

Fig. 2



APPLICATION OF SPLINES TO THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS

Donald C. Todd and John C. Adams, Jr.
ARO, Inc., a Sverdrup Corporation Company

VON KARMAN GAS DYNAMICS FACILITY
ARNOLD ENGINEERING DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
ARNOLD AIR FORCE STATION, TENNESSEE 37389

December 1978

Final Report for Period October 1977 - August 1978

Approved for public release; distribution unlimited.

TECHNICAL REPORTS
FILE COPY

Property of U. S. Air Force

Prepared for

ARNOLD ENGINEERING DEVELOPMENT CENTER/DOTr
ARNOLD AIR FORCE STATION, TENNESSEE 37389

NOTICES

When U. S. Government drawings, specifications, or other data are used for any purpose other than a definitely related Government procurement operation, the Government thereby incurs no responsibility nor any obligation whatsoever, and the fact that the Government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise, or in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture, use, or sell any patented invention that may in any way be related thereto.

Qualified users may obtain copies of this report from the Defense Documentation Center.

References to named commercial products in this report are not to be considered in any sense as an indorsement of the product by the United States Air Force or the Government.

This report has been reviewed by the Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

APPROVAL STATEMENT

This report has been reviewed and approved.



ELTON R. THOMPSON
Project Manager, Research Division
Directorate of Test Engineering

Approved for publication:

FOR THE COMMANDER



ROBERT W. CROSSLEY, Lt Colonel, USAF
Acting Director of Test Engineering
Deputy for Operations

UNCLASSIFIED

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1 REPORT NUMBER AEDC-TR-78-59	2. GOVT ACCESSION NO.	3 RECIPIENT'S CATALOG NUMBER
4 TITLE (and Subtitle) APPLICATION OF SPLINES TO THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS		5 TYPE OF REPORT & PERIOD COVERED Final Report, October 1977 - August 1978
		6 PERFORMING ORG. REPORT NUMBER
7 AUTHOR(s) Donald C. Todd and John C. Adams, Jr., ARO, Inc., a Sverdrup Corporation Company		8 CONTRACT OR GRANT NUMBER(s)
9 PERFORMING ORGANIZATION NAME AND ADDRESS Arnold Engineering Development Center Air Force Systems Command Arnold Air Force Station, Tennessee 37389		10 PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element 65807F
11 CONTROLLING OFFICE NAME AND ADDRESS Arnold Engineering Development Center (OIS) Air Force Systems Command Arnold Air Force Station, Tennessee 37389		12. REPORT DATE December 1978
		13. NUMBER OF PAGES 121
14 MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a DECLASSIFICATION/DOWNGRADING SCHEDULE N/A
16 DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17 DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18 SUPPLEMENTARY NOTES Available in DDC.		
19 KEY WORDS (Continue on reverse side if necessary and identify by block number) Navier-Stokes equations splines boundary value problems differential equations		
20 ABSTRACT (Continue on reverse side if necessary and identify by block number) In the search for better methods to solve the Navier-Stokes equations, this is a preliminary test of spline collocation, or the application of spline collocation to solve two-point boundary-value problems. Pertinent spline theory and the spline collocation method are developed from first principles. The problems considered are nonlinear, third-order, ordinary differential equations. A FORTRAN IV computer program to solve such problems		

UNCLASSIFIED

UNCLASSIFIED

20. ABSTRACT (Continued)

is described, and a source deck listing is included. Several sample problems solved by the program are presented.

UNCLASSIFIED

PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC). The Air Force program manager was Elton R. Thompson. This work was done by ARO, Inc., AEDC Division (a Sverdrup Corporation Company), operating contractor for the AEDC, AFSC, Arnold Air Force Station, Tennessee, under ARO Project Number V32A-P1A. The manuscript was submitted for publication on August 23, 1978.

CONTENTS

	<u>Page</u>
1.0 INTRODUCTION	5
2.0 SPLINE THEORY	
2.1 Cubic Splines	6
2.2 Truncation Error of Cubic Spline Formulas	11
2.3 Higher Order Approximations	14
2.4 Quintic Splines	18
3.0 SPLINE COLLOCATION	
3.1 The Linear Problem	21
3.2 The Nonlinear Problem	30
4.0 THE COMPUTER PROGRAM	31
5.0 EXAMPLE PROBLEMS	
5.1 Example 1	34
5.2 Example 2: The Falkner-Skan Equation	35
5.3 Example 3: Viscoelastic Fluid	36
5.4 Example 4: Chemical Dispersion	39
5.5 Example 5: An Inherently Unstable Problem	40
5.6 Example 6	42
5.7 Example 7: $y'' = e^y$	44
REFERENCES	48

ILLUSTRATIONS

Figure

1. Clarification of the h_i	7
2. Coefficient Matrix for $N = 7$	29
3. Solutions to the Falkner-Skan Equation	37
4. First Derivatives of Solutions to the Falkner-Skan Equation	38
5. Solutions to the Chemical Dispersion Equation	41
6. Solution to the Inherently Unstable Problem	43
7. Solutions of Example 6	46
8. Solution to $y'' = e^y$	47

APPENDIXES

	<u>Page</u>
A. LISTING OF THE COMPUTER PROGRAM	49
B. INITIAL SOLUTION	72
C. CHANGING SPACING	75
D. MODIFICATIONS TO RUN THE EXAMPLE PROBLEMS	77
E. PRINTOUTS OF THE EXAMPLE PROBLEMS	85
NOMENCLATURE	119

1.0 INTRODUCTION

The availability of increased computer power within the past decade has spurred considerable effort toward development of computer programs for the numerical solution of the Navier-Stokes equations. The first and continuing effort involved numerical solution by finite-difference schemes. In recent years, however, other methods have come under investigation; two of these methods are finite elements and spline collocation. A preliminary test of spline collocation, namely application to numerically solve various two-point boundary-value problems, is the subject of the present report. Pertinent spline theory is developed from first principles. The spline collocation method, which is similar to that used by Rubin and Khosla (Ref. 1), is also fully developed. The problems considered are nonlinear, third-order, ordinary differential equations. A FORTRAN IV computer program to solve such problems is described, together with a source deck listing. Several example problems solved by the program are presented.

A spline interpolates between points by polynomials determined by various conditions of continuity. If a spline using polynomials of degree p is continuous along with its first q derivatives, $0 \leq q \leq p$, then the spline is said to have a deficiency of $p - q$. For example, a cubic spline with continuous second derivatives has a deficiency of one. Rubin and Khosla (Ref. 1) have shown that classical three-point difference formulas are equivalent to a quadratic spline with a deficiency of zero. The method presented in this report uses a quintic spline of deficiency three.

Three-point, finite-difference methods, when applied to a linear equation, produce a tridiagonal algebraic system to be solved (Ref. 2), whereas spline collocation (for second-order equations) produces a tridiagonal system of 2 by 2 blocks. End conditions involving first and second derivatives are naturally imposed using spline collocation. For uniform grid spacing the three-point, finite-difference formula for the second derivative yields a second-order approximation, whereas the quintic spline gives third-order accuracy.

Splines were originally invented for interpolation, which means that interpolation of spline solutions is immediately available. This plus the suitability of spline collocation to handle nonuniform spacing makes it feasible to change grid spacing between iterations. This can be used effectively to improve accuracy of the resulting solution.

It is concluded that splines have many useful features and should have a role in future techniques for solving complicated problems.

2.0 SPLINE THEORY

Given points (x_i, y_i) with $i = 1, 2, \dots, N$, a spline has the following properties:

1. It is a function defined on $[x_1, x_N]$.
2. It passes thru the points.
3. It is continuous. The splines considered in this report have continuous first and second derivatives.
4. It is defined by $(N - 1)$ polynomials, one for each interval $[x_i, x_{i+1}]$. The polynomials are determined by the conditions of continuity and the end conditions.

It is convenient for the succeeding analysis to make the following definitions.

$$h_i = x_{i+1} - x_i \quad (1)$$

$$\sigma_i = \frac{h_{i+1}}{h_i} \quad (2)$$

$$\eta_i = \frac{1}{h_i} (x - x_i) \quad (3)$$

$$\theta_i = 1 - \eta_i \quad (4)$$

Figure 1 shows the first two and last three points, (x_i, y_i) , and their related h_i .

2.1 CUBIC SPLINES

A cubic spline is defined by

$$s(x) = p_i(x) \quad (5)$$

for $x_i \leq x \leq x_{i+1}$ with $i = 1, 2, \dots, N - 1$ where the $p_i(x)$ are cubics. Since $s(x)$ passes thru the points,

$$s(x_i) = y_i \quad (6)$$

Since $s(x)$ is assumed to have continuous first and second derivatives, one can make the definitions

$$m_i = s'(x_i) \quad (7)$$

and

$$M_i = s''(x_i) \quad (8)$$

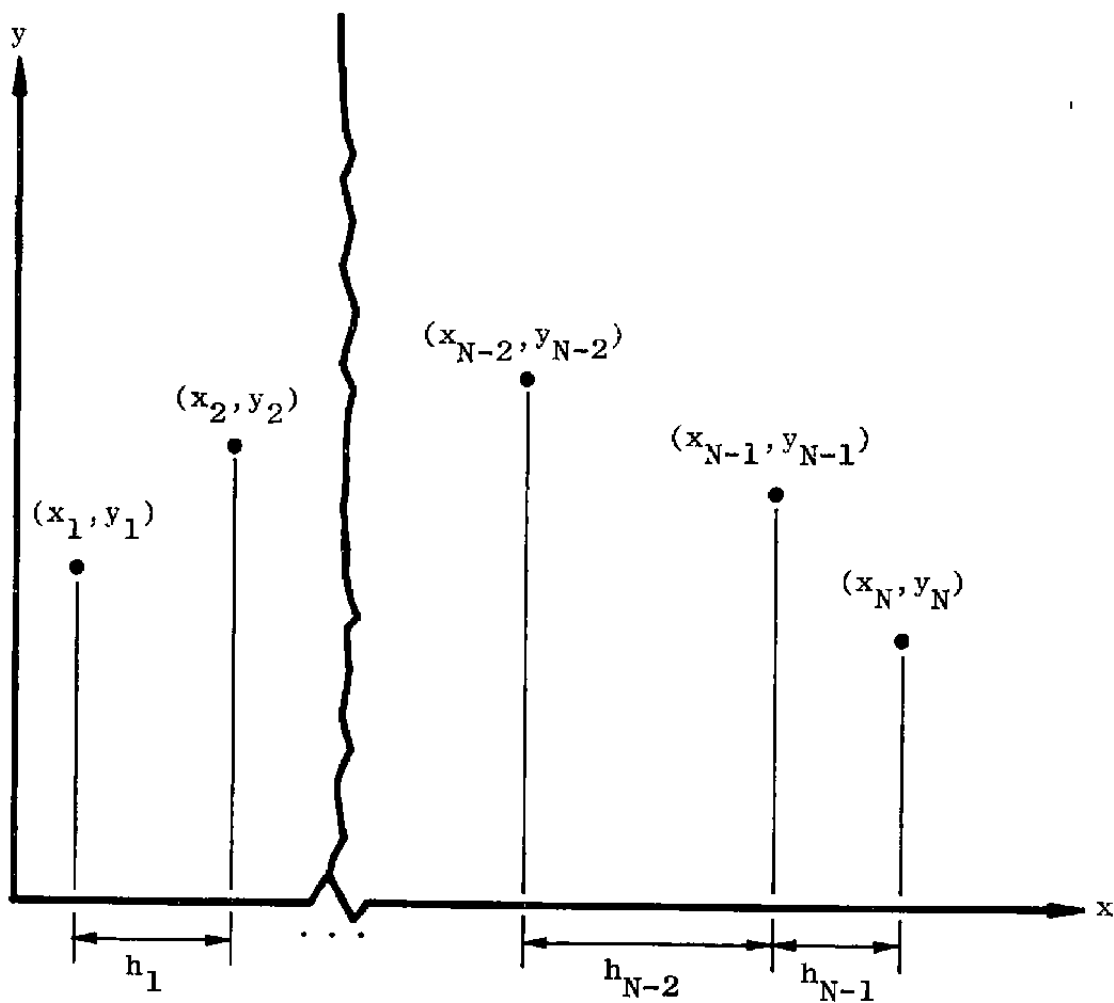


Figure 1. Clarification of the h_i .

The $p_i(x)$ can be written in terms of the m_i as follows. Define

$$y_{Ai} = y_i + \frac{1}{3} h_i m_i \quad (9)$$

and

$$y_{Bi} = y_{i+1} - \frac{1}{3} h_i m_{i+1} \quad (10)$$

then $p_i(x)$ can be written

$$p_i(x) = y_i \theta_i^3 + 3 y_{Ai} \eta_i \theta_i^2 + 3 y_{Bi} \eta_i^2 \theta_i + y_{i+1} \eta_i^3 \quad (11)$$

The first derivative is

$$\begin{aligned} p_i'(x) = \frac{3}{h_i} [& -y_i \theta_i^2 + y_{Ai} (\theta_i - 2\eta_i) \theta_i \\ & + y_{Bi} \eta_i (2\theta_i - \eta_i) + y_{i+1} \eta_i^2] \end{aligned} \quad (12)$$

and the second derivative is

$$p_i''(x) = \frac{6}{h_i^2} [y_i \theta_i + y_{Ai} (\eta_i - 2\theta_i) + y_{Bi} (\theta_i - 2\eta_i) + y_{i+1} \eta_i] \quad (13)$$

The $p_i(x)$ are cubics in x , and it can be confirmed that, as defined,

$$p_i(x_i) = y_i \quad (14)$$

$$p_i(x_{i+1}) = y_{i+1} \quad (15)$$

$$p_i''(x_i) = m_i \quad (16)$$

and

$$p_i'(x_{i+1}) = m_{i+1} \quad (17)$$

The $p_i(x)$ have been written in terms of the m_i which, as yet, have not been determined. A relation among the m_i can be derived by the requirement that the second derivative be continuous; thus,

$$p_{i-1}''(x_i) = p_i''(x_i) = M_i \quad (18)$$

From Eq. (13),

$$p_{i-1}''(x_i) = \frac{6}{h_{i-1}^2} [y_{A,i-1} - 2y_{B,i-1} + y_i] \quad (19)$$

Thus after substituting Eqs. (9) and (10) into (19) and simplifying,

$$M_i = \frac{2}{h_{i-1}} (2m_i + m_{i-1}) - \frac{6}{h_{i-1}^2} (y_i - y_{i-1}) \quad (20)$$

Again, from Eq. (13),

$$p_i''(x_i) = \frac{6}{h_i^2} [y_i - 2y_{Ai} + y_{Bi}] \quad (21)$$

which, after substitution of Eqs. (9) and (10), yields

$$M_i = \frac{6}{h_i^2} (y_{i+1} - y_i) - \frac{2}{h_i} (2m_i + m_{i+1}) \quad (22)$$

Subtracting Eq. (22) from Eq. (20) and simplifying gives the desired relation among the m_i ,

$$\begin{aligned} \frac{1}{h_{i-1}} m_{i-1} + 2 \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) m_i + \frac{1}{h_i} m_{i+1} \\ = -\frac{3}{h_{i-1}^2} y_{i-1} + 3 \left(\frac{1}{h_{i-1}^2} - \frac{1}{h_i^2} \right) y_i + \frac{3}{h_i^2} y_{i+1} \end{aligned} \quad (23)$$

Assume end conditions

$$A_i y_i + B_i m_i + C_i M_i = D_i \quad (24)$$

for $i = 1$ and N where A_i , B_i , C_i , and D_i are given constants. In each case, $i = 1$ or $i = N$, at least one of the coefficients, B_i or C_i , must be nonzero. Since the y_i are given, the A_i are unnecessary and can be assumed zero. They were included for purposes of Section 3 on spline collocation where the y_i are unknown. With $i = 1$, substitution of Eq. (22) into (24) yields

$$\begin{aligned} \left(B_1 - \frac{4}{h_1} C_1 \right) m_1 - \frac{2}{h_1} C_1 m_2 \\ = D_1 - \left(A_1 - \frac{6}{h_1^2} C_1 \right) y_1 - \frac{6}{h_1^2} C_1 y_2 \end{aligned} \quad (25)$$

Similarly, with $i = N$, substitution of Eq. (20) into (24) yields

$$\begin{aligned} \left(B_N + \frac{4}{h_{N-1}} C_N \right) m_N + \frac{2}{h_{N-1}} C_N m_{N-1} \\ = D_N - \left(A_N - \frac{6}{h_{N-1}^2} C_N \right) y_N - \frac{6}{h_{N-1}^2} C_N y_{N-1} \end{aligned} \quad (26)$$

Equations (25), (23) with $i = 2, 3, \dots, N-1$, and Eq. (26) form a tridiagonal system which can be solved for the m_i . When the solution for the m_i has been effected, the spline is known.

It is a straightforward matter to derive formulas for the integral of the spline. If one has a formula for

$$J_i(a, b) = \int_a^b p_i(x) dx \quad (27)$$

where $x_i \leq a < b \leq x_{i+1}$ then the integral over an arbitrary subinterval of $[x_1, x_N]$ can be obtained by summing the appropriate $J_i(a,b)$ with the proper arguments. When Eq. (11) is integrated,

$$J_i(a, b) = h_i \left[-\frac{1}{4} y_i \theta_i^4 - y_{Ai} \left(\eta_i + \frac{1}{4} \theta_i \right) \theta_i^3 + y_{Bi} \eta_i^3 \left(\theta_i + \frac{1}{4} \eta_i \right) + \frac{1}{4} y_{i+1} \eta_i^4 \right]_a^b \quad (28)$$

Of particular interest is

$$J_i(x_i, x_{i+1}) = \frac{h_i}{4} [y_i + y_{Ai} + y_{Bi} + y_{i+1}] \quad (29)$$

When one has the given points, (x_i, y_i) , and the calculated m_i , then the M_i can be computed from Eqs. (20) and (22). In analysis, in order not to give preferential treatment to one formula or the other at interior points, it is expedient to average the two equations to arrive at

$$M_i = \frac{1}{h_{i-1}} m_{i-1} + 2 \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) m_i - \frac{1}{h_i} m_{i+1} + \frac{3}{h_{i-1}^2} y_{i-1} - 3 \left(\frac{1}{h_{i-1}^2} + \frac{1}{h_i^2} \right) y_i + \frac{3}{h_i^2} y_{i+1} \quad (30)$$

The fundamental relation, Eq. (23), was found by subtracting Eq. (20) from (22). Equation (20) was obtained by adding Eqs. (20) and (22). So, in that sense, there is a symmetrical relation between Eqs. (23) and (30).

By eliminating the y_i from Eqs. (20) and (22), a relation between the m_i and M_i is obtained.

$$M_{i-1} + M_i = \frac{2}{h_i} (m_{i+1} - m_i) \quad (31)$$

The spline relations have been written in terms of unknown m_i . Analogous formulas in terms of unknown M_i can be derived. Replacing i with $(i + 1)$ in Eq. (20) and eliminating m_{i+1} from Eqs. (20) and (22) yields

$$m_i = \frac{1}{h_i} (y_{i+1} - y_i) - \frac{h_i}{6} (2M_i + M_{i+1}) \quad (32)$$

Replacing i with $(i - 1)$ in Eq. (22) and eliminating m_{i-1} from Eqs. (22) and (20) yields

$$m_i = \frac{1}{h_{i-1}} (y_i - y_{i-1}) + \frac{h_{i-1}}{6} (2M_i + M_{i-1}) \quad (33)$$

Averaging Eqs. (32) and (33) gives

$$m_i = -\frac{1}{2h_{i-1}} y_{i-1} + \frac{1}{2} \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) y_i + \frac{1}{2h_i} y_{i+1} + \frac{h_{i-1}}{12} M_{i-1} + \frac{1}{6} (h_{i-1} - h_i) M_i - \frac{h_i}{12} M_{i+1} \quad (34)$$

Subtracting gives

$$\begin{aligned} & \frac{h_{i-1}}{6} M_{i-1} + \frac{1}{3} (h_{i-1} + h_i) M_i + \frac{h_i}{6} M_{i+1} \\ &= \frac{1}{h_{i-1}} y_{i-1} - \left(\frac{1}{h_{i-1}} + \frac{1}{h_i} \right) y_i + \frac{1}{h_i} y_{i+1} \end{aligned} \quad (35)$$

The end conditions, Eq. (24), in terms of unknown M_i are found to be

$$\begin{aligned} & \left(C_1 - \frac{h_1}{3} B_1 \right) M_1 - \frac{h_1}{6} B_1 M_2 \\ &= D_1 - \left(A_1 - \frac{1}{h_1} B_1 \right) y_1 - \frac{1}{h_1} B_1 y_2 \end{aligned} \quad (36)$$

and

$$\begin{aligned} & \left(C_N + \frac{h_{N-1}}{3} B_N \right) M_N + \frac{h_{N-1}}{6} B_N M_{N-1} \\ &= D_N - \left(A_N + \frac{1}{h_{N-1}} B_N \right) y_N + \frac{1}{h_{N-1}} B_N y_{N-1} \end{aligned} \quad (37)$$

As an alternative to solving for the m_i , Eqs. (36), (35) with $i = 2, 3, \dots, N-1$, and Eq. (37) form a tridiagonal system which can be solved for the M_i .

The preceding formulas on cubic splines are adequate for the present analysis. For a complete treatment and proofs, see Ref. 3.

2.2 TRUNCATION ERROR OF CUBIC SPLINE FORMULAS

Fyfe, Ref. 4, used operator methods to obtain a Taylor expansion of cubic spline relations with uniform spacing. His work can be generalized for unequal spacing as follows. Define the operator E_i by

$$E_i y_i = y_{i+1} \quad (38)$$

Let $y(x)$ be a function such that $y(x_i) = y_i$. Indicate the j th derivative by $y^{(j)}(x)$ and define $y_i^{(j)} = y^{(j)}(x_i)$. For the purpose of this error analysis, the tenth derivative will be assumed continuous. Expanding the right-hand side of Eq. (38) in a Taylor series,

$$E_i y_i = y_i + h_i y_i' + \frac{1}{2} h_i^2 y_i'' + \dots \quad (39)$$

Using the operator D to denote differentiation, from Eq. (39),

$$E_i = 1 + h_i D + \frac{1}{2} h_i^2 D^2 + \dots \quad (40)$$

and it is interesting to observe that

$$E_i = \exp(h_i D) \quad (41)$$

where the exponential is defined by its expansion.

Since

$$E_{i-1} y_{i-1} = y_i \quad (42)$$

then (assuming existence)

$$y_{i-1} = E_{i-1}^{-1} y_i \quad (43)$$

and from Eq. (41),

$$E_{i-1}^{-1} = \exp(-h_{i-1} D) \quad (44)$$

Some consider the last step presumptuous; however, understanding that operator expressions are merely convenient representations of series, and assuming linearity, the distributive law, etc., rigorous justification is possible. This section will lean heavily on such algebraic analogies.

In operator notation, Eq. (23) becomes

$$\begin{aligned} & \left[\frac{1}{h_{i-1}} E_{i-1}^{-1} + 2 \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) + \frac{1}{h_i} E_i \right] m_i \\ &= \left[-\frac{3}{h_{i-1}^2} E_{i-1}^{-1} - 3 \left(\frac{1}{h_{i-1}^2} - \frac{1}{h_i^2} \right) + \frac{3}{h_i^2} E_i \right] y_i \end{aligned} \quad (45)$$

and solving for m_i ,

$$m_i = \frac{\left[-\frac{3}{h_{i-1}^2} E_{i-1}^{-1} + 3 \left(\frac{1}{h_{i-1}^2} - \frac{1}{h_i^2} \right) + \frac{3}{h_i^2} E_i \right]}{\left[\frac{1}{h_{i-1}} E_{i-1}^{-1} + 2 \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) + \frac{1}{h_i} E_i \right]} y_i \quad (46)$$

or, if one expands Eq. (46) into a series,

$$\begin{aligned} m_i &= \left[1 - \frac{1}{72} \sigma_{i-1} (\sigma_{i-1} - 1) (h_{i-1} D)^3 \right. \\ &\quad - \frac{\sigma_{i-1}}{180} \frac{\sigma_{i-1}^3 + 1}{\sigma_{i-1} + 1} (h_{i-1} D)^4 \\ &\quad \left. - \frac{1}{2,160} \sigma_{i-1} (\sigma_{i-1} - 1) (3\sigma_{i-1}^2 - 5\sigma_{i-1} + 3) (h_{i-1} D)^5 + \dots \right] y_i' \end{aligned} \quad (47)$$

where σ_{i-1} is defined by Eq. (2). Thus it is found that m_i is a second-order-accurate approximation to y_i' (third-order error). For uniform spacing ($\sigma_{i-1} = 1$), it is third-order accurate.

In operator notation, Eq. (31) becomes

$$(E_i + 1) M_i = \frac{2}{h_i} (E_i - 1) m_i \quad (48)$$

so

$$M_i = \frac{2}{h_i} \tanh\left(\frac{1}{2} h_i D\right) m_i \quad (49)$$

or

$$M_i = \frac{2}{h_i} \left[\left(\frac{1}{2} h_i D\right) - \frac{1}{3} \left(\frac{1}{2} h_i D\right)^3 + \frac{2}{15} \left(\frac{1}{2} h_i D\right)^5 - \frac{17}{315} \left(\frac{1}{2} h_i D\right)^7 + \dots \right] m_i \quad (50)$$

Combining with Eq. (47) yields

$$M_i = \left[1 - \frac{1}{12} \sigma_{i-1}^2 (h_{i-1} D)^2 - \frac{1}{72} \sigma_{i-1} (\sigma_{i-1} - 1) (h_{i-1} D)^3 + \frac{1}{360} \sigma_{i-1} (3\sigma_{i-1}^3 - 2\sigma_{i-1}^2 + 2\sigma_{i-1} - 2) (h_{i-1} D)^4 - \frac{1}{4,320} \sigma_{i-1} (\sigma_{i-1} - 1) (\sigma_{i-1}^2 - 10\sigma_{i-1} + 6) (h_{i-1} D)^5 + \dots \right] y_i'' \quad (51)$$

Thus, M_i is a first-order-accurate approximation to y_i'' (second-order error). For uniform spacing ($\sigma_{i-1} = 1$) it is a second-order-accurate approximation.

Equations (47) and (51) do not agree exactly with Rubin and Khosla (Ref. 1); however, as Ref. 1 points out, the expression derived depends on which equation one begins with. This discrepancy is explained by the limited information conveyed by N points. The N points determine a polynomial of degree $(N - 1)$, but there are infinitely many polynomials of higher degree which pass thru the points. Thus a Taylor expansion of the method is not unique. Considering the nonuniqueness, one may wonder about the value of the error analysis. The value comes from the conclusions drawn on the order of the method, which are valid and useful. From the error analysis, higher order approximations will be derived.

Substituting Eqs. (9) and (10) into (29) gives

$$J_i(x_i, x_{i+1}) = \frac{1}{2} h_i (y_i + y_{i+1}) - \frac{1}{12} h_i^2 (m_{i-1} - m_i) \quad (52)$$

This is seen to be the trapezoidal rule plus an extra term. In operator notation,

$$J_i(x_i, x_{i+1}) = \frac{1}{2} h_i (E_i + 1) y_i - \frac{1}{12} h_i^2 (E_i - 1) m_i \quad (53)$$

Substituting Eq. (47) into (53) and expanding gives

$$\begin{aligned}
 J_1(x_i, x_{i+1}) = & \left[1 + \frac{1}{2} h_i D + \frac{1}{6} (h_i D)^2 + \frac{1}{24} (h_i D)^3 \right. \\
 & + \frac{1}{144} (h_i D)^4 + \frac{3\sigma_{i-1}^2 + 5\sigma_{i-1} - 5}{(12)(360)\sigma_{i-1}^2} (h_i D)^5 \\
 & \left. + \frac{9\sigma_{i-1}^2 - 9\sigma_{i-1} + 4}{(12)(720)\sigma_{i-1}^3} (h_i D)^6 + \dots \right] h_i y_1
 \end{aligned} \quad (54)$$

Expanding $y(x)$ in a Taylor series and integrating, one finds that

$$\int_{x_i}^{x_{i+1}} y(x) dx = \left[\sum_{j=0}^{\infty} \frac{(h_i D)^j}{(j+1)!} \right] h_i y_1 \quad (55)$$

Comparing Eqs. (54) and (55),

$$\begin{aligned}
 J_1(x_i, x_{i+1}) = & \int_{x_i}^{x_{i+1}} y(x) dx - \frac{1}{720} (h_i D)^5 \left[1 \right. \\
 & + \frac{3\sigma_{i-1}^2 - 5\sigma_{i-1} + 5}{6\sigma_{i-1}^2} h_i D \\
 & \left. + \frac{12\sigma_{i-1}^3 - 63\sigma_{i-1}^2 + 63\sigma_{i-1} - 28}{84\sigma_{i-1}^3} (h_i D)^2 + \dots \right] D^{-1} y_1
 \end{aligned} \quad (56)$$

Thus, the formula is found to be fourth-order accurate (fifth-order error).

2.3 HIGHER ORDER APPROXIMATIONS

From Eq. (51),

$$y_i'' = M_i + \frac{1}{12} \sigma_{i-1}^2 h_{i-1}^2 y_1^{iv} + \dots \quad (57)$$

If an approximation of zero-order accuracy (first-order error) were known for y_i^{iv} , then substitution into Eq. (57) would yield a second-order-accurate approximation for y_i'' . Consider the conjecture

$$y_i^{iv} \approx \frac{2}{\sigma_{i-1} (\sigma_{i-1} + 1) h_{i-1}^2} \left[\sigma_{i-1} M_{i-1} - (\sigma_{i-1} + 1) M_i + M_{i+1} \right] \quad (58)$$

which is the three-point, finite-difference formula applied to the M_i . Substitution of Eq. (58) into Eq. (57) yields, as the approximation for y_i'' ,

$$\mathfrak{M}_i = M_i - \frac{\Delta_i}{6} \left[\sigma_{i-1} M_{i-1} - (\sigma_{i-1} + 1) M_i + M_{i+1} \right] \quad (59)$$

where

$$\Delta_i = \frac{\sigma_{i-1}}{\sigma_{i-1} + 1} \quad (60)$$

Note that if $\Delta_i = 0$ then $\mathfrak{M}_i = M_i$. In operator notation,

$$\mathfrak{M}_i = \left\{ 1 + \frac{\Delta_i}{6} \left[\sigma_{i-1} E_{i-1}^{-1} - (\sigma_{i-1} + 1) + E_i \right] \right\} M_i \quad (61)$$

or, expanding the operator,

$$\begin{aligned} \mathfrak{M}_i = & \left\{ 1 + \frac{\Delta_i}{6} \left[\frac{1}{2} \sigma_{i-1} (\sigma_{i-1} + 1) (h_{i-1} D)^2 \right. \right. \\ & + \frac{1}{3!} \sigma_{i-1} (\sigma_{i-1}^2 - 1) (h_{i-1} D)^3 + \frac{1}{4!} \sigma_{i-1} (\sigma_{i-1}^3 + 1) (h_{i-1} D)^4 \\ & \left. \left. + \frac{1}{5!} \sigma_{i-1} (\sigma_{i-1}^4 - 1) (h_{i-1} D)^5 - \dots \right] \right\} M_i \end{aligned} \quad (62)$$

Substituting Eq. (51) into (62) gives

$$\begin{aligned} \mathfrak{M}_i = & \left\{ 1 + \frac{\sigma_{i-1}^2}{12} \left(\Delta_i \frac{\sigma_{i-1} + 1}{\sigma_{i-1}} - 1 \right) (h_{i-1} D)^2 \right. \\ & + \frac{\sigma_{i-1} (\sigma_{i-1} - 1)}{72} \left[2\Delta_i (\sigma_{i-1} - 1) - 1 \right] (h_{i-1} D)^3 \\ & + \frac{\sigma_{i-1}}{720} \left[2(3\sigma_{i-1}^3 - 2\sigma_{i-1}^2 + 2\sigma_{i-1} - 2) - 5\Delta_i (\sigma_{i-1}^2 - 1) \right] (h_{i-1} D)^4 \\ & - \frac{\sigma_{i-1} (\sigma_{i-1} - 1)}{(30)(144)} \left[\Delta_i (\sigma_{i-1} + 1) (4\sigma_{i-1}^2 + 5\sigma_{i-1} - 6) - (\sigma_{i-1}^2 - 10\sigma_{i-1} + 6) \right] (h_{i-1} D)^5 \\ & \left. + \dots \right\} y_i'' \end{aligned} \quad (63)$$

By design, when Δ_i is given by Eq. (60), $\mathbb{M}_i$ is a second-order-accurate approximation to y'_i (third-order error) and for uniform spacing ($\sigma_{i-1} = 1$), it is third-order accurate. This is somewhat different from that of Rubin and Khosla, Ref. 1, who reached the same results but with

$$\Delta_i = \frac{\sigma_{i-1}^3 + 1}{\sigma_{i-1}(\sigma_{i-1} + 1)^2} \quad (64)$$

This discrepancy is not caused by error but is explained in the discussion following Eq. (51). Note for uniform spacing ($\sigma_{i-1} = 1$) that Eqs. (60) and (64) both give $\Delta_i = 1/2$. In the limit as σ_{i-1} approaches infinity, both equations give $\Delta_i = 1$; however, as σ_{i-1} approaches zero, Eq. (60) yields $\Delta_i = 0$, whereas Eq. (64) gives $\Delta_i = \text{infinity}$.

