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FEATURES OF ENERGY-BUDGET CLIMATE MODELS: AN EXAMPLE OF WEATHER--ETC(U)

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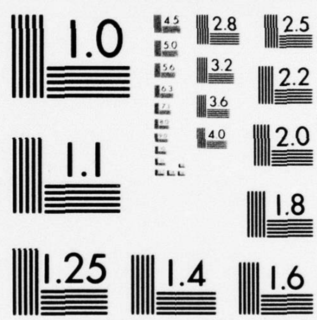
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FEATURES OF ENERGY-BUDGET CLIMATE MODELS: AN EXAMPLE OF WEATHER DRIVEN CLIMATE STABILITY

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The essential features of zonally-averaged energy budget models of the earth's climate are reviewed and abstracted to give an "effective potential" for the appropriate climate variables.

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ABSTRACT

The essential features of zonally-averaged energy budget models of the earth's climate are reviewed and abstracted to give an "effective potential" for the appropriate climate variables. The effective potential permits the multiple valued insolation versus ice latitude behavior common in such models to be considered from the point of view of a smoothly varying energy function. Next we view weather as a short time scale driving mechanism for the long time scale variations of climate and give a formalism for explicitly realizing a climate effective potential by integrating out or averaging over the short time scale weather variations. The effective potential so realized embodies many of the features abstracted from the energy budget models.

CONTENTS

LIST OF FIGURES

I INTRODUCTION

In the past several years there has been an active consideration of models of the growth and recession of the earth's ice caps. These models have their bases in equilibrium states dictated by an energy balance between insolation, radiation from the earth after differential absorption by ice covered or by ice free regions, and energy transported poleward by the atmosphere and oceans. The pioneering work in these "energy budget" models was done by Budyko (1969) and Sellers (1969). It has been followed by a number of papers including Schneider and Gal-Chen (1973), Gal-Chen and Schneider (1976), North (1975a, 1975b), Held and Suarez (1974), Sellers (1973), Su and Hseih (1976), and Lindzen and Farrell (1977).

The continuing interest in these simplified models stems from the fact that they consistently exhibit a great sensitivity to small changes in the solar constant. Precisely what level of sensitivity is to be believed remains a matter of active discussion, Sellers (1977) and Dickinson (1977); nevertheless, since it is common for these models to permit an uncontrolled growth of the earth's ice caps after a few percent decrease in the solar constant, the significance of pursuing them is clear. The physical mechanism for this instability is not hard to identify: as one lowers the solar constant the average temperature at any latitude decreases and the ice pack grows. The ice pack has a higher reflectivity (albedo) than the sea or land so more sunlight is reflected which in turn tends to cool off the earth, form more ice, etc. This instability is countered by the transport of energy poleward by the atmosphere and oceans. When that transport is insufficient, the ice cap will grow. A review of this mechanism in the general context of climate models is given by Schneider and Dickinson (1974).

In this paper, after a review of the essential points of energy budget models, I discuss how one might interpret these models within the general framework of theories such as those discussed by Hasselman (1976). In such theories the climate-weather system is divided into short time scale phenomena called weather and long time scale phenomena called climate. The time scales involved in weather may be as short as a few days or weeks; while climate time scales range from several months or years to many thousands of years in the case of ice ages. It is not absolutely certain that one can cleanly divide the time scales of weather and climate since there may be a very large number of important time scales ranging from days to thousands of years; though it does seem to be generally agreed upon, Lorenz (1977), that such a division is attractive.

Our procedure here will take the energy-budget models and cast them into the language of an "effective potential" whose smooth behavior as a function of external parameters, such as insolation, can give rise to apparently discontinuous or multiple valued equilibrium states. For the ice cap problem the potential will be a function of the cosine of the ice cap latitude, y, and the insolation, q, normalized to current values. The potential V(q,y) will be represented by a polynomial in y with coefficients dependent on q (and thus, if q depends on time, dependent on time):

$$V(q,y) = \frac{a(q)y^2}{2} + \frac{b(q)y^3}{3} + \frac{c(q)y^4}{4} + \ldots \quad . \qquad (1)$$

Three terms will suffice for the problem at hand. The time dependence of y will come from

$$\frac{\partial y}{\partial t} = \frac{-\partial V(q,y)}{\partial y} \quad , \qquad (2)$$

which accounts for the name "effective potential". Clearly the inflection points of $V(y)$ with positive curvature as well as the end points, $y = 0$ or 1, with negative slope are the possible equilibrium states. As q varies, the coefficients $a, b, c, \ldots$ will vary smoothly changing the character of the equilibrium states in a dramatic way.

The nature of $V(q,y)$ will come from a study of the equilibrium states a'la Budyko-Sellers. Once we have V, however, the time dependence of $y(t)$, that is, the climate, is rather easily extracted from the elementary equation (2). The potential itself is an abstract formulation of the energy-budget models; its usefulness is then in the study of the dynamic properties of climate. In the ice cap example the effect of time variations in q on $y(t)$ are simply followed by solving (2). So if one imagines variations in insolation to be at the source of ice ages, various formulations of this hypothesis may be straightforwardly examined.

In addition to providing an effective potential I will give a discussion of its derivation from the interaction of short and long time scale phenomena. Essentially I will treat a two time scale system where weather is represented by a gaussian statistical field. Averaging the climate (long time scale), which is driven by the weather, over the weather variables (that is, "integrating out" the weather) yields an effective potential for climate with features similar to those abstracted from energy budget models.

II ENERGY BUDGET MODELS FOR ICE CAPS

In this section will be a quick review of energy budget models of Budyko-Sellers type. The treatment follows North (1975a) in spirit. We consider a zonally averaged earth with a surface temperature T depending on latitude θ and time t. The earth has ice caps extending to θ_o and open sea or land lying between θ_o and $-\theta_o$. The ice is assigned an albedo $\alpha_i \sim 0.6$ for $|\theta|>\theta_o$, and the ice-free earth is assigned a lower albedo $\alpha \approx 0.4$ for $|\theta|<\theta_o$. The net incoming radiation absorbed will then be

$$A(\theta,\theta_o[t]) = Qs(\theta)\begin{cases} 1-\alpha_i & \theta > \theta_o \\ 1-\alpha & \theta < \theta_o \end{cases} \tag{3}$$

where we consider only $0 \le \theta \le \pi/2$ because of symmetry about the equator. In (3) we have

Q = global average insolation = 1/4 solar constant = 0.35 kwatt/m^2 ,

$$Qs(\theta) = \text{annual mean insolation/unit area, so } \int_0^{\pi/2} d\theta \cos\theta\, s(\theta) = 1 \quad . \tag{4}$$

After absorption this radiation is either reradiated or transported poleward by the ocean and atmosphere. All workers seem to agree that the real black body radiation, σT^4, is very well represented by the linear form $A + BT$, which is essentially an expansion of σT^4 about $T = 0°C$ and is acceptable since changes in T are not large. The coefficients are $A \approx 200$ watts/m^2 and $B \approx 1.5$ watts/m^2-°C.

