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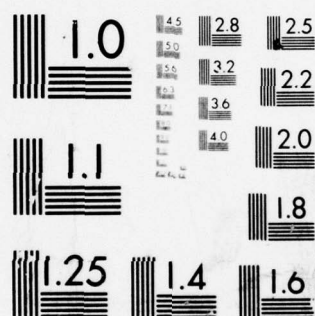
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CHARACTERIZATION OF NONPARAMETRIC CLASSES OF LIFE DISTRIBUTIONS

Interim rept., by

Naftali A. Langberg¹, Ramón V. León^{2,3} and Frank Proschan³

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Characterization of Nonparametric Classes of Life Distributions

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Naftali A. Langberg¹, Ramón V. León^{2,3}, and Frank Proschan³

ABSTRACT

In this paper we obtain characterizations of large classes of nonparametric life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, etc., classes. The methods used differ from the usual functional equation methods used for the far more common characterizations of parametric families of life distributions.

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Characterization of Nonparametric Classes of Life Distributions

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1. Introduction and Summary. Characterizations of particular parametric families of life distributions are quite common in the literature (see, for example, Kagen, Linnik, and Rao, 1973, and Patil, Kotz, and Ord, 1975). In this paper, by contrast, we present characterizations of large classes of nonparametric life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, etc., classes. (See Section 2 for exact definitions.) Such characterizations are far less common and generally require quite different mathematical and statistical techniques.

Our characterizations are based on order statistics, weighted spacings between order statistics, and total time on test transforms; in most cases inequalities among limiting expected values determine the characterizations. Related results concerning total time on test transforms had been obtained earlier by Barlow and colleagues (exact references are given for each of these results as they appear in the text below), but not necessarily under the weakest assumptions on the distributions. Since in characterization, emphasis is placed on obtaining results under the weakest assumptions on the distributions being characterized, we have found it useful to prove stronger versions of a number of these known results--for example, a characterization which requires that a distribution be differentiable is not as appealing as one that requires that it only be continuous.

In Section 2, we present preliminaries consisting of definitions and notation. In Section 3, we present properties of the total time on test transform and a characterization of the IFR(DFR) class of life distributions in terms of the concavity (convexity) of the total time on test transform. In Section 4, we present characterizations of the IFR(DFR) classes based on the monotonicity of the expected values of the weighted spacings between successive order statistics; the number of sample sizes required is infinite. By using the fact that the exponential distribution is both IFR and DFR, we are able to obtain a strengthened characterization of the exponential distribution, as compared with the earlier Saleh (1976) characterization. We also obtain additional characterizations of the IFR(DFR) distribution requiring only a single sample size; of course, we must compensate by making the stronger assumption of stochastic monotonicity rather than expected value monotonicity. In Section 5, we present characterizations of distributions such as IFRA, NBU, NBUE, and their duals, which are similar in spirit to those in Section 4 for the IFR(DFR) classes.

One final remark should be made. Chandra and Singpurwalla (1978) have pointed out the close relationship between the total time on test transform and the Lorenz curve used by econometrists. Thus, some of our results of Section 3 concerning total time on test transforms can be used to obtain analogous results for the Lorenz curve, and may thus be of interest and value in fields other than reliability.

2. Preliminaries. Let F be a life distribution, that is $F(0-) = 0$. We use the following notation and conventions: $F^{-1}(t) \equiv \inf\{x: F(x) > t\}$, $t \in [0, 1)$; $F^{-1}(1) \equiv \sup\{x: F(x) < 1\}$; $\bar{F} \equiv 1 - F$; $R \equiv -\ln \bar{F}$. We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nonincreasing".

Next we define the classes of life distributions to be considered in the sequel.

Definition 2.1. (a) F is increasing failure rate (IFR) if $\bar{F}(t+x)/\bar{F}(t)$ is decreasing in t ($-\infty < t < F^{-1}(1)$) for each $x > 0$.

(b) F is (shifted) decreasing failure rate (SDFR) if $\bar{F}(t+x)/\bar{F}(t)$ is increasing in t ($F^{-1}(0) \leq t < \infty$) for each $x > 0$.

(c) F is increasing failure rate average (IFRA) if $\frac{1}{t}R(t)$ is increasing in t ($0 < t < F^{-1}(1)$).

(d) F is decreasing failure rate average (DFRA) if $\frac{1}{t}R(t)$ is decreasing in $t > 0$.

(e) F is new better than used (NBU) if $\bar{F}(x+y) \leq \bar{F}(x)\bar{F}(y)$ for $x > 0, y > 0$.

