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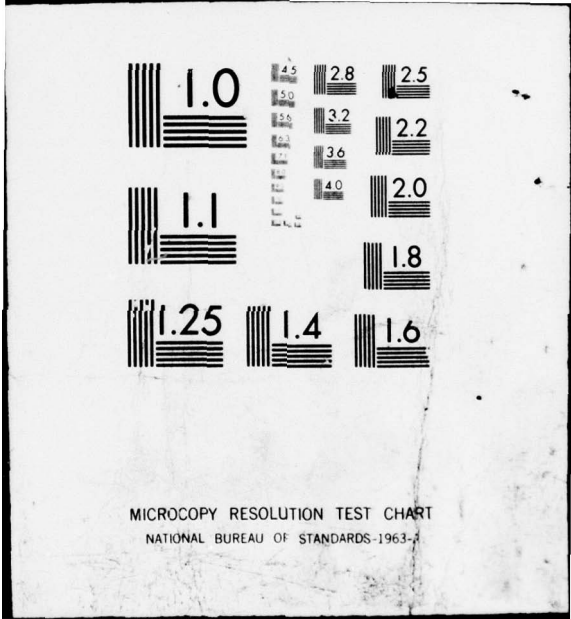
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The Conditional Probability of Random Harmonic Sets

D. A. SWICK

*Advanced Projects Group
Acoustics Division*

June 1978

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THE CONDITIONAL PROBABILITY OF RANDOM HARMONIC SETS

Introduction

The approximate probability that m random variables out of a set of n uniformly and independently distributed on (a,b) are harmonically related to within a tolerance of $100t\%$ has been calculated in [1]. If accidental harmonic multiples are to be avoided, the need for high accuracy in frequency measurements is indicated by these results. Application of them has revealed both a computational and a conceptual problem which are the subjects of this paper. The computational problem is discussed in connection with Eq. 2, below. A key element in its elimination is a by-product of the solution of the conceptual problem: The derivation in [1] is valid when the smallest member of a "harmonically related" subset is a low-order harmonic. (It is compared there with the results of a small-scale simulation for the cases where the smallest member is the fundamental or the second harmonic.) If the smallest member is a harmonic of much higher order, however, asking a posteriori for the probabilities of multiples of the indicated fundamental may be "data-snooping", and too dependent on the way the data fall out. It is more appropriate to ask for the conditional probability that all elements of a subset be multiples of a fundamental, given that the smallest member is such a multiple. Before addressing this problem, the results of [1] will be summarized to establish the notation and to disclose the computational difficulty.

Unconditional Probability: Summary of Results

We consider a set of n random variables represented by X_1, \dots, X_n , independently and uniformly distributed on (a,b) , and an ordered subset of them, $Y_1 \leq Y_2 \leq \dots \leq Y_m$, where $m=2,3,\dots,n$. If the Y_i are all multiples of Y_1/k_1 , to within a tolerance of $100t\%$, where k_1 is the harmonic number of the smallest member of the subset, we say that they form a chain of harmonics of length m . The probability of this is given by

$$\begin{aligned} & P_m(k_1, k_2, \dots, k_m) \\ &= \Pr \{ (Y_1, \dots, Y_m) : (1-t)k_i Y_1/k_1 \leq Y_i \leq (1+t)k_i Y_1/k_1, i=2, \dots, m \} \\ &\approx \frac{(m-1)!}{(b-a)^m} \left(\frac{2t}{k_1 k_m} \right)^{m-1} \left(\prod_{i=2}^{m-1} k_i \right) \left[(k_1 b)^m - (k_m a)^m \right] \end{aligned} \quad (1)$$

where $k_1 < k_2 < \dots < k_m$, and terms of $O(t^2)$ have been dropped.

Note: Manuscript submitted May 15, 1978.

If $L = [bk_1/a(1+t)]$ is the largest harmonic permitted by the range and tolerance, then the probability that a particular subset forms a chain of length m for some set of k_1 is

$$P_m(k_1) = \sum_{k_2=k_1+1}^{L-m+2} \sum_{k_3=k_2+1}^{L-m+3} \cdots \sum_{k_m=k_{m-1}+1}^L P_m(k_1, \dots, k_m). \quad (2)$$

The approximate probability of at least one chain of length m among the n random variables is given by

$$P(n, m, k_1) = \sum_{r=1}^M (-1)^{r+1} \binom{M}{r} [P_m(k_1)]^r, \quad (3)$$

where $M = \binom{n}{m}$.

