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A BILINEAR-QUADRATIC DIFFERENTIAL GAME IN ADVERTISING.(U)

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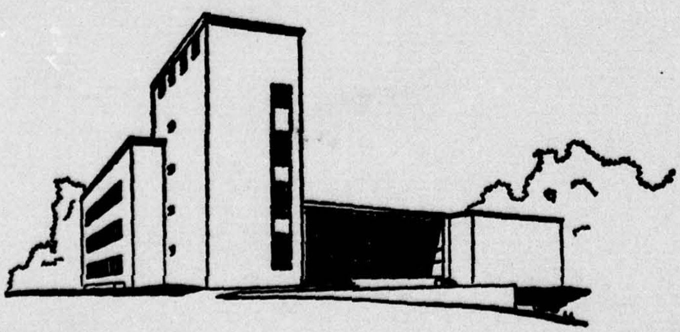

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by

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ABSTRACT

A duopolistic extension of the Vidale-Wolfe advertising model is formulated as a problem in differential games. An important feature of the problem is the presence of reaction terms driven by the difference in the advertising expenditures of the two duopolistic firms under consideration. Formula for the Nash turnpike is derived using the maximum principle. Several competitive encounters with different parameters of the model are numerically solved. Tentative conclusions are drawn by analyzing these results.

Key Words:

Advertising, Duopolistic Competition, Differential Game, Maximum Principle, Nash Turnpike, Bilinear Quadratic Problem, Boundary-Value Problem.

A BILINEAR-QUADRATIC DIFFERENTIAL GAME IN ADVERTISING

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1. INTRODUCTION

As one of the earliest management science applications to marketing, Vidale and Wolfe [1] developed a simple model of sales response to advertising that was consistent with their experimental observations. They argued that changes in rate of sales of a product depend on two effects, response to advertising which acts (via the response constant b) on the unsold portion of the market, and loss due to forgetting which acts (via the decay constant a) on the sold portion of the market. Thus

$$\dot{x} = bu(1-x) - ax, \quad x(0) = x_0 \tag{1.1}$$

where x is the market share (i.e., the rate of sales expressed as a fraction of the market potential or saturation level) and u is the rate of advertising expenditure (a control variable) at time t.

Whereas Vidale and Wolfe offered their model primarily as a description of actual market phenomena represented by cases which they had observed, Sethi [2,3] formulated an optimal advertising model with the Vidale-Wolfe model as its dynamics. He obtained complete solutions for many variants of the optimal control problem so formulated; see also Sethi [4].

Deal and Zionts [5] and Deal [6] offered a duopolostic extension of the Vidale-Wolfe advertising model and treated the problem by the differential game approach. With indices 1 and 2 denoting the two duopolists, respectively, we can describe their dynamics as follows:

$$\dot{x}_1 = b_1u_1(1-x_1-x_2) - a_1x_1, \quad x_1(0) = x_{10}$$
$$\dot{x}_2 = b_2u_2(1-x_1-x_2) - a_2x_2, \quad x_2(0) = x_{20}. \tag{1.2}$$

The main problem with their dynamics is the absence of a competitive term, i.e., in their model, the competitor's advertising expenditure has no direct effect on a firm's market share. This drawback has led us to the modified differential game model to be described in the next section.

The plan of this paper is as follows. Section 2 contains our formulation of the bi-linear quadratic differential game. In Section 3 we apply the appropriate maximum principle [7] to obtain necessary optimality conditions. Section 4 deals with the analysis of the infinite horizon version of the problem and obtains the steady state Nash equilibrium solution, which we term as Nash turnpike. Computational analysis of finite horizon versions of the problem is presented in Section 5. These problems are numerically solved by applying a two-point-boundary-value algorithm. We perform sensitivity analysis of the answers by varying the parameters of the problem. Section 6 concludes the paper.

2. THE BILINEAR QUADRATIC MODEL

The state equations of the bilinear model are modifications of (1.2); they are

$$\dot{x}_1 = b_1u_1(1-x_1-x_2) + e_1(u_1-u_2)(x_1+x_2) - a_1x_1, \quad x_1(0) = x_{10}$$
$$\dot{x}_2 = b_2u_2(1-x_1-x_2) + e_2(u_2-u_1)(x_1+x_2) - a_2x_2, \quad x_2(0) = x_{20}. \tag{2.1}$$

Subject to these equations, firm 1 wants to maximize

$$J_1 = w_1 e^{-\rho T} x_1(T) + \int_0^T (c_1 x_1 - u_1^2) e^{-\rho t} dt \tag{2.2}$$

while firm 2 wants to maximize

$$J_2 = w_2 e^{-\rho T} x_2(T) + \int_0^T (c_2 x_2 - u_2^2) e^{-\rho t} dt. \tag{2.3}$$

where ρ is the constant discount rate, c_1, c_2, w_1, w_2 are constants, and T is the horizon time. The problem defined by (2.1) - (2.3) is a non-zero sum differential game, see [7, 8].

