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COMPUTABLE ERROR BOUNDS FOR THE NYSTRÖM METHOD

J. W. Hilgers* and L. B. Rall

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ABSTRACT

The classical Nyström method for the numerical solution of Fredholm integral equations of second kind consists of numerical integration, collocation, and interpolation. The approximate solution obtained by this procedure is shown to be identical to the solution of certain finite-rank integral equations with kernels belonging to a specified class $\langle \hat{K}_n \rangle$, and thus has minimal error with respect to approximation of the original equation over this class. A computable (but in general nonoptimal) error bound for the Nyström approximate solution can be obtained on the basis of how well a specific finite-rank integral operator with kernel in $\langle \hat{K}_n \rangle$ approximates the integral operator in the Fredholm equation being solved numerically.

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SIGNIFICANCE AND EXPLANATION

Many problems in applied mathematics (such as boundary-value problems for ordinary and partial differential equations) can be solved if a function $x(s)$ satisfying a Fredholm integral equation of second kind,

$$x(s) - \lambda \int_0^1 K(s,t)x(t)dt = y(s), \quad 0 \leq s \leq 1 , \tag{1}$$

can be found, where $y(s)$ and $K(s,t)$ are known. In most cases, $x(s)$ cannot be found explicitly, so numerical methods must be used. E. J. Nyström developed a simple method of this type, using numerical integration to replace the integral in (1) by the finite sum

$$\sum_{j=1}^{n} K(s,t_j)w_j z_j ,$$

where z_j are approximations to the values $x(t_j)$, the points $t_1,t_2,\ldots,t_n$ are the nodes and $w_1,w_2,\ldots,w_n$ the weights of the rule of numerical integration (Simpson, Gauss, etc.) being used. In order to determine the approximate values $z_1,z_2,\ldots,z_n$, collocation at the points $t_1,t_2,\ldots,t_n$ is used to obtain the system of equations

$$z_i - \lambda \sum_{j=1}^{n} K(t_i,t_j)w_j z_j = y(t_i), \quad i = 1,2,\ldots,n , \tag{2}$$

to be solved for these values. Finally, interpolation by the formula

$$z(s) = y(s) + \lambda \sum_{j=1}^{n} K(s,t_j)w_j z_j \tag{3}$$

gives an approximate solution of equation (1) on the entire interval $0 \leq s \leq 1$. This method is simple in concept and easy to implement on electronic computers. The question remains, how accurate are the results? In fact, the Nyström method is known to produce good approximate solutions in many applications, but why is this so? In this report, a number of ways of analyzing the accuracy of the Nyström method are compared, and it is shown that error can be estimated by a simple computational procedure based on approximation of $K(s,t)$ by kernels of the form

$$K_n(s,t) = \sum_{j=1}^{n} K(s,t_j)w_j v_j(t) . \tag{4}$$

It is also shown that the error of the Nyström method depends on the best possible approximation of $K(s,t)$ by kernels $K_n(s,t)$ from a given class, and thus the observed accuracy of this method has a theoretical explanation.

COMPUTABLE ERROR BOUNDS FOR THE NYSTRÖM METHOD

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Dedicated to Professor J. Barkley Rosser on his 70th birthday.

1. THE NYSTRÖM METHOD. The linear Fredholm integral equation of second kind,

$$x(s) - \lambda \int_0^1 K(s,t)x(t)dt = y(s), \qquad 0 \le s \le 1 , \tag{1.1}$$

arises in the solution of boundary value problems for ordinary and partial differential equations, and other important applications. Given the function $y(s)$ and the kernel $K(s,t)$, one problem in connection with (1.1) is to obtain the solution function $x(s)$, at least for values of the parameter λ for which it is unique. Another problem, usually posed for the homogeneous equation ($y(s) \equiv 0$), is to find the eigenvalues and eigenfunctions of the kernel $K(s,t)$, that is, values λ^* of the parameter λ for which the homogeneous equation has corresponding nontrivial solutions $x^*(s) \not\equiv 0$. Fredholm [6] obtained solutions to these problems in terms of infinite series in λ, with coefficients expressed as repeated integrals of larger and larger determinants. As in the case of the analogous Cramer's rule for linear algebraic systems, Fredholm's formulas are satisfactory from a theoretical standpoint, but are usually unsuitable for practical computation. To remedy this situation, various numerical methods to obtain approximate solutions have been developed [4], including the simple and effective procedure due to Nyström [13]. Nyström's method consists of three steps: (i) numerical integration, which replaces the integral in (1.1) by a finite sum to obtain an approximating functional equation; (ii) collocation, which gives a finite linear algebraic system for values $z_1, z_2, \ldots, z_n$ of an approximate solution of (1.1) at n points $t_1, t_2, \ldots, t_n$; and (iii) interpolation, which uses the values found by collocation to construct a function $z(s)$ on the entire interval $0 \le s \le 1$ which is an approximate solution of (1.1) such that $z(t_i) = z_i$, $i = 1,2,\ldots,n$. These steps will be explained in greater detail in the following sections.

The simplicity of the Nyström method stems from the fact that it is basically an interpolatory procedure; only values of the kernel $K(s,t)$ and the function $y(s)$ are needed at the nodes of the numerical integration formula being used. In terms of accuracy, on the other hand, it will be shown that the Nyström method is optimal with respect to a certain class of approximation procedures. Estimates will be derived for the norm $\|x - z\|$ of the error in a normed linear function space. In this connection, it will be convenient to use operator notation, in terms of which (1.1) may be written in the form

$$(I - \lambda K)x = y , \tag{1.2}$$

where I denotes the identity operator, and K the linear integral operator with kernel $K(s,t)$.

2. NUMERICAL INTEGRATION. A rule for numerical integration with nodes $t_1, t_2, \ldots, t_n$ and weights $w_1, w_2, \ldots, w_n$ may be expressed in terms of the linear functional

$$R_n = R_n\begin{pmatrix} t_1 & t_2 & \cdots & t_n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix} , \tag{2.1}$$

where

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(2.2) $$R_n[f] = \sum_{j=1}^{n} f(t_j)w_j .$$

Thus, one has the quadrature formula

(2.3) $$\int_0^1 f(t)dt = R_n[f] + E_n[f]$$

for the integral of a function $f(t)$ over the interval $0 \le t \le 1$, where E_n denotes the (linear) error functional associated with the rule R_n. Attention will be restricted here to quadrature formulas of order m and interpolation type, for which $E_n[f] = 0$ if $f(t)$ is a polynomial in t of degree $m - 1$ or less [10, p. 162]. In addition, it will be assumed that the rules for numerical integration considered have positive weights $w_i > 0$, $i = 1,2,\ldots,n$. This will guarantee that the quadrature formula of interpolation type is convergent in the sense that $\lim_{n\to\infty} E_n[f] = 0$, at least for continuous integrand $f(t)$ [10, p. 186].

Applying the quadrature formula (2.3) to the integral equation (1.1) gives the equivalent equation

(2.4) $$x(s) - \lambda \sum_{j=1}^{n} K(s,t_j)w_j x(t_j) = y(s) + \lambda E_n[Kx](s) ,$$

where the error term has been moved to the right-hand side. The form of (2.4) suggests that an approximate solution $z(s)$ of the integral equation can be obtained by solving the functional equation

(2.5) $$z(s) - \lambda \sum_{j=1}^{n} K(s,t_j)w_j z(t_j) = y(s) ,$$

which will be called the Nyström equation resulting from the application of the rule of numerical integration R_n to the Fredholm integral equation (1.1).

