

AD-A060 660

WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER

F/G 12/1

A POISSON STRUCTURE ON SPACES OF SYMBOLS.(U)

APR 78 W SYMES

DAAG29-75-C-0024

MRC-TSR-1847

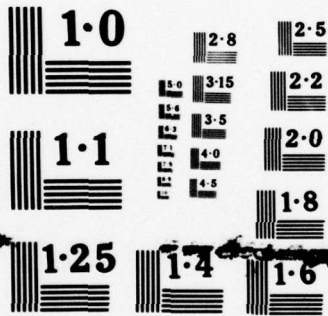
NL

UNCLASSIFIED

| OF |
ADA
060660



END
DATE
FILMED
1-79
DDC



NATIONAL BUREAU OF STANDARDS
MICROCOPY RESOLUTION TEST CHART

AD A 0 6 0 6 6 0

DDC FILE COPY

MRC Technical Summary Report # 1847

A POISSON STRUCTURE ON SPACES
OF SYMBOLS

W. Symes

Mathematics Research Center
University of Wisconsin-Madison
610 Walnut Street
Madison, Wisconsin 53706

*See back page
for 1473*

April 1978

(Received January 3, 1978)

DDC
RECEIVED
NOV 1 1978
REGISTERED
A

Approved for public release
Distribution unlimited

Sponsored by

U.S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, D. C.

20550

78 09 12 052

UNIVERSITY OF WISCONSIN-MADISON
MATHEMATICS RESEARCH CENTER

A POISSON STRUCTURE ON SPACES OF SYMBOLS

W. Symes

Technical Summary Report #1847

April 1978

ABSTRACT

ACCESSION NO.	
MCS	WRITE SECTION <input checked="" type="checkbox"/>
OND	OUT SECTION <input type="checkbox"/>
REPRODUCED	<input type="checkbox"/>
JUSTIFICATION	<input type="checkbox"/>
BY	
DISTRIBUTION/AVAILABILITY COPY	
DIST.	AVAIL. AND SPECIAL
A	

A Poisson structure (antisymmetric bilinear local operator on functionals, obeying the Jacobi identity) is established on certain function spaces (spaces of symbols of pseudodifferential operators on \mathbb{R}^n). The spaces of functionals thus become (infinite-dimensional) Lie Algebras. This type of Lie algebra structure has been established previously for functionals of functions of a single variable ($n = 1$) only. For $n = 1$, the theorem of Gardner, as generalized by Gel'fand, Dikii, and others, is proved: that is, the residues of the zeta-function of the elliptic symbol

$$\xi^{m+2} + \sum_{j=0}^m g_j(x) \xi^j$$

are in involution with respect to the appropriate Poisson bracket. In contrast, it is shown by explicit example that the residues of the zeta functions of higher-dimensional elliptic symbols are generally not in involution.

AMS (MOS) Subject Classification: 35A25

Key Words: Poisson Structures, Integrals in Involution, Korteweg-de Vries Equation, Calculus of Symbols

Work Unit Number 1 (Applied Analysis)

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS75-17385 A01.

78 09 12 052

SIGNIFICANCE AND EXPLANATION

In recent years several nonlinear partial differential equations of applied mathematics have been discovered to have the extraordinary property known as complete integrability: that is, roughly speaking, they possess the maximum number of constants of motion possible for the type of system considered. These equations arise in the study of shallow water waves, acoustic waves in plasmas (Korteweg-deVries equation) nonlinear optics, Josephson junction theory, (sine-Gordon equation), other plasma phenomena (nonlinear Schrödinger equation), and other areas. The complete integrability property - again, roughly speaking - allows unusually explicit solution of these equations.

This report is concerned with two other noteworthy properties of these equations. First, all of the above-mentioned partial differential equations involve only two independent variables, hence model systems whose spatial variation is essentially restricted to a single linear dimension: one independent variable thus represents space (location), and the other variable represents time, in applications. Second, each of the above-mentioned examples is, in some sense, a Hamiltonian system. Hamiltonian systems commonly occur in mechanics (both classical and quantum), and are characterized by the existence of special coordinates, called canonical coordinates, in the state space of the system, in which the equations of motion take a particularly simple form, called Hamilton's Equation. The examples of the first paragraph belong to continuum mechanics, hence manifest infinitely many degrees of freedom. Nonetheless, a set of canonical coordinates (infinitely many, of course) may be chosen for each of those examples, so that each partial differential equation, expressed in terms of these special coordinates, becomes a Hamilton's equation. It should be remarked that Hamiltonian systems are very special dynamical systems.

An obvious question is, whether the phenomenon represented by the examples of the first paragraph is somehow restricted to partial differential equations in two independent variables - or, in terms of applications, to systems with (essentially) one linear dimension. In approaching this question, one may choose to emphasize some aspects of the phenomenon over others. In this report, we concentrate on the Hamiltonian structure suggested by our examples, especially the Korteweg-deVries equation. We show that the Hamiltonian structure is independent of the number of spatial dimensions: that is, we give canonical coordinates on spaces of functions of arbitrarily many (space) variables, so that the Hamiltonian way of writing the Korteweg-deVries equation appears as a special case.

Thus Hamiltonian systems of a type represented by the examples of the first paragraph, are present in any number of (space) dimensions. In contrast, the complete integrability property (which should be considered more special than the Hamiltonian property) seems to fail in dimension greater than one. Precisely, we give an example in which the rather obvious generalizations of the constants of motion for the Korteweg-deVries equation (dim. 1) fail to be constant in dimension greater than one. This is certainly not to say that there are no partial differential equations in many independent variables which represent completely integrable Hamiltonian systems, or even that the Hamiltonian systems constructed in this paper do not have other, more cleverly chosen, constants of motion - only that the obvious choices fail to work. The significance of this apparent counterexample, and, more generally, the importance of Hamiltonian systems of Korteweg-deVries type in modeling phenomena with several independent spatial dimensions, is unclear, and suggests the need for further study.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

W. Symes

§1. Introduction

The purpose of this paper is to explore a Poisson structure on certain function spaces (truncated symbols of nonnegative order), which arose in a search for a multidimensional generalization of the Korteweg-deVries equation. The search must be declared a failure, in the sense that no isospectral deformations of higher-dimensional linear differential operators were found (it is in this connection that the Korteweg-deVries equation is mentioned here). However, the entire formal Hamiltonian apparatus, which has been associated to the Korteweg-deVries and other "completely integrable" systems, does appear in all dimensions of the underlying Euclidean space, and we develop it here.

The theorems of Lax-Gardner-Gel'fand-Dikii, concerning the Hamiltonian nature and existence of local constants of motion for various isospectral deformations of linear ordinary differential operators, are immediate products of the machinery developed below. In particular, the proof of a certain crucial Jacobi identity, which is not even given in [3] in view of the "awkwardness" of the computations, becomes easy in our framework.

On the other hand, the impossibility of finding, within the mechanical structures developed here, systems with "spectral" constants of motion when the dimension of the underlying Euclidean space is greater than one, also becomes particularly clear. We have certainly not proven that there can be no isospectral deformations of linear partial differential operators, governed by systems of partial (or pseudo-) differential equations. What we have shown is that certain simple mechanical systems, which contain all of the known isospectral flows in one dimension, when generalized in the obvious way, fail to generate further examples of this phenomenon.

We should remark that, on the one hand, isospectral deformations of pseudo-differential operators, cum local conservation laws, abound in any number of dimensions; and that, on the other hand, there is some feeling that the paraphernalia of completely integrable systems

in one dimension, especially solitons and the like, ought to be peculiar to one-dimensional problems. Indeed, the only published studies of multi- (two-) dimensional isospectral flows to come to the author's attention (all having to do with the Kadomtsev-Petvishvili equation), are, in the rather crude aspect considered here, merely glorified one-dimensional constructions.

The immediate inspiration for this work was the article [3] by I. M. Gel'fand and L. A. Dikii, in which a formal version of the pseudo differential operator calculus was used to give a particularly elegant construction of many (in fact, all) isospectral deformations of scalar ordinary differential operators, generalizing and clarifying previous work of Lax [4] and Gardner [2]. They formulate local spectral conservation laws (residues of zeta function) for these deformations in the framework of a formal variational calculus. Our approach is a straightforward generalization of theirs, except that we give a smoother treatment of the Jacobi identity for the Poisson bracket.

We now describe the organization of this paper, and explain some of the main terms in it.

A Poisson structure is a prescription of a Lie algebra structure (Poisson bracket) on some space of functions on a manifold. Thus a Poisson structure provides the necessary means to do Hamiltonian mechanics, although - as we explain - the notion is slightly weaker than that of symplectic structure, which is usually the framework of mechanics. Simple facts about garden-variety Poisson structures on finite-dimensional smooth manifolds are explained in §2. Our Poisson structures are constructed from certain Lie algebras of vector fields; this construction is also explained in §2.

The "phase" spaces of our systems are collections of truncated formal symbols. A formal symbol is, roughly speaking, an asymptotic series of a symbol of Hörmander's class $S_{1,0}^m(\mathbb{R}^n)$ (see [6] Ch. 2). Formal symbols are subject to a formal version of the symbol calculus of pseudodifferential operators, which we call symbol algebra, and which we explain in §3.

In §§4 and 5 we give a formal version of the construction, due to Seeley [5], of a zeta-function for elliptic symbols. This meromorphic function is not actually well-defined in our setting, but its residues are, and we compute these. The residues provide important examples and motivation for further constructions; they are integrals of local densities.

In §6 we present the rudiments of an exterior ("variational") calculus on symbol spaces, suitable for computing derivatives of various functionals on symbol spaces, of which the above-mentioned residues are prime examples (§7). The point is that one wants to define differentiation at the level of densities, in order to avoid the imposition of boundary conditions irrelevant to our results, which have a local, algebraic character.

The results of §§5 and 7 motivate the introduction of a projection functional, which in effect establishes pairings between various truncated symbol spaces, and enables us to replace differentials with "gradients". This functional, which is analogous to the trace functional of matrices, was introduced by Adler for dimension one ([1]); he calls it "the trace". We prefer a different name.

In §9 we discuss the algebra of Lax vector fields. These are infinitesimal similarities on spaces of symbols, hence naturally have an isospectral property.

In §10 we introduce Poisson vector fields, show that they form a closed algebra under Lie bracket (Thm 10.2) and hence define a Poisson structure, according to the model of §2. This is the main result of the paper. These Poisson Vector Fields are vector fields on the spaces \mathfrak{S}_+^m of symbols consisting of terms of non-negative integral order $\leq m$ only, which include symbols of differential operators.

In §11 we give a brief treatment of the above-mentioned results of Gel'fand-Dikii. The main step is to notice that the Poisson vector fields corresponding to the residues of the zeta function are in fact Lax vector fields, a circumstance peculiar to dimension one. From this observation it follows trivially that the residues are in involution, i.e. their Poisson brackets vanish in the sense of the formal variational calculus.

In §12 we give an explicit computation which shows that the residues of the zeta-function are generally not in involution in dimension greater than one. It is clear that

the argument of §11 must fail in higher dimensions, since it depends crucially on the fact that, in dimension one, homogeneous functions of degree zero are (essentially) constant. It was not so apparent, however, that the result would fail, too.

The principal shortcoming of this paper is that Poisson vector fields are introduced more-or-less out of the blue, so that their key property (Thm 10.2) seems miraculous. In fact, for $n = 1$ Mark Adler has shown in his recent paper [1] that the Poisson structures considered here may be identified as the Kostant-Kirillov symplectic structure on orbits of the coadjoint action of a certain formal Lie group. His paper thus contains all of the results of the present paper for $n = 1$. We have no doubt that his construction, carried out for dimension one in [1], is also possible in the higher-dimensional setting of this paper, and would reveal the result of Thm 10.2 as inevitable.

