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MIXED BOUNDARY VALUE PROBLEMS FOR THE ELASTIC STRIP. THE EIGENF--ETC(U)

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MIXED BOUNDARY VALUE PROBLEMS FOR THE ELASTIC STRIP: THE EIGENFUNCTION EXPANSION

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ABSTRACT

The stress function for a semi-infinite elastic strip with free edges is expanded in eigenfunctions of Papkovitch-Fad'le type, and biorthogonal functions are constructed by use of the adjoint operator. To establish completeness, the same expansion is obtained by a Fourier transform solution of the biharmonic equation.

When end data is supplied in the form of prescribed tractions or prescribed displacements, the coefficients in the expansion must be found by truncation of an infinite set of linear equations. It is noted that the methods for formulating such equations that have been proposed in the past are unstable with respect to the order of truncation.

A systematic search among all possible weighting functions in the biorthogonal family for use in a Galerkin method leads to a unique choice of optimal weighting functions securing maximum stability by generating a diagonally-dominated matrix. Calculations are presented for a number of test cases using this method together with those of Benthem (1963) and Johnson and Little (1965), and with a method of direct collocation. The method of optimal weighting functions produces considerably more stable values for coefficients as the order of truncation is changed. This opens the way for an examination of the convergence of the eigenfunction expansions for data of this class, and should also prove useful for problems involving cracks.

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Key Words: Elastic end strip problem, Biorthogonal eigenfunction expansion, Optimal weighting functions

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SIGNIFICANCE AND EXPLANATION

The problem of calculating stresses in an elastic strip $|x_1| < 1, x_2 > 0$ from given end conditions has proved difficult despite its practical importance. The Airy stress function Ω possesses an expansion in eigenfunctions, but the coefficients in this expansion can be found explicitly only in certain particular cases. When the end data takes the form in which the tractions $\Omega_{,11}$, $\Omega_{,12}$ are prescribed, the coefficients have to be found from the truncated form of an infinite set of linear equations. These equations can be formulated in many ways, but it has been found that those proposed in the past lead to matrices which fail to satisfy the criteria for stability of the solution as the order of truncation increases.

A set of optimal weighting functions for use in a Galerkin method of formulating the linear equations has been found by systematically searching among possible members of the family of eigenfunctions. These functions secure maximum stability by generating a diagonally-dominated matrix. Calculations are presented for a number of test cases using this method together with those of Benthem (1963) and Johnson and Little (1965), and with a method of direct collocation. The method of optimal weighting functions produces considerably more stable values for coefficients as the order of truncation is changed. This opens the way for an examination of the convergence of the eigenfunction expansions for data of this class, and should also prove useful for problems involving cracks.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

MIXED BOUNDARY VALUE PROBLEMS
FOR THE ELASTIC STRIP: THE EIGENFUNCTION EXPANSION

D. A. Spence

1. Introduction

The stress function $\Omega(x,y)$ in a semi-infinite elastic strip $y > 0$ with free edges $x = \pm 1$ satisfies the biharmonic equation $\Delta^2\Omega = 0$, and possesses an eigenfunction expansion of the form

$$\Omega = \sum c_k e^{-\lambda_k y} \omega_k(x) \qquad (1)$$

from which the stress components are found as the second derivatives Ω_{xx}, Ω_{xy}, Ω_{yy}. The λ_k are the zeros with positive real parts of

$$\lambda \pm \sin\lambda \cos\lambda \left(\begin{matrix} \text{even} \\ \text{odd} \end{matrix} \text{ eigenfunctions} \right) .$$

Such expansions, associated with the names of Papkovitch (1940) and Fad'le (1940), are discussed by Lur'e (1964), Buchwald (1964), Buchwald & Doran (1965) and many other authors.

For certain combinations of boundary data on the edge $y = 0$, namely when either (i) Ω_{xx} and Ω_{yy}, or (ii) Ω_y and $\Delta\Omega_y$ are both given on this boundary, the coefficients c_k can be written down as quadratures of the boundary data, using certain functions biorthogonal to appropriate derivatives of the ω_k. The first of these combinations was treated by Smith (1952); recently Joseph (1977) has discussed the convergence of the resulting expansion in terms of the boundary data for this case.

In the present paper, we consider combinations of edge data for which explicit solutions are not available. First, however, expressions for the displacement and stresses in terms of Ω are given in section 2, and the eigenfunction expansion (1.1) is developed in section 3, the stated biorthogonality properties of the functions ω_k and their derivatives being derived formally from properties of the differential operator associated with the biharmonic equation and of its adjoint operator. This derivation does not prove that the expansion (1.1) is complete. However the solution of the biharmonic equation for the strip is again obtained in section 4 by use of the Fourier sine or cosine transform, as appropriate for the boundary conditions. It is

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found that the transform of Ω (even) has poles at the zeros of $\lambda + \sin\lambda \cos\lambda$, and at no other point. The residues at these points produce the expansion (1.1), and its completeness for $y > 0$ can thereby be established.

In the practically-important case when both tractions (i.e. Ω_{xx} and $-\Omega_{xy}$) are prescribed on the edge $y = 0$, an explicit solution is not possible. From (1.1) we have the expansions

$$(\Omega_{xy})_{y=0} = \sum c_k \phi_k^{(1)}(x) \qquad (1.2a)$$

$$(\Omega_{xx})_{y=0} = \sum c_k \phi_k^{(2)}(x) \; ; \qquad (1.2b)$$

where $\phi_k^{(1)}(x) = -\lambda_k \omega_k'(x)$, $\phi_k^{(2)}(x) = \omega_k''(x)$, but the c_k must now be found by solving a truncated set of equations derived from these. Various ways of deriving such a set have been suggested: Benthem (1964) equates Fourier sine and cosine coefficients of the two sides of (1.2a) and (1.2b) respectively; Gaydon and Shepherd (1964) also form Fourier coefficients of the two sides, but evaluate the c_k by optimizing the fit to a larger number of coefficients in least squares approximation.

A different approach involving eigenfunctions of the adjoint operator is due to Johnson and Little (1965). In effect they use certain of these eigenfunctions, say $\psi_m^{(1)}$, $\psi_m^{(2)}$, as weighting functions in a Galerkin method, thus obtaining an infinite set of equations of the form

$$c_m = \sum F_{mk} c_k + d_m \qquad (1.3)$$

where

$$F_{mk} = -\int_0^1 (\phi_k^{(1)} \psi_m^{(1)} + \phi_k^{(2)} \psi_m^{(2)})\,dx \qquad (m \neq k) \; ,$$

$F_{mm} = 0$, and d_m is known from the boundary data.

Johnson and Little are led naturally to their functions $\psi_m^{(1)}$, $\psi_m^{(2)}$ by considering the biorthogonality properties of a larger set of functions $\phi_k^{(i)}, \psi_m^{(i)}$, $i = 1,2,3,4$ which gives rise to the explicit solutions mentioned earlier, but are not able to obtain an explicit solution for the present case, and it can be shown that <u>there is no possible choice of weighting functions for which</u> $F_{mk} = 0$ <u>for every</u> $m \neq k$, as would be necessary for an explicit solution.

To ensure that the solution $\xi^{(N)}$ of a truncated set of 2N of the above equations should approach a definite limit, the solution vector of the infinite system, it is necessary for the matrix to be diagonally-dominated, in the sense that

$$\sum_{k \neq m} |F_{mk}| < |1-F_{mm}| \qquad (1.4)$$

for each m . A detailed examination in section 5 shows that this requirement is not satisfied by the matrices arising in either the Benthem or the Johnson-Little methods.

Accordingly we seek weighting functions that will at least comply with (1.4). For this purpose we examine the matrix F when $\psi_m^{(1)}$, $\psi_m^{(2)}$ are replaced by functions of the form

$$X_m^{(1)} = A\phi_m^{(1)} + B\phi_m^{(3)}$$

$$X_m^{(2)} = C\phi_m^{(2)} + D\phi_m^{(4)} .$$

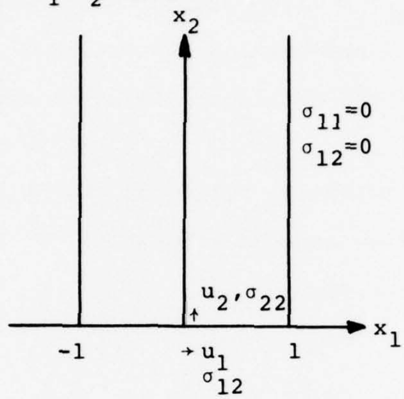
It turns out that the criterion (1.4) can be satisfied with just one choice of the four constants A to D . The weighting functions so found are referred to as "optimal".

The methods mentioned, and in addition direct collocation, i.e. satisfying the equations (1.2) at a particular set of points on (0,1), have been tested by application to three representative sets of initial data for Ω_{xx} and Ω_{xy} on y = 0; the results are compared and discussed in section 6.

2. Stress function and displacements

To define the problem we use the standard suffix notation. Elsewhere in the paper x,y will replace x_1,x_2 as coordinates.

Consider as in figure 1, a semi-infinite elastic beam occupying the strip $-1 < x_1 < 1$, $x_2 > 0$ of the x_1,x_2 plane.



The edges $x_1 = \pm 1$ are treated as stress-free throughout, and we shall consider only distributions of stress such that

$$\left.\begin{array}{ll}\sigma_{11},\sigma_{22} & \text{are even}\\ \sigma_{12} & \text{is odd}\end{array}\right\} \text{ with respect to } x_1 . \tag{2.1}$$

The traction on the end $x_2 = 0$ is assumed to be equilibrated by a force $2N$ in the x_2 direction at infinity, where

$$N = \int_0^1 (\sigma_{22})_{x_2=0}\, dx_1 .$$

The stresses may then be written in terms of a stress function Ω satisfying the biharmonic equation in the form

$$\sigma_{11} = \Omega_{,22} , \quad \sigma_{12} = -\Omega_{,12}, \sigma_{22} = \Omega_{,11} + N \tag{2.2}$$

We shall write $\Delta\Omega = P$

and the biharmonic equation is $\Delta P = 0$. (2.3)

where Δ is the Laplacian

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

The displacement gradients in plane strain are given by

$$2\mu u_{1,1} = (1-\nu)\sigma_{11} - \nu\sigma_{22}, \quad 2\mu u_{2,2} = -\nu\sigma_{11} + (1-\nu)\sigma_{22}$$
$$\mu(u_{1,2} + u_{2,1}) = \sigma_{12} \tag{2.4}$$

To treat boundary value problems, it is convenient to introduce in addition to P, its harmonic conjugate Q linked by the Cauchy-Riemann equations

$$P_{,1} = Q_{,2} \quad P_{,2} = -Q_{,1} \tag{2.5}$$

Q may be defined as zero on $x_1 = 0$, and is then an odd function of x_1. The shear strains can then be written

$$2\mu \begin{pmatrix} u_{1,2} \\ u_{2,1} \end{pmatrix} = -\Omega_{,12} \mp (1-\nu)\, Q \,. \tag{2.6}$$

The values of σ_{12}, $\sigma_{22} - N$, Q and P on the boundary $x_2 = 0$ will be denoted by a 4-vector $f(x_1)$ with components

$$\begin{pmatrix} -\sigma_{12} \\ \sigma_{22} - N \\ -Q \\ P \end{pmatrix}_{x_2=0} = \begin{pmatrix} \Omega_{,12} \\ \Omega_{,11} \\ \int_0^{x_1} (\Delta\Omega)_{,2}\, dx_1 \\ \Delta\Omega \end{pmatrix}_{x_2=0} = \begin{pmatrix} f^{(1)}(x_1) \\ f^{(2)}(x_1) \\ f^{(3)}(x_1) \\ f^{(4)}(x_1) \end{pmatrix} \equiv f(x_1) \tag{2.7}$$

$f^{(1)}$, $f^{(3)}$ are odd functions of x_1, $f^{(2)}$, $f^{(4)}$ are even.

In terms of these functions, the gradients of displacement along the edge are

$$2\mu \begin{pmatrix} u_{1,1} \\ u_{2,1} \end{pmatrix}_{x_2=0} = \begin{pmatrix} -f^{(2)} + (1-\nu) f^{(4)} - \nu N \\ -f^{(1)} \mp (1-\nu) f^{(3)} \end{pmatrix} \tag{2.8}$$

The corresponding expressions for the case of plane stress are obtained by writing $\frac{\nu}{1+\nu}$ in place of ν in the above. These are equivalent to the expressions (2.11 d) of Johnson and Little.

The biharmonic equation can be solved explicitly when either (a) $f^{(1)}$ and $f^{(3)}$ are given (i.e. σ_{12} and u_2) or (b) $f^{(2)}$ and $f^{(4)}$ are given (i.e. σ_{22} and u_1). We proceed to find the solutions for case (a) and case (b) in turn, by use of Fourier cosine and sine transforms respectively, and then combine the solutions to treat (c) (the stress problem): σ_{12} and σ_{22} given. (d) (displacement problem) u_1 and u_2 given, for both of which a unique solution also exists, but can be obtained only in terms of an infinite set of linear equations.

3. The eigen function expansion

From now on we shall replace x_1, x_2 by x, y, and suffixes will be used to denote derivatives.

The biharmonic equation $\Delta^2 \Omega = 0$ in $y > 0$ with stress-free boundary conditions

$$\Omega_{xy} = 0, \quad \Omega_{yy} = 0 \quad \text{on the edges } x = \pm 1 \tag{3.1}$$

and symmetry about $x = 0$ possesses eigen-solutions of the form

$$e^{-\lambda y}\, \omega(x)$$

($\mathrm{Re}\,\lambda > 0$), where ω satisfies the fourth order ordinary differential equation

$$\left.\begin{aligned} &(D^2 + \lambda^2)^2\, \omega = 0 \qquad (D \equiv \tfrac{d}{dx}) \\ &\text{with boundary conditions} \\ &x = 0: \quad D\omega = D^3\omega = 0 \\ &x = 1: \quad \omega = D\omega = 0\,. \end{aligned}\right\} \tag{3.2}$$

These are satisfied by[†]

$$\omega = (\lambda \cos\lambda)^{-1} (\tan\lambda \cos \lambda x - x \sin \lambda x) \tag{3.3}$$

when λ satisfies the equation

$$\lambda + \sin\lambda \cos\lambda = 0 \tag{3.4}$$

In addition to the root $\lambda = 0$, the equation possesses a conjugate pair of roots $\lambda_n, \bar{\lambda}_n$ in each interval $(n - \frac{1}{2})\pi < \mathrm{Re}\,\lambda < (n - \frac{1}{4})\pi$. $-\lambda_n, -\bar{\lambda}_n$ are also roots. The asymptotic location of the roots for large n is

$$\lambda_n, \bar{\lambda}_n \sim \frac{1}{4}\xi_n - \frac{\ell n\, \xi_n}{\xi_n} \pm \frac{1}{2} i\, \ell n\, \xi_n + o\left(\frac{1}{\xi_n}\right) \tag{3.5}$$

where $\xi_n = (4n-1)\pi$. The first 10 roots are listed in Table 1.

Accordingly we seek a solution of the biharmonic equation subject to (3.1) in the form

$$\Omega(x,y) = \sum_k c_k\, e^{-\lambda_k y}\, \omega_k(x) \tag{3.6a}$$

where $\omega_k(x) = \omega(x, \lambda_k)$, and the c_k are coefficients to be determined from the remaining boundary data on $x = 0$. In this summation, λ_k ranges over all the roots of (3.4) with $\mathrm{Re}\,\lambda > 0$, i.e. the sum means

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[†] The factor $(\lambda\cos\lambda)^{-1}$ in the expression for ω has been introduced to simplify later expressions, and ensures that the L_2 norm $\|\omega\| \to 0$ as $\lambda \to \infty$.

Table 1. Roots of (3.4) in first quadrant

n	Re λ_n	Im λ_n
1	2.106196	1.125364
2	5.356269	1.551574
3	8.536683	1.775544
4	11.699178	1.929405
5	14.854060	2.046853
6	18.004933	2.141891
7	21.153413	2.221723
8	24.300342	2.290552
9	27.446203	2.351048
10	30.591295	2.405013

The asymptotic expression $\frac{1}{4}\xi_n - \frac{\ell n \xi_n}{\xi_n}$ gives 6 figure agreement with Re λ_n at $n = 4$, and $\frac{1}{2}\ell n\xi_n + [(\ell n\xi_n)^2 - 2\ell n\xi_n - 1]/\xi_n^2$ agrees with Im λ_n to 4×10^{-5} at $n = 9$.

$$\sum_{n=1}^{\infty} \{c_n e^{-\lambda_n y} \omega_n(x) + \bar{c}_n e^{-\bar{\lambda}_n y} \overline{\omega_n(x)}\} \qquad (3.6b)$$

but will be written from now on in the form (3.6a).