The computational problem implicit in (3) is discussed in [1] and is resolved by a simple recursion formula. A computational problem not resolved there, however, is that of Eq. 2. For a "reasonable" interval (a, b) , L can be large, and the straightforward computation of the nested sums can become economically prohibitive for m larger than five or six. Even for small m , repeated calculations can be costly; they are not feasible for chains of ten or more.

A partial solution to this difficulty is obtained by reformulating the problem in terms of differences between successive variables rather than multiples of a common "fundamental". Since successive differences can be the same, unlike successive multiples, it is possible to effect a computational advantage in an analog of (2) by combining terms. This technique shifts the computational problem to longer chains, but it does not solve it. It is eliminated in the conditional formulation which follows.

Conditional Probability of Harmonic Chains

As before, let $Y_1 \leq Y_2 \leq \dots \leq Y_m$ be an ordered subset of the n random variables X_1, X_2, \dots, X_n which are independently and uniformly distributed on (a, b) . The conditional probability of a chain of harmonics of length m , i.e., the conditional probability that all Y_i are multiples of Y_1/k_1 to within a tolerance of $100t\%$, given that $f_0 = Y_1/k_1$ to within the same tolerance is given by

$$\begin{aligned}
& p_m(k_2, \dots, k_m | k_1) \\
&= \Pr \left[\{ (Y_1, \dots, Y_m) : (1-t)k_i Y_1 / k_1 \leq Y_i \leq (1+t)k_i Y_1 / k_1; i=2, \dots, m \} \right. \\
&\quad \left. \cap \{ (Y_1) : (1-t)k_1 f_0 \leq Y_1 \leq (1+t)k_1 f_0 \} \right] \\
&\quad \cdot \left[\Pr \{ (Y_1) : (1-t)k_1 f_0 \leq Y_1 \leq (1+t)k_1 f_0 \} \right]^{-1} \\
&= \frac{m!}{(b-a)^m} \frac{\int_{(1-t)k_1 f_0}^{(1+t)k_1 f_0} \prod_{i=2}^m \left\{ \int_{(1-t)k_i y_1 / k_1}^{(1+t)k_i y_1 / k_1} dy_i \right\} dy_1}{\frac{1}{b-a} \int_{(1-t)k_1 f_0}^{(1+t)k_1 f_0} dy} \\
&= m! \left(\frac{2t f_0}{b-a} \right)^{m-1} \prod_{i=2}^m k_i, \tag{4}
\end{aligned}$$

where $k_1 < k_2 < \dots < k_m$, and $a/(1-t) \leq k_i f_0 \leq b/(1+2t) \forall i$, neglecting terms of $O(t^2)$.

Now let $L = [b/f_0(1+t)^2]$ be the largest possible harmonic. The conditional probability that a particular subset forms a chain of length m becomes

$$P_m(k_1) = m! \left(\frac{2t f_0}{b-a} \right)^{m-1} \sum_{k_2=k_1+1}^{L-m+2} \sum_{k_3=k_2+1}^{L-m+3} \dots \sum_{k_m=k_{m-1}+1}^L \prod_{i=2}^m k_i. \tag{5}$$

The simplicity of (4), as contrasted with (1), permits an analytic evaluation of (5), thus avoiding the computational problem of (2). The approximate conditional probability of at least one chain of

length m among the n random variables is then given by (3).

Writing (5) as a sequence of nested sums, we have

$$P_m(k_1) = m! \left(\frac{2tf_0}{b-a} \right)^{m-1} S_2 \quad (6)$$

where

$$S_2 = \sum_{k_2=k_1+1}^{L-m+2} S_3, \quad S_3 = \sum_{k_3=k_2+1}^{L-m+3} S_4, \dots, S_{m-1} = \sum_{k_{m-1}=k_{m-2}+1}^{L-1} S_m. \quad (7)$$

The innermost sum is

$$S_m = \binom{m-1}{\prod_{i=2}^{m-1} k_i} [L(L+1) - k_{m-1} - k_{m-1}^2] / 2. \quad (8)$$

