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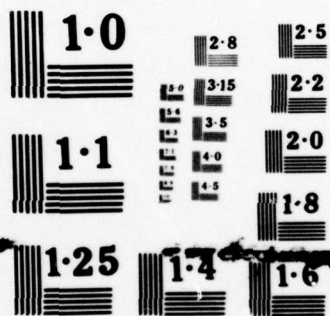
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10 Marc/Mangel

9 Professional Paper, No. 227

11 May 1978

14 CNA-PP-227

12 56 p.

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UNIFORM TREATMENT OF FLUCTUATIONS
AT CRITICAL POINTS

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May, 1978

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ABSTRACT

A generalized critical point is characterized by the vanishing of certain linear relationships. In particular, the dynamics near such a point are completely non-linear. ~~In this paper, we analyze~~ fluctuations at such points of spatially homogeneous systems.

are discussed ~~We discuss~~ thermodynamic critical points as a special case; but the main emphasis is on stochastic kinetic equations. ~~We show that~~ fluctuations at a critical point cannot be characterized by a Gaussian density, but more sophisticated densities yield reasonable results. ~~The~~ ^{Our} theory is applied to the critical harmonic oscillator.

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UNIFORM TREATMENT OF FLUCTUATIONS AT CRITICAL POINTS

SECTION 1. INTRODUCTION

A generalized critical point can be characterized by the vanishing of certain linear relationships. Such a point may be thermodynamic or kinetic. For example, at the liquid-vapor transition temperature, it is well known that

$$\left. \frac{\partial P}{\partial V} \right|_{T_c} = 0 . \quad (1.1)$$

In general, if $\phi(x)$ is some generalized potential and $\phi(0) = 0$ then the Onsager theory of non-equilibrium processes indicates that perturbations from $x = 0$ evolve according to

$$\dot{x} = -L \frac{\partial \phi}{\partial x} . \quad (1.2)$$

In the vicinity of equilibrium, x_{eq} , since $\phi'(x_{eq}) = 0$,

$$\phi(x) = \phi_{eq} + \frac{\phi''(x_{eq})(x - x_{eq})^2}{2} + \dots \quad (1.3)$$

Defining $1/\chi = \phi''(x_{eq})$ to be the reciprocal of a "susceptibility" we find

$$\dot{x} = \frac{-L}{\chi} (x - x_{eq}) + o\left((x - x_{eq})^2\right) . \quad (1.4)$$

At a generalized critical point, $\phi'' = 0$. Hence the dynamics in the vicinity of a critical point are highly nonlinear. Namely, one must use extra terms in the Taylor expansion (1.3), and obtain nonlinear dynamics at the critical point.

In this paper, we consider thermodynamic and kinetic critical points. As the term is usually used, "critical point" refers to a point in parameter space that is characterized by i) a slowing down of the dynamics and ii) long range spatial correlations. In this paper, we do not consider the long range order, but restrict the problem to spatially homogeneous systems, so that only critical slowing down will be evidenced. These equilibria (thermodynamic case) or steady states (kinetic case) are characterized by nonlinear dynamics. This point is given much emphasis in the paper. Due to the nonlinear dynamics the analysis of systems near critical points is quite difficult. Some analysis has been done, by Kubo et. al.(1) and Nitzan et. al.(2). In the first paper, two types of critical steady states were defined for one dimensional systems. We briefly review these. Let $x=0$ be the steady state. The steady state is of the marginal type(1) if perturbations from $x=0$ behave as

$$\dot{x} \sim \pm x^2. \quad (1.5)$$

The origin is a critical type steady state(1) if perturbations from $x=0$ behave as

$$\dot{x} \sim \pm x^3 . \quad (1.6)$$

Recently, the terminology of Kubo et. al. has been generalized to multidimensional systems(3). The paper of Nitzan et. al.(2) is contained as a special case of this paper and the accompanying one(4).

In this paper, we will analyze fluctuations at critical points. Often one reads that fluctuations "become unbounded" or "grow anomalously" at critical points. These statements are meant in the following sense: if one tries to describe fluctuations at a critical point by a Gaussian approximation, then the second moment $\langle x^2 \rangle$ is infinite. We will show that the Gaussian description of fluctuations implicitly assumes linear dynamics. Since critical point dynamics are nonlinear, one should not expect the Gaussian approximation to be valid. Hence, the anomaly is not in the physics, but in the improper use of mathematical approximations. We show that although the Gaussian approximation is not valid, more complicated densities are appropriate. The technique to demonstrate this will utilize formal asymptotic methods.

Our results are analogous to problems in optics (at a caustic) and wave mechanics (at a classical turning point). In those cases, the geometrical optics and WKB solutions break down, yielding

infinite amplitudes. In reality, the intensity of light at a caustic is not infinite, but is large(5). At a caustic, geometrical optics must be replaced by Airy or Pearcey integrals(5,6). Similar analyses hold at the classical turning point (e.g.(7),(8)).

In §2, we give the uniform treatment of fluctuations at the thermodynamic critical point of a homogeneous system. As an example, we calculate volume fluctuations of van der Waals gas at the critical point. Our theory is equivalent to the classical theory of phase transitions(8). The main focus of this paper, however, is kinetic critical points, which have a much richer dynamical behavior. In §3, we introduce the stochastic kinetic equation and diffusion approximation. The theory given here is a variation of the mode-mode coupling theory(10). We discuss a possible resolution of the present controversy regarding a "proper" expansion of the Master Equation to obtain a Fokker-Planck equation(11). A small parameter arises in the derivation of the diffusion approximation; it characterizes the intensity of fluctuations. The fluctuations are characterized by a density that satisfies the forward or Fokker-Planck equation. In this paper, techniques for the construction of solutions of the forward equation are given when the underlying deterministic dynamics exhibit critical behavior. In §4 we derive solutions of the one dimensional time invariant Fokker-Planck equation. We obtain an exact result, which is then analyzed by asymptotic methods. We obtain a Gaussian density at a

non-critical steady state, an Airy density at a marginal type steady state, and a Pearcey density at a critical type steady state. In §5, these densities are used in a general ansatz ("ray method" (12)) to provide asymptotic solutions of the time dependent multidimensional Fokker-Planck equation. We construct densities in which susceptibilities (i.e. first derivatives) at the critical point are large, but finite. The same result applies to variances. In §6, we show how our results can be used to construct time-dependent correlation functions. In §7, we discuss an example of the critical harmonic oscillator (13) and show how the correlation function is constructed.

Antecedents to this work are found in Kubo et. al. (1), Kitahara (14), Keizer (15) and Nitzan et. al. (2). The present work generalizes the results of the above papers.

SECTION 2. UNIFORM THERMODYNAMIC THEORY

In this section, we derive the uniform theory for thermodynamic critical points of spatially homogeneous systems. The ideas which arise here are very similar to the more complex ones that arise in the kinetic case.

2.1. GENERAL THEORY

The thermodynamic theory proceeds from the Einstein fluctuation-entropy formula. We assume that the entropy of the system can be characterized by a parameter x . The equilibrium entropy is $S_0 = S(x_0)$, Let

$$v(x)dx = \text{Pr} \left\{ \begin{array}{l} \text{system reaches a state in which} \\ x \in (x, x+dx) \end{array} \right\} . \quad (2.1)$$

Then (9)

$$v(x)dx = c \exp \left[\frac{S(x) - S_0}{k} \right] dx = c \exp \left[\frac{\Delta S}{k} \right] dx, \quad (2.2)$$

where c is a normalizing constant. As is usually done, we have eliminated the time variable from the discussion of thermodynamic problems (16). This elimination has certain conceptual drawbacks when one tries to describe the time evolution of a system. However, we shall follow standard notation here. We also follow the standard procedure of dividing the universe into the sub-system of interest, characterized by a variable y and certain intensive

parameters $(\alpha_1, \dots, \alpha_n) = \alpha$, and an external reservoir which is assumed to remain in thermodynamic equilibrium (17, page 274). Then

$$v(y)dy = \tilde{c} \exp \left[\frac{-\Delta W(y)}{kT} \right] dy, \quad (2.3)$$

where $\Delta W(y)$ is the work done on the subsystem by an external source. Let $\phi(y, \alpha)$ denote the potential of the system, so that (with $y=y_0$ denoting equilibrium):

$$\Delta W(y) = \phi(y, \alpha) - \phi(y_0, \alpha). \quad (2.4)$$

Then

$$v(y)dy = c \exp \left[\frac{-\phi(y, \alpha)}{kT} \right] dy. \quad (2.5)$$

Usually (9,16,17) $\phi(y, \alpha)$ is expanded in a Taylor series, keeping terms of second order

$$\begin{aligned} \phi(y, \alpha) = & \phi(y_0, \alpha) + \phi'(y_0, \alpha)(y-y_0) \\ & + \phi'' \frac{(y_0, \alpha)}{2} (y-y_0)^2 + \dots \end{aligned} \quad (2.6)$$

At equilibrium $\phi'(y_0, \alpha) = 0$ and $\phi''(y_0, \alpha) > 0$. Thus we obtain

$$v(y)dy \sim \exp \left[\frac{-1}{2kT} \phi''(y_0, \alpha) (y-y_0)^2 \right] dy. \quad (2.7)$$

Equation (2.7) gives a locally Gaussian density with variance

$$\sigma^2 = \frac{kT}{\phi''(y_0, \alpha)} \quad (2.8)$$

The Onsager-Machlup theory of irreversible processes(18) proceeds from this point.

Suppose, however, that there is a value $\alpha = \alpha_m$ such that

$$\phi'(y_0, \alpha_m) = \phi''(y_0, \alpha_m) = 0 \quad \phi'''(y_0, \alpha_m) \neq 0. \quad (2.9)$$

Such a point might correspond to a second order phase transition(2,9). A third term is needed in the Taylor expansion (2.6).