3. COLLOCATION. The left-hand side of equation (2.4) involves the values $x(t_j)$ of the solution $x(s)$ of the integral equation (1.1) at the nodes $t_1,t_2,\ldots,t_n$ of the rule of numerical integration R_n. Setting $s = t_i$ in equation (2.4) gives the system of equations

(3.1) $$x(t_i) - \lambda \sum_{j=1}^{n} K(t_i,t_j)w_j x(t_j) = y(t_i) + \lambda E_n[Kx](t_i) ,$$

$i = 1,2,\ldots,n$, for these values. Approximations z_i to $x(t_i)$ may be obtained by discarding the error term in (3.1) and solving the resulting collocation equations,

(3.2) $$z_i - \lambda \sum_{j=1}^{n} K(t_i,t_j)w_j z_j = y_i , \qquad i = 1,2,\ldots,n ,$$

where $y_i = y(t_i)$. Given the rule of numerical integration R_n, setting up and solving the finite linear algebraic system (3.2) (or the corresponding eigenvalue-eigenvector problem) can be carried out readily with the aid of an electronic computer, even for moderately large values of n.

Some other formulations of the system (3.2) may be useful in particular instances. For example, the quantities $\xi_i = w_i z_i$ may be needed for numerical or theoretical purposes. Multiplying the equations (3.2) by $w_1,w_2,\ldots,w_n$ in turn and setting $\eta_i = w_i y_i$ gives the system

(3.3) $$\xi_i - \lambda \sum_{j=1}^{n} w_i K(t_i,t_j)\xi_j = \eta_i , \qquad i = 1,2,\ldots,n ,$$

which can be solved directly for $\xi_1,\xi_2,\ldots,\xi_n$. Another case would be that the kernel $K(s,t)$ of the integral equation (1.1) is symmetric, $K(s,t) = K(t,s)$ (or Hermitian, $K(s,t) = \overline{K(t,s)}$), and it is desired to carry this property over to the coefficient matrix of the finite system, which might be particularly convenient when calculating approximate eigenvalues and eigenfunctions. One way to do this is to use numerical integration rules of Chebyshev type with equal weights $w_i = \frac{1}{n}$, $i = 1,2,\ldots,n$ [10, pp. 213-216]. A symmetrization procedure which works for arbitrary rules with positive weights is to multiply the ith equation of (3.2) by $\sqrt{w_i}$. In terms of $\zeta_i = \sqrt{w_i}z_i$, $\theta_i = \sqrt{w_i}y_i$, the resulting system is

(3.4) $$\zeta_i - \lambda \sum_{j=1}^{n} \sqrt{w_i}K(t_i,t_j)\sqrt{w_j}\zeta_j = \theta_i , \qquad i = 1,2,\ldots,n ,$$

which has a symmetric (or Hermitian) coefficient matrix if $K(s,t)$ has the corresponding property.

4. INTERPOLATION. In order to keep the system of collocation equations small, Nyström [13] recommended the use of highly accurate rules of numerical integration, such as Gaussian quadrature. Even when using electronic computers,

there are advantages in speed and accuracy to be gained in this way. However, the Gaussian nodes (see [9, p. 288] for a table based on the interval $0 \le t \le 1$) are not always the points at which approximate values of the solution of the integral equation are desired. In other applications, what is needed is not just a *finite* set of values $z_1, z_2, \ldots, z_n$, but rather a function $z(s)$ which is an approximate solution on the entire interval $0 \le s \le 1$. These requirements can be met by some method for interpolation (or extrapolation) from the values computed at the collocation points to other points of the interval. Frequently used procedures for interpolation are based on piecewise linear functions, polynomials, or spline functions, which lead to various representations for an approximate solution of (1.1). Nyström's interpolation formula is simply

$$z(s) = y(s) + \lambda \sum_{j=1}^{n} K(s,t_j) w_j z_j \tag{4.1}$$

in terms of the solutions $z_1, z_2, \ldots, z_n$ of the system of equations (3.2). It follows from the Nyström equation (2.5) that $z(t_i) = z_i$, $i = 1,2,\ldots,n$; in fact, (4.1) is the unique solution of (2.5) which interpolates the values computed from (3.2) [4, pp. 88-89]. Formula (4.1) is natural from the standpoint of simplicity and the fact that it makes use of information from the original integral equation (values of $y(s)$ and $K(s,t)$) at all points of the interval, while an arbitrary interpolation formula would only use the values $z_1, z_2, \ldots, z_n$. This suggests that the Nyström method should be fairly accurate, as is usually observed in actual practice.

Other forms of (4.1) may be useful if equations (3.3) or (3.4) are used instead of (3.2). In terms of the solutions $\xi_1, \xi_2, \ldots, \xi_n$ of (3.3), the Nyström interpolation formula becomes simply

$$z(s) = y(s) + \lambda \sum_{j=1}^{n} K(s,t_j) \xi_j \,. \tag{4.2}$$

If equations (3.4) are solved for $\zeta_1, \zeta_2, \ldots, \zeta_n$ instead, then (4.1) may be written

$$z(s) = y(s) + \lambda \sum_{j=1}^{n} K(s,t_j) \sqrt{w_j}\, \zeta_j \,. \tag{4.3}$$

5. SOME METHODS OF ERROR ESTIMATION. In addition to computational results as furnished by application of Nyström's method, some indication of their reliability is desired in many instances. Such error estimates may range from heuristic to rigorous, and be pointwise, usually for $|x(t_j) - z_j|$, $j = 1,2,\ldots,n$, or global, in which case a bound is given for the norm $\|x - z\|$ of the function $x(s) - z(s)$ in some appropriate space. Among the possible techniques for error estimation are (i) recovery of known solutions; (ii) analysis of the error functional E_n; (iii) use of the theory of collectively compact operator approximation [1]; and (iv) approximation of the integral equation by an equation of finite rank. The method presented in this paper falls into the last category; before going into details, a brief description of the other procedures will be given.

5.1. Recovery of known solutions. For a given kernel $K(s,t)$, it may be that equation (1.1) has known solutions $x(s)$ corresponding to particular choices of the function $y(s)$. In this situation, the approximate solution $z(s)$ obtained by the Nyström method can be compared directly with the exact solution. If the observed accuracy is good, then the numerical results computed for right-hand sides $y(s)$ corresponding to unknown solutions $x(s)$ may be viewed with some confidence. As pointed out by Nyström [13], integral equations with known solutions may be constructed using functions $x(s)$ for which the transformed function $Kx(s)$ can be calculated explicitly.

The computational effort required for this type of error estimation is not particularly great; with a given coefficient matrix, the system (3.2) (or one of the alternative forms (3.3) or (3.4)) may be solved simultaneously for several right-hand sides, corresponding to known and unknown solutions of the integral equation. The accuracy of the approximation obtained for the known solutions may then be used as an indication of reliability of the results calculated for the unknown solutions. It should be emphasized that this procedure is entirely heuristic, it being possible that a certain choice of R_n would work well for some manufactured equations, but not give accurate results for an actual problem. It is comforting to note that the illustrations given by Nyström [13] show good performance of his method applied to boundary-value problems arising in mathematical physics, rather than just to some contrived examples.