(f) F is new worse than used (NWU) if $\bar{F}(x+y) \geq \bar{F}(x)\bar{F}(y)$ for $x > 0, y > 0$.

(g) F is new better than used in expectation (NBUE) if (i) $\int_0^\infty x dF(x) < \infty$;
(ii) $\int_t^\infty \bar{F}(x) dx \geq (\int_0^\infty x dF(x))\bar{F}(t)$ for $t > 0$.

(h) F is new worse than used in expectation (NWUE) if $\int_t^\infty \bar{F}(x) dx \leq (\int_0^\infty x dF(x))\bar{F}(t)$ for $t > 0$.

The chain of implications $IFR \Rightarrow IFRA \Rightarrow NBU \Rightarrow NBUE$ is readily established (see Marshall and Proschan, 1972).

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from F . The k -th weighted spacing, $W_{k:n}$, between order statistics $X_{k-1:n}$ and $X_{k:n}$ is defined by $W_{k:n} \equiv (n - k + 1) \cdot (X_{k:n} - X_{k-1:n})$ for $k = 1, 2, \dots, n$, where $X_{0:n} \equiv 0$. The total time on test up to the k -th order statistic, $T(X_{k:n})$, is defined by $T(X_{k:n}) = \sum_{i=1}^k W_{i:n}$ for $k = 1, 2, \dots, n$, and $T(X_{0:n}) \equiv 0$. If we assume that n items are placed on test at time 0 and that successive failures are observed at times $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$, then $W_{k:n}$ represents the total test time observed between $X_{k-1:n}$ and $X_{k:n}$, and $T(X_{k:n})$ represents the total test time observed between 0 and $X_{k:n}$ (see Barlow and Proschan, 1975, p. 61).

3. Properties of the Total Time on Test and Its Transform. Let $H_F^{-1}(t) \equiv \int_0^{F^{-1}(t)} \bar{F}(u) du$ for $0 \leq t \leq 1$. Barlow and Campo (1975) call H_F^{-1} the total time on test transform. In this section we develop some of the properties of H_F^{-1} .

Before starting the first theorem we need two definitions.

Definition 3.1. A point x is a point of increase of F if $F(x - h) < F(x) < F(x + h)$ for every $h > 0$.

Definition 3.2. A sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$ of ordered pairs of natural numbers is a t -sequence ($0 \leq t \leq 1$) if (i) $1 \leq k_r \leq n_r < n_r + 1$ for all r , and (ii) $k_r/n_r \rightarrow t$ as $r \rightarrow \infty$.

Theorem 3.3. Let $F^{-1}(t)$ be a point of increase of F , and let (k, n) range over a t -sequence. Then as $n \rightarrow \infty$,

$$\frac{1}{n} T(X_{k:n}) \rightarrow H_F^{-1}(t) \text{ a.s. .}$$

Proof. Let F_n denote the empirical distribution function of F . Then $T(X_{k:n}) = n H_{F_n}^{-1}\left(\frac{k-1}{n}\right) = \int_0^{X_{k:n}} \bar{F}_n(u) du$ (see Barlow and Campo, 1975). Also for (k, n) ranging over a t -sequence, $X_{k:n} \rightarrow F^{-1}(t)$ a.s. as $n \rightarrow \infty$ since $F^{-1}(t)$ is a point of increase of F (see Rao, 1973, p. 423). The desired result follows by the Glivenko-Cantelli Theorem (Chung, 1974, p. 123). ||

Next we note that if EX_1 is finite, then $EX_{k:n}$, $EW_{k:n}$, and $ET(X_{k:n})$ are also finite, since $0 \leq X_{k:n} \leq T(X_{k:n}) \leq \sum_{i=1}^n X_i \equiv n \bar{X}_n$. This observation can be used to show that whenever EX_1 is finite, $\{\frac{1}{n_r} T(X_{k_r:n_r})\}_{r=1}^{\infty}$ is uniformly integrable for every t -sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$. Since a uniformly convergent sequence which converges almost surely converges in mean (see Breiman, 1973, p. 91), we can state the following result.

Theorem 3.4. Let t , k , and n be as in Theorem 3.3 and let EX_1 be finite. Then $E|\frac{1}{n}T(X_{k:n}) - H_F^{-1}(t)| \rightarrow 0$ as $n \rightarrow \infty$. In particular, $\frac{1}{n}ET(X_{k:n}) \rightarrow H_F^{-1}(t)$ as $n \rightarrow \infty$.