In an analogous manner, a higher order approximation, m_i to y'_i , can be obtained. Substituting Eq. (58) into Eq. (47), let

$$m_i = m_i + \frac{\delta_i h_{i-1}}{36} [\sigma_{i-1} M_{i-1} - (\sigma_{i-1} + 1) M_i + M_{i+1}] \quad (65)$$

where

$$\delta_i = \frac{\sigma_{i-1} - 1}{\sigma_{i-1} + 1} \quad (66)$$

Note for uniform spacing ($\sigma_{i-1} = 1$) that $\delta_i = 0$ and $m_i = m_i$. Therefore, m_i is an improvement over m_i only for nonuniform spacing. However, for uniform spacing, m_i was already third-order accurate. In the limit, when $\sigma_{i-1} = 0$, then $\delta_i = -1$, and when $\delta_{i-1} = \text{infinity}$, then $\delta_i = 1$. In operator notation,

$$m_i = m_i + \frac{\delta_i h_{i-1}}{36} [\sigma_{i-1} E_{i-1}^{-1} - (\sigma_{i-1} + 1) + E_i] \mathbb{M}_i \quad (67)$$

Expanding the operator and substituting Eqs. (47) and (51) into Eq. (67) yields

$$\begin{aligned} m_i = & \left\{ 1 + \frac{\sigma_{i-1}}{72} [\delta_i (\sigma_{i-1} + 1) - (\sigma_{i-1} - 1)] (h_{i-1} D)^3 \right. \\ & + \frac{\sigma_{i-1}}{180} \left[5\delta_i (\sigma_{i-1}^2 - 1) - 6 \frac{\sigma_{i-1}^3 + 1}{\sigma_{i-1} + 1} \right] (h_{i-1} D)^4 \\ & - \frac{\sigma_{i-1} (\sigma_{i-1} - 1)}{4,320} [5\delta_i (\sigma_{i-1} + 1) + 2 (\sigma_{i-1}^2 - 5\sigma_{i-1} + 3)] (h_{i-1} D)^5 \\ & + \dots \left. \right\} y'_i \end{aligned} \quad (68)$$

By design, when δ_i is given by Eq. (66), m_i is a third-order-accurate approximation to y'_i (fourth-order error).

The analysis on m_i and M_i is valid only at interior points ($i = 2, 3, \dots, N - 1$). At the end points ($i = 1$ and N) they will be defined as $m_i = m_i$ and $M_i = M_i$.

In the next section an approximation, $K_i(x_i, x_{i+1})$, to the integral of Eq. (55) will be derived as follows:

$$\begin{aligned} K_i(x_i, x_{i+1}) &= \frac{1}{2} h_i (y_i + y_{i+1}) - \frac{1}{10} h_i^2 (m_{i+1} - m_i) \\ &+ \frac{1}{120} h_i^3 (M_i + M_{i+1}) \end{aligned} \quad (69)$$

It is convenient to include the error analysis of the formula at this time. In operator notation

$$\begin{aligned} K_i(x_i, x_{i+1}) &= \frac{1}{2} h_i (E_i + 1) y_i - \frac{1}{10} h_i^2 (E_i - 1) m_i \\ &+ \frac{1}{120} h_i^3 (E_i + 1) M_i \end{aligned} \quad (70)$$

Expanding the operators and substituting Eqs. (68) and (63) into Eq. (70), one can arrive at

$$\begin{aligned} K_i(x_i, x_{i+1}) &= \left\{ 1 + \frac{1}{2} h_i D + \frac{1}{6} (h_i D)^2 + \frac{1}{24} (h_i D)^3 \right. \\ &+ \frac{1}{120} (h_i D)^4 + \left[\frac{1}{720} + \frac{(\sigma_{i-1} - 1)(2\sigma_{i-1} - 1)}{(36)(120)\sigma_{i-1}^2} \right] (h_i D)^5 \\ &+ \left[\frac{1}{4800} + \frac{16\sigma_{i-1}^3 + 5\sigma_{i-1}^2 + 95\sigma_{i-1} + 25}{(360)(120)\sigma_{i-1}^3} \right] (h_i D)^6 \\ &\left. + \dots \right\} h_i y_i \end{aligned} \quad (71)$$

Comparing Eqs. (71) and (55),

$$\begin{aligned}
 K_i(x_i, x_{i+1}) &= \int_{x_i}^{x_{i+1}} y(x) dx \\
 &+ (h_i D)^6 \left\{ \frac{(\sigma_{i-1} - 1)(2\sigma_{i-1} - 1)}{(36)(120)\sigma_{i-1}^2} \right. \\
 &+ \left[\frac{16\sigma_{i-1}^3 + 5\sigma_{i-1}^2 + 95\sigma_{i-1} + 25}{(360)(120)\sigma_{i-1}^3} + \frac{1}{100,800} \right] (h_i D) \\
 &\left. + \dots \right\} D^{-1} y_i
 \end{aligned} \tag{72}$$

Thus Eq. (69) is found to be fifth-order accurate (sixth-order error) and for uniform spacing ($\sigma_{i-1} = 1$), it is sixth-order accurate. If m_i and M_i were exact, then the formula would be sixth-order accurate.

2.4 QUINTIC SPLINES

Once the higher order approximations, m_i and M_i , have been found, a spline, $z(x)$, is needed for which the first and second derivatives at x_i are m_i and M_i , respectively. That is,

$$z(x) = q_i(x) \tag{73}$$

for $x_i \leq x \leq x_{i+1}$, where the $q_i(x)$ are polynomials such that

$$z'(x_i) = m_i \tag{74}$$

and

$$z''(x_i) = M_i \tag{75}$$

for $i = 1, 2, \dots, N$. For a particular $q_i(x)$, this requires the conditions

$$q_i(x_i) = y_i \tag{76}$$

$$q_i'(x_i) = m_i \tag{77}$$

$$q_i''(x_i) = M_i \tag{78}$$

$$q_i(x_{i+1}) = y_{i+1} \quad (79)$$

$$q'_i(x_{i+1}) = m_{i+1} \quad (80)$$

$$q''_i(x_{i+1}) = M_{i+1} \quad (81)$$

Since there are six conditions, $q_i(x)$ is quintic. Now one can define

$$y_{ai} = y_i + \frac{1}{5} h_i m_i \quad (82)$$

$$y_{bi} = y_i + \frac{2}{5} h_i m_i + \frac{1}{20} h_i^2 M_i \quad (83)$$

$$y_{ci} = y_{i+1} - \frac{2}{5} h_i m_{i+1} + \frac{1}{20} h_i^2 M_{i+1} \quad (84)$$

$$y_{di} = y_{i+1} - \frac{1}{5} h_i m_{i+1} \quad (85)$$

then

$$\begin{aligned} q_i(x) = & y_i \theta_i^5 + 5y_{ai} \eta_i \theta_i^4 + 10y_{bi} \eta_i^2 \theta_i^3 \\ & + 10y_{ci} \eta_i^3 \theta_i^2 + 5y_{di} \eta_i^4 \theta_i + y_{i+1} \eta_i \end{aligned} \quad (86)$$

The first derivative is

$$\begin{aligned} q'_i(x) = & \frac{5}{h_i} \left[-y_i \theta_i^4 + y_{ai} (\theta_i - 4\eta_i) \theta_i^3 + 2y_{bi} (2\theta_i - 3\eta_i) \eta_i \theta_i^2 \right. \\ & \left. + 2y_{ci} (3\theta_i - 2\eta_i) \eta_i^2 \theta_i - y_{di} (4\theta_i - \eta_i) \eta_i^3 + y_{i+1} \eta_i^4 \right] \end{aligned} \quad (87)$$

and the second derivative is

$$\begin{aligned} q''_i(x) = & \frac{20}{h_i^2} \left[y_i \theta_i^3 + y_{ai} (3\eta_i - 2\theta_i) \theta_i^2 \right. \\ & + y_{bi} (\theta_i^2 - 6\eta_i \theta_i + 3\eta_i^2) \theta_i \\ & + y_{ci} (3\theta_i^2 - 6\eta_i \theta_i + \eta_i^2) \eta_i \\ & \left. + y_{di} (3\theta_i - 2\eta_i) \eta_i^2 + y_{i+1} \eta_i^3 \right] \end{aligned} \quad (88)$$

It can be verified that Eq. (86) does satisfy the six conditions, (76) thru (81).

To compute the integral of the quintic spline, if one has a formula for

$$K_i(a, b) = \int_a^b q_i(x) dx \quad (89)$$

where $x_i \leq a < b \leq x_{i+1}$, then one can obtain the integral over an arbitrary subinterval of $[x_1, x_N]$ by summing the appropriate $K_i(a,b)$ with the proper arguments. Integrating Eq. (86),

$$\begin{aligned}
 K_i(a, b) = & h_i \left[-\frac{1}{6} y_i \theta_i^6 - y_{a1} \left(\eta_i + \frac{1}{6} \theta_i \right) \theta_i^5 \right. \\
 & - y_{b1} \left(\frac{5}{2} \eta_i^2 + \eta_i \theta_i + \frac{1}{6} \theta_i^2 \right) \theta_i^4 \\
 & + y_{c1} \left(\frac{5}{2} \theta_i^2 + \eta_i \theta_i + \frac{1}{6} \eta_i^2 \right) \eta_i^4 \\
 & \left. + y_{d1} \left(\theta_i + \frac{1}{6} \eta_i \right) \eta_i^5 + \frac{1}{6} y_{i+1} \eta_i^6 \right]_a^b
 \end{aligned} \quad (90)$$

Of particular interest is

$$K_i(x_i, x_{i+1}) = \frac{1}{6} h_i (y_i + y_{a1} + y_{b1} + y_{c1} + y_{d1} + y_{i+1}) \quad (91)$$

If one substitutes Eqs. (82) thru (85) into (91), then one obtains Eq. (69).

3.0 SPLINE COLLOCATION

The general solution to a second-order differential equation involves two arbitrary constants; thus two conditions must be made to determine a particular solution. If the two conditions are the specification of the values of the solution and its derivative at a particular x-value, then the problem is called an initial value problem, and one-step methods such as Runge-Kutta can be applied, with nonlinearity presenting no special problem. When the two conditions are given at two separate x-values, the problem is called a two-point boundary-value problem, and several techniques are applicable, all requiring iteration to handle nonlinearity. Some methods allow the problem to be left in its nonlinear form, for example shooting methods for which each iteration is an initial value problem. However, probably the best methods are those where the problem is linearized and each iteration solves a linear problem. That is the approach used in this report. Section 3.1 describes a spline collocation method to solve linear problems, and Section 3.2 shows how to linearize a problem and iterate to obtain a solution to the nonlinear problem.

A collocation method for solving a differential equation involves writing an approximate solution in terms of a number of parameters, then determining the parameters by requiring the approximate solution to satisfy the differential equation at certain discrete points. The approximate solution and the differential equation collocate

at those points. In the case of spline collocation, the approximate solution is the spline, and the parameters are the y_i and M_i . Since there are $2N$ unknowns, the Y_i and M_i , $2N$ equations are needed. Because the unknowns are spline parameters, N spline relations are in force. The other N equations come from the collocation requirements that the spline satisfy the differential equation at the x_i .

In the following analysis, quintic splines are assumed throughout; however, the equations for cubic splines are found by setting $\Delta_i = \delta_i = 0$. In Section 2, since the spline passed thru the points, z_i equaled y_i [$z_i = z(x_i)$]. In this section, the requirement is for the spline to satisfy the differential equation at the x_i ; therefore, z_i does not necessarily equal y_i . For this reason, when reference is made to equations in Section 2, it will be assumed that the y_i are replaced by z_i without further explanation.

3.1 THE LINEAR PROBLEM

Given the differential equation

$$y'' = \alpha(x) + \beta(x)y + \gamma(x)y' + \epsilon(x)I \quad (92)$$

where

$$I = I_1 + \int_{x_1}^x y(x') dx' \quad (93)$$

where I_1 is a given constant; and given end conditions, Eq. (24), it is desired to find a numerical solution for $x_1 \leq x \leq x_N$ by spline collocation.

Equation (92) should be classed as a third-order differential equation in I , since $y = I'$, $y' = I''$, and $y'' = I'''$. Originally, the method to be presented was developed for second-order equations; later, it was extended to solve third-order equations by the above device. Spline collocation determines the spline, $z(x)$, which satisfies the differential equation, Eq. (92), at the x_i . Therefore, for $i = 1, 2, \dots, N$,

$$M_i = \alpha_i + \beta_i z_i + \gamma_i m_i + \epsilon_i g_i \quad (94)$$

where the subscript, i , on the functions α , β , γ , and ϵ indicates their evaluation at x_i . The g_i are defined by

$$g_1 = I_1 \quad (95)$$

and for $i > 1$,

$$g_i = g_{i-1} + K_{i-1}(x_{i-1}, x_i) \quad (96)$$

which by Eq. (69) becomes

$$\begin{aligned}
 g_i &= g_{i-1} + \frac{1}{2} h_{i-1} (Z_i - Z_{i-1}) \\
 &- \frac{1}{10} h_{i-1}^2 (m_i - m_{i-1}) \\
 &+ \frac{1}{120} h_{i-1}^3 (\mathfrak{M}_i + \mathfrak{M}_{i-1})
 \end{aligned} \tag{97}$$

for $1 < i \leq N$. If one defines

$$Q_i = \sigma_{i-1} M_{i-1} - (\sigma_{i-1} + 1) M_i + M_{i+1} \tag{98}$$

then Eqs. (65) and (59) can be written

$$m_i = \begin{cases} m_1, & \text{for } i = 1 \text{ and } i = N \\ m_1 + \frac{\delta_1 h_{i-1}}{36} Q_i, & \text{for } 1 < i < N \end{cases} \tag{99}$$

$$\mathfrak{M}_i = \begin{cases} M_1, & \text{for } i = 1 \text{ and } i = N \\ M_1 - \frac{\Delta_1}{6} Q_i, & \text{for } 1 < i < N \end{cases} \tag{100}$$

Ordinarily one would choose the z_i and m_i as the principal unknowns using Eqs. (23), (25), and (26). However, in light of Eq. (98) and its exclusive involvement of the M_i , the analysis can be simplified by choosing the z_i and M_i as the principal unknowns and using Eqs. (36), (35), and (37), which can be written

$$\begin{aligned}
 &\left(A_1 - \frac{B_1}{h_1}\right) z_1 - \frac{B_1}{h_1} z_2 + (C_1 \\
 &- \frac{1}{3} B_1 h_1) M_1 - \frac{1}{6} B_1 h_1 M_2 = D_1 \\
 &\frac{1}{h_{i-1}} z_{i-1} - \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) z_i + \frac{1}{h_i} z_{i+1} \\
 &- \frac{1}{6} h_{i-1} M_{i-1} - \frac{1}{3} (h_{i-1} + h_i) M_i - \frac{1}{6} h_i M_{i+1} = 0 \\
 &\quad \text{for } 1 < i < N \\
 &\left(A_N + \frac{B_N}{h_{N-1}}\right) z_N - \frac{B_N}{h_{N-1}} z_{N-1} + (C_N \\
 &+ \frac{1}{3} B_N h_{N-1}) M_N + \frac{1}{6} B_N h_{N-1} M_{N-1} = D_N
 \end{aligned} \tag{101}$$

The m_i values come from Eqs. (32), (34), and (33), which can be written

$$m_i = \begin{cases} \frac{1}{h_1} (z_2 - z_1) - \frac{h_1}{6} (2M_1 + M_2) & \text{for } i = 1 \\ -\frac{1}{2h_{i-1}} z_{i-1} + \frac{1}{2} \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) z_i + \frac{1}{2h_i} z_{i+1} \\ + \frac{h_{i-1}}{12} M_{i-1} + \frac{1}{6} (h_{i-1} - h_i) M_i - \frac{h_i}{12} M_{i+1} & \text{for } 1 < i < N \\ \frac{1}{h_{N-1}} (z_N - z_{N-1}) + \frac{h_{N-1}}{6} (2M_N + M_{N-1}) & \text{for } i = N \end{cases} \quad (102)$$

From Eqs. (94) thru (102) one could form a tridiagonal system with 7 by 7 blocks with unknowns z_i , M_i , m_i , Q_i , σ_i , $\mathcal{M}_i$, and g_i . However, for the sake of numerical efficiency, it is best to eliminate some of the unknowns before defining the system. The m_i and $\mathcal{M}_i$ can be eliminated by substitution of Eqs. (99) and (100) into Eqs. (94) and (97). Equation (94) becomes

$$\begin{aligned} \beta_i z_i - M_i - \gamma_i m_i - \epsilon_i g_i &= -a_i \\ &\text{for } i = 1 \text{ and } i = N \\ \beta_i z_i - M_i + \gamma_i m_i + \epsilon_i g_i + \frac{1}{36} (\gamma_i \delta_i h_{i-1} - 6\Delta_i) Q_i &= -a_i \\ &\text{for } 1 < i < N \end{aligned} \quad (103)$$

If one defines

$$a_{1i} = \frac{h_{i-1}^3}{720} (\Delta_i - 2\delta_i) \quad (104)$$

and

$$a_{2i} = \frac{h_{i-1}^3}{720} \left(\Delta_i + \frac{2\delta_{i-1}}{\sigma_{i-2}} \right) \quad (105)$$

then, after m_i and M_i are eliminated, the g_i can be written

$$g_i = \begin{cases} I_1 & \text{for } i = 1 \\ g_1 + \frac{1}{2} h_1 (z_2 + z_1) - \frac{1}{10} h_1^2 (m_2 - m_1) \\ + \frac{1}{120} h_1^3 (M_2 + M_1) + a_{12} Q_2 & \text{for } i = 2 \\ g_{i-1} + \frac{1}{2} h_{i-1} (z_i + z_{i-1}) - \frac{1}{10} h_{i-1}^2 (m_i - m_{i-1}) \\ + \frac{1}{120} h_{i-1}^3 (M_i + M_{i-1}) + a_{1i} Q_i + a_{2i} Q_{i-1} & \text{for } 2 < i < N \\ g_{N-1} + \frac{1}{2} h_{N-1} (z_N + z_{N-1}) - \frac{1}{10} h_{N-1}^2 (m_N - m_{N-1}) \\ + \frac{1}{120} h_{N-1}^3 (M_N + M_{N-1}) + a_{2N} Q_{N-1} & \text{for } i = N \end{cases} \quad (106)$$

The m_i and Q_i can be eliminated from the system by substitution of Eqs. (102) and (98) into Eqs. (103) and (106). The results, after substitution, can be called Eqs. (103)' and (106)', which will not be written out explicitly. The system then consists of Eqs. (106)', (103)', and (101), which can be written in the form

$$\begin{aligned} D_{01} V_1 + D_{11} V_2 &= R_1 \\ D_{-12} V_1 - D_{02} V_2 + D_{12} V_3 &= R_2 \\ D_{-2i} V_{i-2} + D_{-1i} V_{i-1} + D_{0i} V_i + D_{1i} V_{i+1} &= R_i \quad \text{for } 2 < i < N \\ D_{-2N} V_{N-2} + D_{-1N} V_{N-1} + D_{0N} V_N &= R_N \end{aligned} \quad (107)$$

where

$$V_i = \begin{bmatrix} g_i \\ M_i \\ z_i \end{bmatrix} \quad (108)$$

$$R_i = \begin{bmatrix} -I_1 \\ -a_i \\ D_i \end{bmatrix} \quad \begin{bmatrix} 0 \\ -a_i \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -a_N \\ D_N \end{bmatrix} \quad \begin{matrix} \text{for } i=1 & \text{for } 1 < i < N & \text{for } i=N \end{matrix} \quad (109)$$

and

$$D_{mi} = (d_{mijk}) \quad (110)$$

where all the d_{mijk} are zero, except for ones which will be listed below. A brief clarification of the subscripts may be useful. The subscript m takes values -2, -1, 0, and 1. The value 0 refers to the principal diagonal; -1 to the first lower diagonal; -2 to the second lower; and 1 to the upper diagonal. The subscript i indicates the i th point with $1 \leq i \leq N$. The subscripts j and k take values 1, 2, and 3. The j subscript refers to Eqs. (106)', (103)', and (101), respectively. The k subscript refers to g_i , M_i , and Z_i , respectively.

The nonzero d_{mijk} are as follows:

$$d_{-2i12} = \frac{h_{i-2} h_{i-1}^2}{120} + a_{2i} \sigma_{i-2} \quad \text{for } 2 < i \leq N \quad (111)$$

$$d_{-2i13} = -\frac{\sigma_{i-2} h_{i-1}}{20} \quad \text{for } 2 < i \leq N \quad (112)$$

$$d_{-1i11} = 1 \quad \text{for } 1 < i \leq N \quad (113)$$

$$d_{-1i12} = \begin{cases} -\frac{h_1^3}{30} + a_{12} \sigma_1 & \text{for } i = 2 \\ \frac{h_{i-1}^3}{60} \left(\frac{1}{\sigma_{i-2}} - 1 \right) + a_{1i} \sigma_{i-1} - a_{2i} (\sigma_{i-2} + 1) & \text{for } 2 < i < N \\ \frac{h_{N-1}^3}{120} \left(\frac{2}{\sigma_{N-2}} - 3 \right) - a_{2N} (\sigma_{N-2} + 1) & \text{for } i = N \end{cases} \quad (114)$$

$$d_{-1i13} = \begin{cases} \frac{9h_1}{20} & \text{for } i = 2 \\ \frac{h_{i-1}}{20} (\sigma_{i-2} + 10) & \text{for } 2 < i < N \\ \frac{h_{N-1}}{20} (\sigma_{N-2} + 11) & \text{for } i = N \end{cases} \quad (115)$$

$$d_{0i11} = -1 \quad \text{for } 1 \leq i \leq N \quad (116)$$

$$d_{0i12} = \begin{cases} \frac{h_1^3}{120} (2\sigma_1 - 3) - a_{12} (\sigma_1 + 1) & \text{for } i = 2 \\ \frac{h_{i-1}^3}{60} (\sigma_{i-1} - 1) - a_{1i} (\sigma_{i-1} + 1) + a_{2i} & \text{for } 2 < i < N \\ -\frac{h_{N-1}^3}{30} + a_{2N} & \text{for } i = N \end{cases} \quad (117)$$

$$d_{0i13} = \begin{cases} \frac{h_1}{20} \left(\frac{1}{\sigma_1} + 11 \right) & \text{for } i = 2 \\ \frac{h_{i-1}}{20} \left(\frac{1}{\sigma_{i-1}} + 10 \right) & \text{for } 2 < i < N \\ \frac{9h_{N-1}}{20} & \text{for } i = N \end{cases} \quad (118)$$

$$d_{1i12} = \frac{h_{i-1}^2 h_i}{120} + a_{1i} \quad \text{for } 1 < i < N \quad (119)$$

$$d_{1i13} = -\frac{h_{i-1}}{20\sigma_{i-1}} \quad \text{for } 1 < i < N \quad (120)$$

$$d_{-1i22} = \begin{cases} \frac{\gamma_i h_{i-1}}{12} + \frac{\sigma_{i-1}}{36} (\gamma_i \delta_i h_{i-1} - 6\Delta_i) & \text{for } 1 < i < N \\ \frac{\gamma_N h_{N-1}}{6} & \text{for } i = N \end{cases} \quad (121)$$

$$d_{-1i23} = \begin{cases} -\frac{\gamma_i}{2h_{i-1}} & \text{for } 1 < i < N \\ -\frac{\gamma_N}{h_{N-1}} & \text{for } i = N \end{cases} \quad (122)$$

$$d_{0i21} = \epsilon_i \quad \text{for } 1 \leq i \leq N \quad (123)$$

$$d_{0i22} = \begin{cases} -1 - \frac{\gamma_1 h_1}{3} & \text{for } i = 1 \\ -1 + \frac{\gamma_1}{6} (h_{i-1} - h_i) - \frac{1}{36} (\gamma_i \delta_i h_{i-1} - 6\Delta_i) (\sigma_{i-1} + 1) & \text{for } 1 < i < N \\ -1 + \frac{\gamma_N h_{N-1}}{3} & \text{for } i = N \end{cases} \quad (124)$$

$$d_{0i23} = \begin{cases} \beta_1 - \frac{\gamma_1}{h_1} & \text{for } i = 1 \\ \beta_i + \frac{\gamma_1}{2} \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) & \text{for } 1 < i < N \\ \beta_N + \frac{\gamma_N}{h_{N-1}} & \text{for } i = N \end{cases} \quad (125)$$

$$d_{1i22} = \begin{cases} -\frac{\gamma_1 h_1}{6} & \text{for } i = 1 \\ -\frac{\gamma_i h_i}{12} + \frac{1}{36} (\gamma_i \delta_i h_{i-1} - 6\Delta_i) & \text{for } 1 < i < N \end{cases} \quad (126)$$

$$d_{1i23} = \begin{cases} \frac{\gamma_1}{h_1} & \text{for } i = 1 \\ \frac{\gamma_i}{2h_i} & \text{for } 1 < i < N \end{cases} \quad (127)$$

$$d_{-1i32} = \begin{cases} -\frac{h_{i-1}}{6} & \text{for } 1 < i < N \\ \frac{B_N h_{N-1}}{6} & \text{for } i = N \end{cases} \quad (128)$$

$$d_{-1i33} = \begin{cases} \frac{1}{h_{i-1}} & \text{for } 1 < i < N \\ -\frac{B_N}{h_{N-1}} & \text{for } i = N \end{cases} \quad (129)$$

$$d_{0i32} = \begin{cases} C_1 - \frac{B_1 h_1}{3} & \text{for } i = 1 \\ -\frac{1}{3} (h_{i-1} + h_i) & \text{for } 1 < i < N \\ C_N + \frac{B_N h_{N-1}}{3} & \text{for } i = N \end{cases} \quad (130)$$

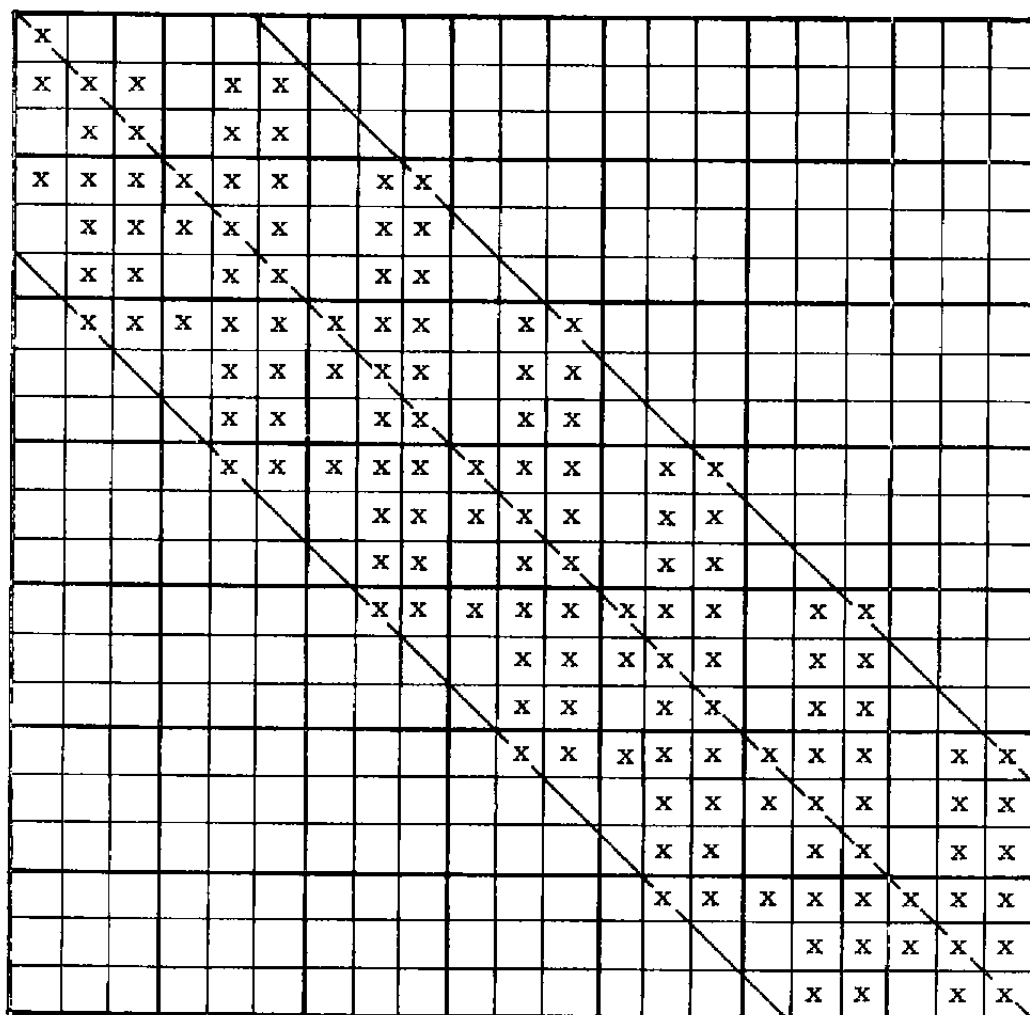
$$d_{0i33} = \begin{cases} A_1 - \frac{B_1}{h_1} & \text{for } i = 1 \\ -\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) & \text{for } 1 < i < N \\ A_N - \frac{B_N}{h_{N-1}} & \text{for } i = N \end{cases} \quad (131)$$

$$d_{1i32} = \begin{cases} -\frac{B_1 h_1}{6} & \text{for } i = 1 \\ -\frac{h_i}{6} & \text{for } 1 < i < N \end{cases} \quad (132)$$

$$d_{1i33} = \begin{cases} \frac{B_1}{h_1} & \text{for } i = 1 \\ \frac{1}{h_i} & \text{for } 1 < i < N \end{cases} \quad (133)$$

Figure 2 shows the coefficient matrix for $N = 7$. Each small square corresponds to a d_{mijk} , and each large square corresponds to a D_{mi} . The X's mark the nonzero elements. The most straightforward way to solve the system is to use a banded system solver. The matrix is seen to have five diagonals above and below the principal diagonal for a bandwidth of 11.