5.2. Analysis of the error functional E_n. Essentially, the error in the approximate solution $z(s)$ obtained by the Nyström method is due to neglect of terms involving the error functional E_n; that is, the replacement of (2.4) by (2.5) and using (3.2) instead of (3.1). (More precisely, this is the truncation error of the method; in this discussion, roundoff error in the actual computation and errors in the data $K(s,t)$ and $y(s)$ will be ignored. An expression for the truncation error $x(s) - z(s)$ will now be obtained. From (2.4) and (2.5),

$$x(s) - z(s) = \lambda\left\{E_n[Kx](s) + \sum_{j=1}^{n} K(s,t_j)w_j[x(t_j) - z(t_j)]\right\} . \tag{5.1}$$

For simplicity of notation, set

$$E_n(s) = E_n[Kx](s), \qquad 0 \le s \le 1 . \tag{5.2}$$

The errors $x(t_i) - z(t_i)$ at the collocation points satisfy the linear system of equations

$$[x(t_i) - z(t_i)] - \lambda \sum_{j=1}^{n} K(t_i,t_j)w_j[x(t_j) - z(t_j)] = \lambda E_n(t_i) , \tag{5.3}$$

$i = 1,2,\ldots,n$, by (3.1) and (3.2). Suppose that the coefficient matrix A of (5.3),

$$A = (\delta_{ij} - \lambda K(t_i,t_j)w_j) , \tag{5.4}$$

where δ_{ij} is the Kronecker delta ($\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = 1$), has the inverse

$$B = (\beta_{ij}) = A^{-1} . \tag{5.5}$$

In terms of the coefficients of B, the solutions of (5.3) may be written

$$x(t_i) - z(t_i) = \lambda \sum_{j=1}^{n} \beta_{ij}E_n(t_j), \qquad i = 1,2,\ldots,n , \tag{5.6}$$

and (5.1) becomes

$$x(s) - z(s) = \lambda\left\{E_n(s) + \lambda \sum_{j=1}^{n}\sum_{k=1}^{n} K(s,t_j)w_j\beta_{jk}E_n(t_k)\right\} . \tag{5.7}$$

On the basis of some assumption about the magnitude of $E_n(s) = E_n[Kx](s)$, for example, $|E_n(s)| \le M$, $0 \le s \le 1$, (5.6) may be used to derive pointwise error bounds at the collocation points $t_1,t_2,\ldots,t_n$, and (5.7) will furnish a global error estimate, or pointwise bounds at points other than the nodes of the rule of numerical integration.

If the integrand $f(t)$ is smooth enough, then the error term $E_n[f]$ of a formula of order m and interpolation type can be expressed as

$$E_n[f] = C_m(n) \cdot f^{(m)}(\tau_n), \qquad 0 < \tau_n < 1 , \tag{5.8}$$

where $C_m(n)$ is a known function of n for which $\lim_{n\to\infty} C_m(n) = 0$ in the case of convergent rules, and the point τ_n is unknown [9, pp. 108-116; 5, pp. 217-223]. Thus, the problem of bounding $E[Kx](s)$ can be reduced to estimating $\frac{\partial^m}{\partial t^m} K(s,t)x(t)$. This has the obvious drawback of requiring guesses for the size of derivatives of the unknown function $x(t)$. Another hindrance to this method for obtaining error bounds is the possibility that the integrand $K(s,t)x(t)$ of the integral transform has a low degree of continuity in t, as occurs, for example, in the important applications in which $K(s,t)$ is a Green's function. Thus, even when $x(t)$ is known to be fairly smooth, one may have to settle for a small value of m in (5.8), and a correspondingly large majorant function $C_m(n)$ [5, pp. 257-260].

To avoid some of these difficulties, use may be made of a device which gives good results in many cases, although it is not completely rigorous. If the same quadrature formula is applied with two different values of n, say p and q, then

$$E_p[f]/E_q[f] \simeq [C_m(p)/C_m(q)]\,\frac{f^{(m)}(\tau_p)}{f^{(m)}(\tau_q)} . \tag{5.9}$$

Assuming that $f^{(m)}(\tau_p)$ and $f^{(m)}(\tau_q)$ are approximately equal, the ratio of the errors $E_p[f]$ to $E_q[f]$ can be estimated by the computable value of $C_m(p)/C_m(q)$. For example, commonly

$$C_m(n) = c\left(\frac{1}{n}\right)^m , \tag{5.10}$$

where c is a known constant, and

(5.11) $$C_m(q)/C_m(p) = (p/q)^m .$$

A frequent choice is $q = 2p$, which leads to the rule of thumb that doubling the nodes of the rule of numerical integration reduces the truncation error by a factor of 2^{-m} or, in other words, gives m additional binary digits of accuracy. To apply this or other estimates obtained from (5.11), one may simply compare the number of digits which agree in the two answers computed, or apply a suitable extrapolation formula [11, pp. 231-237]. In order to use this method of error estimation, the system of equations (3.2) (or one of its equivalent forms) must be set up and solved for at least two values of n. This will involve considerably more labor than the corresponding operation for integration of a function of a single variable. In addition, more than two numerical solutions of the integral equation for different values of n may be required to provide assurance that a theoretical convergence rate as predicted, for example, by (5.10), is actually being observed empirically.

5.3. Collectively compact operator approximation theory. Error estimates for the Nyström method can be obtained on the basis of the theory of collectivel compact operator approximation developed by Anselone [1]. As this application of the general theory is also explained clearly and in detail in the book by Atkinson [4, pp. 88-104], only the essential features of this approach will be summarized here. The setting for this technique, which provides rigorous results, is some Banach space of functions $x(s)$ defined on the interval $0 \le s \le 1$. Usual examples are the space $C[0,1]$ of continuous functions with the maximum norm, or $L_2[0,1]$, the Hilbert space of functions with Lebesgue integrable squares and the norm defined in the ordinary way as the square root of the integral of the square of the function. The norm of an element x of the space will be denoted in the customary fashion by $\|x\|$. In order to avoid possible confusion, a different notation will be used for the operator norm of a linear operator L that maps the space into itself, which is defined by

(5.12) $$M(L) = \sup_{\|x\|=1} \|Lx\| .$$

(Often $\|L\|$ is written for $M(L)$, as in the literature cited [1,4], it being clear from the context whether the element or the operator norm is meant.) For example, if K is a linear integral operator in the space $C[0,1]$ with

continuous kernel $K(s,t)$, then

(5.13) $$M(K) = \max_{0\le s\le 1} \int_0^1 |K(s,t)|dt .$$

In $L_2[0,1]$, if $K(s,t)$ is symmetric and positive definite with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \ldots$, then

(5.14) $$M(K) = \frac{1}{\lambda_1} .$$

For arbitrary integral operators K in $L_2[0,1]$, one has

(5.15) $$M(K) \le \left(\int_0^1 \int_0^1 |K(s,t)|^2 dsdt \right)^{\frac{1}{2}} ,$$

which is computationally more tractable than expressions of the form (5.14); however, the right-hand side of inequality (5.15) may overestimate $M(K)$ grossly.

Fundamental to the Anselone theory is the definition of the numerical integration operator Q_n by

(5.16) $$Q_n x = R_n[Kx] ,$$

or

(5.17) $$Q_n x(s) = \sum_{j=1}^{n} K(s,t_j) w_j x(t_j) ,$$

for the given nodes and weights of the rule of numerical integration considered. A straightforward application of approximation theory is not possible at this point, as the operator Q_n does not approximate K in the operator norm; in fact [4, p. 90]

(5.18) $$M(K - Q_n) \ge M(K)$$

independently of the value of n. However, provided that the set of numerical integration operators $\{Q_n\}$ has a technical property known as collective compactness [4, pp. 3-4], error estimates may be obtained in terms of the operator norm of

(5.19) $$\Delta_n = (Q_n - K)K = Q_nK - K^2$$

which has the kernel

(5.20) $$\Delta_n(s,t) = \sum_{j=1}^{n} K(s,t_j)w_jK(t_j,t) - \int_0^1 K(s,r)K(r,t)dr \ .$$

In this formulation, the Nyström approximation $z(s)$ is obtained by solving the equation

(5.21) $$(I - \lambda Q_n)z = y \ .$$

In the nonsingular case, $(I - \lambda Q_n)^{-1}$ exists, and

(5.22) $$z = (I - \lambda Q_n)^{-1}y \ .$$

Furthermore [1, pp. 11-12], if

(5.23) $$\delta_n = M((I - \lambda Q_n)^{-1})M(\lambda^2\Delta_n) < 1 \ ,$$

then $(I - \lambda K)^{-1}$ exists (that is, the original integral equation (1.1) has a unique solution $x(s)$), and

(5.24) $$\|z - x\| \leq \frac{M((I - \lambda Q_n)^{-1})\|\lambda Q_n y - \lambda K y\| + \delta_n\|z\|}{1 - \delta_n} \ .$$

As the property of collective compactness of the set of operators $\{Q_n\}$ and the fact that $\lim_{n\to\infty} \delta_n = 0$ (including a rate of convergence) may be verified fairly easily for convergent numerical integration rules commonly used in practice, inequality (5.24) provides an error bound which is both rigorous and computable. Some difficulty may be encountered in obtaining precise bounds, particularly if the transformed function Ky and the kernel of the iterated operator K^2 cannot be calculated explicitly; however, satisfactory overestimates for the quantities on the right-hand side of inequality (5.24) can usually be obtained. It is worth noting that if $K(s,t)$ is symmetric (or Hermitian), then it follows directly from (5.20) that $\Delta_n(s,t)$ has the same property. This may be useful in $L_2[0,1]$, as sharper bounds for the operator norm $M(\Delta_n)$ than given by (5.15) may be computable in terms of approximate eigenvalues of $\Delta_n(s,t)$.