I take great pleasure in thanking Mark Adler for many very illuminating conversations, and Charles Conley for the privilege of presenting some of this material in his dynamical systems seminar at the University of Wisconsin-Madison.

§2. Poisson Structures

In this section we give the simple properties of Poisson structures. In particular, we show how to construct Poisson structures from certain Lie algebras of vector fields, which we call Poisson vector fields.

In §10 we shall develop Poisson structures on function spaces. Here we discuss Poisson structures on smooth finite-dimensional manifolds instead; these remarks will be used as a guide to the more involved function-space constructions.

A Poisson structure on a smooth manifold M is an antisymmetric bilinear map $C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ (= real-valued smooth functions on M), denoted by brackets $\{ \}$, for which the Jacobi identity holds:

$$0 = \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} .$$

We postulate also the locality condition: if g vanishes to second order at $x \in M$, then

$$\{f, g\}(x) = 0$$

for all $f \in C^\infty(M)$. Thus the map

$$g \mapsto \{f, g\}$$

is effected by a differential operator of first order, which we shall shortly name.

The main example of a Poisson structure is that associated to a symplectic structure. A manifold M is said to have a symplectic structure if nondegenerate closed 2-form ω is defined on M ; the symplectic structure is the pair (M, ω) . Suppose such a structure is given; for any $f \in C^\infty(M)$, define the Hamiltonian vector field X_f by

$$X_f \lrcorner \omega = df .$$

The vector field X_f is well-defined because ω is nondegenerate. For $f, g \in C^\infty(M)$, set

$$\{f, g\}_\omega = \omega(X_f, X_g)$$

The map $\{ \}_\omega : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ defined in this way is obviously antisymmetric and bilinear, and the Jacobi identity is implied by $d\omega = 0$; hence a symplectic structure

(M, ω) generates a Poisson structure.

Suppose on the other hand that M is provided with a Poisson bracket $\{ \}$. The equation

$$\{f, g\} \equiv X_f \cdot g = dg(X_f) \quad (2.1)$$

then defines a vector field X_f for each $f \in C^\infty(M)$, by declaring the action of X_f on an arbitrary smooth function $g \in C^\infty(M)$. (The locality condition ensures that X_f is a vector field.) Suppose that the collection $\{X_f(x) : f \in C^\infty(M)\}$ spans the tangent space $T_x M$ for each $x \in M$. Then the form ω (well-) defined by

$$\omega(X_f, X_g) \equiv \{f, g\}$$

is antisymmetric, bilinear, and - by virtue of Jacobi's identity - closed. Hence a Poisson structure also gives rise to a symplectic structure, provided that there are "enough" vector fields of the form (2.1).

It is not at all obvious that there are always "enough" vector fields: in fact, the Poisson structures developed in this paper do not give rise to symplectic structures in this way, since the collections $\{T_f(x) : f \in C^\infty(M)\}$ demonstrably do not span the various tangent spaces. Nonetheless, we can still "do mechanics" with such "degenerate" Poisson structures, since Hamiltonian vector fields are still defined by (2.1).

The Poisson structures of this paper are constructed in the following way. For each $f \in C^\infty(M)$, a vector field $\mathcal{D}f$ on M is assigned. This complex-linear assignment is required to have the following two properties, each of which is necessary for $\mathcal{D}f$ to be the "Hamiltonian" vector field constructed from f by way of a Poisson bracket, as outlined above:

- (i) The distribution in TM , spanned by $\{\mathcal{D}f : f \in C^\infty(M)\}$ must be integrable;
- (ii) for $f, g \in C^\infty(M)$, $[\mathcal{D}f, \mathcal{D}g] = \mathcal{D}h$ with $h = dg(\mathcal{D}f) = -df(\mathcal{D}g)$.

(Of course (i) follows from (ii).)

Proposition 2.1. Under hypotheses (i) and (ii), the assignment

$$C^\infty(M) \ni f, g \mapsto \{f, g\} \equiv dg(\delta f)$$

defines a Poisson structure on M .

Proof. The antisymmetric bilinearity of $\{, \}$, so defined, is obvious from (ii). The locality property is equally obvious, since g vanishes to second order at $x \in M$ means precisely that $dg(x) = 0$. As for the Jacobi identity, for $f, g, h \in C^\infty(M)$,

$$\begin{aligned} & \{f, \{g, h\}\} + \{h\{f, g\}\} + \{g\{h, f\}\} \\ &= d\{g, h\}(\delta f) + d\{f, g\}(\delta h) = d\{h, f\}(\delta g) \\ &= d(dh(\delta g))(\delta f) + d(dg(\delta f))(\delta h) + d(df(\delta h))(\delta g) \\ &= -df(\delta dh(\delta g)) + d(df(\delta h))(\delta g) - d(df(\delta g))(\delta h) \\ &= -df([\delta g, \delta h]) + d(df(\delta h))(\delta g) - d(df(\delta g))(\delta h) \\ &= 0 \end{aligned}$$

where we have repeatedly used all of (ii).

q.e.d.

§3. Symbol Algebra

In this section we review the formal symbol calculus of pseudodifferential operators. The basic objects are certain formal series. We point out that these formal series may be regarded as asymptotic series of symbols of class $S_{0,1}^M$, as explained in Ch. 2. of [6].

By a symbol we shall mean a formal sum

$$A = \sum_{\ell=0}^{\infty} A_{\ell}(x, \xi) \quad (3.1)$$

where A_{ℓ} is a smooth complex-valued function of $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ homogeneous of degree d_{ℓ} in ξ , with

$$d_0 > d_1 > d_2 > \dots \rightarrow -\infty$$

a sequence of real numbers. The collection of such formal sums, denoted by \mathfrak{S} , forms a vector space over \mathbb{C} , and a module over $C^{\infty}(\mathbb{R}^n)$ (functions of x), with the operations defined pointwise and term-by-term. \mathfrak{S} forms an algebra with the product

$$A \circ B = \sum_{|\nu| \geq 0} \frac{1}{\nu!} \partial^{\nu} A D^{\nu} B \quad (3.2)$$

Here

$$\partial = \left(\frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_n} \right)$$

$$D = \left(-i \frac{\partial}{\partial x_1}, \dots, -i \frac{\partial}{\partial x_n} \right), \quad i = \sqrt{-1}$$

and we have used the common multi-index notation for partial derivatives.

The sum on the right hand side of (3.2) is not in the form (3.1); however, since only finitely many terms appear of each homogeneous degree in the term-by-term product inside the summation, rearrangement into a unique sum of form (3.1) is possible, and (3.2) thus defines a symbol.

The order of a symbol A is the homogeneous degree of the sum and of highest homogeneous degree appearing in (3.1) - that is, $\text{ord } A = d_0$.

Note that

$$\text{ord } A \circ B = \text{ord } A + \text{ord } B \quad .$$

The symbol $1(x, \xi) \equiv 1$ is the multiplicative identity for this algebra. The symbol commutator $[A, B]$ is defined by

$$[A, B] = A \circ B - B \circ A .$$

Clearly

$$\text{ord } [A, B] \leq \text{ord } A + \text{ord } B - 1 .$$

Note for future reference that the following two subsets of \mathfrak{S} form subalgebras:

1) The collection \mathfrak{S}^m of symbols of order $\leq m$ (m an integer) for which the degrees d_ℓ appearing in the definition are all integral;

2) The collection $\mathfrak{S}^m(d)$ of symbols in \mathfrak{S}^m , $d_0 = m$, $d = (d_0, \dots, d_k) \in \mathbb{Z}^{k+1}$, $k \leq \infty$, $d_j > d_{j+1}$ for $j = 0, \dots, k-1$ whose summands are polynomials in a collection $\vec{Q} = (Q_0(x, \xi), \dots, Q_k(x, \xi))$ of smooth functions (Q_j homogeneous in ξ of degree d_j) and their derivatives, with coefficients which are polynomial in ξ , $|\xi|$, and $|\xi|^{-1}$.

The collection defined by 2), as \vec{Q} runs over all possible collections (Q_0, \dots, Q_k) of such functions, will be called the algebra of Q -symbols. This algebra appears quite naturally, as will be noted in the next section. Observe that a Q -symbol may be considered a function of the symbol

$$Q = \sum_{j=0}^k Q_j$$

so that $\mathfrak{S}^m(d)$ is a class of \mathfrak{S}^m -valued functions on the space of symbols of this type.

§4. Resolvent Symbols and complex powers of elliptic symbols

In this section we compute resolvents and complex powers of certain elliptic symbols. The results of this and the next section are formal versions of Seeley's work [5]. The proofs are there also, so we omit them.

Let $Q \in \mathfrak{S}^m$, and denote by L the symbol

$$L = |\xi|^{n+m+1} + Q$$

(the reasons for the exponent $n+m+1$ will be explained later). L is an elliptic symbol: that is, its principal part (highest order summand) is nonvanishing for $\xi \neq 0$.

The resolvent symbol $R(\lambda)$ of L is the (symbol) inverse of $L - \lambda$. The resolvent symbol exists for λ not nonnegative real. To see this, write

$$\begin{aligned} 1 &= R(\lambda) \circ (L - \lambda) = R(\lambda) \circ (|\xi|^{n+m+1} - \lambda + Q) \\ &= (|\xi|^{n+m+1} - \lambda)R(\lambda) + R(\lambda) \circ Q \end{aligned}$$

in the form

$$R(\lambda) = (|\xi|^{n+m+1} - \lambda)^{-1} - (|\xi|^{n+m+1} - \lambda)^{-1} R(\lambda) \circ Q.$$

It follows easily from this that $R(\lambda)$ exists for $\lambda \in \mathbb{C}$ $\{\lambda \geq 0\}$, is a Q -symbol, and admits an expansion of the form

$$R(\lambda) = \sum_{\ell=n+m+1}^{\infty} R_{\ell}(\lambda) \tag{4.1}$$

where $R_{\ell}(\lambda)$ is homogeneous in ξ and $\lambda^{\frac{1}{n+m+1}}$ of degree $-\ell$. In fact,

$$R_{\ell} = \sum_{k=2}^{\ell-n-1} B_{\ell k} (|\xi|^{n+m+1} - \lambda)^{-k} \tag{4.2}$$

where $B_{\ell k}$ is a polynomial in ξ and x -derivatives of finitely many terms of $Q = \sum_{p=0}^{\infty} Q_p$, homogeneous of degree $-\ell + k(n+m+1)$ in ξ . The $B_{\ell k}$ are computed by a rather messy recursion; see §12 for sample computations.

Note that (4.1) does not have the form of the definition (3.1). Nonetheless, rearrangement into the form (3.1) is easy, using (4.2) and the obvious expansion of $(|\xi|^{n+m+1} - \lambda)^{-k}$

in powers of $|\xi|$; thus $R(\lambda) \in \mathfrak{S}^{-n-m-1}(d)$, $d = (m, m-1, m-2, \dots)$.

It is easy to show that R satisfies the usual resolvent equation: that is,

$$R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda) \circ R(\mu) .$$

On the basis of this properly, we mimic the usual construction of functions of L via the resolvent and Cauchy's formula. We wish in particular to define L^s for $s \in \mathfrak{C}$.

We would like to write

$$L^s = \frac{1}{2\pi i} \oint d\lambda \lambda^s R(\lambda)$$

where the integration is over some suitable contour but the formal nature of the objects we have called symbols renders this formula meaningless without interpretation. The following procedure works.

