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ANALYTICAL HAZARD REPRESENTATION FOR USE
IN RELIABILITY, MORTALITY, AND
SIMULATION STUDIES. Part I.

by

⑩ D. P. / Gaver
and
M. / Acar

⑪ August 1978

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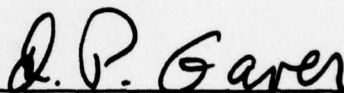
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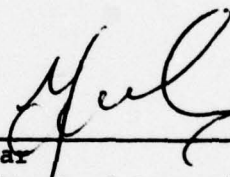
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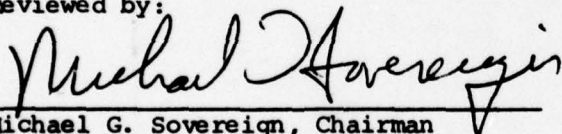


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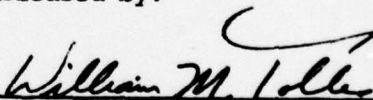
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TABLE OF CONTENTS

	Page
I. INTRODUCTION	1
II. SYSTEM FAILURE PATTERNS	2
III. MODELS FOR THE HAZARD FUNCTION	5
IV. MATHEMATICAL PROPERTIES OF THE "BATHTUB" HAZARD MODEL	10
V. MATHEMATICAL PROPERTIES OF THE DECREASING FAILURE RATE MODEL	18
VI. AN ALTERNATIVE "BATHTUB" HAZARD REPRESENTATION	20
VII. OBTAINING SPECIFIED HAZARD BEHAVIOR BY SIMPLE SAMPLING	23
VIII. CONCLUSION	26
REFERENCES	27

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ANALYTICAL HAZARD REPRESENTATIONS FOR USE
IN RELIABILITY, MORTALITY, AND SIMULATION STUDIES

Part I

D. P. Gaver *

M. Acar

I. INTRODUCTION

The failure rate function, or hazard function (hazard for short) may be described as the conditional probability of an equipment's failing at operating age t , having survived to that age. The reliabilities of a variety of electronic and mechanical items are conveniently and naturally described in terms of the appropriate hazard function, and so is the longevity of human beings. The term force of mortality replaces hazard in the latter context.

This paper is devoted to a study of several simple analytical representations for hazard functions. These representations are in turn based upon representations of random variables having certain required properties, in terms of others having familiar distributions--in particular the exponential. Similar ideas are due to Tukey (1976) and recently have been examined by Parzen (1978). The hazard representations proposed are quite

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expeditiously used in simulation studies, e.g. of system reliability or availability in terms of component lifetimes. They may also be used in data analysis studies, in order to parsimoniously describe data sets in terms of perturbations of convenient and familiar standard distributions. Their use in data analysis and simulation is also described in Gaver, Lavenberg, and Price (1976), and in Gaver and Chu (1977).

