

AD-A057 307

ARIZONA UNIV TUCSON DEPT OF AEROSPACE AND MECHANICA--ETC F/G 12/1  
DECOUPLING OF TWO-TIME-SCALE LINEAR SYSTEMS.(U)  
AUG 78 L R ANDERSON

N00014-76-C-0326

UNCLASSIFIED

23

NL

1 OF 1  
ADA  
057307



END  
DATE  
FILMED  
9-78  
DDC

AD A057307

*R*  
*See back page*  
*for 1473*

**LEVEL II**



DECOUPLING OF TWO-TIME-SCALE LINEAR SYSTEMS

Leonard Anderson  
Aerospace and Mechanical Engineering Department  
University of Arizona  
Tucson, Arizona 85721

DDC  
RECEIVED  
AUG 10 1978  
D

ABSTRACT

Recent studies of two-time-scale linear systems have used a simple transformation of variables to block-diagonalize the system so that fast and slow modes are decoupled. A matrix used in this transformation satisfies an algebraic Riccati equation. In this study it is shown that this transformation is unique and its eigenstructure is described. Also, a new method of computing the decoupling transformation is presented and demonstrated using the 16th-order linear model of a turbofan engine.

INTRODUCTION

Consider the system of linear first-order differential equations

$$\dot{x} = Ax \tag{1}$$

where  $x$  is an  $n$ -dimensional vector and  $A$  is a real, constant matrix. As proposed by Chow and Kokotovic (1976), the system (1) will be called two-time-scale if the eigenvalues of  $A$ , represented as  $\lambda(A)$ , can be separated by absolute value into nonempty sets  $S$  and  $F$  so that

$$|s_i| \ll |f_j| \text{ for all } s_i \in S \text{ and } f_j \in F. \tag{2}$$

Let  $n_1$  be the number of eigenvalues in  $S$ , and order them so that

$$|s_i| \leq |s_{i+1}|, \quad i = 1, 2, \dots, n_1 - 1$$

$$|f_j| \leq |f_{j+1}|, \quad j = 1, 2, \dots, n_2 - 1,$$

where  $n_2 = n - n_1$ .

For any two-time-scale system the ratio

$$u = \frac{|s_{n_1}|}{|f_1|} \ll 1 \tag{3}$$

will define the system small parameter needed to measure the system's time-scale separation. Such systems are also described as stiff, ill-conditioned or singularly perturbed.

Given the system with  $n_1$  slow modes and  $n_2$  fast modes, partition the state vector  $x$  into subvectors  $x_1$  and  $x_2$  of dimension  $n_1$  and  $n_2$ , and partition the  $A$  matrix accordingly so (1) can be written

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{4}$$

The following two step transformation has been used by Chang (1972) and Kokotovic (1975) to first reduce (4) to block-triangular form and then to block-diagonal form. If the  $n_2 \times n_1$  matrix  $L$  satisfies the nonsymmetric algebraic Riccati equation

AD No. \_\_\_\_\_  
DDC FILE COPY

**DISTRIBUTION STATEMENT A**  
Approved for public release;  
Distribution Unlimited

78 08 09 053



where  $Q = M^{-1}$ . The  $M$ ,  $J$  and  $Q$  matrices are partitioned compatibly with the partitioned system (4).

If the  $A$  matrix is nondefective, i.e., has a full set of linearly independent eigenvectors, then  $J$  is diagonal with eigenvalues appearing along the diagonal. The  $M$  matrix is the modal matrix whose columns are the corresponding (right) eigenvectors of  $A$ . If  $A$  is defective then  $J$  contains one or more Jordan blocks and some of the columns of  $M$  are generalized eigenvectors.  $Q$  is the matrix of reciprocal basis vectors whose rows are the left eigenvectors and left generalized eigenvectors of  $A$ . The existence and uniqueness of  $M$  and  $J$  are discussed in many linear algebra texts such as Stewart (1973).

For the case where  $A$  has multiple eigenvalues it can be difficult to get a unique representation of  $M$ . However, if  $\lambda$  is an eigenvalue with multiplicity  $m$ , there is a unique  $m$ -dimensional subspace spanned by the eigenvectors and generalized eigenvectors corresponding to  $\lambda$ .

Narasimhamurthi and Wu (1977) and Veljko (1977) have described the eigenstructure of the solution to the algebraic Riccati equation (5). Those results are given below as Theorem 1 with a simplified proof. Potter (1966), Martensson (1971) and Kucera (1972) presented similar results for the symmetric algebraic Riccati equation which arises from the linear regulator problem.

In Theorem 1 we need not assume that  $A$  is two-time-scale, but only that partition (14) is possible. That is, that the Jordan matrix  $J$  can be partitioned into submatrices  $J_1$ ,  $J_2$  without splitting a Jordan block. Also, the  $L$  matrix may be complex.