The representation of the heat transport by ocean currents and by the atmosphere is complicated in detail. It seems quite reasonable, however, to adopt the usual form for diffusion

$$\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\ D\frac{\partial T}{\partial\theta}\right) \qquad (5)$$

where D is a diffusion coefficient sometimes taken to depend on θ.

All together these considerations lead to the diffusion equation

$$C\frac{\partial T(\theta,t)}{\partial t} = \text{energy absorbed} - \text{energy radiated} + \text{energy transported}$$

$$= Qs(\theta)A(\theta,\theta_o[t]) - (A + BT[\theta,t]) + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\ D\frac{\partial}{\partial\theta}T[\theta,t]\right) \quad , \qquad (6)$$

where C is some "heat capacity" and really serves to set the time scale on which the system approaches equilibrium.

Taking D to be a constant, North (1975a) is able to provide an analytic solution to the time independent version of (6). That is to say, he seeks the possible equilibrium states governed by the energy balance indicated on the right hand side of (6). The boundary conditions used are

1) $$(1-x^2)^{\frac{1}{2}}\frac{\partial T(x)}{\partial x}\bigg|_{\substack{x=0\\ \text{and}\\ x=1}} = 0 \quad , \quad x = \sin\theta \quad ,$$

that is, no heat is transported across the equator or poles.

2) $$\lim_{\varepsilon\to 0} T(x_o+\varepsilon) = \lim_{\varepsilon\to 0} T(x_o-\varepsilon) = T_o \quad , \quad x_o = \sin\theta_o \quad ,$$

that is, the temperature at the ice front (at θ_o or x_o) is the same approached from below or above in latitude. This temperature is taken to be a convenient number, say -10°C, where water is likely to be frozen.

3) $$\left.\frac{\partial T}{\partial x}\right|_{x=x_o+0} = \left.\frac{\partial T}{\partial x}\right|_{x=x_o-0} ,$$

that is, no net heat is transported across the ice front.

The result of this kind of model calculation is quite clearly presented in terms of a graph of $x_o = \sin\theta_o$ versus q, the solar constant normalized to the contemporary value (see Figure 1). This figure is sketched from North (1975a) but is similar to that found, for example, by Held and Suarez (1974). As demonstrated explicitly by North (1975a) and by Su and Hsieh (1976) the regions with $dq/dx_o > 0$ are stable in the sense that small changes in q lead to exponentially damped changes in $x_o(t)$ computed from (6). When $dq/dx_o < 0$, (6) has solutions for $x_o(t)$ which grow until the ice front reaches the equator, $x_o = 0$.

These models are quite sensitive to the connection albedo and temperature or equivalently the magnitudes of α_i and α. Clearly the larger α_i becomes, the more reflection there will be from ice and the stronger will be the driving mechanism for unstable ice front growth.

There are many serious simplifications in this kind of model. Perhaps the most blatant are the treatment of the atmosphere as homogeneous and essentially cloud free and the use of quantities averaged over longitude as well as over seasons. Corrections such as these seriously complicate the calculations in the model and yet result in rather similar patterns for the