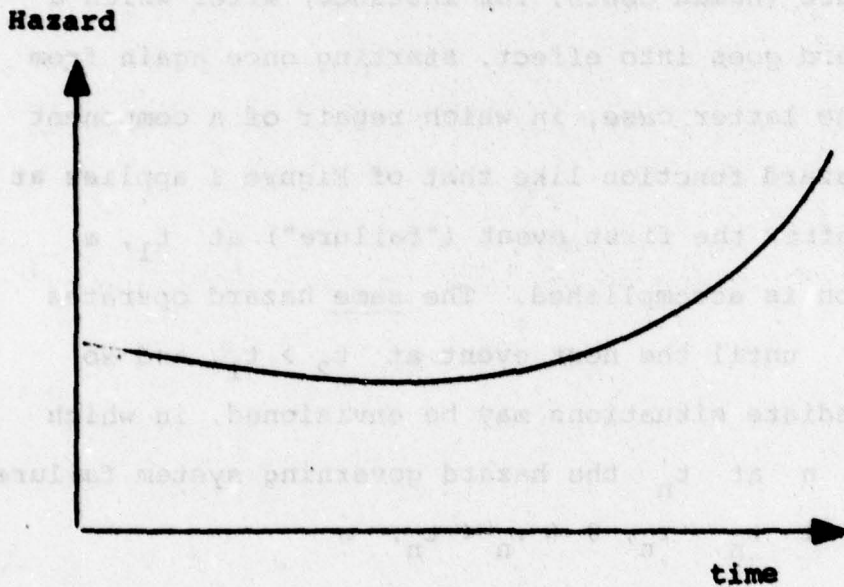
II. SYSTEM FAILURE PATTERNS

It is plausible to think that the time series of failures in a system may involve these stages.

1. Early failures. There may be a relatively large number of failure soon after a system is introduced because of design defects, production errors, or errors stemming from maintenance personnel inexperience. This situation is characterized by a hazard function that is initially large, but that decreases with time. "Infant mortality" is in evidence.
2. Random Failures. Following the early failure period there may be a period during which failures occur at an essentially constant rate for a rather prolonged time. During this period the hazard function is nearly constant, so the times between failures are close to being exponentially distributed. The effect of age or wearout is not yet apparent.

3. Wearout Failures. Eventually following the period during which a constant hazard is evident there is likely to be a period of ever-increasing failure rate caused by wearout of system components.

A graphical representation of a hazard function that exhibits the behavior described is given below. Note that it has the legendary "bath tub" shape.



Some comments on the above follow:

- A. The term "failure" may refer to an event that is analogous to human death, after which the entire system is replaced. On the other hand repair or component replacement may occur after failure; the system is only repaired, not entirely replaced. In the former case, a hazard function of the kind depicted in Figure 1 applies to each system event ("death"); when the system is installed (or is born) that hazard operates starting from scratch at $t = 0$ until system failure (human death, for instance) after which a similar hazard goes into effect, starting once again from zero. In the latter case, in which repair of a component occurs, a hazard function like that of Figure 1 applies at $t = 0$, but after the first event ("failure") at t_1 , a repair action is accomplished. The same hazard operates for $t \geq t_1$ until the next event at $t_2 > t_1$, and so on. Intermediate situations may be envisioned, in which after event n at t_n the hazard governing system failure $n+1$ starts at $t_n - \tau_n$, $0 < \tau_n < t_n$.

- B. Although there is reason to assume that hazards somewhat like that of Figure 1 occur in general for systems, the possibility exists that the system hazard is "bumpy" because wearout failures of components or subsystems may well occur at intermediate times.

C. If the theory is applied to systems with little or no wearout propensity, as should be the case when dealing with computer software modules, then the hazard function may well exhibit the initial falloff of Figure 1 but not the rise at later times. In fact, a constant decline as bugs are found and removed could be (optimistically) anticipated for software. The right-hand side of the bath tub vanishes, and the picture is that of a ski slope.

III. MODELS FOR THE HAZARD FUNCTION

In this section mathematical models are presented for the failure rate or hazard function. Recall that the hazard may be defined as follows.

Definition. Suppose that the time to failure, X , is a random variable with distribution function $F(x)$, where $F(0) = 0$; the latter possesses the density function $f(x)$, $f(x) = dF/dx$, such that for any positive x ,

$$F(x) = \int_0^x f(y)dy . \quad (3.1)$$

Then the hazard function, or failure rate at age x , is given by

$$h(x) = \frac{f(x)}{1 - F(x)} . \quad (3.2)$$

The interpretation of $h(x)dx$ is that it is the conditional probability of failure in the interval $(x, x + dx)$, given that there has been no failure up to age x .

Express the hazard as

$$h(x) = \frac{dF/dx}{1 - F(x)}$$

$$h(x)dx = \frac{dF}{1 - F(x)} = - d\{\log[1 - F(x)]\},$$

it then follows after integration that

$$F(x) = 1 - \exp\left[-\int_0^x h(y)dy\right]. \quad (3.3)$$

Thus if the hazard is specified, so is the distribution function, and conversely.