**Theorem 1.** Matrix  $L$  satisfies the algebraic Riccati equation (5) if and only if  $L$  satisfies the linear equation

$$LM_{11} = -M_{21} \quad (15)$$

for  $A = M J Q$  partitioned as (14).

Proof: Assume first that (5) is satisfied. Rewrite (5) as  $L(A_{11} - A_{12}L) = -A_{21} + A_{22}L$  and let  $(A_{11} - A_{12}L)$  have Jordan form  $XGX^{-1}$ . Setting  $Y = -LX$ , it follows that

$$\begin{aligned} A_{11} - A_{12}L &= XGX^{-1} \\ A_{21} - A_{22}L &= -LXGX^{-1}. \end{aligned}$$

Post-multiply by  $X$  to obtain

$$\begin{aligned} A_{11}X + A_{12}Y &= XG \\ A_{21}X + A_{22}Y &= YG \end{aligned}$$

or

$$A \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} G.$$

Thus the diagonal elements of the Jordan form  $G$  are  $n_1$  of the  $n$  eigenvalues of  $A$ , and

$$\begin{bmatrix} X \\ Y \end{bmatrix}$$

are  $n_1$  corresponding eigenvectors of  $A$ . Also  $LX = -Y$ , completing the first half of the proof.

Suppose  $LM_{11} = -M_{21}$ . We need to show that the lower left block of the product

$$\begin{bmatrix} L & 0 \\ L & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -L & I \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}L & A_{12} \\ LA_{11} + A_{21} - LA_{12}L - A_{22}L & A_{22} + LA_{12} \end{bmatrix}$$

is zero.

Since  $M$  is full rank, the matrices

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} = \begin{bmatrix} I \\ -L \end{bmatrix} M_{11} \quad \text{and} \quad M_{11}$$

are both full rank and  $L$  can be represented as  $-M_{21}M_{11}^{-1}$ . Also, since  $QM = I$ , it follows that

$$Q_{21}M_{11} + Q_{22}M_{21} = 0$$

so the  $L$  matrix can also be represented as

$$Q_{22}L = Q_{21} \quad \text{or} \quad L = Q_{22}^{-1}Q_{21}. \quad (16)$$

Write  $A$  in the Jordan form and complete the product so that

$$\begin{bmatrix} I & 0 \\ -M_{21}M_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Q_{22}^{-1}Q_{21} & I \end{bmatrix} = \\ \begin{bmatrix} M_{11}J_1M_{11}^{-1} & M_{11}J_1Q_{12} + M_{12}J_2Q_{22} \\ 0 & Q_{22}^{-1}J_2Q_{22} \end{bmatrix} = \begin{bmatrix} B_1 & A_{12} \\ 0 & B_2 \end{bmatrix},$$

completing the proof.

In the final expression we use the identities

$$M_{11}^{-1} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{21}$$

and

$$Q_{22}^{-1} = M_{22} - M_{21}M_{11}^{-1}M_{12}.$$

As this theorem illustrates, if  $L$  satisfies (15), then the eigenvalues and eigenvectors of  $B_1$  and  $B_2$  are  $M_{11}$ ,  $J_1$  and  $Q_{22}^{-1}$ ,  $J_2$  respectively. The case of generalized eigenvectors causes no difficulty in the proof since any Jordan block appearing in  $G$  also appears in  $J_1$ , and vice versa.

The conditions which insure that  $L$  is real also follow quite readily from this theorem. That is, if  $A$  has complex eigenvalue  $\lambda$  appearing in  $J_1$ , then  $\lambda$  and  $\bar{\lambda}^\dagger$  must only appear in  $J_1$ . Under this restriction there will always exist a nonsingular complex matrix  $C$  such that

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} C = \begin{bmatrix} M'_{11} \\ M'_{21} \end{bmatrix}$$

is real. If  $L$  satisfies  $LM_{11} = -M_{21}$  then  $L$  also satisfies  $LM'_{11} = M'_{21}$  so that  $L$  is real.

The general form of the block-diagonalization transformation (12) can be determined from Theorem 1. Comparing the Jordan forms  $A = MJQ$  and

$$\begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} = \begin{bmatrix} M_{11} & 0 \\ 0 & Q_{22}^{-1} \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} M_{11}^{-1} & 0 \\ 0 & Q_{22} \end{bmatrix}$$

it is apparent that

$$\begin{bmatrix} I+KL & K \\ L & I \end{bmatrix} = \begin{bmatrix} M_{11}Q_{11} & M_{11}Q_{12} \\ Q_{22}^{-1}Q_{21} & I \end{bmatrix}.$$

<sup>†</sup> complex conjugate

### UNIQUENESS OF THE L AND K MATRICES

As shown by Narasimhamurthi and Wu (1977), only under very restrictive conditions will the algebraic Riccati equation have a unique solution. Namely, only when the quadratic term is zero and the equation becomes linear. However, there is a unique solution which decouples a two-time-scale system with slow and fast modes separated. This is proven in the following theorem.