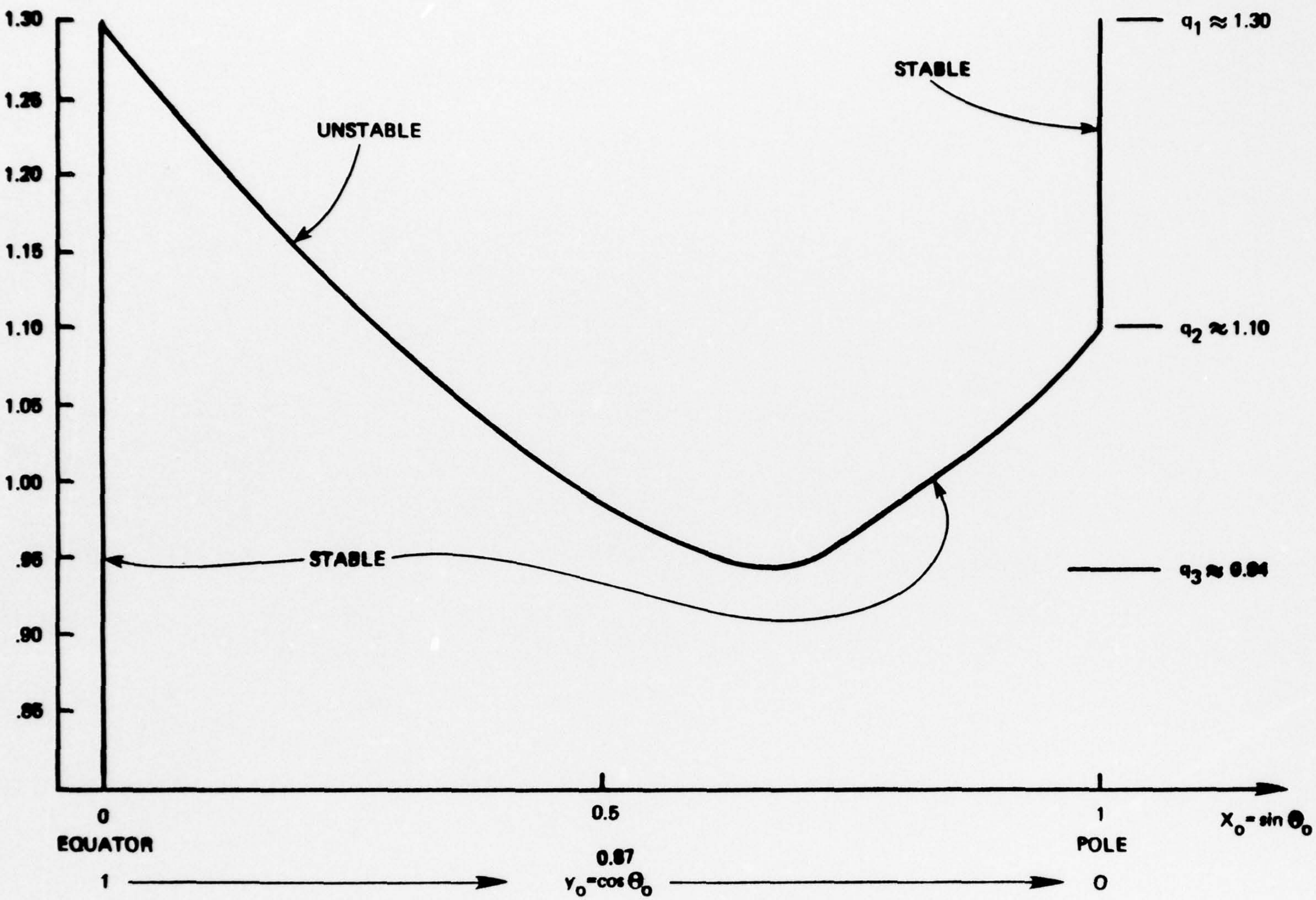

Figure 1 Cosine or sine of the ice front latitude as a function of isolation. This curve is more or less taken from North (1975 a), but it is representative of energy budget model calculations.

q versus x_o behavior, Sellers (1977). For the present we will accept the results of Figure 1 as representative, if perhaps absent of fine structure, and attempt to give a general framework for these models. This occurs next.

III GENERALITIES ABOUT THE ENERGY BUDGET MODELS : EFFECTIVE POTENTIALS

We begin by examining Figure 1. When q is very large, $q > q_1 \approx 1.3$, there is a single allowed equilibrium state which has $y_o = \cos\theta_o = 0$, that is, no ice cover at all. When q is lowered to below q_1 but above $q_2 \approx 1.10$, we have three possible equilibrium states: $y_o = 1$ (ice covered earth), $y_o = 0$, and a set of intermediate values depending on q. The states $y_o = 0$ or 1 are stable; the other is unstable. When q goes below $q_2 \approx 1.10$ but is above $q_3 \approx 0.94$, there are still two stable states $y_o = 1$ and y_o in the region 0 to 0.8. The present ice cap has $y_o \approx 0.32$ and clearly lies in this region. There is also a possible unstable state in this regime. Finally, when $q < q_3 \approx 0.94$, there is only one possible equilibrium state $y_o = 1$, and the ice cap grows in an uncontrolled way to cover the earth.

Such a situation is reminiscent of a system undergoing various phase transitions. The external parameter in most phase transitions is the temperature, or the concentration of some impurity or perhaps the magnitude of an external magnetic field. Here we have the possible phases of the system governed by the insolation q. In the theory of phase transitions it has proved convenient to characterize the system by an effective free energy taken to be a polynomial in a mean field and whose coefficients depend on the external parameter. Such a phenomenological approach was initiated by Landau and co-workers (see Landau and Lifshitz, 1968) and has proven especially useful as a compact description of an intrinsically complicated system, for example, a ferromagnet.

In the present case we want to think of $y_o(t)$ as a "mean field". In a more realistic model the ice front would depend on the longitude λ and the mean field would be $y_o(\lambda,t)$, as computed from $T(\theta,\lambda,t) = T_o$. We now assign an effective potential $V(q,y_o)$ to the system and determine the time dependence of $y_o(t)$ from

$$\frac{dy_o(t)}{dt} = -\frac{\partial V(q,y_o)}{\partial y_o} \quad . \tag{7}$$

In the present case it will be sufficient to choose

$$V(q,y_o) = \frac{1}{2}\, a(q)y_o^2 + \frac{1}{3}\, b(q)y_o^3 + \frac{1}{4}\, c(q)y_o^4 \quad , \tag{8}$$

so

$$\frac{dy_o}{dt} = -y_o(t)\left[a(q) + b(q)y_o + c(q)y_o^2\right] \quad . \tag{9}$$

To describe the situation in Figure 1 we need the following

1) When $q > q_1$ $\qquad a(q), b(q), c(q) > 0$. (10)

This means $V(q,y_o)$ appears as in Figure 2. Whatever the initial value of y_o it will "roll down the hill" toward $y_o = 0$ (no ice) at a rate