In order for the error bound given by (5.24) to be small, the function Q_ny must be a good approximation to the transformed function Ky, as measured by the norm of their difference $\|Q_ny - Ky\|$, and the finite rank operator Q_nK must be close to the iterated operator K^2 in the operator norm topology. It will be shown below that the Nyström method can also be formulated in terms of finite rank operators K_n which approximate K directly, leading to simpler error bounds than (5.24).

5.4. Approximation by equations of finite rank. A standard procedure in the numerical analysis of the integral equation (1.1) is to approximate it by an equation

(5.25) $$z(s) - \lambda \int_0^1 L(s,t)z(t)dt = y(s), \quad 0 \leq s \leq 1 \ ,$$

which can be solved for $z(s)$ [15]. Here, one looks for estimates of $\|x - z\|$ in terms of $M(K - L)$, $M((I - \lambda L)^{-1})$, and perhaps $\|z\|$, it being assumed that these values or upper bounds for them are computable. For example, in the nonsingular case that $(I - \lambda K)^{-1}$ and $(I - \lambda L)^{-1}$ exist, it is easy to verify the identity

(5.26) $$(I - \lambda K)^{-1} - (I - \lambda L)^{-1} = \lambda(I - \lambda L)^{-1}(K - L)(I - \lambda K)^{-1} \ .$$

Operating on y with both sides of (5.26) gives

(5.27) $$x - z = \lambda(I - \lambda L)^{-1}(K - L)x \ ,$$

and thus

(5.28) $$\frac{\|x - z\|}{\|x\|} \leq |\lambda|M((I - \lambda L)^{-1})M(K - L)$$

gives a computable bound for the relative error $\|x - z\|/\|x\|$. By symmetry,

(5.29) $$\frac{\|x - z\|}{\|z\|} \leq |\lambda|M((I - \lambda K)^{-1})M(K - L) \ ,$$

which requires an estimate for $M((I - \lambda K)^{-1})$ to be computable. However, if

(5.30) $$\theta(K - L) = M((I - \lambda L)^{-1})M(K - L) < \frac{1}{|\lambda|} \ ,$$

then $(I - \lambda K)^{-1}$ exists, and

(5.31) $$M((I - \lambda K)^{-1}) \leq \frac{M((I - \lambda L)^{-1})}{1 - |\lambda|\theta(K - L)}$$

[16, pp. 50-57]. Substitution of (5.31) into (5.29) yields

(5.32) $$\|x - z\| \leq \frac{|\lambda|\theta(K - L)}{1 - |\lambda|\theta(K - L)} \|z\| ,$$

which provides a computable bound for the absolute error $\|x - z\|$. The error bounds (5.28) and (5.32) are simpler than (5.24).

The error analysis of the Nyström method conducted here will be based on approximation of (1.1) by equations (5.25) with kernels of finite rank, that is, $L(s,t) = F_n(s,t)$, where

(5.33) $$F_n(s,t) = \sum_{j=1}^{n} u_j(s)v_j(t) ,$$

and $\{u_1(s),u_2(s),\ldots,u_n(s)\}$, $\{v_1(t),v_2(t),\ldots,v_n(t)\}$ are sets of linearly independent functions. More particularly, the choice $u_j(s) = K(s,t_j)w_j$, $j = 1,2,\ldots,n$ will be made for the Nyström method, which leads to approximate kernels of the form

(5.34) $$K_n(s,t) = \sum_{j=1}^{n} K(s,t_j)w_j v_j(t) ,$$

and error bounds corresponding to (5.28) and (5.32) with $L = K_n$.

Error analysis of a number of methods for the numerical solution of integral equations can be carried out on the basis of approximation by finite rank equations with kernels (5.33) or (5.34), including collocation-interpolation procedures which have the Nyström method as a special case. The papers by Phillips [14], Noble [12], and Sloan [17] describe various methods for which the present approach is suitable. One of the principal results of this paper is to show that the Nyström method is of optimal accuracy with respect to a class of approximations of the kernel $K(s,t)$ by finite rank kernels of the form (5.34).

6. SOLUTION OF FINITE RANK EQUATIONS. In 1907, Goursat [8] gave the recipe for solving Fredholm integral equations of second kind with finite rank kernels (5.33), that is, equations of the form

(6.1) $$z(s) - \lambda \int_0^1 \sum_{j=1}^{n} u_j(s)v_j(t)z(t)dt = y(s) .$$

In terms of the ordinary inner product

(6.2) $$(f,g) = \int_0^1 f(t)g(t)dt ,$$

which will be assumed to be defined whether or not the function space considered is a Hilbert space, equation (6.1) may be written

(6.3) $$z(s) - \lambda \sum_{j=1}^{n} u_j(s)(v_j,z) = y(s) .$$

By taking the inner products of equation (6.3) with $v_1(s),v_2(s),\ldots,v_n(s)$ in turn, one obtains the finite system of linear algebraic equations

(6.4) $$(v_i,z) - \lambda \sum_{j=1}^{n} (v_i,u_j)(v_j,z) = (v_i,y) ,$$

$i = 1,2,\ldots,n$, for the unknown inner products $(v_1,z),(v_2,z),\ldots,(v_n,z)$. In terms of the solutions of (6.4), the solution of (6.3) and hence of the finite rank integral equation (6.1) may be written as

(6.5) $$z(s) = y(s) + \lambda \sum_{j=1}^{n} u_j(s)(v_j,z) .$$

There is an obvious similarity between (6.3) and the Nyström equation (2.5), the linear algebraic system (6.4) and the collocation equations (3.2), and finally between the solution (6.5) and the interpolation formula (4.1). These similarities will be exploited below to obtain error bounds. The principal difference between the two sets of equations is that the Nyström method makes use of interpolation data, that is, values of the functions involved at specified points, while the inner products (6.2) involved in the Goursat equations (6.3)-(6.5) require values of the functions considered on the entire interval $0 \leq t \leq 1$ (except possibly for sets of measure zero), which will be called approximation data.