We remark that neither Theorem 3.3 nor Theorem 3.4 is true if t is not a point of increase of F . In this case a counterexample to Theorem 3.3 and to Theorem 3.4 can easily be constructed using the facts that

$\lim_{n \rightarrow \infty} X_{[nt]:n} = F^{-1}(t-)$ and $\overline{\lim}_{n \rightarrow \infty} X_{[nt]:n} = F^{-1}(t-)$ and that $F^{-1}(t-) \neq F^{-1}(t)$. ($[\cdot]$ denotes the greatest integer function). To show that $\lim_{n \rightarrow \infty} X_{[nt]:n} = F^{-1}(t)$ and $\overline{\lim}_{n \rightarrow \infty} X_{[nt]:n} = F^{-1}(t)$, use the fact that $P[X_{[nt]:n} > x] = P[B(n, \bar{F}(x)) > n - [nt] + 1]$, where $B(n, \bar{F}(x))$ denotes a binomial random variable (see Mood, Graybill, and Boes, 1974, p. 252), and the law of the iterated logarithm (see Breiman, 1968, p. 291).

Let ${}^+f(x_0)$ denote the right-hand derivative of f at the point x_0 . We have the following lemma.

Lemma 3.5. Let x be a point of increase and of continuity of F . Then ${}^+H_F^{-1}(F(x))$ exists and is nonzero if and only if ${}^+R(x)$ exists and is nonzero. In either case, ${}^+H_F^{-1}(F(x)) {}^+R(x) = 1$.

Proof. Note that in a neighborhood of x , F^{-1} behaves like the usual inverse function of F . The result follows using standard differentiation results. ||

The next proposition is easily verified.

Proposition 3.6. The life distribution F is (i) IFR if and only if either F is degenerate or $R(x)$ is convex on $(F^{-1}(0), F^{-1}(1))$ and $F(F^{-1}(0)) = 0$; (ii) SDFR if and only if $R(x)$ is concave on $(F^{-1}(0), \infty)$.

The following simple properties of H_F^{-1} will be needed in the proof of our next theorem.

$$(3.1) \quad H_F^{-1}(0) = F^{-1}(0).$$

$$(3.2) \quad H_F^{-1}(t+) = H_F^{-1}(1) = EX_1.$$

$$(3.3) \quad H_F^{-1}(1-) = H_F^{-1}(1) = EX_1.$$

$$(3.4) \quad H_F^{-1} \text{ is increasing on } [0, 1].$$

(3.5) For $y \in [0, \infty)$, the set $\{s: H_F^{-1}(s) = y\} = [a, b]$, where $0 < a < b \leq 1$, if and only if $P(X = F^{-1}(a)) = b - a$.

(3.6) For $0 \leq a < 1$, $H_F^{-1}(a-) = H_F^{-1}(a)$ if and only if $F(F^{-1}(a-)) = F(F^{-1}(a))$. In particular H_F^{-1} is continuous on $[a, b]$ if and only if every point in $(F^{-1}(a), F^{-1}(b-))$ is a point of increase of F .

Theorem 3.7. (Barlow and Campo, 1975). The life distribution F is IFR(SDFR) if and only if H_F^{-1} is concave (convex) on $[0, 1]$.

Proof. Let H_F^{-1} be concave on $[0, 1]$. Since H_F^{-1} is increasing on $[0, 1]$, there exists a real number A in $[0, 1]$ such that H_F^{-1} is strictly increasing on $[0, A]$ and constant on $[A, 1]$. If $A = 0$, H_F^{-1} is constant on $[0, 1]$ and consequently, F is the IFR distribution degenerate at $F^{-1}(0)$. Next suppose that $A > 0$. It follows that $H_F^{-1}(t) > 0$ for all $t \in (0, A)$. Equivalently, ${}^+H(F(x)) > 0$ for x in $(F^{-1}(0), F^{-1}(1))$ since $F^{-1}(A-) = F^{-1}(1)$. By (3.5) and (3.6), every point of $(F^{-1}(0), F^{-1}(1))$ is a point of increase and of continuity of F . Hence, by Lemma 3.5, the concavity of H_F^{-1} implies that ${}^+R(x)$ exists and is increasing on $(F^{-1}(0), F^{-1}(1))$, that is, R is convex on $(F^{-1}(0), F^{-1}(1))$. Since by (3.6) $F(F^{-1}(0)) = 0$, then F is IFR by Proposition 3.6.