For α near α_m , instead of (2.7), one obtains

$$v(y)dy \sim \exp \left[\frac{-1}{kT} \left\{ \frac{\phi''(y_0, \alpha)}{2} (y-y_0)^2 + \frac{\phi'''(y_0, \alpha)}{6} (y-y_0)^3 \right\} \right] dy. \quad (2.10)$$

A simple change of variables converts (2.10) to

$$v(y)dy \sim \exp \left[\frac{-1}{kT} \left\{ z(y)^3 - \tilde{\alpha} z(y) + \beta \right\} \right] dz(y), \quad (2.11)$$

where $z(y)$ is a regular function of y and $\tilde{\alpha}(\alpha)$ is a regular function of α with the property that $\tilde{\alpha}(\alpha_m) = 0$. A density of the form (2.11) is called an Airy density.

At thermodynamic critical points, if $\phi''(y_0, \alpha)$ vanishes, then usually $\phi'''(y_0, \alpha)$ also will vanish. This can be shown by using free energy arguments(17). Such will not be the case for kinetic equations, however.

It is also possible that at a different value of $\alpha, \alpha = \alpha_c$.

$$\phi''(y_0, \alpha_c) = \phi'''(y_0, \alpha_c) = 0; \phi^{(iv)}(y_0, \alpha_c) \neq 0. \quad (2.12)$$

Such a point corresponds to a first order phase transition(2). We take another term in the Taylor expansion (2.6) and obtain, for α near α_c :

$$v(y)dy \sim \exp \left[\frac{-1}{kT} \left\{ \frac{\phi^{iv}(y_0, \alpha)(y-y_0)^4}{24} + \frac{\phi'''(y_0, \alpha)(y-y_0)^3}{6} + \frac{\phi''(y_0, \alpha)(y-y_0)^2}{2} \right\} \right] dy. \quad (2.13)$$

Equation (2.13) can be put into the form

$$v(y)dy \sim \exp \left[\frac{-1}{kT} \left\{ \frac{z(y)^4}{4} + \frac{\tilde{\alpha}_1(\alpha)z(y)^2}{2} + \tilde{\alpha}_2(\alpha) \right\} \right] dz(y). \quad (2.14)$$

In (2.14), $z(y)$ is a regular function of y ; $\tilde{\alpha}_1, \tilde{\alpha}_2$ are regular functions of α and vanish at $\alpha = \alpha_c$. We call the density (2.14) a Pearcey density.

Equations (2.10,14) represent a formal extension of Onsager's theory to critical point phenomena. In light of (1.4), we are lead to totally nonlinear dynamics when $\alpha=\alpha_c$ or $\alpha=\alpha_m$. Our result is, of course, purely formal and is applicable to small deviations from equilibrium only, which is the best that one can expect of a thermodynamic theory.

2.2. VOLUME FLUCTUATIONS OF A VAN DER WAALS GAS

As an example of the above analysis, we consider the volume fluctuations of a gas at the liquid-vapor critical point, using a van der Waals model. Levich(17) shows that in this case

$$\Delta W = P_0 \Delta V + \Delta F, \quad (2.15)$$

where P_0 is the equilibrium pressure of the reservoir. Expanding ΔF gives

$$\begin{aligned} \Delta W = P_0 \Delta V + \left(\frac{\partial F}{\partial V} \right)_T \Delta V + \left(\frac{\partial^2 F}{\partial V^2} \right)_T \frac{(\Delta V)^2}{2} \\ + \left(\frac{\partial^3 F}{\partial V^3} \right)_T \frac{(\Delta V)^3}{6} + \left(\frac{\partial^4 F}{\partial V^4} \right)_T \frac{(\Delta V)^4}{24}, \end{aligned} \quad (2.16)$$

where $\Delta V = V - V_0$ is the deviation from the equilibrium volume.

Usually, only two terms are of (2.16) are used. If we set

$$-\left(\frac{\partial F}{\partial V} \right)_T = P = P_0, \text{ we find}$$

$$E((\Delta V)^2) = \frac{kT}{|(\partial P / \partial V)_T|} \quad (2.17)$$

The value of $E(\Delta V)^2$ becomes infinite at $T=T_c$. The cause of the divergence is purely mathematical: only two terms of the expansion were used. At the critical temperature(17)

$$\left(\frac{\partial P}{\partial V}\right)_{T_c} = \left(\frac{\partial^2 P}{\partial V^2}\right)_{T_c} = 0 \text{ but } \left(\frac{\partial^3 P}{\partial V^3}\right)_{T_c} \neq 0. \quad (2.18)$$

Then (2.16) becomes

$$\Delta W = -\left(\frac{\partial^3 P}{\partial V^3}\right)_{T_c} \frac{(\Delta V)^4}{24} \quad (2.19)$$

Equation (2.19) is general. We now specialize to a van der Waals gas, for which

$$P = \frac{RT}{V-b} - \frac{a}{V^2} \quad (2.20)$$

The conditions (2.18) lead to the following values of the critical parameters:

$$T_c = \frac{8a}{27bR} \quad V_c = 3b \quad P_c = \frac{a}{27b^2} \quad (2.21)$$

Then

$$\left(\frac{\partial^3 P}{\partial V^3}\right)_{T_c} \equiv -\gamma = \frac{.0123457a}{b^5} . \quad (2.22)$$

Then we obtain

$$v(\Delta V)d(\Delta V) = c \exp \left[\frac{-\gamma(\Delta V)^4}{24kT_c} \right] d(\Delta V), \quad (2.23)$$

where c is the normalization constant. Hence we obtain

$$\begin{aligned} E((\Delta V)^2) &= \frac{\int_{-\infty}^{\infty} x^2 \exp \left[\frac{-\gamma x^4}{24kT_c} \right] dx}{\int_{-\infty}^{\infty} \exp \left[\frac{-\gamma x^4}{24kT_c} \right] dx} \\ &= \left(\frac{kT_c}{\gamma} \right)^{1/2} \frac{\int_{-\infty}^{\infty} y^2 e^{-y^4/24} dy}{\int_{-\infty}^{\infty} e^{-y^4/24} dy} . \end{aligned} \quad (2.24)$$

Thus, the van der Waals theory (i.e. the classical theory, see below) predicts $E((\Delta V)^2) \propto T_c^{1/2}$. In Table 1, we give the results of calculations of $E((\Delta V)^2)$ for a number of gases. The above theory yields values of $E((\Delta V)^2)$ which are large, but not infinite.

The thermodynamic theory presented in this section is a "classical" theory(19) and thus will not predict the "correct" critical exponents. This is a fault of the use of thermodynamic theory per se. It is not clear how the thermodynamic theory given above could be modified to yield the correct exponents. In a recent work Mou et. al.(34), using the master equation, also derived the classical result.

TABLE 1

VOLUME FLUCTUATIONS AT THE CRITICAL POINT

<u>Gas</u>	<u>V_c (cc/mole) *</u>	<u>$E ((\Delta V)^2)/V_c$</u>
He	57.6	4.416×10^2
H ₂	65.0	4.336×10^2
N ₂	90.0	4.832×10^2
CO	90.0	5.060×10^2
O ₂	74.4	4.702×10^2
C ₂ H ₄	127.5	5.168×10^2
CO ₂	95.7	5.147×10^2
NH ₃	72.4	6.780×10^2
H ₂ O	45	1.187×10^2

*from (20, page 18)

SECTION 3. STOCHASTIC KINETIC EQUATIONS, DIFFUSION APPROXIMATION AND FOKKER-PLANCK EQUATION

The thermodynamic theory of §2 can not be used to treat highly nonequilibrium kinetic phenomena, which are of interest in many areas of chemistry, physics and biology. Let $\tilde{x}(t)$ denote the statistical variables. Often we can postulate an equation for the mean value of $\tilde{x}(t)$, $x(t)$:

$$\begin{aligned} \frac{dx^i}{dt} &= b^i(x, \alpha) & x^i(0) &= x_0^i & i &= 1, \dots, n \\ \alpha &= \{\alpha_1, \dots, \alpha_m\}. \end{aligned} \tag{3.1}$$

In order to treat fluctuations, we need to know the kinetic equation that $\tilde{x}(t)$ satisfies. Ideally, we would start with the Liouville equation and derive the kinetic equation. Such a derivation is possible for only the simplest system(33). Instead, we shall use a generalization of the Langevin method. We will add a zero-mean stochastic term to (3.1). The stochastic function $\tilde{y}(\tau)$ is characterized by a microscopic time scale, τ , small compared to the macroscopic scale on which measurements are made.

