in (10.1). We, however, did not try to learn how we can define the analytical continuation of $\nu_k(r)$ in (10.1) when the above proposition is not applicable, although we must anticipate the need of the analytical continuation of such cases. In this study we were lucky to have discovered that $\nu_k(r)$ is expressible with a hypergeometric series that is summable in a closed form.

APPENDIX B. TENSORIAL TRANSFORMATIONS

We used dyadic tensor expressions to transform tensor components. The dyadic expressions can be organized in such a way that the geometric or mechanical meanings of the tensor notations are clear at any stage of the transformations. We shall explain the derivations of the tensor formulas we introduced by this method for use in this paper.

B1. Transformations of (1.10) to (1.11)

Figure B1 illustrates Q_x and Q_y, the magnitude per unit length of the shears acting on the *yz*- and *xz*-planes, respectively. The first step for achieving our objective is to find vector **Q** that has components Q_x and Q_y.

Let $\mathbf{c}_x$ and $\mathbf{c}_y$ be the unit vectors in the *x*- and *y*-directions, respectively. Let $\mathbf{c}_n$ be the unit vector outwardly normal in the *xy*-plane to the hypotenuse *AB* (see Fig. B2). Because

$$\mathbf{c}_n \cdot \mathbf{c}_x = dy/ds$$

and

$$\mathbf{c}_n \cdot \mathbf{c}_y = dx/ds$$

$\mathbf{c}_n$ is given by

$$\mathbf{c}_n\, ds = \mathbf{c}_x\, dy + \mathbf{c}_y\, dx. \qquad \text{(i)}$$

Let Q_s be the shear defined per unit length along the negative *z* direction on the plane hanging downward from the hypotenuse *AB* (Fig. B1). The shear Q_s is given by

$$Q_s\, ds = Q_x\, dy + Q_y\, dx. \qquad \text{(ii)}$$

Define vector **Q** by

$$\mathbf{Q} \cdot \mathbf{c}_x = Q_x \qquad \text{(iii)}$$

$$\mathbf{Q} \cdot \mathbf{c}_y = Q_y \qquad \text{(iv)}$$

which are equivalent to

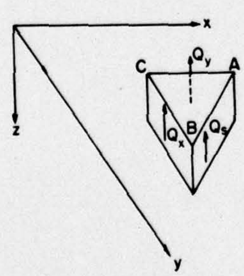

Figure B1. Definition of Q_x and Q_y.

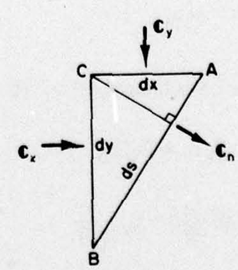

Figure B2. Definition of $\mathbf{c}_n$.

$$\mathbf{Q} = Q_x \mathbf{c}_x + Q_y \mathbf{c}_y . \tag{B.1}$$

Dotting (i) and (B.1) and using (ii) yields

$$\mathbf{Q} \cdot \mathbf{c}_n = Q_s \tag{v}$$

which has the same form with (iii) and (iv). Therefore, (B.1) defines the desired vector **Q**.

Substitution of Q_x and Q_y in (1.10) into (B.1) yields an invariant form:

$$Q = \nabla \cdot \mathbf{M} + \nabla w \cdot \mathbf{N} \tag{B.2}$$

where ∇ is an operator

$$\nabla = \mathbf{c}_x \frac{\partial}{\partial x} + \mathbf{c}_y \frac{\partial}{\partial y} \tag{B.3}$$

and **M** and **N** are tensors

$$\mathbf{M} = M_{xx} \mathbf{c}_x \mathbf{c}_x + M_{xy} (\mathbf{c}_x \mathbf{c}_y + \mathbf{c}_y \mathbf{c}_x) + M_{yy} \mathbf{c}_y \mathbf{c}_y \tag{B.4}$$

$$\mathbf{N} = N_{xx} \mathbf{c}_x \mathbf{c}_x + N_{xy} (\mathbf{c}_x \mathbf{c}_y + \mathbf{c}_y \mathbf{c}_x) + N_{yy} \mathbf{c}_y \mathbf{c}_y . \tag{B.5}$$