The matrix can be treated as four diagonals of 3 by 3 blocks. The upper diagonal can be eliminated and the solution obtained by forward substitution. Some advantage can be taken of the sparseness of D_{2i} (two nonzero elements). One must be careful in programming such a procedure if the procedure is intended to improve on the banded solver.

Figure 2. Coefficient matrix for $N = 7$.

When $\epsilon(x) \equiv 0$, then Eqs. (103) and (101) become uncoupled from Eq. (106) and a tridiagonal of 2 by 2 blocks can be solved for the M_i and z_i . The integral can then be computed from Eq. (69) if it is needed.

3.2 THE NONLINEAR PROBLEM

Most problems are nonlinear. Instead of Eq. (92) one may have

$$y'' = f(x, y, y', I) \quad (134)$$

Methods of solution of such problems often involve a process called quasi-linearization. The process requires an "in-hand" solution, which to begin with may be just a guess. Indicating the in-hand solution by a bar,

$$y'' \approx f(x, \bar{y}, \bar{y}', \bar{I}) + f_2(x, \bar{y}, \bar{y}', \bar{I}) (y - \bar{y}) \\ + f_3(x, \bar{y}, \bar{y}', \bar{I}) (y' - \bar{y}') - f_4(x, \bar{y}, \bar{y}', \bar{I}) (I - \bar{I}) \quad (135)$$

where f_j indicates the partial derivative of f with respect to the j th argument. Equation (135) can be put in the form of Eq. (92) where

$$\alpha(x) = f(x, \bar{y}, \bar{y}', \bar{I}) - f_2(x, \bar{y}, \bar{y}', \bar{I}) \bar{y} \\ - f_3(x, \bar{y}, \bar{y}', \bar{I}) \bar{y}' - f_4(x, \bar{y}, \bar{y}', \bar{I}) \bar{I} \quad (136)$$

$$\beta(x) = f_2(x, \bar{y}, \bar{y}', \bar{I}) \quad (137)$$

$$\gamma(x) = f_3(x, \bar{y}, \bar{y}', \bar{I}) \quad (138)$$

and

$$\epsilon(x) = f_4(x, \bar{y}, \bar{y}', \bar{I}) \quad (139)$$

With α , β , γ , and ϵ defined by Eqs. (136) thru (139), a solution can be obtained by the method of Section 3.1. This solution becomes the new in-hand solution. Iteration continues until a converged solution is obtained or until it becomes apparent that the method (for a particular case) fails to converge.

4.0 THE COMPUTER PROGRAM

The computer program was written in FORTRAN for the IBM 370/165 computer. To apply the program to a particular problem, the user must provide the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $\epsilon(x)$ of Eq. (92). This is done via a subroutine, QLDE. Only the text of the subroutine need be changed, the interface having already been programmed. The user must also provide the end condition constants of Eq. (24); the constant, I_1 , of Eq. (93); a starting solution; and various logic control variables. These inputs are made in subroutine DATA. The different parameters are described by comments, and the user sets their values as desired. One other routine, CHEKBM, is problem dependent; however, it is not essential to the solution. CHEKBM is evoked after a solution has been found and the user can check or process the solution as desired. The use procedure of the computer program is demonstrated in Section 5 by example problems.

A listing of the program is given in Appendix A. A conscientious effort was made to make the program clean and modular, and to use comments liberally. Below are notes on some of the subroutines. If a subroutine is not included below, then it is adequately documented by comments. Line numbers refer to sequence numbers in columns 73 thru 80.

PROCED

The general logic of the program is contained in this routine. The DO loop of Line 340 provides the option of solving multiple problems. Exit from this loop is effected by a RETURN 2 from DATA (Line 350). Iteration is performed by the DO loop of Line 380. A converged solution (and branch-out of the loop) is indicated by a RETURN 1 from UPDATE (Line 430). The user is given the opportunity to check or process the solution as desired in CHEKBM, Line 530. The solution will be written on tape or disk, Line 540, or printed, Line 550, if the proper indicators (JUNIT and LPRNT) are set in DATA.

CHEKBM

The user can check or process the solution as desired in this routine. If no such need exists, then Lines 1770 thru 1960 can be deleted or, alternately, Line 530 of PROCED can be deleted.

CHEKDE

If the solution at the collocation points is the input arguments to this routine, then in effect the routine merely checks the validity of the solution from BANDED. This

routine can be used to see how well the solution satisfies the differential equation at points other than the collocation points, as follows:

1. Call QUINTS to determine the solution at the points of interest.
2. Call NTEGRL to compute the integral at the points.
3. Call QLDE to compute the α_i , β_i , γ_i , and ϵ_i .
4. Call CHEKDE and it checks the solution with the differential equation.

DATA

The logic control variables, end condition constants, and starting solution are input via this routine. Note the availability to UNIFORM to compute uniform spacing and GUESS to compute a quintic solution which satisfies the end conditions.

DELTAS

This routine evaluates Eqs. (60) and (66) for Δ_i and δ_i and has been made a separate routine with the idea that it might be useful to experiment with different values. It was pointed out that the expressions for Δ_i and δ_i are not unique, thus it would be interesting to determine their influence. Also, setting $\Delta_i = \delta_i = 0$ reduces the higher order quintic spline method to a cubic, so that the refinement of the method can be evaluated.

DIFRNC AND ERROR

The user should be aware of the availability of these routines which have been found useful in the various checking routines.

GUESS

GUESS is available to the user to compute a starting solution. It determines a quintic solution which is consistent with the end conditions. The method used by GUESS is presented in Appendix B.

NORM

NORM is used to check for convergence. The user has a choice of two tests indicated by the variable LNORM set in DATA. The first test finds the greatest difference between the two solutions relative to the range of the first solution. The second test finds the greatest difference between the two solutions relative to the solutions' local values. The second test is more stringent, especially for solutions which approach an asymptote.

RESPAC

RESPAC computes a new spacing with smaller steps where the solution changes more rapidly. The method used is presented in Appendix C. If no respacing is desired, then the parameter RSC can be set to a large value in DATA.

RESULT

The solutions returned by BANDED are the g_i , M_i , and z_i [See Eq. (108)]. RESULT computes m_i (in the temporary variable, T1) from Eq. (102) and then m_1 and M_1 from Eqs. (99) and (100).

SYSTEM

SYSTEM computes the d_{mijk} from Eqs. (111) thru (133).

PAGER

The user should replace Line 7560 with his own identification.

TIMCHK AND GETNOW

These routines are cosmetic rather than essential. Dummy routines are supplied in order to present a complete working program.

The user can replace Line 7800 with his own job identification or ideally would automate the routine to supply the job number, time, and date, as designed, via the mechanisms of his computer configuration.

5.0 EXAMPLE PROBLEMS

The computer program as listed in Appendix A solves the first example problem (Section 5.1). The program as listed will be called the reference program. Since most of the input must be programmed, for example the functions, $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $\epsilon(x)$, different problems are run by modification of the reference program.

The modifications to run the example problems are listed in Appendix D. They are presented in the format of an in-house update program which has three command statements:

```
-DEL  L1  [ , L2]
-INS  L1
-REP  L1  [ , L2]
```

The L1 and L2 are sequence numbers (i.e., the numbers in columns 73 thru 80 of the reference program). The brackets indicate that the inclusion of L2 is optional. The command -DEL deletes lines L1 thru L2. If L2 is omitted, then only line L1 is deleted. The command -INS inserts lines after line L1. The insertion lines follow the command statement and continue to the next command or else to the end of the data. The command -REP replaces lines L1 thru L2. If L2 is omitted, then only line L1 is replaced. The replacement lines follow the command statement and continue to the next command or to the end of the data.

At AEDC the reference program is kept on a permanent file. The operating procedure to run a particular problem is to create the modified program on a temporary file via the update program, then to compile and execute the modified program.

The printed output of the example problems is given in Appendix E.

5.1 EXAMPLE 1

The first example comes from Ref. 5, as follows:

$$y'' + y^3 y' - y y' \sqrt{4y' + y^4} = 0 \quad (140)$$

with end conditions

$$(x_1, y_1) = (0, 0)$$

$$(x_N, y_N) = (\tan^{-1} 7, 7)$$

It can be verified that the analytic solution is

$$y = \tan x \quad (141)$$

By Eq. (24), the end conditions are specified by

$$A_1 = 1, B_1 = 0, C_1 = 0, D_1 = 0$$

$$A_N = 1, B_N = 0, C_N = 0, D_N = 7$$

Equations (136) thru (139) yield

$$\alpha(x) = 3 y y' \left(y^2 - \frac{2y' + y^4}{\sqrt{4y' + y^4}} \right) \quad (142)$$

$$\beta(x) = y' \left(\frac{4y' - 3y^4}{\sqrt{4y' + y^4}} - 3y^2 \right) \quad (143)$$

$$\gamma(x) = y \left(\frac{6y' + y^4}{\sqrt{4y' + y^4}} - y^2 \right) \quad (144)$$

$$\epsilon(x) = 0 \quad (145)$$

The computer program as listed in Appendix A solves the first example problem. The program as listed will be referred to as the reference program.

5.2 EXAMPLE 2: THE FALKNER-SKAN EQUATION

A well-known equation in boundary-layer similarity flows, Ref. 6, is the Falkner-Skan equation,

$$y'' - Iy' + k(1 - y^2) = 0 \quad (146)$$

where k is a parameter. Equation (87) defines I , where

$$I_1 = 0$$

The end conditions are

$$(x_1, y_1) = (0, 0)$$

$$(x_N, y_N) = (\infty, 1)$$

For numerical computation, the ∞ must be replaced by an appropriate finite number.

By Eq. (24), the end conditions are specified by

$$A_1 = 1, B_1 = 0, C_1 = 0, D_1 = 0$$

$$A_N = 1, B_N = 0, C_N = 0, D_N = 1$$

Equations (136) thru (139) yield

$$\alpha(x) = Iy' - k(1 + y^2) \quad (147)$$

$$\beta(x) = 2ky \quad (148)$$

$$\gamma(x) = -I \quad (149)$$

$$\epsilon(x) = -y' \quad (150)$$

It took three attempts at solving this problem to obtain satisfactory results in all cases. The goal was to recompute a table in Ref. 6 which is printed in the first page of printout for this example (Appendix E). One could expect difficulty for $k < 0$, since multiple solutions exist, but only the one solution presented in the table was of interest. The first attempt was with $x_N = 6$. Good solutions were obtained for $k \geq 0$, but for $k < 0$, the iteration converged to a wrong solution. The second attempt was to choose (using the table) a more appropriate x_N for each k . This improved the accuracy of the solution for $k = 10$, but otherwise it had little effect. The first two attempts used GUESS (Section 4) to obtain the initial solution of the iteration, which produced a straight line. The third attempt was to use as the initial solution

$$y = 1 - e^{-3x} \quad (151)$$

With this as the starting solution, the iteration converged to the correct solution in each case.

The solutions were not printed (LPRNT = 0), but were written on disk (JUNIT = 20). A second program read the file on disk and plotted the solutions and the first derivatives of the solutions. These plots are shown in Figs. 3 and 4 and can be compared with similar plots in Ref. 6.

5.3 EXAMPLE 3: VISCOELASTIC FLUID

The third example comes from Ref. 7 and describes the steady flow of a viscoelastic fluid parallel to an infinite plane surface with uniform suction

$$k y'' + y' + y = 0 \quad (152)$$

with end conditions

$$x_1 = 0 \quad x_N = \infty$$

$$I_1 = 0 \quad I_N = 1$$

$$y_I = \lambda$$

where

$$\lambda = \frac{1 - \sqrt{1 - 4k}}{2k} \quad (153)$$

It can be verified that the analytic solution is

$$I = 1 - e^{-\lambda x} \quad (154)$$

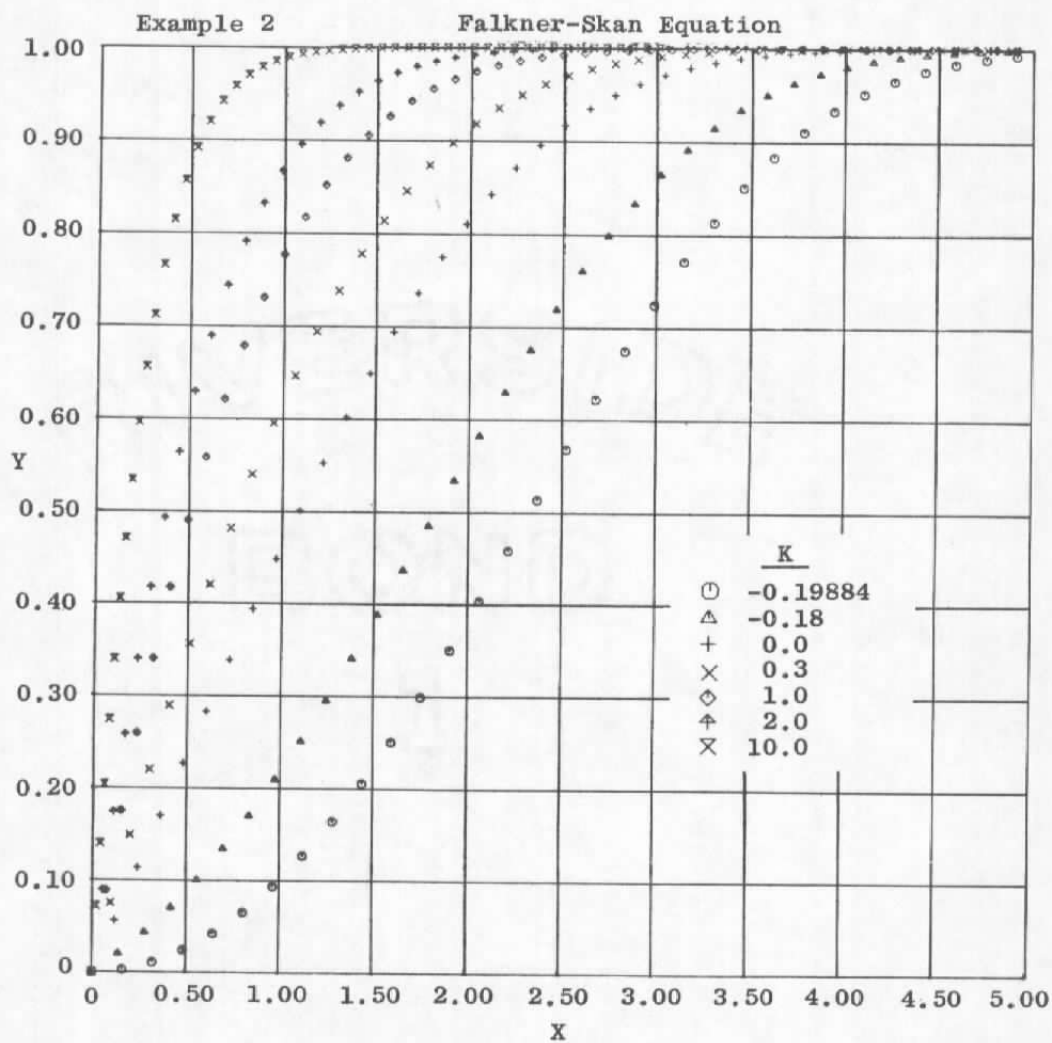


Figure 3. Solutions to the Falkner-Skan equation.

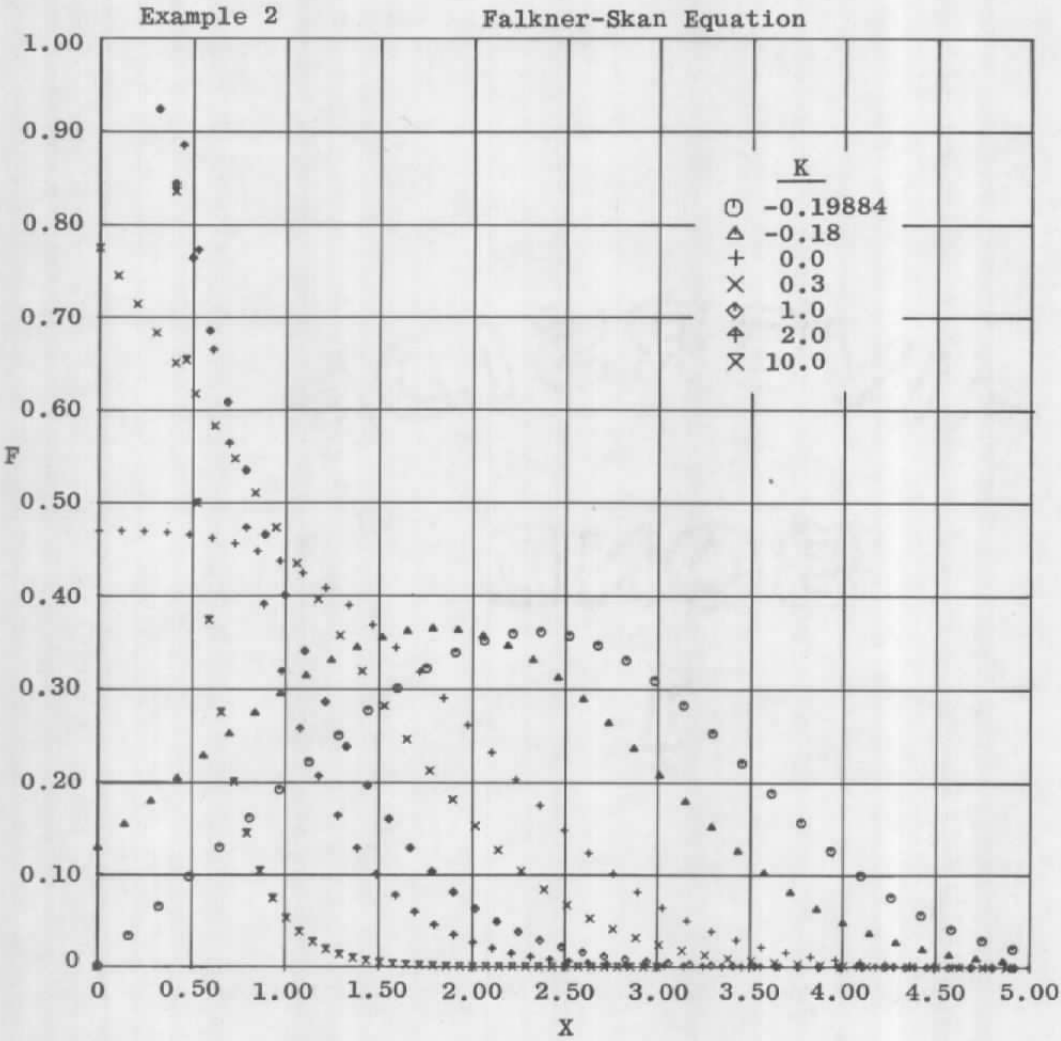


Figure 4. First derivatives of solutions to the Falkner-Skan equation.

The end condition, Eq. (24) with $i = 1$, is specified by

$$A_1 = 1, B_1 = 0, C_1 = 0, D_1 = \lambda$$

For numerical purposes, x_N must be set to an appropriate finite number. Also, to be noted, the end condition at $x = x_N$ is not of the assumed form, Eq. (24) with $i = N$. This deviation requires a modification in SYSTEM, replacement of Lines 6750 thru 6780 with

$$D(3,N,4) = 1.D0$$

and replacement of Line 6810 with

$$R(3,N) = 1.D0$$

i.e., replacement of Eq. (24) at $i = N$ with

$$I_N = 1$$

Equation (152) is linear, and from Eq. (92),

$$\alpha = 0 \quad (155)$$

$$\beta = -\frac{1}{k} \quad (156)$$

$$\gamma = -\frac{1}{k} \quad (157)$$

$$\epsilon = 0 \quad (158)$$

Since Eq. (152) is linear, iteration is unnecessary; however, the accuracy is improved by respacing. Solutions were computed for $k = 0.001, 0.01$, and 0.1 . Accuracy improved for larger k values. This was expected since the equation becomes stiff for small k values (Ref. 7). The stiffness is further evidenced by the oscillation of signs in the numerical solution for y'' for the smaller k and larger x values.

5.4 EXAMPLE 4: CHEMICAL DISPERSION

The fourth example,

$$k y'' = y' + \frac{1.25 y}{1 + 0.1 y} \quad (159)$$

with end conditions

$$x_1 = 0 \quad y_1 - k y'_1 = 1$$

$$x_N = 1 \quad y'_N = 0$$

comes from Ref. 7 and is a special case of a chemical dispersion equation. Checkpoints for the numerical solution (taken from Ref. 7) are given below.

x	y(x)		
	k = 1	k = 0.1	k = 0.001
0	0.620042	0.905407	0.999068
0.5	0.473143	0.532646	0.599183
1.0	0.418415	0.339965	0.307476

Using Eq. (24), the end conditions are specified by

$$A_1 = 1, B_1 = -k, C_1 = 0, D_1 = 1$$

$$A_N = 0, B_N = 1, C_N = 0, D_N = 0$$

Equations (136) thru (139) yield

$$\alpha(x) = \frac{0.125}{k} \left(\frac{y}{1 + 0.1y} \right)^2 \quad (160)$$

$$\beta(x) = \frac{1}{k} \frac{1.25}{(1 + 0.1y)^2} \quad (161)$$

$$\gamma(x) = \frac{1}{k} \quad (162)$$

$$\epsilon(x) = 0 \quad (163)$$

No difficulties were encountered in obtaining the checkpoint values. A plot of the solutions is shown in Fig. 5.

5.5 EXAMPLE 5: AN INHERENTLY UNSTABLE PROBLEM

The fifth example,

$$y'' = x^2 y + \frac{1}{1+x} \quad (164)$$

with end conditions

$$(x_1, y_1) = (1, 1)$$

$$(x_N, y_N) = (\infty, 0)$$

comes from Ref. 7, where it is called "an inherently unstable problem."

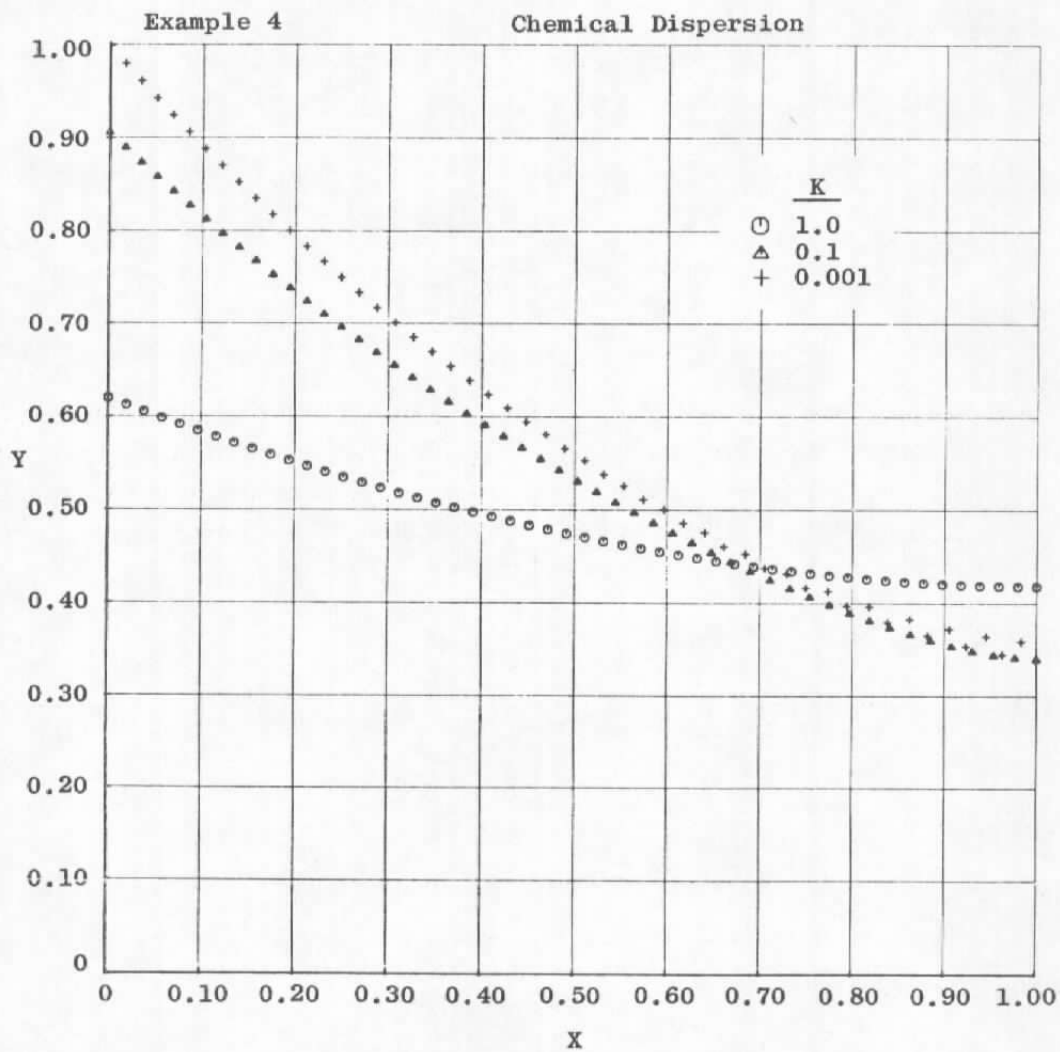


Figure 5. Solutions to the chemical dispersion equation.

By Eq. (24) the end conditions are specified by

$$A_1 = 1, B_1 = 0, C_1 = 0, D_1 = 1$$

$$A_N = 1, B_N = 0, C_N = 0, D_N = 0$$

Equation (164) is linear, and from Eq. (92),

$$\alpha(x) = \frac{1}{1+x} \quad (165)$$

$$\beta(x) = x^2 \quad (166)$$

$$\gamma = 0 \quad (167)$$

and

$$\epsilon = 0 \quad (168)$$

On the first page of printout for this example (Appendix E), the column labeled YA is a tabular solution presented in Ref. 7. The column labeled YB is the computed solution, and the column labeled DIFF is the difference between the two. The accuracy is as good as the accuracy claimed for the table in Ref. 7. A plot of the solution is shown in Fig. 6.

5.6 EXAMPLE 6

The sixth example comes from Ref. 8:

$$y'' = k^2 y + (k^2 + 4\pi^2) \cos^2 \pi x - 2\pi^2 \quad (169)$$

with end conditions

$$(x_1, y_1) = (0, 0)$$

$$(x_N, y_N) = (1, 0)$$

The analytic solution is

$$y = \frac{e^{k(x-1)} + e^{-kx}}{1 + e^{-k}} - \cos^2 \pi x \quad (170)$$

By Eq. (24) the end conditions are specified by

$$A_1 = 1, B_1 = 0, C_1 = 0, D_1 = 0$$

$$A_N = 1, B_N = 0, C_N = 0, D_N = 0$$

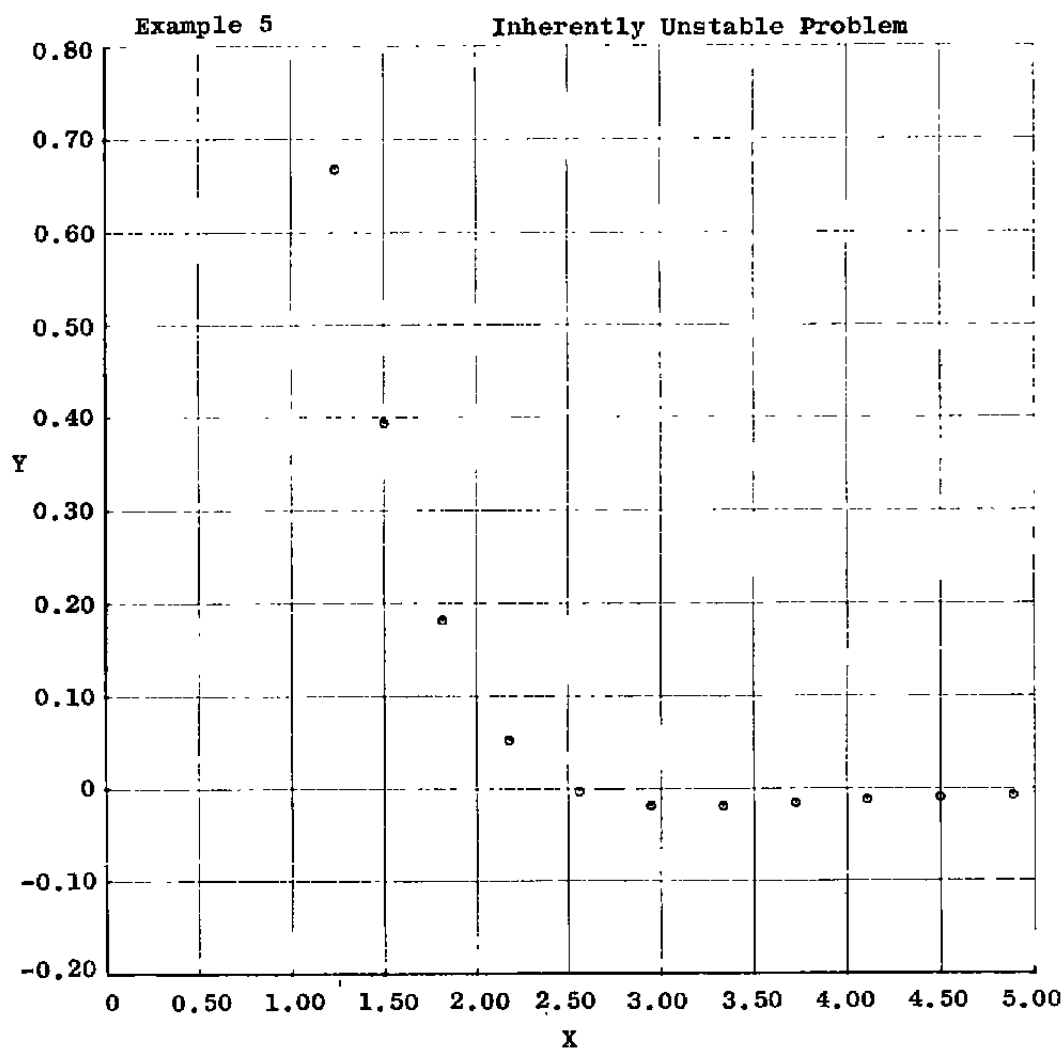


Figure 6. Solution to the inherently unstable problem.

