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A PROCEDURE FOR ESTIMATING AN OBJECT'S POSITION BASED ON TWO OR--ETC(U)
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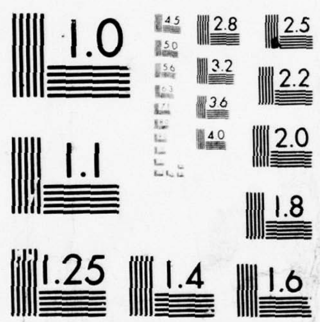

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NPS55-77-34 (Revised)

NAVAL POSTGRADUATE SCHOOL

Monterey, California

A PROCEDURE FOR ESTIMATING AN OBJECT'S POSITION BASED ON TWO OR MORE BEARINGS WITH A PROGRAM FOR A TI-59 CALCULATOR. Revision

by

R. Neagle Forrest

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September 1977

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Approved for public release; distribution unlimited.

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

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Elliptical Confidence Region
TI-59 Calculator

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Report provides a procedure for estimating an object's position based on bearings taken from or on the object for two or more stations. The Report also provides a program for the TI-59 calculator to implement the procedure.

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In this revision, the TI-59 program has been modified so that a user may now revise a location estimate by entering additional bearing data.

An example of the use of this option is given on Page 16.

TABLE OF CONTENTS

The programs in this report are for use within the Department of the Navy, and they are presented without representation or warranty of any kind.

A PROCEDURE FOR ESTIMATING AN OBJECT'S POSITION BASED ON TWO OR MORE BEARINGS WITH A PROGRAM FOR A TI-59 CALCULATOR

I. Introduction

A procedure for estimating an object's position with bearings taken on or from two or more stations is developed in Section IV of this report. In the development of the procedure, the following things are assumed: The object and the stations are fixed on the surface of a flat earth and the position of each station is known. The error in the bearing taken on or from a station is a normal random variable with a known standard deviation e and a mean of zero (if bias exists, it is known and removed); and station bearing errors are independent. The user instructions for a TI-59 program to implement the procedure are given in Section II, and the program listing is given in Section III.

As an example to illustrate a use of the program, suppose bearings are taken on an object from three stations (1, 2 and 3) as illustrated in Figure 1. Also, suppose that the assumptions stated above are satisfied and that an initial estimate of the object's position is made and that it is relatively near the object. This assumption is discussed in Section IV.

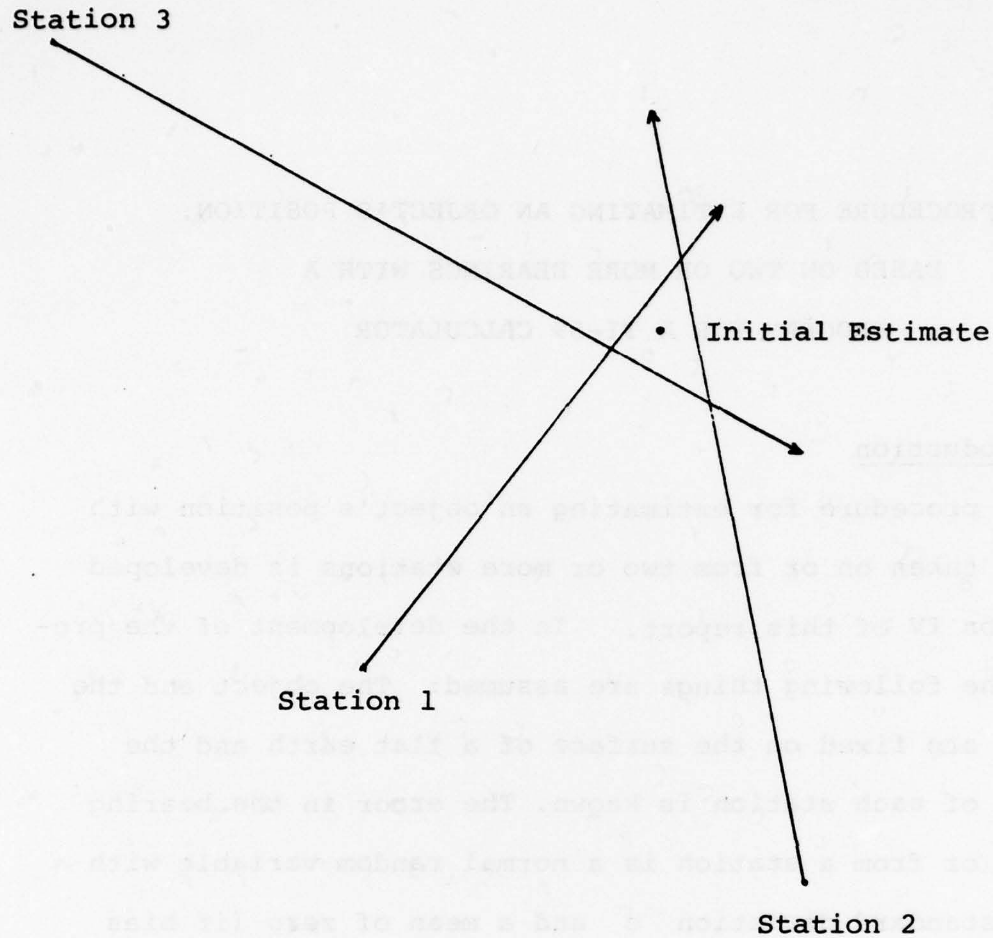

FIGURE 1. Geometry for the Example

Let the measured bearings and bearing errors (standard deviations) be:

$$\theta_1 = 35° \qquad e_1 = 4°$$

$$\theta_2 = 351° \qquad e_2 = 7°$$

$$\theta_3 = 131° \qquad e_3 = 5°$$

And let the ranges and bearings of the initial estimate be:

$r_1 = 10{,}000$ meters, $\beta_1 = 38°$

$r_2 = 15{,}000$ meters, $\beta_2 = 346°$

$r_3 = 12{,}000$ meters, $\beta_3 = 127°$.