In the nonsingular case, it is customary to express the solution of the integral equation (1.1) as

(6.6) $$x(s) = y(s) + \lambda \int_0^1 \Gamma(s,t;\lambda)y(t)dt ,$$

where $\Gamma(s,t;\lambda)$ is called the resolvent kernel of the kernel $K(s,t)$. The resolvent kernel $G_n(s,t;\lambda)$ of the finite rank kernel $F_n(s,t)$ defined by (5.33) may be expressed in terms of the inverse matrix $B = (\beta_{ij}) = A^{-1}$ of the coefficient matrix

$$A = (\delta_{ij} - \lambda(v_i,u_j)) \tag{6.7}$$

of the linear system (6.4) as

$$G_n(s,t;\lambda) = \sum_{j=1}^{n}\sum_{k=1}^{n} u_j(s)\beta_{jk}v_k(t) \ . \tag{6.8}$$

Equation (6.8) follows directly by substitution of the solutions

$$(v_j,z) = \sum_{j=1}^{n} \beta_{jk}(v_k,y), \qquad j = 1,2,\ldots,n \ , \tag{6.9}$$

of the system (6.4) into (6.5). There are a number of ways to express the resolvent kernel (6.8) as a kernel of rank n in the form (5.33). For example, defining

$$V_j(t) = \sum_{k=1}^{n} \beta_{jk}v_k(t), \qquad j = 1,2,\ldots,n \ , \tag{6.10}$$

one obtains

$$G_n(s,t;\lambda) = \sum_{j=1}^{n} u_j(s)V_j(t) \ . \tag{6.11}$$

Other expressions may be obtained by summing over j first, or by manipulation of the matrix B such as reduction to a canonical form, LU or singular value decomposition, etc., in order to write the double sum in (6.8) as a single summation of products of functions $U_j(s)V_j(t)$, $j = 1,2,\ldots,n$. In some applications, one of these alternative forms may be more useful than (6.11).

In the singular case, λ is an eigenvalue of the kernel $F_n(s,t)$, and the homogeneous system

$$(v_i,z) - \lambda \sum_{j=1}^{n} (v_i,u_j)(v_j,z) = 0, \qquad i = 1,2,\ldots,n \ , \tag{6.12}$$

corresponding to (6.4) has d linearly independent sets of solutions $(v_1,z)_k,(v_2,z)_k,\ldots,(v_n,z)_k$, $k = 1,2,\ldots,d$, where $d = n - r$ is called the defect of the matrix A of coefficients of (6.12), r being its rank. In terms of these solutions of (6.12), d linearly independent (right) eigenfunctions $z_1(s),z_2(s),\ldots,z_d(s)$ of $F_n(s,t)$ are

$$z_k(s) = \lambda \sum_{j=1}^{n} u_j(s)(v_j,z)_k, \qquad k = 1,2,\ldots,d \ . \tag{6.13}$$

Any given right eigenfunction of $F_n(s,t)$ may be expressed as a linear combination of the functions (6.13).

By the Fredholm theory [6], if λ is an eigenvalue, then the transposed homogeneous system

$$(p,u_j) - \lambda \sum_{i=1}^{n} (p,u_i)(v_i,u_j) = 0, \qquad j = 1,2,\ldots,n \tag{6.14}$$

has d linearly independent sets of solutions $(p,u_1)_k,(p,u_2)_k,\ldots,(p,u_n)_k$, $k = 1,2,\ldots,d$, corresponding to which

$$p_k(t) = \lambda \sum_{i=1}^{n} (p,u_i)_k v_i(t), \qquad k = 1,2,\ldots,d \ , \tag{6.15}$$

form a complete set of linearly independent left eigenfunctions of the kernel $F_n(s,t)$; that is, any solution $p(t)$ of the transposed homogeneous integral equation

$$p(t) - \lambda \int_0^1 p(s) \sum_{i=1}^{n} u_i(s)v_i(t)dt = 0, \qquad 0 \le t \le 1 \ , \tag{6.16}$$

can be expressed as linear combinations of the functions (6.15). In this case, the inhomogeneous system (6.4) has solutions if and only if

$$\sum_{i=1}^{n} (p,u_i)_k(v_i,y) = 0, \qquad k = 1,2,\ldots,d \ , \tag{6.17}$$

or, in integral form,

$$\int_0^1 \left[\sum_{i=1}^{n} (p,u_i)_k v_i(t)\right] y(t)dt = \int_0^1 p_k(t)y(t)dt = 0 \ , \tag{6.18}$$

$k = 1,2,\ldots,d$. In other words, the necessary and sufficient condition that (6.1) be solvable in case λ is an eigenvalue is that $y(t)$ be orthogonal to all solutions of the transposed homogeneous integral equation (6.16). Assuming that this holds, given a particular solution $z_0(s)$ of (6.1), the general solution may be written

$$z(s) = z_0(s) + \sum_{k=1}^{d} a_k z_k(s) , \tag{6.19}$$

where $a_1, a_2, \ldots, a_d$ are arbitrary.

Thus, approximation of the integral equation (1.1) by a finite rank equation (6.1) provides a way to construct an approximate solution and resolvent kernel in the nonsingular case, and approximate eigenvalues and left and right eigenfunctions from the corresponding homogeneous equations. To handle the inhomogeneous equation in the singular case, it may be necessary to approximate also the right-hand side of (1.1) by a function which satisfies (6.18).

7. IDENTIFICATION OF THE NYSTRÖM METHOD WITH FINITE RANK APPROXIMATIONS. On the basis of the above, it can be seen that the results of the Nyström method are identical to those obtained by approximation of (1.1) by finite rank equations with kernels $K_n(s,t)$ of the form (5.34), provided that the functions $v_1(t), v_2(t), \ldots, v_n(t)$ are chosen to satisfy the $n^2 + n$ conditions

$$(v_i, K_j) = \int_0^1 v_i(t) K(t,t_j)\,dt = K(t_i,t_j) , \tag{7.1}$$

$i,j = 1,2,\ldots,n$, where $K_j(t) = K(t,t_j)$, and

$$(v_i, y) = \int_0^1 v_i(t) y(t)\,dt = y(t_i) , \tag{7.2}$$

$i = 1,2,\ldots,n$.

There are many ways to find functions $v_i(t)$, $i = 1,2,\ldots,n$, which satisfy (7.1) and (7.2). For example, suppose that a reproducing kernel $R(s,t)$ [3; 7, pp. 146-160] is known for a space containing the functions $K_j(s)$, $j = 1,2,\ldots,n$, (or, more generally, the functions $K_t(s) = K(s,t)$ for $0 \le t \le 1$) and $y(s)$, then one may take

$$v_i(t) = R(t_i,t), \quad i = 1,2,\ldots,n . \tag{7.3}$$

A less exotic way to determine suitable $v_1(t), v_2(t), \ldots, v_n(t)$ would be as linear combinations of functions for which the integrals in (7.1) and (7.2) can be calculated explicitly. The space Ω of functions $\omega(t)$ orthogonal to $K_1(t), K_2(t), \ldots, K_n(t)$, and $y(t)$ is infinite, with codimension at most $n + 1$, and if $v_i(t)$, $i = 1,2,\ldots,n$, satisfy the linear constraints (7.1) and (7.2), then so do the functions $v_i(t) + \omega_i(t)$ for arbitrary $\omega_i \in \Omega$, $i = 1,2,\ldots,n$. Thus, given the integral equation (1.1) and the rule of numerical integration R_n, the class of finite rank kernels for which (7.1) and (7.2) hold will be denoted by

$$\{K_n(s,t)\} = \{K(s,t), R_n, y(s)\} . \tag{7.4}$$

A special notation will be used for the homogeneous case $y(s) \equiv 0$, namely

$$\{K_n(s,t)\}_0 = \{K(s,t), R_n, 0\} , \tag{7.5}$$

for which $K_n(s,t)$ has to satisfy only the conditions (7.1). The notations $\{K_n\} = \{K, R_n, y\}$ and $\{K_n\}_0 = \{K, R_n, 0\}$ will be used for the corresponding classes of finite rank linear integral operators K_n with kernels $K_n(s,t)$.