**Theorem 2.** For a given two-time-scale system A there is a unique decoupling matrix L satisfying both (5) and (11).

**Proof:** Assume that L and L' both satisfy (5) and (11). Applying Theorem 1 there exist full rank matrices

$$\begin{bmatrix} X \\ Y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} X' \\ Y' \end{bmatrix} \quad (17)$$

such that  $LX = -Y$  and  $L'X' = -Y'$ . Moreover, the columns are (generalized) eigenvectors of A corresponding to eigenvalues S. Since there is a unique  $n_1$ -dimensional subspace spanned by these (generalized) eigenvectors, there exists matrix C such that

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix} C.$$

Post-multiply  $L'X' = -Y'$  by C to obtain  $L'X = -Y$ . As shown in the proof of Theorem 1,  $X^{-1}$  exists. Thus  $L' = L = -YX^{-1}$ , completing the proof.

Although the decoupling matrix L is unique for a given two-time-scale system, there are generally many real solutions to the algebraic Riccati equation (5). Thus any effort to obtain L should avoid these other solutions. In a special case the number of distinct solutions can be enumerated by

**Corollary 1.** If A has distinct real eigenvalues, then there are  $\binom{n}{n_1}$  distinct real solutions of (5) corresponding to the number of distinct ways of partitioning the n eigenvalues of A into sets of  $n_1$  and  $n_2$  members.

**Proof:** Since M is real, all solutions of (5) are real. We will assume that  $\lambda(A_{11} - A_{12}L) \neq \lambda(A_{11} - A_{12}L')$  and prove that this implies  $L \neq L'$ . Let  $LX = -Y$  and  $L'X' = -Y'$ . The linear independence of the eigenvectors corresponding to the  $n_1 + 1$  or more eigenvalues  $\lambda(A_{11} - A_{12}L) \cup \lambda(A_{11} - A_{12}L')$  implies that the augmented matrix

$$\begin{bmatrix} X & X' \\ Y & Y' \end{bmatrix} \quad (18)$$

has rank at least  $n_1 + 1$ . Then  $L = -YX^{-1}$  and  $L' = -Y'(X')^{-1}$ . If we assume that  $L = L'$  so that  $YX^{-1} = Y'(X')^{-1}$  and  $Y = Y'(X')^{-1}X$ , then

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y'(X')^{-1}X \end{bmatrix} = \begin{bmatrix} X' \\ Y' \end{bmatrix} (X')^{-1}X$$

which implies that the  $n_1$  columns of  $\begin{bmatrix} X \\ Y \end{bmatrix}$  are linear combinations of the columns of  $\begin{bmatrix} X' \\ Y' \end{bmatrix}$  and the rank of (18) equals  $n_1$ , a contradiction. Thus  $L \neq L'$  and the proof is complete.

The uniqueness of the K matrix is insured by the two-time-scale property. As shown by Gantmacher (1959), the linear equation (8) has a unique solution provided  $B_1$  and  $B_2$  have no common eigenvalues.

### THE CHOICE OF SLOW AND FAST VARIABLES

An important question which arises in the decoupling and reducer-order modeling of large linear systems is how should state variables x be reordered so that states which contain predominately slow mode are placed in  $x_1$ . If we reorder the state variables then the rows of the modal matrix N are reordered in the same manner. From our previous results it is apparent that  $\|L\|$  is approximately proportional to  $\|M_{21}\|$  and inversely proportional to  $\|M_{11}\|$ . So if we can choose an ordering which minimizes  $\|M_{21}\|$ , this will insure that the

slow modes appear primarily in variables  $x_1$ , and  $\|L\|$  will be small. In a similar fashion, the  $K$  matrix can be written as

$$K = -M_{12}^{-1}Q_{22}$$

so that choosing an ordering of states which minimizes  $\|M_{12}\|$  will place primarily fast variables in  $x_2$  and yield a small  $\|K\|$ .

This process is clarified by introducing slow mode and fast mode coupling ratios  $\rho_s$  and  $\rho_f$ . Assume that the columns of  $M$  are normalized so that each is of length one, and define

$$\rho_s = \frac{\|M_{21}\|}{\|M_{11}\|}, \quad \rho_f = \frac{\|M_{12}\|}{\|M_{22}\|}.$$

Here  $\|\cdot\|$  refers to the Euclidean norm and, where necessary, take the modulus of complex entries. Either  $\rho_s$  or  $\rho_f$  can be minimized by searching over all orderings of the row indices  $\{i\}$  of the modal matrix  $M = [m_{ij}]$ . If it should happen that all rows of

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$$

have equal norm, then the minimum value of  $\rho_s$  will be  $(n_2/n_1)^{1/2}$ . On the other hand, if  $x$  can be ordered so that  $M_{21} = 0$ , then  $\rho_s = 0$  and the slow modes are naturally decoupled from the  $x_2$  states. For this ordering of  $x$ , the  $A$  matrix will be block-triangular. Likewise, if  $M_{12} = 0$ , then  $\rho_f$  is zero and the resulting  $A$  will be block-triangular.