Use of the position estimation program with this data gives a final position estimate (fix) determined by:

$$x = -512 \text{ meters}$$

$$y = -75 \text{ meters}$$

where x is its East-West distance and y is its North-South distance from the initial position estimate. The East-West, North-South xy-coordinate system with its origin at the initial estimate is shown in Figure 2. So the final position estimate is 512 meters to the West and 75 meters to the South of the initial position estimate.

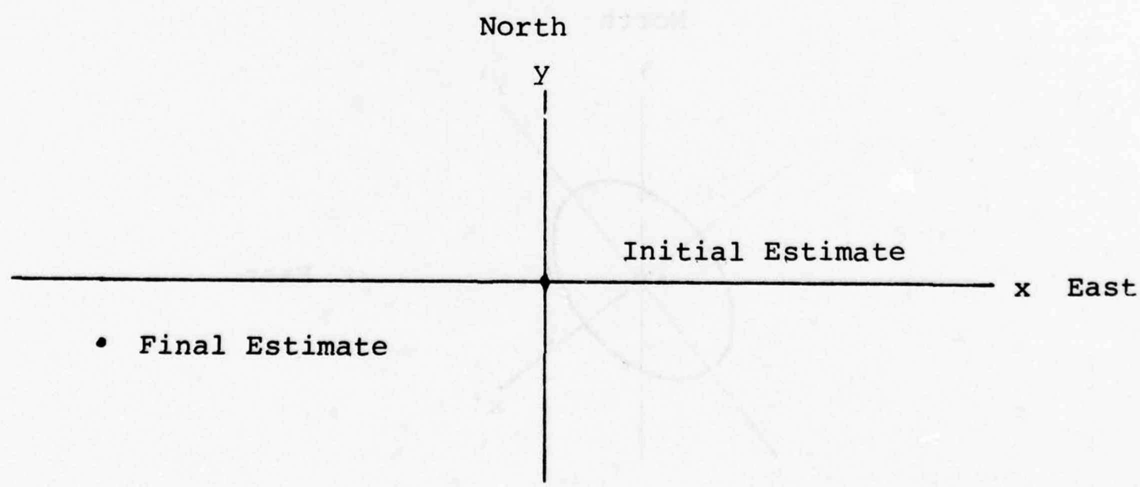

FIGURE 2. The Location of the Final Position Estimate with Respect to the Initial Position Estimate.

Minimum area elliptical confidence regions for an object's position can also be found by using the TI-59 program. The centers of the regions are at the fix, and their axes lie along the x' and y' axes of the coordinate system obtained by rotating the East-West, North-South xy-coordinate system with origin at the fix through an angle γ. The angle γ is defined so that it is positive for a rotation in the counterclockwise direction.

With the data from the above example, the program gives $\gamma = -31°$; so, the x' axis is directed 31° South of East. For a confidence region with minimum area and a confidence level of .9000, the ellipse bounding the region has a semi-major axis of 2064 meters, and a semi-minor axis of 1453 meters. The area of the region is 9.43 square killometers or 2.75 square nautical miles. The region is shown in Figure 3.

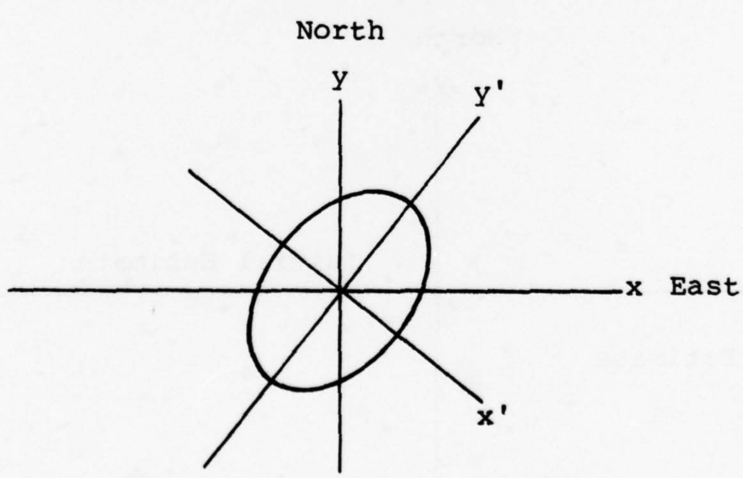

FIGURE 3. A .9000 Confidence Region for an Object's Position.

In the example discussed above, the position of the initial estimate is an input to the program. If this is not desirable, the program can be used to determine a position for the initial estimate. The position is the intersection of the two bearing lines corresponding to the first two bearings entered in the program. Both options are illustrated in Section II.

Since, in general, the smaller the bearing errors, the more likely that the initial estimate will be relatively near the object; small bearing errors can be considered to be a condition on the use of the procedure.

Note, if the length of the base line joining the first two stations is small enough and their bearing errors are large enough, observed bearing lines from the two stations may not intersect. If they do not intersect, the initial estimate determined by the program will be at the intersection of the reciprocal bearing lines, and a gross error can result.

Encl (2)

II. User Instructions

The TI-59 program to which the user instructions in this section apply can be used to calculate the quantities described in Section I.

The program requires the following inputs:

1. the observed bearing from or on an object for two or more stations;
2. station positions relative to a reference position; and
3. the bearing error (standard deviation) for each observed bearing.

Station positions can be specified in either of two ways. In the first way, Mode A, each station's position is specified in terms of its bearing α and its range ρ from a reference position. In the second way, Mode B, each station's position is specified in terms of its East-West distance x (plus for East) and its North-South distance (plus for North) from a reference position. The reference position can be any convenient location. For example, if it were at a station, then for that station $\alpha = 0$ and $\rho = 0$ or $x = 0$ and $y = 0$.

The program also requires an initial estimate of the object's position. The user has two options:

1. Let the program provide an estimate, or
2. Provide one with the input data.

For Option 1, the initial estimate is at the intersection of the bearing lines determined by the first two observed bearings entered into the program. For this reason, if this option is chosen, the first and second groups of data entered should correspond to the two stations estimated to have the smallest products $r_i e_i$. Although in this option the reference position cannot be at the initial estimate, it can be at one of the stations. If only two stations are involved, the final estimate is at the intersections of the bearing lines. (If the second option of either mode is used with an initial estimate which is not at the intersection of the two bearing lines, the coordinates of the final estimate will differ from coordinates of the intersection to the degree of the approximations involved in the estimation procedure.)