8. THE NYSTRÖM CONSTANT AND ERROR BOUNDS. As approximation of the integral equation (1.1) by any finite rank equation

$$z(s) - \lambda \int_0^1 K_n(s,t) z(t)\,dt = y(s), \quad 0 \le s \le 1 , \tag{8.1}$$

with kernel $K_n(s,t)$ chosen from $\{K_n(s,t)\}$ gives precisely the same results as the Nyström method with the rule of numerical integration R_n, the accuracy of the Nyström solution can be studied in terms of how well the integral operator K can be approximated by finite rank operators K_n with kernels $K_n(s,t)$ belonging to $\{K_n(s,t)\}$. To this end, define

$$\nu = \nu(K, R_n, y) = \inf_{K_n \in \{K_n\}} M(K - K_n) \tag{8.2}$$

to be the Nyström constant for the given integral equation and rule of numerical integration. In the homogeneous case, the notation

$$\nu_0 = \inf_{K_n \in \{K_n\}_0} M(K - K_n) \tag{8.3}$$

will be used. As the constraints (7.2) are automatically satisfied in the homogeneous case, one has $\{K_n\} = \{K,R_n,y\} \subset \{K_n\}_0$ for arbitrary y, and thus

$$\nu_0 \le \nu = \nu(K,R_n,y) . \tag{8.4}$$

The minimal Nyström constant ν_0 is appropriate for the eigenvalue-eigenfunction problem for $K(s,t)$, and estimation of the accuracy with which the resolvent kernel $\Gamma(s,t;\lambda)$ of $K(s,t)$ can be approximated by resolvent kernels

$$\Gamma_n(s,t;\lambda) = \sum_{j=1}^{n}\sum_{k=1}^{n} K(s,t_j)w_j\beta_{jk}v_k(t) \tag{8.5}$$

of $K_n(s,t)$ belonging to $\{K_n(s,t)\}_0$, as the function $y(s)$ is not involved in this calculation.

In the nonsingular case, for

$$B_0 = M((I - \lambda K)^{-1}) , \tag{8.6}$$

the inequality

$$\frac{\|x - z\|}{\|z\|} \le |\lambda|\nu B_0 \tag{8.7}$$

follows from (5.29) and (8.2). Hence, the accuracy of the Nyström method is optimal with respect to approximation of the integral operator K by finite rank operators K_n belonging to the class $\{K_n\}$. This supports the observation that good results are usually obtained in practice, as in the examples cited by Nyström [13] and Atkinson [4, pp. 102-104].

It is also interesting to note that either $(I - \lambda K_n)^{-1}$ exists for all $K_n \in \{K_n\}_0$ or λ is an eigenvalue of all the kernels $K_n(s,t)$ belonging to $\{K_n(s,t)\}_0$. This is true because the invertibility of $I - \lambda K_n$ is equivalent by construction to that of the matrix A given by (5.4) for all kernels $K_n(s,t) \in \{K_n(s,t)\}_0$ (and hence for all $K_n(s,t) \in \{K(s,t),R_n,y(s)\}$ for arbitrary $y(s)$). If $A^{-1} = B$ exists, then the classes $\{K_n\}_0$ and $\{K_n\}$ are said to be nonsingular. A sufficient condition for nonsingularity of these classes is that $(I - \lambda K)^{-1}$ exist, and

$$|\lambda|\nu_0 B_0 < 1 . \tag{8.8}$$

The Nyström method determines the approximate solution $z(s)$ uniquely, but not the approximate resolvent kernel (8.5), as the functions $v_1(t),v_2(t),\ldots,v_n(t)$ are only required to satisfy (7.1). In terms of the resolvent operators Γ of K and Γ_n of K_n, the identity (5.26) may be written

$$\Gamma - \Gamma_n = (I + \lambda\Gamma_n)(K - K_n)(I + \lambda\Gamma) \tag{8.9}$$

in the nonsingular case. Supposing that (8.8) holds, choose $\varepsilon > 0$ and $K_n^{\varepsilon} \in \{K_n\}_0$ such that

$$M(K - K_n^{\varepsilon}) \le \nu_0 + \varepsilon < \frac{1}{|\lambda|B_0} . \tag{8.10}$$

Then,

$$M(I + \lambda\Gamma_n^{\varepsilon}) = M((I - \lambda K_n^{\varepsilon})^{-1}) \le \frac{B_0}{1 - |\lambda|(\nu_0 + \varepsilon)B_0} , \tag{8.11}$$

and, from (8.9),

$$M(\Gamma - \Gamma_n^{\varepsilon}) \le \frac{(\nu_0 + \varepsilon)B_0^2}{1 - |\lambda|(\nu_0 + \varepsilon)B_0} . \tag{8.12}$$

Thus, the distance γ_0 from Γ to the class $\{\Gamma_n\}_0$ of resolvent operators of the finite rank operators K_n belonging to $\{K_n\}_0$ satisfies

$$\gamma_0 = \inf_{\Gamma_n \in \{\Gamma_n\}_0} M(\Gamma - \Gamma_n) \le \frac{\nu_0 B_0^2}{1 - |\lambda|\nu_0 B_0} \tag{8.13}$$

in the operator norm. The use of a resolvent kernel (8.5) selected from $\{\Gamma_n(s,t;\lambda)\}_0$ to solve equations (8.1) for various choices of $y(s)$ actually amounts to a modification of the Nyström method by the use of approximation data

$$y_i = \langle v_i,y\rangle, \quad i = 1,2,\ldots,n , \tag{8.14}$$

on the right-hand side of (3.2) instead of the interpolation data $y_i = y(t_i)$, $i = 1,2,\ldots,n$. The two sets of data will be identical, of course, if $\Gamma_n(s,t;\lambda)$ is the resolvent kernel of a kernel $K_n(s,t)$ belonging to the class $\{K_n(s,t)\} = \{K(s,t),R_n,y(s)\}$.

$k = 1,2,\ldots,d$. In other words, the necessary and sufficient condition that (6.1) be solvable in case λ is an eigenvalue is that $y(t)$ be orthogonal to all solutions of the transposed homogeneous integral equation (6.16). Assuming that this holds, given a <u>particular solution</u> $z_0(s)$ of (6.1), the <u>general solution</u> may be written

$$z(s) = z_0(s) + \sum_{k=1}^{d} a_k z_k(s) , \tag{6.19}$$

where $a_1, a_2, \ldots, a_d$ are arbitrary.

Thus, approximation of the integral equation (1.1) by a finite rank equation (6.1) provides a way to construct an approximate solution and resolvent kernel in the nonsingular case, and approximate eigenvalues and left and right eigenfunctions from the corresponding homogeneous equations. To handle the inhomogeneous equation in the singular case, it may be necessary to approximate also the right-hand side of (1.1) by a function which satisfies (6.18).

7. <u>IDENTIFICATION OF THE NYSTRÖM METHOD WITH FINITE RANK APPROXIMATIONS</u>. On the basis of the above, it can be seen that the results of the Nyström method are identical to those obtained by approximation of (1.1) by finite rank equations with kernels $K_n(s,t)$ of the form (5.34), provided that the functions $v_1(t), v_2(t), \ldots, v_n(t)$ are chosen to satisfy the $n^2 + n$ conditions

$$(v_i, K_j) = \int_0^1 v_i(t) K(t,t_j)\,dt = K(t_i,t_j) , \tag{7.1}$$

$i,j = 1,2,\ldots,n$, where $K_j(t) = K(t,t_j)$, and

$$(v_i, y) = \int_0^1 v_i(t) y(t)\,dt = y(t_i) , \tag{7.2}$$

$i = 1,2,\ldots,n$.