Substituting (3.6) in (2.6) gives

$$f^{(i)}(x) = \sum_k \phi_k^{(i)}(x), \qquad i = 1,2,3,4$$

where

$$\phi_k = \begin{pmatrix} \phi_k^{(1)} \\ \phi_k^{(2)} \\ \phi_k^{(3)} \\ \phi_k^{(4)} \end{pmatrix} = \begin{pmatrix} -\lambda D\omega \\ D^2\omega \\ (\lambda^{-1}D^3+\lambda D)\omega \\ (D^2+\lambda^2)\omega \end{pmatrix}_{\lambda=\lambda_k} = \begin{pmatrix} \cos\lambda\sin\lambda x - x\sin\lambda\cos\lambda x \\ -\lambda^2\omega - (2/\cos\lambda)\cos\lambda x \\ (2/\cos\lambda)\sin\lambda x \\ -(2/\cos\lambda)\cos\lambda x \end{pmatrix}_{\lambda=\lambda_k} \qquad (3.7)$$

Biorthogonal functions

Functions biorthogonal to the 2-vector

$$\begin{pmatrix} \phi_k^{(1)} \\ \phi_k^{(3)} \end{pmatrix} \equiv \phi_k^{(1,3)}$$

can be constructed by looking on (3.2) as the system

$$L\,\phi^{(1,3)} = \lambda^2 \phi^{(1,3)} \qquad (3.8a)$$

where L is the operator $\begin{pmatrix} -D^2 & D^2 \\ 0 & -D^2 \end{pmatrix}$, with associated boundary conditions

$$\left.\begin{aligned} y = 0: &\quad \phi^{(1)} = \phi^{(3)} = 0 \\ y = 1: &\quad \phi^{(1)} = 0, \quad D(\phi^{(1)}-\phi^{(3)}) = 0 \end{aligned}\right\} \qquad (3.8b)$$

The eigen functions $\phi_k^{(1,3)}$ are then biorthogonal to the eigen functions $\psi_m^{(1,3)}$ of the adjoint operator L^* which is defined by

$$(\psi^{(1,3)}, L\phi^{(1,3)}) = (L^*\psi^{(1,3)}, \phi^{(1,3)}). \qquad (3.9)$$

Here we have introduced an inner product notation

$$(f^{(1,3)}, g^{(1,3)}) = \int_0^1 (f^{(1)}g^{(1)} + f^{(3)}g^{(3)})dx \qquad (3.10)$$

We shall also use the notation

$$\langle f \rangle \equiv \int_0^1 f(x)\,dx$$

We find that $L^* = L^T$, with boundary conditions

$$y = 0: \quad \psi^{(1)} = \psi^{(3)} = 0$$
$$y = 1: \quad \psi^{(3)} = 0, \quad D(\psi^{(1)} - \psi^{(3)}) = 0 . \tag{3.11}$$

These are satisfied by

$$\begin{pmatrix} \psi^{(1)} \\ \psi^{(3)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi^{(3)} \\ \phi^{(1)} \end{pmatrix} \tag{3.12}$$

where λ is a root of (3.4). The normalizing factor $\frac{1}{2}$ has been introduced so that the scalar product

$$(\psi_m^{(1,3)}, \phi_n^{(1,3)}) = \delta_{mn} . \tag{3.13}$$

The vector $\phi^{(2,4)}$ is also an eigenfunction of the same matrix operator with the different boundary conditions

$$y = 0: \quad D\phi^{(2)} = D\phi^{(4)} = 0$$
$$y = 1: \quad \phi^{(2)} - \phi^{(4)} = D(\phi^{(2)} - \phi^{(4)}) = 0. \tag{3.14}$$

In this case the adjoint is again L^T, with side conditions

$$y = 0: \quad D\psi^{(2)} = D\,\psi^{(4)} = 0$$
$$y = 1: \quad \psi^{(4)} = 0, \quad D\,\psi^{(4)} = 0$$

whence

$$\begin{pmatrix} \psi^{(2)} \\ \psi^{(4)} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi^{(4)} \\ \phi^{(2)} - \phi^{(4)} \end{pmatrix} \tag{3.15}$$

and likewise

$$(\psi_m^{(2,4)}, \phi_n^{(2,4)}) = \delta_{mn} . \tag{3.16}$$

The biorthogonality relations can be confirmed by direct calculation (Appendix A). Writing $\langle \psi_m^{(i)} \phi_n^{(i)} \rangle = K_{mn}^{(i)}$, $i = 1,2,3,4$ we find that

$$K_{mm}^{(i)} = \frac{1}{2} \tag{3.17a}$$

while for $m \neq n$

$$K_{mn}^{(1)} = -K_{mn}^{(3)} = \frac{2\lambda_m \lambda_n c_{mn}}{\lambda_m^2 - \lambda_n^2} , \quad K_{mn}^{(2)} = -K_{mn}^{(4)} = \frac{2\lambda_m^2 c_{mn}}{\lambda_m^2 - \lambda_n^2} \tag{3.17b}$$

where

$$c_{mn} \equiv (\cos \lambda_m \cos \lambda_n)^{-1} \langle \cos \lambda_m x \cos \lambda_n x \rangle$$
$$= \begin{cases} (\lambda_m \tan \lambda_m - \lambda_n \tan \lambda_n)/(\lambda_m^2 - \lambda_n^2) & m \neq n \\ 0 & m = n \end{cases} \tag{3.17c}$$

From the equation $\tan^2\lambda + \lambda^{-1}\tan\lambda + 1 = 0$, which is a form of (3.4), we deduce that

$$\tan\lambda_m = \pm i\,(1 - 1/4\lambda_m^2)^{1/2} - 1/2\lambda_m,$$

the upper and lower signs corresponding to $\mathrm{Im}\,\lambda_m \gtrless 0$. Thus $\tan\lambda_m \to \pm i$ in upper and lower half planes, and

$$|c_{mn}| \sim \frac{1}{\pi(m+n)},\ \frac{1}{\pi(m-n)}$$

accordingly as the imaginary parts of λ_m, λ_n have the same or opposite signs.

Explicit solutions

If the functions $f^{(1)}$ and $f^{(3)}$ are prescribed, the biorthogonality relation (3.13) immediately gives the coefficients in the expansion (3.6) as

$$c_k = \langle \psi_k^{(1)} f^{(1)} + \psi_k^{(3)} f^{(3)} \rangle . \tag{3.18a}$$

Likewise when $f^{(2)}$ and $f^{(4)}$ are prescribed

$$c_k = \langle \psi_k^{(2)} f^{(2)} + \psi_k^{(4)} f^{(4)} \rangle . \tag{3.18b}$$

In other cases, biorthogonality does not lead to closed expressions for the $\{c_k\}$, although the $\{\psi_k^{(i)}\}$ provide useful candidates for weighting functions for use in a Galerkin method. We return to these problems in Section 5, but before this in section 4 we demonstrate the completeness of the expansion (3.6) by use of Fourier transformation methods.

Eigenvectors of the Fredholm determinants $K_{mn}^{(i)}$

We conclude this section by noting a set of identities among the coefficients $K_{mn}^{(i)} = \langle \psi_m^{(i)} \phi_n^{(i)} \rangle$ that can be inferred as follows from the expansions (3.7) and the explicit solutions just obtained. In the case $i = 1$, we have

$$f^{(1)}(x) - \int_0^1 (\sum \phi_m^{(1)}(x)\psi_m^{(1)}t)f^{(1)}(t)dt = \int_0^1 \sum \phi_m^{(1)}(x)\psi_m^{(3)}(t)f^{(3)}(t)dt. \tag{3.19}$$

This is in the form of a second kind Fredholm equation for $f^{(1)}(x)$, given $f^{(3)}$. However, $f^{(1)}$ and $f^{(3)}$ represent independent boundary information. It is physically obvious that neither can imply the other. By the Fredholm alternative theorem however, the equation (3.9) has a solution for every $f^{(3)}$ unless the homogeneous equation obtained by setting $f^{(3)} = 0$ possesses a *non*-trivial solution. We therefore expect such an eigen-solution to exist.

From the form of the kernel the solution must be expressible as

$$f^{(1)}(x) = \sum \alpha_m^{(1)} \phi_m^{(1)} \qquad (3.20)$$

where $\underset{\sim}{\alpha}^{(1)}$ satisfies $\alpha_m^{(1)} = \sum K_{mn}^{(1)} \alpha_n^{(1)}$; i.e. the Fredholm determinant $K_{mn}^{(1)}$ has unity as an eigenvalue.

The eigenvector $\underset{\sim}{\alpha}^{(1)}$ can be found by use of the result

$$\sum \frac{\lambda_n^2 \, c_{mn}}{\cos^2\lambda_n(\lambda_m^2-\lambda_n^2)} = -\sum \frac{\lambda_m^2 \, c_{mn}}{\cos^2\lambda_n(\lambda_m^2-\lambda_n^2)} = 1/4 \cos^2\lambda_m \qquad (3.21)$$

which is found by contour integration in Appendix C.

Using (3.17) and (3.21) we find $K^{(1)}\alpha^{(1)} = \alpha^{(1)}$ where $\alpha_n^{(1)} = \lambda_n/\cos^2\lambda_n$.

Similar reasoning shows that each of the determinants $K^{(i)}$ has a unit eigenvalue, i.e. $K^{(i)}\alpha^{(i)} = \alpha^{(i)}$ where

$$\alpha_n^{(2)} = (\lambda_n/\cos\lambda_n)^2, \quad \alpha_n^{(3)} = 1/\lambda_n\cos^2\lambda_n, \quad \alpha_n^{(4)} = 1/\cos^2\lambda_n \, . \qquad (3.22)$$

We also find that $K^{(1)}\alpha^{(3)} = K^{(3)}\alpha^{(1)} = K^{(2)}\alpha^{(4)} = K^{(4)}\alpha^{(2)} = 0$. (3.23)

4. Fourier transform solution

In this section we seek the solution of the biharmonic equation in the strip $|x| \leq 1$, $y > 0$ by use of a Fourier transform, subject to the condition

$$\Omega = 0 \ , \quad \Omega_{xy} = 0 \quad \text{on} \quad x = \pm 1 \tag{4.1}$$

and given data on the end $y = 0$ in either of the forms

(a) $$\Omega_{xy} = f^{(1)}(x), \quad \int_0^x (\Delta\Omega)_y dx = f^{(3)}(x) \tag{4.2}$$

or

(b) $$\Omega_{xx} = f^{(2)}(x), \quad \Delta\Omega = f^{(4)}(x) \tag{4.3}$$

Only solutions for Ω even in x will be treated, and we assume that the boundary data is consistent with this, i.e. that $f^{(1)}$ and $f^{(3)}$ are odd with respect to x, and $f^{(2)}$ and $f^{(4)}$ even, and further that $\int_0^1 f^{(2)}(x)dx = 0$, in accordance with (2.6).

We carry out the solution in turn in the two cases.

Case (a): Multiply the biharmonic equation on both sides by $\cos \xi y$ and integrate with respect to y from 0 to ∞, obtaining

$$(D^2 - \xi^2)^2 \ \Omega^{(c)} = (\Omega_{yyy} + 2\Omega_{xxy} - \xi^2\Omega_y)_{y=0} \tag{4.4}$$

where

$$\Omega^{(c)}(x;\xi) = \int_0^\infty (\cos\xi y)\Omega(x,y)dy$$

and $D \equiv \frac{d}{dy}$ as before. The solution of this equation even in y is

$$\Omega^{(c)} = -\frac{1}{2\xi^2}\int_0^x k^{(1,3)}(\xi x-\xi t)f^{(1,3)}(t)dt + \frac{1}{\xi^2}\int_x^1 f^{(1)}(t)dt \tag{4.5}$$
$$+ A(\xi)\cosh\xi x + B(\xi)x\sinh\xi x$$

where $k^{(1)}(s) = -2\cosh s$, $k^{(3)}(s) = -s\sinh s$,

and $k^{(1,3)}f^{(1,3)}$ denotes the scalar product $k^{(1)}f^{(1)} + k^{(3)}f^{(3)}$. The functions A, B are to be determined from the conditions (4.1), which can be applied in the form

$$\Omega^{(c)} = 0 \ , \quad \Omega_x^{(c)} = 0 \quad \text{on} \quad x = 1$$

giving

$$M(\xi)\binom{A}{B} = \frac{1}{2\xi^2}\int_0^1 \begin{pmatrix} k^{(1,3)}(\xi-\xi t) \\ k'^{(1,3)}(\xi-\xi t)\end{pmatrix} f^{(1,3)}(t)dt \qquad (4.6)$$

where

$$M(\xi) = \begin{pmatrix} \cosh\xi & \sinh\xi \\ \sinh\xi & \cosh\xi+\xi^{-1}\sinh\xi \end{pmatrix}.$$

Thus $\Omega^{(c)}$ is given by (4.5), with the last two terms expressible as

$$A\cosh\xi x + Bx\sinh\xi x = \frac{1}{2\xi^2}\int_0^1 (\cosh\xi x,\ x\sinh\xi x)\, M^{-1}(\xi)\begin{pmatrix} k^{(1,3)}(\xi-\xi t) \\ k'^{(1,3)}(\xi-\xi t)\end{pmatrix} f^{(1,3)}(t)dt, \qquad (4.7)$$

The inverse matrix in the integrand is

$$M^{-1}(\xi) = (\det M)^{-1}\begin{pmatrix} \cosh\xi + \xi^{-1}\sinh\xi & -\sinh\xi \\ -\sinh\xi & \cosh\xi \end{pmatrix},$$

with $\qquad \det M = 1 + (1/\xi)\cosh\xi\sinh\xi.$

$\Omega^{(c)}$ is an even function of ξ, and a detailed examination shows that it is regular at $\xi = 0$. The inverse cosine transform is therefore

$$\Omega(x,y) = \frac{2}{\pi}\int_0^\infty \Omega^{(c)}(\cos\xi y)d\xi = \frac{1}{\pi}\int_{-\infty}^\infty \Omega^{(c)} e^{i\xi y}d\xi \qquad (4.8)$$

As $\xi \to \infty$ $\Omega^{(c)} = 0(\xi^{-2})$. For future use the asymptotic form is quoted in full:

$$\Omega^{(c)} \sim \frac{1}{4\xi^2}\int_0^1 [e^{-\xi|x-t|}(-2f^{(1)}(t) + \xi|x-t|f^{(3)}(t))\operatorname{sgn}(t-x)$$
$$+ e^{-\xi(x+t)}(-2f^{(1)}(t) + \xi(x+t)f^{(3)}(t))]dt \qquad (4.9)$$
$$+ \frac{1}{\xi^2}\int_x^1 f^{(1)}(t)dt .$$

The integral (4.8) can therefore be evaluated for $y > 0$ as 2i times the sum of residues of the integrand at its poles in $\operatorname{Im}\xi > 0$. The only such poles are those of $M^{-1}(\xi)$ at the zeros of $\det M(\xi)$, i.e. at $\xi = i\lambda$, where

$$C(\lambda) \equiv 2\lambda + \sin 2\lambda = 0\ , \quad \operatorname{Re}\lambda > 0 \qquad (4.10)$$

Each of the eigenvalues λ_n, $\bar{\lambda}_n$ given in (3.5) produces a contribution

$$e^{-\lambda_n y}\left(\frac{1}{i\lambda^2}\right)\int_0^1 (\cos\lambda x,\ i\ x\sin\lambda x)(\mathrm{res}M^{-1})\begin{pmatrix}k(i\lambda-i\lambda t\\ k'(i\lambda-i\lambda t)\end{pmatrix}f(t)dt \tag{4.11}$$

The residue of M^{-1} can be expressed as the matrix product

$$(\mathrm{res}\ M^{-1})_{\xi=i\lambda} = \frac{i\lambda}{\cos\lambda}\begin{pmatrix}\tan\lambda\\ i\end{pmatrix}\left(-\frac{1}{2}\ \tan\lambda, -\frac{i}{2}\right) \tag{4.12}$$

and the integrand (4.11) then factorizes since

$$\left.\begin{aligned}
&(\lambda\cos\lambda)^{-1}(\cos\lambda x, -\ ix\ \sin\lambda x)\begin{pmatrix}\tan\lambda\\ i\end{pmatrix} = \omega(x,\lambda)\\
&\left(-\frac{1}{2}\tan\lambda, -\frac{i}{2}\right)\begin{pmatrix}k^{(1,3)}(i\lambda-i\lambda t)\\ k'^{(1,3)}(i\lambda-i\lambda t)\end{pmatrix} = (\psi^{(1)}(t,\lambda),\ \psi^{(3)}(t,\lambda))\\
\text{where}\quad &\omega = (\lambda\cos\lambda)^{-1}(\tan\lambda\ \cos\lambda x - x\sin\lambda x)\\
&\psi^{(1)}(t,\lambda) = (\cos\lambda)^{-1}\sin\lambda t\\
&\psi^{(3)}(t,\lambda) = \frac{1}{2}(\cos\lambda\ \sin\lambda t - t\sin\lambda\ \cos\lambda t)
\end{aligned}\right\} \tag{4.13}$$

are precisely the functions already encountered in Section 3. Thus the solution of the (1,3) problem in $y > 0$ is

$$\left.\begin{aligned}
&\Omega(x,y) = \sum c(\lambda_k)\ e^{-\lambda_k y}\ \omega(x,\lambda_k)\\
\text{where}\quad &\\
&c(\lambda_k) = (\psi^{(1,3)}(t,\lambda_k),\ f^{(1,3)}(t))
\end{aligned}\right\} \tag{4.14}$$

λ_k in the summation ranges over $\lambda_n, \bar{\lambda}_n$, $n = 1,2,\ldots$ and the $\omega(y,\lambda_k)$ form a complete set of eigen functions on $(0,1)$.