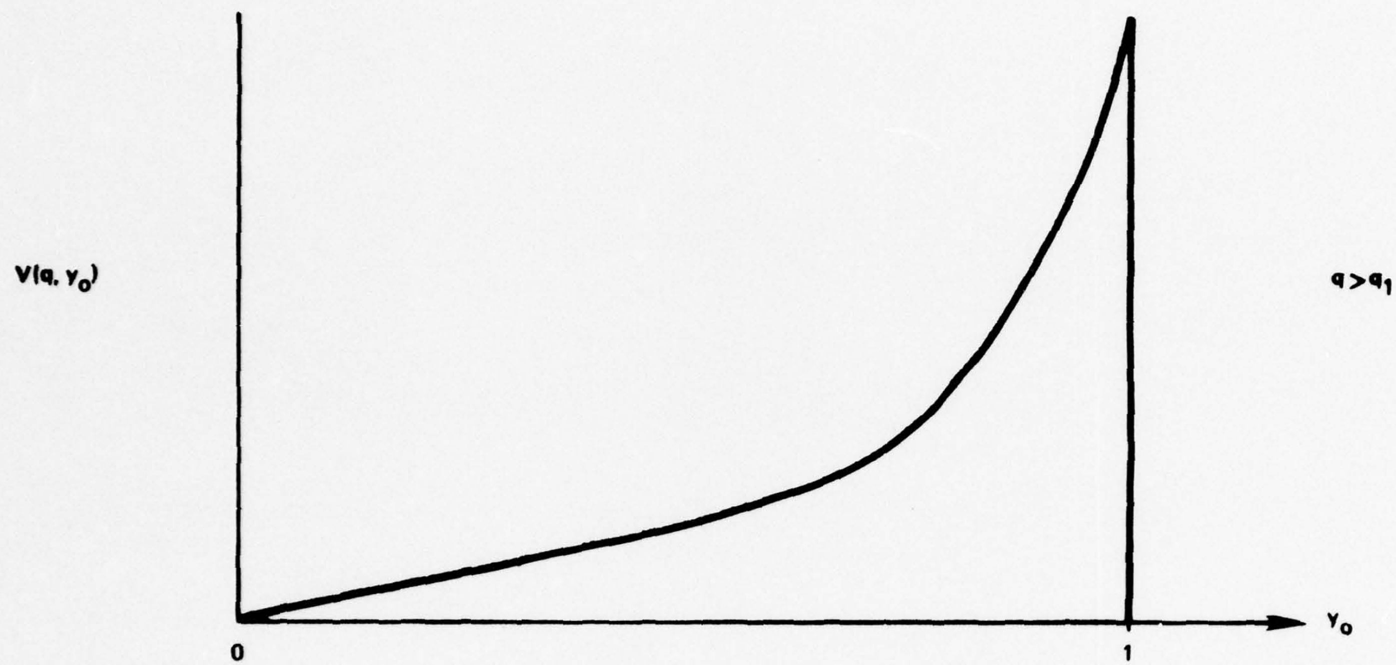

Figure 2 The climate effective potential in the stable ice free regime. q>q1, see Figure 1.

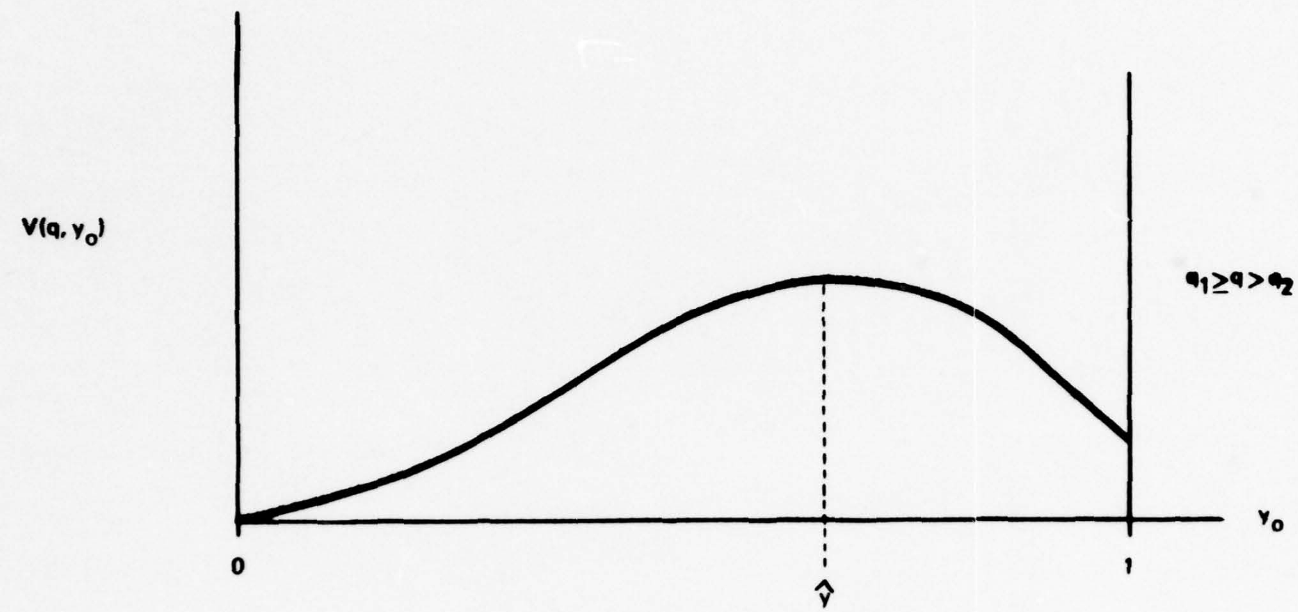

Figure 3 The climate effective potential in the regime $q_1 \geq q > q_2$. $y = o$ and $y = 1$ are stable equilibrium states; $y = \hat{y}$ is unstable.

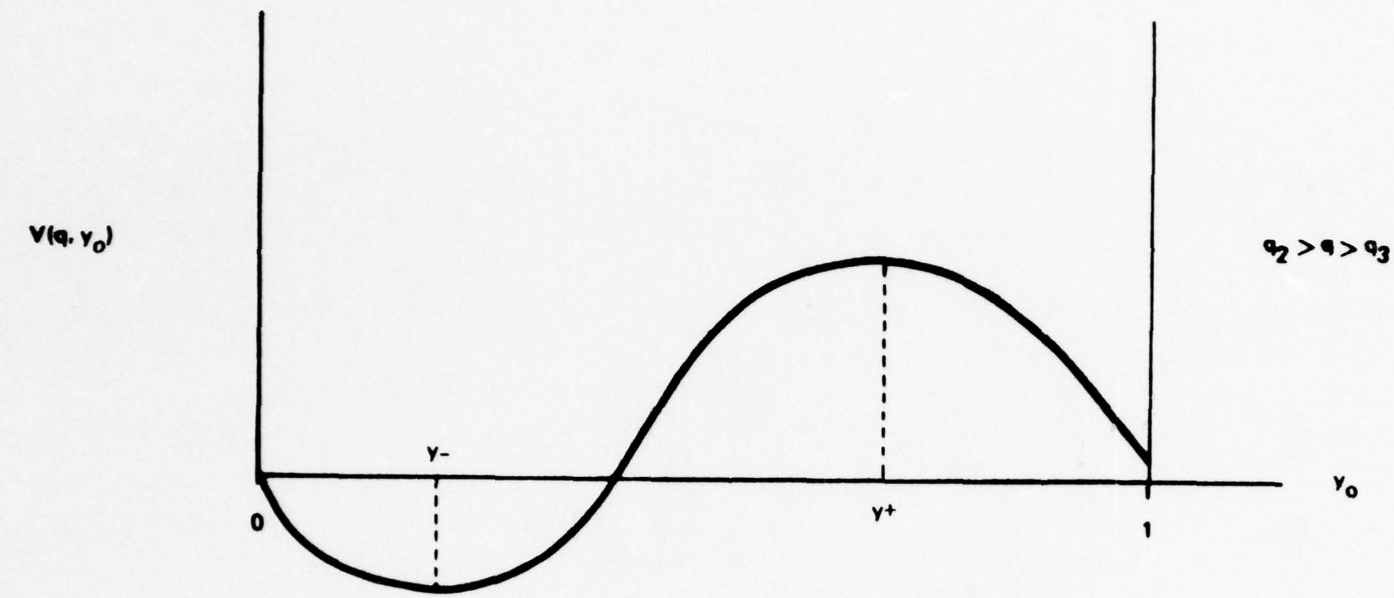

Figure 4 The climate effective potential when $q_2 > q > q_3$. $y = y-$ and $y = 1$ are stable; $y = y+$ is unstable.