Next let F be IFR. By Proposition 3.6 either F is degenerate in which case H_F^{-1} is constant and thus concave or $R(x)$ is convex on $S \equiv (F^{-1}(0), F^{-1}(1))$ and $R(F^{-1}(0)) = 0$. Assuming the latter, let $x \in S$ and $h > 0$. Then

$$0 < \frac{R(x) - R(F^{-1}(0))}{x - F^{-1}(0)} \leq \frac{R(x+h) - R(x)}{h}.$$

Consequently ${}^+R$ exists and is positive on S . Now since ${}^+F = \bar{F} {}^+R$ is positive on S , S contains only points on increase and of continuity of F . Thus by Lemma 3.5, ${}^+H_F^{-1} \circ F$ is decreasing on S , that is, ${}^+H_F^{-1}$ is decreasing on $(0, 1)$. Hence H_F^{-1} is concave on $(0, 1)$. Since an IFR distribution has a finite mean by (3.2) and (3.3), H_F^{-1} is concave on $[0, 1]$.

The counterpart results for the SDFR case can be proved similarly. ||

Theorem 3.7 is due to Barlow and Campo, 1975 (see also Barlow, 1977), but our proof is new. Our proof avoids some technical difficulties which arise in the limiting argument used in the Barlow and Campo proof of the "if" part of Theorem 3.7.

4. Characterizations of the IFR(SDFR) Class. Barlow and Proschan (1966)

have shown that if F is IFR(SDFR), then for all $x > 0$, $P(W_{k:n} > u)$ is decreasing (increasing) in k ($k = 2, 3, \dots, n$) for all $n \geq 2$. If F has a finite mean, then $E W_{k:n} < \infty$ for all choices of k and n and consequently $E W_{k:n}$ is decreasing (increasing) in k ($k = 2, 3, \dots, n$) for all $n \geq 2$. In this section we prove that a slightly weaker version of the last condition is sufficient for F to be IFR(SDFR). Then we use this result to obtain a characterization of the shifted exponential obtained by Saleh (see Kotz, 1974), who required stronger regularity conditions on F than we require. Two other characterizations of the IFR(SDFR) are given.

The main result of this section follows.

Theorem 4.1. Let F be a continuous life distribution with finite mean. Then F is IFR(SDFR) if and only if $E W_{k:n}$ is decreasing (increasing) in k ($k = 2, \dots, n$) for infinitely many n .

The "only if" part has already been shown. To prove the "if" part, we need the following lemma. This lemma shows that every point in the support of F is a point of increase of F . Thus Theorem 3.4 can be used at every point of the support of F to show that H_F^{-1} is concave (convex), that is, F is IFR(SDFR) by Theorem 3.7. We remark that this proof avoids all assumptions on F other than continuity. [If we are willing to assume the existence of a positive continuous density f everywhere in the support of F , we can obtain a more direct proof of the "if" part of Theorem 4.1 by using the fact that in this case $W_{[nt]:n}$ converges in distribution to an exponential distribution with failure rate $r(F^{-1}(t))$, where $r(t) \equiv \frac{f(t)}{\bar{F}(t)}$ is the failure rate of F (see Pyke, 1965).]

Lemma 4.2. Let F be a continuous life distribution with finite mean. Let $E W_{k:n}$ be decreasing (increasing) in k ($k = 2, \dots, n$) for infinitely many n . Then the support of F is the interval $[F^{-1}(0), F^{-1}(1)]$.

Proof. The support of a continuous distribution is a closed set without isolated points (see Chung, 1974, p. 10). It follows that if S , the support of F , is not an interval we can find a, b , and ϵ such that $(a - \epsilon, a] \subset S$, $(a, b) \subset \sim S \equiv \{x: x \notin S\}$, and $[b, b + \epsilon) \subset S$. Let $t = F(a) = F(b)$, $t_1 = \frac{t + F(a - \epsilon)}{2}$ and $t_2 = (t + F(b + \epsilon))/2$. Also let $h > 0$ be small enough so that $[t_1 - h, t_1 + h] \subset (F(a - \epsilon), t)$ and $[t_2 - h, t_2 + h] \subset (t, F(b + \epsilon))$. Since $T(X_{k:n}) \equiv \sum_{i=1}^k W_{i:n}$, we obtain for each one of the infinitely many n that

$$\begin{aligned}
 & ET(X_{([n(t_1-h)]+[n \cdot 2h]):n}) - ET(X_{[n(t_1-h)]:n}) \\
 (4.1) \quad & \geq (\leq) ET(X_{([n(t-h)]+[n \cdot 2h]):n}) - ET(X_{[n(t-h)]:n}) \\
 & \geq (\leq) ET(X_{([n(t_2-h)]+[n \cdot 2h]):n}) - ET(X_{[n(t_2-h)]:n}).
 \end{aligned}$$