Hence

$$\Delta\tau = \eta^2 \Delta t \tag{3.2}$$

where η is a small parameter. We will not assume that $\tilde{y}$ has a δ -correlation function and let

$$\gamma^{kl} = \int_0^\infty E(\tilde{y}^k(s) \tilde{y}^l(0)) ds.$$

We assume that $\tilde{x}(t) = \tilde{x}_\eta(t)$ satisfies the stochastic kinetic equation

$$\frac{dx_\eta^i}{dt} = b^i(\tilde{x}_\eta) + \frac{\sqrt{\epsilon}}{\eta} f_j^i(\tilde{x}_\eta) \tilde{y}^j(t/\eta^2) \quad (3.3)$$

In equation (3.3), ϵ is a small parameter characterizing the size of the system and related to the intensity of the fluctuations (1,2,13,15). Hence $\epsilon \rightarrow 0$ corresponds to the thermodynamic limit. The field $f_j^i(x)$ is a given deterministic field. Ideally, one would like to calculate f_j^i from basic principles. Since (3.3) is somewhat ad-hoc, a prescription is needed for the calculation of f_j^i . (One such prescription is the fluctuation-description theorem. Another is given by Keizer(22)). As $\eta \rightarrow 0$, $x_\eta(t) \rightarrow x(t)$, a diffusion process (22). We set

$$u(x) = E_x \left\{ u_0(\tilde{x}(t)) \mid \tilde{x}(0) = x \right\} \quad (3.4)$$

Then $u(x)$ satisfies

$$\frac{\partial u}{\partial t} = \frac{\epsilon a^{ij}}{2} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i} + c^i \epsilon \frac{\partial u}{\partial x^i} \equiv Lu \quad (3.5)$$

In (3.5), we use the convention that repeated indices are summed from 1 to n and

$$a^{ij} = f_k^i f_l^j (\gamma^{kl} + \gamma^{lk}) \quad (3.6)$$

$$c^i = \gamma^{kl} f_k^j \frac{\partial f_l^i}{\partial x^j} = \frac{1}{4} \frac{\partial}{\partial x^j} a^{ij}. \quad (3.7)$$

On the other hand, if $\tilde{Y}$ has a δ -correlation function (white noise), then we obtain

$$\frac{\partial u}{\partial t} = \frac{\epsilon a^{ij}}{2} \frac{\partial^2 u}{\partial x^i \partial x^j} + b^i \frac{\partial u}{\partial x^i}. \quad (3.8)$$

Numerical work(3) indicates that if the boundaries are non-singular, then equation (3.5) and (3.8) yield equivalent solutions for $\epsilon < .1$.

We also note that equation (3.3) is a stochastic equation with correlations and hence is a more reasonable representation than a white noise equation. Furthermore, equation (3.5) [or (3.8)] is derived rigorously--no expansion procedure is needed (compare(11)). Equation (3.8) is the backward equation. Usually, in the physical literature the forward or Fokker-Planck equation is used. This equation can not be derived rigorously. No expansion procedure will rigorously give the Fokker-Planck equation(1,11). Instead, we will obtain the Fokker-Planck equation by using the theory of

partial differential equations. In (24, 25) it is shown now this can be achieved. Let

$$v(x, t) dx = \Pr \left\{ x \leq \tilde{x}(t) \leq x + dx \right\}. \quad (3.9)$$

Then $v(x, t)$ satisfies (at least weakly) the adjoint equation

$$v_t = L^* v = \frac{\varepsilon}{2} (a^{ij} v)_{ij} - (b^i v)_i - \varepsilon (c^i v)_i. \quad (3.10)$$

In the derivation of (3.10), there is a question of boundary terms for u, v as $|x| \rightarrow \infty$ (25). For the problems considered here these questions are relatively unimportant. In (3.10) subscripts indicate differentiation.

SECTION 4. CANONICAL DENSITIES

In this section, we consider the time independent, one dimensional Fokker-Planck equation

$$\epsilon \frac{(av)_{xx}}{2} - (bv)_x = 0 \quad (4.1)$$

subject to

$$\int_{-\infty}^{\infty} v(s) ds = 1 \quad \lim_{|s| \rightarrow \infty} v(s) = 0. \quad (4.2)$$

The solution of (4.1,2) gives the steady state (but not necessarily equilibrium) density for a process satisfying (3.3). Our results are valid for $\epsilon \rightarrow 0$ ("thermodynamic limit") and generalize the thermodynamic results of §2. In later sections, we generalize the solutions obtained here to solve time dependent, multidimensional problems.

When (4.1) is integrated twice and (4.2) is applied, we find

$$v(x) = k \left[\exp \left\{ \int^x \frac{2b}{\epsilon a} ds \right\} \right], \quad (4.3)$$

where k is the normalization constant

$$k = \int_{-\infty}^{\infty} \exp \left\{ \int^x \frac{2b}{\epsilon a} ds \right\} dx. \quad (4.4)$$

The main contribution to (4.3) comes from the maximum of the function

$$\phi(x) = \int^x \frac{2b}{a} ds. \quad (4.5)$$

We now assume that there is a steady state (i.e. $b(x)=0$ has a solution), x_0 . The steady state is classified according to its dynamic behavior.

The normal type steady state x_0 is characterized by

$$b(x_0) = 0, \quad b'(x_0) \neq 0. \quad (4.6)$$

We are interested in stable steady states, so that we assume $b'(x_0) < 0$. Thus, perturbations from x_0 decay exponentially. When $\phi(x)$ is expanded about x_0 , we obtain:

$$\begin{aligned} v(x) \sim k & \left[\exp \left\{ \frac{-|b'(x_0)| (x-x_0)^2}{\epsilon a(x_0)} \right\} \right] \\ & + o \left[\exp \left\{ \frac{-|b'(x_0)| (x-x_0)^2}{\epsilon a(x_0)} \right\} \right]. \end{aligned} \quad (4.7)$$

Thus, we obtain a locally Gaussian density, for small ϵ . This result has also been derived by Kubo et. al.(1) and Keizer(23) by different arguments. It is the standard result in the theory of nonequilibrium thermodynamics (18).

In the marginal case, b depends on one parameter α such that when $\alpha = \alpha_c$ the marginal type steady state satisfies

$$b(x_0, \alpha_c) = 0 \quad b'(x_0, \alpha_c) = 0 \quad b''(x_0, \alpha_c) \neq 0. \quad (4.8)$$

The canonical dynamics corresponding to the marginal case are (4)

$$\dot{x} = x^2 - \alpha. \quad (4.9)$$

The flow of such dynamics is sketched in Fig. 1. We need to replace the conditions (4.2) by:

$$\lim_{s \rightarrow -\infty} v(s) = 0, \quad \int_{-\infty}^{x_F} v(s) ds = 1 \quad (4.10)$$

where $x_F < \infty$ is an end value for x (see also 3).

Since $b'(x_0, \alpha_c) = 0$, the expansion used to obtain (4.7) breaks down. Hence the Gaussian density breaks down. In particular, from (4.7) we have

$$E(x^2) \propto 1/|b'(x_0, \alpha)|. \quad (4.11)$$

Thus, when $\alpha = \alpha_c$, the Gaussian density yields an infinite variance. This is, of course, a purely mathematical divergence and has nothing to do with the physical problem. We take an extra term in the expansion of $\phi(x, \alpha)$ to obtain

$$v(x) \sim k \exp \left[\frac{b'(x_0, \alpha) (x-x_0)^2}{\epsilon a(x_0)} + \frac{b''(x_0, \alpha) (x-x_0)^3}{3\epsilon a} \right]. \quad (4.12)$$

A change of variables converts (4.12) to the Airy density (2.11).

The critical type steady state is characterized by two parameters, α, β such that when $\alpha = \alpha_c$ and $\beta = \beta_c$

$$b(x_0, \alpha_c, \beta_c) = b'(x_0, \alpha_c, \beta_c) = b''(x_0, \alpha_c, \beta_c) = 0$$

$$b'''(x_0, \alpha_c, \beta_c) \neq 0. \quad (4.13)$$

The canonical dynamics of a critical type dynamical system are (4)

$$\dot{x} = \pm x^3 + \alpha x + \beta. \quad (4.14)$$

In this case $\alpha_c = \beta_c = 0$. The flow of (4.14) is sketched in Fig. 1.

It is clear that the Airy and Gaussian densities both break down when $\alpha = \alpha_c$ and $\beta = \beta_c$. In this case, we take one more term in the Taylor expansion of $\phi(x)$ and obtain the Pearcy density (2.14).

The results of this section can be obtained by direct use of Levinson's theorem(26). It is clear that the Gaussian approximation will be valid whenever $|b'(x_0, \alpha)/b''(x_0, \alpha)| \gg 1$.