$\exp -\int^{t} a(q[t'])dt'$ for large t .

2) When q reaches some value $q_o > q_1$, the coefficient $c(q)$ becomes negative and at $q = q_1$ the slope of $V(q,y_o)$ at $y_o = 1$ is zero. For $q \leq q_1$, but $q > q_2$ we have

$$a(q),\ b(q) > 0\ ,\quad c(q) < 0\ ,\quad \text{and}\ \left.\frac{\partial V}{\partial y_o}\right|_{y_o = 1} \leq 0\ , \tag{11}$$

as shown in Figure 3. If $y_o > \hat{y}$ for $q_2 < q < q_1$, the ice cap will grow in time to $y_o = 1$. If $y_o < \hat{y}$, the ice will vanish in time. The point $y_o = \hat{y}$ is a possible equilibrium state, though it's clearly unstable since

$$\left.\frac{\partial^2 V}{\partial y_o^2}\right|_{y_o = \hat{y}} < 0\ .$$

3) When $q < q_2$, there develops a stable equilibrium y_o away from $y_o = 0$. In the domain $q_3 < q < q_2$ we have

$$a(q)\ ,\ c(q) < 0\ ,\quad \text{and}\ b(q)^2 > 4a(q)c(q) \tag{12}$$

so the roots of $\partial V/\partial y_o$,

$$-2c(q)y_{\pm} = b(q) \pm \sqrt{b(q)^2 - 4a(q)c(q)}\ ,$$

are real and distinct. Now the potential appears as in Figure 4.

If the starting value of y_o is less than y_+ , then $y_o(t)$ will move toward $y_o = y_-$ as time goes on. The rate of approach will be governed for long times by

$$\left.\frac{\partial^2 V}{\partial y_o^2}\right|_{y_o = y_-}\ .$$

The value $y_o = y_+$ is a possible equilibrium value, but again from Figure 4 is clearly unstable.

4) Finally when $q < q_3$, the transport of heat poleward is not sufficient to overcome the ice albedo instability and the ice grows to $y_o = 1$. In this case

$$a(q),\ c(q) < 0 \text{ and } b(q)^2 \leq 4a(q)c(q) \quad . \tag{15}$$

Figure 5 shows this growth of y_o, whatever its initial value, in graphical form.

A possible simple parametrization of a, b, and c is

$$\begin{aligned} a(q) &= \alpha(q-q_2) \\ b(q) &= \sqrt{\beta(q-q_3) + 4\alpha\gamma(q-q_2)(q-q_o)} \\ c(q) &= \gamma(q-q_o) \end{aligned} \tag{16}$$

with $\alpha, \beta, \gamma > 0$, $q_o > q_1$ and q_1 defined by

$$\left.\frac{\partial V}{\partial y_o}\right|_{\substack{y_o=1 \\ q=q_1}} = 0 = a(q_1) + b(q_1) + c(q_1) \tag{17}$$

The introduction of the effective potential $V(q,y_o)$ has thus far only been a transcription of Figure 1 into other terms. It proves useful when we wish to follow the <u>time dependence</u> of y_o by solving (9). Taking q to be time dependent we need to solve

Figure 5 The climate effective potential for very low insolation; $q < q_3$. Only $y = 1$, an ice covered earth is stuble.

$$\frac{dy_o(t)}{dt} = -y_o(t)\left[a(t) + b(t)y_o(t) + c(t)y_o(t)^2\right] \qquad (18)$$

for the parametrization of a, b, and c in (16) and for the chosen time dependence of q. It is clear that as q changes we move from regime to regime in $V(q,y_o)$ and even small changes in q can result in significantly different equilibrium states. We appear today to be in the regime (see Fig. 4) $q \geq q_3$, $y < y_+$; if q were to dip below q_3, the ice caps would grow from their present value. If, during this growth, q moved back to be $> q_3$, but at that time y_o were greater than y_+, the ice caps would keep on growing. If y were less than y_+ when the switch occurred, then y would go back to y_-. Following $y_o(t)$ for any choice of $q(t)$ involves the rather straightforward solution of (18).

This equation is known as the Abel equation of the first kind (Murphy, 1960, p. 23-25) and is, in general, not soluble in closed form. However, near $b = 0$ or $c = 0$ it may be easily solved. As an example of an oversimplified ice cap model consider a system where $a(q) = \alpha\left(q(t) - q_a\right)$, $b(q) = 0$, and $c(q) = \gamma\left(q(t) - q_c\right)$ with $q_a > q_c$. When $q > q_a$ the system has a single stable state at $y_o = 0$; when $q < q_c$, $y_o = 1$ is the only stable state. For $q_c < q < q_a$, various intermediate states exist. In this situation $y_o(t)$ is easily found from (18) to be

$$y_o(t) = \frac{y_o(0)\exp - \int_0^t a(x)dx}{\left[1 + 2y_o^2(0)\int_0^t F(x)dx\right]^{1/2}} \quad , \qquad (19)$$

with

$$F(x) = \left(\exp - 2 \int_0^x du\, a(u)\right) c(x) \quad . \qquad (20)$$

When $c > 0$, $a > 0$, $y_o(t)$ tends to $y_o = 0$ for t large. When $c > 0$, $a < 0$, y_o tends to $\approx \sqrt{\frac{|a|}{c}}$. When a and c are both negative, y_o grows until $y_o = 1$. This can all be qualitatively traced by drawing the effective potential for these cases. The implication of any hypothesis for the time dependence of the insolation, $q(t)$, on the growth and recession of the ice caps can now be directly drawn from (19) and (20).