Depending upon the structure of  $M$ , the ordering which minimizes  $\rho_s$  may or may not also minimize  $\rho_f$ . Depending upon whether we are primarily interested in studying the slow behavior or fast behavior of the system, we could choose the ordering of variables which minimizes either  $\rho_s$  or  $\rho_f$ .

If both  $\rho_s$  and  $\rho_f$  are small, we could describe the system as being "weakly coupled." This definition of weakly coupled systems is similar to one proposed by Milne (1965). However, his definition also required that  $\mu$  be small. Even though a system is strongly two-time-scale, it need not therefore be weakly coupled.

Provided one is able to compute eigenvectors, the following procedure is suggested for re-indexing  $x$ ,  $A$  and  $M$  before obtaining the decoupling transformation.

#### Algorithm 1.

1. Normalize the columns of

$$\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix} \tag{19}$$

so that  $\sum_{i=1}^n |m_{ij}|^2 = 1$ ,  $j = 1, \dots, n_1$ .

2. If (19) contains one or more complex eigenvectors  $v_i$ , replace  $v_i$  with  $\text{Re}(v_i)$  and replace  $\bar{v}_i$  with  $\text{Im}(v_i)$ .

3. Evaluate the norm of each of the  $n$  rows of (19) and let

$$r_i = \left( \sum_{j=1}^{n_1} m_{ij}^2 \right)^{1/2}, \quad i = 1, \dots, n.$$

4. Order the set  $\{r_i\}$  so that  $r_{\alpha_1} \geq r_{\alpha_2} \geq \dots \geq r_{\alpha_n}$  and premultiply  $x$ ,  $A$  and  $M$  by the permutation matrix

$$\begin{bmatrix} e_{\alpha_1} & e_{\alpha_2} & \dots & e_{\alpha_n} \end{bmatrix} \tag{20}$$

where  $e_i$  is the  $i$ -th column of the  $n \times n$  identity matrix.

5. Post-multiply  $A$  by the inverse permutation matrix

$$\begin{bmatrix} e_{\alpha_1} & e_{\alpha_2} & \dots & e_{\alpha_n} \end{bmatrix}^T$$

where  $[\cdot]^T$  indicates transpose.

This general method for systematically picking slow variables should be very useful in applications as discussed by Teneketzis and Sandell (1976). In general not all eigenvectors are required, but only those corresponding to the slow modes. However, one question which needs further exploration is how scaling of the state variables influences this method.

#### COMPUTING THE L AND K MATRICES

Kokotovic (1975) proposed a method of computing the L matrix starting with initial approximation

$$L_0 = A_{22}^{-1} A_{21} \quad (21)$$

and using the iteration

$$L_{i+1} = A_{22}^{-1} (L_i (A_{11} - A_{12} L_i) + A_{21}). \quad (22)$$

He proved that if a certain criterion involving matrix norms is satisfied then this iteration will converge to L. As he noted, the criterion is quite restrictive and is not satisfied by many typical two-time-scale systems.

If efficient eigenanalysis programs are available, then a computational procedure for finding accurate numerical L and K matrices based on the results of Section 2 is quite straightforward. Examples of such programs are EISPACK from the Argonne National Laboratory, described by Smith et al. (1976), or EIGRF from the International Mathematics and Statistics Library (IMSL), Houston, Texas. These eigenanalysis programs are well documented and easy to implement on mainframe computers. Typical execution times for finding all eigenvalues and eigenvectors of matrices of order 40 and 80 are .66 and 4.6 seconds using an IBM 370/195 computer, cf. Smith et al. (1976). Both programs are based upon the QR algorithm, cf. Wilkinson (1965).

Since the L matrix can be expressed as either  $-M_1 M_1^{-1}$  or  $Q_{22}^{-1} Q_{21}$ , not all eigenvectors of A need be evaluated to solve for L. Two procedures could be used, one for the case  $n_1 \leq n_2$  and another for  $n_1 > n_2$ . For  $n_1 \leq n_2$ , obtain  $n_1$  (right) eigenvectors corresponding to the slow eigenvalues S and solve the set of linear equations (15). Such equations can be efficiently solved by some Gaussian elimination scheme such as LU decomposition, e.g., Stewart (1973). For  $n_1 > n_2$ , find the  $n_2$  left eigenvectors  $(Q_{21}, Q_{22})$  corresponding to the fast eigenvalues F and solve the linear system (16). In the event that complex eigenvectors are encountered, apply step 2 of Algorithm 1 to eliminate complex entries in (15) or (16).