There are many ways to find functions $v_i(t)$, $i = 1,2,\ldots,n$, which satisfy (7.1) and (7.2). For example, suppose that a reproducing kernel $R(s,t)$ [3; 7, pp. 146-160] is known for a space containing the functions $K_j(s)$, $j = 1,2,\ldots,n$, (or, more generally, the functions $K_t(s) = K(s,t)$ *for* $0 \le t \le 1$) *and* $y(s)$, *then one may take*

$$v_i(t) = R(t_i,t), \quad i = 1,2,\ldots,n . \tag{7.3}$$

A less exotic way to determine suitable $v_1(t), v_2(t), \ldots, v_n(t)$ would be as linear combinations of functions for which the integrals in (7.1) and (7.2) can be calculated explicitly. The space Ω of functions $\omega(t)$ orthogonal to $K_1(t), K_2(t), \ldots, K_n(t)$, and $y(t)$ is infinite, with codimension at most $n + 1$, and if $v_i(t)$, $i = 1,2,\ldots,n$, satisfy the linear constraints (7.1) and (7.2), then so do the functions $v_i(t) + \omega_i(t)$ for arbitrary $\omega_i \in \Omega$, $i = 1,2,\ldots,n$. *Thus, given the integral equation* (1.1) *and the rule of* numerical integration R_n, the class of finite rank kernels for which (7.1) and (7.2) hold will be denoted by

$$\{K_n(s,t)\} = \{K(s,t), R_n, y(s)\} . \tag{7.4}$$

A special notation will be used for the homogeneous case $y(s) \equiv 0$, namely

$$\{K_n(s,t)\}_0 = \{K(s,t), R_n, 0\} , \tag{7.5}$$

for which $K_n(s,t)$ has to satisfy only the conditions (7.1). The notations $\{K_n\} = \{K, R_n, y\}$ and $\{K_n\}_0 = \{K, R_n, 0\}$ will be used for the corresponding classes of finite rank linear integral operators K_n with kernels $K_n(s,t)$.

8. <u>THE NYSTRÖM CONSTANT AND ERROR BOUNDS</u>. As approximation of the integral equation (1.1) by any finite rank equation

$$z(s) - \lambda \int_0^1 K_n(s,t) z(t)\,dt = y(s), \quad 0 \le s \le 1 , \tag{8.1}$$

with kernel $K_n(s,t)$ chosen from $\{K_n(s,t)\}$ gives precisely the same results as the Nyström method with the rule of numerical integration R_n, the accuracy of the Nyström solution can be studied in terms of how well the integral operator K can be approximated by finite rank operators K_n with kernels $K_n(s,t)$ belonging to $\{K_n(s,t)\}$. To this end, define

$$\nu = \nu(K, R_n, y) = \inf_{K_n \in \{K_n\}} M(K - K_n) \tag{8.2}$$

to be the <u>Nyström constant</u> for the given integral equation and rule of numerical integration. In the homogeneous case, the notation

$$\nu_0 = \inf_{K_n \in \{K_n\}_0} M(K - K_n) \tag{8.3}$$

As noted above, the Nyström method may also be applied to the homogeneous integral equation

$$x(s) - \lambda \int_0^1 K(s,t)x(t)dt = 0, \quad 0 \le s \le 1 , \tag{8.15}$$

to obtain approximate eigenvalues and eigenfunctions of $K(s,t)$ by solving the homogeneous system

$$z_i - \lambda \sum_{j=1}^{n} K(t_i,t_j)w_j z_j = 0, \quad i = 1,2,\ldots,n , \tag{8.16}$$

and using the interpolation formula

$$z(s) = \lambda \sum_{j=1}^{n} K(s,t_j)w_j z_j \tag{8.17}$$

for the corresponding right eigenfunctions. By construction, all the kernels $K_n(s,t)$ in the class $\{K_n(s,t)\}_0$ have the same sets of eigenvalues $\lambda_1^{(n)},\lambda_2^{(n)},\ldots,\lambda_n^{(n)}$ and corresponding right eigenfunctions $z_1^{(n)}(s)$, $z_2^{(n)}(s),\ldots,z_n^{(n)}(s)$, including the possibilities of multiplicity of eigenvalues and the existence of generalized eigenfunctions. This is because the functions $v_1(t),v_2(t),\ldots,v_n(t)$ do not enter into these calculations explicitly.

It is also possible to obtain $O(\nu_0)$ error estimates for $|\lambda - \lambda^{(n)}|$ and $\|z - z^{(n)}\|$, for example, by setting up the eigenvalue-eigenfunction problem as a nonlinear operator equation, and use of the Kantorovich theorem [2] or some other technique [12, pp. 228-231]. As these are outside the scope of this paper, explicit error bounds will not be given here.

The calculation of approximate left eigenfunctions $p_1^{(n)}(t),p_2^{(n)}(t)$, $\ldots,p_n^{(n)}(t)$ is carried out on the basis of the transposed homogeneous system

$$p_j - \lambda \sum_{i=1}^{n} p_i K(t_i,t_j)w_j = 0, \quad j = 1,2,\ldots,n , \tag{8.18}$$

and the interpolation formula

$$p(t) = \lambda \sum_{i=1}^{n} p_i v_i(t) \tag{8.19}$$

for the left eigenfunctions of $K_n(s,t)$, which are hence linear combinations of $v_1(t),v_2(t),\ldots,v_n(t)$, and depend on the particular kernel chosen, unlike (8.17). In this case, the inhomogeneous equation (8.1) will be solvable without further approximation of $y(s)$ if

$$(v_i,y) = 0, \quad i = 1,2,\ldots,n , \tag{8.20}$$

or, if $K_n(s,t) \in \{K(s,t),R_n,y(s)\}$, one has

$$y(t_i) = 0, \quad i = 1,2,\ldots,n . \tag{8.21}$$

Conditions (8.20) and (8.21) are sufficient that $(p,y) = 0$ for all solutions $p(t)$ of the transposed homogeneous equation

$$p(t) - \lambda \int_0^1 p(s)K_n(s,t)ds = 0, \quad 0 \le t \le 1 , \tag{8.22}$$

for the eigenvalue λ.

9. COMPUTABLE ERROR BOUNDS. The error bounds depending on the Nyström constant given by (8.7) or (8.13), for example, are theoretical in character, as the unknown (but fixed) quantity B_0 is involved. However, for some given operator $K_n \in \{K_n\}$, the quantities

$$\nu_n = M(K - K_n), \quad B_n = M((I - \lambda K_n)^{-1}) \tag{9.1}$$

may be computed in the nonsingular case, and, if $|\lambda|\nu_n B_n < 1$, then $(I - \lambda K)^{-1}$ exists, and B_0 may be estimated by

$$B_0 \le \frac{B_n}{1 - |\lambda|\nu_n B_n} , \tag{9.2}$$

as follows from (5.31). The known value ν_n may, of course, be used as an upper bound for ν or ν_0. One has also

$$\frac{\|x - z\|}{\|x\|} \le |\lambda|\nu_n B_n , \tag{9.3}$$

directly from (5.28).

Finding the Nyström constant ν (or ν_0) requires the solution of a nonlinear optimization problem subject to linear constraints (7.1) and (7.2) (or (7.1) only). This could be of comparable or greater difficulty than the

original task of solving the linear integral equation (1.1). In some circumstances, however, it might be desired to estimate the Nyström constant directly, rather than use a particular kernel $K_n(s,t)$ belonging to $\{K_n(s,t)\}$ or $\{K_n(s,t)\}_0$. It would also be useful if the estimation process furnishes information concerning good choices of the functions $v_1(t),v_2(t),\ldots,v_n(t)$. This can be done in $L_2[0,1]$ by classical calculus of variations technique applied to the upper bound (5.15) for the functional $M(K)$. Setting $v = (v_1,v_2,\ldots,v_n)$, this approximating optimization problem may be posed as

$$\text{(9.4)} \qquad \textit{minimize } F[v] = \int_0^1 \int_0^1 |K(s,t) - K_n(s,t)|^2 dsdt ,$$

with v subject to (7.1) and (7.2). Here, the fully constrained case will be treated in detail, with appropriate modifications indicated for the problem in $\{K_n(s,t)\}_0$ corresponding to only the constraints (7.1), and also the unconstrained problem of minimizing $F[v]$ over all v with $v_i \in L_2[0,1]$, $i = 1,2,\ldots,n$. Introducing Lagrange multipliers A_{ij} for the constraints (7.1) and A_i for (7.2), the kernels of the Gâteaux derivatives of the functional

$$\text{(9.5)} \qquad \Phi[v] = F[v] + \sum_{i=1}^n \sum_{j=1}^n A_{ij}((v_i,K_j) - K(t_i,t_j)) + \sum_{i=1}^n A_i((v_i,t) - y(t_i))$$

with respect to $v_1,v_2,\ldots,v_n$ will vanish at the solution of the optimization problem. This necessary condition is equivalent to the system of equations

$$\text{(9.6)} \qquad -2\int_0^1 K_i(s)K(s,t)w_i ds + 2\sum_{j=1}^n \left(\int_0^1 K_i(s)K_j(s)w_i w_j ds\right)v_j(t) +$$

$$+ \sum_{j=1}^n A_{ij}K_j(t) + A_i y(t) = 0, \qquad i = 1,2,\ldots,n .$$

This can be simplified somewhat by writing

$$\text{(9.7)} \qquad \psi_i(t) = \int_0^1 K_i(s)K(s,t)w_i ds, \qquad i = 1,2,\ldots,n ,$$

and

$$\text{(9.8)} \qquad c_{ij} = \int_0^1 K_i(s)K_j(s)w_i w_j ds, \qquad i,j = 1,2,\ldots,n .$$