The salient feature of this model is that it contains reaction terms of the form $e_1(u_1-u_2)(x_1+x_2)$ and $e_2(u_2-u_1)(x_1+x_2)$. To explain these terms assume $e_1 = e_2$ and $u_1 > u_2$. The term $e_1(u_1-u_2)x_2$ represents the persons who switched from firm 2 to firm 1 because of the excess advertising of firm 1; the term $e_1(u_1-u_2)x_1$ is an approximation (and simplification) of the Ozga type term $e_1(u_1-u_2)x_1x_2$ representing the number of people switching from firm 2 to firm 1 as a result of the indirect or word of mouth advertising stimulated by the excess advertising of firm 1. Since the total number of people going from firm 2 to firm 1 is $e_1(u_1-u_2)(x_1+x_2)$, these people must be subtracted from the second state equation. A similar analysis works when $u_1 < u_2$. Another reason for making these assumptions is to end up with a simple bilinear model.

3. APPLICATION OF THE MAXIMUM PRINCIPLE

To apply the maximum principle, we formulate a Hamiltonian for each player as follows:

$$H_1 = c_1x_1-u_1^2 + \lambda_1[b_1u_1(1-x_1-x_2) + e_1(u_1-u_2)(x_1+x_2) - a_1x_1]$$
$$+ \lambda_2[b_2u_2(1-x_1-x_2) + e_2(u_2-u_1)(x_1+x_2) - a_2x_2] \quad (3.1)$$

$$H_2 = c_2x_2-u_2^2 + \gamma_1[b_1u_1(1-x_1-x_2) + e_1(u_1-u_2)(x_1+x_2) - a_1x_1]$$
$$+ \gamma_2[b_2u_2(1-x_1-x_2) + e_2(u_2-u_1)(x_1+x_2) - a_2x_2] \quad (3.2)$$

where, the current-value adjoint variables λ_1 and λ_2 and γ_1 and γ_2 satisfy the following differential equations:

$$\dot{\lambda}_1 = \rho\lambda_1 - \frac{\partial H_1}{\partial x_1} = -c_1 + \lambda_1[\rho+a_1+b_1u_1 - e_1(u_1-u_2)]$$
$$+ \lambda_2[b_2u_2 - e_2(u_2-u_1)] \quad (3.3)$$

$$\dot{\lambda}_2 = \rho\lambda_2 - \frac{\partial H_1}{\partial x_2} = \lambda_1[b_1u_1 - e_1(u_1-u_2)]$$
$$+ \lambda_2[\rho + a_2 + b_2u_2 - e_2(u_2-u_1)] \quad (3.4)$$

$$\dot{\gamma}_1 = \rho\gamma_1 - \frac{\partial H_2}{\partial x_1} = \gamma_1[\rho + a_1 + b_1u_1 - e_1(u_1-u_2)]$$
$$+ \gamma_2[b_2u_2 - e_2(u_2-u_1)] \quad (3.5)$$

$$\dot{\gamma}_2 = \rho\gamma_2 - \frac{\partial H_2}{\partial x_2} = -c_2 + \gamma_1[b_1u_1 - e_1(u_1-u_2)]$$
$$+ \gamma_2[\rho + a_2 + b_2u_2 - e_2(u_2-u_1)]. \quad (3.6)$$

Also, these adjoint variables satisfy the transversality conditions:

$$\lambda_1(T) = w_1 \quad , \quad \lambda_2(T) = 0$$
$$\gamma_1(T) = 0 \quad , \quad \gamma_2(T) = w_2 \, . \tag{3.7}$$

Since the Hamiltonian maximizing conditions are necessary and sufficient for the bilinear quadratic problem (see [7]), we obtain the form of the optimal controls as:

$$\frac{\partial H_1}{\partial u_1} = 0 \Rightarrow u_1 = \frac{1}{2}\,[\lambda_1\{b_1(1-x_1-x_2) + e_1(x_1+x_2)\} - \lambda_2 e_2(x_1-x_2)] \tag{3.8}$$

$$\frac{\partial H_2}{\partial u_2} = 0 \Rightarrow u_2 = \frac{1}{2}\,[\gamma_2(b_2(1-x_1-x_2) + e_2(x_1+x_2)) - \gamma_1 e_1(x_1+x_2)] \; . \tag{3.9}$$

If we substitute (3.8) and (3.9) into (2.1) and (3.3) - (3.7), we obtain the two-point-boundary value problem, which when solved, yields optimal advertising for the two firms. This will be carried out in Section 5. In the next section, we take up the analysis of the infinite horizon problem and obtain the expression for the Nash turnpike.

4. DERIVATION OF THE NASH TURNPIKE

For this derivation we assume $T = \infty$ and modify the objective functions of (2.2) and (2.3) as follows:

$$J_1 = \int_0^\infty (c_1 x_1 - u_1^2)e^{-\rho t}dt \tag{4.1}$$

$$J_2 = \int_0^\infty (c_2 x_2 - u_2^2)e^{-\rho t}dt \; . \tag{4.2}$$

The differential game problem involves firm 1 trying to maximize (4.1) and firm 2 trying to maximize (4.2) subject to conditions (2.1).