Note that if

$$h(x) = \lambda > 0, \quad 0 \leq x$$

then

$$F(x) = 1 - e^{-\lambda x}, \quad 0 \leq x, \quad (3.4)$$

so a constant hazard function implies the exponential distribution of the random variable X , and conversely.

Obviously a constant hazard representation does not describe the bath tub hazard shape of Figure 1, nor does it represent a situation in which hazards decline, possibly because design defects or "bugs" are occasionally removed. Here are two hazard representations likely to be useful for such purposes.

(1) A Bath tub Model

Define the random variable Z in terms of X , X being exponentially distributed with mean λ^{-1} , as follows:

$$Z = G(X) = XL(X) R(X) \tag{3.5}$$

or

$$= X\phi(X), \quad \phi(X) = L(X)R(X)$$

where

- a) $L(x)$ is convex in x , $L(0) < 1$, $L(\infty) = 1$,
- b) $R(x)$ is concave in x , $R(0) = 1$, $R(\infty) > R(0)$.

Then the hazard of Z may be made to exhibit a bath tub shape, as in Figure 1, by proper choice of the function L and R .

Example. Suppose

$$\begin{aligned} L(x) &= \frac{\alpha x}{1 + \alpha x} & \alpha > 0, 0 \leq x \\ R(x) &= \frac{1}{1 + \beta x} & \beta > 0, \end{aligned} \tag{3.6}$$

Clearly

$$z = xL(x) R(x) = x \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x} = \frac{\alpha x}{1 + \alpha x} \cdot \frac{x}{1 + \beta x}$$

is a monotonically increasing function of x . Furthermore, choose α large (e.g. $\alpha = 10$) and β small (e.g. $\beta = 10^{-3}$). Then it is intuitively clear that (i) small x -values transform into even smaller z -values, e.g. $x = 1$ corresponds to $z = 0.91$ and $x = 2$ corresponds to $z = 1.90$, but (ii) this effect dwindles as x increases, so $x = 10$ corresponds to $z = 9.8$ and $x = 50$ to $z = 47.5$ and the z -values closely resemble the x 's percentage-wise, but (iii) as x increases still further the z 's do not follow suit: $x = 10^3$ corresponds to $z = 500$. This suggests that if x is a value assumed by X , that Z shares the properties of X in mid-range, i.e. for intermediate x -values, but differs from X by having a disproportionate probability of assuming small values (near zero), or large values (near, but less than, $1/\beta$). Thus the hazard of Z will appear to be a "bath tubbed" version of X , particularly if X is exponential.

We focus attention on the representation (3.6) in what follows, mainly for analytical and computational convenience. Of course there are many other possibilities, such as

$$L(x) = 1 - e^{-\alpha x} \quad (3.7)$$

$$R(x) = e^{\beta x};$$

these latter may be adjusted to provide sharper-edged bath tubs than can (3.6), but iteration of (3.6) may be induced to accomplish the same purpose.

(2) A Decreasing Failure Rate Model

Define the random variable W in terms of X , X again being exponential with parameter λ^{-1} :

$$W = XT(X) \quad (3.8)$$

where $T(x)$ is an increasing function of x , $L(0) = 1$. Then the hazard may be made to exhibit a decreasing behavior.

Example. Suppose

$$T(x) = 1 + cx, \quad c > 0, \quad 0 \leq x. \quad (3.9)$$

Then

$$z = x(1 + cx) \quad (3.10)$$

is monotonic, and small x -values lead to comparable z -values (especially when c is small), but larger x -values are "amplified" by $1 + cx$ to yield increasingly large z -values.

Attention will be focused upon (3.9), although other possibilities exist that accomplish the same purpose, namely that of lengthening the right tail of the distribution of X (simulating outliers, for instance) while leaving the body of the distribution virtually unchanged.

IV. MATHEMATICAL PROPERTIES OF THE "BATH TUB" HAZARD MODEL

Various analytical properties of the previously described models will now be recorded. These provide useful insights into the behavior of the random variables Z and the underlying (generating) variables X .