Thus, (9.6) becomes

$$\text{(9.9)} \qquad \sum_{j=1}^n c_{ij}v_j(t) = \psi_i(t) - \tfrac{1}{2}\sum_{j=1}^n A_{ij}K_j(t) - \tfrac{1}{2}A_i y(t) ,$$

$i = 1,2,\ldots,n$. If now for $C = (c_{ij})$, one has that $C^{-1} = D = (d_{ij})$ exists, then

$$\text{(9.10)} \qquad v_i(t) = \sum_{j=1}^n d_{ij}\psi_j(t) - \tfrac{1}{2}\sum_{j=1}^n \sum_{k=1}^n d_{ij}A_{jk}K_k(t) -$$

$$- \tfrac{1}{2}\sum_{j=1}^n d_{ij}A_j y(t), \qquad i = 1,2,\ldots,n ,$$

which expresses the functions $v_1(t),v_2(t),\ldots,v_n(t)$ as linear combinations of the unknown Lagrange multipliers, with known functions as coefficients. The unconstrained solutions $\hat{v}_i(t)$ of (9.4), $i = 1,2,\ldots,n$, may be read directly from (9.10) as

$$\text{(9.11)} \qquad \hat{v}_i(t) = \sum_{j=1}^n d_{ij}\psi_j(t) = \sum_{j=1}^n d_{ij}\int_0^1 K_j(s)K(s,t)w_j ds ,$$

which does not involve the Lagrange multipliers. The kernel

$$\text{(9.12)} \qquad \hat{K}_n(s,t) = \sum_{j=1}^n K(s,t_j)w_j\hat{v}_j(t)$$

is the best approximation to $K(s,t)$ in the sense of (9.4), not necessarily in the operator norm. To use (9.12), one must set up and solve the linear system (6.4), which requires approximation data on both sides, and gives results which will differ from the Nyström solution, except in special cases. Linear systems of equations for the Lagrange multipliers may be obtained by substituting (9.10) into the constraint equations (7.1) and (7.2). From (7.1),

$$\text{(9.13)} \qquad K(t_h,t_i) = \sum_{j=1}^n d_{hj}(\psi_j,K_i) - \tfrac{1}{2}\sum_{j=1}^n \sum_{k=1}^n d_{hj}(K_k,K_i)A_{jk} -$$

$$- \tfrac{1}{2}\sum_{j=1}^n d_{hj}(y,K_i)A_j, \qquad h,i = 1,2,\ldots,n ,$$

and (7.2) becomes

$$(9.14)\qquad y(t_i) = \sum_{j=1}^{n} d_{ij}(\psi_j,y) - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} d_{ij}(K_k,y)\Lambda_{jk} -$$

$$- \frac{1}{2}\sum_{j=1}^{n} d_{ij}(y,y)\Lambda_j, \qquad i = 1,2,\ldots,n .$$

If one sets $\Lambda_1 = \Lambda_2 = \ldots = \Lambda_n = 0$ in (9.13) and solves for $\Lambda_{hi} = \Lambda^0_{hi}$, $h,i = 1,2,\ldots,n$, the corresponding functions

$$(9.15)\qquad v_i^0(t) = \sum_{j=1}^{n} d_{ij}\int_0^1 K_j(s)K(s,t)w_j\,ds - \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} d_{ij}\Lambda^0_{jk}K_k(t)$$

give the kernel $K_n^0(s,t)$ which minimizes the functional $F[v]$ over $\{K_n(s,t)\}_0$. By solving (9.13) and (9.14) for the Lagrange multipliers, one can find $v_1(t),v_2(t),\ldots,v_n(t)$ for the completely constrained problem. By substitution, the estimates $\nu_0 \le F[v^0]$ and $\nu \le F[v]$ can be obtained for the respective Nyström constants.

REFERENCES

1. Anselone, P. M., Collectively Compact Operator Approximation Theory, Prentice-Hall, Englewood Cliffs, N. J., 1971.
2. Anselone, P. M. and Rall, L. B., The solution of characteristic value-vector problems by Newton's method, Numer. Math. 11 (1968), 38-45.
3. Aronszajn, N., Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950), 337-404.
4. Atkinson, K. E., A Survey of Numerical Methods for the Solution of Fredholm Integral Equations of the Second Kind, SIAM, Philadelphia, 1976.
5. Davis, P. J. and Rabinowitz, Philip, Methods of Numerical Integration, Academic Press, New York, 1975.
6. Fredholm, I., Sur une classe d'équations fonctionnelles, Acta Math. 27 (1903), 365-390.
7. Golomb, M. and Weinberger, H. F., Optimal approximation and error bounds, On Numerical Approximation, Ed. by R. E. Langer, pp. 117-190, Univ. of Wisconsin Press, Madison, 1959.
8. Goursat, É., Sur un cas élémentaire de l'équation de Fredholm, Bull. Soc. Math. France 35 (1907), 163-173.

9. Milne, W. E., Numerical Calculus, Princeton Univ. Press, Princeton, N. J., 1949.
10. Mysovskih, I. P., Lectures on Numerical Methods, tr. from Russian by L. B. Rall, Wolters-Noordhoff Publishing, Groningen, The Netherlands, 1969.
11. Noble, B., Numerical Methods II: Differences, Integration and Differential Equations, Oliver and Boyd, London, 1964.
12. Noble, B., Error analysis of collocation methods for solving Fredholm integral equations, Topics in Numerical Analysis, Ed. by John J. H. Miller, pp. 211-232, Academic Press, New York, 1973.
13. Nyström, E. J., Über die praktische Auflösung von Integralgleichungen mit Anwendungen auf Randwertaufgaben, Acta Math. 54 (1930), 185-204.
14. Phillips, J. L., Error analysis for direct linear integral equation methods, Math. Comp. 27 (1973), 849-859.
15. Rall, L. B., Error bounds for iterative solution of Fredholm integral equations, Pacific J. Math. 5 (1955), 977-986.
16. Rall, L. B., Computational Solution of Nonlinear Operator Equations, John Wiley & Sons, New York, 1969.
17. Sloan, I. H., Error analysis for a class of degenerate-kernel methods, Numer. Math. 25 (1976), 231-238.

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The classical Nyström method for the numerical solution of Fredholm integral equations of second kind consists of numerical integration, collocation, and interpolation. The approximate solution obtained by this procedure is shown to be identical to the solution of certain finite-rank integral equations with kernels belonging to a specified class $\{K_n\}$, and thus has minimal error with

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respect to approximation of the original equation over this class. A computable (but in general nonoptimal) error bound for the Nyström approximate solution can be obtained on the basis of how well a specific finite-rank integral operator with kernel in $\{K_n\}$ approximates the integral operator in the Fredholm equation being solved numerically.