Moreover the expansion (4.14) holds on $y = 0$. This can be seen since, for $y < 0$, the inverse Fourier transform (4.8) is evaluated as $(-2i)$ times the sum of the residues in $\mathrm{Im}\,\lambda < 0$. These residues are the negative of those in $\mathrm{Im}\,\lambda > 0$. Hence the expression (4.14) holds with $|y|$ in place of y, and is continuous at $y = 0$.

The biorthogonality already established between the $\psi^{(1,3)}$ and the odd derivatives $\phi^{(1,3)}$ follows immediately from (4.14) since it gives the expansions

$$\left.\begin{aligned}
&f^{(1)} = (\Omega_{xy})_{y=0} = \sum c_k\ \phi_k^{(1)}(x), \qquad \phi^{(1)} = -\lambda D\omega\\
&f^{(3)} = \left(\int_0^x (\Delta\Omega)_y dx\right)_{y=0} = \sum c_k\ \phi_k^{(3)}(x),\ \phi^{(3)} = (\lambda^{-1}D^3 + \lambda D)\omega
\end{aligned}\right\} \tag{4.15}$$

From (4.14) we deduce that $(\psi_i^{(1,3)}, \phi_k^{(1,3)}) = \delta_{ik}$. (4.16)

In case (b) the steps are similar: given $f^{(2)}$ and $f^{(4)}$ we form the sine transform of the biharmonic equation, obtaining

$$(D^2-\xi^2)^2\Omega^{(S)} = (\xi^3\Omega - 2\xi\Omega_{xx} - \xi\Omega_{yy})_{y=0} \tag{4.17}$$

where

$$\Omega^{(S)}(x;\xi) = \int_0^\infty (\sin\xi y)\,\Omega(x,y)\,dx$$

with solution

$$\Omega^{(s)} = -\frac{1}{2\xi^2}\int_0^x k^{(2,4)}(\xi x-\xi t)\,f^{(2,4)}(t)\,dt + \frac{1}{\xi}\int_0^1 \max(x,t)\,f^{(2)}(t)\,dt \tag{4.18}$$
$$+ A\cosh\xi x + Bx\sinh\xi x$$

where $k^{(2)}(s) = 2\sinh s$, $k^{(4)}(s) = s\cosh s - \sinh s$. The term $A\cosh\xi x + Bx\sinh\xi x$ is given in terms of $f^{(2,4)}$ by an expression identical in form to (4.7). In this case $\Omega^{(s)}$ is odd with respect to ξ . For future reference we quote the asymptotic form of $\Omega^{(s)}$: as $\xi \to \infty$

$$\Omega^{(s)} \sim \frac{1}{4\xi^2}\int_0^1 [e^{-\xi|x-t|}(2f^{(2)}-(1+\xi|x-t|)f^{(4)})$$
$$+ e^{-\xi(x+t)}(2f^{(2)} - (1+\xi(x+t))f^{(4)})]dt \tag{4.19}$$
$$+ \frac{1}{\xi}\int_x^1 (t-x)f^{(2)}(t)\,dt\ .$$

When steps identical to those for $\Omega^{(c)}$ are applied to (4.18) the expansion (4.14) is again obtained but now

$$c_k = (\psi_k^{(2,4)},\ f^{(2,4)}) \tag{4.20}$$

$$f^{(2)} = \sum c_k\,\phi_k^{(2)},\quad f^{(4)} \doteq \sum c_k\,\phi_k^{(4)}\ , \tag{4.21}$$

the biorthogonal functions in this case being

$$\psi^{(2)}(t,\lambda) = -(1/\cos\lambda)\cos\lambda t,\qquad \psi^{(4)}(t,\lambda) = -\tfrac{1}{2}\lambda^2\,\omega(t,\lambda)$$

and the second biorthogonality relationship (4.22)

$$(\psi_i^{(2,4)},\ \phi_k^{(2,4)}) = \delta_{ik}$$

is again obtained.

5. Mixed boundary value problems

In this section we consider the problem of finding the coefficients $\{c_n\}$ in the expansion of Ω and its derived functions when the data on the boundary does not fall into one of the types $((f^{(1)},f^{(3)})$ or $(f^{(2)},f^{(4)})$ given) for which an explicit solution was found in the last section.

Two cases are of particular practical importance:

(c) The stress problem: σ_{11} and σ_{12} are prescribed on $y = 0$, i.e.

$$f^{(1)}(x)\,(\text{odd}),\ f^{(2)}(x)\,(\text{even})\ \text{given.} \tag{5.1}$$

(It will be assumed that $f^{(1)}(0) = f^{(1)}(1) = 0$, and $\int_0^1 f^{(2)}(x)dx = 0$, to conform with (2.1).)

(d) The displacement problem in which u and v are prescribed on $y = 0$, i.e. the combinations

$$g^{(1)} = f^{(1)} + (1-\nu)f^{(3)}\ (\text{odd}),\ g^{(2)} = f^{(2)} - (1-\nu)f^{(4)}\ (\text{even}) \tag{5.2}$$

are given.

We note first that explicit solutions to these problems cannot be found, i.e. in case (a) there is no set of functions biorthogonal to the set $(\phi_n^{(1)},\phi_n^{(n)}) \equiv (-\lambda_n D\omega_n, D^2\omega_n)$.

If there were such functions $(Z_m^{(1)},Z_m^{(2)})$, we should have for each $m \neq n$.

$$0 = (Z_m^{(1,2)},\phi_n^{(1,2)}) = -\langle(\lambda_n Z_m^{(1)} + DZ_m^{(2)})D\omega_n\rangle$$

(on integration by parts). This can hold for every n only in the trivial case $Z_m^{(1)} = 0$, $Z_m^{(2)}$ = constant.

The coefficients for case (c) can therefore be found in principle only by deriving an infinite set of linear equations from the expansions

$$f^{(1)}(x) = \sum c_n\ \phi_n^{(1)}(x) \tag{5.3a}$$

$$f^{(2)}(x) = \sum c_n\ \phi_n^{(2)}(x) \tag{5.3b}$$

and hoping to recover the c_n's with sufficient accuracy from a truncated set. In these equations, the summation ranges over each eigenvalue λ_n in the first quadrant, and its conjugate $\bar{\lambda}_n$. The corresponding coefficients and eigenfunctions are also conjugate.

Three essentially different methods for formulating equations have been examined.

(A) Direct collocation: In which a truncated set of the equations (5.3 a,b) are identically satisfied at a suitable set of points $\{x_m\}$ of the interval (0,1).

(B) Galerkin methods Here, sets of weighting functions $W_m^{(1)}, W_m^{(2)}$ are introduced and mean equations

$$(W_m^{(1)}, f^{(1)}) = \sum (W_m^{(1)}, \phi_n^{(1)}) c_n, \quad (W_m^{(2)}, f^{(2)}) = \sum (W_m^{(2)} \phi_n^{(2)}) c_n \tag{5.4}$$

are obtained and solved simultaneously. The simplest such equations are provided by the set

$$W_m^{(1)} = \sin m\pi x, \quad W_m^{(2)} = \cos m\pi x. \tag{5.5a}$$

These were proposed by Benthem (1963), and are equivalent to those used by Gaydon & Shepherd. With our notation the matrix elements are given by

$$\langle \sin m\pi x \ \phi_n^{(1)} \rangle = -\left(\frac{\lambda_n}{m\pi} \right) \quad \langle \cos m\pi x \ \phi_n^{(2)} \rangle = (-1)^{m-1} \frac{2m\pi \lambda_n^2 \tan \lambda_n}{(\lambda_n^2 - m^2\pi^2)^2} \tag{5.5b}$$

(C) Use of biorthogonal functions

Johnson & Little in effect add equations (4.14) and (4.20) to obtain

$$2c_m = (\psi_m^{(1,2)}, f^{(1,2)}) + (\psi_m^{(3,4)}, f^{(3,4)}) . \tag{5.6}$$

Here $f^{(1,2)}$ is known, but $f^{(3,4)}$ must be calculated as $\sum c_n \phi_n^{(3,4)}$. From (3.17)

$$(\psi_m^{(3,4)}, \phi_n^{(3,4)}) = \begin{cases} 1 & (m = n) \\ \dfrac{-2\lambda_m(\lambda_m \tan \lambda_m - \lambda_n \tan \lambda_n)}{(\lambda_m - \lambda_n)^2 (\lambda_m + \lambda_n)} & (m \neq n) \end{cases} \equiv \begin{cases} 1 \\ F_{mn} \end{cases} \tag{5.7}$$

say. This gives the set of equations

$$\left. \begin{aligned} c_m &= \sum F_{mn} c_n + d_m \\ \text{where} \quad & \\ d_m &= (\psi_m^{(1,2)}, f^{(1,2)}) . \end{aligned} \right\} \tag{5.8}$$

It is to be noted however that this set of equations can also be obtained directly by a Galerkin method from (5.3a) and (5.3b), using the weights $W_m^{(1,2)} = \psi_m^{(1,2)}$ and adding .

For an infinite set of equations of the form (5.8), Kantorovich & Krylov show that the solution $\{c_n\}$ is stable in the sense of approaching a limit as the number N of equations become large if the L_∞ norm $\|F\|_\infty \equiv \sup_m \sum_n |F|$ is < 1. However this condition is not satisfied by the coefficients (5.7); in fact the sum $\sum_n |F_{mn}|$ tends to infinity with m.

To discuss the sum, it is convenient to write the equations more precisely as

$$c_m = \sum_{n=1} (F_{mn} c_n + F^*_{mn} \bar{c}_n) + d_m$$

where

$$F^*_{mn} = -2\lambda_m(\lambda_m \tan\lambda_m - \bar{\lambda}_n \tan\bar{\lambda}_n)/(\lambda_m - \bar{\lambda}_n)^2(\lambda_m + \bar{\lambda}_n) \quad . \qquad (5.9)$$

The summation now extends over λ_n in the first quadrant only, and we need only consider λ_m in the first quadrant since the conjugate equation holds for $\bar{\lambda}_m$. The asymptotic expressions (3.5) show that

$$\lambda_m, \lambda_n \sim m\pi, n\pi, \text{ while } \tan\lambda_m, \tan\lambda_n \to i.$$

Therefore

$$|F_{mn}| \sim \frac{2}{\pi} \frac{m}{|m^2-n^2|}, \quad |F^*_{mn}| \sim \frac{2m}{\pi}(m-n)^{-2} \quad (m \neq n) \qquad (5.10)$$

(For $m = n$, $F_{mn} = 0$, $F^*_{mm} \sim 2im\pi/(\ln 4m\pi)^2$). The imaginary part of F^*_{mm} does not appear in the diagonal when the equations (5.8) are formulated in real and imaginary form. Then

$$\sum |F_{mn}| \sim \frac{4}{\pi} \ln 2m, \quad \sum |F^*_{mn}| > \frac{m}{\pi}\left(1 + \frac{1}{2^2} + ..\right) = \frac{m\pi}{3} .$$

Similar behaviour is exhibited by the elements (5.5b) of the matrix formed by sine and cosine weighting functions, and we might expect this to be reflected in instability of the solution of truncated sets of these equations. In fact this was shown in numerical experiments described in the next section.

It is evident that the difficulties arise from the factor $(\lambda_m - \lambda_n)^2$ in the denominator of F_{mn}.

D. Optimal weighting functions

In the hope of producing a diagonally-dominated matrix, we may consider weighting functions of the general form

$$\left.\begin{aligned} \chi_m^{(1)} &= A\,\phi_m^{(1)} + B\,\phi_m^{(3)} \\ \chi_m^{(2)} &= C\,\phi_m^{(2)} + D\,\phi_m^{(4)} . \end{aligned}\right\} \qquad (5.11)$$

This form is a natural extension of the biorthogonal weighting functions $\psi_m^{(1)} = \frac{1}{2}\phi_m^{(2)}$ etc. and contains sufficient disposable constants to ensure the suppression of the factor $(\lambda_m - \lambda_n)^2$ in the denominator of the matrix elements.

If we write

$$f^{(1)} + \alpha f^{(3)} = g^{(1)}, \quad f^{(2)} - \alpha f^{(4)} = g^{(2)} \qquad (5.12)$$

then the stress $(\alpha=0)$ and displacement $(\alpha=1-\nu)$ problems may be treated together as cases in which the functions $g^{(1)}$, $g^{(2)}$ are known on $0 \leq x \leq 1$. We shall therefore seek optimal weighting functions in the above sense for $g^{(1)}$ and $g^{(2)}$, and expect the result to depend on the parameter α. These produce the equations

$$\left.\begin{aligned} &c_m = \sum G_{mn} c_n + d_m \\ &\text{where } d_m = (\chi_m^{(1,2)}, g^{(1,2)}) \text{ is known, and} \\ &G_{mn} = \delta_{mn} - \langle \chi_m^{(1)}(\phi_n^{(1)} + \alpha\phi_n^{(3)}) + \chi_m^{(2)}(\phi_n^{(2)} - \alpha\phi_n^{(4)})\rangle . \end{aligned}\right\} \qquad (5.13)$$

Results in appendix A show that for $m \neq n$

$$G_{mn} = \frac{4\lambda_m\lambda_n}{(\lambda_m^2-\lambda_n^2)}\left[\frac{A(\lambda_m^2 + \lambda_n^2) + 2C\lambda_m\lambda_n}{\lambda_m^2 - \lambda_n^2} - B + A + \frac{\lambda_m^2 D + \alpha\lambda_n^2 C}{\lambda_m\lambda_n}\right]c_{mn}$$

$$- 4\alpha\,(B-D)c_{mn} + 4\alpha B\,t_{mn}$$

where $\quad c_{mn} = (\lambda_m \tan\lambda_m - \lambda_n \tan\lambda_n)/(\lambda_m^2 - \lambda_n^2)$

as before, and

$$t_{mn} = (\tan\lambda_m + \tan\lambda_n)/(\lambda_m + \lambda_n).$$

To suppress the factor $(\lambda_m - \lambda_n)^3$ in the denominator we must have $A = -C$. Without loss of generality we can choose $C = 1$, $A = -1$ when the expression reduces to

$$- 4 \left[\frac{\lambda_m \lambda_n}{(\lambda_m + \lambda_n)^2} + \frac{2(B+\alpha)\ \lambda_m \lambda_m + \lambda_m^2 D + \lambda_n^2 \alpha}{\lambda_m^2 - \lambda_n^2} + \alpha(B-D) \right] c_{mn} + 4\alpha B \ \frac{\tan\lambda_m + \tan\lambda_n}{\lambda_m + \lambda_n}$$

and if $B = \frac{1}{2} - \alpha$, $D = -\frac{1}{2} - \alpha$ the remaining factors of $(\lambda_m - \lambda_n)$ in the denominator are suppressed leaving

$$G_{mn} = 2\lambda_m(\lambda_m \tan \lambda_m - \lambda_n \tan \lambda_n)/(\lambda_m + \lambda_n)^3 + 2\alpha(1-2\alpha)(\tan \lambda_m + \tan \lambda_n)/(\lambda_m + \lambda_n). \tag{5.14}$$

This formula holds when $\lambda_m \neq \lambda_n$. Further results in the appendix show that the diagonal term is zero when $\alpha = 0$, and otherwise is

$$G_{mm} = 4\alpha + 2\alpha(1-2\alpha) \left(\frac{\tan\lambda_m}{\lambda_m} \right) . \tag{5.15}$$

The weighting functions are then

$$\chi_m^{(1)} = -\phi_m^{(1)} + (\tfrac{1}{2} - \alpha)\phi_m^{(3)}$$
$$\chi_m^{(2)} = \phi_m^{(2)} - (\tfrac{1}{2} + \alpha)\ \phi_m^{(4)} . \tag{5.16}$$

For the case $\alpha = 0$, i.e. the stress problem, case (c), the diagonal elements are all unity, and the sum of the absolute off diagonal elements is strictly less than 1. This can be seen as follows: Again, denote by G^*_{mn} the value of (5.14) with $\bar{\lambda}_n$ in place of λ_n. Then for $\alpha = 0$, the asymptotic values of G_{mn} , G^*_{mn} are respectively

$$\frac{2im}{\pi} \frac{m-n}{(m+n)^3} , \frac{2im}{\pi} (m + n)^{-2} , \tag{5.17}$$

whence

$$\sum_n (|G_{mn}| + |G^*_{mn}|) \sim \frac{2m}{\pi} \sum_{n=1}^{m} \frac{2m}{(m+n)^3} + \frac{2m}{\pi} \sum_{n=m+1}^{\infty} \frac{2n}{(m+n)^3}$$

$$= \frac{4m^2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(m+n)^3} + \frac{4m}{\pi} \sum_{n=m+1}^{\infty} \left\{ \frac{1}{(m+n)^2} - \frac{2m}{(m+n)^3} \right\} \sim \frac{3}{\pi} \quad \text{as} \quad m \to \infty .$$

The estimate $\frac{3}{\pi}$ is improved to $\frac{2}{\pi}$ when the real and imaginary parts of the equation are separated.