A convention is made in (B.2) that $\mathbf{a} \cdot \mathbf{bc}$ means $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Let $\mathbf{u}_r$ and $\mathbf{u}_\theta$ be the unit vecotrs in the r- and θ-directions; we choose vector $\mathbf{u}_r$ to make angle θ from vector $\mathbf{c}_x$. They are given by

$$\begin{aligned} \mathbf{u}_r &= \mathbf{c}_x \cos\theta + \mathbf{c}_y \sin\theta \\ \mathbf{u}_\theta &= -\mathbf{c}_x \sin\theta + \mathbf{c}_y \cos\theta. \end{aligned} \tag{B.6}$$

Differentiations of these unit vectors yield

$$\begin{aligned} \frac{\partial \mathbf{u}_r}{\partial r} &= 0 \\ \frac{\partial \mathbf{u}_\theta}{\partial r} &= 0 \\ \frac{\partial \mathbf{u}_r}{\partial \theta} &= \mathbf{u}_\theta \\ \frac{\partial \mathbf{u}_\theta}{\partial \theta} &= -\mathbf{u}_r . \end{aligned} \tag{B.7}$$

In the polar coordinates, (B.1), (B.3), (B.4) and (B.5) become

$$\mathbf{Q} = Q_r \mathbf{u}_r + Q_\theta \mathbf{u}_\theta \tag{B.8}$$

$$\nabla = \mathbf{u}_r \frac{\partial}{\partial r} + \mathbf{u}_\theta \frac{\partial}{r \partial \theta} \tag{B.9}$$

$$\mathbf{M} = M_{rr} \mathbf{u}_r \mathbf{u}_r + M_{r\theta} (\mathbf{u}_r \mathbf{u}_\theta + \mathbf{u}_\theta \mathbf{u}_r) + M_{\theta\theta} \mathbf{u}_\theta \mathbf{u}_\theta \tag{B.10}$$

$$\mathbf{N} = N_{rr}\,\mathbf{u}_r\mathbf{u}_r + N_{r\theta}\,(\mathbf{u}_r\mathbf{u}_\theta + \mathbf{u}_\theta\mathbf{u}_r) + N_{\theta\theta}\,\mathbf{u}_\theta\mathbf{u}_\theta , \tag{B.11}$$

respectively.

We substitute (B.9), (B.10), and (B.11) into (B.2), carry out the differentiations by use of (B.7), execute the dottings of the unit vectors $\mathbf{u}_r$ and $\mathbf{u}_\theta$, and identify the components with (B.8); thus we find (1.11).

B2. Transformation of (1.1) to (1.2)

We shall prove that in the general polar coordinates

$$N_{xx}\frac{\partial^2 w}{\partial x^2} + 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} + N_{yy}\frac{\partial^2 w}{\partial y^2} \tag{B.12}$$

transforms to

$$= N_{rr}\frac{\partial^2 w}{\partial r^2} + 2N_{r\theta}\frac{\partial w}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right) + N_{\theta\theta}\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) . \tag{B.13}$$

Expression (B.12) transforms to an invariant form, $\mathbf{N}\cdot\nabla\nabla w$, where $\mathbf{N}$ and ∇ are defined by (B.5) and (B.3), respectively, $\nabla\nabla$ a dyadic operator (Wilson[9]), and the double-dotting $\mathbf{ab}\cdot\cdot\,\mathbf{cd}$ means $(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})$. In the polar coordinates $\nabla\nabla w$ becomes

$$\nabla\nabla w = \left(\mathbf{u}_r\frac{\partial}{\partial r} + \mathbf{u}_\theta\frac{\partial}{r\partial\theta}\right)\left(\mathbf{u}_r\frac{\partial}{\partial r} + \mathbf{u}_\theta\frac{\partial w}{r\partial\theta}\right) .$$

Carrying out the differentiations as indicated by (B.7), *the above transforms to*

$$= \frac{\partial^2 w}{\partial r^2}\mathbf{u}_r\mathbf{u}_r + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)(\mathbf{u}_r\mathbf{u}_\theta + \mathbf{u}_\theta\mathbf{u}_r) + \left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\mathbf{u}_\theta\mathbf{u}_\theta .$$

Double-dotting this expression with $\mathbf{N}$ in (B.11), we find (B.13).