In the remainder of this paper, understand $\lambda^s = \exp(s \log \lambda)$, where the principal branch of \log is selected with the branch cut down the negative imaginary axis.

For $\operatorname{Re} s < -1$, set

$$A_p(s, \xi) = \frac{1}{2\pi i} \oint d\lambda \lambda^s R_{n+m+1+p}(\lambda, \xi) \quad (4.3)$$

where the contour is, say, up $\operatorname{Re} \lambda = \frac{1}{2}$ from $-i\infty + \frac{1}{2}$, around the upper semicircle $|\lambda| = \frac{1}{2}$, and down $\operatorname{Re} \lambda = -\frac{1}{2}$. For $|\xi| > 1$, this integral converges absolutely, and $A_p(s, \xi)$ is homogeneous in ξ , $|\xi| > 1$, of (complex) degree $s(n+m+1) - p$ (the introduction of complex degree is a slight broadening of the class of symbols, but leads to no complications). Extend $A_p(s, \xi)$ to all non-zero ξ by homogeneity, and define

$$L^s = \sum_{p=0}^{\infty} A_p(s, \xi) . \quad (4.4)$$

The contour integral (4.3) may be evaluated by the residue theorem for $\operatorname{Re} s < -1$; and the result is

$$A_p(s, \xi) = \sum_{k=2}^{n+p} B_{n+m+1+p, k}(\xi) (-1)^k \binom{s}{k-1} |\xi|^{(n+m+1)(s-k-1)} \quad (4.5)$$

where

$$\binom{s}{\ell} = \frac{1}{\ell!} \prod_{j=0}^{\ell-1} (s-j) .$$

The formula (4.5) allows continuation of $A_p(s, \xi)$ to an entire function of s , and consequently defines L^s for any complex s by means of (4.4). Note that the A 's inherit from the B 's the property of depending polynomially on Q and derivatives, ξ , and $|\xi|^{\pm 1}$; thus L^s is a Q -symbol.

It is clear from (4.5) that

$$L^0 = 1 \quad L^1 = L .$$

Furthermore, the resolvent equation implies, (via) a mimic of the usual argument (see [5]) that

$$L^s L^t = L^t L^s = L^{s+t} .$$

In particular, $\{L^s : s \in \mathbb{C}\}$ is a commuting family of symbols.

§5. Residue Densities of the Zeta-Function

Suppose L is a self-adjoint operator on a Hilbert space, semibounded below and with a purely discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$. Then the zeta-function of L is defined by

$$\zeta(s) \equiv \text{tr } L^s = \sum_j \lambda_j^s$$

provided that L^s is of trace class for s in some open domain in the complex plane.

We will define something similar to the zeta-function for the symbol L , following Seeley. Of course, our symbols are not even operators, let alone self-adjoint, and in fact there will be no well-defined zeta-function. The "poles" and "residues" will, however, be well-defined: the residues will have values which are functions of x , and represent residues of a zeta-function in the following sense. Suppose that the open set $U \subset \mathbb{R}^n$ is fixed, and suppose that L^s is a family of pseudodifferential operators on U whose symbols (in the usual sense) have asymptotic expressions given by the local expressions (4.3) - (4.5) which define L^s , and suppose that the symbol of L is required to be sufficiently tame at the boundary of U (say, vanishing to infinite order at infinity if $U = \mathbb{R}^n$). Then L^s will be a trace-class operator on various Sobolev spaces for $\text{Re } s \ll 0$, and the zeta function $\text{tr } L^s$ will have residues which are integrals over U of the "residues" computed below. Thus we are computing local densities of residues of the zeta-function of an elliptic pseudodifferential operator L ; it is the remarkable result of Seeley that these residue densities depend only on the asymptotic series L , and are independent of the boundary behaviour of L .

To fix ideas, suppose that K is a trace-class integral operator on $L^2(\mathbb{R}^n)$, with kernel $K(x,y)$. Then

$$\begin{aligned} \text{tr } K &= \int dx K(x,x) \\ &= \int dx \text{diag } K \end{aligned}$$

On the other hand, a pseudodifferential operator of order $< -n$, with symbol $p(x,\xi)$, is an Carleman integral operator with kernel

$$K_p(x, y) = \frac{1}{(2\pi)^n} \int d\xi e^{i\xi \cdot (x-y)} p(x, \xi) .$$

This leads us to define the diagonal of a symbol p by the formula

$$\text{diag } p(x) = \frac{1}{(2\pi)^n} \int d\xi p(x, \xi)$$

provided that the right-hand side exists. Equally well, we define the diagonal of an (asymptotic) symbol $A \in \mathcal{S}^m$ by

$$\text{diag } A(x) = \frac{1}{(2\pi)^n} \int_{|\xi| > 1} d\xi A(x, \xi) \quad (5.1)$$

again provided that the right-hand-side makes sense - term-by-term, of course, so $\text{diag } A$ is a formal sum of functions of x . This of course makes no sense in general, so we define $\text{diag } A$ by (5.1) only for those symbols A which are finite sums of the form (3.1). We define $\text{diag } A$ by an integral excluding the origin, of course. We wish to isolate that aspect of the diagonal of a pseudodifferential operator which depends on the asymptotic expansion of its symbol, and is therefore independent of the behaviour of its symbol near the origin.

Thus set

$$L_N^s = \sum_{p=0}^N A_p(s, \xi) .$$

For $\text{Re } s < -1$, $\text{diag } L_N^s$ exists and is given by

$$\begin{aligned} & \sum_{p=0}^N \int_{|\xi| > 1} d\xi A_p(s, \xi) \\ &= \sum_{p=0}^N \int_{S^{n-1}} d\omega A_p(s, \omega) \int_1^\infty dr r^{n-1+s(n+m+1)-p} \\ &= - \sum_{p=0}^N \left(\int_{S^{n-1}} d\omega A_p(s, \omega) \right) \frac{1}{(n+m+1)s-p+n} . \end{aligned}$$

Thus $\text{diag } L_N^s$ is a meromorphic function of s with poles at $s = \frac{p-n}{n+m+1}$, $p = 0, \dots, N$, and residue

$$\frac{-1}{n+m+1} \int_{S^{n-1}} d\omega A_p \left(\frac{p-n}{n+m+1}, \omega \right).$$

The poles and residues are obviously stable under increase of N , so we summarize this information in

$$\begin{aligned} \text{Res} \quad \text{diag } L^s \\ \Big|_{s = \frac{p-n}{n+m+1}} &= \frac{-1}{n+m+1} \int_{S^{n-1}} d\omega A_p \left(\frac{p-n}{n+m+1}, \omega \right) \end{aligned} \quad (5.2)$$

bearing in mind that the left-hand side is a priori meaningless. This is the formal content of one of Seeley's theorems in [6], for the simple scalar "operators" considered here.

Remark 1. We note for future reference that $A_p \left(\frac{p-n}{n+m+1}, \xi \right)$ is the term in the symbol $L^s, s = \frac{p-n}{n+m+1}$, of homogeneous order $-n$ precisely.

§6. Formal Calculus of Variations

In this section we develop a formal calculus of variations, somewhat along the lines of [3], its predecessor articles, and many others, although with somewhat different emphasis - though there is certainly nothing new here. The point is, to regard functionals on spaces of vector-valued functions as "smooth" if they are integrals of local densities, and then to define directional derivatives (differentials) at the density level. At the same time, we wish to reflect the ultimate fate of these densities: to be integrated over some domain, with some sort of boundary conditions. This leads us to consider the quotient of the space of local densities by the subspace of those which are divergences of other local densities.

Suppose F is a functional of a vector of functions $Q = (Q_0, \dots, Q_m)$ of x , of the form

$$F[Q] = \int dx f(Q_0, \dots, Q_m, \dots)$$

where f is a polynomial in Q_0, \dots, Q_m and their partial derivatives. The differential δF is a functional of pairs (Q, \dot{Q}) , which we think of as tangent vectors, defined by

$$\begin{aligned} \delta F[Q, \dot{Q}] &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (F[Q + \epsilon \dot{Q}] - F[Q]) \\ &= \int dx \delta f(Q_0, \dots, Q_m, \dot{Q}_0, \dots, \dot{Q}_m, \dots) \end{aligned}$$

for a suitable polynomial δf in Q, \dot{Q} , and partial derivatives, which we call the differential of the density f .

δf is computed according to the rule

$$\delta f(Q, \dot{Q}) = \sum_{\substack{j=0 \\ |v| \geq 0}}^m D^v Q_j \frac{\delta f}{\delta (D^v Q_j)} .$$

Remark 1. None of this makes sense unless restrictions are imposed on Q, \dot{Q} , of the nature of boundary conditions. For instance, if the domain of Q is \mathbb{R}^n , the components Q_0, \dots, Q_n can be required to vanish to infinite order at infinity; or, the definition of $F[Q]$ may be made in an invariant way, so that Q may be regarded as a function on a compact manifold.

The point is, these boundary conditions play no role in the computation of δf . Boundary conditions, which must be introduced as remarked above, lead us to introduce a local equivalence relation among densities and their differentials, as follows.

Suppose for instance that F is regarded as a functional on the space of functions Q vanishing to infinite order at infinity in \mathbb{R}^n . Then we should identify densities $f dx$ which differ by an exact form $d\phi$, where the $(n-1)$ form $\phi = \phi[Q]$ depends polynomially on Q and derivatives, since then $f dx$ and $f dx + d\phi$ give the same value F upon integration, according to Stoke's theorem. Thus, we should identify the aggregate of functionals $F[Q]$ with the quotient

$$\Omega^n[Q]/d\Omega^{n-1}[Q] \equiv I[Q]$$

where $\Omega^p[Q] = p$ -forms whose coefficients depend polynomially on Q and derivatives.

Similar considerations apply to the compact manifold case.

More conventionally, two polynomials f and g in Q and derivatives yield the same functional upon integration iff they differ by a divergence $D \cdot \vec{\phi}$, where $\vec{\phi} = (\phi_1, \dots, \phi_n)$ depends polynomially on Q and derivatives, and in this case f and g should be identified. Thus we can also identify I with the quotient

$$I = \frac{\mathbb{C}[Q, DQ, D^2Q, \dots]}{D \cdot \mathbb{C}^n[Q, DQ, \dots]}$$

Similarly, we identify differentials of densities f which differ by appropriate divergences. We shall write $f \stackrel{\cdot}{=} g$ (following Adler) when densities f and g differ by a divergence of a vector polynomial in their arguments.

The point is, the equivalence $\stackrel{\cdot}{=}$ is purely local and algebraic, although its introduction is motivated by global considerations.

The preceding considerations apply almost without modification in case Q depends on an auxiliary parameter ω varying over some smooth parameter set S . The case of interest for this paper is when $S = S^{n-1} \subset \mathbb{R}^n$ is the unit sphere. We consider functionals F of $Q = (Q_0(x, \xi) \dots Q_m(x, \xi))$ of the form

$$F(Q) = \int_{\mathbb{R}^n} dx \int_{|\xi|=1} d\xi f[\xi, Q]$$

where f is a polynomial in ξ and the derivatives $D^{\nu} \partial^{\mu} Q_j$, $|\mu|, |\nu| \geq 0$, $j = 0, \dots, m$. Denote the class of such densities by \mathfrak{F} .