When α is different from zero, it is possible to find a positive constant γ depending on α such that $\sum_n |(\frac{m}{n})^\gamma G_{mn}| < |1-4\alpha|$; this implies that provided $|m^\gamma d_m| \to 0$ as $m \to \infty$, the equations (5.13) possess a unique bounded inverse.

6. Calculations

6.1. End Stresses Prescribed

The four methods described in the previous section:

A Collocation

B Sine and cosine weighting functions

C Biorthogonality relations

D Optimal weighting functions

were programmed and applied to the stress problem (c) for three test cases:

$$\begin{aligned}
&(1) \qquad f^{(1)} = 0 \;, \quad f^{(2)} = 1 - 3x^2 \\
&(2) \qquad f^{(1)} = 0 \;, \quad f^{(2)} = \begin{cases} 1 & (0 \le x < \frac{1}{2}) \\ -1 & (\frac{1}{2} < x \le 1) \end{cases} \\
&(3) \qquad f^{(1)} = x - x^3 \;, \quad f^{(2)} = 0
\end{aligned} \tag{6.1}$$

$(f^{(1)} = (\sigma_{12})_{y=0},\ f^{(2)} = (\sigma_{22})_{y=0})$, using the 2N equations obtained from the eigenvalues $\lambda_1 \dots \lambda_N,\ \bar{\lambda}_1 \dots \bar{\lambda}_N$. The real and imaginary parts of the equation were separated to produce real equations

$$\underset{(2N\times 2N)}{M} \; \underset{\sim}{z} = \underset{\sim}{b} \tag{6.2}$$

where now

$$z_{2n-1} + i\, z_{2n} = c_n \;. \tag{6.3}$$

Similarly the right hand sides were separated as $d_m = b_{2m-1} + i\, b_{2m}$. Thus in method C, (5.9) takes the form (6.2) with

$$\left.\begin{aligned}
M_{2m-1\ 2n-1} &= \delta_{mn} - \mathrm{Re}(F + F^*)_{mn} \\
M_{2m-1\ 2n} &= \mathrm{Im}(F - F^*)_{mn} \\
M_{2m\ \ 2n-1} &= -\mathrm{Im}(F + F^*)_{mn} \\
M_{2m\ \ 2n} &= \delta_{mn} - \mathrm{Re}(F - F^*)_{mn}
\end{aligned}\right\} \tag{6.4}$$

N was given the values 5, 10, 20 and 40 in turn. The resulting coefficients c_n are listed in Table 2 for cases 1 and 2. Case 3 produces results qualitatively very similar to those in case 1. The coefficients have been used to calculate the distributions of normal and shear stress at distances c from the boundaries, using the augmented expansions

$$\sigma_{12}(x,y) = \sum_n c_n e^{-\lambda_n y} \phi_n^{(1)}(x)$$
$$\sigma_{22}(x,y) = \sum_n c_n e^{-\lambda_n y} \phi_n^{(2)}(x) \tag{6.5}$$

truncated to 2N terms. The results of these calculations with $N = 40$ with the various methods are indistinguishable except at $x = 0$, and are plotted in figures 3, 4, 5.

At $y = 0$, the accuracy with which the expansion (5.3) fits the input data was assessed by computing σ, where

$$\sigma^2 = \frac{1}{M}\sum_{i=1}^{M}\left\{\left(f^{(1)}(x_i) - \sum c_n\phi_n^{(1)}(x_i)\right)^2 + \left(f^{(2)}(x_i) - \sum c_n\phi_n^{(2)}(x_i)\right)^2\right\}$$

with $x_i = \frac{i-\frac{1}{2}}{M}$, for $M = 100$. The resulting values of σ are also included in Table 2.

6.2 End displacements prescribed

In addition, the optimal weighting function approach was applied to the displacement problem for a beam bonded to a wall with $u = 0$, $v = 0$ on $x = 0$, $0 < y < 1$, under unit tension at infinity. For these boundary conditions (2.7) and (2.8) give

$$g^{(1)} \equiv f^{(1)} + (1-\nu)f^{(3)} = 0$$
$$g^{(2)} \equiv f^{(2)} - (1-\nu)f^{(4)} = -\nu \ . \tag{6.6}$$

We denote this case by E4, and have obtained the coefficients when $\nu = 0.3$. These are listed in Table 3.

Discussion of results

For cases 1 and 3 in which the data is smooth, with $N = 40$ methods B and D produce virtually identical results, with agreement to 4 or 5 significant figures for the earlier coefficients and the same rate of decay. Methods A also agrees to 3 or 4 figures for the first $N/2$ coefficients, but is less accurate beyond this point. Method C (Johnson and Little) is significantly less accurate than either of these, and in fact becomes worse as N is increased. There is a significant difference in the rate of convergence of the solution vector $\underset{\sim}{z}^{(N)}$ with methods A, B and D: for method D the first 10 coefficients are already close to their final values when computed with $N = 10$. This is not so with collocation, method A, and the stability is less marked with the fourier coefficients method B. The biorthogonality equations C are again significantly worse from this point of view.

The corresponding values of σ are of the same order of magnitude for given N for methods A, B and D, (i.e, 10^{-2}, 10^{-2}, 10^{-3}, 10^{-4} for N = 5, 10, 20, 40) but substantially larger in the case of method C , and in fact increasing with N in this case. For case 2, in which $f^{(2)}$ is discontinuous at $y = 0.5$, the coefficients c_n decrease very slowly. The convergence is discussed in the next section. The optimal weighting function method D again provides the most stable solution as n increases, and the Johnson-Little method C the least stable, but in this case the collocation method A is more successful than method B in its agreement with the earlier coefficients, for which we may assume that method D gives the best estimate.

With all four methods in case 2 the values of the goodness-of-fit parameter σ are much larger than in case 1, but are smaller $(0(10^{-1}))$ with methods A and B than with method D, where σ is 0(1). For method C , σ becomes larger as N increases, and is 6.6 for N = 40. A print out of the sums of the series (5.3) using the calculated coefficients shows that with method A the fit is extremely close (3-4 figures accuracy) except in the immediate neighbourhood of the discontinuity, whereas with method D the fit is poor. This reflects the fact that method D obtains the coefficients in the expansion of the biharmonic function Ω , but that, although this expansion is convergent at $y = 0$, the expansions (6.6) of the stress functions are convergent only for $y > 0$. However the remaining methods provide interpolations at $y = 0$, of which collocation is the most successful.

Convergence

A print out of the ratio of sums of absolute off-diagonal elements to the diagonal element in each row was made for method D and E, and confirms as expected from theoretical considerations the ratio

$$\sum_{j,j\neq i} |M_{ij}|/|M_{ii}|$$

in all cases decreases steadily, e.g. from about .6 to .2 for the stress problem (2) when $N = 40$ (80 equations).

The right hand side of the equations for method D case 2 is also included in table 2, and shows that $c_n \sim d_n$ as n becomes large, because of the diagonal dominance. The effect would be much more marked for elements that decay more rapidly, and it appears that the asymptotic behaviour of the c_n can be estimated from that of the d_n in this case. Thus the biorthogonal weighting functions provide the basis for a discussion of the convergence of series (5.3). In the three cases, the expression for d_m for this method are

$$d_m = \langle \chi_m^{(1)} f^{(1)} + \chi_m^{(2)} f^{(2)} \rangle$$

$$= \begin{cases} -2\left(\dfrac{\tan\lambda_m}{\lambda_m}\right)\left(1 - \dfrac{9}{\lambda_m^2}\right) - \dfrac{6}{\lambda_m^2} \sim O(m^{-1}) & \text{(case 1)} \\ \left(\dfrac{\tan\lambda_m}{\lambda_m}\right)\left(\tfrac{3}{2}\cos\tfrac{1}{2}\lambda_m - \tfrac{1}{2}\cos\tfrac{3}{2}\lambda_m - 1\right) & \text{(case 2)} \\ \qquad \sim m^{-\frac{1}{4}} & \\ -6\left(\dfrac{\tan\lambda_m}{\lambda_m^2}\right)\left(1 - \dfrac{5}{\lambda_m^2}\right) - \dfrac{18}{\lambda_m^3} & \text{(case 3)} \\ \qquad \sim O(m^{-2}) & \\ (-\nu)(1+2\alpha)\left(\dfrac{\tan\lambda_m}{\lambda_m}\right) \sim O(m^{-1}) & \text{(case E4)} \end{cases}$$

In Appendix B, expressions are listed for the L_2 norms of the eigenfunctions. The asymptotic behaviour is

$$\| \phi_n^{(1)} \| , \| \phi_n^{(2)} \| \sim \tfrac{1}{2}n\pi/(\tfrac{1}{2}\ln 4n\pi)^{3/2} ,$$

while $$\| \omega_n \| \sim (2n\pi)^{-1}(\tfrac{1}{2}\ln 4n\pi)^{-3/2} .$$

Hence when $|c_n| \sim n^{-2.5}$ as in case (1), the majorant series

$$\sum |c_n| \| \phi_n^{(1)} \|$$

is convergent; in case (2) on the other hand, $|c_n|$ converges only like $n^{-0.8}$ and the individual terms of the series increase. However the exponentials $e^{-\lambda_n y}$ ensure that in all cases the series (6.6) is convergent for $x > 0$.

Conclusions

The conclusions to be drawn from these results are that:

(i) The optimal weighting functions (5.16), leading to the equations (5.13) with matrix G given in (5.14), (5.15) provide the most stable system of equations from which to determine the coefficients c_n.

(ii) The use of unmodified biorthogonal weighting functions is unsatisfactory, and becomes more inaccurate as N is increased.

(iii) In cases involving discontinuities in edge data, collocation provides a better interpolation than Galerkin methods.

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Appendix A : Quadratures of eigenfunctions

The quadratures quoted in the paper are derived from the following results, all of which are obtained by integration by parts and use of the boundary conditions (3.2) on ω and its derivatives, and of the fact that $\lambda_i = -\sin\lambda_i \cos\lambda_i$. Here $\phi^{(1)} = -\lambda D\omega$, $\phi^{(2)} = D^2\omega = -\lambda^2\omega + \phi^{(4)}$, $\phi^{(3)} = (2/\cos\lambda)\sin\lambda x$, $\phi^{(4)} = -(2/\cos\lambda)\cos\lambda x$.

(i) for $\lambda_i \neq \lambda_j$

All the results are expressible in terms of

$$c_{ij} = (\cos\lambda_i \cos\lambda_j)^{-1}\langle\cos\lambda_i x \cos\lambda_j x\rangle = (\lambda_i \tan\lambda_i - \lambda_j \tan\lambda_j)/(\lambda_i^2-\lambda_j^2). \tag{A1}$$

A double integration by parts of $\omega_i \cos\lambda_j x$ gives

$$\langle\omega_i \cos\lambda_j x\rangle = -\frac{1}{\lambda_j^2}\langle\cos\lambda_j x\, D^2\omega_i\rangle = \frac{\lambda_i^2}{\lambda_j^2}\langle\omega_i\cos\lambda_j x\rangle + \frac{2}{\lambda_j^2}(\cos\lambda_j)c_{ij}$$

whence

$$(\cos\lambda_j)^{-1}\langle\omega_i \cos\lambda_j x\rangle = -\frac{2c_{ij}}{\lambda_i^2-\lambda_j^2}. \tag{A2}$$

This result can now be used to evaluate

$$\begin{aligned}\langle\phi_i^{(1)}\phi_j^{(1)}\rangle &= \lambda_i\lambda_j\langle D\omega_i\, D\omega_j\rangle = -\lambda_i\lambda_j\langle\omega_i\, D^2\omega_j\rangle = \\ &\quad \lambda_i\lambda_j^3\langle\omega_i\omega_j\rangle + 2\lambda_i\lambda_j(\cos\lambda_j)^{-1}\langle\omega_i\cos\lambda_j x\rangle \\ &= \lambda_i\lambda_j^3\langle\omega_i\omega_j\rangle - 4\lambda_i\lambda_j\, c_{ij}/(\lambda_i^2-\lambda_j^2) \\ &= \lambda_i^3\lambda_j\langle\omega_i\omega_j\rangle + 4\lambda_i\lambda_j\, c_{ij}/(\lambda_i^2-\lambda_j^2) \quad \text{by symmetry.}\end{aligned}$$

Therefore

$$\langle\omega_i\omega_j\rangle = -\frac{8c_{ij}}{(\lambda_i^2-\lambda_j^2)^2} \tag{A3}$$

and

$$\langle\phi_i^{(1)}\phi_j^{(1)}\rangle = -4\lambda_i\lambda_j(\lambda_i^2+\lambda_j^2)\,c_{ij}/(\lambda_i^2-\lambda_j^2)^2. \tag{A4}$$

Similar steps using (A2) and (A3) give the results

$$\langle\phi_i^{(2)}\phi_j^{(2)}\rangle = -8\lambda_i^2\lambda_j^2\,c_{ij}/(\lambda_i^2-\lambda_j^2)^2 \tag{A5}$$

$$\langle\phi_i^{(1)}\phi_j^{(3)}\rangle = -4\lambda_i\lambda_j\,c_{ij}/(\lambda_i^2-\lambda_j^2) \tag{A6}$$

$$\langle \phi_i^{(2)} \; \phi_j^{(4)} \rangle = - 4\lambda_j^2 \; c_{ij}/(\lambda_i^2 - \lambda_j^2) \; . \tag{A7}$$

Also by direct integration

$$\langle \phi_i^{(3)} \; \phi_j^{(3)} \rangle = 4 \; c_{ij} - 4(\tan\lambda_i + \tan\lambda_j)/(\lambda_i + \lambda_j) \tag{A8}$$

$$\langle \phi_i^{(4)} \; \phi_j^{(4)} \rangle = 4 \; c_{ij} . \tag{A9}$$

(ii) $i = j$

Direct integration gives

$$\left.\begin{array}{l} \langle \cos^2\lambda_i y \rangle = 0 \; , \; (\cos\lambda_i)^{-1} \langle \omega_i \; \cos\lambda_i y \rangle = 1/2\lambda_i^2 \\ \langle \omega_i^2 \rangle = - 1/3(\lambda_i \cos\lambda_i)^2 - 1/2\lambda_i^4 \; . \end{array}\right\} \tag{A10}$$

Using these results we find

$$\langle \phi_i^{(1)} \; \phi_i^{(1)} \rangle = \frac{1}{2} - \frac{1}{3} \left(\frac{\lambda_i}{\cos\lambda_i}\right)^2 = \langle \phi_i^{(2)} \; \phi_i^{(2)} \rangle - 1 \tag{A11}$$

$$\langle \phi_i^{(3)} \; \phi_i^{(3)} \rangle = 4/\cos^2\lambda_i , \; \langle \phi_i^{(4)} \; \phi_i^{(4)} \rangle = 0 \tag{A12}$$

$$\langle \phi_i^{(1)} \; \phi_i^{(3)} \rangle = \langle \phi_i^{(2)} \; \phi_i^{(4)} \rangle = 1 . \tag{A13}$$

The expressions quoted for $(\psi_i^{(\alpha,\beta)}, \phi_j^{(\alpha,\beta)})$ in Section 3 are obtained immediately from these results since

$$\psi_i^{(1)} = \tfrac{1}{2}\phi_i^{(3)} , \; \psi_i^{(2)} = \tfrac{1}{2}\phi_i^{(4)} , \; \psi_i^{(3)} = \tfrac{1}{2}\phi_i^{(1)} , \; \psi_i^{(4)} = \tfrac{1}{2}\phi_i^{(2)} - \tfrac{1}{2}\phi_i^{(4)} .$$

Appendix B. Norms of eigenfunctions

The norms are found from the expressions of Appendix A by writing $\lambda_j = \bar{\lambda}_i$.