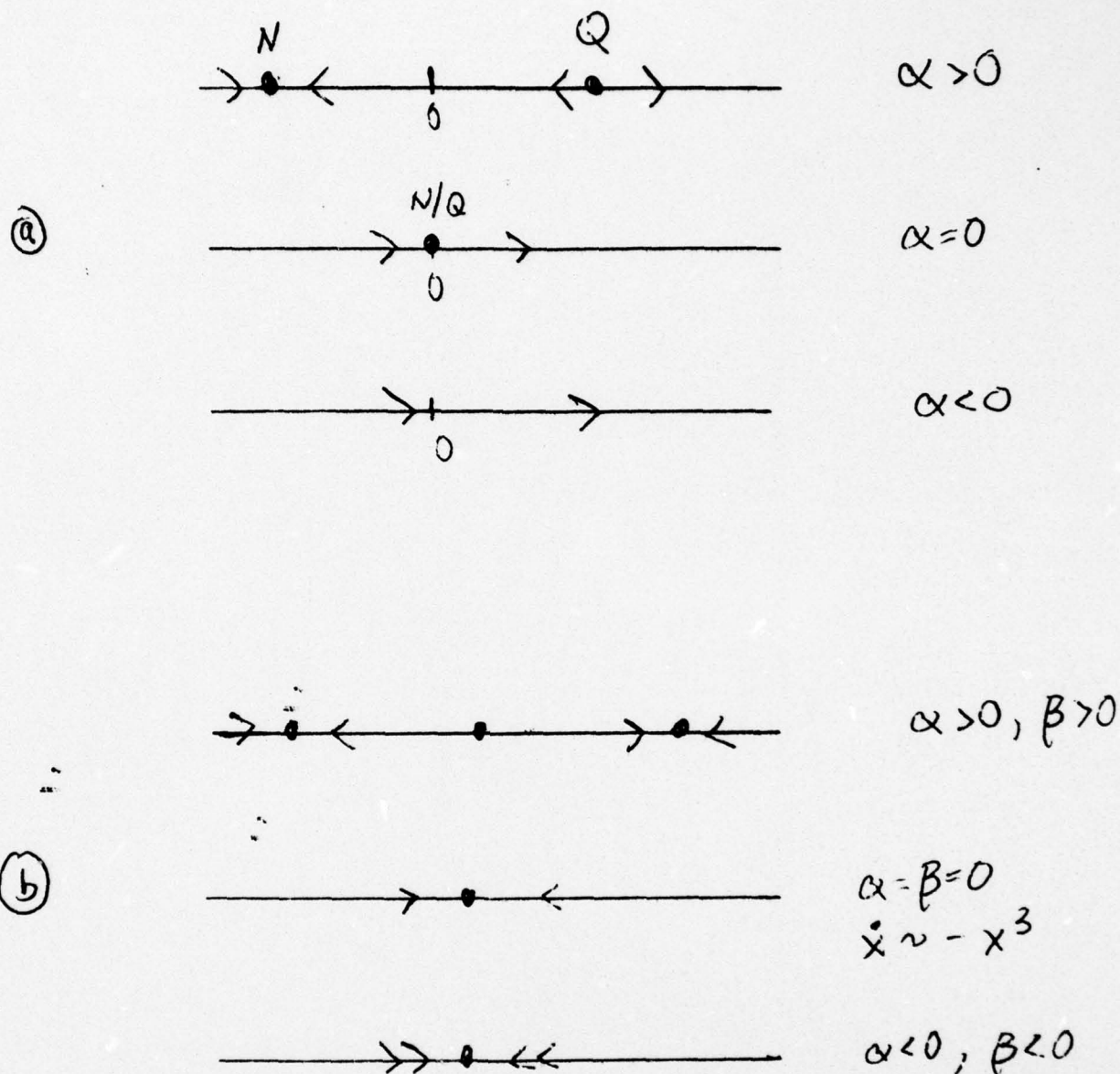


FIG. 1: A) THE MARGINAL TYPE DYNAMICAL SYSTEM HAS TWO STEADY STATES WHEN A PARAMETER $\alpha > 0$, ONE DEGENERATE STEADY STATE WHEN $\alpha = 0$ AND NO STEADY STATES WHEN $\alpha < 0$. B) THE CRITICAL TYPE DYNAMICAL SYSTEM HAS THREE STEADY STATES WHEN $\alpha > 0, \beta > 0$, ONE DEGENERATE STEADY STATE WHEN $\alpha = \beta = 0$ AND ONE STEADY STATE WHEN $\alpha < 0, \beta < 0$.

SECTION 5.

TIME DEPENDENT, MULTIDIMENSIONAL FOKKER-PLANCK EQUATION

In this section, we construct regular solutions of the time dependent Fokker-Planck equation (3.5). Since Ludwig(5) has given the construction for the normal case, we only consider marginal and critical type steady states. Our goal is to construct densities that have finite second moments. Exact definitions of marginal and critical type steady states in multidimensional systems are given in Appendix A.

5.1. MARGINAL TYPE STEADY STATE

We seek a solution of (3.5) of the form

$$v(x,t) = \exp \left[\frac{-1}{\epsilon} \left(\frac{\psi(x,t)^3}{3} - \alpha \psi(x,t) \right) \right] \sum_{n=0} \epsilon^n z^n(x,t). \quad (5.1)$$

The form of (5.1) is a "ray ansatz"(27). In it, $\psi(x,t)$, α , and the functions $z^0(x,t), z^1(x,t) \dots$ are to be determined. In practice we are often interested in just the first term of (5.1). After derivatives are evaluated, terms are collected according to powers of ϵ . We obtain

$$\begin{aligned}
0 = & \exp \left[-\frac{1}{\varepsilon} \left(\frac{\Psi^3}{3} - \tilde{\alpha} \Psi \right) \right] / \varepsilon \left\{ \Psi_t + b^i \Psi_i \right. \\
& + (\tilde{\alpha} - \Psi^2) \frac{a^{ij}}{2} \Psi_i \Psi_j \left. \right\} z^0 (\tilde{\alpha} - \Psi^2) + \exp \left[-\frac{1}{\varepsilon} \left(\frac{\Psi^3}{3} - \tilde{\alpha} \Psi \right) \right] \left\{ b^i_{,j} z^0 \right. \\
& + b^i z^0_i + \frac{a^{ii}_{,ij}}{2} z^0 + \frac{a^{ij}_{,j}}{2} (\tilde{\alpha} - \Psi^2) \Psi_i z^0 \left. \right\} \quad (5.2)
\end{aligned}$$

$$\begin{aligned}
& z_t + a^{ij} (-2\Psi \Psi_i \Psi_j z^0 + (\tilde{\alpha} - \Psi^2) \Psi_{ij} z^0 \\
& + 2(\tilde{\alpha} - \Psi^2) \Psi_j z^0_i - c^i (\tilde{\alpha} - \Psi^2) \Psi_i z^0) \left. \right\} \\
& + 0 \left(\varepsilon \exp \left[-\frac{1}{\varepsilon} \left(\frac{\Psi^3}{3} - \tilde{\alpha} \Psi \right) \right] \right) .
\end{aligned}$$

The leading coefficient of ε vanishes if

$$\Psi_t + b^i \Psi_i + \frac{(\tilde{\alpha} - \Psi^2)}{2} a^{ij} \Psi_i \Psi_j = 0 \quad (5.3)$$

Equation (5.3) is a generalized eikonal equation(3). In the rest of this paper, we shall assume that the initial data for v are concentrated at a point

$$v(x, 0) = \delta(x - x_0) \quad (5.4)$$

At the deterministic steady states, N, Q (see Fig. 2) we expect that $d\Psi/dt = \Psi_t + b^i \Psi_i = 0$. We set $\Psi^2 = \tilde{\alpha}$ at those points. The

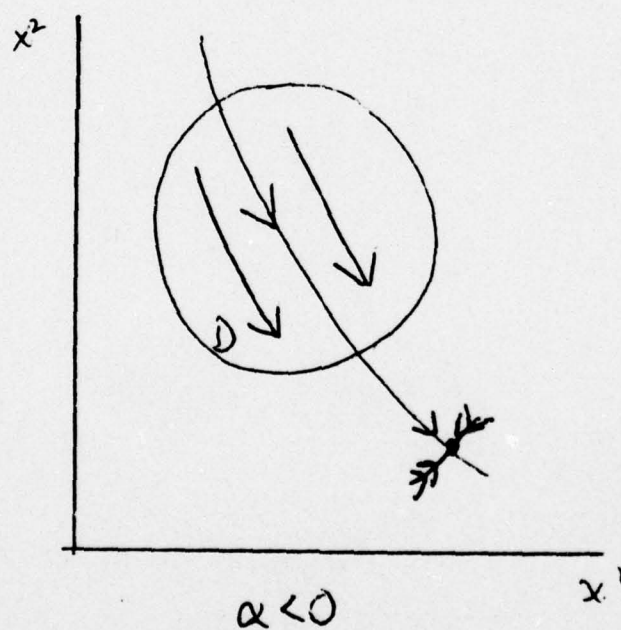
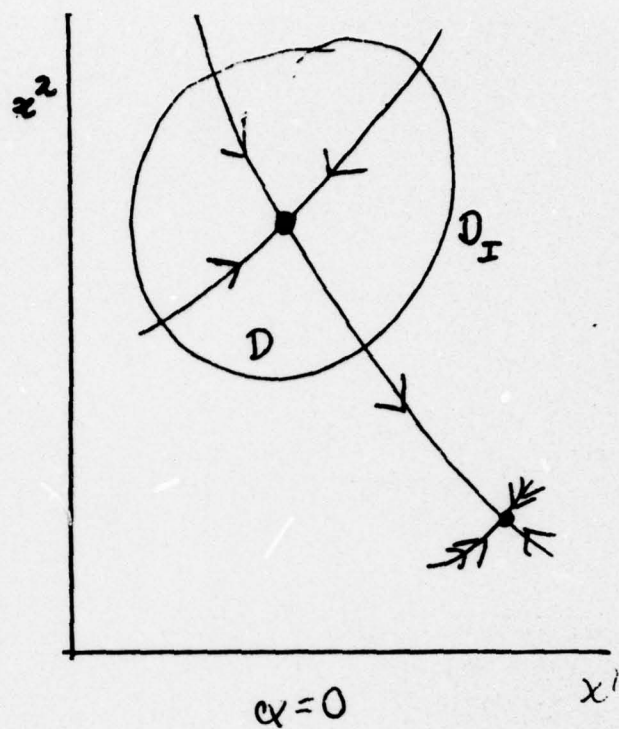
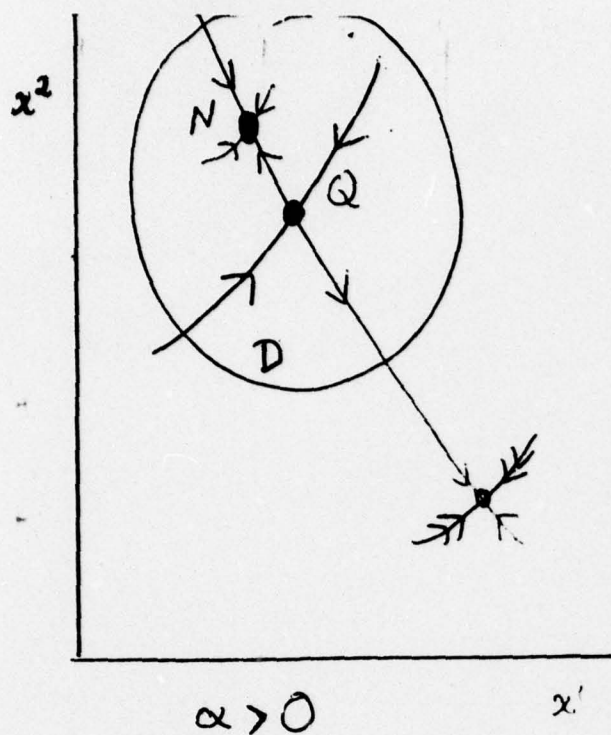


FIG. 2: A TWO DIMENSIONAL MARGINAL DYNAMICAL SYSTEM IN A DOMAIN D IN $\mathbb{R}^2$. WHEN $\alpha < 0$ THE DETERMINISTIC FLOW IS ALWAYS ACROSS D , A PROBLEM FIRST STUDIED BY LEVINSON (28).

node N should correspond to a local maximum for $v(x, t)$. Hence we set $\Psi(N) = \sqrt{\tilde{\alpha}}$. Similar reasoning leads to $\Psi(Q) = -\sqrt{\tilde{\alpha}}$.