Either directly from (5), or from (6) and (8), we have

$$P_2(k_1) = \frac{2tf_0}{b-a} [L(L+1) - k_1 - k_1^2].$$

For $m > 2$, we write the general term in (7) as

$$S_{m-i} = \sum_{k_{m-i}=k_{m-i-1}+1}^{L-i} S_{m-i+1}; i=1, \dots, m-2. \quad (9)$$

To evaluate this sequence of sums we assume a relationship of the form

$$S_{m-i+1} = \binom{m-i}{\prod_{\alpha=2}^{m-i} k_\alpha} \sum_{\beta=1}^{2i+1} A_{i\beta} (k_{m-i})^{\beta-1}; i=1, \dots, m-1. \quad (10)$$

For $i = m-1$ this becomes

$$S_2 = \sum_{\beta=1}^{2m-1} A_{m-1,\beta} k_1^{\beta-1}; m=2, 3, \dots, \quad (11)$$

from which (6) can be evaluated once the coefficients $A_{m-1,\beta}$ have been determined. For $i=1$, (10) becomes

$$S_m = \left(\prod_{\alpha=2}^{m-1} k_\alpha \right) [A_{11} + A_{12}k_{m-1} + A_{13}k_{m-1}^2].$$

Comparison with (8) yields

$$A_{11} = L(L+1)/2, \quad A_{12} = A_{13} = -1/2, \quad (12)$$

valid for all m .

Substituting (10) in (9) and letting $k = k_{m-i}$ we have

$$\begin{aligned} S_{m-i} &= \left(\prod_{\alpha=2}^{m-i-1} k_\alpha \right) \sum_{k=k_{m-i-1}+1}^{L-i} \sum_{\beta=1}^{2i+1} A_{i\beta} k^\beta \\ &= \left(\prod_{\alpha=2}^{m-i-1} k_\alpha \right) \sum_{\beta=1}^{2i+1} A_{i\beta} \left[\sum_{k=1}^{L-i} k^\beta - \sum_{k=1}^{k_{m-i-1}} k^\beta \right]. \quad (13) \end{aligned}$$

In terms of the Bernoulli polynomials $B_n(x) = \sum_{k=0}^n b_k^{(n)} x^k$ [2, Chap. 23] we can write

$$\sum_{k=1}^s k^\beta = \frac{B_{\beta+1}(s+1) - B_{\beta+1}(0)}{\beta+1} = \frac{1}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_\gamma^{(\beta+1)} (s+1)^\gamma,$$

so that (13) becomes

$$\begin{aligned} S_{m-i} &= \left(\prod_{\alpha=2}^{m-i-1} k_\alpha \right) \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_\gamma^{(\beta+1)} [(L-i+1)^\gamma - (k_{m-i-1}+1)^\gamma] \\ &= \left(\prod_{\alpha=2}^{m-i-1} k_\alpha \right) \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_\gamma^{(\beta+1)} \left[(L-i+1)^\gamma - \sum_{\delta=1}^{\gamma+1} \binom{\gamma}{\delta-1} (k_{m-i-1})^{\delta-1} \right]. \quad (14) \end{aligned}$$

We collect powers of k_{m-i-1} . For $\delta=1$ we identify the coefficient of $(k_{m-i-1})^0 = 1$ as

$$A_{i+1,1} = \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_{\gamma}^{(\beta+1)} [(L-i+1)^{\gamma}-1]; \quad (15)$$

otherwise the coefficient of $(k_{m-i-1})^{\delta-1}$ is

$$A_{i+1,\delta} = - \sum_{\beta=1}^{2i+1} \frac{A_{i\beta}}{\beta+1} \sum_{\gamma=1}^{\beta+1} \binom{\gamma}{\delta-1} b_{\gamma}^{(\beta+1)}, \quad \delta=2, \dots, 2i+3, \quad (16)$$

both for $i=1, \dots, m-2$. Using these coefficients, (14) becomes

$$S_{m-i} = \left(\prod_{\alpha=2}^{m-i-1} k_{\alpha} \right) \sum_{\delta=1}^{2i+3} A_{i+1,\delta} (k_{m-i-1})^{\delta-1}; \quad i=0, 1, \dots, m-2,$$

which has the same form as (10) with i replaced by $i+1$. The values of $A_{i\delta}$ are given by (12). For $1 \leq i \leq m-2$ $A_{i+1,\delta}$ may be evaluated recursively using (15) and (16), with $A_{m-1,\delta}$, $\delta=1, \dots, 2m-1$, used to evaluate S_2 in (11). From (15)

$$A_{i+1,1} = \sum_{\beta=1}^{2i+1} d_{i\beta} A_{i\beta},$$

where

$$d_{i\beta} = \frac{1}{\beta+1} \sum_{\gamma=1}^{\beta+1} b_{\gamma}^{(\beta+1)} [(L-i-1)^{\gamma}-1], \quad (17)$$

for $i=1, \dots, m-2$ and $\beta=1, \dots, 2i+1$.