We write $f \stackrel{\cdot}{=} g$ for densities of this type iff

$$f - g = D \cdot \vec{\phi}$$

where ϕ is a vector of densities in \mathfrak{F} ; densities equivalent in this sense give the same functional integrated over $\mathbb{R}^n \times S^{n-1}$, provided this is possible. Define

$$\delta f(Q, \dot{Q}) = \sum_{j=0}^m \sum_{|\mu|, |\nu| \geq 0} D^{\nu} \partial^{\mu} \dot{Q}_j \frac{\partial f}{\partial (D^{\nu} \partial^{\mu} Q_j)} .$$

Make a similar definition of $\stackrel{\cdot}{=}$ for densities depending on Q, \dot{Q} . Then we have the following important (but trivial)

Lemma 6.1. Suppose f and g are densities in \mathfrak{F} , and $f \stackrel{\cdot}{=} g$. Then $\delta f \stackrel{\cdot}{=} \delta g$. Moreover, if $X(Q)$ is a vector of polynomials $X = (X_0, \dots, X_m)$ in ξ and $D^{\nu} \partial^{\mu} Q_j$, $|\mu|, |\nu| \geq 0$, $j = 0, \dots, m$, and we define

$$X \cdot f(Q) \equiv \delta f(Q, X(Q))$$

and similarly for g , then

$$X \cdot f \stackrel{\cdot}{=} X \cdot g .$$

Proof. We have $f - g = D \cdot \phi$ as above. We can write

$$\delta f(Q, \dot{Q}) = \left. \frac{d}{dt} f(Q+t\dot{Q}) \right|_{t=0}$$

whence

$$\begin{aligned} \delta f(Q, \dot{Q}) - \delta g(Q, \dot{Q}) &= \left. \frac{d}{dt} D \cdot \vec{\phi}(Q+t\dot{Q}) \right|_{t=0} \\ &= D \cdot \left. \frac{d}{dt} \vec{\phi}(Q+t\dot{Q}) \right|_{t=0} \\ &= D \cdot \delta \vec{\phi}(Q, \dot{Q}) \Big|_{t=0} . \end{aligned}$$

To see the second part, set $\hat{Q} = X(Q)$.

q.e.d.

Thus the exterior derivative δ is well-defined on the quotient

$$\mathcal{J} = \mathcal{F}/D\mathcal{F}^n$$

of \mathcal{F} by the equivalence relation $\dot{=}$. If we think of $Q \mapsto (Q, X(Q))$ as a vector field on the space of functions Q , we can interpret the second part of the lemma by saying that such vector fields act on \mathcal{J} . This observation will be important.

§7. Exterior derivative of Zeta Function Residues

According to (5.2) and the results of §4, the functions

$$T_p(L) \equiv \operatorname{Res}_{s = \frac{p-n}{n+m+1}} \operatorname{diag} L^s$$

are each polynomial in finitely many summands Q_ℓ of $Q = \sum Q_\ell \in \mathfrak{S}^m$ and their x -derivatives, with coefficients which are polynomial in ξ for $|\xi| = 1$. Thus T_p is a density of the type discussed in §6.

Following Gel'fand and Dikii, we derive an expression for the differential δT_p , or rather its equivalence class under the relation $\dot{=}$. The starting point is the following lemma, which will be useful in other contexts:

Lemma 7.1. Let A, B be symbols. Then

$$[A, B] = \partial \cdot \vec{\Psi} + D \cdot \vec{\Phi}$$

where

$$\vec{\Psi} = \sum_{\ell=0}^{\infty} \vec{\Psi}_\ell, \quad \vec{\Phi} = \sum_{\ell=0}^{\infty} \vec{\Phi}_\ell$$

are n -vector symbols whose summands depend polynomially on the summands of A and B and their x, ξ -derivatives.

The proof is an exercise in the Leibnitz rule.

Next, note that the resolvent symbol satisfies

$$\begin{aligned} 0 &= \{(R(Q+\varepsilon\dot{Q}) - R(Q, \lambda)) \circ (L(Q+\varepsilon\dot{Q}) - \lambda)\}^{-1} \\ &= (R(Q+\varepsilon\dot{Q}, \lambda) - R(Q, \lambda)) \circ (L(Q) - \lambda) \\ &\quad + R(Q, \lambda) \circ \varepsilon\dot{Q} + (R(Q+\varepsilon\dot{Q}, \lambda) - R(Q, \lambda)) \circ \varepsilon\dot{Q} \end{aligned}$$

(where we have emphasized the dependence of R on Q). Hence (multiplying by $R(Q, \lambda)$ on the right),

$$\begin{aligned} &\frac{1}{\varepsilon} (R(Q+\varepsilon\dot{Q}, \lambda) - R(Q, \lambda)) \\ &= -R(Q, \lambda) \circ \dot{Q} \circ R(Q, \lambda) \\ &\quad - (R(Q+\varepsilon\dot{Q}, \lambda) - R(Q, \lambda)) \circ \dot{Q} \circ R(Q, \lambda) \end{aligned} \quad (7.1)$$

Now for $\text{Re } s \ll 0$,

$$\begin{aligned} & \frac{1}{\varepsilon} \{ \text{diag } L_N^S(Q + \varepsilon \dot{Q}) - \text{diag } L_N^S(Q) \} \\ &= \frac{1}{(2\pi)^n} \int_{|\xi| \geq 1} \frac{d\xi}{2\pi i} \oint d\lambda \lambda^s \\ & \quad + \frac{1}{\varepsilon} \{ R(Q + \varepsilon \dot{Q}, \lambda) - R(Q, \lambda) \}_{N+n+m+1} \end{aligned} \quad (7.2)$$

according to (4.3). Here we have introduced the notation $\{A\}_t$ to mean the truncation of a symbol A to order $-t$, which in the case of the resolvent only means to order $-t$ in ξ and $\lambda^{\frac{1}{n+m+1}}$.

It is easy to show that, for fixed Q, \dot{Q} , $p \geq 0$, the integral

$$\int_{|\xi| \geq 1} \frac{d\xi}{2\pi i} \oint d\lambda \lambda^s \{ [R(Q + \varepsilon \dot{Q}, \lambda) - R(Q, \lambda)] \circ \dot{Q} \circ R(Q, \lambda) \}_p$$

approaches zero uniformly for x in compacta in \mathbb{R}^n . (In fact, this limit is even uniform in a certain locally convex topology on $\mathfrak{S}^m \ni Q, \dot{Q}$, but this will not concern us here.) It follows from (7.1), (7.2) that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \{ \text{diag } L_N^S(Q + \varepsilon \dot{Q}) - \text{diag } L_N^S(Q) \} \\ &= - \frac{1}{(2\pi)^n} \int_{|\xi| \geq 1} \frac{d\xi}{2\pi i} \oint d\lambda \lambda^s \{ R(Q, \lambda) \circ \dot{Q} \circ R(Q, \lambda) \}_t \end{aligned}$$

where $t = N+n+m+1$ until further notice. Now Lemma 7.1 implies

$$\begin{aligned} &= - \frac{1}{(2\pi)^n} \int_{|\xi| \geq 1} \frac{d\xi}{2\pi i} \oint d\lambda \lambda^s \{ R(Q, \lambda) \circ R(Q, \lambda) \circ \dot{Q} \}_t \\ &+ \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega \frac{1}{2\pi i} \oint d\lambda \lambda^s \{ \omega \cdot \vec{\Psi}(\lambda, \omega) \}_t \\ &+ \frac{1}{(2\pi)^n} D \cdot \left(\int_{|\xi| \geq 1} \frac{d\xi}{2\pi i} \oint d\lambda \lambda^s \{ \vec{\Phi}(\lambda, \xi) \}_t \right). \end{aligned} \quad (7.3)$$

The second integral becomes an integral over S^{n-1} via Stokes theorem and the decrease of $\vec{\Psi}(\lambda, \omega)$ as $|\xi| \rightarrow \infty$; the divergence $(D \cdot)$ could be extracted from $\{ \}$ since differentiation

with respect to x does not change the order of a term in a symbol, and from under the integral signs for the usual reasons.

Lemma 7.2

(i) $\frac{1}{2\pi i} \oint d\lambda \lambda^s \{\omega \cdot \vec{\Psi}(\lambda, \omega)\}_t$ is an entire function of s , smooth in $\omega \in S^{n-1}$

(ii) The integral

$$\int_{|\xi| \geq 1} d\xi \frac{1}{2\pi i} \oint d\lambda \lambda^s \{\vec{\Phi}(\lambda, \xi)\}_t$$

converges for $\text{Re } s \ll 0$ and continues to a vector-valued function of s , meromorphic with simple poles at (finitely many) points $s = \frac{p-n}{n+m+1}$, $p \in \mathbb{Z}^+$. The residues at these poles are integrals over S^{n-1} of polynomials in ξ , and partial derivatives of Q with respect to x , and ξ .

Proof. (i) Since, according to Lemma 7.1, the components of $\vec{\Psi}$ are polynomials in x and ξ derivatives of summands of $R(Q, \lambda) \circ \dot{Q}$ and $R(Q, \lambda)$, all summands of $\vec{\Psi}$ must be of the form

$$f(Q, \dot{Q}, \xi, |\xi|) (|\xi|^{n+m+1} - \lambda)^{-\ell}$$

for some $\ell > 0$. The assertion follows from this observation, since such terms can be explicitly integrated and give entire (in s) integrals. Smoothness in ω is obvious.

(ii) Summands of $\vec{\Phi}$ must also have the form given above; the assertion now follows from an argument similar to that which established (5.2).

g.e.d.

It follows that, for $p \in \mathbb{Z}^+$

$$\begin{aligned} \delta \text{Res}_s &= \frac{p-n}{n+m+1} \text{diag } L_N^s(Q, \dot{Q}) \\ &= \delta T_p(Q, \dot{Q}) \\ &= \frac{-1}{(2\pi)^n} \text{Res}_s = \frac{p-n}{n+m+1} \int_{|\xi| \geq 1} d\xi \frac{1}{2\pi i} \oint d\lambda \lambda^s \\ &\quad \times \{R(Q, \lambda) \circ R(Q, \lambda) \circ \dot{Q}\}_t \end{aligned}$$

for N sufficiently large. Here the relation $\dot{=}$ is as defined in §6, with the parameter set being the unit sphere S^{n-1} .

Finally, the resolvent formula has an infinitesimal version:

$$R(\lambda) \circ R(\lambda) = \frac{\partial R}{\partial \lambda}.$$

Since

$$\left\{ \frac{\partial}{\partial \lambda} R(Q, \lambda) \circ \dot{Q} \right\}_t$$

and

$$\frac{\partial}{\partial \lambda} \{R(Q, \lambda) \circ \dot{Q}\}_t$$

differ, for t sufficiently large, in higher-order terms which do not affect the residue at $s = \frac{p-n}{n+m+1}$, we can integrate by parts to obtain

$$\begin{aligned} &= \frac{1}{(2\pi)^n} \operatorname{Res}_{s = \frac{p-n}{n+m+1}} \int_{|\xi| \geq 1} \frac{d\xi}{2\pi i} \oint d\lambda \times s\lambda^{s-1} \{R(Q, \lambda) \circ \dot{Q}\}_t \\ &\approx \frac{1}{(2\pi)^n} \operatorname{Res}_{s = \frac{p-n}{n+m+1}} \int_{|\xi| \geq 1} d\xi \{sL^{s-1} \circ \dot{Q}\}_N \\ &= \operatorname{Res}_{s = \frac{p-n}{n+m+1}} \operatorname{diag}\{sL^{s-1} \circ \dot{Q}\}_N. \end{aligned}$$

We summarize this information in

$$\delta \operatorname{Res}_{s = \frac{p-n}{n+m+1}} \operatorname{diag} L^s [Q, \dot{Q}] = \delta T_p(Q, \dot{Q}) \dot{=} \operatorname{Res}_{s = \frac{p-n}{n+m+1}} \operatorname{diag}(sL^{s-1}(Q) \circ \dot{Q}). \quad (7.4)$$

Remark. Note once again that

$$\operatorname{Res}_{s = \frac{p-n}{n+m+1}} \operatorname{diag}(sL^{s-1}(Q) \circ \dot{Q}) = - \frac{p-n}{(n+m+1)^2} \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega C_p$$

where C_p denotes the summand in $L^{s-1} \circ \dot{Q}$, $s = \frac{p-n}{n+m+1}$, of order (homogeneous degree) $-n$ precisely. This follows from (7.4) and (5.2), in the same way as the remark at the end of §5.