Equation (169) is linear, and by Eq. (92),

$$\alpha(x) = (k^2 + 4\pi^2) \cos^2 \pi x - 2\pi^2 \quad (171)$$

$$\beta = k^2 \quad (172)$$

$$\gamma = 0 \quad (173)$$

$$\epsilon = 0 \quad (174)$$

A plot of solutions to Eq. (169) is shown in Fig. 7.

5.7 EXAMPLE 7: $y'' = e^y$

The seventh example comes from Ref. 8.

$$y'' = e^y \quad (175)$$

with end conditions

$$(x_1, y_1) = (0, 0)$$

$$(x_N, y_N) = (1, 0)$$

The analytic solution is

$$y = \ln \left\{ \lambda \sec^2 \left[\sqrt{\frac{\lambda}{2}} \left(x - \frac{1}{2} \right) \right] \right\} \quad (176)$$

where λ is the root of

$$\lambda = \cos^2 \left(\frac{1}{2} \sqrt{\frac{\lambda}{2}} \right) \quad (177)$$

the numerical value of which can be found in the first page of printout for this example (Appendix E).

Using Eq. (24), the end conditions are specified by

$$A_1 = 1, B_1 = 0, C_1 = 0, D_1 = 0$$

$$A_N = 1, B_N = 0, C_N = 0, D_N = 0$$

Equations (136) thru (139) yield

$$\alpha(x) = (1 - y) e^y \quad (178)$$

$$\beta(x) = e^y \quad (179)$$

$$\gamma = 0 \quad (180)$$

$$\epsilon = 0 \quad (181)$$

Of note for this problem was how quickly and accurately the iteration converged to the solution. A plot of the solution is given in Fig. 8.

Example 6

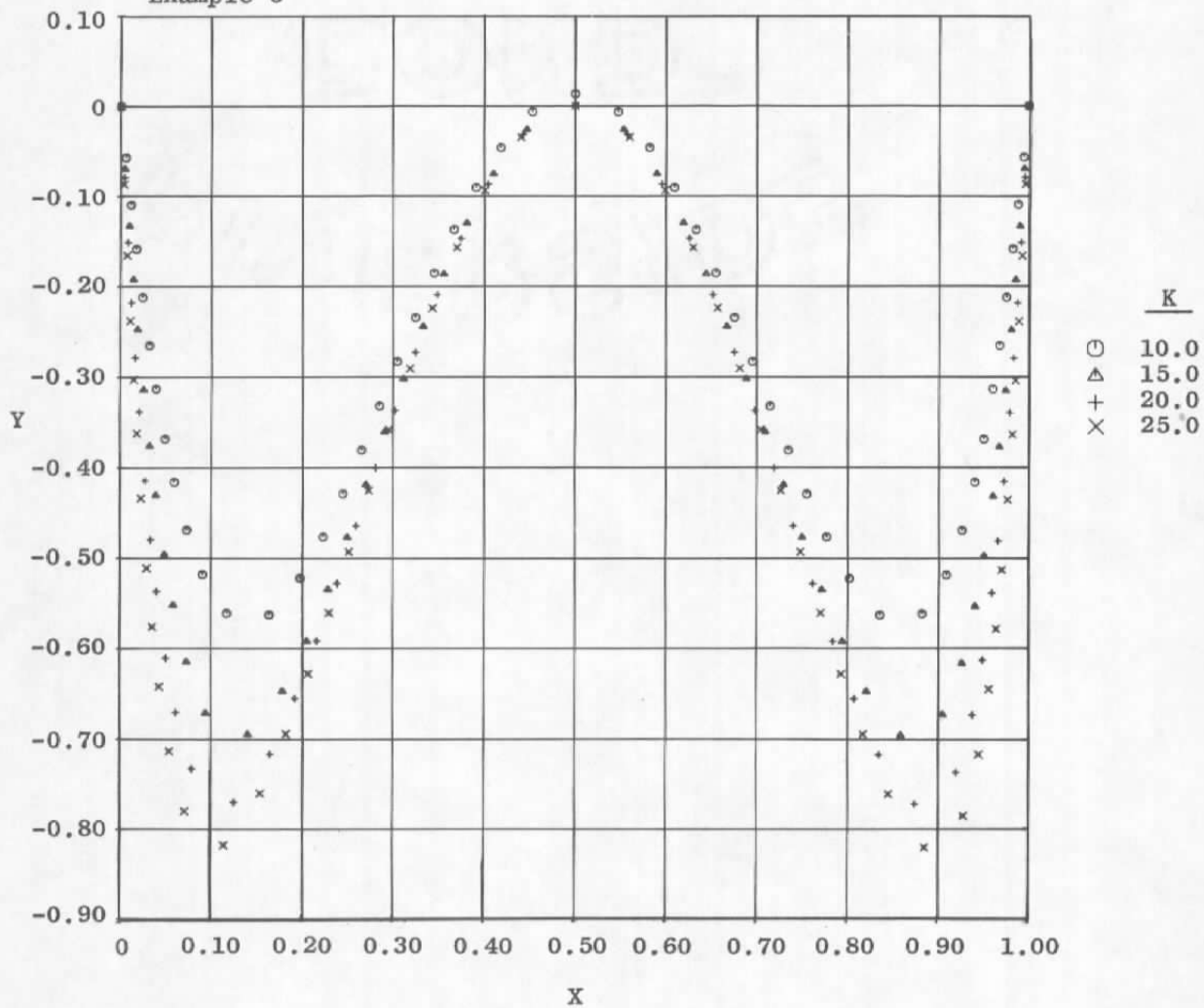


Figure 7. Solutions of example 6.

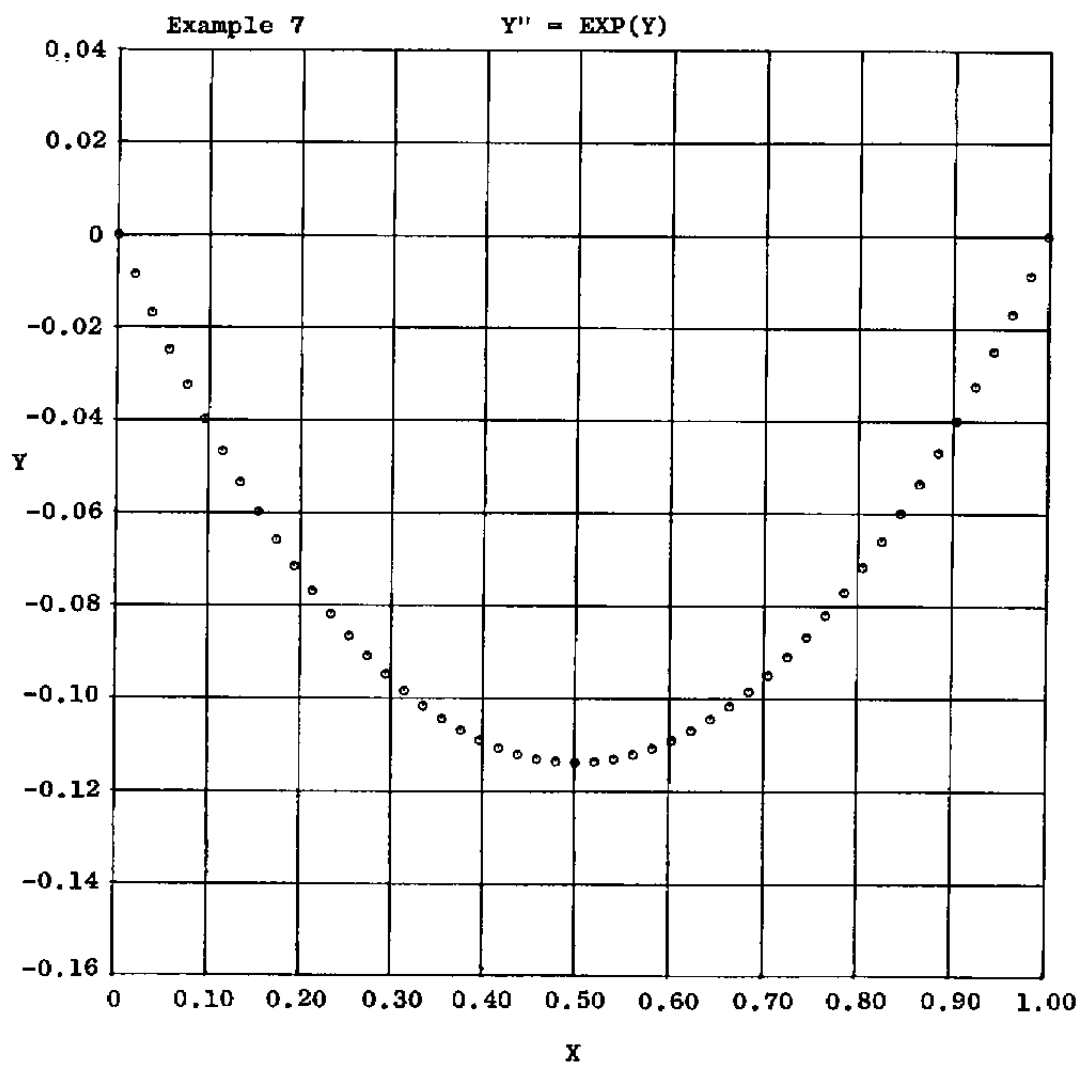


Figure 8. Solution to $y'' = e^y$.

REFERENCES

1. Rubin, S. G. and Khosla, P. K. "Higher Order Numerical Solutions Using Cubic Splines." NASA CR-2653, February 1976.
2. Adams, John C., Jr. "Numerical Calculation of the Subsonic and Transonic Turbulent Boundary Layer on an Infinite Yawed Airfoil." AEDC-TR-73-112 (AD763730), July 1973, Appendixes III and IV.
3. Ahlberg, J. H., Nilson, E. N., and Walsh, J. L. The Theory of Splines and Their Applications. Academic Press, New York, 1967.
4. Fyfe, D. J. "The Use of Cubic Splines in the Solution of Two-Point Boundary Value Problems." The Computer Journal, Vol. 12, No. 2, May 1969, pp. 188-192.
5. Kamke, E. Differentialgleichungen Lösungsmethoden and Lösungen, I, 6. Auflage, p. 558, Eq. 6.67, Akademische Verlagsgesellschaft Geest & Portig K.-G. Leipzig, 1959.
6. White, F. M. Viscous Fluid Flow. McGraw-Hill, New York, 1974, pp. 273-280.
7. Serth, R. W. "Solution of Stiff Boundary Value Problems by Orthogonal Collocation." International Journal for Numerical Methods in Engineering, Vol. 9, 1975, pp. 691-699.
8. Scott, M. R. and Watts, H. A. "A Systematized Collection of Codes for Solving Two-Point Boundary Value Problems," from L. Lapidus and W. E. Schiesser, Numerical Methods for Differential Systems, Academic Press, New York, 1976, pp. 197-227.

APPENDIX A LISTING OF THE COMPUTER PROGRAM

LISTING OF REFERENCE PROGRAM

```

C... SEV00628      V32A-P1A      FOR JOHN C. ADAMS      00000010
C... SOLUTION OF 2-POINT BOUNDARY VALUE PROBLEMS BY SPLINE COLLOCATION 00000020
C... WRITTEN BY DON TODD APRIL 7 1978      00000030
C...      00000040
C... THE MAIN PROGRAM ALLOCATES MEMORY      00000050
C... ALL REDIMENSIONING CAN BE DONE HERE      00000060
C...      00000070
      IMPLICIT REAL*8 (A-H,O-Z)      00000080
      COMMON /ARRAYS/ A(250,45)      00000090
      COMMON /FIXED/ NDIM      00000100
      NDIM=250      00000110
      CALL PPOCED (A(1,1),A(1,2),A(1,3),A(1,4),A(1,5),A(1,6),A(1,7),A(1,8),A(1,9),A(1,10),A(1,10),A(1,11),A(1,12),A(1,13)) 00000120
      WRITE (6,10)      00000130
      STOP      00000140
      10 FORMAT ('*STOP*')      00000150
      END      00000160
      00000170

```

LISTING OF REFERENCE PROGRAM

```

      SUBROUTINE PROCD (X,Y,F,S,G,ALFA,BETA,GAMA,EPSA,D,XA,YA,FA,GA) 00000180
      IMPLICIT REAL*8 (A-H,U-Z) 00000190
      DIMENSION X(1),Y(1),F(1),S(1),G(1),ALFA(1),BETA(1),GAMA(1),EPSA(1) 00000200
      I,J(1),XA(1),YA(1),FA(1),GA(1) 00000210
      COMMON /FIXED/ NDIM,JCASE,JT,NT 00000220
C... PROCEDURE FOR THE 00000230
C... SOLUTION OF 2-POINT BOUNDARY VALUE PROBLEMS BY SPLINE COLLOCATION 00000240
C... WRITTEN BY DON TODD APRIL 7 1978 00000250
C... 00000260
C... F IS THE FIRST DERIVATIVE 00000270
C... S IS THE SECOND DERIVATIVE 00000280
C... G IS THE INTEGRAL 00000290
C... D IS THE COEFFICIENT ARRAY FOR THE SYSTEM 00000300
C... XA,YA,FA, & GA ARE THE NEWLY COMPUTED SOLUTION. NOTE THEY OCCUPY 00000310
C... THE SAME MEMORY AS D (SEE CALL PROCD IN THE MAIN PGM) 00000320
      CALL PAGES 00000330
      DO 50 JCASF=1,100000 00000340
      CALL DATA (N,X,Y,F,S,G,&50,&60) 00000350
      M=3*N 00000360
      JR1=11*M+1 00000370
      DO 10 JT=1,NT 00000380
      CALL QLDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA) 00000390
      CALL SYSTEM (N,X,ALFA,BETA,GAMA,EPSA,D,D(JR1)) 00000400
      CALL BANDED (11,12,M,b,D,&20) 00000410
      CALL RESULT (N,D(JR1),X,YA,FA,S,GA) 00000420
      CALL UPDATE (N,Y,F,G,YA,FA,GA,&30) 00000430
      CALL RESPAC (N,X,Y,F,S,G,XA,YA,FA,GA) 00000440
10  CONTINUE 00000450
      WRITE (6,70) 00000460
      GO TO 30 00000470
20  WRITE (6,80) 00000480
      GO TO 50 00000490
30  CONTINUE 00000500
      CALL QLDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA) 00000510
      CALL CHEKDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA) 00000520
      CALL CHEKBM (N,X,Y,F,S,G) 00000530
      CALL WRITES (N,X,Y,F,S,G) 00000540
      CALL PRINTS (N,X,Y,F,S,G) 00000550
50  CONTINUE 00000560
60  RETURN 00000570
70  FORMAT ('0MAXIMUM NUMBER OF ITERATIONS') 00000580
80  FORMAT ('0BANDED FAILED') 00000590
      END 00000600

```

LISTING OF REFERENCE PROGRAM

```

SUBROUTINE BANDED (NB,N,M,JP,A,*)
  IMPLICIT REAL*8 (A-H,O-Z)
  DIMENSION A(M,N)
C... BANDED SYSTEM SOLVER BY GAUSSIAN ELIMINATION
C... WRITTEN BY DON TODD APRIL 5 1978
C...
C... NB IS THE NUMBER OF DIAGONALS (BAND WIDTH)
C... N IS NUMBER OF EQUATIONS
C... M IS NB + NUMBER OF RIGHT HAND SIDES (RHS'S)
C... A IS COEFFICIENT ARRAY AUGMENTED BY RHS'S
C... SOLUTIONS REPLACE RHS'S COEFFICIENTS ARE DESTROYED
  NB1=NB+1
  MP1=M+1
  MM1=M-1
  NU=NB-JP
  NL=JP-1
  IF (M.EQ.1) GO TO 300
  IF (NU.EQ.0) GO TO 300
  IF (NL.EQ.0) GO TO 100
  IF (NL.GT.NU) GO TO 200
C... ELIMINATE LOWER TRIANGLE
  DO 30 J=1,MM1
    P=A(J,JP)
    IF (P.EQ.0.D0) GO TO 400
    Q=1.00/P
    J1=J+1
    J2=MIN0(J+NL,M)
    J3=MIN0(J+NU,M)
    DO 30 K=J1,J2
      L1=JP-K+J
      F=A(K,L1)
      IF (F.EQ.0.D0) GO TO 30
      F=F*Q
      L2=JP
      DO 10 L=J1,J3
        L1=L1+1
        L2=L2+1
10      A(K,L1)=A(K,L1)-F*A(J,L2)
        DO 20 L=NB1,N
20      A(K,L)=A(K,L)-F*A(J,L)
30      CONTINUE
C... BACK SUBSTITUTE
100  CONTINUE
    DO 140 L=NB1,N
      J=MP1
110  J1=J
      J=J-1
      P=A(J,JP)
      IF (P.EQ.0.D0) GO TO 400
      Q=A(J,L)

```

LISTING OF REFERENCE PROGRAM

	J2=MIN0(J+NU,M)	00001110
	IF (J2.LT.J1) GO TO 130	00001120
	L2=JP	00001130
	DO 120 K=J1,J2	00001140
	L2=L2+1	00001150
120	Q=Q-A(J,L2)*A(K,L)	00001160
130	A(J,L)=Q/P	00001170
	IF (J.GT.1) GO TO 110	00001180
140	CONTINUE	00001190
	RETURN	00001200
C...	ELIMINATE UPPER TRIANGLE	00001210
200	J=M	00001220
210	P=A(J,JP)	00001230
	IF (P.EQ.0.00) GO TO 400	00001240
	Q=1.00/P	00001250
	J1=J-1	00001260
	J2=MAX0(J-NU,1)	00001270
	J3=MAX0(J-NL,1)	00001280
	K=J1	00001290
220	L1=JP+J-K	00001300
	F=A(K,L1)	00001310
	IF (F.EQ.0.00) GO TO 250	00001320
	F=F*Q	00001330
	DO 230 L=NB1,N	00001340
230	A(K,L)=A(K,L)-F*A(J,L)	00001350
	L2=JP	00001360
	L=J1	00001370
240	L1=L1-1	00001380
	L2=L2-1	00001390
	A(K,L1)=A(K,L1)-F*A(J,L2)	00001400
	IF (L.EQ.J3) GO TO 250	00001410
	L=L-1	00001420
	GO TO 240	00001430
250	IF (K.EQ.J2) GO TO 260	00001440
	K=K-1	00001450
	GO TO 220	00001460
260	IF (J.EQ.2) GO TO 270	00001470
	J=J-1	00001480
	GO TO 210	00001490
270	CONTINUE	00001500
C...	FORWARD SUBSTITUTE	00001510
300	CONTINUE	00001520
	DO 330 L=NB1,N	00001530
	DO 330 J=1,M	00001540
	P=A(J,JP)	00001550
	IF (P.EQ.0.00) GO TO 400	00001560
	Q=A(J,L)	00001570
	J2=MAX0(J-NL,1)	00001580
	L2=JP	00001590
	K=J-1	00001600

LISTING OF REFERENCE PROGRAM

310	IF (K,LT,J2) GO TO 320	00001610
	L2=L2-1	00001620
	Q=Q-A(J,L2)*A(K,L)	00001630
	K=K-1	00001640
	GO TO 310	00001650
320	A(J,L)=Q/P	00001660
330	CONTINUE	00001670
	RETURN	00001680
400	RETURN 1	00001690
	END	00001700

LISTING OF REFERENCE PROGRAM

	SUBROUTINE CHEKBM (N,X,Y,F,S,G)	00001710
	IMPLICIT REAL*4 (A-H,O-Z)	00001720
	DIMENSION X(1),Y(1),F(1),S(1),G(1)	00001730
	DIMENSION A(4),E(4),EA(4),KA(4)	00001740
C...	USER PROVIDES THIS ROUTINE TO CHECK SOLUTION AS DESIRED.	00001750
C...	IF NO CHECKS ARE REQUIRED THEN RETURN	00001760
	DO 10 J=1,4	00001770
	EA(J)=0.00	00001780
10	KA(J)=0	00001790
	DO 30 K=1,N	00001800
	SX=1.00/DCOS(X(K))	00001810
	A(1)=DTAN(X(K))	00001820
	A(2)=SX**2	00001830
	A(3)=2.00*A(1)*A(2)	00001840
	A(4)=DLOG(SX)	00001850
	CALL ERROR (A(1),Y(K),E(1))	00001860
	CALL ERROR (A(2),F(K),E(2))	00001870
	CALL ERROR (A(3),S(K),E(3))	00001880
	CALL ERROR (A(4),G(K),E(4))	00001890
	DO 20 J=1,4	00001900
	IF (E(J).LE.EA(J)) GO TO 20	00001910
	EA(J)=E(J)	00001920
	KA(J)=K	00001930
20	CONTINUE	00001940
30	CONTINUE	00001950
	WRITE (6,47) KA,EA	00001960
	RETURN	00001970
40	FORMAT ('0CHEKBM',4I5,1P4E10.2)	00001980
	END	00001990

LISTING OF REFERENCE PROGRAM

```

SUBROUTINE CHEKDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA)      00002000
IMPLICIT REAL*8 (A-H,O-Z)                               00002010
DIMENSION X(1),Y(1),F(1),S(1),G(1),ALFA(1),BETA(1),GAMA(1),EPSA(1) 00002020
COMMON /ENDS/G1,A1,B1,C1,D1,AN,BN,CN,ON                00002030
C... CHECK SATISFACTION OF DIFFERENTIAL EQUATION AND END CONDITIONS 00002040
C... WITH SOLUTION                                     00002050
JM=0                                                     00002060
DM=0.D0                                                 00002070
DO 10 J=1,N                                             00002080
T=ALFA(J)+BETA(J)*Y(J)+GAMA(J)*F(J)+EPSA(J)+G(J)      00002090
CALL DIFRNC (S(J),T,D)                                00002100
IF (D.LE.DM) GO TO 10                                  00002110
DM=D                                                    00002120
JM=J                                                    00002130
10 CONTINUE                                             00002140
CALL ERROR (G1,G(1),DG)                                00002150
T=A1*Y(1)+B1*F(1)+C1*S(1)                             00002160
CALL DIFRNC (D1,T,DA)                                  00002170
T=AN*Y(N)+BN*F(N)+CN*S(N)                             00002180
CALL DIFRNC (DN,T,DB)                                  00002190
WRITE (6,20) JM,DM,DG,DA,DB                           00002200
RETURN                                                  00002210
20 FORMAT('0CHEKDE',I5,I4E10,2)                        00002220
END                                                     00002230

```

LISTING OF REFERENCE PROGRAM

```

SUBROUTINE DATA (N,X,Y,F,S,G,*,*)          00002240
IMPLICIT REAL*8 (A-H,O-Z)                  00002250
DIMENSION X(1),Y(1),F(1),S(1),G(1)         00002260
COMMON /ALPHA/ LAB(18)                      00002270
COMMON /ENDS/ G1,A1,B1,C1,D1,AN,BN,CN,DN    00002280
COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT,LPRNT,LNORM 00002290
COMMON /FLOAT/ TOL,RSC                      00002300
C... DATA IS SUPPLIED TO PROGRAM BY USER VIA THIS ROUTINE 00002310
IF (JCASE.GT.1) GO TO 590                    00002320
C... PROVIDE ID FOR PROBLEM                  00002330
CALL LABELS (LAB,*,*PROBLEM FROM KAMKE      *) 00002340
CALL LABELS (LAB(10),*)                      *) 00002350
C... N IS NUMBER OF POINTS                  00002360
C... NT IS MAXIMUM NUMBER OF ITERATIONS     00002370
C... WILL WRITE SOLUTION ON UNIT JOUT IF JOUT > 0 00002380
C... WILL PRINT SOLUTION IF LPRNT IS NOT ZERO 00002390
C... LNORM CONTROLS WHICH NORM IS USED IN CONVERGENCE TEST (SEE NORM) 00002400
C... ITERATION STOPS IF RELATIVE CHANGE OF SOLUTION IS < TOL 00002410
C... NEW SPACING IS COMPUTED IF CHANGE IN Y IS > RSC 00002420
N=S1                                         00002430
NT=30                                        00002440
JOUT=0                                       00002450
LPRNT=1                                     00002460
LNORM=1                                     00002470
TOL=5.D-4                                   00002480
RSC=.01D0                                   00002490
C... SPECIFY END CONDITIONS                  00002500
G1=0.D0                                     00002510
A1=1.D0                                     00002520
B1=0.D0                                     00002530
C1=0.D0                                     00002540
D1=0.D0                                     00002550
AN=1.D0                                     00002560
BN=0.D0                                     00002570
CN=0.D0                                     00002580
DN=7.D0                                     00002590
C... PRINT DATA                            00002600
CALL PAGER (100)                           00002610
WRITE (6,603)                               00002620
WRITE (6,601) NDIM,JCASE,N,NT,JOUT,LPRNT,LNORM 00002630
WRITE (6,604)                               00002640
WRITE (6,602) G1,TOL,RSC                    00002650
WRITE (6,605)                               00002660
WRITE (6,602) A1,B1,C1,D1                  00002670
WRITE (6,602) AN,BN,CN,DN                  00002680
C... PROVIDE STARTING SOLUTION              00002690
IF (N.GT.NDIM) GO TO 570                    00002700
X(1)=0.D0                                   00002710
X(N)=DATAN(DN)                             00002720
CALL UNIFORM (N,X)                         00002730

```

LISTING OF REFERENCE PROGRAM

CALL GUESS (N,X,Y,F,S,G,2580)	00002740
WRITE (6,606)	00002750
RETURN	00002760
C... TO ABORT A CASE, RETURN 1.	00002770
570 WRITE (6,607)	00002780
580 RETURN 1	00002790
C... WHEN A JOB IS FINISHED THEN RETURN 2	00002800
590 RETURN 2	00002810
601 FORMAT (I2I10)	00002820
602 FORMAT (1P10E12.4)	00002830
603 FORMAT (1H0,5X,4HNDIM,5X,5HJCASE,9X,1HN,8X,2HNT,6X,4HJOUT,5X,5HLP	00002840
INT,5X,5HLCR)	00002850
604 FORMAT (5H0 I1,10X,3HTOL,9X,3HRSC)	00002860
605 FORMAT (4H0 A,11X,1HB,11X,1HC,11X,1HD)	00002870
606 FORMAT (1H)	00002880
607 FORMAT (9HON > NDIM)	00002890
END	00002900

LISTING OF REFERENCE PROGRAM

	SUBROUTINE DELTAS (S,DL,DS)	00002910
	IMPLICIT REAL*8 (A-H,O-Z)	00002920
C...	COMPUTE CAP DELTA, DL, & SMALL DELTA, DS, FROM SIGMA(I-1), S	00002930
	DL=S/(S+1.D0)	00002940
	DS=(S-1.D0)/(S+1.D0)	00002950
	RETURN	00002960
	END	00002970
	SUBROUTINE DIFRNC (A1,A2,D)	00002980
	IMPLICIT REAL*8 (A-H,O-Z)	00002990
C...	COMPUTE RELATIVE DIFFERENCE, D, BETWEEN A1 & A2	00003000
	D=DABS(A1-A2)	00003010
	A=.500*DABS(A1+A2)	00003020
	IF (A.LE.0) RETURN	00003030
	D=D/A	00003040
	RETURN	00003050
	END	00003060
	SUBROUTINE ERROR (A,B,E)	00003070
	IMPLICIT REAL*8 (A-H,O-Z)	00003080
C...	COMPUTE RELATIVE ERROR, E, OF B FROM EXACT VALUE, A	00003090
	E=DABS(A-B)	00003100
	D=DABS(A)	00003110
	IF (D.LE.0) RETURN	00003120
	E=E/D	00003130
	RETURN	00003140
	END	00003150

LISTING OF REFERENCE PROGRAM

```

SUBROUTINE GUESS (N,X,Y,F,S,G,*)
IMPLICIT REAL*8 (A-H,D-Z)
DIMENSION X(1),Y(1),F(1),S(1),G(1)
DIMENSION XA(2),YA(2),FA(2),SA(2)
COMMON /ENDS/ G1,A1,B1,C1,D1,AN,BN,CN,DN
C... GUESS SOLUTION IN LACK OF A BETTER APPROXIMATION
C... WRITTEN BY DON TODD APRIL 13 1978
H=X(N)-X(1)
HS=H**2
BA=B1/H
BB=BN/H
CA=C1/HS
CB=CN/HS
A11=A1**2+3.D0*(BA**2+CA**2)-2.D0*BA*(A1+CA)
ANN=AN**2+3.D0*(BB**2+CB**2)+2.D0*BB*(AN+CB)
A1N=-A1*BB+AN*BA-BA*CB+BB*CA+2.D0*(BA*BB+CA*CB)
DET=A11*ANN-A1N**2
IF (DET.EQ.0.D0) GO TO 10
E1=(ANN*D1-A1N*DN)/DET
EN=(A11*DN-A1N*D1)/DET
XA(1)=X(1)
XA(2)=X(N)
Q=BA*E1+BB*EN
YA(1)=A1*E1-Q
YA(2)=AN*EN+Q
FT=(YA(2)-YA(1))/H
Q=CA*E1+CB*EN
FA(1)=FT*(BA*E1-Q)/H
FA(2)=FT*(BB*EN+Q)/H
FT=(FA(2)-FA(1))/H
SA(1)=FT*CA*E1/HS
SA(2)=FT*CB*EN/HS
CALL QUINTS (2,XA,YA,FA,SA,N,X,Y,F,S)
CALL NTEGRL (N,G1,X,Y,F,S,G)
WRITE (6,20)
RETURN
10 WRITE (6,30)
RETURN 1
20 FORMAT ('0SOLUTION GUESSED')
30 FORMAT ('0CONTRADICTIONARY END CONDITIONS')
END

```

00003160
00003170
00003180
00003190
00003200
00003210
00003220
00003230
00003240
00003250
00003260
00003270
00003280
00003290
00003300
00003310
00003320
00003330
00003340
00003350
00003360
00003370
00003380
00003390
00003400
00003410
00003420
00003430
00003440
00003450
00003460
00003470
00003480
00003490
00003500
00003510
00003520
00003530
00003540
00003550
00003560

LISTING OF REFERENCE PROGRAM

	SUBROUTINE NORM (N,Y1,Y2,C,K)	00003570
	IMPLICIT REAL*8 (A-H,O-Z)	00003580
	DIMENSION Y1(1),Y2(1)	00003590
	COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT,LPRNT,LNORM	00003600
C...	COMPUTE "DISTANCE" BETWEEN ARRAYS Y1 & Y2	00003610
	K=0	00003620
	C=0.00	00003630
	IF (LNORM.NE.1) GO TO 30	00003640
C...	COMPUTE A GLOBAL NORM	00003650
	D=Y1(1)	00003660
	R=D	00003670
	DO 10 J=1,N	00003680
	D=DMIN1(D,Y1(J))	00003690
	R=DMAX1(R,Y1(J))	00003700
10	CONTINUE	00003710
	D=R-D	00003720
	IF (D.LE.0.00) D=R	00003730
	IF (D.LE.0.00) D=1.00	00003740
	DO 20 J=1,N	00003750
	R=DABS(Y2(J)-Y1(J))	00003760
	IF (R.LE.C) GO TO 20	00003770
	C=R	00003780
	K=J	00003790
20	CONTINUE	00003800
	C=C/D	00003810
	RETURN	00003820
30	IF (LNORM.NE.2) GO TO 50	00003830
C...	COMPUTE A LOCAL NORM	00003840
	DO 40 J=1,N	00003850
	CALL DIFRNC (Y1(J),Y2(J),D)	00003860
	IF (D.LE.C) GO TO 40	00003870
	C=D	00003880
	K=J	00003890
40	CONTINUE	00003900
50	RETURN	00003910
	END	00003920