The accuracy of the matrix  $L_0$  obtained from (15) or (16) can be evaluated by substituting it back into the Riccati equation (5) and evaluating the residual error matrix

$$R_0 = L_0 A_{11} + A_{21} - L_0 A_{12} L_0 - A_{22} L_0.$$

If  $R_0$  is judged to be not sufficiently small, i.e.,  $\|R_0\| > \epsilon$  for some small number  $\epsilon$ , then the accuracy of  $L_0$  can be improved by an iterative procedure. If  $L_0$  is a reasonable approximation to L then the matrix  $(A_{22} + L_0 A_{12})$  has large eigenvalues, is nonsingular, and suggests the linear iteration

$$L_{i+1} = (A_{22} + L_i A_{12})^{-1} (L_i A_{11} + A_{21}).$$

This iteration can be rewritten as

#### Algorithm 2.

1. Obtain initial approximation  $L_0$  from either (15) or (16) and set  $i = 0$ .
2. Evaluate  $R_i = L_i A_{11} + A_{21} - L_i A_{12} L_i - A_{22} L_i$  and stop if  $\|R_i\| \leq \epsilon$ .
3. Solve  $(A_{22} + L_i A_{12}) D_i = R_i$  for  $D_i$  and let  $L_{i+1} = L_i + D_i$ .
4. Let  $i = i + 1$  and go to 2.

It can be shown that for  $L_0$  close to L, the proposed iteration converges with approximate rate  $\mu$ . That is,

$$R_{i+1} = (L_i + D_i)A_{11} + A_{21} - (L_i + D_i)A_{12}(L_i + D_i) - A_{22}(L_i + D_i)$$

$$R_{i+1} = R_i + D_i(A_{11} - A_{12}L_i) - (A_{22} + L_iA_{12})D_i - D_iA_{12}D_i$$

and taking the correction matrix  $D_i$  as in step 3 above

$$\begin{aligned} R_{i+1} &= (A_{22} + L_iA_{12})^{-1}R_i(A_{11} - A_{12}L_i - A_{12}D_i) \\ &= (A_{22} + L_iA_{12})^{-1}R_i(A_{11} - A_{12}L_{i+1}). \end{aligned}$$

The largest eigenvalues of  $(A_{22} + LA_{12})^{-1}$  and  $(A_{11} - A_{12}L)$  are  $1/\epsilon_1$  and  $s_{n_1}$ , so that if  $L_0$  is close to  $L$ , then

$$\frac{|R_{i+1}|}{|R_i|} \sim \frac{|s_{n_1}|}{|\epsilon_1|} = \mu \ll 1.$$

The convergence of the iteration defined by Algorithm 2 could be insured by including a line search in step 3. That is, if  $|R_{i+1}| > |R_i|$ , then let  $L_{i+1} = L_i + hD_i$  and search on  $h$  until  $|R_{i+1}|$  is minimized. However, this safeguard procedure was not needed in the example which follows.

This procedure for obtaining numerical solutions to the nonsymmetric Riccati equation has some similarities to one proposed by Farrar and Di Pietro (1977) for solving the symmetric Riccati equation arising in the linear quadratic regulator problem in that both procedures are initialized using eigenvectors. However, their iterative correction involves a bilinear equation whereas Algorithm 2 involves a linear iterative correction.

Algorithm 3 listed below can be used to compute the  $K$  matrix. Based on the previous argument that the eigenvalues of  $B_2$  are all larger than those of  $B_1$ , it can be shown that the rate of convergence will approximately equal the system small parameter.

**Algorithm 3.**

1. Set  $K_0 = 0$ ,  $R_0 = -A_{12}$ , and  $i = 0$ .
2. Solve  $D_i B_2 = R_i$  for  $D_i$  and let  $K_{i+1} = K_i + D_i$ .
3. Evaluate  $R_{i+1} = K_{i+1} B_2 - B_1 K_{i+1} + A_{12}$  and stop if  $|R_{i+1}| \leq \epsilon$ .
4. Set  $i = i + 1$  and go to step 2.

An advantage of this approach is that  $B_2$  is unchanged in the iteration so that it can be decomposed once into LU form and then used repeatedly in step 2 to find  $D_i$ .

To summarize, the proposed procedure for computing the decoupling transformation for a two-time-scale system involves the following steps.