§8. The Projection Functional

The remarks at the ends of §§5 and 7 lead us to make the following definition: for any symbol $A \in \mathfrak{S}$, set

$$\langle A \rangle = \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega A_{-n}(x, \omega)$$

where A_{-n} denotes the summand of A of order $-n$ precisely. (The notation is due to Adler, $n = 1$.) Thus $\langle A \rangle$ is a function of x , which we shall call the projection functional of A (since we project out the term of order $-n$; Adler calls this the trace, which we find confusing). We call it a functional because we shall be most interested in the case in which A is a Q -symbol, and then we will identify $\langle A \rangle$ with an equivalence class in the quotient \cdot (see §6.2).

In terms of the projection functional, formulae (5.2) and (7.4) become

$$\text{Res}_s = \frac{p-n}{n+m+1} \text{diag } L^s = \frac{1}{n+m+1} \left\langle \frac{p-n}{L^{n+m+1}} \right\rangle \quad (8.1)$$

$$\delta \left\langle \frac{p-n}{L^{n+m+1}} \right\rangle (Q, \dot{Q}) = \left\langle \frac{p-n}{n+m+1} L^{\frac{p-n}{n+m+1} - 1} (Q) \circ \dot{Q} \right\rangle \cdot \quad (8.2)$$

Note that the r.h.s. of (8.1) is well-defined, and may be used to give meaning to the l.h.s.

The most important property of the projection functional follows from Lemma 7.1.

Proposition 8.1. For any $A, B \in \mathfrak{S}$,

$$\langle [A, B] \rangle = D \cdot \vec{\phi}$$

where

$$\vec{\phi}(x) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega \vec{\phi}(x, \omega)$$

and summands of $\vec{\phi}$ are vector-valued polynomials in the summands of A and B and their x and ξ partial derivatives.

Proof. This follows from Lemma 7.1 and

Lemma 8.2. Suppose $\vec{\psi}$ is a vector function of ξ , each component being homogeneous of degree $-n + 1$. Then

$$\int_{S^{n-1}} d\omega (\partial \cdot \vec{\psi}) = 0 .$$

Remark. Adler [1] observes this result for $n = 1$, where it is rather trivial. In the following proof, assume $n > 1$.

Proof (of Lemma). Define

$$\vec{W}(\xi) = |\xi|^{n-1} \vec{\psi}(\xi) .$$

Then

$$\int_{S^{n-1}} d\omega \partial \cdot \Psi = \int_{S^{n-1}} d\omega ((-n+1)\omega \cdot \vec{W}(\omega) + (\partial \cdot \vec{W})(\omega)) . \quad (8.3)$$

If we identify ω with the unit normal to $|\xi| = 1$, and apply Stoke's theorem, we obtain

$$\int_{S^{n-1}} d\omega \omega \cdot \vec{W}(\omega) - \frac{1}{\epsilon} \int_{|\omega'|=\epsilon} d\omega' \omega' \cdot \vec{W}(\omega') = \int_{\epsilon \leq |\xi| \leq 1} d\xi (\partial \cdot \vec{W})(\xi) .$$

Since \vec{W} is homogeneous of degree zero,

$$\frac{1}{\epsilon} \int_{|\omega'|=\epsilon} d\omega' \omega' \cdot \vec{W} \rightarrow 0 , \quad \epsilon \rightarrow 0 .$$

Hence

$$\begin{aligned} \int_{S^{n-1}} d\omega \omega \cdot \vec{W}(\omega) &= \int_{|\xi| < 1} d\xi \partial \cdot \vec{W}(\xi) \\ &= \int_{S^{n-1}} d\omega \int_0^1 dr r^{n-1} (\partial \cdot \vec{W})(r, \omega) \\ &= \int_0^1 dr r^{n-2} \left(\int_{S^{n-1}} d\omega (\partial \cdot \vec{W})(\omega) \right) \\ &= \frac{1}{n-1} \int_{S^{n-1}} d\omega (\partial \cdot \vec{W})(\omega) . \end{aligned} \quad (8.4)$$

Combine (8.3) and (8.4).

q.e.d.

Corollary. Suppose A and B are Q-symbols. Then

$$\langle [A, B] \rangle \stackrel{\cdot}{=} 0 .$$

Thus, in classes of Q-symbols, with the equivalence relation $\stackrel{\cdot}{=}$, the projection functional is "trace-like" in the sense that it vanishes on commutators. It is also trace-like in another sense. The trace of matrices is commonly used to identify $gl(n, \mathbb{C})$ with its complex linear dual by means of the pairing

$$(A, B) \rightarrow \text{tr } AB .$$

In particular, differentials of functionals on $gl(n, \mathbb{C})$ can be represented by "gradients" via this pairing. We can do something similar with the projection functional.

Proposition 8.3. Suppose

$$f(\{Q\}_M) = \int_{S^{n-1}} d\omega P[Q_0, \dots, Q_M]$$

is a functional density of the type discussed in §6, where P is a polynomial in the first M + 1 summands in $Q = \sum_{\ell=0}^{\infty} Q_{\ell} \in \mathfrak{g}^m$ and their x and ξ partial derivatives. Then there exists a finite sum

$$Y_f(x, \xi) = \sum_{\ell=0}^M Y_f^{\ell}(x, \xi)$$

where $Y_f^{\ell}(x, \xi)$ is homogeneous in ξ of degree $N - \ell$, some $N \in \mathbb{Z}^+$, for which

$$\delta f(\{Q\}_M, \dot{Q}) \stackrel{\cdot}{=} \langle \dot{Q} \circ Y_f(Q) \rangle$$

where $\dot{Q} = \sum_{\ell=0}^M \dot{Q}_{\ell} \in \mathfrak{m}$. The "gradient" Y_f is uniquely determined by requiring

$N = M - n - m$; in that case, Y_f^{ℓ} is, for each ℓ , a polynomial in $\{Q\}_M$ and x and ξ partial derivatives.

Proof. Under the stated conditions, f has functional ("L²") gradients of the usual sort, computed by the Euler variational derivative rules (integration by parts on $\mathbb{R}^n \times s^{n-1}$):

$$\delta f(Q, \dot{Q}) = \sum_{\ell=0}^M \int_{S^{n-1}} d\omega \frac{\delta f}{\delta Q_{\ell}} \dot{Q}_{\ell} .$$

It is important to notice that it is exactly correct to write $\dot{=}$ here. The Euler derivatives $\frac{\delta f}{\delta Q_{\ell}}$ are polynomial in Q_0, \dots, Q_M and their x and ξ partial derivatives.

On the other hand,

$$\langle Q \circ Y_f(Q) \rangle = \sum^* \frac{1}{(2\pi)^n} \int_{S^{n-1}} \frac{1}{|\nu|} \partial^{\nu} \dot{Q}_p D^{\nu} Y_f^{\ell}(Q)$$

where the sum \sum^* is over $\ell, |\nu|, p \geq 0$ such that $-n = m+n+N-\ell-p-|\nu|$.

We shall set $N = M-n-k$ until further notice; this implies $M = \ell+|\nu|+p$ in \sum^* .

After integration by parts on the unit sphere (see Lemmas 8.5 and 8.6 below), we see that we can determine the Y_f^{ℓ} recursively by the requirement

$$\int_{S^{n-1}} d\omega \frac{\delta f}{\delta Q_p} \dot{Q}_p = \sum_{\ell=0}^{M-p} \sum_{|\nu|=M-\ell-p} \frac{1}{(2\pi)^n} \int_{S^{n-1}} \partial^{\nu} \dot{Q}_p D^{\nu} Y_f^{\ell}(Q) .$$

Since each of these equations, beginning with the first one

$$\int_{S^{n-1}} d\omega \frac{\delta f}{\delta Q_M} \dot{Q}_M = \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega \dot{Q}_M Y_f^0(Q)$$

has a nonsingular term

$$\frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega \dot{Q}_{M-p} Y_f^p(Q)$$

on the right-hand side, the "gradient" Y_f is uniquely determined by this procedure, and has the stated properties.

g.e.d.

We shall call Y_f the projection gradient of f , under the hypotheses of Prop. 8.2. As a gradient, the correspondence $f \rightarrow Y_f$ is well-defined modulo the equivalence $\dot{=}$: Corollary 8.4. Suppose that the densities f and g of the type discussed in the Prop. above are equivalent mod. divergences: $f \dot{=} g$. Then Y_f and Y_g are equal as gradients: that is,

$$\langle Y_f(Q) \circ \dot{Q} \rangle \dot{=} \langle Y_g(Q) \circ \dot{Q} \rangle$$

identically in Q, \dot{Q} .

Proof follows immediately from the Prop. 8.3 and Lemma 6.1.

q.e.d.

In computing the Y_f , the following facts about integration by parts on the unit sphere are useful.

Lemma 8.5. Suppose that F and G are smooth functions of $\xi \in \mathbb{R}^n - \{0\}$, and $F(G)$ is homogeneous of degree $a(b)$. Then

$$\int_{S^{n-1}} d\omega \left(\frac{\partial F}{\partial \xi_j} \right) G = - \int_{S^{n-1}} d\omega F \cdot \frac{\partial}{\partial \xi_j} (|\xi|^{1-(n+a+b)} G)$$

Proof. Note that

$$\int_{S^{n-1}} d\omega \left(\frac{\partial F}{\partial \xi_j} \right) G = \int_{S^{n-1}} d\xi \left(\frac{\partial F}{\partial \xi_j} \right) (|\xi|^{1-(n+a+b)} G(\xi))$$

and that

$$\frac{\partial}{\partial \xi_j} (FG |\xi|^{1-(n+a+b)})$$

is the divergence of the vector field

$$(0, \dots, FG |\xi|^{1-(n+a+b)}, \dots, 0)$$

the components of which have homogeneous degree $-n+1$. Therefore by Lemma 8.2,

$$0 = \int_{S^{n-1}} d\omega \frac{\partial}{\partial \xi_j} (F(\xi) (|\xi|^{1-(n+a+b)} G(\xi)))$$

and the assertion of the lemma follows from Leibnitz' rule.

q.e.d.

Lemma 8.6. F and G are as above, ν is an n-multi index:

$$\int_{S^{n-1}} d\omega (\partial^\nu F) G = (-1)^{|\nu|} \int_{S^{n-1}} d\omega F \partial^\nu (|\xi|^{|\nu|-(n+a+b)} G)$$

Proof. Suppose that the result is true for multi indices of order $m-1$, and let ν be a multi index of order m ,

$$\nu = \nu' + (0, \dots, \underset{j}{1}, \dots, 0)$$

with ν' some multi index of order $m-1$. Then

$$\begin{aligned} \int_{S^{n-1}} d\omega (\partial^\nu F) G &= \int_{S^{n-1}} d\omega \partial^{\nu'} \left(\frac{\partial}{\partial \xi_j} \right) F G \\ &= (-1)^{|\nu|-1} \int_{S^{n-1}} d\omega \left(\frac{\partial}{\partial \xi_j} F \right) \partial^{\nu'} (|\xi|^{|\nu'|- (n+(a-1)+b)} G) \\ &= (-1)^{m-1} \int_{S^{n-1}} d\omega \left(\frac{\partial}{\partial \xi_j} F \right) \partial^{\nu'} (|\xi|^{m-(n+a+b)} G) \end{aligned}$$

by the induction hypothesis. Now the homogeneous degree of $\partial^{\nu'} (|\xi|^{m-(n+a+b)} G)$ is $1 - (n+a)$, so the previous lemma implies the result.

q.e.d.