LISTING OF REFERENCE PROGRAM

```

SUBROUTINE NTEGR1 (N,G1,X,Y,F,S,G)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(1),Y(1),F(1),S(1),G(1)
C... COMPUTE INTEGRAL OF QUINTIC SPLINE
G(1)=G1
DO 10 J2=2,N
J1=J2-1
H=X(J2)-X(J1)
G(J2)=G(J1)+H*(.5D0*(Y(J1)+Y(J2))+H*(.1D0*(F(J1)-F(J2))
1 +H*(S(J1)+S(J2))/120,D0))
10 CONTINUE
RETURN
END
00003930
00003940
00003950
00003960
00003970
00003980
00003990
00004000
00004010
00004020
00004030
00004040
00004050

SUBROUTINE PRINTS (N,X,Y,F,S,G)
IMPLICIT REAL*8 (A-H,O-Z)
DIMENSION X(1),Y(1),F(1),S(1),G(1)
COMMON /ALPHA/ LAB(36)
COMMON /FIXED/ NOIM,JCASE,JT,NT,JOUT,LPRNT
IF (LPRNT.EQ.0) RETURN
C... PRINT SOLUTION
CALL PAGER (100)
WRITE (6,30)
HN=X(2)-X(1)
N2=N-2
DO 10 J=1,N2
H=HN
HN=X(J+2)-X(J+1)
SIGMA=HN/H
10 WRITE (6,40) J,X(J),Y(J),F(J),S(J),G(J),H,SIGMA
J=N-1
H=HN
WRITE (6,40) J,X(J),Y(J),F(J),S(J),G(J),H
J=N
WRITE (6,40) J,X(J),Y(J),F(J),S(J),G(J)
RETURN
30 FORMAT (1H0,10X,1HX,11X,1HY,11X,2HYP,10X,3HYP,9X,1HI,13X,1HH,10X,00004060
11HS)
40 FORMAT (18,1P5E12.4,E13.3,E10.2)
END
00004070
00004080
00004090
00004100
00004110
00004120
00004130
00004140
00004150
00004160
00004170
00004180
00004190
00004200
00004210
00004220
00004230
00004240
00004250
00004260
00004270
00004280
00004290
00004300
00004310

```

LISTING OF REFERENCE PROGRAM

	SUBROUTINE OLDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA).	00004320
	IMPLICIT REAL*8 (A-H,O-Z)	00004330
	DIMENSION X(1),Y(1),F(1),S(1),G(1),ALFA(1),BETA(1),GAMA(1),EPSA(1)	00004340
C...	USER PROVIDES THIS ROUTINE TO	00004350
C...	EVALUATE COEFFICIENTS OF QUASI-LINEARIZED EQUATION	00004360
	DO 10 J=1,N	00004370
	R=DSORT(4.00*F(J)+Y(J)**4)	00004380
	ALFA(J)=3.00*Y(J)*F(J)*(Y(J)**2-(2.00*F(J)+Y(J)**4)/R)	00004390
	BETA(J)=F(J)*((4.00*F(J)+3.00*Y(J)**4)/R-3.00*Y(J)**2)	00004400
	GAMA(J)=Y(J)*((6.00*F(J)+Y(J)**4)/R-Y(J)**2)	00004410
	EPSA(J)=0.00	00004420
10	CONTINUE	00004430
	RETURN	00004440
	END	00004450

LISTING OF REFERENCE PROGRAM

```

      SJBROUTINE QUINTS (N1,X1,Y1,F1,S1,N2,X2,Y2,F2,S2)      00004460
      IMPLICIT REAL*8 (A-H,O-Z)                             00004470
      DIMENSION X1(1),Y1(1),F1(1),S1(1),X2(1),Y2(1),F2(1),S2(1) 00004480
C...  QUINTIC SPLINE INTERPOLATION                          00004490
C...  WRITTEN BY DON TODD APRIL 11 1978                      00004500
      XR=-1.D50                                              00004510
      J2=1                                                    00004520
      DO 30 J=1,N2                                           00004530
      IF (X2(J).LE.XR) GO TO 20                               00004540
10     J1=J2                                                  00004550
      J2=J2+1                                                 00004560
      XR=X1(J2)                                               00004570
      IF (J2.EQ.N1) XR=1.D50                                  00004580
      IF (X2(J).GT.XR) GO TO 10                               00004590
      H=X1(J2)-X1(J1)                                         00004600
      YA=Y1(J1)+.2D0*H*F1(J1)                                00004610
      YB=Y1(J1)+.4D0*H*F1(J1)+.05D0*H**2*S1(J1)             00004620
      YC=Y1(J2)-.4D0*H*F1(J2)+.05D0*H**2*S1(J2)             00004630
      YD=Y1(J2)-.2D0*H*F1(J2)                                00004640
20     TA=(X2(J)-X1(J1))/H                                     00004650
      TC=1.D0-TA                                              00004660
      Y2(J)=(Y1(J1)*TC**2+5.D0*YA*TA*TC+10.D0*YB*TA**2)*TC**3 00004670
      + (Y1(J2)*TA**2+5.D0*YD*TA*TC+10.D0*YC*TC**2)*TA**3    00004680
      QA=(TC-4.D0*TA)*TC**3                                   00004690
      QD=(4.D0*TC-TA)*TA**3                                   00004700
      QB=(2.D0*TC-3.D0*TA)*TC**2                             00004710
      QC=(3.D0*TC-2.D0*TA)*TA**2                             00004720
      F2(J)=5.D0*(-Y1(J1)*TC**4+QA*YA+2.D0*QB*YB*TA          00004730
      +Y1(J2)*TA**4+QD*YD+2.D0*QC*YC*TC)/H                  00004740
      QA=(TC**2-6.D0*TA*TC+3.D0*TA**2)*TC                    00004750
      QD=(3.D0*TC**2-6.D0*TA*TC+TA**2)*TA                     00004760
      S2(J)=20.D0*(Y1(J1)*TC**3-QB*YA+QA*YB                   00004770
      +Y1(J2)*TA**3+QC*YD+QD*YC)/H**2                        00004780
30     CONTINUE                                               00004790
      RETURN                                                  00004800
      END                                                      00004810

```

LISTING OF REFERENCE PROGRAM

	SUBROUTINE RESPAC (N,X,Y,F,S,G,XA,YA,FA,SA)	00004820
	IMPLICIT REAL*8 (A-H,O-Z)	00004830
	DIMENSION X(1),Y(1),F(1),S(1),G(1),XA(1),YA(1),FA(1),SA(1)	00004840
	COMMON /ENDS/ G1	00004850
	COMMON /FLOAT/ TOL,RSC,DY	00004860
C...	DY IS COMPUTED IN UPDATE. RSC IS SET IN DATA.	00004870
	IF (DY.LE.RSC) RETURN	00004880
C...	COMPUTE NEW SPACING FOR NEXT ITERATION	00004890
C...	WRITTEN BY DON TODD APRIL 13 1978	00004900
	N1=N-1	00004910
	H=0.00	00004920
	DO 10 J=1,N1	00004930
	J1=J+1	00004940
10	H=H+DSQRT((X(J1)-X(J))**2+(Y(J1)-Y(J))**2)	00004950
	DH=H/N1	00004960
	H=DH	00004970
	DR=0.00	00004980
	XA(1)=X(1)	00004990
	J2=1	00005000
	DO 40 J=2,N1	00005010
20	IF (DR.GE.H) GO TO 30	00005020
	H=H-DR	00005030
	XR=X(J2)	00005040
	J1=J2	00005050
	J2=J2+1	00005060
	DX=X(J2)-X(J1)	00005070
	DY=Y(J2)-Y(J1)	00005080
	DR=DSQRT(DX**2+DY**2)	00005090
	CT=DX/DR	00005100
	GO TO 20	00005110
30	XA(J)=XR+H*CT	00005120
	XR=XA(J)	00005130
	DR=DR-H	00005140
	H=DH	00005150
40	CONTINUE	00005160
	XA(N)=X(N)	00005170
	CALL QUINTS (N,X,Y,F,S,N,XA,YA,FA,SA)	00005180
	DO 50 J=2,N1	00005190
	X(J)=XA(J)	00005200
	Y(J)=YA(J)	00005210
	F(J)=FA(J)	00005220
50	S(J)=SA(J)	00005230
	CALL NTEGRL (N,G1,X,Y,F,S,G)	00005240
	WRITE (6,60)	00005250
	RETURN	00005260
60	FORMAT (' NEW SPACING COMPUTED')	00005270
	END	00005280

LISTING OF REFERENCE PROGRAM

```

      SUBROUTINE RESULT (N,R,X,Y,F,S,G)                                00005290
      IMPLICIT REAL*8 (A-H,O-Z)                                       00005300
      DIMENSION R(3,N),X(1),Y(1),F(1),S(1),G(1)                      00005310
C... COMPUTE Y, F, & S FROM RESULT, R, OF BANDED                     00005320
      HI=X(2)-X(1)                                                     00005330
      T1=(R(3,2)-R(3,1))/HI-HI*(2.00*R(2,1)+R(2,2))/6.00           00005340
      Y(1)=R(3,1)                                                      00005350
      F(1)=T1                                                           00005360
      S(1)=R(2,1)                                                       00005370
      G(1)=R(1,1)                                                       00005380
      N1=N-1                                                            00005390
      DO 10 I=2,N1                                                      00005400
      IA=I+1                                                            00005410
      ID=I-1                                                            00005420
      HI1=HI                                                            00005430
      HI=X(IA)-X(1)                                                     00005440
      SI1=HI/HI1                                                        00005450
      CALL DELTAS (SI1,DL,DS)                                           00005460
      T1=.500*(-R(3,ID)/HI1+(1.00/HI1-1.00/HI)*R(3,I)+R(3,IA)/HI)    00005470
      I+(HI1*R(2,ID)+2.00*(HI1-HI)*R(2,I)-HI*R(2,IA))/12.00          00005480
      Q=SI1*R(2,ID)-(SI1+1.00)*R(2,I)+R(2,IA)                         00005490
      Y(I)=R(3,I)                                                       00005500
      F(I)=T1+DS*HI1*Q/36.00                                           00005510
      S(I)=R(2,I)+DL*Q/6.00                                             00005520
      G(I)=R(1,I)                                                       00005530
10  CJNT INUE                                                           00005540
      HI1=HI                                                            00005550
      T1=(R(3,N)-R(3,N-1))/HI1*HI1*(2.00*R(2,N)+R(2,N-1))/6.00      00005560
      Y(N)=R(3,N)                                                       00005570
      F(N)=T1                                                           00005580
      S(N)=R(2,N)                                                       00005590
      G(N)=R(1,N)                                                       00005600
      RETURN                                                            00005610
      END                                                                00005620

```

LISTING OF REFERENCE PROGRAM

```

SUBROUTINE SYSTEM (N,X,ALFA,BETA,GAMA,EPSA,D,R)                                00005630
  IMPLICIT REAL*8 (A-H,O-Z)                                                  00005640
  DIMENSION X(N),ALFA(N),BETA(N),GAMA(N),EPSA(N),D(3,N,12),R(3,N)           00005650
  COMMON /ENDS/ G1,A1,B1,C1,D1,AN,BN,CN,DN                                   00005660
C... COMPUTE COEFFICIENTS AND RHS OF SYSTEM                                  00005670
  DO 10 K=1,12                                                                00005680
    DO 10 I=1,N                                                                00005690
      DO 10 J=1,3                                                              00005700
        D(J,I,K)=0.00                                                         00005710
C... EQUATIONS FOR I=1                                                         00005720
        HI=X(2)-X(1)                                                          00005730
        D(1,1,1)=-1.00                                                        00005740
        D(2,1,5)=EPSA(1)                                                       00005750
        D(2,1,6)=-1.00-GAMA(1)*HI/3.00                                       00005760
        D(2,1,7)=BETA(1)-GAMA(1)/HI                                          00005770
        D(2,1,9)=-GAMA(1)*HI/6.00                                             00005780
        D(2,1,10)=GAMA(1)/HI                                                  00005790
        D(3,1,5)=C1-D1*HI/3.00                                                00005800
        D(3,1,6)=A1-B1/HI                                                     00005810
        D(3,1,8)=-B1*HI/6.00                                                  00005820
        D(3,1,9)=D1/HI                                                         00005830
        R(1,1)=-G1                                                            00005840
        R(2,1)=-ALFA(1)                                                        00005850
        R(3,1)=D1                                                             00005860
C... EQUATIONS FOR I=2                                                         00005870
        HI1=HI                                                                00005880
        HI=X(3)-X(2)                                                          00005890
        S11=HI/HI1                                                            00005900
        CALL DELTAS (S11,DL,DS)                                              00005910
        A11=HI1**3*(DL-2.00*DS)/720.00                                       00005920
        QC=(GAMA(2)*DS*HI1-6.00*DL)/36.00                                    00005930
        D(1,2,3)=1.00                                                         00005940
        D(1,2,4)=-HI1**3/30.00+A11*S11                                       00005950
        D(1,2,5)=9.00*HI1/20.00                                              00005960
        D(1,2,6)=-1.00                                                         00005970
        D(1,2,7)=HI1**3*(2.00*S11-3.00)/120.00-A11*(S11+1.00)              00005980
        D(1,2,8)=HI1*(1.00/S11+11.00)/20.00                                  00005990
        D(1,2,10)=HI1**2*HI/120.00+A11                                       00006000
        D(1,2,11)=-HI1/(20.00*S11)                                           00006010
        D(2,2,3)=GAMA(2)*HI1/12.00+S11*QC                                     00006020
        D(2,2,4)=-GAMA(2)/(2.00*HI1)                                         00006030
        D(2,2,5)=EPSA(2)                                                       00006040
        D(2,2,6)=-1.00+GAMA(2)*(HI1-HI)/6.00-QC*(S11+1.00)                 00006050
        D(2,2,7)=BETA(2)+.500*GAMA(2)*(1.00/HI1-1.00/HI)                   00006060
        D(2,2,9)=-GAMA(2)*HI/12.00+QC                                         00006070
        D(2,2,10)=GAMA(2)/(2.00*HI)                                           00006080
        D(3,2,2)=-HI1/6.00                                                     00006090
        D(3,2,3)=1.00/HI1                                                     00006100
        D(3,2,5)=-{(HI1+HI)/3.00}                                           00006110
        D(3,2,6)=-{(1.00/HI1+1.00/HI)}                                       00006120

```

LISTING OF REFERENCE PROGRAM

```

      D(3,2, 8)=-HI/6.D0                                00006130
      D(3,2,9)=1.D0/HI                                  00006140
      R(1,2)=0.D0                                         00006150
      R(2,2)=-ALFA(2)                                     00006160
      R(3,2)=0.D0                                         00006170
C... EQUATIONS FOR 2 < I < N                             00006180
      N1=N-1                                              00006190
      DO 100 I=3,N1                                       00006200
      A2I=HI**3*(DL+2.D0*DS/SI1)/720.D0                 00006210
      HI2=HI1                                              00006220
      HI1=HI                                               00006230
      HI=X(I+1)-X(I)                                       00006240
      SI2=SI1                                              00006250
      SI1=HI/HI1                                           00006260
      CALL DELTAS (SI1,DL,DS)                             00006270
      A1I=HI1**3*(DL-2.D0*DS)/720.D0                   00006280
      QC=(GAMA(I)*DS*HI1-6.D0*DL)/36.D0                 00006290
      D(1,I, 1)=HI2*HI1**2/120.D0+A2I*SI2               00006300
      D(1,I, 2)=-SI2*HI1/20.D0                          00006310
      D(1,I, 3)=1.D0                                       00006320
      D(1,I, 4)=HI1**3*(1.D0/SI2-1.D0)/60.D0+A1I*(SI1-A2I*(SI2+1.D0)) 00006330
      D(1,I, 5)=HI1*(SI2+10.D0)/20.D0                   00006340
      D(1,I, 6)=-1.D0                                       00006350
      D(1,I, 7)=HI1**3*(SI1-1.D0)/60.D0-A1I*(SI1+1.D0)+A2I 00006360
      D(1,I, 8)=HI1*(1.D0/SI1+10.D0)/20.D0               00006370
      D(1,I,10)=HI1**2*HI/120.D0+A1I                    00006380
      D(1,I, 11)=-HI1/(20.D0*SI1)                       00006390
      D(2,I,3)=GAMA(I)*HI1/12.D0+SI1*QC                  00006400
      D(2,I, 4)=-GAMA(I)/(2.D0*HI1)                     00006410
      D(2,I, 5)=EPSA(I)                                    00006420
      D(2,I,6)=-1.D0+GAMA(I)*(HI1-HI)/6.D0-QC*(SI1+1.D0) 00006430
      D(2,I, 7)=BETA(I)+.5D0*GAMA(I)*(1.D0/HI1-1.D0/HI) 00006440
      D(2,I,9)=-GAMA(I)*HI/12.D0+QC                      00006450
      D(2,I,10)=GAMA(I)/(2.D0*HI)                        00006460
      D(3,I, 2)=-HI1/6.D0                                  00006470
      D(3,I, 3)=1.D0/HI1                                  00006480
      D(3,I, 5)=- (HI1+HI)/3.D0                          00006490
      D(3,I, 6)=- (1.D0/HI1+1.D0/HI)                     00006500
      D(3,I, 8)=-HI/6.D0                                   00006510
      D(3,I, 9)=1.D0/HI                                    00006520
      R(1,I)=0.D0                                           00006530
      R(2,I)=-ALFA(I)                                       00006540
      R(3,I)=0.D0                                           00006550
100 CONTINUE                                              00006560
C... EQUATIONS FOR I=N                                    00006570
      A2I=HI**3*(DL+2.D0*DS/SI1)/720.D0                 00006580
      HI2=HI1                                              00006590
      HI1=HI                                               00006600
      SI2=SI1                                              00006610
      D(1,N, 1)=HI2*HI1**2/120.D0+A2I*SI2               00006620

```

LISTING OF REFERENCE PROGRAM

D(1,N, 2)=-SI2*HI1/20.D0	00006630
D(1,N, 3)=1.D0	00006640
D(1,N, 4)=HI1**3*(2.D0/SI2-3.D0)/120.D0-A2I*(SI2+1.D0)	00006650
D(1,N, 5)=HI1*(SI2+1.D0)/20.D0	00006660
D(1,N, 6)=-1.D0	00006670
D(1,N, 7)=-HI1**3/30.D0+A2I	00006680
D(1,N, 8)=9.D0*HI1/20.D0	00006690
D(2,N, 3)=GAMA(N)*HI1/6.D0	00006700
D(2,N, 4)=-GAMA(N)/HI1	00006710
D(2,N, 5)=EPSA(N)	00006720
D(2,N, 6)=-1.D0+GAMA(N)*HI1/3.D0	00006730
D(2,N, 7)=BETA(N)+GAMA(N)/HI1	00006740
D(3,N, 2)=BN*HI1/6.D0	00006750
D(3,N, 3)=-BN/HI1	00006760
D(3,N, 5)=CN+BN*HI1/3.D0	00006770
D(3,N, 6)=AN+BN/HI1	00006780
R(1,N)=0.D0	00006790
R(2,N)=-ALFA(N)	00006800
R(3,N)=DN	00006810
RETURN	00006820
END	00006830

LISTING OF REFERENCE PROGRAM

```

      SUBROUTINE UNIFORM (N,X)                                00006840
      IMPLICIT REAL*8 (A-H,O-Z)                             00006850
      DIMENSION X(1)                                         00006860
C...  GIVEN N, X(1), & X(N) COMPUTE UNIFORM SPACING        00006870
      NI=N-1                                                  00006880
      DX=(X(N)-X(1))/NI                                       00006890
      DO 10 J=2,N1                                           00006900
10    X(J)=X(1)+(J-1)*DX                                     00006910
      WRITE (6,20)                                           00006920
      RETURN                                                  00006930
20    FORMAT ('0UNIFORM SPACING COMPUTED')                  00006940
      END                                                     00006950

      SUBROUTINE UPDATE (N,Y1,F1,G1,Y2,F2,G2,*)             00006960
      IMPLICIT REAL*8 (A-H,O-Z)                             00006970
      DIMENSION Y1(1),F1(1),G1(1),Y2(1),F2(1),G2(1)        00006980
      DIMENSION KA(3)                                         00006990
      COMMON /FIXED/ NDIM,JCASE,JT                           00007000
      COMMON /FLOAT/ TOL,RSC,E(3)                            00007010
C...  COMPUTE CHANGE IN SOLUTION                             00007020
      CALL NORM (N,Y1,Y2,E(1),KA(1))                         00007030
      CALL NORM (N,F1,F2,E(2),KA(2))                         00007040
      CALL NORM (N,G1,G2,E(3),KA(3))                         00007050
      WRITE (6,6001) JT,KA,E                                 00007060
C...  UPDATE SOLUTION                                       00007070
      DO 50 K=1,N                                             00007080
      Y1(K)=Y2(K)                                             00007090
      F1(K)=F2(K)                                             00007100
50    G1(K)=G2(K)                                             00007110
C...  CHECK CONVERGENCE                                     00007120
      DO 60 J=1,3                                             00007130
      IF (E(J).GT.TOL) GO TO 70                               00007140
60    CONTINUE                                                00007150
C...  IF CONVERGED THEN RETURN 1                             00007160
      WRITE (6,6002)                                          00007170
      RETURN 1                                                00007180
70    RETURN                                                  00007190
6001  FORMAT ('* UPDATE',4I10,1P3E10,2)                    00007200
6002  FORMAT ('0CONVERGED SOLUTION')                         00007210
      END                                                     00007220

```

LISTING OF REFERENCE PROGRAM

	SUBROUTINE WRITES (N,X,Y,F,S,G)	00007230
	IMPLICIT REAL*8 (A-H,O-Z)	00007240
	DIMENSION X(N),Y(N),F(N),S(N),G(N)	00007250
	COMMON /ALPHA/ LAB(36)	00007260
	COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT	00007270
	IF (JOUT.LE.0) RETURN	00007280
C...	WRITE SOLUTION ON UNIT JOUT	00007290
	WRITE (JOUT) LAB,N	00007300
	WRITE (JOUT) X,Y,F,S,G	00007310
	WRITE (6,50) JOUT	00007320
	RETURN	00007330
50	FORMAT ('SOLUTION WRITTEN ON UNIT',I3)	00007340
	END	00007350
	SUBROUTINE PAGER (N)	00007360
	COMMON /ALPHA/ LAB(18),ID1(9),ID2(9),JLINE,NLINE,JPAGE	00007370
C...	LINE COUNTER ROUTINE	00007380
	JLINE=JLINE+N	00007390
	IF (JLINE.LE.NLINE) RETURN	00007400
	JLINE=N	00007410
	JPAGE=JPAGE+1	00007420
	CALL TIMCHK (T)	00007430
	M=T/60.	00007440
	S=T-60*M	00007450
	T=M+S/100.	00007460
	WRITE (6,10) ID1,ID2,JPAGE	00007470
	WRITE (6,20) LAB,T	00007480
	RETURN	00007490
	ENTRY PAGES	00007500
C...	INITIALIZE LINE COUNTER	00007510
	CALL TIMCHK (T)	00007520
	CALL LABELS (LAB,36H	00007530
	CALL LABELS (LAB(10),LAB)	00007540
C...	FOLLOWING CARD SUPPLIES USER ID	00007550
	CALL LABELS (ID1,36HDC TODD CDPS	00007560
	CALL GETNOW (ID2)	00007570
	JPAGE=0	00007580
	NLINE=50	00007590
	JLINE=NLINE+1	00007600
	RETURN	00007610
10	FORMAT (1H1,18A4,38X,4HPAGE,I6)	00007620
20	FORMAT (1H ,18A4,38X,4FTIME,F6.2)	00007630
	END	00007640

LISTING OF REFERENCE PROGRAM

	SUBROUTINE LABELS (L,N)	00007650
	DIMENSION L(1), N(1)	00007660
C...	TRANSFER LITERAL INTO AN ARRAY	00007670
	DO 10 J=1,9	00007680
10	L(J)=N(J)	00007690
	RETURN	00007700
	END	00007710
	SUBROUTINE TIMCHK (T)	00007720
C...	SYSTEM DEPENDENT ROUTINE TO PROVIDE USED CPU TIME IN SECONDS	00007730
	T=0.	00007740
	RETURN	00007750
	END	00007760
	SUBROUTINE GETNOW (N)	00007770
	DIMENSION N(1)	00007780
C...	SYSTEM DEPENDENT ROUTINE TO PROVIDE SHOT ID IN FOLLOWING FORM	00007790
	CALL LABELS (N,'ARO08671 10:29 WED MAY 10, 1978')	00007800
	RETURN	00007810
	END	00007820

APPENDIX B INITIAL SOLUTION

To begin iteration (Section 3.2) an initial "in-hand" solution is needed, and at times it cannot be justified as anything more than a guess. Even a guess should be made as consistent as possible, however. In particular, it should satisfy the end conditions, Eq. (24). Since the spline is a quintic in each interval, $[x_i, x_{i+1}]$, it seems reasonable that a quintic over the whole interval, $[x_1, x_N]$, could possibly be a good guess. Equation (24) with $i = 1$ and $i = N$ gives two equations in six unknowns. If further assumptions are made to fully specify the six unknowns, then the quintic would be determined. The following procedure is proposed to specify a quintic with the least variation in itself and its derivative.

Define

$$H = x_N - x_1 \quad (B-1)$$

$$\tilde{m} = \frac{1}{H} (y_N - y_1) \quad (B-2)$$

$$\tilde{M} = \frac{1}{H} (m_N - m_1) \quad (B-3)$$

$$E = y_1^2 + y_N^2 + H^2 [(m_1 - \tilde{m})^2 + (m_N - \tilde{m})^2] \\ + H^4 [(M_1 - \tilde{M})^2 + (M_N - \tilde{M})^2] \quad (B-4)$$

It is proposed that E be minimized subject to the constraints of Eq. (24) with $i = 1$ and $i = N$. Using the Lagrange method of multipliers to minimize E , one defines

$$F = E - 2\lambda_1 (A_1 y_1 + B_1 m_1 + C_1 M_1 - D_1) \\ - 2\lambda_N (A_N y_N + B_N m_N + C_N M_N - D_N) \quad (B-5)$$

which introduces two extra unknowns, λ_1 and λ_N . Differentiating yields

$$\frac{\partial F}{\partial y_1} = 2y_1 + 2H(m_1 + m_N - 2\tilde{m}) - 2A_1 \lambda_1 = 0 \quad (B-6)$$

$$\frac{\partial F}{\partial y_N} = 2y_N - 2H(m_1 + m_N - 2\tilde{m}) - 2A_N \lambda_N = 0 \quad (B-7)$$

$$\frac{\partial F}{\partial m_1} = 2H^2(m_1 - \tilde{m}) + 2H^3(M_1 + M_N - 2\tilde{M}) - 2B_1 \lambda_1 = 0 \quad (B-8)$$

$$\frac{\partial F}{\partial m_N} = 2H^2(m_N - \tilde{m}) - 2H^3(M_1 + M_N - 2\tilde{M}) - 2B_N \lambda_N = 0 \quad (B-9)$$

$$\frac{\partial F}{\partial M_1} = 2H^4 (M_1 - \tilde{M}) - 2C_1 \lambda_1 = 0 \quad (B-10)$$

$$\frac{\partial F}{\partial M_N} = 2H^4 (M_N - \tilde{M}) - 2C_N \lambda_N = 0 \quad (B-11)$$

$$\frac{\partial F}{\partial \lambda_1} = -2(A_1 y_1 + B_1 m_1 + C_1 M_1 - D_1) = 0 \quad (B-12)$$

$$\frac{\partial F}{\partial \lambda_N} = -2(A_N y_N + B_N m_N + C_N M_N - D_N) = 0 \quad (B-13)$$

The last two equations are the equations of constraint. Equations (B-6) thru (B-13), after substitution of Eqs. (B-2) and (B-3) for $\tilde{m}$ and $\tilde{M}$, are eight linear equations in eight unknowns. One can simplify the system as follows. Adding Eqs. (B-8) and (B-9), one obtains

$$H(m_1 + m_N - 2\tilde{m}) = \frac{1}{H} (B_1 \lambda_1 + B_N \lambda_N) \quad (B-14)$$

Substituting Eq. (B-14) into Eqs. (B-6) and (B-7)

$$y_1 = A_1 \lambda_1 - \frac{1}{H} (B_1 \lambda_1 + B_N \lambda_N) \quad (B-15)$$

$$y_N = A_N \lambda_N + \frac{1}{H} (B_1 \lambda_1 + B_N \lambda_N) \quad (B-16)$$

Substituting Eqs. (B-15) and (B-16) into Eq. (B-2),

$$\tilde{m} = \frac{1}{H} (A_N \lambda_N - A_1 \lambda_1) + \frac{2}{H^2} (B_1 \lambda_1 + B_N \lambda_N) \quad (B-17)$$

Adding Eqs. (B-10) and (B-11),

$$H^2 (M_1 + M_N - 2\tilde{M}) = \frac{1}{H^2} (C_1 \lambda_1 + C_N \lambda_N) \quad (B-18)$$

Substituting Eqs. (B-17) and (B-18) into Eqs. (B-8) and (B-9),

$$\begin{aligned} m_1 = & \frac{1}{H} (A_N \lambda_N - A_1 \lambda_1) + \frac{2}{H^2} (B_1 \lambda_1 + B_N \lambda_N) \\ & - \frac{1}{H^3} (C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^2} B_1 \lambda_1 \end{aligned} \quad (B-19)$$

$$\begin{aligned} m_N = & \frac{1}{H} (A_N \lambda_N - A_1 \lambda_1) + \frac{2}{H^2} (B_1 \lambda_1 + B_N \lambda_N) \\ & + \frac{1}{H^3} (C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^2} B_N \lambda_N \end{aligned} \quad (B-20)$$

Substituting Eqs. (B-19) and (B-20) into Eq. (B-3),

$$\tilde{M} = \frac{2}{H^4} (C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^3} (B_N \lambda_N - B_1 \lambda_1) \quad (B-21)$$

Substituting Eq. (B-21) into Eqs. (B-10) and (B-11),

$$M_1 = \frac{2}{H^4} (C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^3} (B_N \lambda_N - B_1 \lambda_1) + \frac{1}{H^4} C_1 \lambda_1 \quad (B-22)$$

$$M_N = \frac{2}{H^4} (C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^3} (B_N \lambda_N - B_1 \lambda_1) + \frac{1}{H^4} C_N \lambda_N \quad (B-23)$$

If Eqs. (B-15), (B-16), (B-19), (B-20), (B-22), and (B-23) are substituted into Eqs. (B-12) and (B-13), then they take the form

$$A_{11} \lambda_1 + A_{1N} \lambda_N = D_1 \quad (B-24)$$

$$A_{N1} \lambda_1 + A_{NN} \lambda_N = D_N \quad (B-25)$$

where

$$A_{11} = A_1^2 + 3\bar{B}_1^2 + 3\bar{C}_1^2 - 2A_1 \bar{B}_1 - 2\bar{B}_1 \bar{C}_1 \quad (B-26)$$

$$A_{1N} = -A_1 \bar{B}_N + A_N \bar{B}_1 - \bar{B}_1 \bar{C}_N + \bar{B}_N \bar{C}_1 + 2\bar{B}_1 \bar{B}_N + 2\bar{C}_1 \bar{C}_N \quad (B-27)$$

$$A_{N1} = A_{1N} \quad (B-28)$$

$$A_{NN} = A_N^2 + 3\bar{B}_N^2 + 3\bar{C}_N^2 + 2A_N \bar{B}_N + 2\bar{B}_N \bar{C}_N \quad (B-29)$$

$$\bar{B}_i = \frac{1}{H} B_i \quad (B-30)$$

$$\bar{C}_i = \frac{1}{H^2} C_i \quad (B-31)$$

with $i = 1$ and $i = N$. Equations (B-24) and (B-25) can be solved for λ_1 and λ_N . One can then obtain y_1 , y_N , m_1 , m_N , M_1 , and M_N from Eqs. (B-15), (B-16), (B-19), (B-20), (B-22), and (B-23). These values determine the quintic which can be written down directly by analogy with Eq. (86). This is the initial solution suggested, for lack of a better approximation.