1. Obtain approximate eigenvalues  $\lambda(A)$  and choose the number of slow modes  $n_1$ . Note that different choices of  $n_1$  are possible to suit the needs of the study.
2. If  $n_1 \leq n_2$ , obtain approximate eigenvectors  $\begin{bmatrix} M_{11} \\ M_{21} \end{bmatrix}$ . Otherwise, obtain left eigenvectors  $\begin{bmatrix} Q_{21} \\ Q_{22} \end{bmatrix}$ .
3. (Optional) Apply Algorithm 1 to place slow variables in  $x_1$ .
4. Eliminate imaginary parts as needed, and solve either (15) or (16) for  $L_0$ .
5. (Optional) Improve the accuracy of  $L_0$  by Algorithm 2.
6. Find  $K$  by Algorithm 3.

This procedure has been successfully applied to five models of physical systems with orders ranging from 4 to 32. In all cases Algorithms 2 and 3 converged with approximate rate  $\mu$  and it was verified that the eigenvalue decoupling conditions (11) were satisfied. However, for the sake of brevity, only one of these examples will be presented here.

#### TURBOFAN EXAMPLE

This example is a 16th order model of a turbofan engine which was the theme problem for a recent conference on control of linear multivariable systems, i.e., Sain (1977). This model is the linearization of a detailed nonlinear simulation of the turbofan at the sea level static intermediate thrust operating point. Two of the state variables are shaft speeds, three are pressures and eleven are temperatures. The eigenvalues for the linear system are  $-0.65$ ,  $-1.90$ ,  $-2.62$ ,  $-6.72 \pm j1.31$ ,  $-17.8 \pm j4.8$ ,  $-18.6$ ,  $-21.3 \pm j0.8$ ,  $-38.7$ ,  $-47.1$ ,  $-50.7$ ,  $-59.2$ ,  $-175.7$ , and  $-577.0$ . For  $n_1 = 15$ ,  $5$  and  $3$ , the resulting small parameters are  $0.304$ ,  $0.371$ , and  $0.383$ . Since these values are not particularly small relative to one, we might call this system weakly two-time-scale.

The computations were performed at the University of Arizona Computing Center on both the DEC 10 and CYBER 175 computers to compare the effects of computer word length and compare execution times for time sharing against batch processing.

The performance of Algorithms 2 and 3 for the two cases  $n_1 = 3$  and  $n_1 = 5$  is illustrated in Figures 1 through 4. The state variables were not reordered for this computation, i.e., Algorithm 1 was not employed. As shown, in all cases the convergence rate approximately equals the small parameter. The corresponding computation times are listed in Table 1. The six orders-of-magnitude difference between the minimum residual errors obtained with the DEC 10 and CYBER computers can be directly attributed to their 36 and 64 bit word lengths. Both EISPACK and IMSL were tested in the computation of  $L_0$ . Their performance in terms of eigenvalue accuracy was found to be very similar, with IMSL yielding somewhat shorter execution times. The execution times in Table 1 were obtained using EISPACK.

The effects of reordering the state variables before partitioning  $x$  into  $(x_1, x_2)^T$  are illustrated in Figure 5 for the case  $n_1 = 5$ . As shown, the choice of ordering does not in this case affect the rate of convergence of Algorithm 2, but strongly influences  $\rho_s$  and  $\|L\|$ .

The computation of the  $L$  matrix using alternate initial values  $L_0$  is described in Figure 6. It's interesting to note that Algorithm 2 succeeds in converging to the correct solution when initialized with  $L_0 = A_{22}^{-1}A_{21}$ , i.e., equation (21), or with  $L_0 = 0$ . This demonstrates that in some cases the decoupling transformation can be obtained without the use of eigenvectors. However, the  $L$  matrix obtained using eigenvectors provides a much more reliable initial value for Algorithm 2.

#### CONCLUSIONS

This work considers the properties of a transformation used by Chang (1972) and Kokotovic (1975) to transform linear systems into block-diagonal form with slow and fast modes decoupled. The eigenstructure of this transformation is described, and it is shown that for any given two-time-scale linear system the decoupling transformation is unique.

General conditions for the existence of solutions of the nonsymmetric algebraic Riccati equation are established which are more general than existence conditions previously stated by Narasimhamurthi and Wu (1977). It is also shown that when the corresponding  $A$  matrix has distinct real eigenvalues, the distinct real solutions to the Riccati equation can be specifically enumerated.

A new method of computing the decoupling transformation is proposed. With this method the number of slow modes can be chosen to meet the needs of the study provided multiple or complex eigenvalues are not separated. The method includes algorithms for selecting slow system variables, and computing accurate  $L$  and  $K$  matrices. Although system eigenvectors are used in this method, not all eigenvectors need be found, and highly accurate eigenvectors are not required. A 16th order linear model of a turbofan is decoupled by this method to demonstrate computation times, convergence rates, and algorithm stability.