The value of α is still undertermined. It can be obtained by the following iterative procedure(3). (If higher order terms are to be considered, then it is necessary to expand

$\tilde{\alpha} = \sum_{k=0} \epsilon^k \tilde{\alpha}_k$. In that case, all of the parameters are determined

in a manner analogous to the determination of $\tilde{\alpha}$. We start at the node N, where $\Psi = \sqrt{\tilde{\alpha}^0}$, the first estimate for $\tilde{\alpha}$. Equation (5.3) can be solved by the method of characteristics(12). The characteristic equations are:

$$\frac{dt}{ds} = 1 \quad (5.5a)$$

$$\frac{dx^i}{ds} = b^i + (\tilde{\alpha} - \Psi^2) p_j a^{ij} \quad (5.5b)$$

$$\frac{d\Psi}{ds} = p_k \frac{dx^k}{ds} = \frac{1}{2}(\tilde{\alpha} - \Psi^2) a^{ij} p_i p_j \quad (5.5c)$$

$$\frac{dp_k}{ds} = - \left(2p_k (-\Psi a^{ij} p_i p_j) + b_{,k}^i p_i + \frac{(\alpha - \Psi^2)}{2} a_{,k}^{ij} p_i p_j \right). \quad (5.5d)$$

Initial data is given on an ellipsoid surrounding N. As $x \rightarrow Q$, the value of α should approach $-\sqrt{\tilde{\alpha}^0}$. If it does not, then a new estimate $\tilde{\alpha}^{(1)}$ is needed. Elsewhere, we have shown that iterates of $\tilde{\alpha}$ can be determined by using the method of false position and that α is a regular function of the deterministic parameter $\alpha(3)$.

When N and Q coalesce (Fig. 2b), $\tilde{\alpha} = 0$. After the annihilation of N, Q (Fig. 2c), $\tilde{\alpha} < 0$. The stochastic problem for a dynamical system similar to Fig. 2c is an old one, solved by Levinson(28) and Ventcel and Friedlin(29). Consequently, we restrict ourselves to the dynamic cases represented by Figs. 2a,b.

From (5.5b), we see that if $\Psi^2 = \alpha$ on a trajectory, then $dx^i/ds = b^i$, so that the trajectory is a deterministic trajectory. In this way, we will be able to estimate deviations from a given deterministic trajectory. When (5.3) is differentiated with respect to x^k and evaluated on a trajectory we find:

$$\frac{d\Psi_k}{dt} + b_{,k}^i \Psi_k \mp \sqrt{\tilde{\alpha}} a^{ij} \Psi_i \Psi_j \Psi_k = 0 \quad k = 1, 2, \dots, n \quad (5.6)$$

In (5.6) the (-) sign corresponds to trajectories that enter N, the (+) sign to trajectories that enter Q. At either of the steady states, we obtain

$$b_{,k}^i \Psi_k \mp \sqrt{\tilde{\alpha}} a^{ij} \Psi_i \Psi_j \Psi_k = 0 \quad k = 1, \dots, n \quad (5.7)$$

Equation (5.7) can be solved to yield values of Ψ_k at N or Q.

When N and Q coalesce, so that $\alpha = 0$ and conditions (A) hold, it is possible to show that the Ψ_k can be calculated. For example,

consider the case of only one spatial dimension. Then (5.7) becomes:

$$b_{,x} - \Psi(N) a \Psi_x^2 = 0 \quad (5.8)$$

or

$$\Psi_x^2(N) = \frac{b_{,x}}{\Psi(N) a} . \quad (5.9)$$

In obtaining (5.8,9), we have replaced $\sqrt{\alpha}$ by $\Psi(N)$. When N, Q coalesce, $b_x \rightarrow 0$ and $\Psi(N) \rightarrow 0$. One application of l'Hospital's rule gives

$$\Psi_x^3(N) = \frac{b_{,xx}(N)}{a(N)} . \quad (5.10)$$

A similar, but more complicated, calculation holds in the multi-dimensional cases(3).

Thus far, we have given our construction without any boundary conditions. In order to determine z^0 , we need to specify the boundary conditions. As time progresses, the process will tend to concentrate (if it is still in D) near D_I . (Fig. 2).

The $0 \left(\exp \left(\frac{1}{\epsilon} \left(\frac{\Psi^3}{3} - \alpha \Psi \right) \right) \right)$ term in (5.2) yields a "transport" equation for z^0 (25, 27). It takes the form

$$\frac{dz^0}{ds} + f(s) z^0 = 0, \quad (5.11)$$

i.e.

$$z^0(s) = z^0(0) \exp \left[- \int^s f(s') ds \right] \quad (5.11a)$$

When the initial data is concentrated at a point, Ludwig has shown that the appropriate initial data for z is $z^0(0) = \text{constant}$.

5.2. CRITICAL TYPE STEADY STATE

For the critical type steady state, instead of (5.1), we seek a solution of (3.5) of the form

$$v(x,t) = \exp \left[-\frac{1}{\varepsilon} \left(\frac{1}{4} \psi^4 - \frac{\gamma \psi^2}{2} - \gamma \psi \right) \right] \sum_{n=0}^{\infty} \varepsilon^n z^n(x,t). \quad (5.12)$$

In this case, it is possible to impose the conditions on $v(x,t)$ that

$$v \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad \int_{-\infty}^{\infty} v(x,t) dx = 1 \quad (5.13)$$

and take all of R^n as the domain of interest. Instead of (5.3), we obtain

$$\psi_t + b^i \psi_i + \frac{a^{ij}}{2} (\psi^3 - \gamma \psi - \beta) \psi_i \psi_j = 0. \quad (5.14)$$

The value of ψ at the deterministic nodes N_1, N_2 and saddle Q (Fig. 3) is determined in a manner analogous to the one used in §5.1. The values of the parameters are also determined in a

similar fashion. It is possible to show that all constructions remain regular as the steady states coalesce(3). The function $z^0(x,t)$ can also be determined in manner analogous to the previous case.

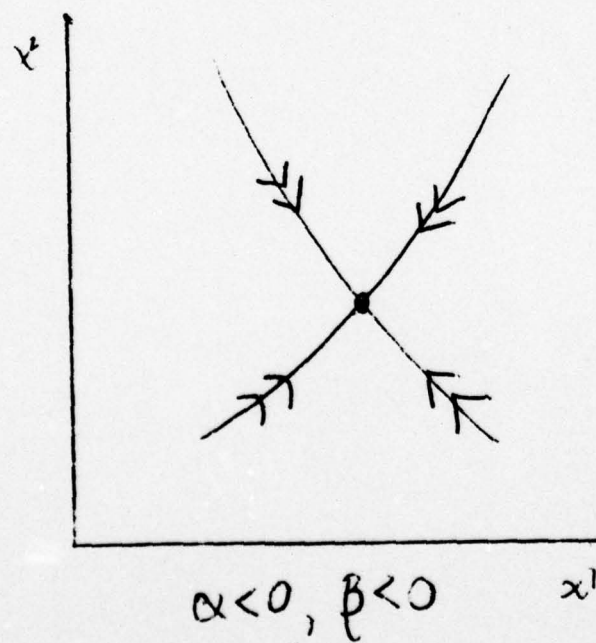
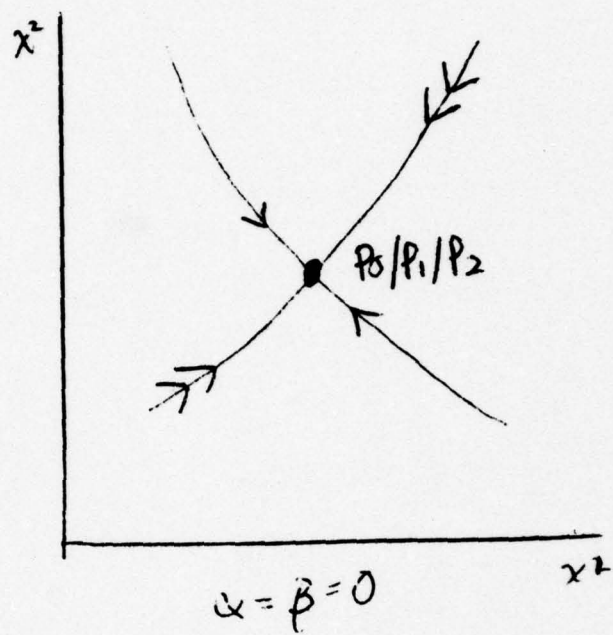
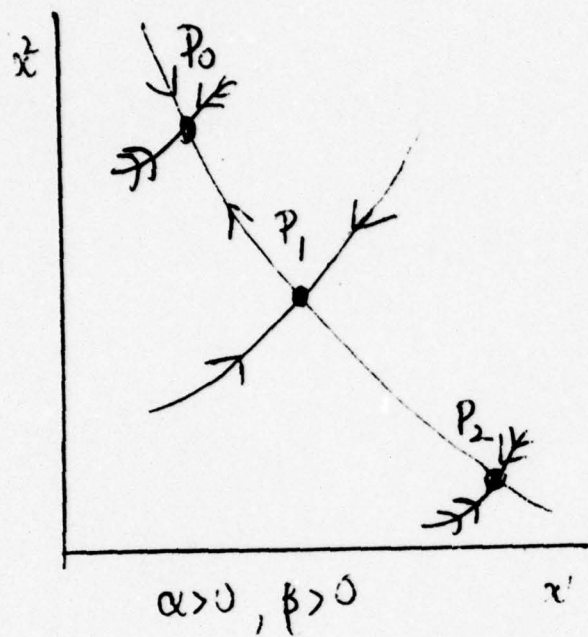