The points at which F equals $t_1 - h, t_1 + h, t - h, t + h, t_2 - h$, and $t_2 + h$ are all in the interior of S and are consequently points of increase of F . Applying Theorem 3.4 to the chain of inequalities (4.1), we conclude that

$$\begin{aligned}
 & H_F^{-1}(t_1 + h) - H_F^{-1}(t_1 - h) \\
 (4.2) \quad & \geq (\leq) H_F^{-1}(t + h) - H_F^{-1}(t - h) \\
 & \geq (\leq) H_F^{-1}(t_2 + h) - H_F^{-1}(t_2 - h).
 \end{aligned}$$

Since $H_F^{-1}(\cdot) = \int_0^{F^{-1}(\cdot)} \bar{F}(u) du$ is continuous at t_1 and t_2 , letting $h \rightarrow 0$ in (4.2), we conclude that $\lim_{h \rightarrow 0} H_F^{-1}(t + h) - H_F^{-1}(t - h) = 0$. But since H_F^{-1} is increasing,

this implies that H_F^{-1} is continuous at t , or equivalently, that F^{-1} is continuous at t . This contradicts the fact that F is constant on (a, b) . It follows that S must be an interval. The desired result follows. ||

We now complete the proof of Theorem 4.1.

Proof of Sufficiency. Let t_1, t_2 , and h be such that $0 \leq t_1 < t_2 < t_2 + h \leq 1$. Using the argument in the proof of Lemma 3.8 yielding (4.2), we obtain:

$$(4.3) \quad H_F^{-1}(t_1 + h) - H_F^{-1}(t_1) \geq (\leq) H_F^{-1}(t_2 + h) - H_F^{-1}(t_2).$$

Since (4.3) is true for all t_1, t_2 , and h satisfying the constraints above, H_F^{-1} must be concave (convex) on $[0, 1]$. By Theorem 3.7, this implies that F is IFR(DFR). ||

It is clear from the proof of Theorem 4.1 that the following characterization of the IFR(SDFR) class is also true.

Theorem 4.3. Let F be a continuous life distribution with finite mean. Then F is IFR(SDFR) if and only if for infinitely many $n \geq N$ and some $\ell (1 \leq \ell < N)$

$$E \sum_{i=k}^{k+\ell} W_{i:n}$$

is decreasing (increasing) in $k (1 \leq k \leq n - \ell)$.

Note that F is both IFR and SDFR if and only if F is shifted exponential, that is,

$$\bar{F}(x) = \begin{cases} e^{-\lambda(x - F^{-1}(0))} & x \geq F^{-1}(0) \\ 1 & x < F^{-1}(0), \end{cases}$$

for some positive λ . Hence as a corollary of Theorem 4.1 we obtain the following:

Corollary 4.4. Let F be a continuous life distribution with finite mean. Then F is shifted exponential with mean μ if and only if for infinitely many $n \geq 2$, $E W_{k:n} = \mu$ for $k = 2, 3, \dots, n$.

A similar characterization was obtained by Saleh (1976) but with the additional (unnecessary) condition that $\inf\{x: F(x) \geq t\}$ is differentiable on $(0, 1)$.

Now we give another characterization of the IFR(SDFR) class which requires conditions for only one sample size. Since

$$(4.4) \quad P(X_{m+1:n} - X_{m:n} > u | X_{m:n} = x) = (\bar{F}(x+u)/\bar{F}(x))^{n-m}$$

(see David, 1970, p. 18), the following result holds:

Theorem 4.5. The distribution F is IFR(SDFR) if and only if for some fixed n and m ($2 \leq m+1 \leq n$), and all $u \geq 0$, $P(X_{m+1:n} - X_{m:n} > u | X_{m:n} = x)$ is decreasing (increasing) in x ($-\infty < x < F^{-1}(1)$) [$(F^{-1}(0) \leq x < \infty)$].

Actually if F is IFR(SDFR), then $P(X_{m+1:n} - X_{m:n} > u | X_{m:n} = x)$ is decreasing (increasing) in x for all n and m , where $2 \leq m+1 \leq n$. However since the emphasis of this paper is on characterizations, we omit this generalization from the statement of Theorem 4.5. A similar remark can be made about other theorems in this paper (see Theorem 5.1 for example).

Recall that a random variable X is stochastically increasing (decreasing) in Y , another random variable, if for all x , $P(X > x | Y = y)$ is increasing (decreasing) in y . Hence Theorem 4.5 can be restated using this language. Similar remarks apply to other theorems in this paper (see for example Theorems 5.1 and 5.3).