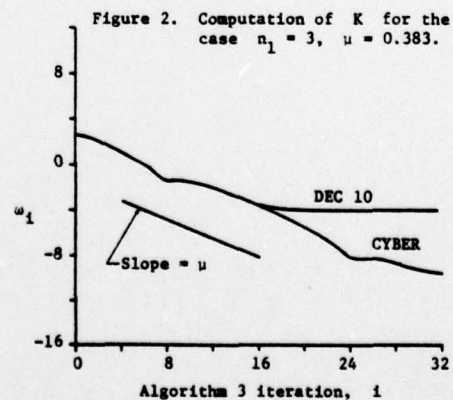
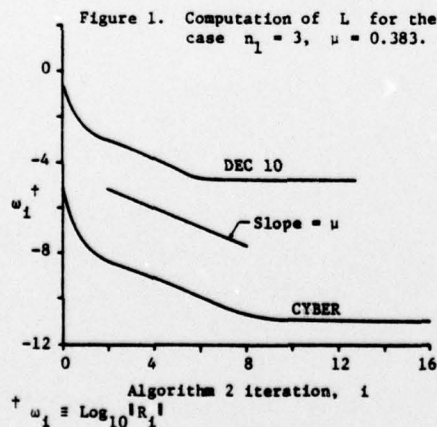
#### ACKNOWLEDGMENT

This research was supported in part by the Office of Naval Research under Contract Number N00014-76-C-0326 and by the Aerospace and Mechanical Engineering Department, University of Arizona.

#### REFERENCES

1. Chang, K. W., "Singular Perturbations of a General Boundary Value Problem," SIAM J. Math. Anal., Vol. 3, August 1972, 520-526.

2. Chow, J. H. and Kokotovic, P. V., "Eigenvalue Placement in Two-Time-Scale Systems," Proceedings of the IFAC Symposium on Large Scale Systems, Udine, Italy, 1976.
3. Farrar, F. A. and Di Pietro, R. C., "Comparative Evaluation of Numerical Methods for Solving the Algebraic Matrix Riccati Equation," Proceedings of the 1977 Joint Automatic Control Conference, San Francisco, June 22-24, 1977, 1543-1548.
4. Gantmacher, F. R., Matrix Theory, Vol. 1, Chelsea Publishing Company, New York, 1959.
5. Kokotovic, P. V., "A Riccati Equation for Block-Diagonalization of Ill-Conditioned Systems," IEEE Trans. Automatic Control, Vol. 20, No. 6, December 1975, 812-814.
6. Kucera, V., "A Contribution to Matrix Quadratic Equations," IEEE Trans. Automatic Control, Vol. 17, June 1972, 344-347.
7. Martensson, K., "On the Matrix Riccati Equation," Information Science, Vol. 3, 1971, 17-49.
8. Milne, R., "The Analysis of Weakly Coupled Dynamic Systems," Int. J. Control, Vol. 2, August 1965, 171-199.
9. Narasimhamurthi, N. and Wu, F. F., "On the Riccati Equation Arising from the Study of Singularly Perturbed Systems," Proceedings of the 1977 Joint Automatic Control Conference, San Francisco, June 22-24, 1977, 1244-1247.
10. Potter, J. E., "Matrix Quadratic Solutions," SIAM J. Applied Math., Vol. 14, No. 3, May 1966, 496-501.
11. Sain, M. K., "Theme Problem," Proceedings of the International Forum on Alternatives for Multivariable Control, Chicago, October 13-14, 1977, 1-12.
12. Smith, B. T., Boyle, J. M., Dongarra, J. J., Garbow, B. S., Ikebe, Y., Klema, V. C., and Moler, C. B., "Matrix Eigensystem Routines - EISPACK Guide," 2nd Ed., Lecture Notes in Computer Science, Vol. 6, Springer-Verlag, Berlin, 1976.
13. Stewart, G. W., Introduction to Matrix Computations, Academic Press, New York, 1973.
14. Teneketzis, D. and Sandell, Jr., N. R., "Linear Regulator Design for Stochastic Systems by a Multiple Time-Scales Method," IEEE Trans. Automatic Control, Vol. 22, No. 4, August 1977, 615-621.
15. Veljko, P., "On the Generalized Riccati Equation and Block-Diagonalization of Linear Systems," personal communication to P. V. Kokotovic, Department of Electrical Engineering, University of Illinois, Champaign-Urbana, Illinois, 1977.
16. Wilkinson, J. H., The Algebraic Eigenvalue Problem, Clarendon Press, Oxford, 1965.