Two ways of providing confidence (probability) region data are available. In the first way, Mode C, the confidence (probability) p is specified. In the second way, Mode D, the multiplier k is specified where $k\sigma_{\hat{x}}$, and $k\sigma_{\hat{y}}$, are the semi-axes of the bounding ellipse.

The values of various quantities calculated by the program are either stored in registers or appear in the display. If a PC-100A printer is used, some of these values will be printed. The location of calculated values and the printing format is given after the user instructions. Those quantities which are not described in Section I are described below in the User Instructions or in Section IV.

All angles required or calculated by the program are in decimal degrees.

Step	Instructions	Enter	Press	Display
1.	If the calculator has been in use and flags have been set or the memory repartitioned, turn the calculator off and then on.			
2.	Read Side 1 and Side 2 of Card 1.			
3.	Read Side 3 of Card 2.			
MODE A:	Station Locations Specified in Terms of Bearing and Range from a Reference Point.			
4a.	If the initial position estimate will be determined by the program, go to Step 7a. See the note on Page 10.			
5a.	Enter the initial estimate's bearing.	α^*	A'	
6a.	Enter the initial estimate's range.	ρ^*	R/S	
7a.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	A	
8a.	Enter the station's bearing.	α_i	R/S	
9a.	Enter the station's range.	ρ_i	R/S	
10a.	Enter the bearing error.	e_i	R/S	i
11a.	Repeat Steps 7a, 8a, 9a and 10a for all stations. The number of repetitions i appears in the display after Step 10a.			

Step	Instructions	Enter	Press	Display
MODE B:	Station Locations Specified in Terms of East-West Distance and North-South Distance from a Reference Point.			
4b.	If the initial position estimate will be determined by the program, go to Step 7b. See the note on Page 10.			
5b.	Enter the initial estimate's East-West distance.	x^*	B'	
6b.	Enter the initial estimate's North-South distance.	y^*	R/S	
7b.	Enter the measured bearing on the object from a station or the reciprocal of the measured bearing on a station from the object.	θ_i	B	
8b.	Enter the station's East-West distance.	x_i	R/S	
9b.	Enter the station's North-South distance.	y_i	R/S	
10b.	Enter the bearing error.	e_i	R/S	i
11b.	Repeat Steps 7b, 8b, 9b and 10b for all stations. The number of repetitions i appears in the display after Step 10b.			
BOTH MODES				
12.	Calculate the East-West distance, the North-South distance, the bearing and the range of the position estimate relative to the reference position. Also calculate the rotation angle γ, and the standard deviations $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$. To include additional bearing measurements after this calculation, go to Step 18.		R/S	

Step	Instructions	Enter	Press	Display
13.	For confidence (probability) region calculations, go to Step 14 if the confidence (probability) for the region is specified. If k is specified where $k\sigma_{\hat{x}}$, and $k\sigma_{\hat{y}}$, are the semi-axes of the bounding ellipse with the larger the major axis, go to Step 16.			
14.	Enter p, the confidence level (probability) and calculate k, $k\sigma_{\hat{x}}$, $k\sigma_{\hat{y}}$, and the area of the region. (The area units correspond to the distance units used.)	p	C	Area
15.	For a different value of p, go to Step 14.			
16.	Enter k and calculate the confidence level (probability) p, $k\sigma_{\hat{x}}$, $k\sigma_{\hat{y}}$, and the area of the region. (The area units correspond to the distance units used.)	k	D	Area
17.	For a different value of k, go to Step 16.			
18.	To include an additional bearing measurement from either a new or old station, go to Step 7a if using Mode A or Step 7b if using Mode B.			

NOTE: If a data entry error occurs in either mode, press RST and then use the following procedure: For Option 1, return to Step 7 and repeat all data entries. For Option 2, return to Step 5 and repeat all data entries.

Also, if a position estimate is to be determined for a new object position or if a new mode is to be used, follow this instruction.

NOTES:

a) The program printing format is given below:

For the initial data, i = 1, 2, ... , n with one space between groups:

Mode A		Mode B
α^*	initial estimate if provided	x^*
ρ^*		y^*
θ_i		θ_i
α_i		x_i
ρ_i		y_i
e_i		e_i

The format for the calculated position data is:

x

y

α

ρ

γ

$\sigma_{\hat{x}}$'

$\sigma_{\hat{y}}$'

For the confidence (probability) region portion of the program the format is after pressing either C or D:

p	
k	
$k\sigma_{\hat{x}'}$	semi-axis
$k\sigma_{\hat{y}'}$	semi-axis
Area	

b) The following data is stored in the indicated registers:

Data	Registers
x*	R38
y*	R39
γ	R29
x	R30
y	R31
α	R32
ρ	R33
$\sigma_{\hat{x}'}$	R16
$\sigma_{\hat{y}'}$	R17
p	R14
k	R15
$k\sigma_{\hat{x}'}$	R18
$k\sigma_{\hat{y}'}$	R19

Four data tapes for a sample problem are given below. Distance units have not been specified, but they could be meters for example. Angles are in degrees. Option 1 (initial estimate not provided) for Mode A and Mode B is indicated by A and by B and Option 2 (initial estimate provided) is indicated by A' and B'.

For each mode and each option, the input data are indicated. The data determine the relative locations of three stations as well as the observed bearing of an object from each station.

For A and B, the reference location is at Station 1 and the initial position estimate (determined by the program) is at the intersection of the bearing lines for Station 1 and Station 2.

The intersection has coordinates $x^* = 906.4853528$ and $y^* = 17296.77092$ with respect to Station 1.

For A' and B', both the initial estimate and the reference location are at the intersection of the bearing lines, so $\alpha^* = 0$ and $\rho^* = 0$ and $x^* = 0$ and $y^* = 0$.

The data for A, B, A' and B' are all equivalent, and each solution gives the same data for a confidence (probability) region calculation. A tape with confidence (probability) region results for both Mode C and Mode D which correspond to A, B, A' and B' is given with the first four data tapes.