To determine the Nash turnpike we set the right hand sides of (2.1) and (3.3) - (3.6) equal to zero and solve these six equations along with (3.8) and (3.9). This gives a system of 8 bilinear equations in 8 variables. We now sketch the reduction of this system to two simultaneous cubic equations in u_1 and u_2. However, one equation has no term in u_1^3 and the other has no term in u_2^3. It is, therefore, possible to solve these equations by an iterative application of the quadratic formula.

The eight equations to be solved are

$$-a_1x_1 + b_1u_1(1-x_1-x_2) + e_1(u_1-u_2)(x_1+x_2) = 0 \quad (4.3)$$

$$-a_2x_2 + b_2u_2(1-x_1-x_2) + e_2(u_1-u_2)(x_1+x_2) = 0 \quad (4.4)$$

$$-c_1 + \lambda_1[\rho+a_1+b_1u_1-e_1(u_1-u_2)] + \lambda_2[b_2u_2-e_2(u_2-u_1)] = 0 \quad (4.5)$$

$$\lambda_2[\rho+a_2+b_2u_2 - e_2(u_2-u_1)] + \lambda_1[b_1u_1-e_1(u_1-u_2)] = 0 \quad (4.6)$$

$$\gamma_1[\rho+a_1+b_1u_1 - e_1(u_1-u_2)] + \gamma_2[b_2u_2-e_2(u_2-u_1)] = 0 \quad (4.7)$$

$$-c_2 + \gamma_2[\rho+a_2+b_2u_2 - e_2(u_2-u_1)] + \gamma_1[b_1u_1 - e_1(u_1-u_2)] = 0 \quad (4.8)$$

$$2u_1 = \lambda_1[b_1(1-x_1-x_2) + e_1(x_1+x_2)] - \lambda_2e_2(x_1+x_2) \quad (4.9)$$

$$2u_2 = \gamma_2[b_2(1-x_1-x_2) + e_2(x_1+x_2)] - \gamma_1e_1(x_1+x_2) . \quad (4.10)$$

Subtracting (4.6) from (4.5) and (4.8) from (4.7) we obtain

$$\lambda_1 = \frac{c_1 + \lambda_2(a_2+\rho)}{a_1 + \rho} \quad (4.11)$$

$$\gamma_2 = \frac{c_2 + \gamma_1(a_1+\rho)}{a_2 + \rho} \tag{4.12}$$

Substituting (4.11) into (4.6) and (4.12) into (4.7) yields

$$\lambda_2 = \frac{c_1[e_1(u_1-u_2) - b_1u_1]}{D(\rho)} \tag{4.13}$$

$$\gamma_1 = \frac{c_2[e_2(u_2-u_1) - b_2u_2]}{D(\rho)} \tag{4.14}$$

where,

$$D(\rho) = (a_1+\rho)(a_2+\rho) + (a_1+\rho)b_2u_2 + (a_2+\rho)b_1u_1$$
$$+ (u_1-u_2)[(a_1+\rho)e_2 - (a_2+\rho)e_1] \tag{4.15}$$

Multiplying (4.3) by a_2 and (4.4) by a_1 and adding gives

$$x_1 + x_2 = \frac{a_2b_1u_1 + a_1b_2u_2}{D(0)} \tag{4.16}$$

Putting (4.11), (4.13) and (4.16) into (4.9) and multiplying by $D(0)D(\rho)(a_1+\rho)$ and putting (4.12), (4.14) and (4.16) into (4.10) and multiplying by $D(0)D(\rho)(a_2+\rho)$ gives the following system of two cubic equations in u_1 and u_2:

$$2D(0)D(\rho)(a_1+\rho)u_1 = c_1b_1D(0)D(\rho) + [(a_2+\rho)c_1(e_1-b_1) - (a_1+\rho)c_1e_2]$$
$$[e_1(u_1-u_2) - b_1u_1][a_2b_1u_1 + a_1b_2u_2] \tag{4.17}$$
$$+ (a_2+\rho)c_1b_1[e_1(u_1-u_2)-b_1u_1]D(0)$$
$$+c_1(e_1-b_1)(a_2b_1u_1 + a_1b_2u_2)D(\rho).$$

$$2D(0)D(\rho)(a_2+\rho)u_2 = c_2b_2D(0)D(\rho) + [(a_1+\rho)c_2(e_2-b_2) - (a_2+\rho)c_2e_1]\cdot$$
$$[e_2(u_2-u_1) - b_2u_2][a_2b_1u_1+a_1b_2u_2]$$
$$+ (a_1+\rho)c_2b_2[e_2 \quad (u_2-u_1)-b_2u_2]D(0) \tag{4.18}$$
$$+ c_2(e_2-b_2)(a_2b_1u_1 + a_1b_2u_2)D(\rho).$$

Multiplying these out and collecting terms, we can rewrite these equations in the following form:

$$\alpha_1u_1^3 + \beta_1u_1^2u_2 + \delta_1u_1u_2^2 + \xi_1u_1^2 + \theta_1u_1u_2 + \Psi_1u_2^2 + \epsilon_1 = 0 \tag{4.19}$$

$$\alpha_2u_2^3 + \beta_2u_1u_2^2 + \delta_2u_1^2u_2 + \xi_2u_2^2 + \theta_2u_1u_2 + \Psi_2u_1^2 + \epsilon_2 = 0, \tag{4.20}$$

where coefficients α_i, β_i, δ_i, ξ_i, θ_i, Ψ_i, and ϵ_i for $i = 1,2$ can be expressed as lengthy expressions in the original parameters.