A. Monotonicity; Quantiles

It is convenient to focus on monotonic increasing transformations, i.e. if

$$z = G(z) = z\phi(z) \quad (4.1)$$

then in order that the above function be monotonically increasing, $dz/dx > 0$. Observe that logarithmic differentiation of (4.1) provides

$$\frac{dz}{z} = \frac{dx}{x} + \frac{\phi'(x)}{\phi(x)} dx \quad (4.2)$$

and thus $dz/dx > 0$ if and only if

$$\frac{1}{x} + \frac{\phi'(x)}{\phi(x)} > 0 \quad (4.3)$$

Alternatively, the condition is, in terms of $L(x)$ and $R(x)$,

$$\frac{1}{x} + \frac{L'(x)}{L(x)} + \frac{R'(x)}{R(x)} > 0 \quad (4.4)$$

It is easily seen that the important example (3.6),

$$\phi(x) = \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x},$$

yields a monotonic relationship between z and x . The fact that this transformation can be easily and explicitly inverted (solved for x in terms of z) will be exploited subsequently.

Of course if $z(x)$ is monotonically increasing then so is $x(z)$, the inverse function. The events $(Z \leq z)$ and $(X \leq x(z))$ are equivalent, and so

$$P\{Z \leq z\} = P\{X \leq x(z)\}, \quad (4.5)$$

from which it follows that if $x_p \equiv x(p)$ is the p -100% quantile of X , i.e.

$$P\{X \leq x(p)\} = p, \quad (4.6)$$

then

$$P\{Z \leq z(p)\} = P\{Z \leq z(x(p))\} = p \quad (4.7)$$

and so $z(p)$, the $p \cdot 100\%$ quantile of Z is simply obtained from

$$z(p) = x(p) \phi(x(p)) = x(p) L(x(p)) R(x(p)) \quad (4.8)$$

In other words we very easily translate from (points on) the inverse distribution of X to the inverse distribution of Z . Explicit representation of the distribution of Z is however, not often easily possible.

B. Hazard and Density Function Relationships

In order to investigate the relationship between the hazards of Z and X , begin by writing

$$p = F_X(x(p)) = 1 - \exp\left[-\int_0^{x(p)} h_X(u) du\right] \quad (4.9)$$

or

$$\int_0^{x(p)} h_X(u) du = -\ln(1-p)$$

Now differentiate with respect to p to find

$$h_X(x(p)) \frac{dx(p)}{dp} = \frac{1}{1-p} \quad (4.10)$$

or

$$h_X(x(p)) = \frac{dp}{dx(p)} \cdot \frac{1}{1-p} = f_X(x(p)) \cdot \frac{1}{1-p}; \quad (4.11)$$

here h_X and f_X are the hazard and density functions of the r.v. X . The relationship (4.11) holds for any distribution, of course.

Differentiation of (3.5) reveals the connection between h_z and h_x . From (4.2)

$$\begin{aligned} \frac{dz(p)}{dp} &= z(p) \left[\frac{1}{x(p)} + \frac{\phi'(x(p))}{\phi(x(p))} \right] \frac{dx(p)}{dp} & (4.12) \\ &= [\phi(x(p)) + x(p) \phi'(x(p))] \frac{dx(p)}{dp} \end{aligned}$$

From (4.11), applied now to the z-hazard, there results

$$\frac{1}{h_z(z(p))} = \frac{1}{h_x(x(p))} [\phi(x(p)) + x(p) \phi'(x(p))] , \quad (4.13)$$

so

$$h_z(z(p)) = h_x(x(p)) \frac{1}{\phi(x(p)) + x(p) \phi'(x(p))} \quad (4.14)$$

Multiplication of both sides by $1-p$ then shows, in view of (4.11), that the density functions are similarly related:

$$f_z(z(p)) = f_x(x(p)) \frac{1}{\phi(x(p)) + x(p) \phi'(x(p))}$$

Example.

