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THE ASYMPTOTIC DISTRIBUTION OF THE ORDER OF ELEMENTS IN ALTERNA--ETC(U)
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THE ASYMPTOTIC DISTRIBUTION OF THE ORDER OF ELEMENTS IN ALTERNATING SEMIGROUPS AND IN PARTIAL TRANSFORMATION SEMIGROUPS

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ABSTRACT

The asymptotic distributions of the order of elements in alternating semigroups and in partial transformation semigroups are obtained and shown to coincide with that for the symmetric semigroup.

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SIGNIFICANCE AND EXPLANATION

The alternating semigroup may be regarded as the semigroup of transformations represented by matrices of "0"'s and "1"'s, each of whose rows has exactly one "1", and whose determinants are 0 or +1. Correspondingly the partial transformation semigroup is representable by matrices of "0"'s and "1"'s each of whose rows has at most one "1" .

These are among the basic semigroups of transformations and have many important applications in chemistry and physics. This paper gives some basic properties of these semigroups.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

THE ASYMPTOTIC DISTRIBUTION OF THE ORDER OF ELEMENTS

IN ALTERNATING SEMIGROUPS AND IN PARTIAL TRANSFORMATION SEMIGROUPS

Bernard Harris

1. Introduction. In the papers by P. Erdös and P. Turán [9,10,11, 12], various questions concerning statistical characteristics of the symmetric group on n letters, S_n, for large n, are studied. In particular let P denote a generic element of S_n and let $O(P)$ be the order of P. Then if $K(n,x)$ is the number of elements of S_n satisfying

(1) $$\log O(P) \le \frac{1}{2}\log^2 n + \frac{x}{\sqrt{3}}\log^{3/2} n \quad ,$$

we have

(2) $$\lim_{n\to\infty} \frac{K(n,x)}{n!} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt \quad .$$

Equivalently, if P is "chosen at random" from S_n, define $Z = (\log O(P) - \frac{1}{2}\log^2 n)/\frac{1}{\sqrt{3}}\log^{3/2} n$; then the distribution of Z converges to the standard normal distribution as $n \to \infty$. In J. Dénes, P. Erdös and P. Turán [8], this result was extended to the alternating group on n letters A_n. Letting $L(n,x)$ be the number of elements of A_n satisfying (1), they showed that

(3) $$\lim_{n\to\infty} \frac{L(n,x)}{n!/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt \quad .$$

In [11], the analogous result to (1) and (2) for the symmetric semigroup T_n was obtained. Specifically T_n is the set of all mappings of $X_n = \{x_1, x_2, \ldots, x_n\}$ into X_n. With no loss of generality, we can take $X_n = \{1, 2, \ldots, n\}$. If $\alpha, \beta \in T_n$, then the product $\alpha\beta \in T_n$ is defined by $(\alpha\beta)x = \alpha(\beta(x))$ for all $x \in X_n$. There is a one-to-one correspondence between the elements of T_n and the class of labeled directed graphs on n vertices, with each vertex having exactly one edge leaving it. Such a graph may be constructed by

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the following procedure: if $\alpha(x) = x_1$, draw the directed edge from x to x_1 (if $x = x_1$; draw a loop at x_1). For each $\alpha \in T_n$, we divide X_n into two classes, cyclical and non-cyclical elements. An $x \in X_n$ is said to be cyclical under α if there is an $m > 0$ with $\alpha^m(x) = x$. The set of cyclical elements under α will be denoted by C_α. The restriction of α to C_α will be denoted by α^*. α^* is a one-to-one mapping of C_α onto C_α and thus permutes the elements of C_α. Dénes [3,4,5,7] has called α^* the main permutation of α. Further for each $x \in X_n$, there is a least integer $r = r_\alpha(x) \geq 0$ such that $\alpha^r(x) \in C_\alpha$. We call r the α-height of x, defining the height of α by $h(\alpha) = \max_{x \in X_n} r_\alpha(x)$.

Then for any $\alpha \in T_n$, the order of $\alpha, O(\alpha)$, is defined as the number of distinct elements of T_n in the set $\{\alpha,\alpha^2,\ldots\}$. It is easily seen that if α is a permutation, this definition reduces to the usual definition. Dénes [3,4,5] has shown that this is equivalent to

$$O(\alpha) = O(\alpha^*) + \max(0,h(\alpha)-1)$$

and also equivalent to defining $O(\alpha)$ as the least m such that for $0 < q \leq m$, $\alpha^q = \alpha^{m+1}$.