From (16),

$$A_{i+1,\delta} = \sum_{\beta=\epsilon}^{2i+1} C_{\beta\delta} A_{i\beta},$$

where

$$C_{\beta\delta} = -\frac{1}{\beta+1} \sum_{\gamma=1}^{\beta+1} \binom{\gamma}{\delta-1} b_{\gamma}^{(\beta+1)} \quad (18)$$

for $i=1, \dots, m-2$, $\delta=2, \dots, 2i+3$, $\epsilon=\max(1, \delta-2)$, $\beta=\epsilon, \dots, 2i+1$, and $C_{\beta\delta} = 0$ if $\beta < \delta-2$. A storage advantage may be obtained from the observation that the $A_{i\delta}$ may be stored in a "lower triangular" two-dimensional array and the $C_{\beta\delta}$ may be stored in an "upper triangular" array. Thus they may share a single $2m$ by $2m-1$ array.

The coefficients $b_k^{(n)}$ of the Bernoulli polynomials $B_n(x)$ are given in Table 23.1 of [2] for $0 \leq n \leq 15$. They may be obtained in general from

$$b_k^{(n)} = \binom{n}{k} B_{n-k}, \quad k=0, \dots, n,$$

where $B_n = B_n(0)$, $B_0=1$, and [2]

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0.$$

The computation of the conditional probability of harmonic chains has been incorporated into a Fortran IV subroutine called PROHARM which is described and listed in the Appendix. The required binomial coefficients are obtained using subroutine BINO, also listed, as is CALLPRO, used to call the subroutines.

APPENDIX: Fortran implementation

Subroutine PROHARM, Figure 1, is a Fortran IV implementation of the algorithm for the computation of the conditional probability of harmonic chains. All parameters in the calling argument follow the notation of this paper. In principle there is no limit to the length of the chain; it is restricted here to $m \leq 20$ because of core limitations, with $n \geq m$ unrestricted. The arrays are dimensioned accordingly.

Terms of order t^2 were neglected in the text. Although they make no difference in the result, it is not much more difficult to calculate the exact coefficient as

$$(m-1)! \left(\frac{f_0}{b-a} \right)^{m-1} (2t)^{m-2} [(1+t)^m - (1-t)^m]$$

rather than the close approximation $m! (2tf_0/(b-a))^{m-1}$. This is done on lines 120-300 of PROHARM. L_1 (line 310) is $L+1$. If m harmonics will not "fit" in the given range, $Pr=0$ is returned. Array AC is to be shared by $A_{i\delta}$ and $C_{\beta\delta}$, with the initial values given by (12) stored in row 22. Throughout the subroutine, the parameter MM is used to avoid repeating a previously-completed calculation; only those coefficients needed for $M > MM$ are calculated.

The coefficients $b_k^{(n)}$ of the Bernoulli polynomials are calculated via lines 400-600, and are stored in array BB. Array CO stores the d_k , which are calculated by lines 610-700 using (17). Equation 18 yields $C_{\beta\delta}$, lines 720-840. These are stored in the upper right portion of array AC, while the lower left of AC holds the $A_{i\beta}$ which are calculated recursively, lines 850-950.

Equations 11 and 6 are implemented in lines 960-1010 and 1020, respectively, to obtain $P_m(k_1)$. Finally, the approximate probability of at least one chain, Eq. 3, is calculated as discussed in [1] via lines 1030-1170. This calculation usually converges rapidly, and is stopped when the next term changes the result by less than 0.0001%. When the probability is actually very close to unity, however, the finite word length of the computer may produce divergence in this calculation. In this case, $Pr=1$ is returned. The validity of this procedure has been established by observations of the changes in the probability introduced by small perturbations of the parameters.