Then $$c_{i\bar{i}} = \frac{\mathrm{Im}(\lambda_i \tan \lambda_i)}{2(\mathrm{Im}\,\lambda_i)(\mathrm{Re}\,\lambda_i)} .$$

From (A4)

$$\|\phi_i^{(1)}\|^2 = \langle\phi_i^{(1)}\ \phi_{\bar{i}}^{(1)}\rangle = -4|\lambda_i|^2 \left[\frac{2(\mathrm{Re}\,\lambda_i^2)\,c_{i\bar{i}}}{(2i\mathrm{Im}\lambda_i)^2(2\mathrm{Re}\lambda_i)^2}\right]$$

whence

$$\|\phi_i^{(1)}\| = \frac{1}{2}|\lambda_i|\left[\frac{(\mathrm{Re}\lambda_i^2)\ \mathrm{Im}(\lambda_i \tan \lambda_i)}{(\mathrm{Im}\,\lambda_i)^3\ (\mathrm{Re}\,\lambda_i)^3}\right]^{1/2} .$$

Similarly from (A5)

$$\|\phi_i^{(2)}\| = \frac{1}{2}|\lambda_i|^2\left[\frac{\mathrm{Im}\ (\lambda_i \tan \lambda_i)}{(\mathrm{Im}\,\lambda_i)^3(\mathrm{Re}\,\lambda_i)^3}\right]^{1/2}$$

and from (A8) and (A9)

$$\|\phi_i^{(3)}\| = \left[\frac{2\ \mathrm{Im}(\bar{\lambda}_i \tan \lambda_i)}{(\mathrm{Im}\,\lambda_i)(\mathrm{Re}\,\lambda_i)}\right]^{1/2}$$

$$\|\phi_i^{(4)}\| = \left[\frac{2\ \mathrm{Im}\ (\lambda_i \tan \lambda_i)}{(\mathrm{Im}\,\lambda_i)(\mathrm{Re}\,\lambda_i)}\right]^{1/2}$$

Also from (A3)

$$\|\omega_i\| = |\lambda_i|^{-2}\ \|\phi_i^{(2)}\|$$

Asymptotically, using (3.5), we find

$$\|\phi_k^{(1)}\|,\ \|\phi_k^{(2)}\| \sim \frac{1}{2}k\pi/(\frac{1}{2}\ \ell n 4k\pi)^{3/2}$$

$$\|\phi_k^{(3)}\|,\ \|\phi_k^{(4)}\| \sim 2/(\ell n 4k\pi)^{1/2}$$

and

$$\|\omega_k\| \sim (2k\pi)^{-1}(\frac{1}{2}\ell n 4k\pi)^{-3/2} .$$

for large k (in fact these are close approximations for $k > 2$).

Appendix C(1) Contour integral to evaluate $\sum_{\substack{j \\ (j \neq k)}} \frac{\sin^2\lambda_j - \sin^2\lambda_k}{(\lambda_j^2 - \lambda_k^2)\cos^2\lambda_j}$

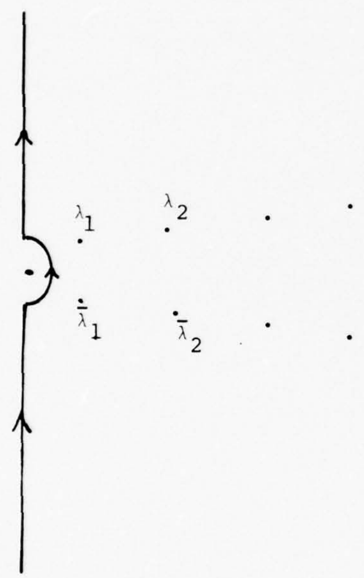
Let

$$f(z) = \frac{2\sin^2 z - 2\sin^2\lambda_k}{(z^2-\lambda_k^2)\, C(z)}$$

where

$$C(z) = z + \sin z \cos z$$

and consider $\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(z)\,dz$ along a contour as indicated passing to the right of the origin but to the left of all the zeros λ_k of $C(z)$ with positive real parts.



Since $f(z)$ is an odd function of z, the contributions to the integral from the two halves of the imaginary axis vanish and

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(z)dz = \frac{1}{2}\text{ (residue of } f \text{ at } z = 0).$$

The integral also equals minus the sum of the residues at the poles in $\text{Re } z > 0$, i.e. at the zeros λ_j of $C(z)$

Since $C'(\lambda_j) = 2\cos^2\lambda_j$, the residues at the points $z = \lambda_j \neq \lambda_k$ give the terms of the required series

The residue at $z = 0$ is $\sin^2\lambda_k/\lambda_k^2 = \frac{1}{\cos^2\lambda_k}$, and that at $z = \lambda_k$ is $-1/\cos^2\lambda_k$. Combining, we have

$$\sum_{\substack{j \\ j\neq k}} \frac{\sin^2\lambda_j - \sin^2\lambda_k}{(\lambda_j^2 - \lambda_k^2)\cos^2\lambda_j} = \frac{1}{2\cos^2\lambda_k}$$

In terms of $c_{jk} = (\lambda_j \tan \lambda_j - \lambda_k \tan \lambda_k)/(\lambda_j^2 - \lambda_k^2)$, the result is

$$\sum_{j(j\neq k)} \frac{c_{jk}}{\cos^2\lambda_j} = -\frac{1}{2\cos^2\lambda_k}$$

(2) The second result

$$\sum_{j(j\neq k)} \frac{c_{jk}}{(\lambda_j^2-\lambda_k^2)\cos^2\lambda_j} = 1/4\ \lambda_k^2 \cos^2 \lambda_k$$

is obtained by the same method applied to the function $(z^2 - \lambda_k^2)^{-1} f(z)$.

DAS/db

TABLE 2. COMPUTED COEFFICIENTS

A (Collocation) Case 1

N = 5

(σ = .191259,-01)

(c_n) real	(c_n) imag.
.61063369+00	.61392605+00
-.71418810-01	.13028745-01
-.34231232-01	-.16279706-01
-.26835435-01	-.42052902-01
.21851789-01	.59572337-02

N = 10

(σ = .745534,-02)

(c_n) real	(c_n) imag.
.61197602+00	.61710115+00
-.68466181-01	.15751234-01
-.25967889-01	-.12011278-01
-.90125573-02	-.90003107-02
-.21360312-02	-.48307601-02
.11323487-02	-.10557592-02
.19263364-02	.32729996-02
-.34329129-02	.67216359-02
-.14023023-01	-.15237260-01
.80038874-02	-.40843952-03

N = 20

(σ = .559897,-03)

(c_n) real	(c_n) imag.
.61143314+00	.61566541+00
-.69403359-01	.14323567-01
-.27184322-01	-.13490598-01
-.10645565-01	-.10669075-01
-.43879625-02	-.70334242-02
-.17982206-02	-.45784616-02
-.64708763-03	-.30236889-02
-.11129095-03	-.20325576-02
.14400096-03	-.13870711-02
.26660636-03	-.95678757-03
.32713498-03	-.66326487-03
.36255633-03	-.45761730-03
.39570522-03	-.30660446-03
.44519822-03	-.18073945-03
.52774147-03	-.39125830-04
.64102925-03	.19997518-03
.65523510-03	.68723676-03
-.55776629-04	.13202449-02
-.13495410-02	-.77284494-03
.38477966-03	-.17611918-03

N = 40

(σ = .470479,-03)

(c_n) real	(c_n) imag.
.61140499+00	.61558381+00
-.69447217-01	.14243374-01
-.27231731-01	-.13570348-01
-.10694934-01	-.10748246-01
-.44391638-02	-.71118366-02
-.18516945-02	-.46559352-02
-.70369482-03	-.31000388-02
-.17236297-03	-.21075600-02
.76410530-04	-.14605170-02
.18931400-03	-.10285257-02
.23496628-03	-.73360839-03
.24703267-03	-.52814852-03
.24252237-03	-.38252730-03
.23038100-03	-.27772044-03
.21524926-03	-.20135130-03
.19965022-03	-.14516082-03
.18496796-03	-.10350489-03
.17197961-03	-.72389746-04
.16106763-03	-.48971940-04
.15244672-03	-.31242143-04
.14634454-03	-.17615650-04
.14291931-03	-.68936447-05
.14246104-03	.19386252-05
.14538541-03	.99778445-05
.15208382-03	.18407256-04
.16302043-03	.28586092-04
.17876976-03	.42484144-04
.19964429-03	.62930762-04
.22533701-03	.94023941-04
.25400598-03	.14192214-03
.28001656-03	.21530492-03
.28974349-03	.32537540-03
.25280907-03	.48271942-03
.10628609-03	.68294941-03
-.26422401-03	.85689188-03
-.99298726-03	.72895106-03
-.18573515-02	-.43119921-03
-.78824002-03	-.29922575-02
.24485574-02	.64005520-03
-.39475704-03	.39739060-03

B (Fourier Coefficients) Case 1

N = 5

(σ = .124842,-01)

.61196943+00	.61740379+00
-.68648842-01	.16008436-01
-.26453999-01	-.11834770-01
-.99103196-02	-.90707846-02
-.32725613-02	-.55189619-02

N = 10

(σ = .394030,-02)

.61154232+00	.61602638+00
-.69252766-01	.14674471-01
-.27044944-01	-.13140750-01
-.10524048-01	-.10317183-01
-.42858740-02	-.66758288-02
-.17166301-02	-.42103435-02
-.58801331-03	-.26383389-02
-.79233923-04	-.16199345-02
.13855134-03	-.93073679-03
.19537313-03	-.42479420-03

N = 20

(σ = .619719,-03)

.61141044+00	.61560272+00
-.69440008-01	.14261664-01
-.27225579-01	-.13552206-01
-.10690154-01	-.10730131-01
-.44358049-02	-.70936223-02
-.18497402-02	-.46374686-02
-.70312456-03	-.30811398-02
-.17320371-03	-.20880438-02
.74122276-04	-.14402050-02
.18557761-03	-.10071175-02
.22967359-03	-.71073227-03
.23996520-03	-.50344894-03
.23344906-03	-.35550524-03
.21891356-03	-.24779171-03
.20090414-03	-.16767390-03
.18164388-03	-.10649439-03
.16190454-03	-.57950053-04
.14100743-03	-.16897873-04
.11492843-03	.21634687-04
.65443454-04	.57553125-04

N = 40

(σ = .311031,-03)

.61139631+00	.61555892+00
-.69460719-01	.14218871-01
-.27246382-01	-.13594793-01
-.10710136-01	-.10772619-01
-.44547421-02	-.71361037-02
-.18675741-02	-.46800459-02
-.71985317-03	-.31239312-02
-.18884488-03	-.21311882-02
.59547002-04	-.14838687-02
.17205230-03	-.10515030-02
.21719621-03	-.75609051-03
.22855693-03	-.55009736-03
.22316833-03	-.40385635-03
.20988085-03	-.29839940-03
.19335072-03	-.22131338-03
.17601934-03	-.16430600-03
.15914360-03	-.12172110-03
.14332582-03	-.89639736-04
.12881870-03	-.65302295-04
.11568607-03	-.46732710-04
.10388763-03	-.32501460-04
.93335177-04	-.21562100-04
.83921507-04	-.13138785-04
.75533786-04	-.66488299-05
.68059953-04	-.16534424-05
.61396804-04	.21797728-05
.55452139-04	.51096406-05
.50140227-04	.73344634-05
.45386793-04	.90118913-05
.41123137-04	.10264347-04
.37290717-04	.11192245-04
.33834981-04	.11879548-04
.30704110-04	.12397569-04
.27848840-04	.12812735-04
.25215059-04	.13197701-04
.22727375-04	.13638738-04
.20258110-04	.14261768-04
.17504634-04	.15274646-04
.13497920-04	.16939768-04
.46135928-05	.17642350-04

<u>C (Biorthogonal weighting functions) Case 1</u>

N = 5

(σ = .101530,-01)

.61135616+00	.61506104+00
-.66043414-01	.22716303-01
-.42069022-01	-.30551851-01
-.44612018-02	-.14961896-01
.52242245-02	.33942899-04

N = 10

(σ = .234948,00)

.61139172+00	.61534385+00
-.67526625-01	.18427741-01
-.36762522-01	-.62455493-02
-.53164362-01	-.13168849+00
.18378647+00	.13980365+00
-.30478820-01	.30754357+00
-.49385241+00	-.67001963+00
.55974969+00	.32367657+00
-.18598923+00	-.20900748-02
.11616366-01	-.12764101-01

N= 20

(σ= .173240,00)

.61139365+00	.61559773+00
-.69515024-01	.12498844-01
-.23046318-01	.45942632-02
-.44470041-01	-.82477171-01
.93779008-01	.92207912-01
-.76023699-01	.83889777-01
-.20072066+00	-.42793861+00
.46102554+00	.41931194+00
-.30125758+00	-.83994967-01
.29581786-01	-.45013340-01
.18476482-01	-.27760241-02
.73103741-02	.50718615-01
-.80956984-01	-.47188486-01
.11736559-02	-.10843889+00
.12660224+00	.43565862-01
.18472532-01	.11413889+00
-.95335673-01	-.39114366-01
.16368511-01	-.19683665-01
.77247485-02	.28738946-02
-.73226797-03	.93369492-03

N = 40

(σ = .353545,00)

.61138740+00	.61577331+00
-.70821297-01	.72895306-02
-.85014262-02	.21413861-01
-.53730856-01	-.31473638-01
-.41468516-01	-.79426311-01
.67799826-01	-.11353737+00
.26504809+00	.44509070+00
-.41282712+00	-.22819515+00
.79425870-01	-.97633028-01
.62999298-01	.71562967-01
-.65732111-01	-.11796643+00
.14099306+00	.11325470+00
-.67857298-01	-.16853370-02
.27470010-02	.56969925-01
-.78382837-01	-.10058266+00
.49289715-01	.16432132+00
-.26295310+00	-.31843493+00
.27991951+00	-.72133556-01
.15114690+00	.28860681+00
-.21247444+00	-.14503381+00
.15410253+00	.59546073-01
-.21342719-01	-.23445248-01
.85220357-01	.11968018+00
-.59676480-01	-.34035228-01
.95739569-01	.16831081+00
-.24286525+00	.16911579+00
-.22744965+00	-.69862766+00
.67420488+00	.76496596+00
-.73232653+00	-.32548673+00
.31215408-01	.38959813+00
-.80186294+00	-.15218450+01
.16166298+01	.22461629-01
.50750915+00	.11626197+01
-.51709488+00	.22315959+00
-.31367386+00	.78136203+00
-.12208629+01	-.14669324+01
.11750934+01	.27751912-01
-.29149578-01	.31095753+00
-.11234190+00	-.42669927-02
.56994109-02	-.14144059-01

N = 5

(σ = .169764,-01)

.61159911+00	.61575004+00
-.69100317-01	.14443838-01
-.26839107-01	-.13360062-01
-.10296426-01	-.10534172-01
-.40524908-02	-.68974781-02

N = 10

(σ = .775839,-02)

.61141186+00	.61557733+00
-.69429025-01	.14239222-01
-.27206685-01	-.13575782-01
-.10666254-01	-.10755842-01
-.44089450-02	-.71218203-02
-.18213547-02	-.46682608-02
-.67402825-03	-.31145654-02
-.14402424-03	-.21241040-02
.10305702-03	-.14789114-02
.21401887-03	-.10485142-02

N = 20

(σ = .184497,-02)

.61139748+00	.61556317+00
-.69459264-01	.14222693-01
-.27245032-01	-.13591299-01
-.10708896-01	-.10769471-01
-.44535988-02	-.71333188-02
-.18665912-02	-.46776211-02
-.71894261-03	-.31218762-02
-.18804684-03	-.21294947-02
.60276574-04	-.14825254-02
.17268593-03	-.10504967-02
.21771170-03	-.75541359-03
.22901988-03	-.54973397-03
.22357400-03	-.40379472-03
.21020583-03	-.29862122-03
.19361435-03	-.22180386-03
.17624872-03	-.16505200-03
.15935618-03	-.12271264-03
.14345350-03	-.90859679-04
.12891212-03	-.66740681-04
.11577716-03	-.48379693-04

N = 40

(σ = .535508,-03)