The use of the effective potential is clearly not limited to the over-simplified zonally averaged models considered to this point. Whenever we have a climate variable $\phi(\vec{x},t)$, we may find a potential for it from the underlying dynamics or from a study of the equilibrium states where $\frac{\partial\phi}{\partial t} = 0$ as in the Budyko-Sellers approach. The time dependence will then be given by

$$\frac{\partial\phi}{\partial t}(\vec{x},t) = \frac{-\partial V}{\partial\phi}(\phi, \text{external parameters}) \quad , \qquad (21)$$

and the various equilibrium states will be classified by the solutions of $\partial V/\partial\phi = 0$ and their stability determined by the sign of $\partial^2 V/\partial\phi^2$ at these inflection points.

IV WEATHER AS A FEEDBACK MECHANISM FOR CLIMATE

In this section we give a formalism which allows one to determine the behavior of long time scale variables (climate) in interaction with a set of variables whose time scales are much shorter (weather). This idea has been stressed by Hasselmann (1976) who has discussed a "slow" system in the presence of a "fast" statistical background with no feedback. Feedback, or non-linear effects, are essential to the issues considered above, so we will have to treat them here.

The idea is that there is a field, $W(\vec{x},t)$, of weather variables whose time scale and perhaps length scale are small compared to those of another field $\phi(\vec{x},t)$ of climate variables. Many variations of $W(\vec{x},t)$ will occur during any significant motion of $\phi(\vec{x},t)$, and we will assume that $W(\vec{x},t)$ acts as a statistical background for the variation of $\phi(\vec{x},t)$ and further that $W(\vec{x},t)$ is a random variable at each point of space, $\vec{x}$, and time, t, with a gaussian distribution. The field $\phi(\vec{x},t)$ will be driven by its interaction with $W(\vec{x},t)$, and our goal is to find the behavior of $\phi(\vec{x},t)$ once the variations in $W(\vec{x},t)$ have been averaged out.

To begin we require a description of the weather; our rapidly varying phenomenon. We give this in terms of a Lagrangian density for the field $W(\vec{x},t)$

$$L_w = \tau_1^2 \frac{\partial \tilde{W}(\vec{x},t)}{\partial t} \frac{\partial W(\vec{x},t)}{\partial t} - \frac{\tau_2}{2}\left[\tilde{W}(\vec{x},t) \frac{\partial W(\vec{x},t)}{\partial t} - \frac{\partial \tilde{W}(\vec{x},t)}{\partial t} W(\vec{x},t)\right]$$

$$- \ell_w^2 \nabla\tilde{W}(\vec{x},t) \cdot \nabla W(\vec{x},t) - a_w \tilde{W}(\vec{x},t) W(\vec{x},t) \quad . \qquad (22)$$

In this Lagrangian the coefficients τ_1 and τ_2 are time scales for the oscillation and diffusion of weather, and ℓ_w is a length scale governing weather; a_w is a dimensionless number. All of these coefficients are in general functions of external parameters such as the insolation. In the Lagrangian appears not only the weather field $W(\vec{x},t)$ but also a mirror image weather $\tilde{W}(\vec{x},t)$ which is necessary because of the dissipative term proportional to τ_2 (Morse and Feshbach, 1953, Section 3.3). The dynamics of weather come from the extremum of the action

$$S_w(W,\tilde{W}) = \int d^D x \, dt \; L_w \qquad (23)$$

where D is the number of space dimensions. Variations of S_w yield

$$\left(\tau_1^2 \frac{\partial^2}{\partial t^2} + \tau_2 \frac{\partial}{\partial t} - \ell_w^2 \nabla^2 + a_w\right) W(\vec{x},t) = 0 \qquad (24)$$

and

$$\left(\tau_1^2 \frac{\partial^2}{\partial t^2} - \tau_2 \frac{\partial}{\partial t} - \ell_w^2 \nabla^2 + a_w\right) \tilde{W}(\vec{x},t) = 0 \qquad (25)$$

which exhibits the different roles played by W and $\tilde{W}$.

Only quadratic terms appear in L_w since we are assuming that weather acts as a gaussian random field (see below). In some sense the non-linearities of weather have been hidden in the term proportional to τ_1 in L_w since this allows oscillations of the system. If one is truly interested in the weather, then additional non-linearities may be represented by quartic and higher order terms in L_w.

We want to treat W(x,t) as a statistical variable at each point of space and time. To this end we must assign a probability density to this distribution. It is natural to choose this to be

$$P_w\left(W(\vec{x},t),\ \tilde{W}(\vec{x},t)\right) = \exp - S_w\ (W,\tilde{W}) \tag{26}$$

so that the most probable situation is that which has minimum action; that is, the state of the system which follows Newton's laws is most likely. Furthermore, with only quadratic terms in L_w, this probability density now describes a set of gaussian random variables. Averages of any function $F(W,\tilde{W})$ are given as usual as

$$\langle F(W,\tilde{W})\rangle = \frac{\prod_{\vec{x},t} \int dW(\vec{x},t)\ d\tilde{W}(\vec{x},t)\ F(W,\tilde{W})\ P_w}{\prod_{\vec{x},t} \int dW(\vec{x},t)\ d\tilde{W}(\vec{x},t)\ P_w} \quad . \tag{27}$$

For example, the "average weather" has been chosen so

$$\langle W(\vec{x},t)\rangle = \langle \tilde{W}(\vec{x},t)\rangle = 0 \quad .$$

So if the weather variable under consideration were the temperature $T(\vec{x},t)$, we would be setting $W(\vec{x},t) = T(\vec{x},t) - \bar{T}$, with $\bar{T}$ the actual mean temperature.