A sample Fortran calling program, CALLPRO, is listed as Fig. 2. This program retrieves the subroutines PROHARM and BINO. The latter, used to calculate the binomial coefficients required by PROHARM, is listed as Fig. 3. Fig. 4 is a run of CALLPRO in which the conditional probability of at least one chain of length m , $2 \leq m \leq 20$, given

```

00100 SUBROUTINE PROHARM(A,B,N,M,T,F0,K1,PR)
00110 DIMENSION BB(38,38),AC(42,39),C0(18,37)
00120 M1=M-1
00130 M2=M-2
00140 TM=1.
00150 MIF=M1
00160 F=F1=F0/(B-A)
00170 TMM=TD=1.-T
00180 TMM=TMM*TD
00190 TPM=TP=1.+T
00200 TPM=TPM*TP
00210 IF(M.EQ.2)GOTO 2
00220 T2=2*T
00230 DO 1 K=1,M2
00240 MK=M1-K
00250 TM=TM*T2
00260 F=F*F1
00270 TMM=TMM*TD
00280 TPM=TPM*TP
00290 1 MIF=MIF*MK
00300 2 CC=MIF*TM*F*(TPM-TMM)
00310 LI=INT(B/(F0*TP*TP))+1
00320 IF(LI.GE.K1+M)GOTO 3
00330 PR=0.
00340 RETURN
00350 3 JM=2*M-1
00360 AC(22,1)=FLOAT(LI)*FLOAT(LI-1)/2.
00370 IF(M.LE.MM)GOTO 6
00380 AC(22,2)=AC(22,3)=-.5
00390 IF(M.EQ.2)GOTO 14
00400 MC=2*M1
00410 JM0=2*MM-1
00420 BB(1,1)=-.5
00430 I0=3
00440 IF(MM.NE.0)I0=2*(MM-2)+3
00450 DO 4 I=I0,MC
00460 I2=I-2
00470 C1=1.
00480 X=-1.
00490 DO 4 K=1,I2
00500 C1=C1*(I-K+1)/K
00510 X=X-C1*BB(1,K)
00520 4 BB(1,I-1)=X/I
00530 IF(MM.EQ.0)I0=2
00540 DO 5 I=I0,MC
00550 I1=I-1
00560 C1=1.
00570 BB(1,I)=1.
00580 DO 5 K=1,I1
00590 C1=C1*(I-K+1)/K
00600 5 BB(1,K)=C1*BB(1,I-K)
00610 6 DO 8 I=1,M2
00620 KM=2*I+1
00630 DO 8 K=1,KM
00640 KP=K+1
00650 X=0.
00660 FLI=FL0=FLOAT(LI-I)
00670 DO 7 L=1,KP
00680 X=X+BB(KP,L)*(FLI-1.)
00690 7 FLI=FLI*FL0

```

```

00700 8   C0(I,K)=X/KP
00710   IF(M.LE.MM)GOTO 11
00720   KM=JM-2
00730   MM=M
00740   DO 10 J=2,JM
00750   J1=J-1
00760   K0=MAX0(1,J-2)
00770   IF(J.LE.JM0)K0=JM0-1
00780   DO 10 K=K0,KM
00790   LM=K+1
00800   X=0.
00810   DO 9 L=J1,LM
00820   CALL BINO(L,J1,IG)
00830 9   X=X-IG*BB(LM,L)
00840 10  AC(J-1,K)=X/LM
00850 11  DO 13 I=2,M1
00860   KM=2*I-1
00870   JI=KM+2
00880   AC(I+21,1)=0.
00890   DO 12 K=1,KM
00900 12  AC(I+21,1)=AC(I+21,1)+C0(I-1,K)*AC(I+20,K)
00910   DO 13 J=2,JI
00920   AC(I+21,J)=0.
00930   K0=MAX0(1,J-2)
00940   DO 13 K=K0,KM
00950 13  AC(I+21,J)=AC(I+21,J)+AC(J-1,K)*AC(I+20,K)
00960 14  F=1.
00970   S2=0.
00980   FL=FLOAT(K1)
00990   DO 15 J=1,JM
01000   S2=S2+AC(M1+21,J)*F
01010 15  F=F*FL
01020   PM=CC*S2
01030   CALL BINO(N,M,NCM)
01040   KS=1
01050   S=PR=NCM*PM
01060   IF(NCM.EQ.1)RETURN
01070   DO 16 K=2,NCM
01080   KS=-1*KS
01090   S=((S*PM)/K)*(NCM-K+1)
01100   IF(S.LT.1.E-6*PR)RETURN
01110   IF(S.GT.1.E+12)GOTO 17
01120 16  PR=PR+KS*S
01130   IF(PR.LT.0)PR=1.
01140   IF(PR.GT.1.)PR=1.
01150   RETURN
01160 17  PR=1.
01170   RETURN
01180   END