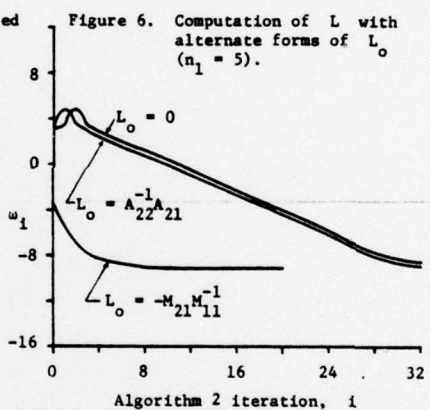
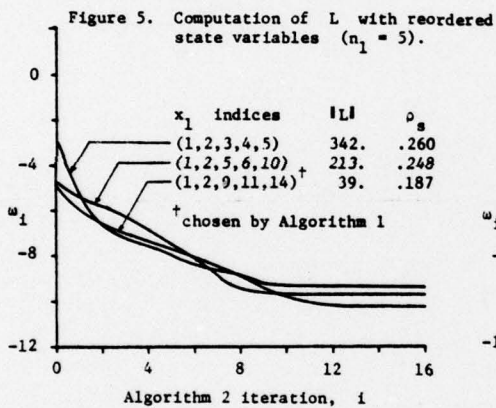
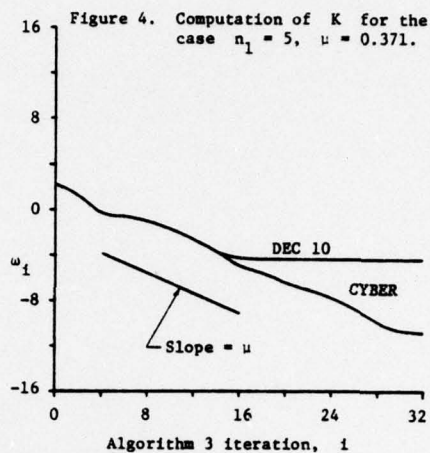
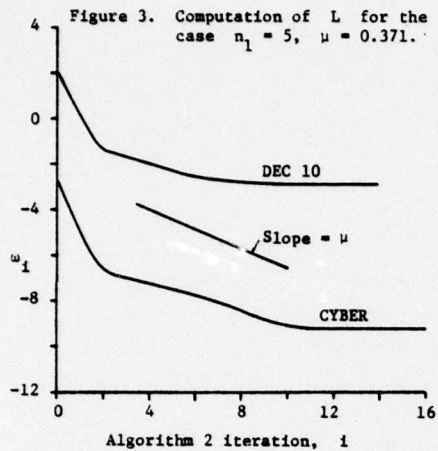


Table 1. Typical Execution Times (CPU seconds)

|  | Case $n_1 = 5$ |        | Case $n_1 = 3$ |        |
|--|----------------|--------|----------------|--------|
|  | CYBER          | DEC 10 | CYBER          | DEC 10 |
| Time to compute $L_0$                  | .109           | .615   | .096           | .599   |
| Time per $L_i$ iteration (Algorithm 2) | .023           | .187   | .021           | .171   |
| Time per $K_i$ iteration (Algorithm 3) | .015           | .126   | .012           | .091   |

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE  |                                       | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                                |
|--|---------------------------------------|--|
| 1. REPORT NUMBER<br>Twenty-Three   | 2. GOVT ACCESSION NO.<br><b>14 23</b> | 3. RECIPIENT'S CATALOG NUMBER  |
| 4. TITLE (and Subtitle)<br><b>Decoupling of Two-Time-Scale Linear Systems.</b>   |                                       | 5. TYPE OF REPORT & PERIOD COVERED<br><b>Technical Report</b>              |
| 7. AUTHOR(s)<br><b>R./Anderson</b>   |                                       | 6. PERFORMING ORG. REPORT NUMBER   |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Department of Mathematics<br>University of Arizona<br>Tucson, Arizona 85721   |                                       | 8. CONTRACT OR GRANT NUMBER(s)<br><b>15</b> <b>14-76-C-0326</b>            |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Mathematics Branch<br>Office of Naval Research<br>Arlington, Virginia   |                                       | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br>NR041-466   |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)  |                                       | 12. REPORT DATE<br><b>11</b> <b>August 1978</b>                            |
|  |                                       | 13. NUMBER OF PAGES<br>Eleven (11)   |
|  |                                       | 15. SECURITY CLASS. (of this report)<br>Unclassified <b>12</b> <b>12p.</b> |
|  |                                       | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE                                 |
| 16. DISTRIBUTION STATEMENT (of this Report)<br><br>Approved for public release. Distribution unlimited.  |                                       |  |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)   |                                       |  |
| 18. SUPPLEMENTARY NOTES  |                                       |  |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br><br>Singular perturbations, reduced-order modeling, modal decoupling, large scale systems  |                                       |  |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br><br>Recent studies of two-time-scale linear systems have used a simple transformation of variables to block-diagonalize the system so that fast and slow modes are decoupled. A matrix used in this transformation satisfies an algebraic Riccati equation. In this study it is shown that this transformation is unique and its eigenstructure is described. Also, a new method of computing the decoupling transformation is presented and demonstrated using the 16th-order linear model of a turbofan engine. |                                       |  |

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

402 192

Am