A

3.	θ_1
0.	α_1
0.	ρ_1
4.	e_1
33.	θ_2
273.	α_2
10000.	ρ_2
3.	e_2
303.	θ_3
33.	α_3
14000.	ρ_3
8.	e_3
573.5878933	x
16462.71223	y
1.995471725	α
16472.70157	ρ
-7.325392245	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

A'

0.	α^*
0.	ρ^*
3.	θ_1
183.	α_1
17320.50808	ρ_1
4.	e_1
33.	θ_2
213.	α_2
20000.	ρ_2
3.	e_2
303.	θ_3
129.5867755	α_3
8717.797886	ρ_3
8.	e_3
-332.8974567	x
-834.0586835	y
201.7584019	α
898.0393111	ρ
-7.325392259	γ
787.3663757	$\sigma_{\hat{x}'}$
1233.080776	$\sigma_{\hat{y}'}$

B

3.	θ_1
0.	x_1
0.	y_1
4.	e_1
33.	θ_2
-9986.295348	x_2
523.3595624	y_2
3.	e_2
303.	θ_3
7624.94649	x_3
11741.38795	y_3
8.	e_3
573.5878927	x
16462.71223	y
1.995471723	α
16472.70157	ρ
-7.325392245	γ
787.3663756	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

B'

0.	x^*
0.	y^*
3.	θ_1
-906.4853528	x_1
-17296.77092	y_1
4.	e_1
33.	θ_2
-10892.7807	x_2
-16773.41136	y_2
3.	e_2
303.	θ_3
6718.461137	x_3
-5555.382969	y_3
8.	e_3
-332.8974589	x
-834.0586906	y
201.7584019	α
898.0393184	ρ
-7.325392239	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

C or D

0.9	p
2.145966026	k
1689.661493	$k\sigma_{\hat{x}'}$
2646.149454	$k\sigma_{\hat{y}'}$
14046364.97	Area
.8646647168	p
2.	k
1574.732751	$k\sigma_{\hat{x}'}$
2466.161553	$k\sigma_{\hat{y}'}$
12200517.6	Area

The following data tape illustrates the effects of using only the bearings for the first two stations. The tape is for Mode A, Option 1. The values stored in Registers 38 and 39 (the x and y coordinates of the intersection of the bearing lines from Station 1 and Station 2 with reference, in this case, to Station 1) are also listed on the tape (as well as given above). And, as can be seen, the initial estimate and final estimate correspond.

The data tape also illustrates the use of additional bearing data to revise a position estimate. The data for Station 3 printed after the first confidence (probability) region calculation results was entered by again repeating Step 11a, and the remaining results were obtained by next repeating Step 12 and then Steps 14 and 16. Note, these results are the same as the corresponding results for Mode A on page 14.

A

3.	θ_1
0.	α_1
0.	ρ_1
4.	e_1

33.	θ_2
273.	α_2
10000.	ρ_2
3.	e_2

906.4853528	x
17296.77092	y
3.	α
17320.50808	ρ

-20.35750198	γ
818.886822	$\sigma_{\hat{x}'}$
3092.663848	$\sigma_{\hat{y}'}$

0.9	p
2.145966026	k
1757.303299	$k\sigma_{\hat{x}'}$
6636.751549	$k\sigma_{\hat{y}'}$
36639720.91	Area

.8646647168	p
2.	k
1637.773644	$k\sigma_{\hat{x}'}$
6185.327696	$k\sigma_{\hat{y}'}$
31824857.22	Area

303.	θ_3
33.	α_3
14000.	ρ_3
8.	e_3

573.5878933	x
16462.71223	y
1.995471725	α
16472.70157	ρ

-7.325392245	γ
787.3663755	$\sigma_{\hat{x}'}$
1233.080777	$\sigma_{\hat{y}'}$

0.9	p
2.145966026	k
1689.661492	$k\sigma_{\hat{x}'}$
2646.149455	$k\sigma_{\hat{y}'}$
14046364.98	Area

.8646647168	p
2.	k
1574.732751	$k\sigma_{\hat{x}'}$
2466.161554	$k\sigma_{\hat{y}'}$
12200517.6	Area

906.4853528	R38
17296.77092	R39

To obtain the results given in Section I, use A' and take the reference position at the initial estimate ($\alpha^* = 0$, $\rho^* = 0$). Then $\alpha_1 = 218°$, $\alpha_2 = 166°$ and $\alpha_3 = 307°$. The data tape for the calculation is given below.

A'

0.	α^*
0.	ρ^*
35.	θ_1
218.	α_1
10000.	ρ_1
4.	e_1
351.	θ_2
166.	α_2
15000.	ρ_2
7.	e_2
131.	θ_3
307.	α_3
12000.	ρ_3
5.	e_3
-511.961856	x
-75.43753883	y
261.617789	α
517.4898687	ρ
-31.23492683	γ
677.2632305	$\sigma_{\hat{x}'}$
961.6888632	$\sigma_{\hat{y}'}$
0.9	p
2.145966026	k
1453.383883	$k\sigma_{\hat{x}'}$
2063.751628	$k\sigma_{\hat{y}'}$
9422966.381	Area
.8646647168	p
2.	k
1354.526461	$k\sigma_{\hat{x}'}$
1923.377726	$k\sigma_{\hat{y}'}$
8184684.605	Area

III. Program Listing

Before entering the program, press 2nd and then CP or turn the calculator off and then on. Next enter 5 in the display, press 2nd and then Op 17. This repartitions the calculator's memory so that the complete program can be entered.