§9. The Algebra of Lax Vector Fields

In this section we demonstrate the analogue, in our context, of the main fact about similarities of matrices: if two matrices are similar, their spectra are the same. Our similarities are formal and infinitesimal, and, instead of spectral invariance, we establish the invariance, in the sense of $\dot{=}$, of the residue densities of the zeta function.

We also show that these infinitesimal similarities, which we call Lax Vector Fields, form a Lie algebra of vector fields on \mathfrak{g}^m , and establish a formula for the commutator.

In order to avoid inessential topological complications, we define the tangent bundle of the linear space \mathfrak{g}^m to be

$$T\mathfrak{g}^m = \mathfrak{g}^m \times \mathfrak{g}^m .$$

A vector field on \mathfrak{g}^m is thus a map $\mathfrak{g}^m \rightarrow \mathfrak{g}^m \times \mathfrak{g}^m$ of the form

$$Q \mapsto (Q, X(Q))$$

where X depends somehow on Q .

A Lax vector field on \mathfrak{g}^m is a vector field of the form

$$Q \mapsto (Q, [L(Q), N(Q)])$$

where:

$$L(Q) = |\xi|^{n+m+1} + Q$$

and $N : \mathfrak{g}^m \rightarrow \mathfrak{g}^{-n}$ is a Q -symbol, that is, each summand is a polynomial in (finitely many) summands of Q and their partial derivatives.

Note that

$$X_N(Q) \equiv [L(Q), N(Q)]$$

is also a Q -symbol $X_N : \mathfrak{g}^m \rightarrow \mathfrak{g}^m$.

Denote by $\mathcal{F}(\mathfrak{g}^m)$ the class of functional densities of the form

$$f(Q) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega P[Q, \omega]$$

where the polynomial P depends on some truncation $\{Q\}_M$ and x and ξ partial derivatives, with coefficients which are functions of ω .

Denote by $\mathcal{J}(\mathfrak{S}^m)$ the quotient of \mathcal{F} by the equivalence relation $\dot{=}$.

Denote by $\mathcal{X}(\mathfrak{S}^m)$ the class of vector fields $Q \mapsto (Q, X(Q))$ with X a Q -symbol.

We define an action of $\mathcal{X}(\mathfrak{S}^m)$ on $\mathcal{J}(\mathfrak{S}^m)$ in the obvious way: if $f \in \mathcal{F}(\mathfrak{S}^m)$, $v \in \mathcal{X}(\mathfrak{S}^m)$

$$v(Q) = (Q, X(Q))$$

then

$$\delta f(Q, X(Q)) \equiv v \cdot f(Q)$$

is again a member of \mathcal{F} . If $f \dot{=} g$, then

$$v \cdot f \dot{=} v \cdot g .$$

So \mathcal{X} acts on \mathcal{J} . (Lemma 6.1)

If $v_1, v_2 \in \mathcal{X}$, it is easy to verify that there is a vector field $[v_1, v_2]^V \in \mathcal{X}$ for which

$$v_1 \cdot (v_2 \cdot f) - v_2 \cdot (v_1 \cdot f) = [v_1, v_2]^V \cdot f$$

for all $f \in \mathcal{F}$, and in fact $[v_1, v_2]^V$ is uniquely defined by this requirement. The bracket $[]^V$ is antisymmetric bilinear and satisfies the Jacobi identity, hence $(\mathcal{X}(\mathfrak{S}^m), []^V)$ is a Lie algebra.

The collection of Lax vector fields is denoted by $\mathcal{L}(\mathfrak{S}^m)$.

Proposition 9.1. $\mathcal{L} \subset \mathcal{X}$ is a subalgebra.

Proof. For any Q -symbols $N: \mathfrak{S}^m \rightarrow \mathfrak{S}^{-n}$, and $X: \mathfrak{S}^m \rightarrow \mathfrak{S}^m$, note that

$$\delta N(X)(Q) \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (N(Q + \epsilon X(Q)) - N(Q))$$

exists (pointwise in x, ξ) and defines a Q -symbol

$$\delta N(X) : \mathfrak{S}^m \rightarrow \mathfrak{S}^{-n} .$$

In this notation, it is quite easy to show that, if $V_1 = [L, N_1]$ and $V_2 = [L, N_2]$ are Lax vector fields, then

$$[V_1, V_2]^V = [L, N_3] \in \mathcal{L}$$

where

$$N_3 = [N_1, N_2] - \delta N_1([L, N_2]) + \delta N_2([L, N_1]) \quad . \quad (9.1)$$

g.e.d.

The formula (9.1) motivates another string of definitions. Denote by \mathfrak{g} the class of Q -symbols $N : \mathfrak{g}^m \rightarrow \mathfrak{g}^{-n}$, by \mathfrak{g}_0 the collection of those $N \in \mathfrak{g}$ for which

$$[L(Q), N(Q)] \equiv 0$$

(identically in $Q \in \mathfrak{g}^m$: there are such Q -symbols, e.g. $N(Q) = R(Q, \lambda)$). If $N_1, N_2 \in \mathfrak{g}$, define

$$[N_1, N_2]^X \equiv [N_1, N_2] - \delta N_1([L, N_2]) + \delta N_2([L, N_1]) \quad .$$

Now if $N \in \mathfrak{g}_0$, $N_1 \in \mathfrak{g}$, then $[L, [N, N_1]^X]$ is the second component of $[V, V_1]^V$ by (9.1), where

$$V(Q) = (Q, [L(Q), N(Q)]) = (Q, 0)$$

$$V_1(Q) = (Q, [L(Q), N_1(Q)]) \quad .$$

But clearly

$$[V, V_1]^V(Q) \equiv (Q, 0) \quad ,$$

so

$$[L, [N, N_1]^X] \equiv 0 \quad ,$$

$[N, N_1]^X \in \mathfrak{g}_0$. It follows that the cross-bracket $[]^X$ is well-defined on the quotient $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{g}_0$. There is a natural map

$$\phi : \mathfrak{g} \rightarrow \mathcal{L}$$

given by

$$N \mapsto (Q \rightarrow (Q, [L(Q), N(Q)]))$$

and

$$\mathfrak{g}_0 = \phi^{-1}\{Q \rightarrow (Q, 0)\} .$$

Moreover, (9.1) means

$$\phi([N_1, N_2]^X) = [\phi(N_1), \phi(N_2)]^V .$$

Also, ϕ factors via the quotient map:

$$\begin{array}{ccc} & \phi & \\ & \searrow & \nearrow \\ \mathfrak{g} & \xrightarrow{\quad} & \mathfrak{L} \\ & \swarrow & \searrow \\ & \tilde{\mathfrak{g}} & \tilde{\phi} \end{array}$$

and $\tilde{\phi}$ is a bijection. Since

$$\tilde{\phi}([\tilde{N}_1, \tilde{N}_2]^X) = [\tilde{\phi}(\tilde{N}_1), \tilde{\phi}(\tilde{N}_2)]^V$$

for $\tilde{N}_1, \tilde{N}_2 \in \tilde{\mathfrak{g}}$, it follows that $[,]^X$ satisfies the Jacobi identity in $\tilde{\mathfrak{g}}$, so that

$(\tilde{\mathfrak{g}}, [,]^X)$ is a Lie Algebra. We summarize this line of thought in

Proposition 9.2. $\tilde{\phi} : \tilde{\mathfrak{g}} \rightarrow \mathfrak{L}$ is an isomorphism of Lie Algebras.

Now Lax vector fields are formal versions of infinitesimal similarities, so the following proposition is hardly surprising.

Proposition 9.3. (Gel'fand-Dikii for $n = 1$). Let T_p be any of the residue densities of the zeta function, discussed in §5, V any Lax vector field. Then

$$V \cdot T_p \stackrel{\cdot}{=} 0 .$$

Proof. Say $f = \langle L^\sigma \rangle$, where $\sigma = \frac{p-n}{n+m+1}$ for some $p \in \mathbb{Z}^+$, according to (8.1). Then, according to (8.2), if we write

$$V(Q) = (Q, [L(Q), N(Q)])$$

then

$$\begin{aligned} V \cdot f(Q) &\stackrel{\cdot}{=} \langle \sigma L^{\sigma-1}(Q) \circ [L(Q), N(Q)] \rangle \\ &\stackrel{\cdot}{=} - \langle \sigma N(Q) \circ [L(Q), L^{\sigma-1}(Q)] \rangle \\ &= 0 \end{aligned}$$

where we have used:

- (i) that $\langle \cdot \rangle$ vanishes (in the sense of $\stackrel{\cdot}{=}$) on commutators (Prop. 8.1)
- (ii) that $\{L^s : s \in \mathbb{C}\}$ is a commuting family of operators (54).

q.e.d.

§10. The Algebra of Poisson Vector Fields

Denote by \mathfrak{S}_+^m the linear subspace of \mathfrak{S}_0^m composed of symbols with summands of only non-negative homogeneous degree. Thus

$$\mathfrak{S}_+^m \ni Q = \sum_{\ell=0}^m Q_\ell$$

with Q_ℓ homogeneous in ξ of degree $m - \ell$. As before, define $T\mathfrak{S}_+^m = \mathfrak{S}_+^m \times \mathfrak{S}_+^m$, and $\mathcal{X}(\mathfrak{S}_+^m)$ to be the collection of maps $\mathfrak{S}_+^m \rightarrow T\mathfrak{S}_+^m$ of the form

$$Q \mapsto (Q, X(Q))$$

with $X(Q)$ a Q -symbol; $\mathcal{X}(\mathfrak{S}_+^m)$ is a Lie algebra of operators acting as vector fields on the collection $\mathfrak{F}(\mathfrak{S}_+^m)$ of functional densities of the form

$$f(Q) = \frac{1}{(2\pi)^n} \int_{S^{n-1}} d\omega P[\omega, Q]$$

where P is a polynomial in ω and x, ξ , partial derivatives of summands of $Q \in \mathfrak{S}_+^m$. According to Lemma 6.1, the action of \mathcal{X} descends to $\mathfrak{A}(\mathfrak{S}_+^m)$, the quotient of $\mathfrak{F}(\mathfrak{S}_+^m)$ by the appropriate equivalence relation \doteq .

Again denote the commutator of vector fields \mathcal{X} , acting on either \mathfrak{F} or \mathfrak{A} , by $[\cdot, \cdot]^V$. For each $f \in \mathfrak{F}(\mathfrak{S}_+^m)$ we introduce the Poisson vector field \mathfrak{D}_f , defined by

$$\mathfrak{D}_f : Q \mapsto (Q, [L(Q), Y_f(Q)]_+) \quad (10.1)$$

Here we have used the notation

$$A_+ \equiv \{A\}_0$$

for the (truncation) projection of a symbol $A \in \mathfrak{S}^m$ on \mathfrak{S}_+^m .

Denote by $\mathcal{P}(\mathfrak{S}_+^m)$ the aggregate of Poisson vector fields.