.61139783+00	.61556352+00
-.69458680-01	.14223256-01
-.27244363-01	-.13590491-01
-.10708185-01	-.10768378-01
-.44528639-02	-.71319176-02
-.18658388-02	-.46758996-02
-.71817554-03	-.31198313-02
-.18726647-03	-.21271293-02
.61069416-04	-.14798469-02
.17349051-03	-.10475154-02
.21852723-03	-.75214178-03
.22984544-03	-.54618507-03
.22440858-03	-.39998284-03
.21104832-03	-.29456071-03
.19446362-03	-.21750909-03
.17710359-03	-.16053709-03
.16021549-03	-.11799133-03
.14431612-03	-.85945279-04
.12977694-03	-.61645959-04
.11664314-03	-.43116855-04
.10474977-03	-.28927634-04
.94194293-04	-.18034100-04
.84737332-04	-.96600355-05
.76310987-04	-.32227700-05
.68883314-04	.17106726-05
.62184615-04	.54778403-05
.56214307-04	.83321986-05
.50896950-04	.10469243-04
.46140892-04	.12044575-04
.41847435-04	.13176434-04
.38072247-04	.13956383-04
.34622139-04	.14463590-04
.31539193-04	.14750772-04
.28744154-04	.14868894-04
.26228844-04	.14850977-04
.23981761-04	.14728459-04
.21921008-04	.14523604-04
.20033427-04	.14259437-04
.18366402-04	.13946897-04
.16807824-04	.13601162-04

Table 2, continued

A (Collocation) Case 2

N = 5

(σ = .377507,00)

.60708918+00	.66304201+00
-.35188187+00	-.17427991+00
-.28157589+00	-.47760766+00
.47300177+00	.40033579+00
-.19663317+00	-.82746527-01

N = 10

(σ = .219051,00)

.57906948+00	.61243860+00
-.36669589+00	-.20909621+00
-.30052660+00	-.45171404+00
.27748704+00	.20345909-01
.19129017+00	.29230234+00
-.23842074+00	-.28631552-01
-.16859768+00	-.26586103+00
.15557469+00	-.11997603+00
.23094832+00	.23250230+00
-.11012884+00	.13362892-01

N = 20

(σ = .161249,00)

.58713482+00	.62885981+00
-.36300297+00	-.19720814+00
-.30410254+00	-.46249637+00
.31862022+00	.36416865-01
.25062907+00	.37256076+00
-.26245858+00	.83564809-02
-.19685610+00	-.30897051+00
.23409013+00	-.18709555-01
.16493079+00	.26660706+00
-.20342026+00	.27805010-01
-.13808390+00	-.22251441+00
.16880088+00	-.29508198-01
.10778646+00	.17803678+00
-.13921401+00	.24179758-01
-.86363585-01	-.13958594+00
.98589820-01	-.30298234-01
.63020841-01	.77197284-01
-.39639047-01	.15390686-01
-.11717444-01	-.85468226-02
.12554324-02	-.23836171-02

N = 40

(σ = .110522,00)

.58819868+00	.63057330+00
-.36379328+00	-.19661377+00
-.30736237+00	-.46758012+00
.32520469+00	.37528259-01
.25972973+00	.38773020+00
-.27555545+00	.94992820-02
-.21074082+00	-.33426678+00
.25949360+00	-.19895160-01
.18852364+00	.30828986+00
-.23995054+00	.33259461-01
-.16795803+00	-.28014245+00
.22568741+00	-.35661868-01
.15300309+00	.26112026+00
-.21134837+00	.40168331-01
-.13950907+00	-.24058998+00
.19807889+00	-.40232635-01
.12780728+00	.22368137+00
-.18469279+00	.41250500-01
-.11673459+00	-.20569544+00
.17154585+00	-.39967122-01
.10660226+00	.18927678+00
-.15770189+00	.39199884-01
-.96262812-01	-.17173319+00
.14413066+00	-.36781278-01
.86907761-01	.15505973+00
-.12904927+00	.35049330-01
-.76430163-01	-.13659206+00
.11462381+00	-.31208296-01
.67303174-01	.11954374+00
-.97930284-01	.29460705-01
-.56595991-01	-.98454383-01
.81308153-01	-.23017876-01
.45556638-01	.81798684-01
-.65718552-01	.22511513-01
-.39810450-01	-.57370233-01
.37709177-01	-.17224680-01
.16968374-01	.27095119-01
-.19062765-01	-.83537380-02
.96351358-02	.84214337-03
-.18104902-02	.14124047-02

Note: If the 8 points from .465 to .535 are omitted in calculating σ, the values obtained are

N	σ
5	.298667,00
10	.990740,-01
20	.210252,-01
40	.792599,-02

B (Fourier Coefficients) Case 2

N = 5

(σ = .736045,00)

.56995639+00	.55534087+00
-.38407921+00	-.27106706+00
-.31845103+00	-.55005451+00
.33589948+00	-.59793413-01
.33028603+00	.25814066+00

N = 10

(σ = .207997,00)

.58806410+00	.62859458+00
-.36435445+00	-.19886986+00
-.30825322+00	-.47186992+00
.32808891+00	.34885158-01
.26403508+00	.38890459+00
-.27766989+00	.45313066-02
-.21135522+00	-.35069076+00
.27603655+00	-.34000815-01
.21881788+00	.29652846+00
-.16217948+00	-.10051733-02

N = 20

(σ = .191882,00)

.58957268+00	.63449720+00
-.36264388+00	-.19315642+00
-.30713315+00	-.46602409+00
.32853490+00	.41145275-01
.26370712+00	.39600671+00
-.27907255+00	.13194183-01
-.21468521+00	-.33912911+00
.26815637+00	-.16769374-01
.19634257+00	.32537525+00
-.25169142+00	.38864315-01
-.17778908+00	-.29426499+00
.24288676+00	-.33183134-01
.16605129+00	.29165625+00
-.23492995+00	.50471317-01
-.15762143+00	-.26482061+00
.22529559+00	-.35674692-01
.14413953+00	.27715714+00
-.23148645+00	.68114654-01
-.16844438+00	-.22385806+00
.12317142+00	-.16466878-01

N = 40

(σ = .218429,00)

.58975801+00	.63482024+00
-.36226089+00	-.19281731+00
-.30660976+00	-.46568250+00
.32917903+00	.41474951-01
.26445764+00	.39630662+00
-.27822699+00	.13441530-01
-.21375274+00	-.33896343+00
.26917150+00	-.16722648-01
.19744132+00	.32525495+00
-.25050136+00	.38513344-01
-.17648978+00	-.29493297+00
.24432939+00	-.34289289-01
.16769939+00	.28993511+00
-.23296141+00	.47865605-01
-.15510928+00	-.26874551+00
.22882351+00	-.41664174-01
.14980730+00	.26772833+00
-.22051420+00	.52543725-01
-.14094901+00	-.25047254+00
.21755645+00	-.45204845-01
.13744334+00	.25206361+00
-.21102463+00	.55313495-01
-.13074123+00	-.23652538+00
.20871393+00	-.46846117-01
.12817530+00	.24036727+00
-.20346676+00	.57303982-01
-.12299536+00	-.22509122+00
.20139518+00	-.47197168-01
.12078234+00	.23157838+00
-.19735243+00	.59317960-01
-.11704624+00	-.21480368+00
.19490485+00	-.46003304-01
.11428336+00	.22585464+00
-.19290000+00	.62894641-01
-.11341144+00	-.20314605+00
.18710000+00	-.40337969-01
.10449237+00	.22795691+00
-.19881080+00	.77279381-01
-.13695378+00	-.17431433+00
.92258291-01	-.26421636-01

C (Biorthogonal Weighting Functions) Case 2

N = 5

(σ = .296387,00)

.58652741+00	.62435478+00
-.33741776+00	-.16508999+00
-.29684963+00	-.43551654+00
.29403796+00	.14827842+00
-.37653250-01	-.10846512-01

N = 10

(σ = .307518,01)

.58866173+00	.63242231+00
-.38162056+00	-.20100133+00
-.32037079+00	-.87871221+00
.13286031+01	.21618090+01
-.25776729+01	-.17168784+01
.12810420+00	-.40114113+01
.60157050+01	.81086854+01
-.69125748+01	-.43046653+01
.26601005+01	.28426049+00
-.26820242+00	.17901652+00

N = 20

(σ = .240200,01)

.58854527+00	.63018823+00
-.36254239+00	-.16309715+00
-.39442898+00	-.79456420+00
.90945917+00	.11895634+01
-.12321781+01	-.10696868+01
.75953955+00	-.11929522+01
.24599904+01	.53198091+01
-.58505628+01	-.57619814+01
.44671741+01	.18077749+01
-.95020428+00	.51546749+00
-.46288896+00	-.66689474-01
-.33980731+00	-.14919883+01
.19370439+01	.97540758+00
-.31444669-01	.19517805+01
-.21437625+01	-.72133705+00
-.26357984+00	-.19435165+01
.17111665+01	.81203042+00
-.37875147+00	.39581158+00
-.21841977+00	-.12684664+00
.49799360-01	-.27011399-01

N = 40

(σ = .665455,01)

.58848195+00	.63296056+00
-.38304045+00	-.24097565+00
-.19047652+00	-.54153776+00
.75542744+00	.16643047+01
-.24979798+01	-.28580935+01
.22185785+01	-.14801681+01
.39463186+01	.89123588+01
-.94356234+01	-.10759307+02
.10056579+02	.65261164+01
-.41267305+01	.14796249+01
-.26940930+01	-.18084892+01
-.56665776+00	-.37974272+01
.40055308+01	.89648982+00
.12530144+01	.45549932+01
-.44096398+01	-.95201063-01
-.24406790+01	-.41197091+01
.18242068+01	-.10525019+01
.10884015+01	-.22752192+01
.37482455+01	.41324295+01
-.24000814+01	-.17623820+01
.24400731+01	.37477357+01
-.36617273+01	-.36561043+01
.40659974+01	.17940227+01
-.56689912+00	.79464592+00
.82535621+00	.21386242+01
-.29357848+01	.33170363+01
-.42118856+01	-.97371110+01
.89781090+01	.10282159+02
-.10300884+02	-.50393864+01
.33223339+00	.59910794+01
-.12261743+02	-.23590204+02
.25161335+02	.33406487+00
.79291270+01	.18463717+02
-.84301004+01	.29131306+01
-.44095077+01	.12657680+02
-.19545304+02	-.23355228+02
.18855635+02	.60294698+00
-.59048611+00	.49891589+01
-.18217431+01	-.10922078+00
.10474451+00	-.23236669+00

D (Optimal Weighting Functions) Case 2

N = 5

(σ = .121382,01)

.59108404+00	.63404593+00
-.35984104+00	-.19119311+00
-.30363506+00	-.46219734+00
.33234702+00	.46340538-01
.26760696+00	.40214169+00

N = 10

(σ = .862277,00)

.58983302+00	.63217881+00
-.36149827+00	-.19510081+00
-.30532441+00	-.46791507+00
.33073473+00	.39232946-01
.26612317+00	.39403208+00
-.27654766+00	.11122597-01
-.21211596+00	-.34133417+00
.27073417+00	-.19149920-01
.19891346+00	.32276706+00
-.24912561+00	.35961911-01

N = 20

(σ = .133466,01)

.58805975+00	.63042476+00
-.36506118+00	-.19726869+00
-.30974839+00	-.47016876+00
.32587561+00	.36979368-01
.26107277+00	.39181367+01
-.28164125+00	.89569794-02
-.21716120+00	-.34343723+00
.26579367+00	-.21184948-01
.19411167+00	.32080286+00
-.25376931+00	.34069547-01
-.17968707+00	-.29937065+00
.24120909+00	-.38724386-01
.16465991+00	.28549818+00
-.23591712+00	.43421326-01
-.15797897+00	-.27320231+00
.22604069+00	-.46139496-01
.14711094+00	.26322744+00
-.22312404+00	.48008004-01
-.14347210+00	-.25505199+00
.21511944+00	-.49838479-01

N = 40

(σ = .204838,01)

.58833995+00	.63071703+00
-.36445971+00	-.19688687+00
-.30895874+00	-.46976236+00
.32678397+00	.37388938-01
.26205347+00	.39221676+00
-.28062099+00	.93487867-02
-.21612499+00	-.34305917+00
.26682878+00	-.20821836-01
.19513340+00	.32115053+00
-.25276970+00	.34401735-01
-.17871569+00	-.29905374+00
.24214811+00	-.38422405-01
.16556391+00	.28578571+00
-.23504967+00	.43694923-01
-.15714877+00	-.27294209+00
.22683353+00	-.45892067-01
.14786680+00	.26346264+00
-.22240446+00	.48231564-01
-.14278787+00	-.25483952+00
.21576945+00	-.49636553-01
.13570904+00	.24755440+00
-.21270471+00	.50711810-01
-.13236338+00	-.24123280+00
.20714962+00	-.51680766-01
.12667339+00	.23538134+00
-.20489849+00	.52138400-01
-.12435064+00	-.23046878+00
.20012298+00	-.52834260-01
.11960960+00	.22562209+00
-.19840117+00	.52968735-01
-.11793883+00	-.22164478+00
.19421522+00	-.53480232-01
.11388429+00	.21753837+00
-.19285919+00	.53435815-01
-.11265369+00	-.21421830+00
.18913580+00	-.53816912-01
.10911814+00	.21067804+00
-.18804207+00	.53672894-01
-.10819491+00	-.20784126+00
.18469257+00	-.53957823-01

NOTE: Omitting the 8 points between .465 and .53 gives

N	σ
5	.116627,01
10	.842872,00
20	.132231,01
40	.203624,01

RHS for Case D2

$(d_n)_{real}$	$(d_n)_{imag}$
.52712634+00	.44575244+00
-.42918404+00	-.34640071+00
-.34688970+00	-.58352952+00
.30454501+00	-.51253072-01
.24841781+00	.32093542+00
-.28936553+00	-.49650438-01
-.22194791+00	-.39309137+00
.26283596+00	-.64103497-01
.19233676+00	.28309255+00
-.25475471+00	.48408798-03
-.19013182+00	-.32962089+00
.24114218+00	-.66229125-01
.16486112+00	.26028875+00
-.23552364+00	.20156783-01
-.15744683+00	-.29479932+00
.22667289+00	-.66291870-01
.14781493+00	.24433808+00
-.22236924+00	.30231793-01
-.14268221+00	-.27183997+00
.21593248+00	-.65743316-01
.13591914+00	.23225148+00
-.21245583+00	.36135656-01
-.13208243+00	-.25514869+00
.20745715+00	-.64994085-01
.12700295+00	.22262010+00
-.20455069+00	.39884693-01
-.12398774+00	-.24225424+00
.20049833+00	-.64186285-01
.11999510+00	.21467241+00
-.19800745+00	.42393384-01
-.11753854+00	-.23187091+00
.19462068+00	-.63379679-01
.11429367+00	.20794512+00
-.19244695+00	.44130015-01
-.11223951+00	-.22325366+00
.18955116+00	-.62597247-01
.10953394+00	.20213854+00
-.18762645+00	.45361058-01
-.10777996+00	-.21593749+00
.18510641+00	-.61849424-01

TABLE 3

Computed Coefficients for Zero-End Displacement Problem (E4)

N = 5

$(c_n)_{real}$	$(c_n)_{imag}$
.11490602-01	.16880880+00
-.16110192-01	.66001308-01
-.14339124-01	.41379458-01
-.11674771-01	.30448630-01
-.95871504-02	.24290585-01

N = 10

$(c_n)_{real}$	$(c_n)_{imag}$
.96778788-02	.16817558+00
-.18170833-01	.65860470-01
-.16417219-01	.41481841-01
-.13676391-01	.30699111-01
-.11474775-01	.24638308-01
-.97904070-02	.20740474-01
-.84885197-02	.18008303-01
-.74622143-02	.15976204-01
-.66370235-02	.14398375-01
-.59616720-02	.13132877-01

N = 20

$(c_n)_{real}$	$(c_n)_{imag}$
.88332921-02	.16768864+00
-.19208893-01	.65608801-01
-.17557180-01	.41356522-01
-.14865636-01	.30663542-01
-.12680160-01	.24671699-01
-.10990873-01	.20828860-01
-.96706884-02	.18141507-01
-.86177045-02	.16146455-01
-.77607587-02	.14599547-01
-.70507870-02	.13360038-01
-.64534499-02	.12341069-01
-.59441896-02	.11486079-01
-.55050490-02	.10756609-01
-.51226218-02	.10125559-01
-.47867055-02	.95732568-02
-.44894039-02	.90850566-02
-.42245104-02	.86498135-02
-.39870794-02	.82588878-02
-.37731218-02	.79054692-02
-.35793839-02	.75841130-02