The correlation function for weather variables is, however,

$$\langle W(\vec{x}_1,t_1)\ W(\vec{x}_2,t_2)\rangle = \langle \tilde{W}(\vec{x}_1,t_1)\ \tilde{W}(\vec{x}_2,t_2)\rangle = 0 \quad , \tag{28}$$

and

$$\langle W(\vec{x}_1,t_1)\ \tilde{W}(\vec{x}_2,t_2)\rangle = \frac{1}{2}\, C_w\ (\vec{x}_1-\vec{x}_2,\ t_1-t_2) \tag{29}$$

with

$$\left(\tau_1^2 \frac{\partial^2}{\partial t^2} + \tau_2 \frac{\partial}{\partial t} - \ell_w \nabla^2 + a_w\right) C_w(\vec{x},t) = - \delta^D(\vec{x})\delta(t) \quad , \tag{30}$$

so $C_w(\vec{x},t)$ is the Green function for the weather. The derivation of these results follows from the multidimensional gaussian integral

$$\int \prod_{j=1}^{n} dv_j \exp - \left[\sum_{i,j=1}^{n} v_i A_{ij} v_j + \sum_{j=1}^{n} B_j v_j \right]$$

$$= \pi^{n/2} (\det A)^{-1/2} \exp \frac{1}{4} \sum_{i,j=1}^{n} B_i (A^{-1})_{ij} B_j \quad , \tag{31}$$

with the labels i and j being continuous as is appropriate for fields $v_j = v(\vec{x},t)$. Averages are conveniently evaluated by taking derivatives with respect to B_j.

This weather will drive our climate variables, so we want to describe the climate first in the absence of short term variations and then in interaction with them. Without weather we take for a climate Lagrangian density

$$L_c = T_1^2 \frac{\partial}{\partial t} \tilde{\phi}(\vec{x},t) \frac{\partial}{\partial t} \phi(\vec{x},t) - \frac{T_2}{2} \left[\tilde{\phi}(\vec{x},t) \frac{\partial \phi}{\partial t}(\vec{x},t) - \frac{\partial \tilde{\phi}}{\partial t}(\vec{x},t) \phi(\vec{x},t) \right]$$

$$- \ell_c^2 \nabla\tilde{\phi}(\vec{x},t) \cdot \nabla\phi(\vec{x},t) - A_c \tilde{\phi}(\vec{x},t)\phi(\vec{x},t) - \frac{\Lambda}{2} \tilde{\phi}(\vec{x},t)^2 \phi(\vec{x},t)^2 \quad , \tag{32}$$

and for the interaction between climate and weather we'll take

$$L_{\pm} = - \lambda_1 [W(\vec{x},t) + \tilde{W}(\vec{x},t)] [\tilde{\phi}(\vec{x},t) + \phi(\vec{x},t)]$$

$$- \lambda_2 [W(\vec{x},t) + \tilde{W}(\vec{x},t)] \tilde{\phi}(\vec{x},t)\phi(\vec{x},t)$$

$$- \lambda_3 W(\vec{x},t) \tilde{W}(\vec{x},t) [\phi(\vec{x},t) + \tilde{\phi}(\vec{x},t)] \quad . \tag{33}$$

The assumption that weather is gaussian restricts us to quadratic terms in W and $\tilde{W}$; higher order polynomials in ϕ and $\tilde{\phi}$ may appear. Symmetry between W and $\tilde{W}$ and between ϕ and $\tilde{\phi}$ is maintained consistent with their interpretation as diffusing and anti-diffusing fields; see Equations (24) and (25).

At this stage we have a set of coupled fields where clearly the dynamics of the rapidly varying W drives ϕ as we see in the Euler-Lagrange equations for ϕ

$$\left(T_1^2 \frac{\partial^2}{\partial t^2} + T_2 \frac{\partial}{\partial t} - \ell_c^2 \nabla^2 + A_c\right)\phi(\vec{x},t) =$$

$$-[\lambda_1 + \lambda_2 \phi(\vec{x},t)]\ [W(\vec{x},t) + \tilde{W}(\vec{x},t)] - \lambda_3 W(\vec{x},t)\tilde{W}(\vec{x},t) - \Lambda\tilde{\phi}(\vec{x},t)^2\phi(\vec{x},t)\ . \quad (34)$$

The usual Brownian motion treatment of climate as in Hasselmann (1976) (see Wang and Uhlenbeck, 1945), sets $\lambda_2 = \lambda_3 = 0$ and using (34) derives dynamic equations for $\langle\phi(\vec{x},t)\rangle$ or $\langle\phi(\vec{x},t)^2\rangle$ by utilizing the statistical properties of W . The <u>interaction</u> or <u>feedback</u> represented by the λ_2 and λ_3 terms are important. In the ice front-insolation questions of before they are the effective transport or stabilization mechanisms for the climate ϕ . The term in λ_2, for example, provides a space-time dependent contribution to the balance between the oscillation and diffusion terms and directly affects the stability of climate motion.

Through its interaction with statistical weather the climate becomes a statistical quantity itself. The statistical weight for the interacting climate-weather system will be

$$P(W,\tilde{W},\phi,\tilde{\phi}) = \exp - S(W,\tilde{W},\phi,\tilde{\phi}) \tag{35}$$

where S is the full action

$$S(W,\tilde{W},\phi,\tilde{\phi}) = \int d^Dx\, dt[L_w + L_c + L_I] \quad . \tag{36}$$

The statistical properties of the climate alone are given by the effective action

$$\exp - S_{eff}(\phi,\tilde{\phi}) = \int_{\vec{x},t} \Pi\, dW(\vec{x},t) d\tilde{W}(\vec{x},t) \exp - S(W,\tilde{W},\phi,\tilde{\phi}) \tag{37}$$

which amounts to integrating out the weather variables. The statistics of ϕ are now in

$$S_{eff}(\phi,\tilde{\phi}) = \int d^Dx\, dt\, L_{eff}(\phi,\tilde{\phi}) \quad , \tag{38}$$

with

$$L_{eff}(\phi,\tilde{\phi}) = L_c(\phi,\tilde{\phi}) - \frac{1}{2}\int d^Dy d\tau\, U\Big(\phi(\vec{x},t),\, \tilde{\phi}(\vec{x},t)\Big)$$

$$\cdot\, G_w(\vec{x}-\vec{y},\, t-\tau)\, U\Big(\phi(\vec{y},\tau),\, \tilde{\phi}(\vec{y},\tau)\Big) \tag{39}$$

where

$$U(\phi,\tilde{\phi}) = \lambda_1[\phi(\vec{x},t) + \tilde{\phi}(\vec{x},t)] + \lambda_2\tilde{\phi}(\vec{x},t)\phi(\vec{x},t) \quad , \tag{40}$$

and G_W is the interacting weather Green function satisfying

$$\left[\tau_1^2 \frac{\partial^2}{\partial t^2} + \tau_2 \frac{\partial}{\partial t} - \ell_w^2 \nabla^2 + a_w + \lambda_3 \Big(\phi(\vec{x},t) + \tilde{\phi}(\vec{x},t)\Big)\right] G_w(\vec{x},t) =$$

$$- \delta^D(\vec{x})\delta(t) \quad . \tag{41}$$

Looking at (39) we see that the weather has provided non-linear feedback terms for the climate variation which are apparent in the terms of