Equations (4.19) and (4.20) are two simultaneous cubic equations in u_1 and u_2 which can be solved by a number of methods, provided a real solution exists. Note that for fixed u_1, equation (4.19) is quadratic in u_2; and for fixed u_2, equation (4.20) is quadratic in u_1. Thus there is a possibility of developing an algorithm that alternates between the solutions of two quadratic equations.

Instead of solving (4.19) and (4.20) we intend, in a later paper, to extend the numerical solution procedure of the next section to problems with large T, and to observe the steady state values, and check to see whether they satisfy the two equations.

5. NUMERICAL SOLUTION OF THE FINITE COMPETITIVE MODEL

From section 3, the necessary and sufficient conditions for optimality provide two system equations each with a specified initial condition (2.1), four adjoint equations (3.3 - 3.6) with their four terminal conditions (3.7), and the two equations for the control variables (3.8, 3.9).

The algorithm is initiated by guessing values for each competitor's controls for each discrete instant of time during the planning horizon. The system equations are then functions of the state variables only and can be solved numerically forward in time by using the specified initial conditions. At each forward step of the solution, the values of $x_i(t)$ are calculated and stored. The values of the performance indices are calculated during each forward pass.

At terminal time, the terminal conditions on the adjoint variables are calculated and are used in the solution of the adjoint equations backward in time. At each successive time decrement of the backward pass, the appropriate values of the adjoint variables are determined along with a new set of control variable values. When back at the initial time, there is a revised time sequence of values for the control function. These are used in the next forward pass, the objective of which is to evaluate the effect of these new controls on the performance indices. At the end of each forward pass a stopping criterion is evaluated. When the iterative procedure has been terminated, the solution is checked against the Nash equilibrium property. During extensive testing, the algorithm exhibited stability and realistic convergence properties while consistently obtaining Nash solutions.

Using the numerical algorithm, we have investigated many competitive encounters by varying the parameter values individually and in combination. The computer runs provide information on the values of the performance indices, the state and control variables and the adjoint variables at each instant of time during the planning horizon. Computer generated graphs of this information are also obtained.

Limited space does not allow for an extensive presentation of the numerical results. For illustrative purposes, we present two extreme cases.

The first case is a sequence of four encounters where the two competitors are identical except for one attribute, the speed of sales decay. The firms have the same moderate sales response, relatively low competitive reaction and low initial market saturation. Both firms are attempting to maximize profit over the finite planning period. The sales decay parameter for Firm 1 remains at the same moderate level during the four encounters. Firm 2 has a very low rate of decay in encounter 1. This decay increases until it is identical to Firm 1's decay in encounter 3 and is larger than Firm 1's in encounter 4. The parameter values are found in Table 1.