APPENDIX C CHANGING SPACING

Splines were originally invented for interpolation, and interpolation of spline solutions is immediately available. This, plus the suitability of spline collocation to handle nonuniform spacing, makes it feasible to change the spacing between iterations to improve accuracy.

Since smaller steps are needed where the function changes the fastest, the following procedure can be used. Consider straight line segments between points, (x_i, y_i) (that is, a broken-line solution). Compute the length of the broken line and divide by $(N-1)$ to determine a constant step size along its length. Taking constant steps along the broken-line solution, determine the new x_i . Interpolation provides the new in-hand solution for the next iteration.

Figure C-1 illustrates the respacing technique with $N = 6$. Beginning with equal spacing, the old solution is marked by the heavy dots. The new spacing is determined by taking equal steps along the broken line, marked by the symbol " Δ ". The new solution is obtained by interpolation at the new x_i and is marked by the symbol " $\square$ ". Note the shorter h_i at the ends, where the function is changing the fastest, and the longer h_i in the middle, where the function is changing the slowest.

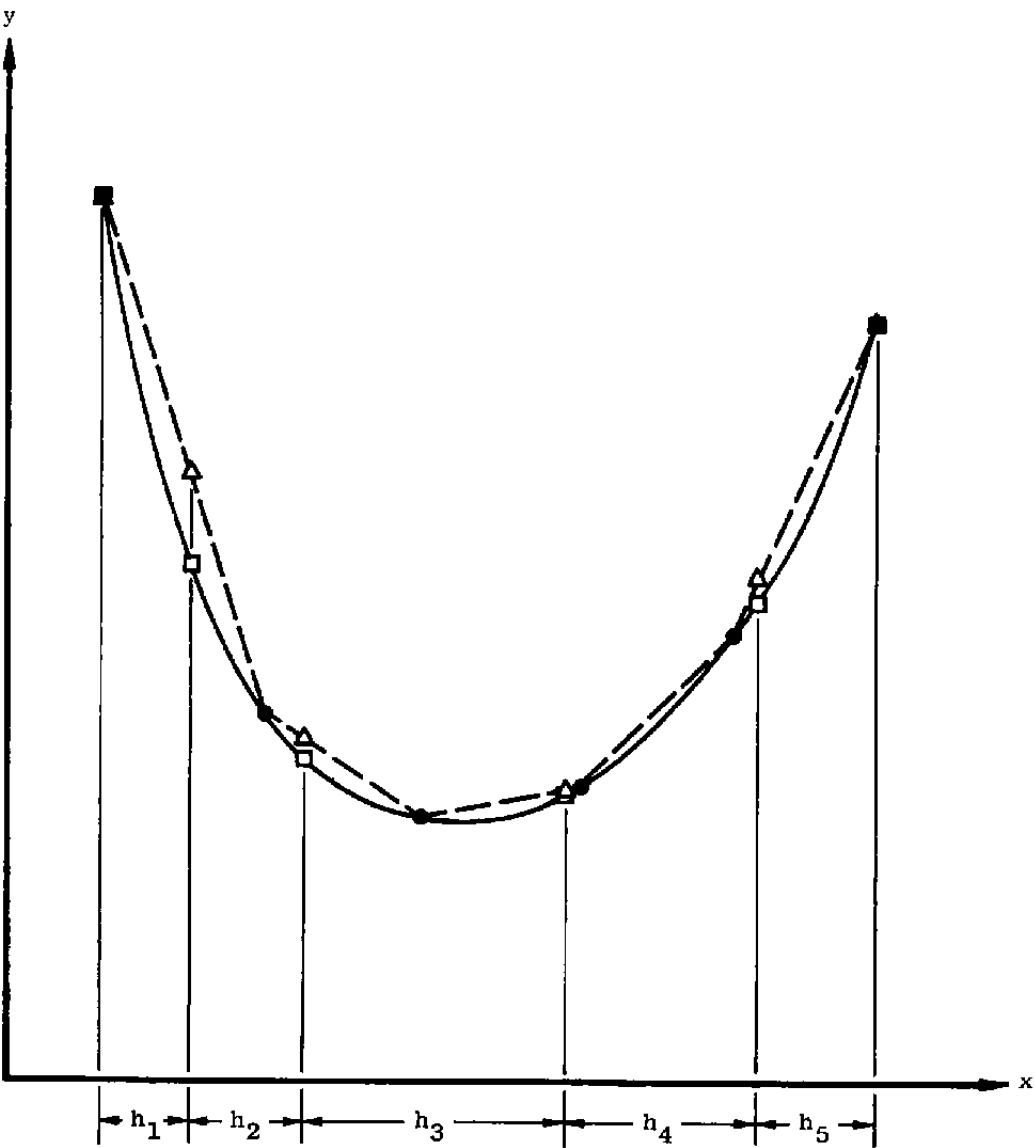


Figure C-1. Respacing technique.

APPENDIX D MODIFICATIONS TO RUN THE EXAMPLE PROBLEMS

MODIFICATIONS TO RUN EXAMPLE 2

-INS	1740	1
	DIMENSION YB(28),FB(28),SB(28)	2
	COMMON /ENDS/ G1	3
	COMMON /FIXED/ NDIM,JCASE	4
	COMMON /PARM/ PA(7),FA(7),XA(28),YA(7,28)	5
-REP	1800,1900	6
	M=15	7
	DO 14 K=16,28	8
	IF (YA(JCASE,K).EQ.1.D0) GO TO 16	9
14	M=M+1	10
	M=28	11
16	CALL QUINTS (N,X,Y,F,S,M,XA,YB,FB,SB)	12
	CALL ERROR (FA(JCASE),F(1),EA(2))	13
	DO 30 K=1,M	14
	CALL ERROR (YA(JCASE,K),YB(K),E(1))	15
	DO 20 J=1,1	16
-REP	2310,2350	17
	COMMON /PARM/ PA(7),FA(7),XA(28),YA(7,28)	18
	DIMENSION XN(7)	19
	DATA XN/8.D0,7.D0,6.SD0,6.D0,5.SD0,4.800,3.D0/	20
	IF (JCASE.GT.7) GO TO 590	21
	IF (JCASE.GT.1) GO TO 30	22
	CALL LABELS (LAB,36HEXAMPLE 2	23
	CALL LABELS (LAB(10),36HFALKNER-SKAN EQUATION	24
	READ (5,5001) PL,PA	25
	READ (5,5001) FL,FA	26
	DO 10 K=1,28	27
10	READ (5,5003) XA(K),(YA(J,K),J=1,7)	28
	DO 15 K=16,28	29
	DO 15 J=5,7	30
	IF (YA(J,K).EQ.0.D0) YA(J,K)=1.D0	31
15	CONTINUE	32
	CALL PAGER (100)	33
	WRITE (6,6001) PL,PA	34
	WRITE (6,6001) FL,FA	35
	WRITE (6,6002)	36
	DO 20 K=1,28	37
20	WRITE (6,6003) K,XA(K),(YA(J,K),J=1,7)	38
30	CONTINUE	39
-REP	2450,2460	40
	JOUT=20	41
	LPRNT=0	42
-REP	2590	43
	DN=1.D0	44
-INS	2610	45
	WRITE (6,6004) PA(JCASE)	46
-REP	2720	47
	X(N)=XN(JCASE)	48
-REP	2740	49
	A=3.D0	50

MODIFICATIONS TO RUN EXAMPLE 2

DO 50 J=1,N	51
EX=DEXP(-A*X(J))	52
Y(J)=1.00-EX	53
F(J)=A*EX	54
S(J)=-EX*A**2	55
G(J)=X(J)+(EX-1.00)/A	56
50 CONTINUE	57
-INS 2890	58
5001 FORMAT (A4,5X,7E9.0)	59
5003 FORMAT (8E9.0)	60
6001 FORMAT (1H0,A4,5X,7F10.5)	61
6002 FORMAT (1H0,7X,42HX Y(X) CORRESPONDING TO ABOVE K VALUES)	62
6003 FORMAT (I3,F7.1,7F10.5)	63
6004 FORMAT ('CK =' ,F10.5)	64
-INS 4340	65
COMMON /FIXED/ NDIM,JCASE	66
COMMON /PARAM/ PA(7),FA(7),XA(28),YA(7,28)	67
-INS 4360	68
P=PA(JCASE)	69
-REP 4380,4420	70
ALFA(J)=G(J)*F(J)-P*(1.00+Y(J)**2)	71
BETA(J)=2.00*P*Y(J)	72
GAMA(J)=-G(J)	73
EPSA(J)=-F(J)	74

MODIFICATIONS TO RUN EXAMPLE 3

-INS	1740	1
	COMMON /PARM/ P,Q	2
-REP	1810,1890	3
	Z=DEXP(-Q*X(K))	4
	A(1)=Q*Z	5
	A(2)=-Q*A(1)	6
	A(3)=-Q*A(2)	7
	A(4)=1.00-Z	8
	E(1)=DABS(A(1)-Y(K))	9
	E(2)=DABS(A(2)-F(K))	10
	E(3)=DABS(A(3)-S(K))	11
	E(4)=DABS(A(4)-G(K))	12
-INS	2300	13
	COMMON /PARM/ P,Q	14
-REP	2320	15
	IF (JCASE.EQ.1) GO TO 20	16
	IF (JCASE.EQ.2) GO TO 10	17
	IF (JCASE.GT.3) GO TO 590	18
	P=.100	19
	GO TO 30	20
10	P=.0100	21
	GO TO 30	22
20	P=.00100	23
30	Q=(1.00-DSQRT(1.00-4.00*P))/(2.00*P)	24
-REP	2340,2350	25
	CALL LABELS (LAB,36EXAMPLE 3	26
	CALL LABELS (LAB(10),36HVISCOELASTIC FLUID	27
-REP	2550	28
	D1=Q	29
-INS	2610	30
	WRITE (6,700) P,Q	31
-REP	2720	32
	X(N)=12.00/Q	33
-REP	2740	34
	DO 40 J=1,N	35
	Y(J)=1.00	36
	F(J)=1.00	37
	S(J)=1.00	38
40	G(J)=1.00	39
-INS	2890	40
700	FORMAT (4HOK =,1PE12.4,5X,8HLAMBDA =,E12.4)	41
-INS	4340	42
	COMMON /PARM/ P,Q	43
-REP	4380,4410	44
	ALFA(J)=0.00	45
	BETA(J)=-1.00/P	46
	GAMA(J)=BETA(J)	47
-REP	6750,6780	48
	D(3,N,4)=1.00	49
-REP	6810	50
	R(3,N)=1.00	51

MODIFICATIONS TO RUN EXAMPLE 4

-REP	1740	1
	DIMENSION XA(3),YA(3,3),YB(3),FB(3),SB(3),EA(4),KA(4)	2
	COMMON /FIXED/ NDIM,JCASE	3
	DATA XA/0.00,.500,1.00/	4
	DATA YA/.62004200,.47314330,.41841500,.90540700,.53264600,.3399650	5
	10,.99906800,.55918300,.30747600/	6
-REP	1800,1950	7
	CALL QUINTS (N,X,Y,F,S,3,XA,YB,FB,SB)	8
	DO 20 J=1,3	9
20	CALL ERROR (YA(J,JCASE),YB(J),EA(J))	10
-INS	2300	11
	COMMON /PARM/ P	12
	DIMENSION PA(3)	13
	DATA PA/1.00,.100,.00100/	14
-REP	2320	15
	IF (JCASE.GT.3) GO TO 590	16
	P=PA(JCASE)	17
-REP	2340,2350	18
	CALL LABELS (LAB,36HEXAMPLE 4	19
	CALL LABELS (LAB(10),36HCHEMICAL DISPERSION	20
-REP	2450,2460	21
	JOUT=20	22
	LPRNT=1	23
-REP	2530,2590	24
	B1=-P	25
	C1=0.00	26
	D1=1.00	27
	AN=0.00	28
	BN=1.00	29
	CN=0.00	30
	DN=0.00	31
-INS	2610	32
	WRITE (6,6004) P	33
-REP	2720	34
	X(N)=1.00	35
-INS	2890	36
6004	FORMAT ('OK =',F10.5)	37
-INS	4340	38
	COMMON /PARM/ P	39
-REP	4380,4410	40
	BETA(J)=1.2500/(P*(1.00+.100*Y(J))**2)	41
	A_LFA(J)=.100*BETA(J)*Y(J)**2	42
	GAMA(J)=1.00/P	43

MODIFICATIONS TO RUN EXAMPLE 5

-REP	1747	1
	DIMENSION XA(20),YA(20),YB(20),DM(20)	2
	DATA XA/1.00,1.200,1.400,1.600,1.800,2.000,2.200,2.400,2.600,2.800	3
	1.3.000,3.200,3.400,4.00,5.000	4
	DATA YA/1.00,.71400,.497100,.314600,.189700,.103900,.047500,.01340	5
	10,-.005500,-.015500,-.012200,-.019200,-.018100,-.013100,-.006800/	6
-REP	1770,1980	7
	CALL QUINTS (N,X,Y,F,5,15,XA,YB,DM,DM)	8
	WRITE (6,400)	9
	DO 50 J=1,15	10
	D=YA(J)-YB(J)	11
	WRITE (6,410) J,XA(J),YA(J),YB(J),D	12
50	CONTINUE	13
	RETURN	14
400	FORMAT ('COMPARISON OF SOLUTION WITH TABLE IN REF 7'/1H0,8X,1HJ,8	15
	1X,1HX,6X,2HYA,8X,2HYB,8X,4HDIFF)	16
410	FORMAT ('111,F10.1,3F10.4)	17
-REP	2340,2350	18
	CALL LABELS (LAB,26)EXAMPLE 5	19
	CALL LABELS (LAB(10),36)INHERENTLY UNSTABLE PROBLEM	20
-REP	2450,2460	21
	JOUT=20	22
	LPRNT=1	23
-REP	2550	24
	DI=1.00	25
-REP	2590	26
	DN=2.00	27
-REP	2710,2720	28
	X(1)=1.00	29
	X(N)=20.00	30
-REP	4380,4410	31
	ALFA(J)=1.00/(1.00+X(J))	32
	BETA(J)=X(J)**2	33
	GAMA(J)=0.00	34

MODIFICATIONS TO RUN EXAMPLE 6

```

-INS 1740 1
COMMON /PARM/ P,PS,PI,TP,C1,C2 2
-INS 1790 3
E1=DEXP(-P) 4
D=1.00+E1 5
C0=(1.00-E1)/(P*D) 6
-REP 1810,1890 7
E1=P*(X(K) 8
CX=DCOS(E1) 9
SX=DSIN(E1) 10
CC=CX**2 11
CS=CX*SX 12
E1=DEXP(P*(X(K)-1.00)) 13
E2=DEXP(-P*X(K)) 14
S1=(E1+E2)/D 15
S2=(E1-E2)/D 16
A(1)=S1-CC 17
A(2)=P*S2+TP*CS 18
A(3)=PS*S1+2.00*C2*CC-C2 19
A(4)=S2/P-CS/TP-.500*X(K)+C0 20
E(1)=DABS(A(1)-Y(K)) 21
E(2)=DABS(A(2)-F(K)) 22
E(3)=DABS(A(3)-S(K)) 23
E(4)=DABS(A(4)-G(K)) 24
-INS 2300 25
COMMON /PARM/ P,PS,PI,TP,CA,C2 26
DIMENSION PA(4) 27
DATA PA/10.00,15.00,20.00,25.00/ 28
-REP 2320 29
IF (JCASE.GT.1) GO TO 10 30
PI=4.00*DATAN(1.00) 31
TP=2.00*PI 32
C2=TP*PI 33
10 IF (JCASE.GT.4) GO TO 590 34
P=PA(JCASE) 35
PS=P**2 36
CA=PS*2.00*C2 37
-REP 2340,2350 38
CALL LABELS (LAB,36,EXAMPLE 6 39
CALL LABELS (LAB(10),36H 40
-REP 2450,2460 41
JOUT=20 42
LPRNT=1 43
-REP 2590 44
DN=0.00 45
-INS 2610 46
WRITE (6,6004) P 47
-REP 2720 48
X(N)=1.00 49
-INS 2890 50

```

MODIFICATIONS TO RUN EXAMPLE 6

6004	FORMAT ('OK =',F10.1)	S1
-INS	4340	S2
	COMMON /PARM/ P,PS,PI,TP,C1,C2	S3
-REP	4380,4410	S4
	ALFA(J)=C1*DCOS(PI*X(J))**2-C2	S5
	BETA(J)=PS	S6
	GAMA(J)=0.00	S7

MODIFICATIONS TO RUN EXAMPLE 7

-INS	1740	1
	COMMON /PARM/ Q,SR	2
-REP	1810,1850	3
	AG=SR*(X(K)-.5D0)	4
	TAG=DTAN(AG)	5
	SECS=TAG**2+1.D0	6
	A(2)=2.D0*SR*TAG	7
	A(3)=Q*SECS	8
	A(4)=G(K)	9
	A(1)=DLOG(A(3))	10
-INS	2300	11
	COMMON /PARM/ Q,SP	12
	Q=.5D0	13
	DO 10 J=1,1000	14
	P=Q	15
	SR=DSORT(.5D0*P)	16
	Q=DCOS(.5D0*SR)**2	17
	IF (P.EQ.Q) GO TO 20	18
10	CONTINUE	19
	J=1000	20
20	CONTINUE	21
-REP	2340,2750	22
	CALL LABELS (LAB,26EXAMPLE 7	23
	CALL LABELS (LAB(10),26HY'' = EXP(Y)	24
-REP	2430,2460	25
	JOUT=20	26
	LPRNT=1	27
-REP	2590	28
	DN=Q.D0	29
-INS	2610	30
	WRITE (6,F000) J,Q,P,Q	31
-REP	2720	32
	X(N)=1.D0	33
-INS	2990	34
6000	FORMAT (4H0J =,15,5X,9HLAMBDA =,1PE22,13,10X,2Z20)	35
-REP	4380,4410	36
	BETA(J)=DEXP(Y(J))	37
	ALFA(J)=(1.D0-Y(J))*BETA(J)	38
	GAMA(J)=Q.D0	39

APPENDIX E

PRINTOUTS OF THE EXAMPLE PROBLEMS

DC TODD COPS
PROBLEM FROM KAMKE

AR008671 10:29 WED MAY 10. 1978

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	1	51	30	0	1	1
11	TOL	RSC				
0.0	5.00000D-04	1.00000D-02				
A	B	C	D			
1.00000D 00	0.0	0.0	0.0			
1.00000D 00	0.0	0.0	7.00000D 00			

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	35	51	51	3.560-01	2.310 15	4.250-01
NEW SPACING COMPUTED							
UPDATE	2	34	51	51	1.850-01	1.300 00	2.920-01
NEW SPACING COMPUTED							
UPDATE	3	38	51	51	3.410-02	2.760-01	3.870-02
NEW SPACING COMPUTED							
UPDATE	4	41	51	51	1.230-03	1.490-02	1.280-03
UPDATE	5	44	51	51	1.250-06	2.710-05	1.790-06

CONVERGED SOLUTION

CHEKDE	51	6.76D-10	0.0		0.0		0.0	
CHEKBM	11	51	51	15	5.01D-04	6.44D-04	1.28D-03	4.82D-04

DC TOJD COPS
PROBLEM FROM KAMKE

ARD08671 10:29 WED MAY 10, 1978

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	9.99570-01	0.0	0.0	1.0360-01	9.890-01
2	1.03600-01	1.03930-01	1.01040 00	2.09970-01	5.37400-03	1.0250-01	9.780-01
3	2.06080-01	2.08960-01	1.04320 00	4.35890-01	2.13770-02	1.0030-01	9.670-01
4	3.06330-01	3.16150-01	1.09950 00	6.95070-01	4.76520-02	9.6900-02	9.540-01
5	4.03230-01	4.26410-01	1.18140 00	1.00730 00	8.35630-02	9.2440-02	9.410-01
6	4.95660-01	5.40440-01	1.29160 00	1.39580 00	1.28170-01	8.6980-02	9.280-01
7	5.82650-01	6.58650-01	1.43340 00	1.88790 00	1.80230-01	8.0740-02	9.160-01
8	6.63390-01	7.81170-01	1.60990 00	2.51490 00	2.38260-01	7.4000-02	9.060-01
9	7.37380-01	9.07860-01	1.82400 00	3.31150 00	3.30650-01	6.7020-02	8.980-01
10	8.04410-01	1.03820 00	2.07790 00	4.31460 00	3.65780-01	6.0220-02	8.930-01
11	8.64630-01	1.17190 00	2.37350 00	5.56310 00	4.32230-01	5.3780-02	8.890-01
12	9.18410-01	1.30830 00	2.71200 00	7.09690 00	4.98850-01	4.7840-02	8.890-01
13	9.56250-01	1.44680 00	3.09390 00	8.95420 00	5.64680-01	4.2520-02	8.890-01
14	1.00880 00	1.58710 00	3.51990 00	1.11750 01	6.29120-01	3.7790-02	8.910-01
15	1.04660 00	1.72870 00	3.98980 00	1.37980 01	6.91730-01	3.3660-02	8.930-01
16	1.08020 00	1.87140 00	4.50380 00	1.68620 01	7.52260-01	3.0050-02	8.960-01
17	1.11030 00	2.01480 00	5.06160 00	2.04040 01	8.10630-01	2.6920-02	8.990-01
18	1.13720 00	2.15890 00	5.66350 00	2.44630 01	8.66740-01	2.4190-02	9.020-01
19	1.16140 00	2.30350 00	6.30900 00	2.90770 01	9.20670-01	2.1820-02	9.060-01
20	1.18320 00	2.44850 00	6.99860 00	3.42870 01	9.72490-01	1.9760-02	9.090-01
21	1.20300 00	2.59380 00	7.73210 00	4.01310 01	1.02230 00	1.7960-02	9.120-01
22	1.22090 00	2.73950 00	8.50960 00	4.66470 01	1.07020 00	1.6380-02	9.150-01
23	1.23730 00	2.88550 00	9.33120 00	5.38770 01	1.11620 00	1.4990-02	9.180-01
24	1.25230 00	3.03170 00	1.01970 01	6.18600 01	1.16060 00	1.3760-02	9.170-01
25	1.26610 00	3.17810 00	1.11070 01	7.06350 01	1.20330 00	1.2620-02	9.220-01
26	1.27870 00	3.32410 00	1.20570 01	8.02000 01	1.24430 00	1.1630-02	9.260-01
27	1.29030 00	3.47700 00	1.30490 01	9.06090 01	1.28380 00	1.0780-02	9.290-01
28	1.30110 00	3.61610 00	1.40850 01	1.01920 02	1.32190 00	1.0010-02	9.310-01
29	1.31110 00	3.76230 00	1.51040 01	1.14170 02	1.35880 00	9.3140-03	9.320-01
30	1.32040 00	3.90870 00	1.62880 01	1.27400 02	1.39460 00	8.6850-03	9.350-01
31	1.32910 00	4.05510 00	1.74550 01	1.41640 02	1.42910 00	8.1150-03	9.360-01
32	1.33720 00	4.20160 00	1.86650 01	1.56940 02	1.46260 00	7.5000-03	9.390-01
33	1.34480 00	4.34810 00	1.99190 01	1.73320 02	1.49510 00	7.1320-03	9.390-01
34	1.35190 00	4.49480 00	2.12160 01	1.90840 02	1.52660 00	6.7010-03	9.410-01
35	1.35860 00	4.64130 00	2.25560 01	2.09510 02	1.55720 00	6.3070-03	9.440-01
36	1.36490 00	4.78790 00	2.39390 01	2.29370 02	1.58700 00	5.9520-03	9.440-01
37	1.37090 00	4.93460 00	2.53660 01	2.50490 02	1.61590 00	5.6190-03	9.460-01
38	1.37650 00	5.08120 00	2.68350 01	2.72880 02	1.64400 00	5.3180-03	9.480-01
39	1.38180 00	5.22790 00	2.83490 01	2.96590 02	1.67150 00	5.0410-03	9.490-01
40	1.38690 00	5.37460 00	2.99060 01	3.21660 02	1.69820 00	4.7830-03	9.510-01
41	1.39170 00	5.52150 00	3.15060 01	3.48130 02	1.72420 00	4.5500-03	9.510-01
42	1.39620 00	5.66850 00	3.31530 01	3.76090 02	1.74970 00	4.3290-03	9.540-01
43	1.40050 00	5.81560 00	3.48440 01	4.05530 02	1.77450 00	4.1320-03	9.560-01
44	1.40470 00	5.96320 00	3.65820 01	4.36560 02	1.79890 00	3.9430-03	9.510-01
45	1.40860 00	6.11090 00	3.83670 01	4.69200 02	1.82270 00	3.7510-03	9.560-01
46	1.41240 00	6.25820 00	4.01900 01	5.03340 02	1.84590 00	3.5840-03	9.580-01
47	1.41590 00	6.40550 00	4.20570 01	5.39130 02	1.86860 00	3.4340-03	9.510-01
48	1.41940 00	6.55320 00	4.39720 01	5.76670 02	1.89080 00	3.3000-03	9.610-01
49	1.42270 00	6.70150 00	4.59390 01	6.16100 02	1.91270 00	3.1710-03	9.640-01
50	1.42580 00	6.85030 00	4.79570 01	6.57450 02	1.93420 00	3.0560-03	
51	1.42890 00	7.00000 00	5.00320 01	7.00890 02	1.95530 00		

STOP

DC TDDO CDPS
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K	-0.19884	-0.18000	0.0	0.30000	1.00000	2.00000	10.00000
F0	0.0	0.12864	0.46960	0.77476	1.23259	1.68722	3.67523
X	Y(X) CORRESPONDING TO ABOVE K VALUES						
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.1	0.00099	0.01376	0.04696	0.07597	0.11826	0.15876
3	0.2	0.00398	0.02933	0.09391	0.14894	0.22661	0.29794
4	0.3	0.00895	0.04668	0.14081	0.21886	0.32524	0.41854
5	0.4	0.01591	0.06582	0.18761	0.28569	0.41446	0.52190
6	0.5	0.02485	0.08673	0.23423	0.34938	0.49465	0.60964
7	0.6	0.03578	0.10937	0.28058	0.40988	0.56628	0.68343
8	0.7	0.04868	0.13373	0.32653	0.46713	0.62986	0.74496
9	0.8	0.06355	0.15975	0.37196	0.52107	0.68594	0.79587
10	0.9	0.08038	0.18737	0.41672	0.57167	0.73508	0.83767
11	1.0	0.09913	0.21651	0.46063	0.61890	0.77787	0.87172
12	1.2	0.14232	0.27899	0.54525	0.70322	0.84567	0.92142
13	1.4	0.19274	0.34622	0.62439	0.77425	0.89681	0.95308
14	1.6	0.24982	0.41691	0.69670	0.83254	0.93235	0.97269
15	1.8	0.31271	0.48946	0.76106	0.87906	0.95683	0.98452
16	2.0	0.38026	0.56205	0.81669	0.91509	0.97322	0.99146
17	2.2	0.45097	0.63269	0.86330	0.94211	0.98385	0.99542
18	2.4	0.52308	0.69942	0.90107	0.96173	0.99055	0.99761
19	2.6	0.59450	0.76048	0.93060	0.97548	0.99463	0.99879
20	2.8	0.66348	0.81449	0.95288	0.98480	0.99705	0.99940
21	3.0	0.72776	0.86061	0.96905	0.99088	0.99842	0.99972
22	3.2	0.78578	0.89853	0.98037	0.99471	0.99919	0.99987
23	3.4	0.83635	0.92854	0.98797	0.99704	0.99959	0.99995
24	3.6	0.87882	0.95138	0.99289	0.99840	0.99980	0.99998
25	3.8	0.91315	0.96605	0.99594	0.99916	0.99991	0.99999
26	4.0	0.93982	0.97575	0.99777	0.99958	0.99995	1.00000
27	4.5	0.97940	0.99449	0.99957	0.99994	0.99999	1.00000
28	5.0	0.99439	0.99997	0.99994	0.99999	1.00000	1.00000

DC TODD CDPS
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = -0.19884

NOIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	1	51	30	20	0	1
11	TOL	RSC				
0.0	5.00000-04	1.00000-02				
A	B	C	D			
1.00000 00	0.0	0.0	0.0			
1.00000 00	0.0	0.0	1.00000 00			

UNIFORM SPACING COMPUTED

UPDATE	1	5	1	41	5.730-01	8.810-01	1.040-01
NEW SPACING COMPUTED							
UPDATE	2	11	5	51	2.690-01	4.410-01	7.670-02
NEW SPACING COMPUTED							
UPDATE	3	14	20	51	1.390-01	2.230-01	5.140-02
NEW SPACING COMPUTED							
UPDATE	4	16	23	51	6.970-02	1.150-01	3.050-02
NEW SPACING COMPUTED							
UPDATE	5	16	23	51	3.470-02	5.520-02	1.640-02
NEW SPACING COMPUTED							
UPDATE	6	16	23	51	1.740-02	2.730-02	8.380-03
NEW SPACING COMPUTED							
UPDATE	7	16	23	51	8.660-03	1.360-02	4.210-03
UPDATE	8	16	23	51	4.280-03	6.750-03	2.090-03
UPDATE	9	16	23	51	2.050-03	3.230-03	1.000-03
UPDATE	10	16	23	51	8.710-04	1.370-03	4.270-04
UPDATE	11	16	23	51	2.450-04	3.860-04	1.200-04

CONVERGED SOLUTION

CHEKDE	16	1.490-06	0.0	0.0	0.0		
CHEKBM	2	0	0	0	7.690-02	7.190-04	0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = -0.18000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	2	51	30	20	0	1

I1	TOL	RSC
0.0	5.00000D-04	1.00000D-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	1.00000 00

UNIFORM SPACING COMPUTED

UPDATE	1	5	1	46	5.64D-01	8.73D-01	1.17D-01
NEW SPACING COMPUTED							
UPDATE	2	12	6	51	2.45D-01	4.05D-01	8.10D-02
NEW SPACING COMPUTED							
UPDATE	3	16	23	51	9.82D-02	1.61D-01	4.19D-02
NEW SPACING COMPUTED							
UPDATE	4	18	26	51	2.09D-02	3.59D-02	1.05D-02
NEW SPACING COMPUTED							
UPDATE	5	19	27	51	1.02D-03	1.68D-03	5.55D-04
UPDATE	6	19	28	51	2.51D-06	3.85D-06	1.40D-06