FIG. 3: A TWO DIMENSIONAL CRITICAL TYPE DYNAMICAL SYSTEM

SECTION 6. CORRELATION FUNCTIONS FOR CRITICAL TYPE SYSTEMS

In many physical problems, the object of interest is the correlation function

$$\begin{aligned} R(\tau) &= E \left\{ \tilde{x}(t) \tilde{x}(t + \tau) \right\} \\ &= \int \int x x^1 \Pr \left\{ \tilde{x}(t) \in (x, x + dx), \tilde{x}(t + \tau) \in (x^1, x^1 + dx^1) \right\} dx dx^1. \end{aligned} \quad (6.1)$$

Since the process in our problem is assumed to be stationary, $R(\tau) = E \left\{ \tilde{x}(0) \tilde{x}(\tau) \right\}$. Now we consider the conditional correlation function:

$$R_{x_0}(\tau) = E \left\{ \tilde{x}(0) \tilde{x}(\tau) \mid \tilde{x}(0) = x_0 \right\}. \quad (6.2)$$

If $v_0(x_0)dx_0$ the initial density for x_0 , then we clearly have

$$R(\tau) = \int R_{x_0}(\tau) v_0(x_0) dx_0. \quad (6.3)$$

However,

$$\begin{aligned} R_{x_0}(\tau) &= \int x^1 \Pr \left\{ \tilde{x}(\tau) \in (x^1, x^1 + dx^1) \mid \tilde{x}(0) = x_0 \right\} dx^1 \\ &= \int x^1 v_{x_0}(x^1, \tau) dx^1, \end{aligned} \quad (6.4)$$

where $v_{x_0}(x^1, \tau) dx^1$ was calculated in the previous section. Thus

$$R_{x_0}(\tau) = \int x^1 \exp \left[-\frac{1}{\varepsilon} \left(\frac{\psi(x^1, \tau)^4}{4} - \frac{\tilde{\alpha}\psi^2}{2} - \beta\psi \right) z(x, \tau) \right] dx \Big|_{x(0) = x_0} \quad (6.5)$$

Namely, we start the ray calculation at $x = x_0$ and integrate the ray equations

$$\frac{dt}{ds} = 1 \quad \frac{dx^i}{ds} = b^i + \psi^3 a^{ij} p_j. \quad (6.6)$$

$$\begin{aligned} \frac{d\psi}{ds} = p_k \frac{dx^k}{ds} \quad \frac{dp_k}{ds} = -(3a^{ij} p_i p_j p_k + b^i_{,k} p_i \\ + \psi^3 \frac{a^{ij}_{,k}}{2} p_i p_j), \end{aligned} \quad (6.7)$$

until $s = \tau$. Thus (6.5) can be evaluated. The full correlation function, obtained from (6.4) is

$$\begin{aligned} R(\tau) = \iint x^1 \exp \left[-\frac{1}{\varepsilon} \left(\frac{\psi(x^1, \tau)^4}{4} - \frac{\tilde{\alpha}\psi^2}{2} - \beta\psi \right) z^0(x, \tau) \right] \\ \Big|_{x(0) = x_0} v_0(x_0) dx^1 dx_0. \end{aligned} \quad (6.8)$$

In the next section, we give an example of such a calculation.

SECTION 7. CRITICAL HARMONIC OSCILLATOR

In this section, we consider a modified Duffing oscillator(30) in contact with a heat bath(13,31). We shall use a stochastic equation of the standard "mode-mode" form, but initially will indicate how a more general analysis would proceed.

The Hamiltonian of the system is

$$H = \frac{k(\eta)x^2}{2} + \frac{\alpha x^4}{4} + \frac{p^2}{2m} + \phi_{\text{int}}(\tilde{r}, x) + \frac{p_i^2}{2m_i}. \quad (7.1)$$

In (7.1), $\phi_{\text{int}}(\tilde{r}, x)$ is the interaction potential of the oscillator with coordinates (x, p) and heat bath with coordinates (r^i, p_i) . The last term represents the kinetic energy of the heat bath. The motion at the full system (oscillator + heat bath) is generated by

$$\dot{x} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial x} \quad (7.2)$$

$$\dot{r}^i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = -\frac{\partial H}{\partial r^i} \quad i = 1, \dots, N$$

The motion of the entire system occurs on a manifold in the phase space given by $H = \bar{E}$, where E is the initial (i.e. constant) energy of the system. This manifold, M , will be bounded and compact. We expect that the full system is ergodic(32). We are interested in a submanifold of M , M^1 , which is the manifold of

(x, p) coordinates. A possible projection operator from M to M^1 is

$$\mathcal{O} \left\{ x, p, r_j^i \dots r^n, p_1, \dots p_n \right\} = \left\{ x, p, 0, \dots, 0 \right\}. \quad (7.3)$$

Namely, we "project" from $M \rightarrow M^1$. On M^1 , we assume that the following measure exists

$$\begin{aligned} \bar{P}(t, x, p, A) = \Pr \left\{ \tilde{x}(t), \tilde{p}(t) \in A \mid \tilde{x}(0) \in (x, x \right. \\ \left. + dx), \tilde{p}(0) \in (p, p + dp) \right\}. \end{aligned} \quad (7.4)$$

We have introduced $\tilde{x}(t), \tilde{p}(t)$ as random variables. This is a result of the elimination of (x, p) from consideration. By averaging, we are treating the latter variables as random; thus x, p become random variables

Next, we assume that if $\tilde{x}(t) = x, \tilde{p}(t) = p$, then

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (\tilde{x} - x) \bar{P}(t + \Delta t, x, p, dx, dp) = p/m \quad (7.5)$$

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (\tilde{p} - p) \bar{P}(t + \Delta t, x, p, dx, dp) = \\ -k(\eta)x - \alpha x^3 - \gamma(x)p \end{aligned} \quad (7.6)$$

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \int (\tilde{p} - p)^2 \bar{P}(t + \Delta t, x, p, dx, dp) = \varepsilon a, \quad (7.7)$$

and that all other moments are zero. These assumptions have yet to be verified for any but the simplest system(33). These assumptions lead to the Langevin equations

$$\dot{x} = p/m \quad (7.8)$$

$$\dot{p} = -k(\eta)x - \alpha x^3 - \gamma x^3 + \sqrt{\epsilon a} \frac{d\tilde{y}}{dt}, \quad (7.9)$$

where $\tilde{y}(t)$ is an approximation to Gaussian white noise (e.g.(22), page 651).

The Fokker-Planck equation for the density $f(t,x,p)$ is

$$\epsilon a p f_{pp} - \frac{p}{m} f_x - ((-k(\eta)x - \alpha x^3 - \gamma p)f)_p = f_t, \quad (7.10)$$

where (see §3)

$$\rho = \int_0^\infty E(\tilde{y}(s)\tilde{y}(0)) ds. \quad (7.11)$$

The equilibrium density is ($\beta^{-1} = kT$)

$$f_{eq} = \exp \left[-\beta \left(\frac{p^2}{2m} + k(\eta)x + \frac{\alpha x^4}{4} \right) \right]. \quad (7.12)$$

We require that f_{eq} be a solution of (7.10) and obtain (the "fluctuation-dissipation" result):

$$\epsilon a \rho = \frac{2\gamma}{\beta} = 2kT\gamma. \quad (7.12)$$

The steady states of the averaged equations (7.8,9) are

$$p = 0 \quad x = 0, \pm i \sqrt{|k(\eta)|_\alpha}, \quad (7.14)$$

where we have made the assumption that $k \geq 0, \alpha > 0$.

We assume that when $\eta = \eta_c$, $k(\eta_c) = 0$. Then $(0,0)$ is a critical type steady state. At $\eta = \eta_c$, we have a critical harmonic oscillator.

We now nondimensionalize (7.9). We let $\epsilon = \frac{kT}{E_0} \ll 1$ be a small parameter, where E_0 is some reference energy. Introducing dimensionless variables by

$$v = \sqrt{\frac{E_0}{m}} v', \quad x = \sqrt{\frac{E_0 m}{\gamma_0^2}} x', \quad t = \frac{mt'}{\gamma_0}, \quad (7.15)$$

$$k = \frac{\gamma}{\sqrt{E_0 m}} k', \quad \alpha = \frac{\gamma^4}{(E_0 m)^{3/2}}, \quad \gamma(x) = \gamma'(x') \gamma_0$$

equation (7.9) becomes (with $v \equiv p/m$)

$$f_{t'} = \epsilon \gamma' f_{v', v'} - (v' f)_{x'}, \\ - \left((-k' x' - \alpha' (x')^3 - \gamma' v') f \right)_{v'}, \quad (7.16)$$

In the sequel, we drop the primes in (7.16). Since $k' \propto k$, when $\eta = \eta_c$, $k'(\eta_c) = 0$.