Note that (9.1) makes sense: according to Prop. 8.3 (and in the notation developed there), for $f \in \mathfrak{S}_+^m$ we have $M = m$ so $N = -n$, so that $Y_f : \mathfrak{S}_+^m \rightarrow \mathfrak{S}^n$, hence $[L, Y_f]$

takes values in \mathfrak{S}^m and $[L, Y_f]_+$ in \mathfrak{S}_+^m (this is the reason for the choice of exponent $n+m+1$ in the principal summand of L).

Lemma 10.1. For $f, g \in \mathfrak{F}(\mathfrak{S}_+^m)$,

$$\delta f(\delta g) \stackrel{\cdot}{=} -\delta g(\delta f) .$$

Proof.
$$\begin{aligned} \delta f(\delta g) &= \langle Y_f \circ [L, Y_g]_+ \rangle \\ &= \langle Y_f \circ [L, Y_g] \rangle \\ &\stackrel{\cdot}{=} -\langle Y_g \circ [L, Y_f] \rangle \\ &= -\langle Y_g \circ [L, Y_f]_+ \rangle \\ &= -\delta g(\delta f) \end{aligned}$$

where in the first and fourth equalities we have observed that, by virtue of the form of the symbol product, terms of negative homogeneous degree in $[L, Y_f \text{ or } g]$ cannot contribute to the projection functional, which singles out the term of order $-n$, since Y_f and Y_g already have order $-n$.

g.e.d.

We come now to the main result of this paper.

Theorem 10.2. $\mathcal{P}(\mathfrak{S}_+^m)$ is a Lie algebra of operators on $\mathcal{A}(\mathfrak{S}_+^m)$.

Proof. This means that, given $h, f, g \in \mathfrak{F}(\mathfrak{S}_+^m)$, there must exist $b \in \mathfrak{F}(\mathfrak{S}_+^m)$ for which

$$\begin{aligned} [\delta f, \delta g]^V h &\equiv \delta h([\delta f, \delta g]^V) \\ &\stackrel{\cdot}{=} \delta h(\delta b) = \delta b \cdot h \end{aligned}$$

we compute

$$\begin{aligned} \delta h([\delta f, \delta g]^V) &= -\delta(\delta h(\delta f))(\delta g) + \delta(\delta h(\delta g))(\delta f) \\ &= -\delta\langle Y_h \circ [L, Y_f]_+ \rangle(\delta g) + \delta\langle Y_h \circ [L, Y_g]_+ \rangle(\delta f) \\ &= -\delta\langle Y_h \circ [L, Y_f] \rangle(\delta g) + \delta\langle Y_h \circ [L, Y_g] \rangle(\delta f) \end{aligned}$$

$$\begin{aligned}
&= - \langle \delta Y_h(\delta g) \circ [L, Y_f] \rangle - \langle Y_h \circ [[L, Y_g], Y_f] \rangle - \langle Y_h \circ [L, \delta Y_f(\delta g)] \rangle \\
&\quad + \langle \delta Y_h(\delta f) \circ [L, Y_g] \rangle + \langle Y_h \circ [[L, Y_f], Y_g] \rangle + \langle Y_h \circ [L, \delta Y_g(\delta f)] \rangle . \quad (10.2)
\end{aligned}$$

Here we have used the order argument of the previous proof several times to drop the subscript "+". Also, we have written

$$\delta Y_g(Q, \dot{Q}) \equiv \frac{d}{dt} Y_g(Q+t\dot{Q}) \Big|_{t=0} .$$

It is clear that $\delta Y_f(\delta g)$ etc. are again polynomial maps: $\mathfrak{S}_+^m \rightarrow \mathfrak{S}^{-n}$.

Now identify $\mathfrak{F}(\mathfrak{S}_+^m)$ with the subset of $\mathfrak{F}(\mathfrak{S}^m)$ consisting of functionals independent of summands of negative homogeneous degree. It is clear that, if $Q \mapsto (Q, X(Q))$ is a vector field on \mathfrak{S}^m , and $a \in \mathfrak{F}(\mathfrak{S}_+^m)$, that

$$\delta a(Q, X(Q)) \equiv \delta a(Q, X(Q)_+) .$$

We can therefore write

$$\begin{aligned}
&- \delta(\delta h(\delta f))(\delta g) + \delta(\delta h(\delta g))(\delta f) \\
&= - \delta(\delta h(\mathcal{V}_f))(\mathcal{V}_g) + \delta(\delta h(\mathcal{V}_g))(\mathcal{V}_f)
\end{aligned}$$

where

$$\mathcal{V}_f(Q) = (Q, [L(Q), Y_f])$$

so

$$(\mathcal{V}_f(Q))_+ = \delta f(Q)$$

and \mathcal{V}_f is a Lax vector field. Therefore $\delta h([\delta f, \delta g]^V) = \delta h([\mathcal{V}_f, \mathcal{V}_g]^V) = \delta h(\mathcal{W})$, where

$$\mathcal{W}(Q) = (Q, [L(Q), [Y_f, Y_g]^X(Q)])$$

according to the formula (9.1) for commutators of Lax vector fields, where

$$[Y_f, Y_g]^X = [Y_f, Y_g] - \delta Y_f(\mathcal{V}_g) + \delta Y_g(\mathcal{V}_f) .$$

Since, according to Prop. 8.3, Y_f and Y_g are (considered as maps: $\mathfrak{S}^m \rightarrow \mathfrak{S}^{-n}$) independent of summands of negative homogeneous degree, we can re-write this as

$$= [Y_f, Y_g] - \delta Y_f(\delta g) + \delta Y_g(\delta f) .$$

Thus we obtain

$$\begin{aligned} \delta h([f, g]^V) &= \delta h([L, [Y_f, Y_g] + \delta Y_f(\delta g) - \delta Y_g(\delta f)]) \\ &= \langle Y_h \circ [L[Y_f, Y_g]] \rangle - \langle Y_h \circ [L, \delta Y_f(\delta g)] \rangle + \langle Y_h \circ [L, \delta Y_g(\delta f)] \rangle . \end{aligned} \quad (10.3)$$

Comparing (10.3) and (10.2) and making use of the Jacobi identity for symbol commutators, we obtain:

$$0 = \langle \delta Y_h(\delta g) \circ [L, Y_f] \rangle - \langle \delta Y_h(\delta f) \circ [L, Y_g] \rangle . \quad (10.4)$$

Now using the identity $\langle A \circ [B, C] \rangle = -\langle C \circ [B, A] \rangle$ which holds for arbitrary $A, B, C \in \mathcal{L}$, rewrite (10.3) as

$$\delta h([f, g]^V) = -\langle ([Y_f, Y_g] - \delta Y_f(\delta g) + \delta Y_g(\delta f)) \circ [L, Y_h] \rangle . \quad (10.5)$$

Since (10.4) holds for an arbitrary triple $h, f, g \in \mathfrak{S}_+^m$, conclude for instance that

$$\langle \delta Y_f(\delta g) \circ [L, Y_h] \rangle = \langle \delta Y_f(\delta h) \circ [L, Y_g] \rangle$$

etc.; re-write (10.5) as

$$\begin{aligned} \delta h([f, g]^V) &= -\langle Y_f \circ [[L, Y_h] Y_g] \rangle \\ &\quad + \langle \delta Y_f(\delta h) \circ [L, Y_g] \rangle - \langle \delta Y_g(\delta h) \circ [L, Y_f] \rangle \\ &= -\langle Y_f \circ [[L, Y_h], Y_g] \rangle + \langle \delta Y_f(\delta h) [L, Y_g] \rangle \\ &\quad + \langle Y_f [L, \delta Y_g(\delta h)] \rangle \\ &= -\delta \langle Y_f \circ [L, Y_g] \rangle(\delta h) \\ &= \delta \langle Y_g \circ [L, Y_f] \rangle(\delta h) \\ &= \delta h(\delta \delta g(\delta f)) . \end{aligned}$$

Hence we should take $b = \delta g(\delta f)$.

q.e.d.

Roughly, the distribution in the tangent bundle $T\mathbb{S}_+^m$, spanned by the vector fields δf , $f \in \mathfrak{F}(\mathbb{S}_+^m)$, is "involutive modulo $\dot{=}$ ". It is not worth the trouble to make this notion precise, but it is valuable as a guiding idea.

For $f, g \in \mathfrak{F}(\mathbb{S}_+^m)$, set

$$\{f, g\} \equiv \delta g(\delta f)$$

and call $\{ \}$ the Poisson bracket on $\mathfrak{F}(\mathbb{S}_+^m)$

Lemma 10.3. If $f_1 \dot{=} f_2$, $g_1 \dot{=} g_2$, then

$$\{f_1, g_1\} \dot{=} \{f_2, g_2\} .$$

Proof. By Lemmas 6.1 and 10.1,

$$\begin{aligned} \{f_1, g_1\} &= \delta g_1(\delta f_1) \\ &\dot{=} \delta g_2(\delta f_1) \\ &\dot{=} -\delta f_1(\delta g_2) \\ &\dot{=} -\delta f_2(\delta g_2) \\ &\dot{=} \delta g_2(\delta f_2) \\ &= \{f_2, g_2\} . \end{aligned}$$

q.e.d.

Thus we can regard $\{ \}$ as being defined on \mathfrak{J} rather than \mathfrak{F} .

Corollary 10.4. The bracket $\{ \}$ defines a Poisson structure on $\mathfrak{A}(\mathbb{S}_+^m)$.

Proof. The meaning of this statement is a direct transliteration of the definition of §2, with $\mathfrak{J}(\mathbb{S}_+^m)$ in place of $C^\infty(M)$. The antisymmetric bilinearity of $\{ \}$ is clear. The locality condition is clearly satisfied in obvious paraphrase, where we now interpret "g vanishes to second order at $Q \in \mathbb{S}_+^m$ " to mean that $\delta g(Q, \dot{Q}) \dot{=} 0$ for all $(Q, \dot{Q}) \in T_Q \mathbb{S}_+^m$. Finally, the Jacobi identity (in the sense of $\dot{=}$) is proven in precisely the same way as in Prop. 2.1.

q.e.d.

§11. Dimension One, Theorem of Gardner-Lax-Gel'fand-Dikii

We now recover the result of Gel'fand-Dikii on the involution of zeta function residues in dimension one - ([3], Theorem 7, Cor. 3). This result goes back to Gardner [2], for the case $L = \xi^2 + q(x)$, and has been given in various forms by Lax [4] and Zakharov and Fadeev [7].

For $n = 1$, we have, on \mathfrak{S}_+^m ,

$$L = |\xi|^{m+2} + \sum_{\ell=0}^m Q_{\ell}(x, \xi)$$

with Q_{ℓ} homogeneous of degree $m - \ell$ in ξ . Up to a sign ambiguity which we do not treat, this can be re-written

$$L = \xi^{m+2} + \sum_{\ell=0}^m q_{\ell}(x) \xi^{m-\ell}$$

which is the symbol of a differential operator. This translation must be made to match our results with those of [3].

Theorem 11.1. (Gel'fand-Dikii). Suppose $n = 1$.

Then the zeta-function residues $\{T_p : p \in \mathbb{Z}^+\}$, regarded as functionals on \mathfrak{S}_+^m , are in involution, i.e.

$$\{T_p, T_{p'}\} \doteq 0$$

for all $p, p' \in \mathbb{Z}^+$.