5. Characterizations of Classes of Life Distributions Other Than the IFR(SDFR). In this section we characterize classes of life distributions other than the IFR(SDFR). The following two characterizations are similar in spirit to the characterization of the IFR(SDFR) class given in Theorem 4.5. By (4.4) we immediately obtain:

Theorem 5.1. The life distribution F is NBU(NWU) if and only if $P(X_{1:n-m} > u) \geq (\leq) P(X_{m+1:n} - X_{m:n} > u | X_{m:n} = x)$ for some fixed n and m ($1 \leq m < n$), and all $u \geq 0$ and $x \geq 0$.

Since $E(X_{n:n} - X_{n-1:n} | X_{n-1:n} = x) = \frac{\int_x^\infty \bar{F}(u) du}{\bar{F}(x)}$, we also have:

Theorem 5.2. Let F be a life distribution with finite mean. Then F is NBUE (NWUE) if and only if $E(X_{n:n} - X_{n-1:n} | X_{n-1:n} = x) \leq (\geq) EX_1$ for some fixed $n \geq 2$, and all $0 \leq x < F^{-1}(1)$.

Next we give a characterization of the IFRA(DFRA) class:

Theorem 5.3. Let F be a life distribution such that $F(0) = 0$. Then F is IFRA if and only if for all $x > 0$, $P(W_{1:n} > x)$ is increasing in $n \geq N$, where N is arbitrary.

Proof. We have for $0 \leq x < \infty$, $P(W_{1:n} > x) = \bar{F}^n(x/n)$.

It follows that $P(W_{1:n} > x)$ is increasing in $n \geq N$ for all $x > 0$ if and only if

$$(5.1) \quad \bar{F}^n(x/n) \leq \bar{F}^m(x/m) \text{ for all } N \leq n < m \text{ and all } 0 \leq x < \infty.$$

Recall that F is IFRA if and only if

$$(5.2) \quad \bar{F}^{1/t_2}(t_2) \leq \bar{F}^{1/t_1}(t_1) \text{ for all } 0 < t_1 < t_2 < F^{-1}(1).$$

We show that (5.1) implies (5.2). Let $0 < t_1 < t_2$, both be rational, and let $N \leq n < m$ be natural numbers such that $\frac{t_2}{t_1} = \frac{m}{n}$. Let $\alpha = (m_1 t_1)^{-1}$.

Then $na = 1/t_2$ and $ma = 1/t_1$, so it is easily seen that (5.1) implies that $\bar{F}^{1/t_2}(xat_2) \leq \bar{F}^{1/t_1}(xat_1)$. Letting $x = a^{-1}$, (5.2) follows since $\bar{F}$ is right continuous and the rationals are dense.

To show (5.2) implies (5.1), let x , n , and m be such that $x \geq 0$ and $N \leq n < m$. If $x = 0$, then $\bar{F}^n(x/n) = \bar{F}^m(x/m) = 1$ since $F(0) = 0$ so (5.1) follows. If $x/m \geq F^{-1}(1)$ then $\bar{F}(x/m) = 1$ and (5.1) also follows. If $0 < x$ and $\frac{x}{m} < F^{-1}(1)$, then set $t_1 = x/m$ and $t_2 = x/n$ in (5.2) to obtain (5.1). ||

The following dual theorem has a similar proof which we omit.

Theorem 5.4. Let F be a life distribution. Then F is DFRA if and only if for all $x > 0$, $P(W_{1:n} > x)$ is decreasing in $n \geq N$, where N is arbitrary.

To prove our next result we need a theorem of Barlow and Proschan (1966).

Let $X(t)$ have distribution $F(G)$. We assume that $F(0) = 0 = G(0)$, and that F and G are continuous. We also assume that the support of F is an interval, possibly infinite, and that G is strictly increasing on its support.

Theorem 5.5. (Barlow and Proschan, 1966, Theorem 3.6). Let $\frac{G_0^{-1}F(x)}{x}$ be increasing in x in the support of F . Then $EX_{i:n}/EY_{i:n}$ is decreasing in i ($i = 1, 2, \dots, n$).

Theorem 5.6. Let F be a continuous life distribution with finite mean. Assume that the support of F is an interval and that $F(0) = 0$. Then F is IFRA(DFRA) if and only if $EX_{i:n}/(\sum_{k=1}^i 1/(n-k+1))$ is decreasing (increasing) in i ($i = 1, 2, \dots, n$) for infinitely many n .

Proof. Let G in Theorem 5.5 be the exponential distribution with mean 1. Then $EY_{i:n} = \sum_{k=1}^i \frac{1}{n-k+1}$ (see Barlow and Proschan, 1975, p. 60). Thus necessity follows from Theorem 5.5 (as does the dual result for the opposite direction of monotonicity) if we note that $G^{-1}(x) = -\ln(1-x)$.