X is exponential(λ). Then

$$h_z(x_p) = \frac{\lambda}{\phi(x(p)) + x(p) \phi'(x(p))} \quad (4.15)$$

Now use the specific $\phi(x)$ of (3.6):

$$\phi(x) = \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x}$$

or, in terms of logarithms,

$$\ln \phi(x) = \ln \alpha x - \ln(1 + \alpha x) - \ln(1 + \beta x),$$

so

$$\frac{\phi'(x)}{\phi(x)} = \frac{1}{x} - \frac{\alpha}{1 + \alpha x} - \frac{\beta}{1 + \beta x} = \frac{1 - \alpha\beta x}{x(1 + \alpha x)(1 + \beta x)}, \quad (4.16)$$

and

$$\phi(x) + x\phi'(x) = \phi(x) \left[\frac{2 + (\alpha + \beta)x}{(1 + \alpha x)(1 + \beta x)} \right]; \quad (4.17)$$

finally

$$h_z(z(p)) = \frac{\lambda(1 + \alpha x(p))^2 (1 + \beta x(p))^2}{\alpha x(p) [2 + (\alpha + \beta)x(p)]} \quad (4.18)$$

Although this expression is not quite explicit, qualitative properties of h_z can be deduced from it.

(a) If $p \rightarrow 0$, $x(p) = -\frac{1}{\lambda} \ln(1-p) \rightarrow 0$, and hence

$$h_z(z(p)) \sim \frac{\lambda}{2\alpha x(p)}, \quad (4.19)$$

or

$$\lim_{p \rightarrow 0} x(p) h_z(z(p)) = \frac{\lambda}{2\alpha} \quad (4.20)$$

Since for $p \rightarrow 0$,

$$z(p) \sim \alpha x^2(p)$$

and hence

$$x(p) \sim [z(p)/\alpha]^{1/2} \quad (4.21)$$

there results

$$h_z(z(p)) \sim \frac{\lambda}{2[z(p)]^{1/2} \sqrt{\alpha}}$$

or

$$\lim_{p \rightarrow 0} \sqrt{z(p)} h_z(z(p)) = \frac{\lambda}{2 \sqrt{\alpha}} \quad (4.22)$$

This shows that $h_z(z) \rightarrow \infty$ as $z \rightarrow 0$, creating the left-hand end of the bath tub of Figure 1.

(b) If $p \rightarrow 1$, $x(p) \rightarrow \infty$, and $z(p) \rightarrow 1/\beta$ so

$$h_z(z(p)) \sim \lambda \frac{\alpha \beta^2 x^2(p)}{(\alpha + \beta)}$$

or

$$\lim_{p \rightarrow \infty} \frac{1}{[x(p)]^2} h_z(z(p)) = \lambda \frac{\alpha \beta^2}{\alpha + \beta} \quad (4.23)$$

For $p \rightarrow 1$

$$1 - \beta z(p) \sim \frac{(\alpha + \beta)}{\alpha \beta x(p)} \quad (4.24)$$

so

$$x(p) \sim \frac{\alpha + \beta}{\alpha \beta} \frac{1}{1 - \beta z(p)} \quad (4.24)$$

and thus

$$h_z(z(p)) \sim \lambda \left(\frac{\alpha + \beta}{\alpha} \right) \frac{1}{(1 - \beta z)^2} \quad (4.26)$$

Once again it appears that the hazard rises rapidly, this time as $x(p) \uparrow \infty$ and $z(p) \uparrow \beta^{-1}$; the other end of the bath tub is thus fashioned.

(c) If $p = 1 - e^{-1}$, then $x(p) = \lambda^{-1}$. Then

$$h_z(z(1-e^{-1})) = \frac{\lambda [1 + \alpha/\lambda]^2 [1 + \beta/\lambda]^2}{\frac{\alpha}{\lambda} [2 + (\alpha + \beta)/\lambda]} \quad (4.27)$$

The bath tub effect is presumably achieved by choosing α large and β small. Let $\alpha \rightarrow 0$ and $\beta \rightarrow 0$ independently in (4.27); it is clear that the limiting value of the hazard is λ . This indicates that the hazard is (approximately) λ for middling values of z .