We now seek a solution of (7.16) of the form

$$f(t, x, v) = \exp \left[-\frac{1}{\epsilon} \left(\frac{1}{4} \Psi^4 - \frac{\alpha \Psi^2}{2} - \beta \Psi \right) \right] \sum \epsilon^n z^n(x, t) \quad (7.17)$$

where Ψ, α, β and z^n are to be determined. Following the procedure in §5, we obtain

$$\begin{aligned} \Psi_t + v \Psi_x - (k(\eta)x + \alpha x^3 + \gamma v) \Psi_v \\ + \gamma(\Psi^3 - \alpha \Psi - \beta) \Psi_v^2 = 0. \end{aligned} \quad (7.18)$$

Let us now specialize to $\eta = \eta_c$, $k = 0$; i.e. the critical harmonic oscillator. Then $\alpha = \beta = 0$ in (7.17) and (7.18). The ray equations become

$$\begin{aligned} \frac{dx}{dt} &= v & \frac{dv}{dt} &= -\alpha x^3 - \gamma v + 2\gamma \Psi_v \Psi^3 \\ \frac{d\Psi}{dt} &= v \Psi_x + \frac{dv}{dt} \Psi_v + \Psi_t \\ \frac{d\Psi_x}{dt} &= -6\Psi_x \Psi_v^2 \Psi^2 - 3\alpha x^2 \Psi_v \\ \frac{d\Psi_v}{dt} &= -6\Psi_v^3 \Psi^2 - \Psi_x + \gamma \Psi_v \end{aligned} \quad (7.19)$$

By integrating the ray equations from an initial point $\hat{x}(t_0) = x_0$, $v(t_0) = v_0$, we obtain the conditional density $f(x, t, v, x_0, v_0)$. Then following the procedure in §6, we can obtain the correlation function. Our solution thus allows the calculation of correlation functions at critical points. Once the correlation function is known, we obtain the spectrum of the oscillator by Fourier transform.

ACKNOWLEDGEMENTS

I thank Professor R.F. Snider for a discussion about a previous version of the manuscript. Professors Donald Ludwig and Neil Fenichel helped me appreciate some of the dynamical problems studied here.

REFERENCES

1. R. Kubo, K. Matsuo and K. Kitahara, J. Stat. Phys 9:51 (1973)
2. A. Nitzan, P. Ortoleva, J. Deutch and J. Ross, J. Chem. Phys. 61:7056 (1974)
3. M. Mangel, Technical Report 77-6, IAMS, Univ. of B.C., Vancouver, Canada
4. M. Mangel, "Relaxation of Critical Points: Deterministic and Stochastic Theory", preprint
5. D. Ludwig, Comm. Pure Applied Math. 19:215 (1966)
6. T. Pearcey, Phil. Mag. 37:311 (1946)
7. J.N.L. Connor, Mol. Phys. 31:33 (1976)
8. See any reference on collision theory
9. L. Landau and E. Lifschetz, "Statistical Physics", Pergamon Press (1959)
10. S. Ma, "Modern Theory of Critical Phenomena," Benjamin (1973)
11. N. Van Kampen, Adv. Chem. Phys. 34:245 (1976)
12. J. Cohen and R. Lewis, J. Inst. Math. Appl. 3:266 (1967)
13. H. Haken, Rev. Mod. Phys. 47:67 (1975)
14. K. Kitahara, Adv. Chem. Phys. 29:85 (1975)
15. J. Keizer, J. Chem. Phys., 63:398 (1975)
16. L.C. Woods, "Thermodynamics," Oxford University Press (1975)
17. B. Levich, "Theoretical Physics, Vol 2" Wiley (1971)
18. L. Onsager and S. Machlup, Phys. Rev., 91:1505 (1953)
19. M.E. Fisher, in Boulder Lectures in Theoretical Physics (1964)
20. W. Moore, "Physical Chemistry," Prentice Hall (1962)
21. Y. Sinai, Russ Math Surveys 25:137 (1970)

REFERENCES (Continued)

22. G.C. Papanicolaou and W. Kohler, Comm. Pure. Appl. Math 27:641 (1974)
23. J. Keizer, J. Chem. Phys 63:5037 (1975)
24. W. Feller, "An Introduction to Probability Theory and Its Applications, Vol 2," Wiley (1971)
25. D. Ludwig, SIAM Rev. 17:605 (1975)
26. N. Levinson, Duke Math. J. 28:345 (1961)
27. Ref 12, page 283
28. N. Levinson, Ann. Math 51:428 (1950)
29. A.D. Ventcel and M. Freidlin, Russ. Math. Surveys 25:1 (1970)
30. A. Nayfeh, "Perturbation Methods," Wiley (1973)
31. K. Nordholm and R. Zwanzig, J. Stat. Phys. 11:143 (1974)
32. D.V. Ansov and Y. Sinai, Russ. Math. Surveys 22:103 (1967)
33. Ref. 21
34. C.Y. Mou, G. Nicolis and R.M. Mazo, J. Stat. Phys., in press

APPENDIX A

MARGINAL AND CRITICAL TYPE DYNAMICAL SYSTEMS

In this appendix, we give exact conditions for marginal and critical type dynamical systems. Our work generalizes the scheme of Kubo et. al.(1973).

MARGINAL TYPE DYNAMICAL SYSTEMS

The deterministic evolution of the macrovariables is governed by

$$\dot{x} = b(x, \eta) \quad (A.1)$$

where η is a parameter. Equation (A.1) may have three steady states, $Q_0(\eta)$, $Q_1(\eta)$ and P_2 . Let B_k be the matrix $(b^i_{,j})$ evaluated at Q_0 , Q_1 or P_2 ($k = 0, 1, 2$). We assume that:

- For all values of η , B_2 has two real negative eigenvalues. Although P_2 may depend upon η , P_2 is always bounded away from the other steady states.

- As $\eta \rightarrow 0$, the distance between $Q_0(\eta)$ and $Q_1(\eta)$ decreases. When $\eta = 0$, Q_0 and Q_1 coalesce and annihilate each other (i.e. when $\eta < 0$, (A.1) has one real and two complex steady states).

- When $\eta > 0$, B_0 has two real negative eigenvalues and B_1 has one real positive and one real negative eigenvalue. When $\eta = 0$, $b_0 = B_1$ has one zero and one real negative eigenvalue. The

eigenvector corresponding to the negative eigenvalue has positive slope. The double point $Q_0(0)/Q_1(0)$ is called a saddle node (3).

A deterministic system satisfying the above assumptions will be structurally similar to the system sketched in Fig. 2.

The above conditions can be reformulated by a change of coordinates. Define the y^1 axis in the direction of the eigenvector of the non-negative eigenvalue of B_1 . The y^2 axis is in the direction of the eigenvector of the negative eigenvalue of B_1 , with the origin at Q_1 . Then

$$\dot{y} = \tilde{b}(y, \eta) \quad (A.2)$$

is the deterministic system in the new coordinates. The system is of the marginal type if:

- 1) $\det(\tilde{b}^i_{,j}(Q_1, 0)) = 0$
- 2) $\tilde{b}^1_{,1}(Q_1, 0) = \tilde{b}^2_{,1}(Q_1, 0) = 0$
- 3) $\tilde{b}^2_{,2}(Q_1, 0) \neq 0$
- 4) $\tilde{b}^1_{,11}(Q_1, 0) - \tilde{b}^2_{,11}(Q_1, 0) \neq 0$.

The conditions (A.3) have the following interpretation. Condition 1) indicates that the original system has a zero eigenvalue. Condition 2) indicates that when $\eta = 0$ the linear dynamics in the

y^1 direction vanish, condition 4) indicates that these dynamics are quadratic. Condition 3) indicates that the second eigenvalue is non-zero.

CRITICAL TYPE DYNAMICAL SYSTEMS

The macrovariables evolve according to a deterministic kinetic equation

$$\dot{x} = b(x, \eta, \delta) \quad (A.4)$$

where η, δ are one dimensional parameters. The entire bifurcation set of equation (A.4) is still unknown (3). The physical systems of interest here motivate the following assumptions:

- For some values of η, δ , (A.4) has three steady states $P_0(\eta, \delta)$, $P_1(\eta, \delta)$ and $P_2(\eta, \delta)$.

If $B_k = (b^i_{,j})$ evaluated at P_k , then when the three steady states are distinct, B_0 and B_2 have real negative eigenvalues. B_1 has one real negative and one real positive eigenvalue. The eigenvector corresponding to the negative eigenvalue has positive slope.

- As η, δ vary, two of the steady states may coalesce and annihilate each other. This behavior is analogous to the marginal bifurcation.

• As η, δ vary, all three steady states may move together and coalesce when $\eta = \delta = 0$. At the critical bifurcation, $B_1 = (b_{,j}^i)$ has a zero eigenvalue. We assume that the steady state remaining after the critical bifurcation is a stable steady state.

A deterministic system satisfying the above postulates will be structurally similar to the one sketched in Fig. 3.

The above properties can be restated in terms of a new coordinate system as follows. The y^1 axis is in the direction of the eigenvector of the non negative eigenvalue of B_1 . The y^2 axis is in the direction of the eigenvector of the negative eigenvalue, with the origin at P_1 . The deterministic evolution is then

$$\dot{y} = \tilde{b}(y, \eta, \delta). \quad (A.5)$$

A dynamical system is a critical type system if:

- 1) $\det(\tilde{b}_{,j}^i(P_1, 0, 0)) = 0$
- 2) $\tilde{b}_{,1}^1(P_1, 0, 0) = \tilde{b}_{,1}^2(P_1, 0, 0)$
 $= \tilde{b}_{,11}^1(P_1, 0, 0) = \tilde{b}_{,11}^2(P_1, 0, 0) = 0$ (A.6)
- 3) $\tilde{b}_{,2}^2(P_1, 0, 0) \neq 0$
- 4) $\tilde{b}_{,111}^1 - \tilde{b}_{,111}^2 \neq 0$.

These conditions have the following interpretation 1) indicates that the system has a zero eigenvalue, while condition 3) indicates that the second eigenvalue is non-zero. Condition 2) indicates that the linear and quadratic dynamics in the y^1 direction vanish, while 4) indicates that the dynamics are cubic.