N = 40

$(c_n)_{real}$	$(c_n)_{imag}$
.84393964-02	.16735048+00
-.19706981-01	.65384974-01
-.18124259-01	.41190661-01
-.15481876-01	.30540044-01
-.13331680-01	.24582611-01
-.11667422-01	.20769061-01
-.10364452-01	.18107270-01
-.93226098-02	.16134865-01
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-.71668816-02	.12384557-01
-.66548045-02	.11544617-01
-.62109432-02	.10828878-01
-.58222854-02	.10210389-01
-.54789477-02	.96696002-02
-.51732931-02	.91919710-02
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-.46522762-02	.83844685-02
-.44282941-02	.80392931-02
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-.37079195-02	.69329957-02
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-.32997837-02	.63055693-02
-.31820646-02	.61240146-02
-.30720113-02	.59539179-02
-.29688973-02	.57941600-02
-.28720870-02	.56437680-02
-.27810212-02	.55018915-02
-.26952048-02	.53677824-02
-.26142000-02	.52407815-02
-.25376154-02	.51203039-02
-.24651012-02	.50058289-02
-.23963454-02	.48968929-02
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-.22099449-02	.45993563-02
-.21536654-02	.45088055-02

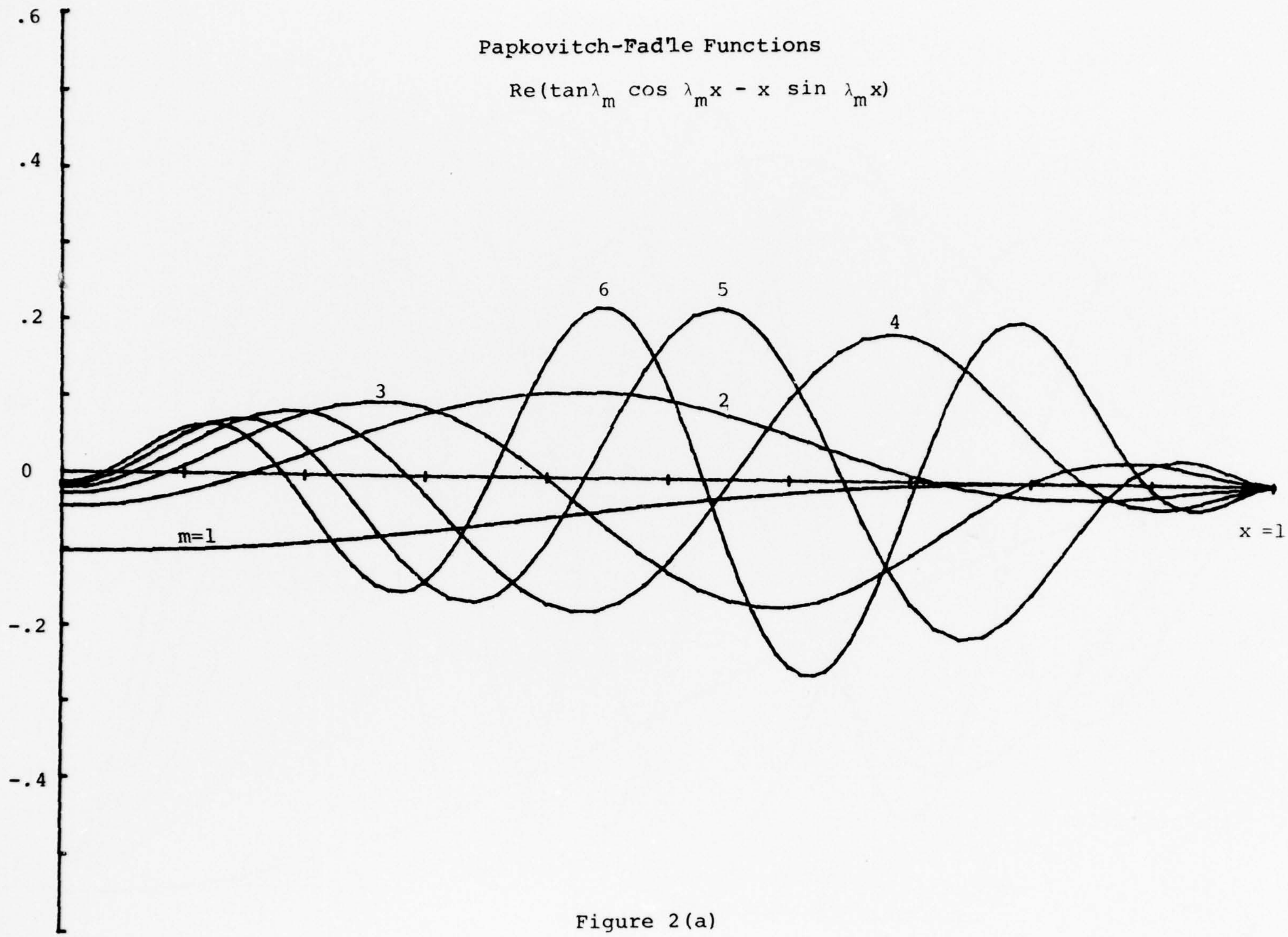

Papkovitch-Fad'le Functions
Re(tanλm cos λmx - x sin λmx)
.6
.4
.2
0
-.2
-.4
6
5
4
3
2
m=1
x =1

Figure 2(a)

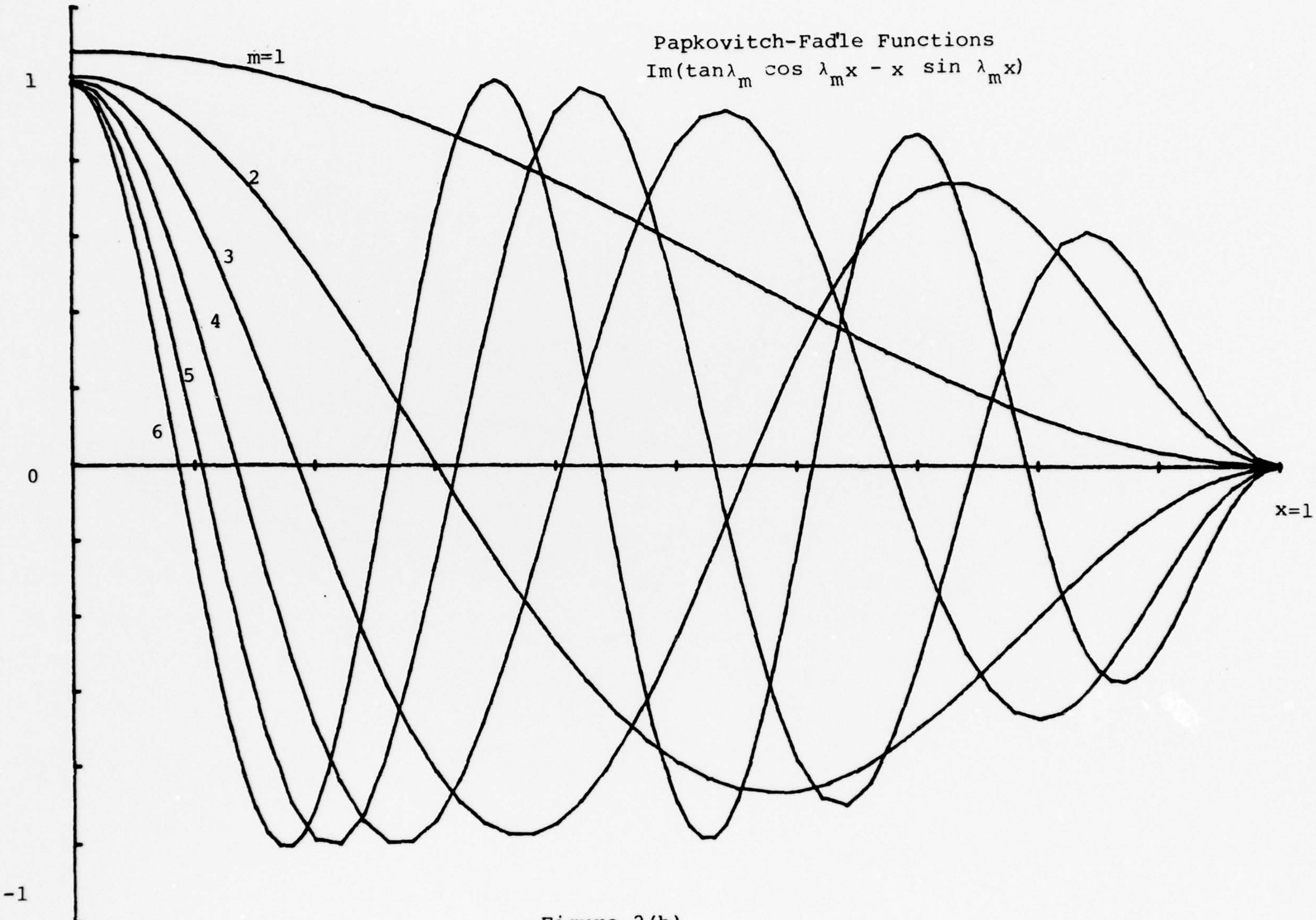

Papkovitch-Fadle Functions
Im(tanλ_m cos λ_m x - x sin λ_m x)
m=1
2
3
4
5
6
1
0
-1
x=1

Figure 2(b)

Figure 3

STRESS DISTRIBUTIONS FOR CASE 1 $(\sigma_{22})_0 = 1-3x^2$, $(\sigma_{12})_0 = 0$

(a) Direct Stress

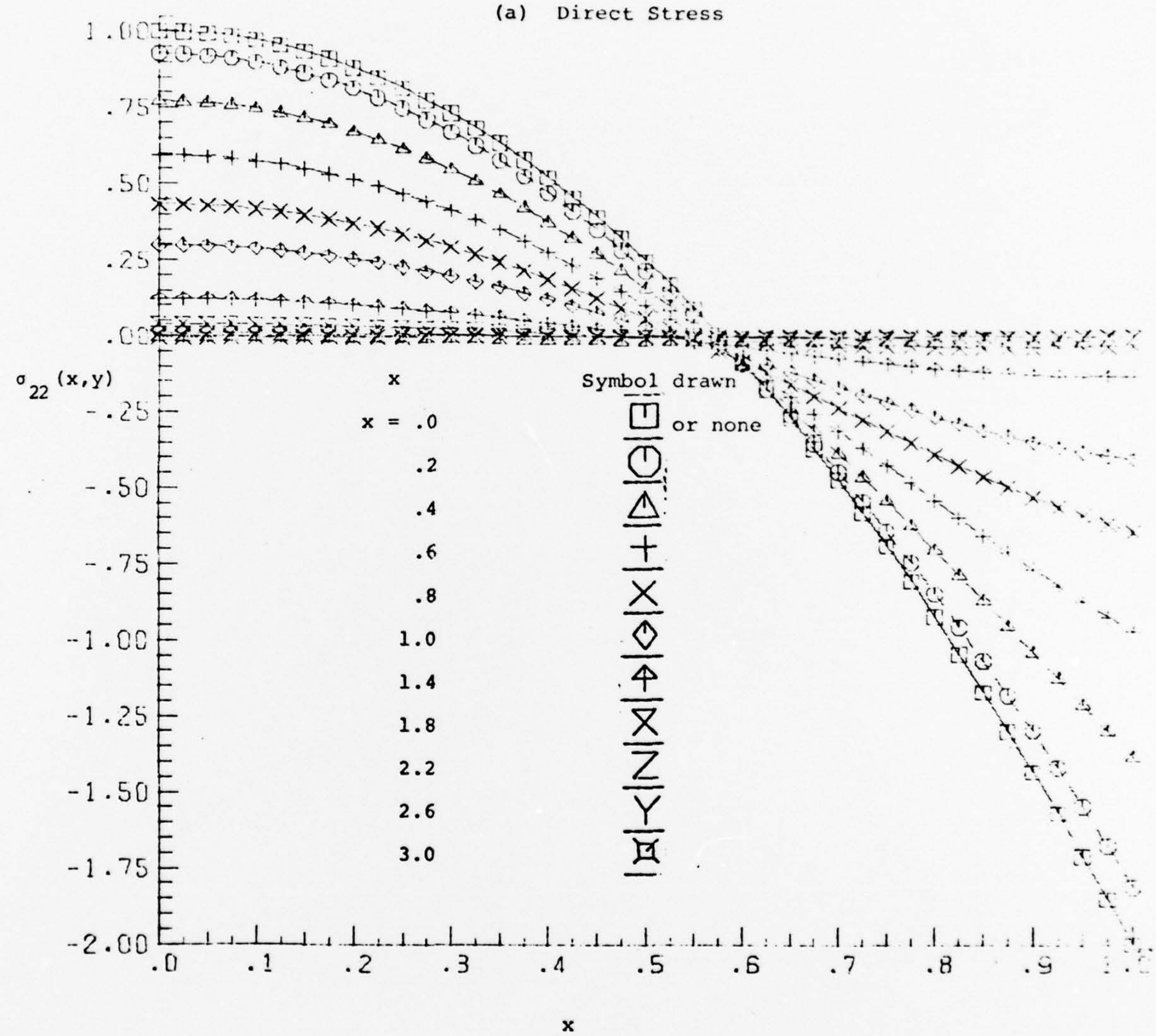


Figure 3

(b) Shear Stress

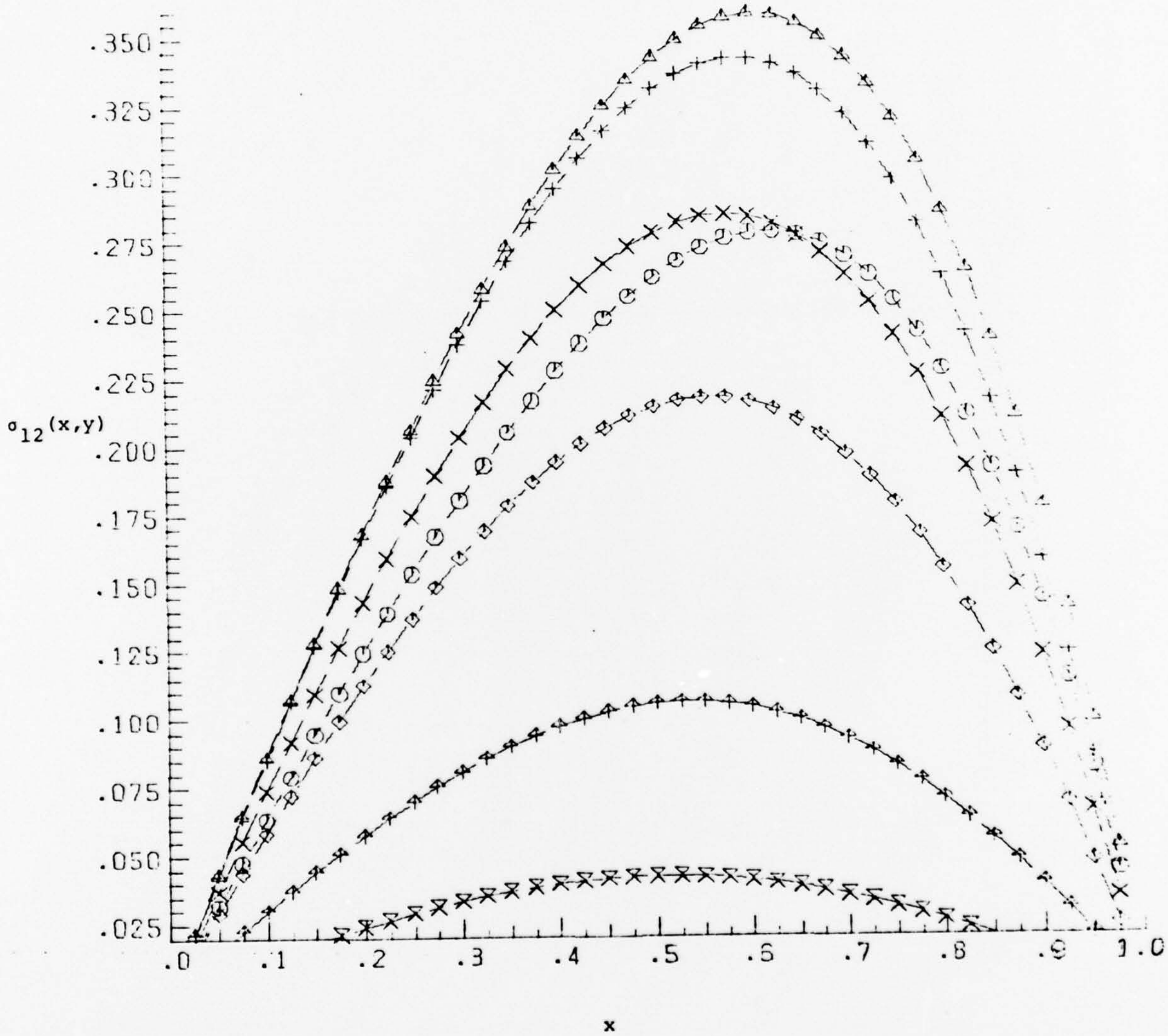


Figure 4

STRESS DISTRIBUTIONS FOR CASE 2: $(\sigma_{22})_0 = \pm 1$, $(\sigma_{12})_0 = 0$