```

FIG. 1 Subroutine PROHARM

```

00100      PROGRAM CALLPRO(INPUT,OUTPUT)
00110      PRINT,*RANGE: A,B*,
00120      READ,A,B
00130 1     PRINT,*N,M,TOL,F0,K1*,
00140      READ,N,M,T,F0,K1
00150      CALL PROHARM(A,B,N,M,T,F0,K1,PR)
00160      PRINT 2,PR
00170      GOTO 1
00180 2     FORMAT(* PR=*,E10.2/)
00190      RETRIEVE(PROHARM)
00200      RETRIEVE(BINO)
00210      END

```

FIG. 2 Program CALLPRO

```

00100      SUBROUTINE BINO(N,M,K)
00120      L=N-M
00130      MM=M
00140      IF(L.GT.M)MM=L
00150      L=N-MM
00160      K=0
00170      IF(L.EQ.0)K=1
00180      IF(L.EQ.1)K=N
00190      IF(L.LE.1)RETURN
00200      K=L
00210      A=1.*N
00220      D=1.*K
00230      B=A/D
00240      DO 1 I=2,K
00250      A=A-1.
00260      D=D-1.
00270 1     B=B*A/D
00280      K=INT(B+.5)
00290      RETURN
00300      END

```

FIG. 3 Subroutine BINO

PROGRAM CALLPRO

RANGE: A,B ? .05,.8
N,M,TOL,F0,K1 ? 20,2,.003,.02,8
PR= 1.00E+00

N,M,TOL,F0,K1 ? 20,4,.003,.02,8
PR= 9.97E-01

N,M,TOL,F0,K1 ? 20,6,.003,.02,8
PR= 9.77E-01

N,M,TOL,F0,K1 ? 20,7,.003,.02,8
PR= 5.76E-01

N,M,TOL,F0,K1 ? 20,8,.003,.02,8
PR= 1.38E-01

N,M,TOL,F0,K1 ? 20,9,.003,.02,8
PR= 1.97E-02

N,M,TOL,F0,K1 ? 20,10,.003,.02,8
PR= 2.06E-03

N,M,TOL,F0,K1 ? 20,11,.003,.02,8
PR= 1.67E-04

N,M,TOL,F0,K1 ? 20,12,.003,.02,8
PR= 1.05E-05

N,M,TOL,F0,K1 ? 20,13,.003,.02,8
PR= 5.08E-07

N,M,TOL,F0,K1 ? 20,14,.003,.02,8
PR= 1.88E-08

N,M,TOL,F0,K1 ? 20,15,.003,.02,8
PR= 5.21E-10

N,M,TOL,F0,K1 ? 20,16,.003,.02,8
PR= 1.06E-11

N,M,TOL,F0,K1 ? 20,17,.003,.02,8
PR= 1.50E-13

N,M,TOL,F0,K1 ? 20,18,.003,.02,8
PR= 2.93E-16

N,M,TOL,F0,K1 ? 20,19,.003,.02,8
PR= 2.52E-19

N,M,TOL,F0,K1 ? 20,20,.003,.02,8
PR= 7.23E-24

FIG. 4 $P(n,m,k_1)$ for $(a,b) = (.05,.8)$, $n=20$, $t=0.003$, $f_0=0.02$, $k_1=8$ and
 $2 \leq m \leq 20$.

$f_0 = 0.02$ is calculated. A tolerance of 0.003, $k_1=8$, and $n=20$ random variables independently and uniformly distributed on (0.05, 0.8) are assumed. With these parameters and range of m , the probability drops from unity to essentially zero. The calculation shown in Fig. 4 took less than 6 CPU seconds, including compilation, on a CDC CYBER 174.

References

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