Before recording the program, enter 6 in the display, press 2nd and then Op 17. This returns the calculator's memory to the normal partition (479.59). Returning the calculator to the normal partition allows the two program cards to be read in the normal partition without forcing. When the program is used, it repartitions the calculator so that Bank 3 registers are program registers.

```
000  76  LBL
001  15  E
002  09  9
003  42  STO
004  00  00
005  42  STO
006  01  01
007  92  RTN
008  76  LBL
009  18  C'
010  69  OP
011  20  20
012  72  ST*
013  00  00
014  92  RTN
015  76  LBL
016  19  D'
017  69  OP
018  21  21
019  73  RC*
020  01  01
021  92  RTN
022  76  LBL
023  10  E'
024  65  ×
025  89  π
026  55  ÷
027  01  1
028  08  8
029  00  0
030  95  =
031  92  RTN
032  76  LBL
033  12  B
034  86  STF
035  01  01
036  76  LBL
037  11  A
038  32  X:T
039  87  IFF
040  00  00
041  00  00
042  59  59
043  87  IFF
044  03  03
045  00  00
046  59  59
047  05  5
048  69  OP
049  17  17
050  47  CMS
051  15  E
052  02  2
053  42  STO
054  09  09
055  69  OP
056  28  28
057  86  STF
058  00  00
059  32  X:T
060  99  PRT
061  18  C'
062  91  R/S
063  99  PRT
064  18  C'
065  91  R/S
066  99  PRT
067  18  C'
068  91  R/S
069  99  PRT
070  10  E'
071  18  C'
072  87  IFF
073  03  03
074  01  01
075  67  67
076  19  D'
077  87  IFF
078  01  01
079  00  00
080  91  91
081  75  -
082  19  D'
083  95  =
084  94  +/-
085  38  SIN
086  65  ×
087  19  D'
088  61  GTO
089  01  01
090  02  02
091  85  +
092  19  D'
093  32  X:T
094  19  D'
095  22  INV
096  37  P/R
097  24  CE
098  95  =
099  39  COS
100  65  ×
101  32  X:T
102  95  =
103  48  EXC
104  19  19
105  22  INV
106  97  DSZ
107  09  09
108  01  01
109  15  15
110  01  1
111  69  OP
112  21  21
113  98  ADV
114  91  R/S
115  42  STO
116  18  18
117  65  ×
118  43  RCL
119  14  14
120  42  STO
121  20  20
122  38  SIN
123  75  -
124  43  RCL
125  10  10
126  22  INV
127  44  SUM
128  20  20
129  38  SIN
130  65  ×
131  43  RCL
132  19  19
133  95  =
134  55  ÷
135  43  RCL
136  20  20
137  38  SIN
138  95  =
139  42  STO
140  38  38
141  43  RCL
142  14  14
143  39  COS
144  65  ×
145  43  RCL
146  18  18
147  75  -
148  43  RCL
149  10  10
```

150	39	COS
151	65	×
152	43	RCL
153	19	19
154	95	=
155	55	÷
156	43	RCL
157	20	20
158	38	SIN
159	95	=
160	42	STO
161	39	39
162	86	STF
163	03	03
164	02	2
165	42	STO
166	09	09
167	15	E
168	19	D'
169	10	E'
170	42	STO
171	18	18
172	19	D'
173	87	IFF
174	01	01
175	01	01
176	81	81
177	32	X:T
178	19	D'
179	32	X:T
180	37	P/R
181	75	-
182	43	RCL
183	38	38
184	95	=
185	94	+/-
186	32	X:T
187	22	INV
188	87	IFF
189	01	01
190	01	01
191	93	93
192	19	D'
193	75	-
194	43	RCL
195	39	39
196	95	=
197	94	+/-
198	32	X:T
199	22	INV
200	37	P/R
201	42	STO
202	19	19
203	10	E'
204	75	-
205	43	RCL
206	18	18
207	95	=
208	94	+/-
209	32	X:T
210	69	OP
211	21	21
212	64	PD*
213	01	01
214	42	STO
215	18	18
216	89	π
217	32	X:T
218	22	INV
219	77	GE
220	02	02
221	26	26
222	75	-
223	32	X:T
224	65	×
225	02	2
226	95	=
227	49	PRD
228	18	18
229	73	RC*
230	01	01
231	35	1/X
232	42	STO
233	27	27
234	42	STO
235	26	26
236	32	X:T
237	43	RCL
238	19	19
239	37	P/R
240	42	STO
241	28	28
242	49	PRD
243	27	27
244	33	X^2
245	44	SUM
246	23	23
247	43	RCL
248	18	18
249	49	PRD
250	26	26
251	49	PRD
252	27	27
253	32	X:T
254	49	PRD
255	28	28
256	49	PRD
257	26	26
258	33	X^2
259	44	SUM
260	21	21
261	43	RCL
262	28	28
263	44	SUM
264	22	22
265	43	RCL
266	26	26
267	44	SUM
268	24	24
269	43	RCL
270	27	27
271	44	SUM
272	25	25
273	22	INV
274	97	DSZ
275	09	09
276	02	02
277	81	81
278	61	GTO
279	01	01
280	68	68
281	69	OP
282	28	28
283	43	RCL
284	08	08
285	98	ADV
286	91	R/S
287	43	RCL
288	21	21
289	42	STO
290	41	41
291	43	RCL
292	22	22
293	42	STO
294	42	42
295	43	RCL
296	23	23
297	42	STO
298	43	43
299	43	RCL

```
300  23  23
301  42  STO
302  10  10
303  65  ×
304  43  RCL
305  21  21
306  22  INV
307  44  SUM
308  10  10
309  75  -
310  43  RCL
311  22  22
312  22  INV
313  49  PRD
314  10  10
315  33  X²
316  95  =
317  35  1/X
318  49  PRD
319  41  41
320  49  PRD
321  42  42
322  49  PRD
323  43  43
324  43  RCL
325  10  10
326  35  1/X
327  65  ×
328  02  2
329  95  =
330  22  INV
331  30  TAN
332  55  ÷
333  02  2
334  95  =
335  42  STO
336  29  29
337  43  RCL
338  43  43
339  65  ×
340  43  RCL
341  24  24
342  75  -
343  43  RCL
344  42  42
345  65  ×
346  43  RCL
347  25  25
348  85  +
349  43  RCL
350  38  38
351  95  =
352  42  STO
353  30  30
354  99  PRT
355  32  X:T
356  43  RCL
357  42  42
358  65  ×
359  43  RCL
360  24  24
361  75  -
362  43  RCL
363  41  41
364  65  ×
365  43  RCL
366  25  25
367  85  +
368  43  RCL
369  39  39
370  95  =
371  42  STO
372  31  31
373  99  PRT
374  32  X:T
375  22  INV
376  37  P/R
377  99  PRT
378  42  STO
379  32  32
380  01  1
381  32  X:T
382  99  PRT
383  42  STO
384  33  33
385  98  ADV
386  43  RCL
387  29  29
388  99  PRT
389  37  P/R
390  42  STO
391  11  11
392  33  X²
393  42  STO
394  12  12
395  42  STO
396  13  13
397  32  X:T
398  49  PRD
399  11  11
400  33  X²
401  42  STO
402  14  14
403  42  STO
404  15  15
405  43  RCL
406  41  41
407  49  PRD
408  12  12
409  49  PRD
410  14  14
411  43  RCL
412  42  42
413  65  ×
414  02  2
415  95  =
416  49  PRD
417  11  11
418  43  RCL
419  43  43
420  49  PRD
421  13  13
422  49  PRD
423  15  15
424  43  RCL
425  15  15
426  85  +
427  43  RCL
428  11  11
429  85  +
430  43  RCL
431  12  12
432  95  =
433  34  √X
434  99  PRT
435  42  STO
436  16  16
437  43  RCL
438  13  13
439  75  -
440  43  RCL
441  11  11
442  85  +
443  43  RCL
444  14  14
445  95  =
446  34  √X
447  99  PRT
448  42  STO
449  17  17
```