To prove sufficiency, first observe that every point in the support of $F(G)$ is a point of increase of $F(G)$. Hence $Y_{[nt]:n} \rightarrow G^{-1}(t) = -\ln(1-t)$ and $X_{[nt]:n} \rightarrow F^{-1}(t)$ a.s. as $n \rightarrow \infty$. We show $\{X_{[nt]:n}\}_{n=1}^{\infty}$ is uniformly integrable. We have

$$P(X_{[nt]:n} > x) = P(B(n, \bar{F}(x)) > n - [nt] + 1),$$

where $B(n, \bar{F}(x))$ denotes a binomial random variable with parameters n and $\bar{F}(x)$. Thus

$$(5.3) \quad P[X_{[nt]:n} \geq \frac{n}{n - [nt] + 1} \bar{F}(x)]$$

since $P(Z > t) \leq \frac{EZ}{t}$ for any nonnegative random variable Z . Hence

$$\begin{aligned} EX_{[nt]:n} I[X_{[nt]:n} \geq A] \\ = \int_A^{\infty} P[X_{[nt]:n} > x] dx + AP[X_{[nt]:n} \geq A] \end{aligned}$$

[by integration by parts]

$$\leq \frac{n}{n - [nt] + 1} \left(\int_A^{\infty} \bar{F}(x) dx + A \bar{F}(A) \right)$$

[by (5.3)]

$$\leq \frac{1}{1 - \frac{[nt]}{n} + \frac{1}{n}} (E X_1 I[X_1 \geq A]).$$

It follows that $X_{[nt]:n}$ (and similarly $Y_{[nt]:n}$) is a uniformly integrable sequence in n . Consequently, $EX_{[nt]:n} \rightarrow F^{-1}(t)$ and $EY_{[nt]:n} \rightarrow G^{-1}(t)$ as $n \rightarrow \infty$. Thus by hypothesis, $F^{-1}(t)/(-\ln(1-t))$ is decreasing (increasing) in t ($0 < t < 1$). Equivalently, $\frac{F^{-1}(F(x))}{-\ln(1 - \bar{F}(x))} = \frac{x}{-\ln \bar{F}(x)}$ is decreasing (increasing) in x ($0 < x < F^{-1}(1)$). Sufficiency follows. ||

Now we characterize the NBU (NWU) class. First we prove a lemma.

Lemma 5.7. Let F be a continuous life distribution. Then F is NBU (NWU) if and only if $F^{-1}(t + s - ts) \leq (\geq) F^{-1}(t) + F^{-1}(s)$ for every t and s such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F .

Proof. Let $F^{-1}(t + s - ts) \leq F^{-1}(t) + F^{-1}(s)$ for all t and s such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F . To show that F is NBU, it is enough to show that $\bar{F}(x + y) \leq \bar{F}(x) \bar{F}(y)$ for all x and $y \in D = \{z > 0: F(z - \epsilon) < F(z) \text{ for all } \epsilon > 0\}$. But since $\bar{F}$ is continuous for each x and $y \in D$, we can find sequences $\{x_n\}$ and $\{y_n\}$ of points of increase of F such that $x_n \uparrow x$ and $y_n \uparrow y$ as $n \rightarrow \infty$. Thus it is enough to show $\bar{F}(x + y) \leq \bar{F}(x) \bar{F}(y)$ for all x and y which are points of increase of F . Let x and y be two such points and let $t = F(x)$ and $s = F(y)$. Then $t + s - ts \leq F(F^{-1}(t + s - ts)) \leq F(F^{-1}(t) + F^{-1}(s))$ by hypothesis. Hence $1 - t - s + ts \geq \bar{F}(F^{-1}(t) + F^{-1}(s))$; that is, $\bar{F}(x) \bar{F}(y) \geq \bar{F}(x + y)$, as desired.

Necessity follows by reversing the steps of the last three sentences above. ||

Theorem 5.8. Let F be a continuous life distribution with finite mean. Then F is NBU (NWU) if and only if for every t and s in $(0, 1)$ such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F , we have

$$E(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} | X_{[nt]:n})$$

a.s.

$$\leq (\geq) EX_{[n(1-t)s]:n-[nt]}$$

for infinitely many n .

Proof. We first prove sufficiency. Let t and s be such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F , let n always range over the infinite

sequence of the hypothesis. By the Markov property of order statistics, we have

$$(5.4) \quad P_n(x) \equiv P(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} > x | X_{[nt]:n})$$

$$\stackrel{\text{a.s.}}{=} P(B(n - [nt]), \frac{\bar{F}(x + X_{[nt]:n})}{\bar{F}(X_{[nt]:n})} > n - [nt] - [n(1-t)s] + 1),$$

where $B(n, p)$ is a binomial random variable with parameters n and p .