(a) Direct Stress

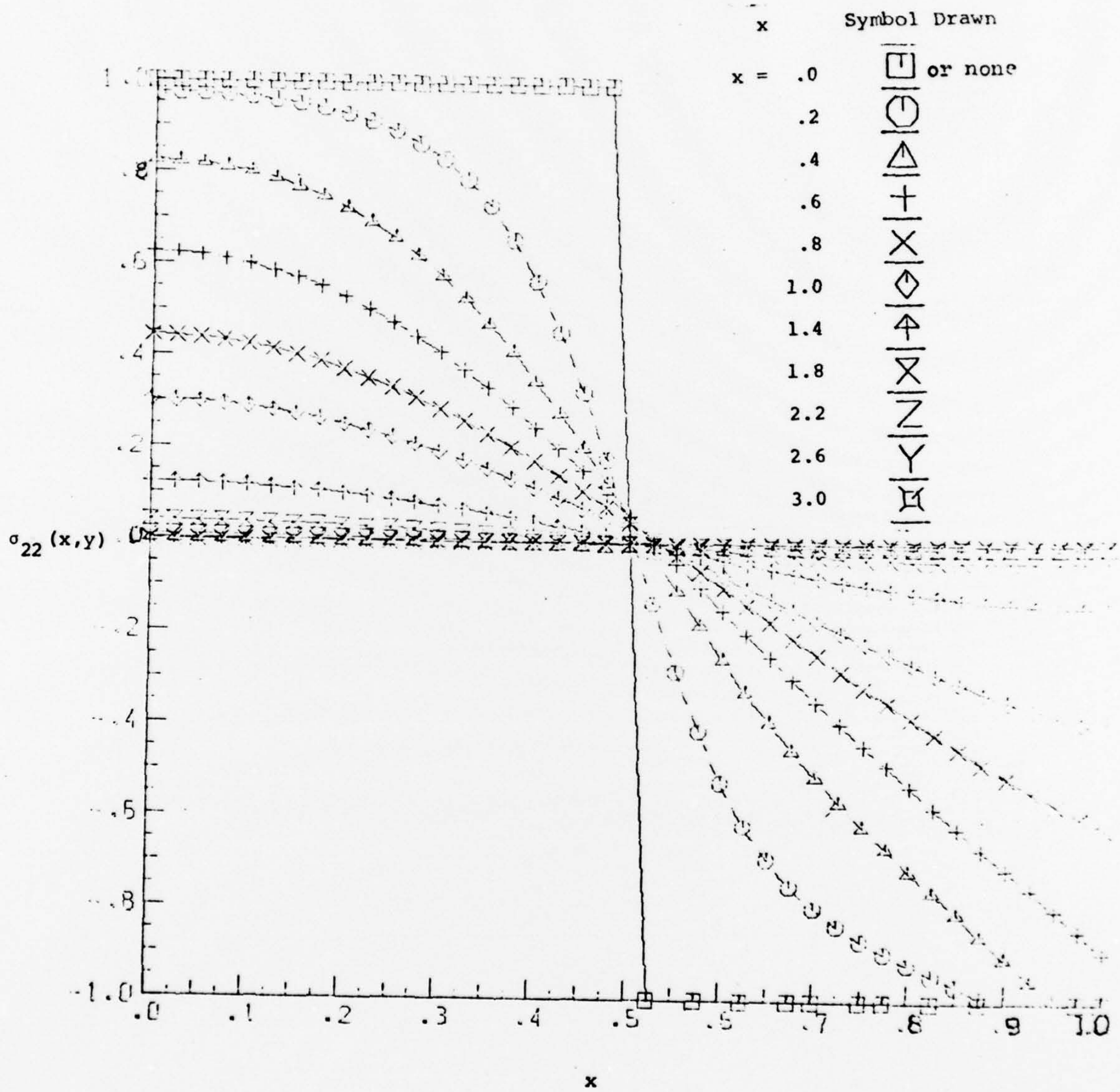


Figure 4

(b) Shear Stress

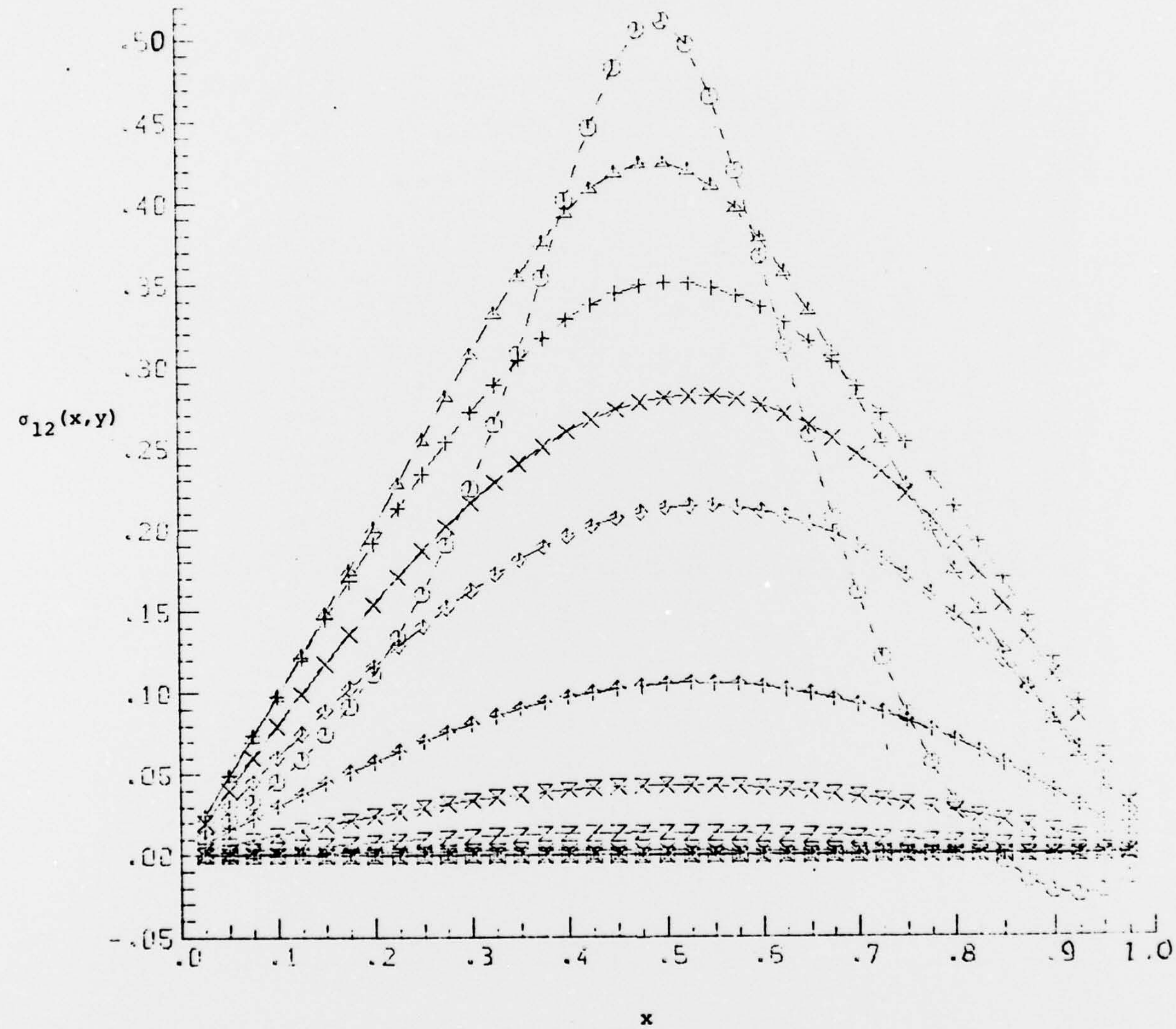


Figure 5

STRESS DISTRIBUTIONS FOR CASE 3: $(\sigma_{22})_0 = 0$, $(\sigma_{12})_0 = x - x^3$

(a) Direct Stress

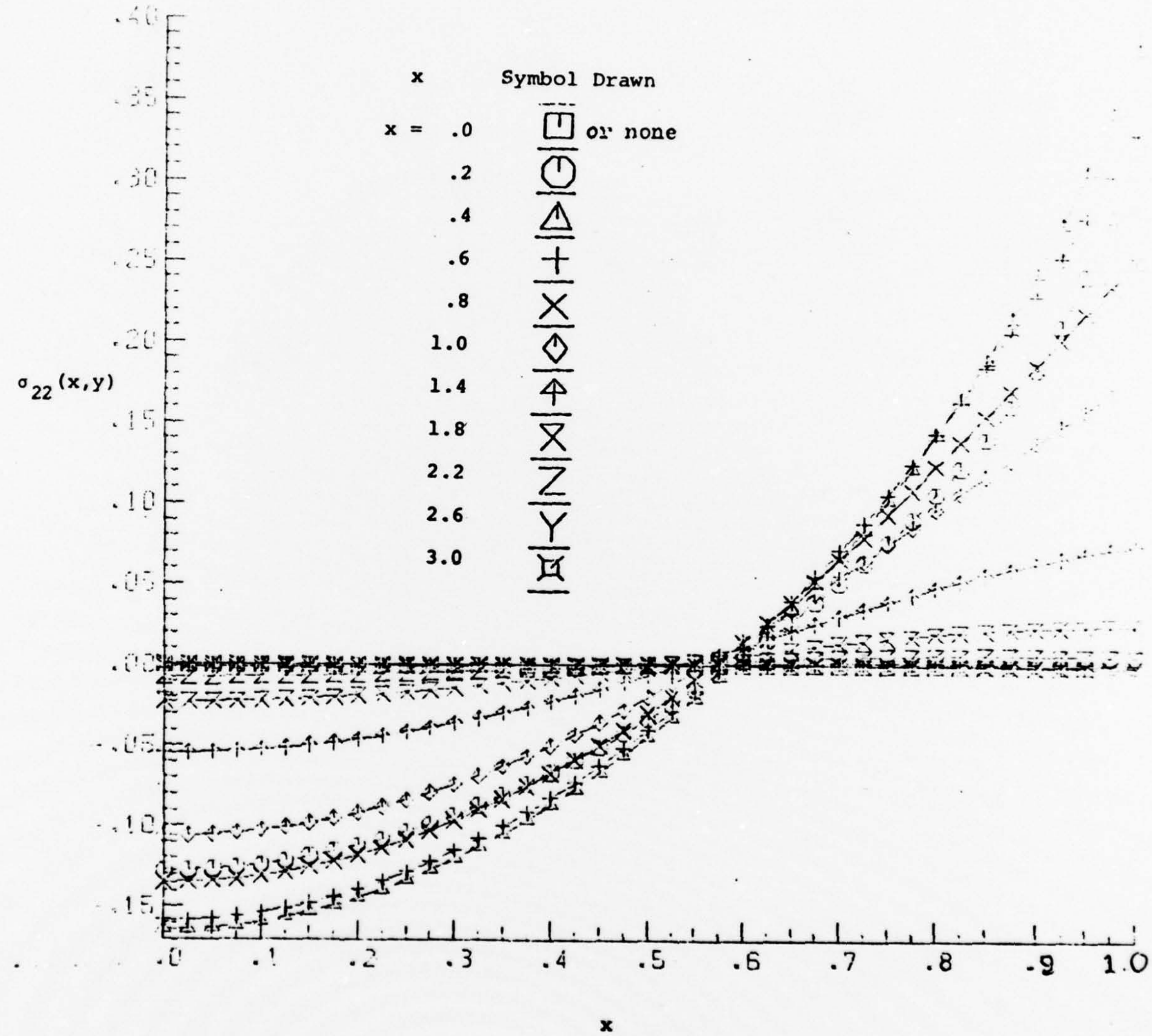
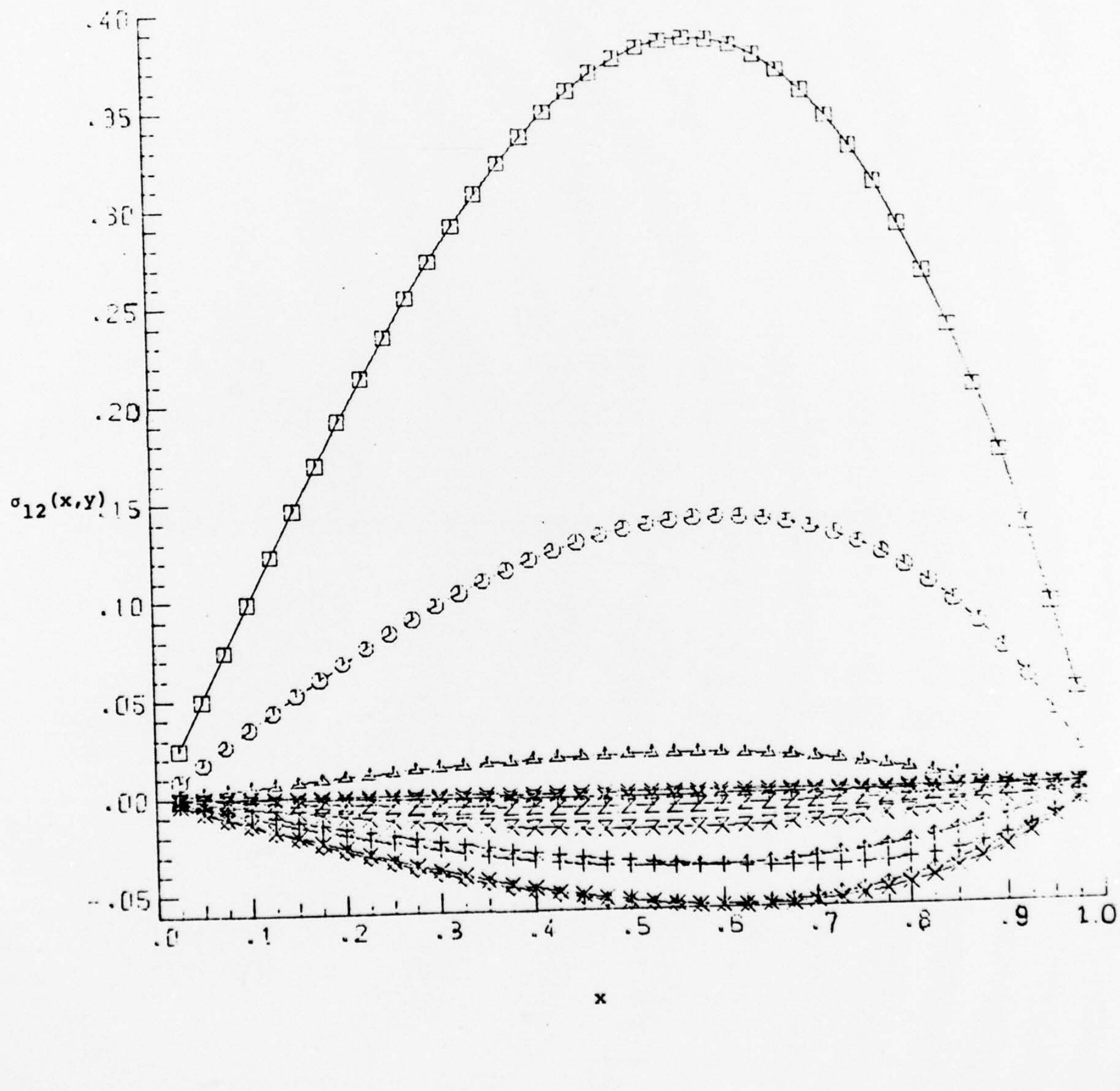


Figure 5

(b) Shear Stress



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The stress function for a semi-infinite elastic strip with free edges is expanded in eigenfunctions of Papkovitch-Fad'le type, and biorthogonal functions are constructed by use of the adjoint operator. To establish completeness, the same expansion is obtained by a Fourier transform solution of the biharmonic equation.

When end data is supplied in the form of prescribed tractions or prescribed displacements, the coefficients in the expansion must be found by truncation of an infinite set of linear equations. It is noted that the methods for formulating → next page

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Abstract (continued)

such equations that have been proposed in the past are unstable with respect to the order of truncation.

A systematic search among all possible weighting functions in the bi-orthogonal family for use in a Galerkin method leads to a unique choice of optimal weighting functions securing maximum stability by generating a diagonally-dominated matrix. Calculations are presented for a number of test cases using this method together with those of Benthem (1963) and Johnson and Little (1965), and with a method of direct collocation. The method of optimal weighting functions produces considerably more stable values for coefficients as the order of truncation is changed. This opens the way for an examination of the convergence of the eigenfunction expansions for data of this class, and should also prove useful for problems involving cracks.