Proof. According to (8.2), for $(Q, \dot{Q}) \in T\mathfrak{S}^m$

$$\begin{aligned} \delta T_p(Q, \dot{Q}) &\doteq \frac{p-1}{m+2} \left\langle L^{\frac{p-1}{m+2}-1} (Q) \circ \dot{Q} \right\rangle \\ &= \frac{p-1}{m+2} \left\langle \left\langle L^{\frac{p-1}{m+2}-1} \right\rangle_{-m-1} (Q) \right\rangle \circ \dot{Q} . \end{aligned} \quad (11.1)$$

Here we have used the notation $\{ \}_{-p}$ for truncation (see (7.2) and following remark), and $(A)_-$ for the symbol consisting of summands of A of negative order. This step is justified by the following observations:

- (i) terms of order $< -m-1$ in $L^{\sigma-1}$, $\sigma = \frac{p-1}{m+2}$, cannot contribute to the term of $L^{\sigma-1} \circ \dot{Q}$ of order -1 , since \dot{Q} has order m , so we might as well throw them out;
- (ii) in dimension one only, it follows from the definition of symbol product that the product of two symbols, whose summands have only non-negative integral orders, again has this property; hence

$$\langle (\{L^{\sigma-1}\}_{-m-1}(Q))_+ \circ \dot{Q} \rangle = 0 .$$

Taken together, (i) and (ii) imply (11.1). Note that (ii) is a consequence of the fact that homogeneous functions of degree zero in one variable are necessarily constant, except possibly for a jump at the origin.

According to the uniqueness part of Prop. 8.3, we therefore have

$$y_{T_p} = \sigma (\{L^{\sigma-1}\}_{-m-1})_- \quad (11.2)$$

hence

$$\delta_{T_p} = \sigma [L, (\{L^{\sigma-1}\}_{-m-1})_-]_+ .$$

Now

$$L^{\sigma-1} = (L^{\sigma-1})_+ + (\{L^{\sigma-1}\}_{-m-1})_- + R$$

where R consists of terms of order $< -m-1$. Hence

$$0 = [L, L^{\sigma-1}]_+ = [L, (L^{\sigma-1})_+]_+ + [L, (\{L^{\sigma-1}\}_{-m-1})_-]_+ + [L, R]$$

where $[L, R]$ consists of terms of negative degree only. Hence

$$0 = \sigma [L, L^{\sigma-1}]_+ = \sigma [L, (L^{\sigma-1})_+]_+ + \delta_{T_p}$$

i.e.

$$\delta_{T_p} = - \sigma [L, (L^{\sigma-1})_+]_+ .$$

But, by another application of (ii) above, we see that

$$[L, (L^{\sigma-1})_+]_+ = [L, (L^{\sigma-1})_+] .$$

Hence δT_p is a Lax vector field. Now Prop. 9.3 gives exactly

$$\begin{aligned} 0 &= \dot{\delta T}_p \cdot T_p = \delta T_p, (\delta T_p) \\ &= \{T_p, T_p\} \end{aligned}$$

for any $p' \in \mathbb{Z}^+$.

g.e.d.

We can also compute the expression ([3], formulae (19), (20)) for δT_p in terms of the Euler variational derivatives $\frac{\delta T}{\delta Q_\ell}$, $\ell = 0, \dots, m$, using formulae (8.2), the results of §§3, 4, and reversing the integration-by-parts used in the proof of Prop. 8.3. The idea is obvious, so we omit the details.

§12. Higher-Dimensions: Non-involution of Zeta-Function Residues.

We now show that a result analogous to Theorem 11.1 does not hold in arbitrary dimensions, by computing a specific Poisson bracket and observing that it does not vanish.

We consider the simplest case: set $m = 0$, so that

$$L = |\xi|^{n+1} + Q_0(x, \xi)$$

where Q_0 is homogeneous in ξ of degree zero. We have

$$T_p(Q_0) = \langle L^{\frac{p-n}{n+1}}(Q_0) \rangle$$

and (assuming $p \geq n + 1$)

$$\begin{aligned} \delta T_p(Q_0, \dot{Q}_0) &= \frac{p-n}{n+1} \langle L^{\frac{p-n}{n+1}-1}(Q_0) \circ \dot{Q}_0 \rangle \\ &= \frac{p-n}{n+1} \frac{1}{(2\pi)^n} \sum_{\ell=0}^{p-n-1} \sum_{|\nu|=p-n-\ell-1} \int_{|\xi|=1} d\xi \frac{1}{\nu!} \partial^{\nu} \dot{Q}_0 D^{\nu} A_{\ell} \\ &= \frac{p-n}{n+1} \frac{1}{(2\pi)^n} \sum_{\ell=0}^{p-n-1} \sum_{|\nu|=p-n-\ell-1} \frac{(-1)^{|\nu|}}{\nu!} \int_{|\xi|=1} d\xi \dot{Q}_0 \partial^{\nu} (|\xi|^{|\nu|-p-n-\ell-1} D^{\nu} A_{\ell}) \\ &= \frac{p-n}{n+1} \frac{1}{(2\pi)^n} \sum_{\ell=0}^{p-n-1} \sum_{|\nu|=p-n-\ell-1} \frac{(-1)^{|\nu|}}{\nu!} \int_{|\xi|=1} d\xi \dot{Q}_0 \partial^{\nu} D^{\nu} A_{\ell} \end{aligned}$$

(using Lemma 8.6).

It follows (uniqueness in Prop. 8.3) that

$$Y_{T_p} = \frac{p-n}{n+1} \sum_{\ell=0}^{p-n-1} \sum_{|\nu|=p-n-\ell-1} \frac{(-1)^{|\nu|}}{\nu!} \partial^{\nu} D^{\nu} A_{\ell} \left(\frac{p-2n-1}{n+1} \right)$$

It is amusing to note that, up to a complex conjugation, Y is the term of order $-n$

in the adjoint symbol of $\frac{p-n}{n+1} L^{\frac{p-n}{n+1}-1}$.

It is now necessary to do some of the computations described in §53, 4.

The resolvent series begins

$$R(\lambda) = (|\xi|^{n+1} - \lambda)^{-1} - (|\xi|^{n+1} - \lambda)^{-2} Q_0 \\ + (n+1) |\xi|^{n-1} (|\xi|^{n+1} - \lambda)^{-3} \sum_{j=1}^n \xi_j D_j Q_0 + \dots$$

Hence, in the notation of §4,

$$A_0(s, \xi) = (|\xi|^{n+1})^s \\ A_{n+1}(s, \xi) = -s (|\xi|^{n+1})^{s-1} Q_0 \\ A_{n+2}(s, \xi) = - (n+1) |\xi|^{n-1} \frac{1}{2} s(s-1) (|\xi|^{n+1})^{s-2} \sum_{j=1}^n \xi_j D_j Q_0$$

These expressions allow us to compute Y_{T_p} for $p = 2n + 2, 2n + 3$. We obtain

$p = 2n + 2$:

$$Y = \frac{n+2}{n+1} A_{n+1} \left(\frac{1}{n+1} \right) \\ = - \frac{n+2}{(n+1)^2} |\xi|^{-n} Q_0$$

$p = 2n + 3$:

$$Y = -3 \frac{(n+3)(n-1)}{(n+1)^2} |\xi|^{-n+1} \sum_{j=0}^n \xi_j D_j Q_0 \\ + \frac{2(n+3)}{(n+1)^2} |\xi|^{-n+1} \sum_{j=0}^n D_j \partial_j Q_0$$

Next we compute

$$\delta T_{2n+2} = [L, Y_{2n+2}]_+ \\ = [|\xi|^{n+1}, Y_{2n+2}]_+ \\ = - \frac{n+2}{(n+1)^2} \sum_{j=1}^n \partial_j (|\xi|^{n+1}) |\xi|^{-n} D_j Q_0 \\ = - \frac{n+2}{n+1} |\xi|^{-1} \sum_{j=1}^n \xi_j D_j Q_0$$

(At this point it is worth remarking that δT_{2n+2} is generally not independent of ξ , even if Q_0 is. Thus the Poisson vector fields are generally not tangent to the (polynomial) symbols of differential operators, except in dimension one.)

Next, note that the only term of order $-n$ in $Y_{2n+3} \circ [L, Y_{2n+2}]_+$ is the leading term in the product formula, involving no differentiations of either factor. Hence

$$\begin{aligned} \{T_{2n+2}, T_{2n+3}\} &= \frac{1}{(2\pi)^n} \frac{3(n+3)(n+2)(n-1)}{(n+1)^3} \int_{|\xi|=1} d\xi \left(\sum_{j=0}^n \xi_j D_j Q_0 \right)^2 \\ &\quad - \frac{1}{(2\pi)^n} \frac{2(n+3)(n+2)}{(n+1)^3} \int_{|\xi|=1} d\xi \left(\sum_{j=0}^n \xi_j D_j Q_0 \right) \left(\sum_{j=0}^n D_j \delta_j Q_0 \right). \end{aligned}$$

Now the right-hand side of this equation is generally not a divergence, unless $n = 1$, in which case it vanishes. For instance, if Q_0 happens to be independent of ξ , the right-hand side is a multiple of $\sum (D_j Q_0)^2$, which is certainly not a divergence. It is amusing to note that the numerical coefficient contains the factor $(n - 1)$.

This concludes our example.

REFERENCES

1. M. Adler, "On the Relationship of Some Results of Gel'fand-Dikii and P. Moerbeke, and a Natural Trace Functional for Formal Symbols", MRC Technical Summary Report # , (1978).
2. C. Gardner, "Korteweg-de Vries Equation and Generalisations: IV", J. Math. Phys. 12, No. 8, pp. 1548-1551 (1971).
3. I. Gel'fand and L. Dikii, "Fractional Powers of Operators and Hamiltonian Systems", Functional Analysis and its Applications, 10, No. 4, pp. 259-273 (1977).
4. P. Lax, "Periodic Solutions of the KdV Equation", Comm. Pure Appl. Math., 28, pp. 141-188 (1975).
5. R. Seeley, "Complex Powers of an Elliptic Operator", Proc. Symp. Pure Math., X, A.M.S., Providence, R. I., pp. 288-307 (1966).
6. M. Taylor, Springer notes on Pseudo-differential Operators.
7. V. Zakharov and L. Fadeev, "The Korteweg-de Vries Equation is a Fully Integrable Hamiltonian System", Funct. Anal. Appl., 5, No. 4, pp. 18-27 (1971).

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER #1847	2. GOVT ACCESSION NO. 9 Summary rept.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A POISSON STRUCTURE ON SPACES OF SYMBOLS.		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
7. AUTHOR(s) W. Symes		6. PERFORMING ORG. REPORT NUMBER
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of Wisconsin 610 Walnut Street Madison, Wisconsin 53706		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024, NSF- MCS75-17385
11. CONTROLLING OFFICE NAME AND ADDRESS See Item 18 below		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 1 - (Applied Analysis)
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 12 52 p.		12. REPORT DATE Apr 1978
		13. NUMBER OF PAGES 46
		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 14 MRC-TSR-1847		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U.S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709 National Science Foundation Washington, D. C. 20550		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Poisson Structures, Integrals in Involution, Korteweg-de Vries Equation, Calculus of Symbols. R to the nth power		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A Poisson structure (anti-symmetric bilinear local operator on functionals, obeying the Jacobi identity) is established on certain function spaces (spaces of symbols of pseudodifferential operators on \mathbb{R}^n). The spaces of functionals thus become (infinite-dimensional) Lie Algebras. This type of Lie algebra structure has been established previously for functionals of functions of a single variable ($n = 1$) only. For $n = 1$, the theorem of Gardner, as generalized by Gel'fand, Dikii, and others, is proved: that is, the residues of the zeta-function of the elliptic symbol (continued) → next page		

Her

ABSTRACT (continued)

$$\xi^{m+2} + \sum_{j=0}^m g_j(x) \xi^j$$

are in involution with respect to the appropriate Poisson bracket. In contrast, it is shown by explicit example that the residues of the zeta functions of higher-dimensional elliptic symbols are generally not in involution.