L_{eff} with no space or time derivations, which we call the effective potential

$$V_{eff}(\tilde{\phi}, \phi) = A_c\tilde{\phi}\phi + \frac{\lambda_1^2}{2}(\phi + \tilde{\phi})G_w(\phi + \tilde{\phi})$$

$$+ \frac{\lambda_1\lambda_2}{2}\left[(\phi + \tilde{\phi})G_w\tilde{\phi}\phi + \tilde{\phi}\phi G_w(\phi + \tilde{\phi})\right] + \frac{1}{2}\tilde{\phi}\phi(\Lambda + \lambda_2^2 G_w)\tilde{\phi}\phi \quad , \qquad (42)$$

using a more or less obvious shorthand notation.

The effective potential is the key result of this section. It shows how the weather, when "removed" from the climate problem by integration over $dWd\tilde{W}$, has provided non-linear feedback terms in the potential governing the motion of ϕ , the climate. To exhibit the connection with the previous section a bit more explicitly, let's look more closely at G_w. The space and time scales in the integration required for (39) or (42) are climate space and time scales and are larger than ℓ_w or τ_1 or τ_2. If we simplify a bit more by neglecting $\lambda_3(\phi + \tilde{\phi})$ in (41) as well as the space and time derivatives because of the previous argument, we may approximate $G_w(\vec{x} - \vec{y}, t - \tau)$ by

$$G_w(\vec{x} - \vec{y}, t - \tau) = -\frac{1}{a_w}\delta^D(\vec{x} - \vec{y})\delta(t - \tau) \quad . \qquad (43)$$

This formula is accurate as long as $a_w \neq 0$. Next imagine the climate dissipation term in T_2 is small so $\phi \approx \tilde{\phi}$. The effective potential becomes

$$V(q,\phi) = \frac{a(q)}{2}\phi^2 + \frac{b(q)}{3}\phi^3 + \frac{c(q)}{4}\phi^4 \quad , \qquad (44)$$

with

$$\frac{a(q)}{2} = A_c(q) - \frac{2\lambda_1(q)^2}{a_w(q)} \quad , \qquad (45)$$

$$\frac{b(q)}{3} = -\frac{\lambda_1(q)\lambda_2(q)}{a_w(q)} \quad , \tag{46}$$

$$\frac{c(q)}{4} = \frac{1}{2}\left(\Lambda(q) - \frac{\lambda_2(q)^2}{a_w(q)}\right) \quad , \tag{47}$$

where we have restored the dependence of the coefficients on the external parameter q .

The discussion of the previous section can now be carried out as before. The climate variable will be identified with y_o, and the dynamics of y_o are as described above. The value of the considerations here are primarily to give a framework for the effective potential formulation. One is able to identify directly the parameters a(q), etc. of the effective potential in terms of the parameters governing the underlying dynamics of short term phenomena which drive the climate. Additionally a certain insight is provided for the meaning of the effective potential. The dynamics of the climate are to be viewed as a most probable path in ϕ space; namely that path which minimizes the effective action S_{eff}. The full motion is much more complicated than given by (44) since it involves the high order non-linearities contained in G_w. One may proceed, however, with some hope that the approximations to G_w will suffice.

V DISCUSSION

In this note we have first of all tried to isolate some general features of energy budget models of the Budyko-Sellers variety by casting them into an effective potential or effective free energy. Extrema of this free energy with respect to the climate variables, ϕ , will yield the equilibrium states of the energy-budget models, and, as explained, above the time dependence of the models may be directly read off from the equation

$$\frac{\partial \phi}{\partial t} = - \frac{\partial}{\partial \phi} \text{(Effective Potential)} \quad .$$

In the ice front problem ϕ is taken to be the cosine of the ice front latitude, $y_o(t)$, and the dynamics then lie in a first order, non-linear equation $y_o(t)$. Such equations may sometimes admit of analytic solutions and may be easily treated numerically. Indeed the major use of the effective potential viewpoint is precisely the ease with which the time dependent problem may be studied. In addition it allows one to see developing in a smoothly varying manner the apparent discontinuous behavior in the y_o versus q plots typified by Figure 1. As noted in the text this is just the situation in the theory of phase transitions where a smoothly varying free energy may give rise to an apparently discontinuous phenomenon such as freezing of a liquid or spontaneous magnetization of a ferromagnet.

The second part of our discussion has concerned the framework in which to place the effective potential. Treating weather as a short time scale gaussian random field, we were able to construct an effective action for climate (long time scale phenomena) in interaction with weather; i.e., driven by it. The key point was to assign a probability density to the weather field

$$P = \exp - (\text{classical action}) \quad ,$$

where

$$\text{classical action} = \int d(\text{space})dt \; [\text{Langrangian density for the field}] \; .$$

In analogy with usual statistical physics we would then interpret the action as an "entropy" equal to $-\log$ (Probability) . The most probable state extremizes the action and is governed by the Euler-Lagrangian equations (Newton's law) for the system. Since weather fluctuates rapidly we determine an effective action for climate by integrating over all weather variations

$$\exp - (\text{effective climate action}) = \int d(\text{weather}) \exp - (\text{total action}) \; .$$

This leads directly to the effective potential point of view.

This paper has repeatedly fallen back on the ice cap-insolation energy budget models as a guide to the formulation and possible arena of use of the effective potential formulation. One may well imagine that in many cases where non-linear phenomena, stable and unstable, are driven by external forces, such as insolation, the effective potential will provide a convenient and tractable strategy for the study of their time dependence. The birth, growth, and decay of dust storms on Mars is just such an example; its study in the framework outlined here will be given in another publication (Vesecky and Abarbanel, 1977).

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