Then, in Harris [13], it was shown that for $M(n,x)$, the number of elements of T_n satisfying

$$\log O(\alpha) \leq \frac{1}{8}\log^2 n + \frac{x}{\sqrt{24}}\log^{3/2} n \quad , \tag{4}$$

we have

$$\lim_{n\to\infty} \frac{M(n,x)}{n^n} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt \quad . \tag{5}$$

In the present paper, the result is extended to the alternating semigroup K_n and to the partial transformation semigroup P_n. The alternating semigroup on n letters is defined by

$$K_n = (T_n - S_n) \cup A_n \quad . \tag{6}$$

The partial transformation semigroup is the semigroup of all transformations whose domain and range are subsets of X_n. The significance of the alternating semigroup and various results concerning it may be found in J. M. Howie [15], J. Dénes [6], K. H. Kim and J. Dénes [17], Chapter IV, N. Ito [16] and O. Ore [18]. Some results for the partial transformation semigroup may be found in J. Baillieul [1] and C. D. Bass and K. H. Kim [2].

In section 2 the asymptotic distribution of the order of elements in alternating semigroups is obtained and in section 3 the corresponding result for the partial transformation semigroup is given.

2. The asymptotic distribution of $O(\alpha)$, $\alpha \in K_n$. We now show that the asymptotic distribution of the order of elements in the alternating semigroup is identical with that for the symmetric semigroup.

Clearly

(7) $$|T_n| = n^n \quad , \quad |K_n| = n^n - n!/2 \quad .$$

Let $K_n(B) = \{\alpha \in K_n : O(\alpha) \in B\}$ and let $T_n(B) = \{\alpha \in T_n : O(\alpha) \in B\}$, where B is any set of positive integers. Then, clearly

(8) $$|T_n(B)| - n!/2 \leq |K_n(B)| \leq |T_n(B)| \quad .$$

It is convenient to proceed using language and notation equivalent to assuming that the elements of K_n are equally likely and selected at random. Then $\alpha \in K_n$ is a random variable taking values in K_n. Using such notation, we have

(9) $$P\{\alpha \in K_n(B)\} = |K_n(B)|/(n^n - n!/2) \quad .$$

Hence, from (8), we have

$$P\{\alpha \in K_n(B)\} \leq |T_n(B)|/(n^n - n!/2) = \frac{|T_n(B)|}{n^n} \; \frac{n^n}{n^n - n!/2} \; ;$$

thus,

(10) $$P\{\alpha \in K_n(B)\} \leq P\{\alpha \in T_n(B)\}(1 - \delta(n))^{-1} \quad ,$$

where $\delta(n) \to 0$ as $n \to \infty$. Also, from (8) and (9),

$$P\{\alpha \in K_n(B)\} \geq \frac{|T_n(B)| - n!/2}{n^n - n!/2} \geq \frac{|T_n(B)| - n!/2}{n^n}$$

yields

(11) $$P\{\alpha \in T_n(B)\} \geq P\{\alpha \in T_n(B)\} - \delta(n) \quad .$$

Thus combining (10) and (11), we get

$$\lim_{n\to\infty} P\{\alpha \in K_n(B)\} = P\{\alpha \in T_n(B)\}$$

and hence if $N(n,x)$ is the number of elements of K_n with

$\log O(\alpha) \leq \frac{1}{8} \log^2 n + \frac{x}{\sqrt{24}} \log^{3/2} n$, then

$$(12) \qquad \lim_{n\to\infty} \frac{N(n,x)}{n^n - n!/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\,dt \ .$$

Remarks. This problem was suggested by Dr. J. Dénes and appears in a list of research problems in combinatorics that he has circulated, where it is identified as problem 4.4. There he conjectured that the asymptotic distribution for the alternating semigroup would be the same as that for the alternating group. There is however an interesting parallel of a different kind. If P belongs either to the alternating group or symmetric group on n letters,

$$\sqrt{3} \log^{-3/2}(\log O(P) - \frac{1}{2} \log^2 n)$$

has the standard normal distribution. Further if α belongs either to the symmetric semigroup or alternating semigroup on n letters, then

$$\sqrt{24} \log^{-3/2} n \ (\log O(\alpha) - \frac{1}{8} \log^2 n)$$

has the standard normal distribution.