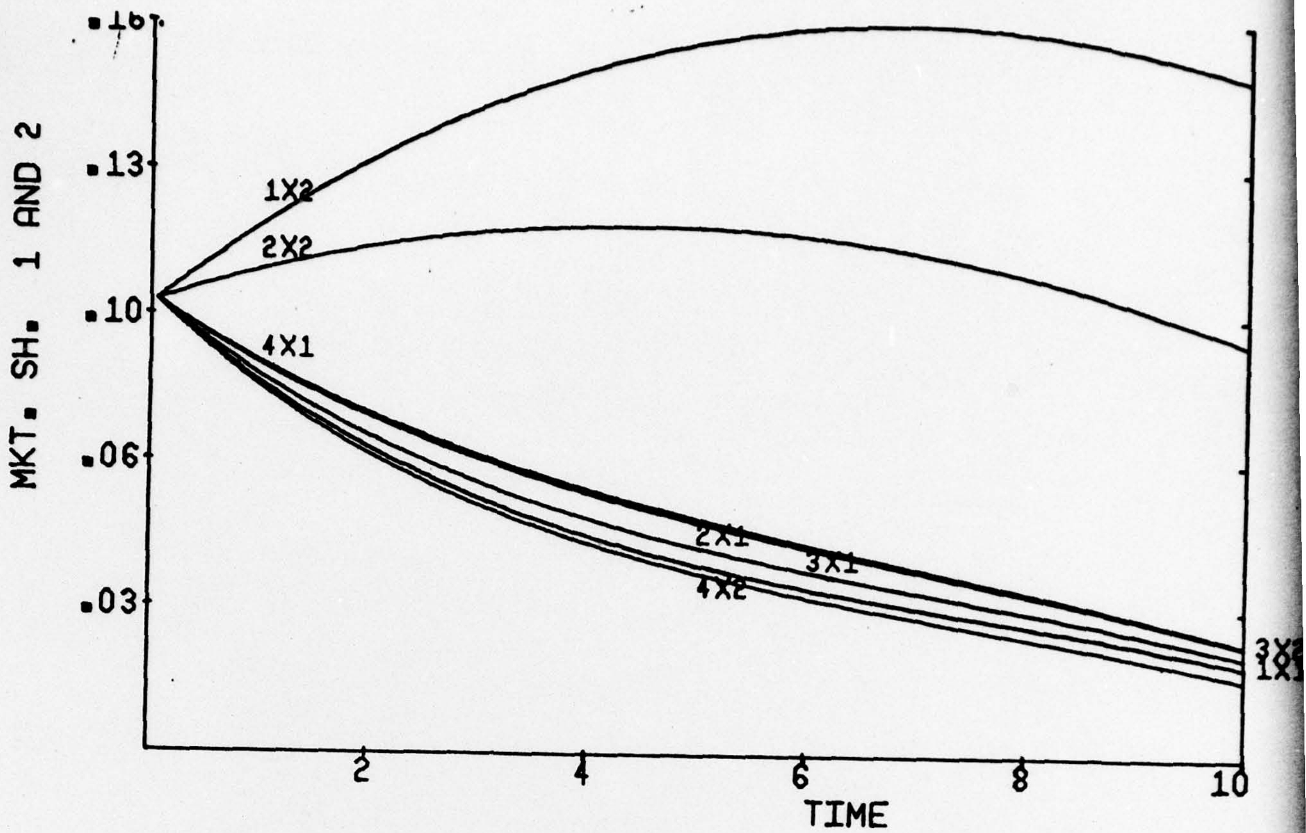
The graphical results are presented in Figure 1. The labelling of the curves is iUj for the advertising curves and iXj for the market share curves where i refers to the encounter number, i = 1,..., 4, and j indicates the firm number, j = 1, 2. For example, in Figure 1a the optimal advertising expenditures path for Firm 2 during encounter 1 is indicated by 1U2. Firm 2's resulting market shares path is labelled 1X2 in Figure 1b.

The time-phased advertising and market share paths should appeal to a realistic explanation from a "marketing common sense" standpoint. Since the firms are concerned with maximizing profit over the current planning period only, it would be expected that the firms should spend their advertising dollars to gain maximum performance index value before the end of the period. To these firms the carryover of market share response to the next period is