CONVERGED SOLUTION

CHEKDE	51	1.37D-09	0.0	0.0	0.0
CHEKBM	28	0	0	0	1.16D-03 5.40D-05 0.0 0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = 0.0

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	3	51	30	20	0	1
I1	TOL	RSC				
0.0	5.00000D-04	1.00000D-02				
A	B	C	D			
1.00000D 00	0.0	0.0	0.0			
1.00000D 00	0.0	3.0	1.00000D 00			

UNIFORM SPACING COMPUTED

UPDATE	1	5	1	47	4.81D-01	8.04D-01	1.02D-01
NEW SPACING COMPUTED							
UPDATE	2	12	5	51	1.24D-01	2.08D-01	4.14D-02
NEW SPACING COMPUTED							
UPDATE	3	17	23	51	1.26D-02	1.88D-02	5.11D-03
NEW SPACING COMPUTED							
UPDATE	4	20	26	51	1.32D-04	1.98D-04	5.96D-05

CONVERGED SOLUTION

CHEKDE	51	2.21D-06	0.0	0.0	0.0		
CHEKBM	3	0	0	0	4.37D-05	3.12D-05	0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD CDPS
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = 0.30000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNDRM
250	4	51	30	20	0	1

II	TOL	RSC
0.0	5.00000-04	1.00000-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	1.00000 00

UNIFORM SPACING COMPUTED

UPDATE	1	5	1	48	3.91D-01	7.17D-01	8.30D-02
NEW SPACING COMPUTED							
UPDATE	2	12	3	51	5.99D-02	8.62D-02	1.99D-02
NEW SPACING COMPUTED							
UPDATE	3	16	8	51	2.03D-03	1.92D-03	8.05D-04
UPDATE	4	20	27	51	2.43D-06	2.27D-06	1.07D-06

CONVERGED SOLUTION

CHEKDE	50	6.07D-09	0.0	0.0	0.0
CHEKBM	2	0	0	0	6.26D-05 3.88D-06 0.0 0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD CD'S
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = 1.00000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	5	51	30	20	0	1

II	TOL	RSC
0.0	5.00000-04	1.00000-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	1.00000 00

UNIFORM SPACING COMPUTED

UPDATE	1	5	1	44	2.69D-01	5.75D-01	5.46D-02
NEW SPACING COMPUTED							
UPDATE	2	11	1	51	2.16D-02	3.28D-02	6.55D-03
NEW SPACING COMPUTED							
UPDATE	3	14	1	51	1.92D-04	1.86D-04	7.03D-05

CONVERGED SOLUTION

CHEKDE	51	3.12D-06	0.0	0.0	0.0
CHEKBM	2	0	0	0	4.62D-05 1.09D-05 0.0 0.0

SOLUTION WRITTEN ON UNIT 20

DC TQDD CDPS
EXAMPLE 2

AR000346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = 2.00000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	6	51	30	20	0	1

II	TOL	RSC
0.0	5.00000D-04	1.00000D-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	1.00000 00

UNIFORM SPACING COMPUTED

UPDATE	1	5	1	39	1.710-01	4.300-01	3.460-02
NEW SPACING COMPUTED							
UPDATE	2	10	2	51	7.950-03	1.250-02	2.220-03
UPDATEF	3	13	1	51	2.210-05	2.390-05	7.330-06

CONVERGED SOLUTION

CHEKDE	51	3.420-06	0.0	0.0	0.0
CHEKBN	2	0	0	0	2.420-05 1.830-05 0.0 0.0

SOLUTION WRITTEN ON UNIT 20

DC T000 COPS
EXAMPLE 2

ARD00346 13:38 MON AUG 07, 1978
FALKNER-SKAN EQUATION

K = 10.00000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNDRM
250	7	51	30	20	0	1
I1	TOL	RSC				
0.0	5.00000-04	1.00000-02				
A	B	C	D			
1.00000 00	0.0	0.0	0.0			
1.00000 00	0.0	0.0	1.00000 00			

UNIFORM SPACING COMPUTED

UPDATE	1	7	1	51	1.180-01	2.310-01	3.610-02
NEW SPACING COMPUTED							
UPDATE	2	14	3	51	5.000-03	5.240-03	1.390-03
UPDATE	3	15	1	51	8.890-06	7.860-06	2.460-06

CONVERGED SOLUTION

CHEKDE	51	1.940-08	0.0	0.0	0.0		
CHEKBN	15	0	0	0	1.610-04	2.530-05	0.0

SOLUTION WRITTEN ON UNIT 20

STOP

DC TODD COPS
EXAMPLE 3

ARD00346 13:42 MON AUG 07, 1978
VISCOELASTIC FLUID

K = 1.0000D-03 LAMBDA = 1.0010D 00

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	1	51	30	0	1	1

II	TOL	RSC
0.0	5.0000D-04	1.0000D-02

A	B	C	D
1.0000D 00	0.0	0.0	1.0010D 00
1.0000D 00	0.0	0.0	7.0000D 00

UNIFORM SPACING COMPUTED

UPDATE	1	51	1	1	1.00D 00	2.00D 00	1.00D 00
NEW SPACING COMPUTED							
UPDATE	2	51	38	38	5.14D-04	3.03D-03	1.92D-05
UPDATE	3	0	0	0	0.0	0.0	0.0

CONVERGED SOLUTION

CHEKDE	18	1.81D-12	0.0	0.0	7.00D 00		
CHEKBM	51	50	51	50	1.29D-03	1.03D-03	2.58D-01 1.80D-05

DC TODD COPS
EXAMPLE 3ARD00346 13:42 MON AUG 07, 1978
VISCOELASTIC FLUID

	X	Y	YP	YPP	I	H	S
1	0.0	1.00100 00	-1.00200 00	1.00820 00	0.0	1.8240-01	1.070 00
2	1.82420-01	8.33980-01	-8.34810-01	8.30440-01	1.66900-01	1.9450-01	1.060 00
3	3.76910-01	6.86390-01	-6.87090-01	6.93660-01	3.14280-01	2.0680-01	1.050 00
4	5.83710-01	5.58100-01	-5.58650-01	5.53140-01	4.42510-01	2.1770-01	1.040 00
5	8.01380-01	4.48760-01	-4.49220-01	4.56430-01	5.51660-01	2.2620-01	1.030 00
6	1.02760 00	3.57910-01	-3.58260-01	3.51450-01	6.42510-01	2.3240-01	1.020 00
7	1.26000 00	2.83520-01	-2.83810-01	2.91920-01	7.16720-01	2.3670-01	1.010 00
8	1.49670 00	2.23820-01	-2.24030-01	2.15890-01	7.76480-01	2.3950-01	1.010 00
9	1.73620 00	1.76000-01	-1.76180-01	1.85390-01	8.24130-01	2.4130-01	1.000 00
10	1.97750 00	1.38350-01	-1.38480-01	1.28940-01	8.61870-01	2.4240-01	1.000 00
11	2.21990 00	1.08410-01	-1.08530-01	1.19050-01	8.91630-01	2.4310-01	1.000 00
12	2.46300 00	8.51340-02	-8.52080-02	7.41270-02	9.15040-01	2.4360-01	1.000 00
13	2.70660 00	6.65600-02	-6.66390-02	7.86990-02	9.33430-01	2.4380-01	1.000 00
14	2.95040 00	5.23090-02	-5.23480-02	3.95380-02	9.47850-01	2.4400-01	1.000 00
15	3.19440 00	4.07960-02	-4.08510-02	5.46980-02	9.59150-01	2.4410-01	1.000 00
16	3.43860 00	3.21410-02	-3.21590-02	1.73830-02	9.68010-01	2.4420-01	1.000 00
17	3.68270 00	2.49680-02	-2.50090-02	4.09240-02	9.74950-01	2.4420-01	1.000 00
18	3.92690 00	1.97710-02	-1.97740-02	2.74980-03	9.80380-01	2.4420-01	1.000 00
19	4.17120 00	1.52490-02	-1.52830-02	3.35850-02	9.84640-01	2.4430-01	1.000 00
20	4.41540 00	1.21920-02	-1.21850-02	-7.41900-03	9.87970-01	2.4430-01	1.000 00
21	4.65970 00	9.27860-03	-9.30890-03	3.03630-02	9.90580-01	2.4430-01	1.000 00
22	4.90400 00	7.55440-03	-7.53940-03	-1.50290-02	9.92630-01	2.4430-01	1.000 00
23	5.14820 00	5.60610-03	-5.63600-03	2.98610-02	9.94230-01	2.4430-01	1.000 00
24	5.39250 00	4.72210-03	-4.70080-03	-2.12750-02	9.95480-01	2.4430-01	1.000 00
25	5.63680 00	3.34150-03	-3.37280-03	3.12480-02	9.96460-01	2.4430-01	1.000 00
26	5.88110 00	2.99880-03	-2.97190-03	-2.69250-02	9.97230-01	2.4430-01	1.000 00
27	6.12530 00	1.93830-03	-1.97240-03	3.40500-02	9.97830-01	2.4430-01	1.000 00
28	6.36960 00	1.95770-03	-1.92520-03	-3.24830-02	9.98310-01	2.4430-01	1.000 00
29	6.61390 00	1.06120-03	-1.09920-03	3.80150-02	9.98670-01	2.4430-01	1.000 00
30	6.85820 00	1.33710-03	-1.29880-03	-3.83000-02	9.98970-01	2.4430-01	1.000 00
31	7.10250 00	5.04100-04	-5.47130-04	4.30300-02	9.99190-01	2.4430-01	1.000 00
32	7.34670 00	9.77250-04	-9.32610-04	-4.46400-02	9.99370-01	2.4430-01	1.000 00
33	7.59100 00	1.40370-04	-1.89450-04	4.90800-02	9.99500-01	2.4430-01	1.000 00
34	7.83530 00	7.80280-04	-7.28560-04	-5.17180-02	9.99620-01	2.4430-01	1.000 00
35	8.07960 00	-1.08120-04	5.19120-05	5.62120-02	9.99700-01	2.4430-01	1.000 00
36	8.32380 00	6.86810-04	-6.27070-04	-5.97310-02	9.99770-01	2.4430-01	1.000 00
37	8.56810 00	-2.89790-04	2.25270-04	6.45250-02	9.99810-01	2.4430-01	1.000 00
38	8.81240 00	6.60910-04	-5.92040-04	-6.88700-02	9.99860-01	2.4430-01	1.000 00
39	9.05670 00	-4.34900-04	3.60750-04	7.41560-02	9.99890-01	2.4430-01	1.000 00
40	9.30100 00	6.81190-04	-6.01850-04	-7.93380-02	9.99920-01	2.4430-01	1.000 00
41	9.54520 00	-5.62680-04	4.77400-04	8.52790-02	9.99930-01	2.4430-01	1.000 00
42	9.78950 00	7.35250-04	-6.43890-04	-9.13530-02	9.99960-01	2.4430-01	1.000 00
43	1.00340 01	-6.85680-04	5.87570-04	9.81040-02	9.99960-01	2.4430-01	1.000 00
44	1.02780 01	8.16280-04	-7.11120-04	-1.05160-01	9.99980-01	2.4430-01	1.000 00
45	1.05220 01	-8.12480-04	6.99600-04	1.12880-01	9.99970-01	2.4430-01	1.000 00
46	1.07670 01	9.21090-04	-8.00050-04	-1.21040-01	9.99990-01	2.4430-01	1.000 00
47	1.10110 01	-9.49360-04	8.19470-04	1.29890-01	9.99980-01	2.4430-01	1.000 00
48	1.12550 01	1.04880-03	-9.09490-04	-1.39310-01	1.00000 00	2.4430-01	1.000 00
49	1.14990 01	-1.10130-03	9.51860-04	1.49470-01	9.99990-01	2.4430-01	1.000 00
50	1.17440 01	1.20010-03	-1.03970-03	-1.60380-01	1.00000 00	2.4430-01	
51	1.19880 01	-1.27930-03	1.02170-03	2.57530-01	1.00000 00		

DC TODD CDPS
EXAMPLE 3

AR000346 13:42 MON AUG 07, 1978
VISCOELASTIC FLUID

K = 1.00000D-02 LAMBDA = 1.0102D 00

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	2	51	30	0	1	1

II	TOL	RSC
0.0	5.00000D-04	1.00000D-02

A	B	C	D
1.00000D 00	0.0	0.0	1.0102D 00
1.00000D 00	0.0	0.0	7.00000D 00

UNIFORM SPACING COMPUTED

UPDATE	1	51	1	1	1.000D 00	2.020D 00	1.000D 00
NEW SPACING COMPUTED							
UPDATE	2	8	3	3	4.27D-05	2.82D-04	2.09D-06

CONVERGED SOLUTION

CHEKDE	26	8.06D-13	0.0	0.0	7.00D 00
--------	----	----------	-----	-----	----------

CHEKBN	2	1	1	7	3.75D-05	5.63D-05	5.63D-03	1.07D-05
--------	---	---	---	---	----------	----------	----------	----------

DC T000 COPS
EXAMPLE 3AR000346 13:42 MON AUG 07, 1978
VISCOELASTIC FLUID

	X	Y	YP	YPP	I	H	S
1	0.0	1.01020 00	-1.02060 00	1.03660 00	0.0	1.7940-01	1.070 00
2	1.79390-01	8.42800-01	-8.51350-01	8.55310-01	1.65750-01	1.9140-01	1.060 00
3	3.70770-01	6.94600-01	-7.01740-01	7.13700-01	3.12410-01	2.0380-01	1.050 00
4	5.74520-01	5.65430-01	-5.71160-01	5.72640-01	4.40320-01	2.1480-01	1.040 00
5	7.89300-01	4.55090-01	-4.59780-01	4.68890-01	5.49490-01	2.2350-01	1.030 00
6	1.01280 00	3.63150-01	-3.66820-01	3.66410-01	6.40560-01	2.2990-01	1.020 00
7	1.24280 00	2.87830-01	-2.90810-01	2.97900-01	7.15060-01	2.3430-01	1.010 00
8	1.47710 00	2.27210-01	-2.29490-01	2.27870-01	7.75120-01	2.3720-01	1.010 00
9	1.71430 00	1.78740-01	-1.80600-01	1.86330-01	8.23050-01	2.3910-01	1.000 00
10	1.95340 00	1.40430-01	-1.41830-01	1.39530-01	8.61020-01	2.4030-01	1.000 00
11	2.19370 00	1.10120-01	-1.11280-01	1.16080-01	8.90970-01	2.4100-01	1.000 00
12	2.43470 00	8.63700-02	-8.72160-02	8.45610-02	9.14540-01	2.4150-01	1.000 00
13	2.67620 00	6.76280-02	-6.83530-02	7.25030-02	9.33040-01	2.4180-01	1.000 00
14	2.91790 00	5.30190-02	-5.35260-02	5.07270-02	9.47550-01	2.4190-01	1.000 00
15	3.15980 00	4.14800-02	-4.19360-02	4.56180-02	9.58920-01	2.4200-01	1.000 00
16	3.40190 00	3.25240-02	-3.28240-02	3.00030-02	9.67830-01	2.4210-01	1.000 00
17	3.64400 00	2.54270-02	-2.57170-02	2.90470-02	9.74810-01	2.4210-01	1.000 00
18	3.88610 00	1.99490-02	-2.01220-02	1.73510-02	9.80280-01	2.4220-01	1.000 00
19	4.12830 00	1.55810-02	-1.57700-02	1.88220-02	9.84560-01	2.4220-01	1.000 00
20	4.37050 00	1.22370-02	-1.23340-02	9.65260-03	9.87910-01	2.4220-01	1.000 00
21	4.61270 00	9.54500-03	-9.66990-03	1.24950-02	9.90540-01	2.4220-01	1.000 00
22	4.85490 00	7.50850-03	-7.55840-03	4.98880-03	9.92590-01	2.4220-01	1.000 00
23	5.09710 00	5.84470-03	-5.93030-03	8.56150-03	9.94200-01	2.4220-01	1.000 00
24	5.33930 00	4.60930-03	-4.63110-03	2.18270-03	9.95460-01	2.4220-01	1.000 00
25	5.58150 00	3.57670-03	-3.63770-03	6.09860-03	9.96450-01	2.4220-01	1.000 00
26	5.82370 00	2.81160-03	-2.83680-03	5.12510-04	9.97220-01	2.4220-01	1.000 00
27	6.06590 00	2.18670-03	-2.23210-03	4.54010-03	9.97820-01	2.4220-01	1.000 00
28	6.30810 00	1.74150-03	-1.73690-03	4.64210-04	9.98300-01	2.4220-01	1.000 00
29	6.55030 00	1.33500-03	-1.37030-03	3.53900-03	9.98670-01	2.4220-01	1.000 00
30	6.79250 00	1.07300-03	-1.06280-03	1.01850-03	9.98960-01	2.4220-01	1.000 00
31	7.03470 00	8.13120-04	-8.41950-04	2.88210-03	9.99190-01	2.4220-01	1.000 00
32	7.27690 00	6.62820-04	-6.49660-04	1.31640-03	9.99360-01	2.4220-01	1.000 00
33	7.51910 00	4.93530-04	-5.17920-04	2.43870-03	9.99500-01	2.4220-01	1.000 00
34	7.76130 00	4.11100-04	-3.96510-04	1.45940-03	9.99610-01	2.4220-01	1.000 00
35	8.00350 00	2.97900-04	-3.19190-04	2.12860-03	9.99700-01	2.4220-01	1.000 00
36	8.24570 00	2.56520-04	-2.41420-04	1.50990-03	9.99760-01	2.4220-01	1.000 00
37	8.48790 00	1.78240-04	-1.97260-04	1.90220-03	9.99820-01	2.4220-01	1.000 00
38	8.73010 00	1.61500-04	-1.46450-04	1.50560-03	9.99860-01	2.4220-01	1.000 00
39	8.97230 00	1.05140-04	-1.22430-04	1.72940-03	9.99890-01	2.4220-01	1.000 00
40	9.21450 00	1.03010-04	-8.83140-05	1.46990-03	9.99920-01	2.4220-01	1.000 00
41	9.45670 00	6.05580-05	-7.64700-05	1.59120-03	9.99940-01	2.4220-01	1.000 00
42	9.69890 00	6.69280-05	-5.27610-05	1.41680-03	9.99950-01	2.4220-01	1.000 00
43	9.94110 00	3.34530-05	-4.82150-05	1.47620-03	9.99960-01	2.4220-01	1.000 00
44	1.01830 01	4.45920-05	-3.10440-05	1.35480-03	9.99970-01	2.4220-01	1.000 00
45	1.04260 01	1.70470-05	-3.08180-05	1.37710-03	9.99980-01	2.4220-01	1.000 00
46	1.06680 01	3.06960-05	-1.78060-05	1.28900-03	9.99990-01	2.4220-01	1.000 00
47	1.09100 01	7.18740-06	-2.00810-05	1.28940-03	9.99990-01	2.4220-01	1.000 00
48	1.11520 01	2.19850-05	-9.76080-06	1.22240-03	9.99990-01	2.4220-01	1.000 00
49	1.13940 01	1.32960-06	-1.34310-05	1.21020-03	1.00000 00	2.4220-01	1.000 00
50	1.16370 01	1.64670-05	-4.86530-06	1.16010-03	1.00000 00	2.4220-01	
51	1.18790 01	-2.46900-06	-1.41580-05	1.66270-03	1.00000 00		

DC TDDO CDPS
EXAMPLE 3

AR000346 13:42 MON AUG 07, 1978
VISCOELASTIC FLUID

K = 1.00000-01 LAMBDA = 1.12700 00

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	3	51	30	0	1	1
I1	TOL	RSC				
0.0	5.00000-04	1.00000-02				
A	B	C	D			
1.00000 00	0.0	0.0	1.12700 00			
1.00000 00	0.0	0.0	7.00000 00			

UNIFORM SPACING COMPUTED

UPDATE	1	51	1	1	1.000 00	2.270 00	1.000 00
NEW SPACING COMPUTED							
UPDATE	2	2	1	5	2.020-05	5.800-04	4.970-06
UPDATE	3	0	0	0	0.0	0.0	0.0

CONVERGED SOLUTION

CHEKDE	26	1.250-14	0.0	0.0	7.000 00		
CHEKBN	3	1	1	7	2.660-05	1.670-04	1.670-03 1.550-05

DC F000 CDP5
EXAMPLE 3AR000346 13:42 MON AUG 07, 1978
VISCOELASTIC FLUID

	X	Y	YP	YPP	I	H	S
1	0.0	1.12700 00	-1.27000 00	1.42980 00	0.0	1.4530-01	1.070 00
2	1.45270-01	9.56840-01	-1.07820 00	1.21390 00	1.51020-01	1.5520-01	1.080 00
3	3.00440-01	8.03320-01	-9.05370-01	1.02050 00	2.87240-01	1.6750-01	1.080 00
4	4.67950-01	6.65120-01	-7.49620-01	8.44940-01	4.09860-01	1.8020-01	1.070 00
5	6.48150-01	5.42870-01	-6.11850-01	6.89750-01	5.18330-01	1.9200-01	1.040 00
6	8.40160-01	4.37230-01	-4.92780-01	5.55500-01	6.12060-01	2.0060-01	1.030 00
7	1.04070 00	3.48770-01	-3.93080-01	4.43100-01	6.90550-01	2.0670-01	1.020 00
8	1.24740 00	2.76300-01	-3.11400-01	3.51010-01	7.54860-01	2.1100-01	1.010 00
9	1.45850 00	2.17810-01	-2.45480-01	2.76690-01	8.06750-01	2.1400-01	1.010 00
10	1.67240 00	1.71130-01	-1.92870-01	2.17390-01	8.48170-01	2.1590-01	1.010 00
11	1.88840 00	1.34170-01	-1.51210-01	1.70430-01	8.80960-01	2.1710-01	1.000 00
12	2.10550 00	1.05040-01	-1.18390-01	1.33430-01	9.06800-01	2.1790-01	1.000 00
13	2.32340 00	8.21710-02	-9.26090-02	1.04360-01	9.27100-01	2.1840-01	1.000 00
14	2.54170 00	6.42440-02	-7.24050-02	8.16070-02	9.43000-01	2.1870-01	1.000 00
15	2.76040 00	5.02120-02	-5.65910-02	6.37830-02	9.55450-01	2.1880-01	1.000 00
16	2.97920 00	3.92370-02	-4.42220-02	4.98420-02	9.65190-01	2.1890-01	1.000 00
17	3.19820 00	3.06570-02	-3.45520-02	3.89430-02	9.72800-01	2.1900-01	1.000 00
18	3.41720 00	2.39520-02	-2.69940-02	3.04250-02	9.78750-01	2.1900-01	1.000 00
19	3.63620 00	1.87120-02	-2.10890-02	2.37690-02	9.83400-01	2.1910-01	1.000 00
20	3.85530 00	1.46180-02	-1.64750-02	1.85690-02	9.87040-01	2.1910-01	1.000 00
21	4.07440 00	1.14200-02	-1.28700-02	1.45060-02	9.89870-01	2.1910-01	1.000 00
22	4.29350 00	8.92110-03	-1.00540-02	1.13320-02	9.92090-01	2.1910-01	1.000 00
23	4.51260 00	6.96910-03	-7.85430-03	8.85250-03	9.93820-01	2.1910-01	1.000 00
24	4.73170 00	5.44420-03	-6.13570-03	6.91540-03	9.95180-01	2.1910-01	1.000 00
25	4.95080 00	4.25290-03	-4.79310-03	5.40220-03	9.96230-01	2.1910-01	1.000 00
26	5.16990 00	3.32230-03	-3.74430-03	4.22020-03	9.97060-01	2.1910-01	1.000 00
27	5.38900 00	2.59530-03	-2.92500-03	3.29670-03	9.97700-01	2.1910-01	1.000 00
28	5.60810 00	2.02740-03	-2.28500-03	2.57530-03	9.98210-01	2.1910-01	1.000 00
29	5.82720 00	1.58380-03	-1.78500-03	2.01180-03	9.98600-01	2.1910-01	1.000 00
30	6.04630 00	1.23720-03	-1.39440-03	1.57160-03	9.98910-01	2.1910-01	1.000 00
31	6.26540 00	9.66510-04	-1.08930-03	1.22770-03	9.99150-01	2.1910-01	1.000 00
32	6.48450 00	7.55020-04	-8.50920-04	9.59060-04	9.99340-01	2.1910-01	1.000 00
33	6.70360 00	5.89810-04	-6.64730-04	7.49200-04	9.99480-01	2.1910-01	1.000 00
34	6.92270 00	4.60750-04	-5.19270-04	5.85270-04	9.99600-01	2.1910-01	1.000 00
35	7.14180 00	3.59930-04	-4.05650-04	4.57200-04	9.99690-01	2.1910-01	1.000 00
36	7.36100 00	2.81170-04	-3.16890-04	3.57160-04	9.99760-01	2.1910-01	1.000 00
37	7.58010 00	2.19650-04	-2.47550-04	2.79000-04	9.99810-01	2.1910-01	1.000 00
38	7.79920 00	1.71580-04	-1.93380-04	2.17950-04	9.99850-01	2.1910-01	1.000 00
39	8.01830 00	1.34040-04	-1.51060-04	1.70260-04	9.99890-01	2.1910-01	1.000 00
40	8.23740 00	1.04710-04	-1.18010-04	1.33010-04	9.99910-01	2.1910-01	1.000 00
41	8.45650 00	8.17960-05	-9.21860-05	1.03900-04	9.99930-01	2.1910-01	1.000 00
42	8.67560 00	6.38980-05	-7.20150-05	8.11660-05	9.99950-01	2.1910-01	1.000 00
43	8.89470 00	4.99160-05	-5.62570-05	6.34060-05	9.99960-01	2.1910-01	1.000 00
44	9.11380 00	3.89940-05	-4.39470-05	4.95320-05	9.99970-01	2.1910-01	1.000 00
45	9.33290 00	3.04610-05	-3.43300-05	3.86930-05	9.99980-01	2.1910-01	1.000 00
46	9.55200 00	2.37960-05	-2.68180-05	3.02260-05	9.99990-01	2.1910-01	1.000 00
47	9.77110 00	1.85890-05	-2.09500-05	2.36120-05	9.99990-01	2.1910-01	1.000 00
48	9.99030 00	1.45210-05	-1.63660-05	1.84460-05	9.99990-01	2.1910-01	1.000 00
49	1.02090 01	1.13440-05	-1.27850-05	1.44090-05	1.00000 00	2.1910-01	1.000 00
50	1.04280 01	8.86150-06	-9.98730-06	1.12570-05	1.00000 00	2.1910-01	1.000 00
51	1.06480 01	6.92260-06	-7.79960-06	8.77030-06	1.00000 00	2.1910-01	1.000 00

STOP

DC TODD COPS
EXAMPLE 4

AR000346 13:45 MON AUG 07, 1978
CHEMICAL DISPERSION

K = 1.00000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	1	51	30	20	1	1
IL	TOL	RSC				
0.0	5.00000-04	1.00000-02				
A	B	C	D			
1.00000 00	-1.00000 00	0.0	1.00000 00			
0.0	1.00000 00	0.0	0.0			

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	51	1	51	1.23D 00	4.23D-01	3.33D 00
NEW SPACING COMPUTED							
UPDATE	2	51	7	51	4.04D-02	1.28D-02	1.45D-02
NEW SPACING COMPUTED							
UPDATE	3	51	13	51	1.39D-05	4.39D-06	4.73D-06

CONVERGED SOLUTION

CHEKDE	39	1.95D-12	0.0	5.72D-15	2.26D-15		
CHEKBM	0	0	0	0	5.07D-08	1.28D-06	7.25D-07 0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD CDP5
EXAMPLE 4AR000346 13:45 MON AUG 07, 1978
CHEMICAL DISPERSION

	X	Y	YP	YPP	I	H	S
1	0.0	6.20040-01	-3.79960-01	3.49840-01	0.0	1.9190-02	1.000 00
2	1.91940-02	6.12810-01	-3.73260-01	3.48530-01	1.18320-02	1.9240-02	1.000 00
3	3.84310-02	6.05700-01	-3.66560-01	3.47320-01	2.35520-02	1.9280-02	1.000 00
4	5.77090-02	5.98700-01	-3.59880-01	3.46220-01	3.51610-02	1.9320-02	1.000 00
5	7.70290-02	5.91810-01	-3.53200-01	3.45230-01	4.66600-02	1.9360-02	1.000 00
6	9.63890-02	5.85030-01	-3.46520-01	3.44350-01	5.80520-02	1.9400-02	1.000 00
7	1.15790-01	5.78380-01	-3.39850-01	3.43590-01	6.93370-02	1.9440-02	1.000 00
8	1.35230-01	5.71830-01	-3.33180-01	3.42950-01	8.05160-02	1.9480-02	1.000 00
9	1.54710-01	5.65410-01	-3.26500-01	3.42440-01	9.15920-02	1.9520-02	1.000 00
10	1.74220-01	5.59100-01	-3.19820-01	3.42050-01	1.02570-01	1.9550-02	1.000 00
11	1.93780-01	5.52910-01	-3.13140-01	3.41790-01	1.13440-01	1.9590-02	1.000 00
12	2.13370-01	5.46840-01	-3.06440-01	3.41670-01	1.24210-01	1.9630-02	1.000 00
13	2.33000-01	5.40900-01	-2.99740-01	3.41690-01	1.34890-01	1.9670-02	1.000 00
14	2.52660-01	5.35070-01	-2.93020-01	3.41850-01	1.45460-01	1.9700-02	1.000 00
15	2.72360-01	5.29360-01	-2.86280-01	3.42160-01	1.55950-01	1.9740-02	1.000 00
16	2.92100-01	5.23780-01	-2.79520-01	3.42610-01	1.66340-01	1.9770-02	1.000 00
17	3.11870-01	5.18320-01	-2.72740-01	3.43230-01	1.76640-01	1.9800-02	1.000 00
18	3.31680-01	5.12980-01	-2.65940-01	3.44000-01	1.86860-01	1.9840-02	1.000 00
19	3.51510-01	5.07780-01	-2.59100-01	3.44940-01	1.96980-01	1.9870-02	1.000 00
20	3.71390-01	5.02700-01	-2.52240-01	3.46050-01	2.07020-01	1.9900-02	1.000 00
21	3.91290-01	4.97740-01	-2.45340-01	3.47340-01	2.16980-01	1.9940-02	1.000 00
22	4.11220-01	4.92920-01	-2.38400-01	3.48810-01	2.26850-01	1.9970-02	1.000 00
23	4.31190-01	4.88230-01	-2.31420-01	3.50460-01	2.36650-01	2.0000-02	1.000 00
24	4.51190-01	4.83670-01	-2.24390-01	3.52310-01	2.46360-01	2.0030-02	1.000 00
25	4.71220-01	4.79250-01	-2.17320-01	3.54350-01	2.56010-01	2.0060-02	1.000 00
26	4.91270-01	4.74960-01	-2.10190-01	3.56600-01	2.65570-01	2.0090-02	1.000 00
27	5.11360-01	4.70810-01	-2.03000-01	3.59050-01	2.75070-01	2.0110-02	1.000 00
28	5.31480-01	4.66800-01	-1.95750-01	3.61730-01	2.84500-01	2.0140-02	1.000 00
29	5.51620-01	4.62930-01	-1.88440-01	3.64630-01	2.93870-01	2.0170-02	1.000 00
30	5.71790-01	4.59210-01	-1.81050-01	3.67760-01	3.03160-01	2.0200-02	1.000 00
31	5.91980-01	4.55630-01	-1.73590-01	3.71120-01	3.12400-01	2.0220-02	1.000 00
32	6.12200-01	4.52190-01	-1.66050-01	3.74740-01	3.21580-01	2.0250-02	1.000 00
33	6.32450-01	4.48910-01	-1.58420-01	3.78600-01	3.30700-01	2.0270-02	1.000 00
34	6.52720-01	4.45770-01	-1.50710-01	3.82730-01	3.39770-01	2.0290-02	1.000 00
35	6.73010-01	4.42790-01	-1.42900-01	3.87130-01	3.48790-01	2.0320-02	1.000 00
36	6.93330-01	4.39970-01	-1.34990-01	3.91800-01	3.57750-01	2.0340-02	1.000 00
37	7.13660-01	4.37310-01	-1.26970-01	3.96760-01	3.66670-01	2.0360-02	1.000 00
38	7.34020-01	4.34810-01	-1.18840-01	4.02020-01	3.75550-01	2.0380-02	1.000 00
39	7.54400-01	4.32470-01	-1.10590-01	4.07590-01	3.84380-01	2.0400-02	1.000 00
40	7.74790-01	4.30300-01	-1.02220-01	4.13470-01	3.93180-01	2.0410-02	1.000 00
41	7.95210-01	4.28300-01	-9.37140-02	4.19670-01	4.01950-01	2.0430-02	1.000 00
42	8.15640-01	4.26470-01	-8.50750-02	4.26210-01	4.10680-01	2.0440-02	1.000 00
43	8.36080-01	4.24820-01	-7.62920-02	4.33100-01	4.19380-01	2.0460-02	1.000 00
44	8.56540-01	4.23350-01	-6.73580-02	4.40340-01	4.28050-01	2.0470-02	1.000 00
45	8.77010-01	4.22070-01	-5.82670-02	4.47950-01	4.36710-01	2.0480-02	1.000 00
46	8.97490-01	4.20970-01	-4.90110-02	4.55940-01	4.45340-01	2.0490-02	1.000 00
47	9.17980-01	4.20060-01	-3.95830-02	4.64320-01	4.53950-01	2.0500-02	1.000 00
48	9.38480-01	4.19350-01	-2.99760-02	4.73110-01	4.62560-01	2.0500-02	1.000 00
49	9.58980-01	4.18830-01	-2.01820-02	4.82310-01	4.71150-01	2.0510-02	1.000 00
50	9.79490-01	4.18520-01	-1.01930-02	4.91940-01	4.79740-01	2.0510-02	
51	1.00000 00	4.18410-01	2.26120-15	5.02010-01	4.88320-01		