3. The asymptotic distribution of the order of elements in partial transformation semigroups. In order to represent $\beta \in P_n$ as a labeled directed graph as described in section one, we replace X_n by $X_n^* = \{0,1,\ldots,n\}$ and represent $\beta \in P_n$ by the mapping α of X_n^* into X_n^* such that if x is in the domain of β, $\alpha(x) = \beta(x)$ and if x is not in the domain of β, $\alpha(x) = 0$. Note that this implies $\alpha(0) = 0$. It is obvious that there is a one-to-one correspondence between the elements of P_n and the mappings obtained in this manner. Furthermore this correspondence preserves the order of the mapping. Thus we can regard P_n as a particular subset of T_{n+1}. The above argument also makes it obvious that

(13) $$|P_n| = (n+1)^n$$

We will proceed by regarding the mappings α defined above as the elements of P_n and assume that they are selected at random with equal probabilities.

In the directed graph representation of α, since $\alpha(0) = 0$, 0 is the vertex of a loop. Thus $O(\alpha^*) = O(\hat{\alpha})$, where $\hat{\alpha}$ is the permutation determined by restricting α to $C_\alpha \setminus \{0\} = C_{\hat{\alpha}}$. Thus

(14) $$O(\alpha) = O(\hat{\alpha}) + \max(0,h(\alpha)-1).$$

If $C_\alpha \setminus \{0\} = \phi$, we set $O(\hat{\alpha}) = 1$.

For $\alpha \in T_n$, $L(\alpha) = |C_\alpha|$ is a random variable taking values in $\{1,2,\ldots,n\}$. In B. Harris [13], it was shown that

(15) $$P\{L(\alpha) = \ell\} = p(\ell,n) = \frac{(n-1)!\,\ell}{(n-\ell)!\,n^\ell}, \quad \ell = 1,2,\ldots,n.$$

For $\alpha \in P_n$, let $L_1(\alpha) = |C_{\hat{\alpha}}|$. $L_1(\alpha)$ is a random variable with values in $\{0,1,\ldots,n\}$ and

(16) $$P\{L_1(\ell) = \ell\} = r(\ell,n) = \frac{n!\,(\ell+1)}{(n-\ell)!\,(n+1)^{\ell+1}}, \quad \ell = 0,1,\ldots,n.$$

We present (16) without proof at this time. This result is contained in a paper under preparation and can easily be established by extending Corollary 2 to Theorem 1 in B. Harris and L. Schoenfeld [14] to the partial transformation semigroup.

Comparing (15) and (16), we see that

(17) $$p(\ell+1,n+1) = r(\ell,n), \quad \ell = 0,1,\ldots,n.$$

This observation permits us to apply the methods of B. Harris [13] with no essential changes. It is readily observed that the hypotheses of all lemmas are satisfied. In particular, if $C_{\hat{\alpha}} = \{x_{i1},x_{i2},\ldots,x_{i\ell}\}$, where the x_{ij}'s are any ℓ distinct elements of X_n . Then

$$P\{\hat{\alpha} = \hat{\alpha}_0 \mid C_{\hat{\alpha}} = \{x_{r1}, x_{r2}, \ldots, x_{r\ell}\}\} = 1/\ell! ,$$

where $\hat{\alpha}_0$ is any specified permutation of the elements of $C_{\hat{\alpha}}$. This symmetry condition is essential to the use of the results in [13] and is the technical reason for using $C_{\hat{\alpha}}$ instead of C_α .

Thus, from (17), we immediately obtain the following:

Theorem. If $R(n,x)$ is the number of elements of P_n with $\log O(\alpha) \leq \frac{1}{8}\log^2 n + \frac{x}{\sqrt{24}}\log^{3/2} n$, then

$$\lim_{n \to \infty} \frac{P(n,x)}{(n+1)^n} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{x} e^{-t^2/2}\, dt .$$

Concluding Remarks. I deem it a singular honor to participate in this volume commemorating the death of Professor Paul Turán, who was a dear friend and who inspired much of my scientific efforts. His death is a great loss to mathematics and to each of us personally.

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