C. An Explicit Formula for a Hazard

The expression (3.6) leads to the relationship

$$z(p) = \frac{\alpha x^2(p)}{(1 + \alpha x(p))(1 + \beta x(p))}, \quad (4.28)$$

and the latter may be explicitly inverted by solving a quadratic equation. The result is

$$x(p) = \frac{(\alpha + \beta) z(p) + \sqrt{(\alpha + \beta)^2 z^2(p) + 4z(p) \alpha(1 - \beta z(p))}}{2\alpha(1 - \beta z(p))} \quad (4.29)$$

Now a direct differentiation of this expression and invocation of (4.11) produces the expression

$$h_z(z) = \frac{\lambda}{2\alpha(1-\beta z)} \left[\frac{\beta\{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}\}}{1 - \beta z} + (\alpha + \beta) + \frac{\{(\alpha-\beta)^2 z + 2\alpha\} \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{(\alpha + \beta)^2 z^2 + 4\alpha z(1 - \beta z)} \right] \quad (4.30)$$

This form, while explicit, provides no particularly useful insights; the bath tub end shapes already noted in (4.22) and (4.26) can be deduced directly from (4.30).

Some graphical plots of h_z are presented later. They illustrate the behavior of the present hazard representation in a more understandable fashion than does the formula itself.

D. An Explicit Formula for the Failure Time Distribution

Because x and z are monotonically related through (4.28) we have

$$\begin{aligned}
P\{Z \leq z\} &= P \left\{ X \leq x = \frac{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{2\alpha(1-\beta z)} \right\} \\
&= 1 - \exp \left\{ -\lambda \left[\frac{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{2\alpha(1-\beta z)} \right] \right\} \quad (4.31)
\end{aligned}$$

Again the explicit formula seems unproductive of insights.

V. MATHEMATICAL PROPERTIES OF THE DECREASING FAILURE RATE MODEL

A. The Hazard Behavior.

The expression (4.14) can be applied to deduce the hazard function of the representation (3.8), advertised to produce a decreasing failure rate. There we specified

$$\phi(x) \equiv T(x) = 1 + cx, \quad (5.1)$$

and thus, from (4.14),

$$h_w(w(p)) = \frac{\lambda}{(1 + cx(p)) + x(p)c} = \frac{\lambda}{1 + 2cx(p)} \quad (5.2)$$

Qualitative properties follow easily.

(a) If $p \rightarrow 0$, $x(p) \rightarrow 0$, and

$$h_w(w(p)) \sim \lambda \quad (5.3)$$

Thus the hazard is approximately λ for small z .

(b) if $p \rightarrow 1$, $x(p) \rightarrow \infty$, and

$$w(p) \sim c[x(p)]^2$$

so

$$h_w(w(p)) \sim \frac{\lambda}{2\sqrt{cw(p)}}, \quad (5.5)$$

which clearly decreases, as claimed. It may be inferred that the distribution of W appears nearly exponential, but possesses an extraordinarily long right tail--possibly the result of outliers.

B. Explicit Formulas for the Hazard and the Distribution Function

Direct solution of the quadratic equation

$$w = x(1 + cx) = x + cx^2$$

presents

$$x(p) = \frac{\sqrt{1 + 4cw(p)} - 1}{2c}, \quad (5.6)$$

which, when differentiated, leads to

$$\begin{aligned} h_w(w) &= \frac{1}{\sqrt{1 + 4cw}} h_x(x) \\ &= \frac{\lambda}{\sqrt{1 + 4cw}} \end{aligned} \quad (5.7)$$

The distribution function is

$$\begin{aligned}
 P(W \leq w) &= P \left\{ X \leq x = \frac{\sqrt{1 + 4cw} - 1}{2c} \right\} & (5.8) \\
 &= 1 - \exp \left\{ -\lambda \left[\frac{\sqrt{1 + 4cw} - 1}{2c} \right] \right\}
 \end{aligned}$$

This distribution bears a close family resemblance to the Weibull distribution $1 - F(w) = \exp\{-k\sqrt{w}\}$, especially for large (right tail) values of w .