CNA Professional Papers – 1973 to Present*

- PP 103
Friedheim, Robert L., "Political Aspects of Ocean Ecology" 48 pp., Feb 1973, published in *Who Protects the Oceans*, John Lawrence Hargrove (ed.) (St. Paul: West Publ'g Co., 1974), published by the American Society of International Law AD 757 936
- PP 104
Schick, Jack M., "A Review of James Cable, Gunboat Diplomacy Political Applications of Limited Naval Forces," 5 pp., Feb 1973, (Reviewed in the *American Political Science Review*, Vol. LXVI, Dec 1972)
- PP 105
Corn, Robert J. and Phillips, Gary R., "On Optimal Correction of Gunfire Errors," 22 pp., Mar 1973, AD 761 674
- PP 106
Stoloff, Peter H., "User's Guide for Generalized Factor Analysis Program (FACTAN)," 35 pp., Feb 1973, (Includes an addendum published Aug 1974) AD 758 824
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Stoloff, Peter H., "Relating Factor Analytically Derived Measures to Exogenous Variables," 17 pp., Mar 1973, AD 758 820
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McConnell, James M. and Kelly, Anne M., "Superpower Naval Diplomacy in the Indo-Pakistani Crisis," 14 pp., 5 Feb 1973, (Published, with revisions, in *Survival*, Nov/Dec 1973) AD 761 675
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Berghoefer, Fred G., "Salaries—A Framework for the Study of Trend," 8 pp., Dec 1973, (Published in *Review of Income and Wealth*, Series 18, No. 4, Dec 1972)
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Augusta, Joseph, "A Critique of Cost Analysis," 9 pp., Jul 1973, AD 766 376
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Ginsberg, Lawrence H., "Propagation Anomalies During Project SANGUINE Experiments," 5 pp., Apr 1974, AD 786 968
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Rehm, Allan S., "An Assessment of Military Operations Research in the USSR," 19 pp., Sep 1973, (Reprinted from *Proceedings, 30th Military Operations Research Symposium (U)*, Secret Dec 1972) AD 770 116
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McWhite, Peter B. and Rattliff, H. Donald,* "Defending a Logistics System Under Mining Attack,"** 24 pp., Aug 1976 (to be submitted for publication in *Naval Research Logistics Quarterly*), presented at 44th National Meeting, Operations Research Society of America, November 1973, AD A030 454
*University of Florida.
**Research supported in part under Office of Naval Research Contract N00014-68-0273-0017
- PP 118
Barfoot, C. Bernard, "Markov Duels," 18 pp., Apr 1973, (Reprinted from *Operations Research*, Vol. 22, No. 2, Mar-Apr 1974)
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Stoloff, Peter and Lockman, Robert F., "Development of Navy Human Relations Questionnaire," 2 pp., May 1974, (Published in *American Psychological Association Proceedings*, 81st Annual Convention, 1973) AD 779 240
- PP 120
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- PP 123
Petersen, Charles C., "The Soviet Port-Clearing Operation in Bangladesh, March 1972-December 1973," 35 pp., Jun 1974, (Published in Michael McGwire, et al. (eds) *Soviet Naval Policy: Objectives and Constraints*, (New York: Praeger Publishers, 1974) AD 780 540
- PP 124
Friedheim, Robert L. and Jehn, Mary E., "Anticipating Soviet Behavior at the Third U.N. Law of the Sea Conference: USSR Positions and Dilemmas," 37 pp., 10 Apr 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 783 701
- PP 125
Weinland, Robert G., "Soviet Naval Operations—Ten Years of Change," 17 pp., Aug 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 783 962
- PP 126 – Classified.
- PP 127
Dragnich, George S., "The Soviet Union's Quest for Access to Naval Facilities in Egypt Prior to the June War of 1967," 64 pp., Jul 1974, AD 786 318
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Stoloff, Peter and Lockman, Robert F., "Evaluation of Naval Officer Performance," 11 pp., (Presented at the 82nd Annual Convention of the American Psychological Association, 1974) Aug 1974, AD 784 012
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Holen, Arlene and Horowitz, Stanley, "Partial Unemployment Insurance Benefits and the Extent of Partial Unemployment," 4 pp., Aug 1974, (Published in the *Journal of Human Resources*, Vol. IX, No. 3, Summer 1974) AD 784 010
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Dismukes, Bradford, "Roles and Missions of Soviet Naval General Purpose Forces in Wartime: PROSSBN Operation," 20 pp., Aug 1974, AD 786 320
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Weinland, Robert G., "Analysis of Gorskov's *Navies in War and Peace*," 45 pp., Aug 1974, (Published in *Soviet Naval Policy*, ed. Michael McGwire; New York: Praeger) AD 786 319
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*Department of Mechanical Engineering, University of Maryland.
- PP 135
Goudreau, Kenneth A.; Kuzmack, Richard A.; Wiedemann, Karen, "Analysis of Closure Alternatives for Naval Stations and Naval Air Stations," 47 pp., 3 Jun 1975 (Reprinted from "Hearing before the Subcommittee on Military Construction of the Committee on Armed Services," U.S. Senate, 93rd Congress, 1st Session, Part 2, 22 Jun 1973)
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*Mathematics, Inc.
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Lockman, Robert F., Jehn, Christopher, and Shughart, William F. II., "Models for Estimating Premature Losses and Recruiting District Performance," 36 pp., Dec 1975 (Presented at the RAND Conference on Defense Manpower, Feb 1976; to be published in the conference proceedings) AD A 020 443
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McConnell, James M., "Military-Political Tasks of the Soviet Navy in War and Peace," 62 pp., Dec 1975 (Published in Soviet Oceans Development Study of Senate Commerce Committee October 1976) AD A022 590
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*Research supported by the National Science Foundation
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Mizrahi, Maurice M., "WKB Expansions by Path Integrals, With Applications to the Anharmonic Oscillator," 137 pp., May 1976, AD A025 440
*Research supported by the National Science Foundation
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Wilson, Desmond P., Jr., "The U.S. Sixth Fleet and the Conventional Defense of Europe," 50 pp., Sep 1976 (Submitted for publication in Adelphi Papers, I.I.S.S., London) AD A030 457
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Horowitz, Stanley A. and Sherman, Allan, "The Characteristics of Naval Personnel and Personnel Performance," 16 pp. April 1977, (Presented at the NATO Conference on Manpower Planning and Organization Design, Stresa, Italy, 20 June 1977), AD A039 050
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Balut, Stephen J. and Stoloff, Peter, "An Inventory Planning Model for Navy Enlisted Personnel," 35 pp., May 1977, (Prepared for presentation at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Science, 9 May 1977, San Francisco, California), AD A042 221
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Murray, Russell, 2nd, "The Quest for the Perfect Study or My First 1138 Days at CNA," 57 pp., April 1977
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- PP 184
Lockman, Robert F., "An Overview of the OSD/ONR Conference on First Term Enlisted Attrition," 22 pp., June 1977, (Presented to the 39th MORS Working Group on Manpower and Personnel Planning, Annapolis, Md., 28-30 June 1977), AD A043 618
- PP 185
Kassing, David, "New Technology and Naval Forces in the South Atlantic," 22 pp. (This paper was the basis for a presentation made at the Institute for Foreign Policy Analyses, Cambridge, Mass., 28 April 1977), AD A043 619
- PP 186
Mizrahi, Maurice M., "Phase Space Integrals, Without Limiting Procedure," 31 pp., May 1977, (Invited paper presented at the 1977 NATO Institute on Path Integrals and Their Application in Quantum Statistical, and Solid State Physics, Antwerp, Belgium, July 17-30, 1977) (Published in Journal of Mathematical Physics 19(1), p. 298, Jan 1978), AD A040 107
- PP 187
Coile, Russell C., "Nomography for Operations Research," 35 pp., April 1977, (Presented at the Joint National Meeting of the Operations Research Society of America and The Institute for Management Services, San Francisco, California, 9 May 1977), AD A043 620
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Coile, Russell C., "Error Detection in Computerized Information Retrieval Data Bases," July, 1977, 13 pp. Presented at the Sixth Cranfield International Conference on Mechanized Information Storage and Retrieval Systems, Cranfield Institute of Technology, Cranfield, Bedford, England, 26-29 July 1977, AD A043 580
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Horowitz, Stanley A., "A Model of Unemployment Insurance and the Work Test," August 1977, 7 pp., (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 4, Jul 1977)
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Classen, Kathleen P., "The Effects of Unemployment Insurance on the Duration of Unemployment and Subsequent Earnings," August 1977, 7 pp., (Reprinted from Industrial and Labor Relations Review, Vol. 30, No. 4, Jul 1977)
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Wells, Anthony R., "The Centre for Naval Analyses," 14 pp., Dec 1977, AD A049 107
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Kassing, David, "Protecting The Fleet," 40 pp., Dec 1977 (Prepared for the American Enterprise Institute Conference on Problems of Sea Power as We Approach the 21st Century, October 6-7, 1977), AD A049 109
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Mangel, Marc, "On Singular Characteristic Initial Value Problems with Unique Solutions," 20 pp., Jun 1978 (To be submitted for publication in *Journal of Mathematical Analysis and Its Applications*)
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Weinland, Robert G., "A Somewhat Different View of The Optimal Naval Posture," 37 pp., Jun 1978 (Presented at the 1978 Convention of the American Political Science Association (APSA/IUS Panel on "Changing Strategic Requirements and Military Posture"), Chicago, Ill., September 2, 1976)
- PP 215
Coile, Russell C., "Comments on: *Principles of Information Retrieval* by Manfred Kochen, 10 pp., Mar 78, (Published as a Letter to the Editor, *Journal of Documentation*, Vol. 31, No. 4, pages 298-301, December 1975)
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Coile, Russell C., "Lotka's Frequency Distribution of Scientific Productivity," 18 pp., Feb 1978, (Published in the *Journal of the American Society for Information Science*, Vol. 28, No. 6, pp. 366-370, November 1977)
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"Portions of this work were completed at the Institute of Applied Mathematics and Statistics, University of British Columbia, Vancouver, Canada."
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