B3. Derivation of (16.3)

Substituting eq (1.4) of Mansfield,[5] we can transform the tensor expression

$$\boldsymbol{\sigma} = \sigma_x\mathbf{c}_x\mathbf{c}_x + \sigma_y\mathbf{c}_y\mathbf{c}_y + \tau_{xy}(\mathbf{c}_x\mathbf{c}_x + \mathbf{c}_y\mathbf{c}_x)$$

to an invariant form

$$\boldsymbol{\sigma} = -[Ez/(1-\nu^2)]\,(\nabla\nabla w + \Delta\Delta w) \tag{B.14}$$

where Δ is an invariant operator

$$\Delta = \mathbf{c}_x\frac{\partial}{\partial y} - \mathbf{c}_y\frac{\partial}{\partial x} ,$$

which, in polar coordinates, becomes

$$\Delta = \mathbf{u}_r\frac{\partial}{r\partial\theta} - \mathbf{u}_\theta\frac{\partial}{\partial r} . \tag{B.15}$$

We substitute ∇ in (B.9) and Δ in (B.15) in (B.14), carry out the differentiations as shown by (B.7) and find that (B.14) becomes

$$\boldsymbol{\sigma} = \sigma_{rr} \mathbf{u}_r \mathbf{u}_r + \sigma_{r\theta} (\mathbf{u}_r \mathbf{u}_\theta + \mathbf{u}_\theta \mathbf{u}_r) + \sigma_{\theta\theta} \mathbf{u}_\theta \mathbf{u}_\theta$$

with σ_{rr}, $\sigma_{r\theta}$ and $\sigma_{\theta\theta}$ given by (16.3), where $h = z/2$.

APPENDIX C. COMPARISON OF THE SEMI-INFINITE PLATE BUCKLING WITH THE ASYMPTOTIC BUCKLING

We may be tempted to infer that the semi-infinite plate buckling solution must be close to the solution of an infinitely large hole that can be found, as explained below, by use of the asymptotic complex fundamental solution $F(x)$ in (12.6). In the following we shall show that this inference is correct in the case of the buckling pressures determined for the free-edge condition – in fact, the pressures are different by only 7% – *however, the forms of deflection are different.* For the simple-edge condition, this inference cannot be justified. The clamped-edge condition does not yield any solutions for either case.

C.1. Comparison of the buckling pressures

Let $-p$ be the pressure per unit thickness on the internal hole in our analysis. By $(1.3)_1$, we have

$$p = a\gamma \ell_0^4 r_0^{-2} \tag{C.1-1}$$

where r_0 is the radius of the internal hole.

By use of D, x_0, and κ, this transforms to

$$p = \frac{\kappa^2 + 1}{x_0^2} \frac{D}{\ell_0^2} .$$

Therefore, for large x_0, p asymptotically approaches

$$p \sim \frac{\kappa^2}{x_0} \frac{D}{\ell_0^2} . \tag{C.1-2}$$

In the case of the free-edge condition, using

$$\zeta_0 = 2 - \sqrt{3}$$

in (14.14) yields

$$p \sim 4(2 - \sqrt{3})(D/\ell_0^2) = 1.0717\,(D/\ell_0^2). \tag{C.1-3}$$

As shown later, the free-edge condition semi-infinite plate buckling pressure is given by

$$p = D/\ell_0^2. \tag{C.1-4}$$

Therefore, the difference of the buckling pressure is only 7%.