VI. AN ALTERNATIVE "BATHTUB" HAZARD REPRESENTATION

The simple parametric model (3.6) leading to a bathtub-shaped hazard is by no means the only possibility. We next describe another simple approach. It is that of defining a hazard function having an appropriate shape, and then deducing the corresponding distribution function, and a procedure for sampling from it, rather than proceeding in reverse order, as before.

Let the hazard be of the form

$$h(z) = g(z) + \lambda + k(z), \quad (6.1)$$

where $g(z) > 0$ is a decreasing function of z such that $\lim_{x \rightarrow \infty} g(z) = 0$, and $k(x)$ is an increasing function of x , such that (preferably) $k(0) = 0$ and $k(\infty) = \infty$. Such a function can yield a bathtubted hazard.

Example.

$$h(z) = \frac{A}{z + \alpha} + Bz + \lambda \quad (6.2)$$

A, B, α , λ all positive.

Clearly, (6.2) has a generally "bath tub-like" appearance, since

$$h'(z) = -\frac{A}{(z + \alpha)^2} + B \quad (6.3)$$

for if

$$-\frac{A}{\alpha} + B < 0, \quad \text{then } h'(0) < 0 \quad (6.4)$$

while for

$$z > z_0 = \left(\frac{A}{B}\right)^{1/2} - \alpha \quad (6.5)$$

$h(z) > 0$.

Detailed behavior is adjustable by choice of the parameters. Now the distribution function of time to failure, Z, is obtained from (6.2):

$$\begin{aligned} P\{Z > z\} &= \exp\left\{-\int_0^z h(x) dx\right\} \\ &= \exp\left\{-\int_0^z \left(\frac{A}{x + \alpha} + Bx + \lambda\right) dx\right\} \\ &= \exp\left\{-\left[A \ln\left(1 + \frac{z}{\alpha}\right) + \frac{B}{2} z^2 + \lambda z\right]\right\} \\ &= \left(\frac{\alpha}{\alpha + z}\right)^A \exp\left(-\frac{B}{2} z^2\right) e^{-\lambda z} \\ &= \bar{F}_1(z) \bar{F}_2(z) \bar{F}_3(z), \end{aligned} \quad (6.6)$$

where

$$\bar{F}_1(z) = \left(\frac{\alpha}{\alpha + z}\right)^A$$

$$\bar{F}_2(z) = \exp\left(-\frac{B}{2} z^2\right) \quad (6.7)$$

$$\bar{F}_3(z) = e^{-\lambda z}$$

All of the above are recognized as being the complements of distribution functions. In effect, the distribution of Z is that of the minimum of three independent random variables:

$$P\{Z > z\} = P\{X_1 > z\} \cdot P\{X_2 > z\} \cdot P\{X_3 > z\}, \quad (6.8)$$

X_i having the distribution $F_i = 1 - \bar{F}_i$ ($i = 1, 2, 3$). This fact leads directly to an easy procedure for simulation of Z by simply obtaining the smallest from among the realization of X_1 , X_2 , and X_3 . The advantage of the previous method, based on (3.6) for instance, is that only one realization--that of an exponential in that specific example--leads to the realization of Z . This is not only computationally attractive, but seems to facilitate the application of such Monte Carlo variance reduction techniques as control and antithetic variables cf. Hammersley and Handscomb (1964).

VII. OBTAINING SPECIFIED HAZARD BEHAVIOR BY SIMPLE SAMPLING

The development of the last section illustrates one manner in which hazard behavior may be conveniently represented and simulated. We now show how such behavior may alternatively be obtained by simple simulation, i.e. from one realization of a basic (possibly exponential) random variable.