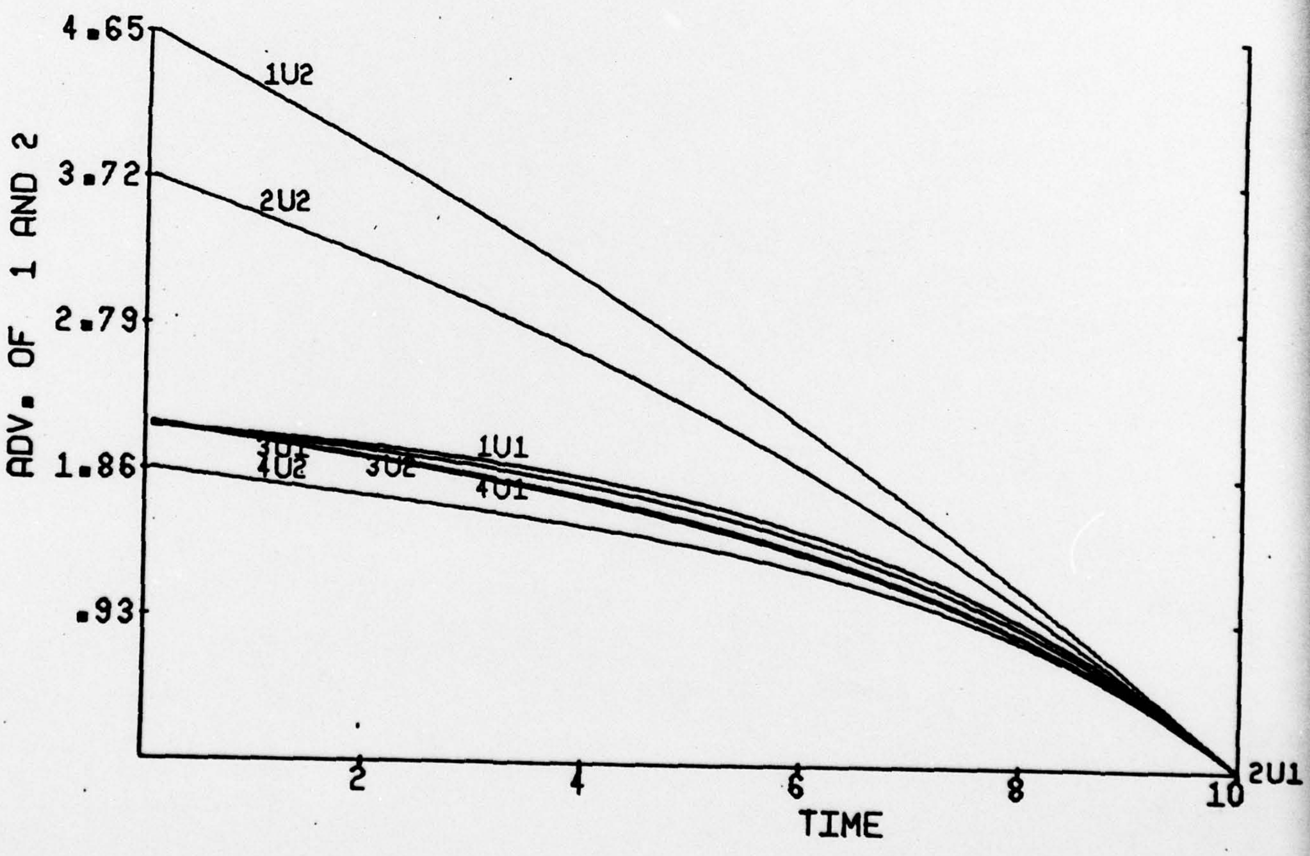
TABLE 1

(Encounters 1 - 4)

i	a_i	b_i	e_i	c_i	w_i	J_i	$x_i(0)$	$x_i(T)$
1	.25	.005	.01	200	0	69.55	.10	.020
2	.05	.005	.01	200	0	211.18	.10	.149
1	.25	"	"	"	"	76.45	"	.022
2	.10					162.45		.091
1	.25	"	"	"	"	86.32	"	.025
2	.25					86.32		.025
1	.25	"	"	"	"	87.90	"	.025
2	.30					72.93		.017



.16
.13
.10
.06
.03
MKT. SH. 1 AND 2
1X2
2X2
4X1
2X1
3X1
4X2
3X2
1X1
2
4
6
8
10
TIME



4.65
3.72
2.79
1.86
.93
ADV. OF 1 AND 2
1U2
2U2
3U1
4U2
3U2
1U1
4U1
2U1
2
4
6
8
10
TIME

Figure 1

wasted money. Consequently, advertising expenditures should decrease to zero by the end of the term. In encounter 1 Firm 2 has a much slower decay of market share (greater carryover) than does Firm 1. Although the immediate response to advertising expenditures is identical, Firm 2 obtains significantly greater total effect for each dollar spent. Firm 1 is overwhelmed by Firm 2. Firm 2 is able to obtain a substantially larger total profit and increase its market share by spending more than twice Firm 1's amount initially and tapering to zero by terminal time. Firm 1 finds that a subsistence level of advertising allows it to "maximize" profit in a decidedly dominated situation.

As shown in Table 1, the decay parameter of Firm 1 increases until it becomes larger than 2's decay in encounter 4. Naturally, the advertising and sales paths are quite different in encounters 1 and 4. The profit maximization objective has been investigated extensively by holding all but one parameter fixed and varying the values of that key parameter over a large range. Also, an experimental design was used to perform model testing.

The parameter values used in the second set of encounters are presented in Table 2. The corresponding graphs are in Figure 2. These graphs illustrate advertising strategies and market results which are practically opposite to those of the first set of encounters. Of course, the key to this difference is the firms' re-orientation to placing heavy importance on terminal market share, as is indicated by the value of the weight, w, in the table.

To illustrate the effect of the competitive interaction term, we present one last set of encounters in Table 3 and Figure 3. As shown in the "e" column of Table 3, Firm 1 is dominant in encounter 1, is identical to Firm 2 in encounter 2, and becomes dominated by Firm 2 in encounters 3

TABLE 2

(Encounters 1 - 4)

i	a_i	b_i	e_i	c_i	w_i	J_i	$x_i(0)$	$x_i(T)$
1	.25	.005	.01	200	2550	-72.40	.10	.039
2	.05	.005	.01	200	2550	749.89	.10	.389
1	.25	"	"	"	"	60.65	"	.062
2	.10					494.13		.240
1	.25	"	"	"	"	188.24	"	.084
2	.25					188.24		.084
1	.25	"	"	"	"	202.28	"	.086
2	.30					147.90		.065