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450  98 ADV
451  91 R/S
452  76 LBL
453  17 B'
454  86 STF
455  02  02
456  76 LBL
457  16 A'
458  99 PRT
459  32 X:T
460  05  5
461  69 OP
462  17  17
463  47 CMS
464  15  E
465  91 R/S
466  99 PRT
467  87 IFF
468  02  02
469  04  04
470  73  73
471  32 X:T
472  37 P/R
473  42 STO
474  38  38
475  32 X:T
476  42 STO
477  39  39
478  86 STF
479  03  03
480  98 ADV
481  91 R/S
482  76 LBL
483  14  D
484  86 STF
485  04  04
486  76 LBL
487  13  C
488  98 ADV
489  87 IFF
490  04  04
491  05  05
492  36  36
493  42 STO
494  14  14
495  99 PRT
496  75  -
497  01  1
498  95  =
499  94 +/-
500  23 LNX
501  65  ×
502  02  2
503  95  =
504  94 +/-
505  34 √X
506  42 STO
507  15  15
508  99 PRT
509  65  ×
510  43 RCL
511  16  16
512  95  =
513  99 PRT
514  42 STO
515  18  18
516  65  ×
517  53  (
518  43 RCL
519  15  15
520  65  ×
521  43 RCL
522  17  17
523  54  )
524  99 PRT
525  42 STO
526  19  19
527  65  ×
528  89  π
529  95  =
530  99 PRT
531  98 ADV
532  22 INV
533  86 STF
534  04  04
535  91 R/S
536  42 STO
537  15  15
538  33 X²
539  55  ÷
540  02  2
541  95  =
542  94 +/-
543  22 INV
544  23 LNX
545  75  -
546  01  1
547  95  =
548  94 +/-
549  42 STO
550  14  14
551  99 PRT
552  43 RCL
553  15  15
554  61 GTO
555  05  05
556  08  08
557  00  0
558  00  0
559  00  0
```

IV. A Development for the Procedure

In the development for the estimation procedure given here, all angles are in radians and the assumptions stated in Section I apply.

Figure 4 shows three bearing lines from the ith of n stations. One is the observed bearing line of an object. One of length κ_i goes to the origin of an xy-coordinate system located at the object's unknown position. And one of length r_i goes to an initial estimate with known position but unknown coordinates (x,y). Note, estimates for $-x$ and $-y$ estimate the object's position. To find estimates $-\hat{x}$ and $-\hat{y}$, consider the arc coordinates $u_i = \kappa_i(\theta_i-\phi_i)$ of the observed bearing line and $v_i = \kappa_i(\beta_i-\phi_i)$ of the bearing line to the point (x,y). They are defined by the three bearing lines and the circle of radius κ_i which goes through the object's position and which is centered on the station as shown in Figure 4.

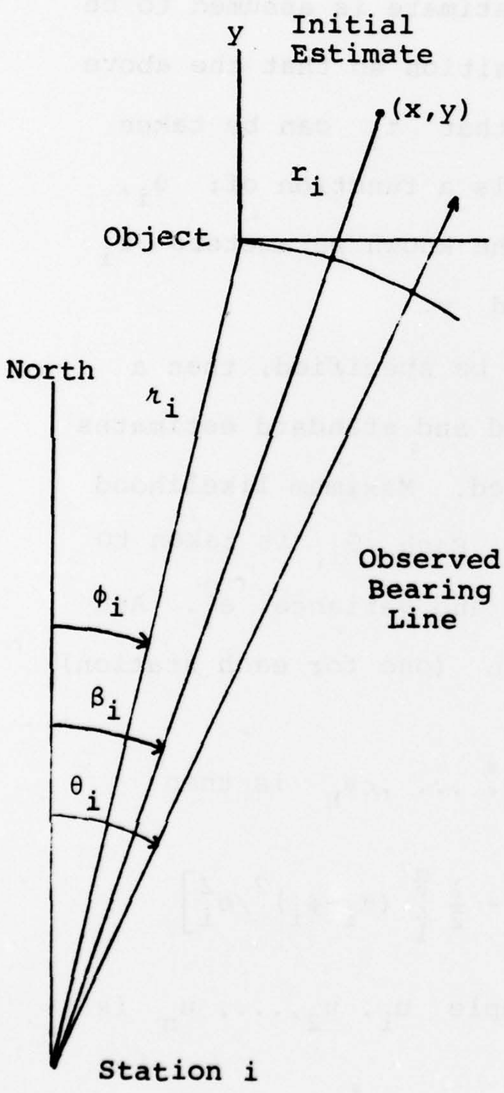


FIGURE 4. Problem Geometry.

By defining $w_i = r_i(\theta_i - \beta_i)$ (all angles in radians), $u_i = v_i + w_i$. Note, $\theta_i - \beta_i$ is known, but $\beta_i - \phi_i$ is not. However, v_i can be expressed in terms of x and y, and, to first order, $v_i = x \cos \beta_i - y \sin \beta_i$; so, if $\tan(\beta_i - \phi_i) \simeq (\beta_i - \phi_i)$ for $i = 1,2,\ldots,n$, that is, if (x,y) is relatively near the object's position, $u_i \simeq r_i(\theta_i - \beta_i) + x \cos \beta_i - y \sin \beta_i$ for $i = 1,2,\ldots,n$.

In this development, the initial estimate is assumed to be relatively close enough to the object's position so that the above approximation for u_i can be used and so that r_i can be taken equal to r_i. With this assumption, u_i is a function of: θ_i, the observed value of a random quantity; the known parameters r_i and β_i; and the unknown parameters x and y.

If a distribution for the Θ_i can be specified, then a distribution for the U_i can be determined and standard estimates $\hat{x}$ and $\hat{y}$ for x and y can be considered. Maximum likelihood estimates are discussed in this section. Each Θ_i is taken to be a normal random variable with mean ϕ_i and variance e_i^2. And the n random variables Θ_i, $i = 1,2,\ldots,n$ (one for each station) are taken to be independent.

The likelihood for a sample $\theta_1, \theta_2, \ldots, \theta_n$ is then

$$L(\theta_1, \theta_2, \ldots, \theta_n) = \prod_1^n \left(\frac{1}{\sqrt{2\pi}\, e_i}\right) \exp\left[-\frac{1}{2} \sum_1^n (\theta_i - \phi_i)^2 / e_i^2\right]$$

and the likelihood for a corresponding sample $u_1, u_2, \ldots, u_n$ is

$$L(u_1,u_2,\ldots,u_n) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi}\,\sigma_i}\right) \exp\left[-\frac{1}{2}\sum_{1}^{n} u_i^2/\sigma_i^2\right]$$

where $\sigma_i = r_i e_i$ (with e_i in radians) since $u_i = r_i(\theta_i - \phi_i)$.

By definition, the maximum likelihood estimates of x and y are the estimates $\hat{x}$ and $\hat{y}$ which make $L(u_1,u_2,\ldots,u_n)$ a maximum. In this case, making $L(u_1,u_2,\ldots,u_n)$ a maximum is equivalent to making $\sum_1^n(u_i^2/\sigma_i^2)$ a minimum. So, to find $\hat{x}$ and $\hat{y}$, solve the following two equations for x and y:

$$\frac{\partial(\ell n\ L)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial(\ell n\ L)}{\partial y} = 0 .$$

The solutions are $x = \hat{x}$ and $y = \hat{y}$, and $\hat{x}$ and $\hat{y}$ are the maximum likelihood estimates. With $w_i = r_i(\theta_i - \beta_i)$ and the conditions assumed above these two equations are linear equations in x and y. And,

$$\sum_{1}^{n} [w_i + \hat{x}\cos\beta_i - \hat{y}\sin\beta_i](\cos\beta_i)/\sigma_i^2 = 0$$

and

$$\sum_{1}^{n} [w_i + \hat{x}\cos\beta_i - \hat{y}\sin\beta_i](\sin\beta_i)/\sigma_i^2 = 0 .$$

And, in terms of the following quantities:

$$A = \Sigma(\cos^2\beta_i)/\sigma_i^2 , \qquad B = \Sigma(\sin\beta_i \cos\beta_i)/\sigma_i^2 ,$$

$$C = \Sigma(\sin^2\beta_i)/\sigma_i^2 , \qquad D = \Sigma(w_i \cos\beta_i)/\sigma_i^2 ,$$

$$E = \Sigma(w_i \sin\beta_i)/\sigma_i^2 ,$$

the equations are:

$$A\hat{x} - B\hat{y} = -D$$

$$B\hat{x} - C\hat{y} = -E \ .$$

So the solutions are:

$$\hat{x} = (BE - CD)/(AC - B^2)$$

$$\hat{y} = (AE - BD)/(AC - B^2) \ .$$

A confidence region can be constructed about an estimated position. In order to indicate how this is done, a probability region about the true position will be considered first.

Note, $\hat{x}$ and $\hat{y}$ are values of random variables. If a new set of bearings $\theta_1, \theta_2, \ldots, \theta_n$ is observed (for a fixed initial estimate and object), in general, a new pair of values $\hat{x}$ and $\hat{y}$ will be obtained.

If $\hat{X}$ and $\hat{Y}$ represent these random variables, then

$$\hat{X} = \frac{1}{(AC-B^2)} \sum_1^n (W_i/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)$$

$$\hat{Y} = \frac{1}{(AC-B^2)} \sum_1^n (W_i/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)$$

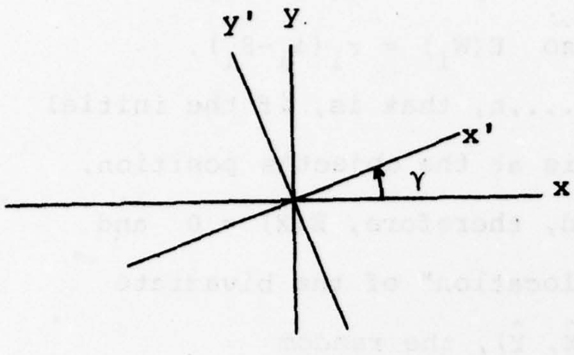
with $W_i = r_i(\Theta_i - \beta_i)$. ($W_i$ is the random distance intercepted along the ith arc between the bearing lines defined by Θ_i and β_i.)

Note, $\hat{X}$ and $\hat{Y}$ have a bivariate normal distribution, since they are a linear combination of the n normal random variables $W_1, W_2, \ldots, W_n$, or equivalently of the n normal random variables $\Theta_1, \Theta_2, \ldots, \Theta_n$. Also $E(W_i) = r_i(\phi_i - \beta_i)$.

If $\beta_i = \phi_i$ for $i = 1,2,\ldots,n$, that is, if the initial estimate of the object's position is at the object's position, $E(W_1) = 0$ for $i = 1,2,\ldots,n$. And, therefore, $E(\hat{X}) = 0$ and $E(\hat{Y}) = 0$. So, in this case, the "location" of the bivariate normal distribution of a point $(\hat{X}, \hat{Y})$, the random coordinates of the object's estimated position, is the same as that for the point $(-\hat{X}, -\hat{Y})$ and both are centered on the object's position. However, the "location" of the distribution of $(-\hat{X}, -\hat{Y})$ is independent of the location of the initial estimate when the coordinates $(-\hat{X}, -\hat{Y})$ refer to a coordinate system with origin at the initial estimate. This fact simplifies the establishment of a confidence region about the location of an estimated position.

A region of minimum area for a given probability of containment of an estimated position can be determined. The region is bounded by an ellipse which is centered on the object's position and whose axes lie along the axes of an x'y'-coordinate system obtained by rotating the xy-coordinate system centered on the object's position through an angle γ. In this system, $\sigma_{\hat{x}\hat{y}} = 0$, that is, $\hat{X}'$ and $\hat{Y}'$ are independent normal random variables.

The two coordinate systems are illustrated in Figure 5. The coordinates of a point in the two systems are related by



$$x' = x\cos\gamma + y\sin\gamma$$

$$y' = -x\sin\gamma + y\cos\gamma$$

These relations, along with $\sigma_{\hat{x}'\hat{y}'} = 0$, imply:

FIGURE 5. Rotation Geometry.

$$\sigma^2_{\hat{x}'} = \sigma^2_{\hat{x}}\cos^2\gamma + 2\sigma_{\hat{x}\hat{y}}\cos\gamma\sin\gamma + \sigma^2_{\hat{y}}\sin^2\gamma\ ,$$

$$\sigma^2_{\hat{y}'} = \sigma^2_{\hat{x}}\sin^2\gamma - 2\sigma_{\hat{x}\hat{y}}\cos\gamma\sin\gamma + \sigma^2_{\hat{y}}\cos^2\gamma$$

and

$$\tan 2\gamma = \frac{2\sigma_{\hat{x}\hat{y}}}{\sigma^2_{\hat{x}} - \sigma^2_{\hat{y}}}$$

where γ, the angle of rotation of the coordinate axes, is positive in the counterclockwise direction.