Refer to (3.5), in which

$$Z = G(X) \quad (7.1)$$

and, if $G(\cdot)$ is monotonically increasing,

$$z(p) = G(x(p)) \quad , \quad (7.2)$$

$z(p)$ and $x(p)$ being the $p \cdot 100\%$ percentiles of Z and X respectively. Then the counterpart to (4.14) that results from differentiation of (7.2) is the expression

$$h_z(z(p)) = h_x(x(p)) \frac{1}{G'(x(p))} = h_x(x(p)) \frac{1}{(dz/dx)} \quad (7.3)$$

Consequently, if one specifies $h_z(z)$ as a suitable function of the "time" z , and specifies the distribution of the stochastic variable X --and hence its hazard, h_x --there results a differential equation for $z(x) \equiv G(x)$:

$$h_z(z) \frac{dz}{dx} = h_x(x) \quad ; \quad (7.4)$$

integration then provides the desired transformation, G . In other words, we seek $z(x)$ satisfying

$$\int_0^z h_z(u) du = \int_0^x h_x(v) dv, \quad (7.5)$$

which can sometimes be carried out in a useful closed form.

Example 7.1. Refer to the example of Section VI, wherein h_z is given by the expression (6.2) and we assume that X is exponential, so h_x is constant. Then in order to determine $G(x) \equiv z(x)$, solve the equation

$$\int_0^z \left[\frac{A}{v + \alpha} + Bv + \lambda \right] dv = \int_0^x du$$

or

$$A \ln\left(1 + \frac{z}{\alpha}\right) + \frac{B}{2} z^2 + \lambda z = x \quad (7.6)$$

Closed-form solution of this expression for z in terms of x is of course impossible. One possible approach is purely numerical: find an approximate solution, $z_0(x)$, e.g. the appropriate solution of the quadratic

$$\frac{B}{2} z^2 + \lambda z = x \quad (7.7)$$

and then correct the result by a few Newton-Raphson iterations.

In other words, put

$$z_1(x) = \frac{-\lambda + \sqrt{\lambda^2 + 2B\lambda x}}{B}; \quad (7.8)$$

now apply Newton to obtain an improved solution

$$z_2(x) = z_1 - \frac{A \ln(1 + z_1/\alpha)}{A/(z_1 + \alpha) + Bz_1 + \lambda} \quad (7.9)$$

which will be feasible if $0 < z_2$. The process can be iterated (the numerator will change after the first iteration). If one wishes to use this model it may actually be desirable to start by solving

$$\frac{B}{2} z^2 + (\lambda + \frac{A}{\alpha})z - x = 0 \quad (7.10)$$

for z_1 , in which case the numerator will not be as shown in (7.9); convergence may be more rapid.

Example 7.2. Change the hazard representation of the previous example as follows: let

$$h_z(v) = \frac{A}{(v + \alpha)^2} + Bv + \lambda; \quad (A, \alpha, B, \lambda > 0) \quad (7.11)$$

then

$$\int_0^z h_z(v) dv = \frac{Az}{\alpha(z + \alpha)} + \frac{B}{2} z^2 + \lambda z \quad (7.12)$$

Now the solution of the equation

$$\frac{Az}{\alpha(z + \alpha)} + \frac{B}{2} z^2 + \lambda z = x \quad (7.13)$$

can be carried out in closed (if cumbersome) form, since (7.13) is a cubic. Hence an explicit representation of $z(p)$ in terms of an exponential's $x(p)$ can be provided. Perhaps more importantly, a direct, simple, simulation of a bathtubted random variable is at hand.

VIII. CONCLUSION

The purpose of this report has been to show that rather complex distributional behavior can be represented in terms of simple standard distributions. We have concentrated here upon alterations of the exponential, but it is obvious that other distributions can be treated similarly.

In a later report we will illustrate the use of this technique in data analysis, and in the simulation of non-homogeneous stochastic point processes.

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