In the case of the simple-edge condition, use of (15.5) yields

$$p = 4D/\ell_0^2. \tag{C.1-5}$$

However, the free-edge condition semi-infinite plate buckling problem yields only a solution that continues to oscillate at $x = \infty$. If we may take this as the solution, we have

$$p = 2D/\ell_0^2 \qquad \text{(C.1-6)}$$

which is half the value of p in (C.1-5).

In the case of the clamped-edge condition, there is no asymptotic root in the case of the asymptotic buckling as discussed in section 15, and there is no acceptable solution in the case of the semi-infinite plate buckling as discussed in the next section.

C.2. Buckling of the semi-infinite plate

Let the y axis be the edge of the semi-infinite plate. We assume that the uniform pressure $-p$ is applied per unit thickness along the y axis. Then, (1.1) reduces to

$$D\frac{d^4 w}{dx^4} + \gamma w = -p\frac{d^2 w}{dx^2} \qquad \text{(C.2-1)}$$

Define the nondimensional length ξ by

$$\xi = x/\ell_0 \qquad \text{(C.2-2)}$$

and the nondimensional parameter b by

$$p = 2b\gamma\ell_0^2 \qquad \text{(C.2-3)}$$

where

$$\ell_0 = (D/\gamma)^{1/4}. \qquad \text{(C.2-4)}$$

Then (C.2-1) becomes

$$\frac{d^4 w}{d\xi^4} + 2b\frac{d^2 w}{d\xi^2} + w = 0. \qquad \text{(C.2-5)}$$

In this equation

$$b \geqq 0$$

because we assume $p \geqq 0$. The solutions of (C.2-5) are

$$w_k = \exp(\lambda_k \xi)$$

where

$$\lambda_k^2 = -b \pm \sqrt{b^2 - 1}$$

In the following, we consider only the fundamental solutions that satisfy condition (1.6) at $x = \infty$.

When $b > 1$, letting

$$b = \cosh 2\mu$$

we find four solutions

$$\cos \nu\xi, \quad \sin \nu\xi, \quad \cos(\xi/\nu), \quad \sin(\xi/\nu),$$

where

$$\nu = \exp(\mu).$$

These solutions do not satisfy the boundary condition (1.6) at $x = \infty$, and there is no fundamental solution in the case of $b > 1$.

When $b = 1$, we find two solutions

$$\cos x, \quad \sin x.$$

Therefore, there is no fundamental solution in the case of $b = 1$.

When $0 \leqq b < 1$, letting

$$b = \cos 2\mu \tag{C.2-6}$$

where μ must be restricted to the range

$$0 < \mu \leqq \pi/4 \tag{C.2-7}$$

we find the fundamental solution, which we write in the complex form

$$W = e^{-(\beta+i\nu)\xi} \tag{C.2-8}$$

where

$$\beta = \sin\mu$$

$$\nu = \cos\mu.$$

The real fundamental solutions w_1 and w_2 are the real and imaginary parts of W

$$W = w_1 + iw_2. \tag{C.2-9}$$

The general solution w is given by

$$w = Aw_1 + Bw_2$$

where A and B are arbitrary constants. Note that β is positive because of (C.2-7).

The *free-edge condition* for the semi-infinite plate is that

$$\frac{d^2w}{d\xi^2} = 0$$

$$\frac{d^3w}{d\xi^3} + 2b\frac{dw}{d\xi} = 0$$

at $\xi = 0$. The latter is derived from $(1.10)_1$, which in this case becomes

$$-D\frac{d^3w}{dx^3} - p\frac{dw}{dx} = 0.$$

The determinant to evaluate the eigenvalue is

$$D_1 = \begin{vmatrix} w_1'' & w_1''' + 2bw_1' \\ w_2'' & w_2''' + 2bw_2' \end{vmatrix}$$

which we transform to the complex form

$$D_1 = \frac{1}{-2i} \begin{vmatrix} W'' & W''' + 2bW' \\ \overline{W}'' & \overline{W}''' + 2b\overline{W}' \end{vmatrix}$$

to facilitate the computation, where the upper bar signifies taking the conjugate complex number. Substituting (C.2-8), we find the eigenvalue

$$b = \frac{1}{2}$$

i.e.,

$$\mu = \frac{\pi}{6}, \ \nu = \frac{\sqrt{3}}{2} .$$

Therefore, (C.2-3) yields (C.1-4) by use of (C.2-4).