But since $F^{-1}(t)$ is a point of increase of F , then by the SLLN and the fact that $X_{[nt]:n} \rightarrow F^{-1}(t)$ a.s. as $n \rightarrow \infty$, we have

$$(5.5) \quad \frac{1}{n - [nt]} B(n - [nt]), \frac{\bar{F}(x + X_{[nt]:n})}{\bar{F}(X_{[nt]:n})} \rightarrow \frac{\bar{F}(x + F^{-1}(t))}{1 - t}$$

a.s. as $n \rightarrow \infty$. Hence by (5.4) and (5.5),

$$(5.6) \quad P(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} > x | X_{[nt]:n})$$

$$\rightarrow \begin{cases} 1 & \text{if } x < F^{-1}(t + s - ts) - F^{-1}(t) \\ 0 & \text{if } x > F^{-1}(t + s - ts) - F^{-1}(t) \end{cases}$$

a.s. as $n \rightarrow \infty$.

Now (5.4) and the inequality $P(Z > t) \leq \frac{EZ}{t}$ ($t > 0$) imply that

$$P_n(x) \stackrel{\text{a.s.}}{\leq} \frac{1}{n - [nt] - [n(1-t)s] + 1} \left[\frac{(n - [nt])\bar{F}(x + X_{[nt]:n})}{\bar{F}(X_{[nt]:n})} \right]$$

$$\stackrel{\text{a.s.}}{\leq} \left[\frac{(n - [nt])}{(n - [nt] - [n(1+t)s])\bar{F}(X_{[nt]:n})} \right] \bar{F}(x).$$

Since $\int_0^\infty \bar{F}(x)dx < \infty$, $X_{[nt]} \rightarrow F^{-1}(t)$ a.s. as $n \rightarrow \infty$, and $\bar{F}(F^{-1}(t)) > 0$, it follows that for all sufficiently large n , $P_n(x) \stackrel{\text{a.s.}}{\leq} g(x)$, where $g(x)$ is integrable. Thus by the Lebesgue dominated convergence theorem,

$$E(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n}) = \int_0^\infty P_n(x)dx \\ \rightarrow F^{-1}(t + s - ts) - F^{-1}(t) \text{ a.s. as } n \rightarrow \infty.$$

Since $X_{[n(1-t)s]:n-[nt]} \rightarrow F^{-1}(s)$ a.s. as $n \rightarrow \infty$, we have by hypothesis that $F^{-1}(t + s - ts) \leq F^{-1}(t) + F^{-1}(s)$. Sufficiency follows from Lemma 5.7.

To show necessity, let $X(H)$ denote a random variable with distribution H . For $y \geq 0$, let $\bar{G}_y \equiv \bar{F}(x + y)/\bar{F}(y)$. Then F is NBU (NWU) if and only if for all $y > 0$, $X(G_y)$ is stochastically smaller (larger) than $X(F)$, written $X(G_y) \stackrel{\text{st}}{\leq} (\stackrel{\text{st}}{\geq}) X(F)$. Hence if F is NBU (NWU), then for all $y \geq 0$, $0 < t < 1$, and $0 < s < 1$,

$$X(G_y)_{[n(1-t)s]:n-[nt]} \stackrel{\text{st}}{\leq} (\stackrel{\text{st}}{\geq}) X(F)_{[n(1-t)s]:n-[nt]}.$$

But by the Markov property of order statistics, the conditional random variable $X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} | X_{[nt]} = y$ has the same distribution as the random variable $X(G_y)_{[n(1-t)s]:n-[nt]}$ necessity follows. ||

Observe that by the Markov property of order statistics,

$$\frac{1}{n - [nt]} E\left(\sum_{k=1}^{n-[nt]} X_{[nt]+k:n} - X_{[nt]:n} | X_{[nt]} = y\right) \\ \stackrel{\text{a.s.}}{=} \frac{\int_0^\infty \bar{F}(x + X_{[nt]:n})ds}{\bar{F}(X_{[nt]:n})}.$$

Hence we can use the methods of the proof of Theorem 5.8 to obtain the following characterization of the NBUE (NWUE) class.

Theorem 5.9. Let F be a continuous life distribution with finite mean. Then F is NBUE (NWUE) if and only if for every t in $(0, 1)$ such that $F^{-1}(t)$ is a point of increase, we have

$$\frac{1}{n - [nt]} E\left(\sum_{k=1}^{n-[nt]} X_{[nt]+k:n} - EX_{[nt]:n} \mid X_{[nt]:n}\right) \leq EX_1$$

for infinitely many n .

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