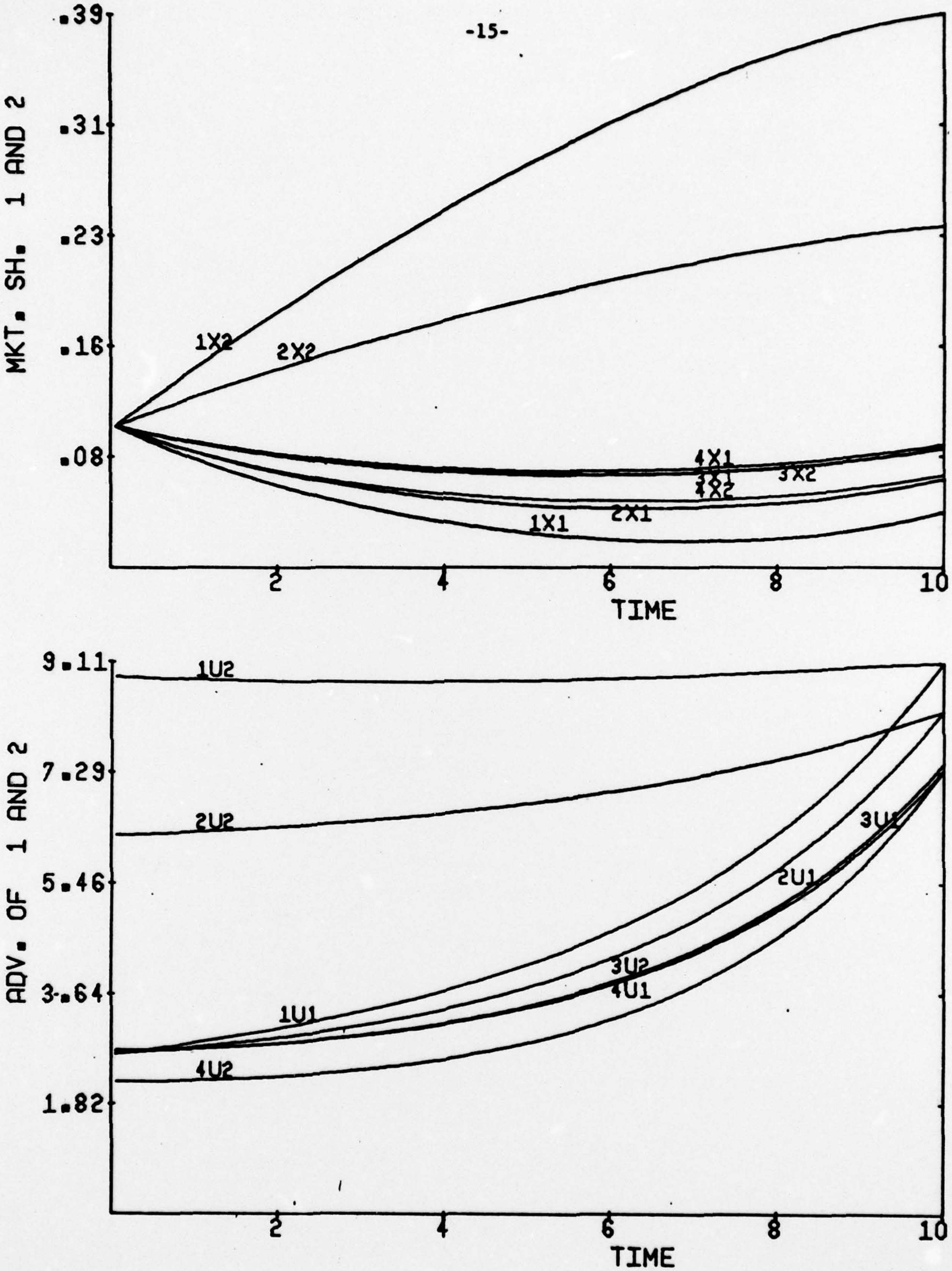

MKT. SH. 1 AND 2
.39
.31
.23
.16
.08
1X2
2X2
4X1
3X2
1X1
2X1
2
4
6
8
10
TIME
ADV. OF 1 AND 2
9.11
7.29
5.46
3.64
1.82
1U2
2U2
3U1
2U1
3U2
4U1
1U1
4U2
TIME

Figure 2

TABLE 3

(Encounters 1 - 4)

i	a_i	b_i	e_i	c_i	w_i	J_i	$x_i(0)$	$x_i(T)$
1	.10	.005	.010	200	2550	703.28	.30	.318
2	.10	.005	.005	200	2500	791.24	.30	.167
1	"	"	.010	"	"	407.91	"	.244
2			.010			407.91		.244
1	"	"	.010	"	"	-53.82	"	.165
2			.015			298.22		.396
1	"	"	.010	"	"	-997.35	"	.039
2			.020			-62.87		.702