To determine the deflection, we must calculate

$$\frac{A}{B} = -\frac{w_2''}{w_1''}$$

from which we find

$$\frac{A}{B} = \tan\frac{\pi}{3} .$$

Thus, (C.2-9) transforms to

$$w = K\, e^{-\xi/2} \cos\left(\frac{\sqrt{3}}{2}\,\xi + \frac{\pi}{6}\right) \qquad \text{(C.2-10)}$$

where K is an arbitrary constant. The extrema of this curve occur when

$$\frac{\sqrt{3}}{2}\,\xi + \frac{\pi}{3} = n\pi \qquad \text{(C.2-11)}$$

where n is a non-negative integer.

The *simple-edge condition* is that

$$w = 0$$

$$\frac{d^2 w}{d\xi^2} = 0.$$

The determinant to evaluate the eigenvalue is

$$D_2 = \begin{vmatrix} w_1 & w_1'' \\ w_2 & w_2'' \end{vmatrix}$$

which we transform to the complex form

$$D_2 = \frac{1}{-2i} \begin{vmatrix} W & W'' \\ \overline{W} & \overline{W}'' \end{vmatrix}$$

to facilitate the computation. The eigenroots thus determined are outside the range of (C.2-7), but one of them is at the border

$$\mu = 0.$$

The general solution for this case

$$w = A \cos \xi + B \sin \xi \tag{C.2-12}$$

does not satisfy the boundary condition at $\xi = \infty$. If we substitute the eigenvalue $b = 1$ of this unacceptable solution into (C.2-3), we find (C.1-6) by use of (C.2-4).

The *clamped-edge condition* is that

$$w = 0$$

$$w' = 0.$$

The determinant to evaluate the eigenvalue is

$$D_3 = \begin{vmatrix} w_1 & w_1' \\ w_2 & w_2' \end{vmatrix}$$

which we transform to the complex form

$$D_3 = \begin{vmatrix} W & W' \\ \overline{W} & \overline{W}' \end{vmatrix}$$

The eigenroots thus determined are

$$\mu = \pi/2 + n\pi$$

where n is an integer. None of them satisfy the condition (C.2-7).

C.3. Asymptotic deflection

The nondimensional internal radius x_0 is evaluated in the case of the free-edge condition by x_h in (14.13) to be

$$x_0 = \kappa/\sqrt{4(2 \mp \sqrt{3})} \tag{C.3-1}$$

and in the case of the simple-edge condition by x in (15.5) to be

$$x_0 = \kappa/2. \tag{C.3-2}$$

We shall derive the asymptotic buckling deflections for the free- and simple-edge conditions by use of the asymptotic complex fundamental solution $F(x)$ in (12.6).

Decomposing $F(\ )$, we find

$$\begin{aligned} w_1 &= R\cos l \\ w_2 &= R\sin l \end{aligned} \tag{C.3-3}$$

where

$$R(x) = \sqrt{\pi/x}\,\exp[-(x/\sqrt{2}) - \kappa^2/(4\sqrt{2}\,x)]$$

$$l(x) = x/\sqrt{2} - \kappa^2/(4\sqrt{2}\,x) + \pi/8.$$

The *simple-edge condition* (1.9) yields the ratio

$$\frac{A}{B} = -\frac{L(w_2)_{x_0}}{L(w_1)_{x_0}} = -\frac{M(w_2)_{x_0}}{M(w_1)_{x_0}}$$

where L and M are operators defined by (14.1) and (14.2), respectively, and the suffix x_0 signifies that those attached with this must be evaluated at x_0. Use of F yields