With the initial estimate of the object's position at the object's position $(\beta_i = \phi_i,\ i = 1,2,\ldots,n)$, so $E(W_i) = 0$ and $Var(W_i) = \sigma_i^2$,

$$\sigma^2_{\hat{x}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)^2,$$

$$\sigma^2_{\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(A \sin \beta_i - B \cos \beta_i)^2$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{1}{(AC-B^2)^2} \sum_1^n (1/\sigma_i^2)(B \sin \beta_i - C \cos \beta_i)(A \sin \beta_i - B \cos \beta_i).$$

Using the definition for A, B and C, the above become

$$\sigma^2_{\hat{x}} = \frac{C}{(AC-B^2)},$$

$$\sigma^2_{\hat{y}} = \frac{A}{(AC-B^2)},$$

and

$$\sigma_{\hat{x}\hat{y}} = \frac{B}{(AC-B^2)}.$$

So, $\tan 2\gamma = 2B/(C-A)$ for $\beta_i = \phi_i$, $i = 1,2,\ldots,n$.

With the object's position known and, hence, ϕ_i known for $i = 1,2,\ldots,n$, the above equations for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be used, since the initial estimate of the object's position can be taken as the object's position.

With values for $\sigma_{\hat{x}}$, $\sigma_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ, values for $\sigma_{\hat{x}'}$ and $\sigma_{\hat{y}'}$ can be found by using the equations in the middle of Page 30. And then, the probability that an estimated position will be within an ellipse of semiaxes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$

which is centered on the object's position can be found. It is $1 - \exp(-k^2/2)$. (This result follows from integrating the bivariate normal density over the ellipse.) And the area of the ellipse is $\pi k^2 \sigma_{\hat{x}'} \sigma_{\hat{y}'}$.

Given estimates $\hat{x}$ and $\hat{y}$ found by using the relations on Page 28, an ellipse with semi-axes $k\sigma_{\hat{x}'}$ and $k\sigma_{\hat{y}'}$ centered on the point with coordinates $(-\hat{x},-\hat{y})$ in a coordinate system with origin at the initial estimate and oriented as indicated by γ is a $1 - \exp(-k^2/2)$ confidence region. This follows from the bivariate normal distribution of $-\hat{X}$ and $-\hat{Y}$ which in this system is centered on the object's position. The ellipse is defined if $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$ and $\sigma_{\hat{x}\hat{y}}$ are known (the covariance matrix is known). And to the degree of the approximations involved, this can be assumed to be the case. In particular, by assuming the initial estimate of the object's position is at the object's position, which is consistent with assuming $(\beta_i - \phi_i)$ is small, values for $\sigma^2_{\hat{x}}$, $\sigma^2_{\hat{y}}$, $\sigma_{\hat{x}\hat{y}}$ and γ can be obtained by using the relations on Page 31. These values can then be used to determine $\sigma^2_{\hat{x}'}$ and $\sigma^2_{\hat{y}'}$ by using the relations on Page 30. And, then, with a value for k, a confidence region can be constructed. To the degree of the approximations involved, the shape of the confidence region is independent of both the object's position and of the initial estimate of the object's position.

For the case where bearings are taken from the object on two or more stations, θ_i is the reciprocal of the bearing taken from the object.

A discussion for this and for other bearings only position estimation procedures for situations similar to the one considered here is given in Reference 1 listed below. Reference 2 gives an equivalent bearings only procedure. It also gives a range only procedure, a range and bearing procedure and HP-9830A programs with which to implement the procedures. Using the fix determined by two lines of bearing as the initial estimate was suggested by this reference.

The equations used in the program to determine (x^*, y^*), the coordinates of the fix, are:

$$x^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \sin\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \sin\theta_1$$

$$y^* \sin(\theta_2-\theta_1) = [\rho_1 \sin(\alpha_1-\theta_1)] \cos\theta_2 - [\rho_2 \sin(\alpha_2-\theta_2)] \cos\theta_1$$

References:

1. Schrader, John Yale, Jr., "An Alternative Approach to Long Range DF Fixing," Naval Postgraduate School Ph.D. Thesis, September 1974.

2. Thompson, K.P. and Kullback, J.H., "Position-Fixing and Position-Predicting Programs for the Hewlett-Packard Model 9830A Programmable Calculator," NRL Memorandum Report 3265, Naval Research Laboratory, Washington, D.C.

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