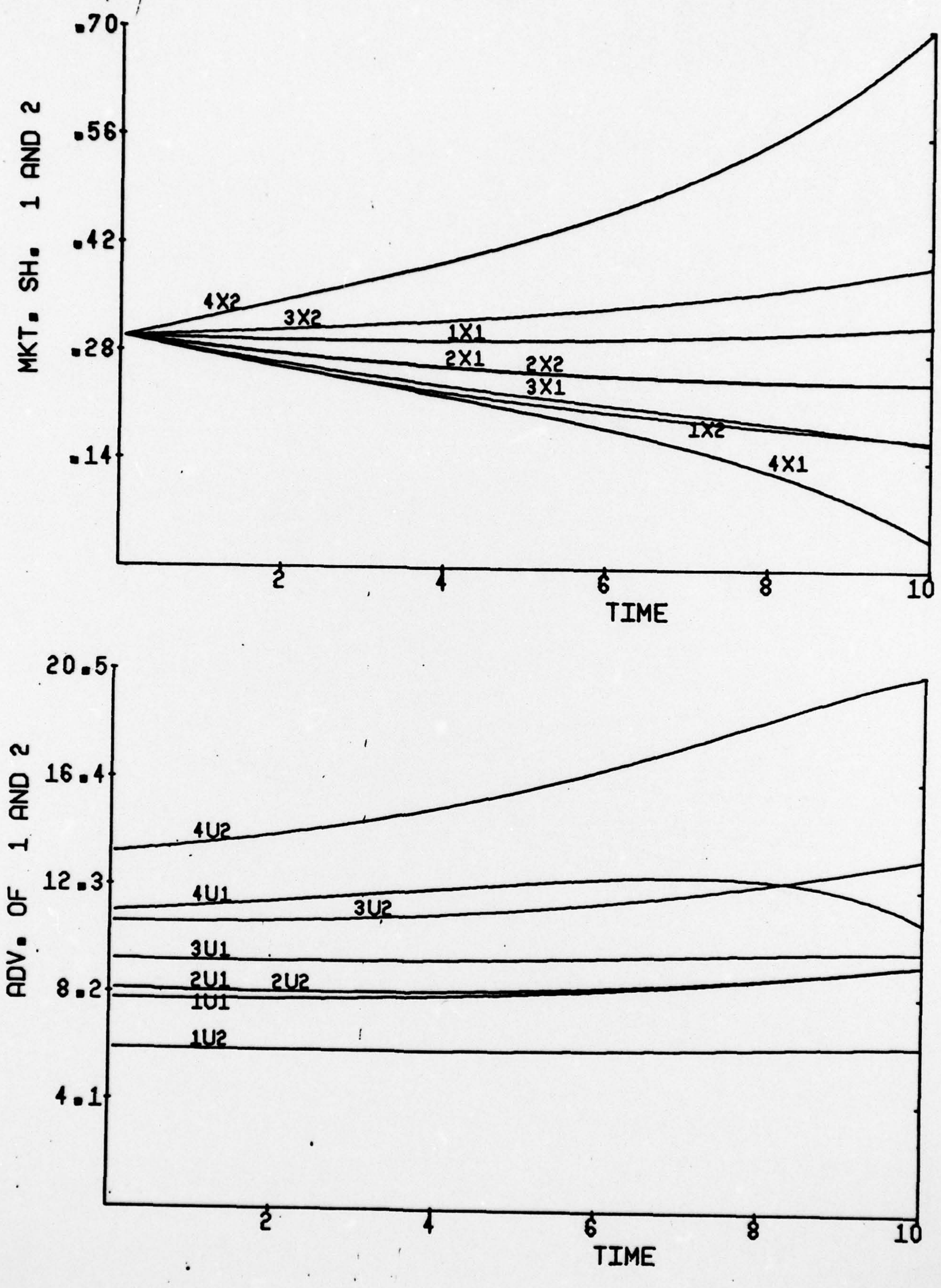

.70
.56
.42
.28
.14
MKT. SH. 1 AND 2
4X2
3X2
1X1
2X1
2X2
3X1
1X2
4X1
2
4
6
8
10
TIME
20.5
16.4
12.3
8.2
4.1
ADV. OF 1 AND 2
4U2
4U1
3U2
3U1
2U1
2U2
1U1
1U2
2
4
6
8
10
TIME

Figure 3

and 4.

Over these encounters, Firm 2 changes position from one in which it cannot maintain its initial market share during encounter 1 to the position where it has a 70% market share compared to Firm 1's 4% market share in encounter 4.

Comparison of the advertising strategies resulting from profit maximization (Figure 1) and those resulting from having a heavy emphasis on terminal market share (Figure 2) indicates the tremendous differences between advertising strategies under these two performance criteria. Naturally, less extreme weighting of the components of the performance indices would produce results which are more realistic for most actual marketing organizations.

Many advertising executives maintain the need to have a very strong impact in a communication media or to stay out of that media except for some bare subsistence level spending. The benefit of dominating an industry through advertising has been illustrated strongly by the last set of encounters which was based on variation of the reaction terms. The effect of the reaction terms will vary depending upon the values of the other parameters and the performance index weighting. However, the benefit of advertising dominance will exist in every case.

6. CONCLUSIONS

The model presented in this paper has the conceptual advantage of combining logically several critical dimensions of advertising competition. A benefit of this model lies in its ability to represent the marketing characteristics of market share response to advertising, decay of market share due to forgetting, the saturation effect, and the effect of sales switching due to advertising dominance within the most natural environment of advertising -- dynamic competition with multiple objective performance indices.

We are encouraged by the numerical results to the finite horizon problem. Our next stage in the investigation will be to solve numerically the infinite horizon problem as proposed in section 4 and to compare the results to the solution of the original problem when a greatly lengthened, but finite, time period is used. We are also looking forward to the research of a particular industrial situation which coincides closely with the assumptions of this model.

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A dopolistic extension of the Vidale-Wolfe advertising model is formulated as a problem in differential games. An important feature of the problem is the presence of reaction terms driven by the difference in the advertising expenditures of the two duopolistic firms under consideration. Formula for the Nash Turnpike is derived using the maximum principle. Several competitive encounters with different parameters of the model are numerically solved. Tentative conclusions are drawn by analyzing these results.