$$L(F) = \left\{\beta^2 - 2\zeta_h^2 + \beta^{-2}\zeta_h^4\right\}F$$

$$M(F) = \left\{(\beta - \beta^{-1}\zeta_h)^3 + 4\zeta_h(\beta - \beta^{-1}\zeta_h)\right\}F$$

where ζ_h is defined by (14.14). Decomposing $L(F)$ and $M(F)$ into the real and imaginary parts and using the relation

$$1 - 4x + x^2 = 0$$

satisfied by $x = \zeta_h$, we find A and B

$$A = K\left\{(1+\zeta_h)\cos[l(x_0)] + (1-\zeta_h)\sin[l(x_0)]\right\}$$

$$B = K\left\{(1+\zeta_h)\sin[l(x_0)] - (1-\zeta_h)\cos[l(x_0)]\right\}$$

where K is arbitrary. Imposing the normalization

$$w(x_0) = 1$$

K is determined

$$K = [R(x_0)(1+\zeta_h)]^{-1}.$$

Thus the normalized deformation $w_N(x)$ is given by

$$w_N(x) = [R(x)/R(x_0)]\left\{\cos[l(x) - l(x_0)] \mp 3^{-1/2}\sin[l(x) - l(x_0)]\right\}.$$

To compare this with the semi-infinite plate solution, let

$$x = x_0 + \eta.$$

Assuming η to be negligible as compared with x_0, $w_N(x)$ transforms to

$$w_N(\eta) = \exp(-\eta/\sqrt{2})\,[\cos(\eta/\sqrt{2}) \mp (\sqrt{3})^{-1} \sin(\eta/\sqrt{2})]$$

$$= \frac{2}{\sqrt{3}}\, e^{-\eta/\sqrt{2}} \cos\left(\frac{\eta}{\sqrt{2}} + \frac{\pi}{6}\right).$$

Letting

$$\eta = \xi/\sqrt{2}$$

this transforms to

$$w_N(\xi) = \frac{2}{\sqrt{3}}\, e^{-\xi/2} \cos\left(\frac{\xi}{2} + \frac{\pi}{6}\right). \qquad \text{(C.3-4)}$$

The extrema of this curve occur when

$$\xi - \frac{\pi}{12} = (2n+1)\pi$$

where n is an integer. This curve is close to the curve of (C.2-10) but is different.

The *simple-edge condition* $(1.8)_1$ yields the ratio

$$\frac{A}{B} = -\tan\frac{\pi}{8}$$

when (C.3-2) and (C.3-3) are used. Letting

$$A = -K\sin(\pi/8)$$

$$B = K\cos(\pi/8)$$

where K is arbitrary, we find the deflection

$$w(x) = KR(x)\sin[x/\sqrt{2} - \kappa^2/(4\sqrt{2}\,x)].$$

To compare this with the semi-infinite plate solution, let

$$x = \kappa/2 + \eta.$$

Assuming η to be negligible as compared with $\kappa/2$, and letting

$$K = \sqrt{\kappa/(2\pi)}\,\exp(\kappa/\sqrt{2})$$

we find

$$w(x) = \sin(\sqrt{2}\,\eta).$$

This is a case of the general solution (C.2-12), as may be seen by letting

$$\sqrt{2}\,\eta = \xi.$$

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Takagi, Shunsuke
The buckling pressure of an elastic plate floating on water and stressed uniformly along the periphery of an internal hole / by Shunsuke Takagi. Hanover, N.H.: U.S. Army Cold Regions Research and Engineering Laboratory, 1978.
1 v.; 55 p; illus.; 27 cm. (CRREL Report 78-14)
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Bibliography: p. 36.
1. Buckling. 2. Deflections. 3. Eigenvalue solutions. 4. Floating elastic plate. 5. Shore structures. I. Takagi, Shunsuke. II. U.S. Army Cold Regions Research and Engineering Laboratory, Hanover, N.H., CRREL Report 78-14.