DC TODD COPS
EXAMPLE 4

AR000346 13:45 MON AUG 07, 1978
CHEMICAL DISPERSION

K = 0.10000

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	2	51	30	20	1	1

I1	TOL	RSC
0.0	5.00000-04	1.00000-02

A	B	C	D
1.00000 00	-1.00000-01	0.0	1.00000 00
0.0	1.00000 00	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	15	1	51	1.340-01	3.740-01	8.370-02
NEW SPACING COMPUTED							
UPDATE	2	50	11	51	2.710-04	3.750-04	2.090-04

CONVERGED SOLUTION

CHEKDE	32	3.730-08	0.0	3.610-16	1.830-16
CHEKBN	0	0	0	0	4.790-08 6.340-07 4.480-05 0.0

SOLUTION WRITTEN ON UNIT 20

DC 1000 COPS
EXAMPLE 4AR000346 13:45 MON AUG 07, 1978
CHEMICAL DISPERSION

	X	Y	VP	YFP	I	H	S
1	0.0	9.0541D-01	-9.4593D-01	9.1866D-01	0.0	1.701D-02	1.01D 00
2	1.7006D-02	8.8945D-01	-9.3042D-01	9.0587D-01	1.5262D-02	1.713D-02	1.01D 00
3	3.4139D-02	8.7364D-01	-9.1500D-01	8.9309D-01	3.0365D-02	1.726D-02	1.01D 00
4	5.1400D-02	8.5798D-01	-8.9970D-01	8.8033D-01	4.5309D-02	1.739D-02	1.01D 00
5	6.8788D-02	8.4247D-01	-8.8450D-01	8.6760D-01	6.0092D-02	1.752D-02	1.01D 00
6	8.6304D-02	8.2711D-01	-8.6942D-01	8.5489D-01	7.4714D-02	1.765D-02	1.01D 00
7	1.0395D-01	8.1190D-01	-8.5444D-01	8.4222D-01	8.9174D-02	1.777D-02	1.01D 00
8	1.2172D-01	7.9685D-01	-8.3959D-01	8.2958D-01	1.0347D-01	1.791D-02	1.01D 00
9	1.3963D-01	7.8195D-01	-8.2485D-01	8.1699D-01	1.1761D-01	1.805D-02	1.01D 00
10	1.5768D-01	7.6719D-01	-8.1022D-01	8.0444D-01	1.3158D-01	1.817D-02	1.01D 00
11	1.7585D-01	7.5260D-01	-7.9571D-01	7.9195D-01	1.4539D-01	1.830D-02	1.01D 00
12	1.9415D-01	7.3817D-01	-7.8133D-01	7.7952D-01	1.5903D-01	1.843D-02	1.01D 00
13	2.1258D-01	7.2390D-01	-7.6708D-01	7.6717D-01	1.7250D-01	1.855D-02	1.01D 00
14	2.3113D-01	7.0980D-01	-7.5296D-01	7.5490D-01	1.8580D-01	1.868D-02	1.01D 00
15	2.4981D-01	6.9587D-01	-7.3898D-01	7.4272D-01	1.9893D-01	1.880D-02	1.01D 00
16	2.6861D-01	6.8211D-01	-7.2513D-01	7.3065D-01	2.1188D-01	1.892D-02	1.01D 00
17	2.8753D-01	6.6852D-01	-7.1141D-01	7.1869D-01	2.2466D-01	1.905D-02	1.01D 00
18	3.0658D-01	6.5510D-01	-6.9784D-01	7.0688D-01	2.3727D-01	1.917D-02	1.01D 00
19	3.2575D-01	6.4185D-01	-6.8440D-01	6.9521D-01	2.4970D-01	1.929D-02	1.01D 00
20	3.4504D-01	6.2878D-01	-6.7110D-01	6.8373D-01	2.6195D-01	1.941D-02	1.01D 00
21	3.6449D-01	6.1588D-01	-6.5794D-01	6.7246D-01	2.7403D-01	1.952D-02	1.01D 00
22	3.8397D-01	6.0316D-01	-6.4492D-01	6.6143D-01	2.8593D-01	1.964D-02	1.01D 00
23	4.0361D-01	5.9062D-01	-6.3203D-01	6.5070D-01	2.9765D-01	1.976D-02	1.01D 00
24	4.2337D-01	5.7826D-01	-6.1928D-01	6.4030D-01	3.0920D-01	1.987D-02	1.01D 00
25	4.4323D-01	5.6608D-01	-6.0666D-01	6.3032D-01	3.2057D-01	1.998D-02	1.01D 00
26	4.6312D-01	5.5409D-01	-5.9416D-01	6.2084D-01	3.3176D-01	2.009D-02	1.01D 00
27	4.8330D-01	5.4228D-01	-5.8178D-01	6.1196D-01	3.4277D-01	2.020D-02	1.01D 00
28	5.0350D-01	5.3065D-01	-5.6950D-01	6.0382D-01	3.5360D-01	2.030D-02	1.01D 00
29	5.2381D-01	5.1921D-01	-5.5732D-01	5.9658D-01	3.6426D-01	2.041D-02	1.01D 00
30	5.4422D-01	5.0796D-01	-5.4521D-01	5.9046D-01	3.7474D-01	2.051D-02	1.01D 00
31	5.6473D-01	4.9690D-01	-5.3315D-01	5.8574D-01	3.8505D-01	2.062D-02	1.00D 00
32	5.8535D-01	4.8603D-01	-5.2110D-01	5.8274D-01	3.9518D-01	2.072D-02	1.00D 00
33	6.0607D-01	4.7536D-01	-5.0904D-01	5.8191D-01	4.0514D-01	2.082D-02	1.00D 00
34	6.2689D-01	4.6489D-01	-4.9691D-01	5.8379D-01	4.1493D-01	2.092D-02	1.00D 00
35	6.4781D-01	4.5462D-01	-4.8465D-01	5.8910D-01	4.2455D-01	2.102D-02	1.00D 00
36	6.6883D-01	4.4456D-01	-4.7217D-01	5.9874D-01	4.3400D-01	2.113D-02	1.00D 00
37	6.8996D-01	4.3472D-01	-4.5937D-01	6.1385D-01	4.4329D-01	2.123D-02	1.01D 00
38	7.1119D-01	4.2511D-01	-4.4612D-01	6.3592D-01	4.5241D-01	2.134D-02	1.01D 00
39	7.3253D-01	4.1573D-01	-4.3224D-01	6.6685D-01	4.6138D-01	2.145D-02	1.01D 00
40	7.5398D-01	4.0662D-01	-4.1751D-01	7.0907D-01	4.7020D-01	2.156D-02	1.01D 00
41	7.7554D-01	3.9778D-01	-4.0163D-01	7.6574D-01	4.7887D-01	2.169D-02	1.01D 00
42	7.9723D-01	3.8926D-01	-3.8425D-01	8.4094D-01	4.8741D-01	2.182D-02	1.01D 00
43	8.1905D-01	3.8108D-01	-3.6487D-01	9.3998D-01	4.9581D-01	2.197D-02	1.01D 00
44	8.4101D-01	3.7330D-01	-3.4286D-01	1.0698D 00	5.0410D-01	2.213D-02	1.01D 00
45	8.6315D-01	3.6599D-01	-3.1738D-01	1.2395D 00	5.1228D-01	2.231D-02	1.01D 00
46	8.8546D-01	3.5924D-01	-2.8736D-01	1.4611D 00	5.2037D-01	2.250D-02	1.01D 00
47	9.0796D-01	3.5316D-01	-2.5137D-01	1.7502D 00	5.2838D-01	2.271D-02	1.01D 00
48	9.3067D-01	3.4793D-01	-2.0752D-01	2.1277D 00	5.3634D-01	2.293D-02	1.01D 00
49	9.5360D-01	3.4378D-01	-1.5333D-01	2.6211D 00	5.4427D-01	2.313D-02	1.01D 00
50	9.7673D-01	3.4098D-01	-8.5605D-02	3.2657D 00	5.5218D-01	2.327D-02	
51	1.0000D 00	3.3995D-01	-1.8301D-16	4.1097D 00	5.6010D-01		

DC TODD CDPS
EXAMPLE 4

ARD00346 13:45 MON AUG 07, 1978
CHEMICAL DISPERSION

K = 0.00100

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	3	51	30	20	1	1

I1	TOL	RSC
0.0	5.00000D-04	1.00000D-02

A	B	C	D
1.00000 00	-1.00000D-03	0.0	1.00000 00
0.0	1.00000 00	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	51	1	51	4.070-01	8.590-01	1.560-01
NEW SPACING COMPUTED							
UPDATE	2	51	48	51	3.400-02	4.320-01	3.480-03
NEW SPACING COMPUTED							
UPDATE	3	50	47	51	1.900-02	6.360-01	3.400-03
NEW SPACING COMPUTED							
UPDATE	4	51	48	51	1.050-02	3.990-01	5.810-04
NEW SPACING COMPUTED							
UPDATE	5	50	48	51	9.800-03	3.280-01	1.770-03
UPDATE	6	51	50	51	6.370-07	4.330-06	5.500-08

CONVERGED SOLUTION

CHEKDE	10	2.850-11	0.0	0.0	2.490-14
CHEKBN	0	0	0	0	2.020-04 4.210-04 9.740-02 0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 4AROC0346 13:45 MON AUG 07, 1978
CHEMICAL DISPERSION

	X	Y	YP	YPP	I	M	S
1	0.0	9.98870-01	-1.13370 00	1.48830 00	0.0	1.6760-02	1.010 00
2	1.67580-02	9.80040-01	-1.11480 00	8.96450-01	1.65810-02	1.5900-02	1.010 00
3	3.36600-02	9.61350-01	-1.09490 00	1.44150 00	3.29870-02	1.7090-02	1.010 00
4	5.07480-02	9.42820-01	-1.07620 00	7.64890-01	4.92560-02	1.7230-02	1.010 00
5	6.79740-02	9.24430-01	-1.05620 00	1.52530 00	6.53380-02	1.7420-02	1.010 00
6	8.53970-02	9.06220-01	-1.03810 00	5.96130-01	8.12850-02	1.7550-02	1.010 00
7	1.02950-01	8.88130-01	-1.01800 00	1.65040 00	9.70330-02	1.7760-02	1.010 00
8	1.20710-01	8.70260-01	-1.00040 00	3.79060-01	1.12650-01	1.7880-02	1.010 00
9	1.38590-01	8.52490-01	-9.80080-01	1.83040 00	1.28050-01	1.8110-02	1.010 00
10	1.56700-01	8.34970-01	-9.63180-01	9.79060-02	1.43330-01	1.8210-02	1.010 00
11	1.74910-01	8.17540-01	-9.42610-01	2.08310 00	1.58370-01	1.8460-02	1.000 00
12	1.93370-01	8.00390-01	-9.26610-01	-2.67930-01	1.73300-01	1.8530-02	1.020 00
13	2.11900-01	7.83300-01	-9.05570-01	2.43150 00	1.87980-01	1.8810-02	1.000 00
14	2.30710-01	7.66540-01	-8.90700-01	-7.45320-01	2.02550-01	1.8850-02	1.020 00
15	2.49560-01	7.49790-01	-8.68970-01	2.90510 00	2.16840-01	1.9170-02	9.990-01
16	2.68730-01	7.33460-01	-8.55540-01	-1.36920 00	2.31060-01	1.9150-02	1.020 00
17	2.87890-01	7.17060-01	-8.32810-01	3.54170 00	2.44960-01	1.9540-02	9.960-01
18	3.07430-01	7.01180-01	-8.21230-01	-2.18480 00	2.58810-01	1.9460-02	1.020 00
19	3.26890-01	6.85120-01	-7.97100-01	4.38970 00	2.72300-01	1.9910-02	9.910-01
20	3.46800-01	6.69720-01	-7.87860-01	-3.25090 00	2.85790-01	1.9740-02	1.030 00
21	3.66540-01	6.54000-01	-7.61800-01	5.51010 00	2.98850-01	2.0290-02	9.860-01
22	3.86830-01	6.39130-01	-7.55560-01	-4.64360 00	3.11970-01	2.0010-02	1.030 00
23	4.06840-01	6.23720-01	-7.26890-01	6.98010 00	3.24610-01	2.0680-02	9.790-01
24	4.27530-01	6.09440-01	-7.24500-01	-6.46160 00	3.37360-01	2.0250-02	1.040 00
25	4.47780-01	5.94300-01	-6.92310-01	8.89620 00	3.49550-01	2.1090-02	9.710-01
26	4.68860-01	5.80680-01	-6.94850-01	-8.83260 00	3.61930-01	2.0470-02	1.050 00
27	4.89330-01	5.65780-01	-6.57980-01	1.13790 01	3.73670-01	2.1500-02	9.600-01
28	5.10830-01	5.52910-01	-6.66850-01	-1.19230 01	3.85690-01	2.0650-02	1.060 00
29	5.31480-01	5.38190-01	-6.23800-01	1.45760 01	3.96960-01	2.1940-02	9.470-01
30	5.53420-01	5.26190-01	-6.40800-01	-1.59470 01	4.08630-01	2.0780-02	1.080 00
31	5.74210-01	5.11560-01	-5.89650-01	1.86710 01	4.19420-01	2.2400-02	9.320-01
32	5.96600-01	5.00600-01	-6.17110-01	-2.11860 01	4.30750-01	2.0870-02	1.100 00
33	6.17470-01	4.85960-01	-5.55410-01	2.38860 01	4.41040-01	2.2870-02	9.130-01
34	6.40340-01	4.76290-01	-5.96300-01	-2.80040 01	4.52050-01	2.0880-02	1.120 00
35	6.61220-01	4.61510-01	-5.20940-01	3.04980 01	4.61830-01	2.3330-02	8.920-01
36	6.84550-01	4.53430-01	-5.79080-01	-3.68760 01	4.72510-01	2.0810-02	1.140 00
37	7.05360-01	4.38370-01	-4.86100-01	3.88550 01	4.81790-01	2.3770-02	8.680-01
38	7.29130-01	4.32300-01	-5.66390-01	-4.84140 01	4.92140-01	2.0630-02	1.170 00
39	7.49760-01	4.16800-01	-4.50740-01	4.94200 01	5.00890-01	2.4110-02	8.430-01
40	7.73880-01	4.13240-01	-5.59460-01	-6.34090 01	5.10900-01	2.0340-02	1.200 00
41	7.94210-01	3.97120-01	-4.14600-01	6.28470 01	5.19140-01	2.4300-02	8.180-01
42	8.18510-01	3.95660-01	-5.59840-01	-8.29310 01	5.28790-01	1.9880-02	1.220 00
43	8.38400-01	3.79750-01	-3.77150-01	8.01720 01	5.36500-01	2.4170-02	7.930-01
44	8.62570-01	3.82950-01	-5.69460-01	-1.08430 02	5.45730-01	1.9180-02	1.230 00
45	8.81750-01	3.65190-01	-3.37380-01	1.03030 02	5.52890-01	2.3510-02	7.770-01
46	9.05260-01	3.72280-01	-5.90380-01	-1.41740 02	5.61570-01	1.8260-02	1.220 00
47	9.23520-01	3.53600-01	-2.92720-01	1.34190 02	5.68190-01	2.2290-02	7.730-01
48	9.45810-01	3.64490-01	-6.25380-01	-1.85790 02	5.76200-01	1.7240-02	1.250 00
49	9.63040-01	3.44590-01	-2.37710-01	1.78680 02	5.82300-01	2.0600-02	7.940-01
50	9.83640-01	3.58880-01	-6.76580-01	-2.43520 02	5.89560-01	1.6360-02	
51	1.00000 00	3.37440-01	2.48690-14	4.08030 02	5.95240-01		

STOP

DC TODD CDPS
EXAMPLE 5

AR000384 19:12 MON AUG 07, 1978
INHERENTLY UNSTABLE PROBLEM

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	1	51	30	20	1	1

II	TOL	RSC
0.0	5.0000D-04	1.0000D-02

A	B	C	D
1.0000D 00	0.0	0.0	1.0000D 00
1.0000D 00	0.0	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	5	1	51	9.200-01	3.66D 16	9.560-01
NEW SPACING COMPUTED							
UPDATE	2	2	1	51	5.19D-04	2.53D-03	6.42D-04
UPDATE	3	0	0	0	0.0	0.0	0.0

CONVERGED SOLUTION

CHEKDE 45 2.62D-12 0.0 0.0 0.0

COMPARISON OF SOLUTION WITH TABLE IN REF 7

J	X	YA	YB	DIFF
1	1.0	1.0000	1.0000	0.0
2	1.2	0.7140	0.7140	0.0000
3	1.4	0.4871	0.4869	0.0002
4	1.6	0.3146	0.3145	0.0001
5	1.8	0.1897	0.1897	0.0000
6	2.0	0.1039	0.1036	0.0003
7	2.2	0.0475	0.0476	-0.0001
8	2.4	0.0134	0.0134	-0.0000
9	2.6	-0.0055	-0.0058	0.0003
10	2.8	-0.0155	-0.0153	-0.0002
11	3.0	-0.0192	-0.0190	-0.0002
12	3.2	-0.0192	-0.0195	0.0003
13	3.4	-0.0181	-0.0183	0.0002
14	4.0	-0.0131	-0.0128	-0.0003
15	5.0	-0.0068	-0.0068	-0.0000

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 5AR000384 19:12 MON AUG 07. 1978
INHERENTLY UNSTABLE PROBLEM

	X	Y	YP	YPP	I	H	S
1	1.00000 00	1.00000 00	-1.57970 00	1.50000 00	0.0	2.3670-01	1.120 00
2	1.23670 00	5.67930-01	-1.22660 00	1.46870 00	1.95770-01	2.6400-01	1.190 00
3	1.51370 00	3.93680-01	-8.59620-01	1.28650 00	3.33750-01	3.1410-01	1.160 00
4	1.81480 00	1.82090-01	-5.05760-01	9.54960-01	4.21250-01	3.6280-01	1.060 00
5	2.17750 00	5.26920-02	-2.31840-01	5.64560-01	4.60840-01	3.8310-01	1.010 00
6	2.56060 00	-2.90690-03	-7.78380-02	2.61790-01	4.68500-01	3.8720-01	1.000 00
7	2.94780 00	-1.84420-02	-1.30470-02	9.30560-02	4.63570-01	3.8750-01	1.000 00
8	3.33530 00	-1.88100-02	6.79380-03	2.14150-02	4.56110-01	3.8750-01	1.000 00
9	3.72290 00	-1.53700-02	9.65750-03	-1.28260-03	4.49450-01	3.8750-01	1.000 00
10	4.11040 00	-1.19100-02	7.99330-03	-5.53660-03	4.44190-01	3.8750-01	1.000 00
11	4.49790 00	-9.22780-03	5.91490-03	-4.80470-03	4.40120-01	3.8750-01	1.000 00
12	4.88550 00	-7.26290-03	4.31770-03	-3.43880-03	4.36940-01	3.8750-01	1.000 00
13	5.27300 00	-5.81900-03	3.20100-03	-2.38520-03	4.34420-01	3.8760-01	1.000 00
14	5.66060 00	-4.73790-03	2.42230-03	-1.67700-03	4.32390-01	3.8760-01	1.000 00
15	6.04810 00	-3.91170-03	1.87670-03	-1.20670-03	4.30720-01	3.8760-01	1.000 00
16	6.43570 00	-3.26850-03	1.46870-03	-8.87940-04	4.29330-01	3.8760-01	1.000 00
17	6.82320 00	-2.75990-02	1.17010-03	-6.66190-04	4.28170-01	3.8760-01	1.000 00
18	7.21060 00	-2.35210-03	5.44260-04	-5.08310-04	4.27180-01	3.8760-01	1.000 00
19	7.59830 00	-2.02120-03	7.70650-04	-3.93600-04	4.26330-01	3.8760-01	1.000 00
20	7.98590 00	-1.74980-03	6.25360-04	-3.08790-04	4.25600-01	3.8760-01	1.000 00
21	8.37340 00	-1.52510-03	5.28620-04	-2.45790-04	4.24970-01	3.8760-01	1.000 00
22	8.76100 00	-1.32730-03	4.43450-04	-1.96600-04	4.24420-01	3.8760-01	1.000 00
23	9.14850 00	-1.17920-03	3.74810-04	-1.59210-04	4.23930-01	3.8760-01	1.000 00
24	9.53610 00	-1.04510-03	3.18980-04	-1.30070-04	4.23500-01	3.8760-01	1.000 00
25	9.92370 00	-9.30670-04	2.73200-04	-1.07120-04	4.23120-01	3.8760-01	1.000 00
26	1.03110 01	-8.32360-04	2.35350-04	-8.88670-05	4.22780-01	3.8760-01	1.000 00
27	1.06990 01	-7.47430-04	2.03850-04	-7.42320-05	4.22470-01	3.8760-01	1.000 00
28	1.10860 01	-6.73690-04	1.77460-04	-6.24000-05	4.22190-01	3.8760-01	1.000 00
29	1.14740 01	-6.09350-04	1.55200-04	-5.27640-05	4.21950-01	3.8760-01	1.000 00
30	1.18610 01	-5.52950-04	1.36330-04	-4.48630-05	4.21720-01	3.8760-01	1.000 00
31	1.22490 01	-5.03320-04	1.20250-04	-3.83410-05	4.21520-01	3.8760-01	1.000 00
32	1.26370 01	-4.59450-04	1.06470-04	-3.29260-05	4.21330-01	3.8760-01	1.000 00
33	1.30240 01	-4.20540-04	9.46160-05	-2.84040-05	4.21160-01	3.8760-01	1.000 00
34	1.34120 01	-3.85900-04	8.43650-05	-2.46080-05	4.21000-01	3.8760-01	1.000 00
35	1.37990 01	-3.54970-04	7.54660-05	-2.14050-05	4.20860-01	3.8760-01	1.000 00
36	1.41870 01	-3.27260-04	6.77110-05	-1.86900-05	4.20730-01	3.8760-01	1.000 00
37	1.45740 01	-3.02360-04	6.09280-05	-1.63780-05	4.20610-01	3.8760-01	1.000 00
38	1.49620 01	-2.79930-04	5.49730-05	-1.44010-05	4.20490-01	3.8760-01	1.000 00
39	1.53490 01	-2.59660-04	4.97300-05	-1.27030-05	4.20390-01	3.8760-01	1.000 00
40	1.57370 01	-2.41300-04	4.50950-05	-1.12450-05	4.20290-01	3.8760-01	1.000 00
41	1.61240 01	-2.24640-04	4.09940-05	-9.96210-06	4.20200-01	3.8760-01	1.000 00
42	1.65120 01	-2.09470-04	3.73260-05	-8.94030-06	4.20120-01	3.8760-01	1.000 00
43	1.69000 01	-1.95640-04	3.41560-05	-7.66420-06	4.20040-01	3.8760-01	1.000 00
44	1.72870 01	-1.83010-04	3.09110-05	-6.16870-06	4.19970-01	3.8760-01	1.000 00
45	1.76750 01	-1.71420-04	2.97200-05	-1.85710-06	4.19900-01	3.8760-01	1.000 00
46	1.80620 01	-1.60870-04	2.15610-05	-2.43540-05	4.19830-01	3.8760-01	1.000 00
47	1.84500 01	-1.50830-04	4.33900-05	7.17660-05	4.19770-01	3.8760-01	1.000 00
48	1.88370 01	-1.42970-04	-5.64990-05	-3.20720-04	4.19720-01	3.8760-01	1.000 00
49	1.92250 01	-1.30280-04	3.41680-04	1.29200-03	4.19660-01	3.8760-01	1.000 00
50	1.96120 01	-1.19280-04	-1.28170-03	-5.05940-03	4.19630-01	3.8760-01	
51	2.00000 01	0.0	5.79360-03	4.76190-02	4.19520-01		

STOP

DC T000 COPS
EXAMPLE 6

AR000384 19:16 MON AUG 07, 1978

K = 10.0

NDIM	JCASE	N	NT	JDUT	LPRNT	LNORM
250	1	51	30	20	1	1
I1	TOL	RSC				
0.0	5.00000-04	1.00000-02				
A	B	C	D			
1.00000 00	0.0	0.0	0.0			
1.00000 00	0.0	0.0	0.0			

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	8	51	51	5.720-01	1.000 01	3.000-01
NEW SPACING COMPUTED							
UPDATE	2	12	1	26	3.560-04	2.150-04	1.060-04

CONVERGED SOLUTION

CHEKDE	19	2.410-15	0.0	0.0	0.0		
--------	----	----------	-----	-----	-----	--	--

CHEKBM	12	39	12	26	2.030-04	3.420-03	2.030-02	3.090-05
--------	----	----	----	----	----------	----------	----------	----------

SOLUTION WRITTEN ON UNIT 20

QC TODD COPS
EXAMPLE 6

ARD00384 19:16 MON AUG 07, 1978

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-9.99710 00	1.19740 02	0.0	5.8820-03	1.000 00
2	5.88200-07	-5.87650-02	-9.30570 00	1.14020 07	-1.68930-04	5.8820-03	1.000 00
3	1.17640-12	-1.39580-01	-8.65520 00	1.08590 02	-6.60050-04	5.8820-03	1.180 00
4	1.76460-02	-1.58650-01	-8.03170 00	1.03450 02	-1.45070-03	6.9130-03	1.100 00
5	2.45590-02	-2.11750-01	-7.33650 00	9.77360 01	-2.73380-03	7.6010-03	1.000 00
6	3.21610-12	-2.64750-11	-6.61620 07	9.18450 01	-4.54830-03	7.6010-03	1.320 00
7	3.97620-02	-3.12440-01	-5.93930 00	8.63300 01	-6.74530-03	1.0010-02	1.010 00
8	4.97740-07	-3.87700-01	-5.10920 00	7.95870 01	-1.01570-02	1.0090-02	1.370 00
9	5.98640-12	-4.15330-11	-4.33830 07	7.33230 01	-1.41140-02	1.3860-02	1.260 00
10	7.37290-02	-4.68660-01	-3.37690 00	6.55230 01	-2.02570-02	1.7400-02	1.530 00
11	9.11270-02	-5.17970-01	-2.31450 00	5.68180 01	-2.88700-02	2.6590-02	1.750 00
12	1.17730-01	-5.60880-01	-9.63530-01	4.54270 01	-4.32940-02	4.6630-02	7.200-01
13	1.64350-01	-5.62690-01	7.65420-01	2.94750 01	-6.98000-02	3.5560-02	7.520-01
14	1.97920-01	-5.22190-01	1.59600 00	2.01980 01	-8.80850-02	2.5230-02	8.730-01
15	2.23140-01	-4.76160-01	2.02750 07	1.40950 01	-1.00700-01	2.2010-02	9.310-01
16	2.45160-01	-4.28510-01	2.28390 00	9.27030 00	-1.10670-01	2.0490-02	9.640-01
17	2.65650-01	-3.80060-01	2.42110 00	5.14770 00	-1.18960-01	1.9750-02	9.890-01
18	2.85400-01	-3.31280-01	2.49610 07	1.48790 07	-1.25990-01	1.9540-02	1.010 00
19	3.04940-01	-2.82440-01	2.49220 00	-1.84210 00	-1.31980-01	1.9780-02	1.040 00
20	3.24710-01	-2.23720-01	2.42480 00	-4.92170 00	-1.37080-01	2.0490-02	1.060 00
21	3.45200-01	-1.85280-01	2.29390 07	-7.80290 00	-1.41370-11	2.1790-02	1.100 00
22	3.66990-01	-1.37370-01	2.09370 00	-1.05080 01	-1.44880-01	2.3940-02	1.140 00
23	3.90930-01	-9.05230-02	1.81080 00	-1.30460 01	-1.47590-01	2.7360-02	1.270 00
24	4.18290-01	-4.61740-02	1.42040 00	-1.53660 01	-1.49440-01	3.4760-02	1.350 00
25	4.53050-01	-6.56200-03	8.46870-01	-1.73830 01	-1.50300-01	4.6950-02	1.000 00
26	5.00000-01	1.34600-02	-1.49560-03	-1.83930 01	-1.49980-01	4.6950-02	7.400-01
27	5.46950-01	-6.67420-03	-8.47990-01	-1.73940 11	-1.49660-01	3.4760-02	7.870-01
28	5.81710-01	-4.62970-02	-1.41990 00	-1.53780 01	-1.50530-01	2.7360-02	8.750-01
29	6.09070-01	-9.06250-02	-1.81010 00	-1.30560 01	-1.52370-01	2.3940-02	9.100-01
30	6.33010-01	-1.37460-01	-2.09300 00	-1.05170 01	-1.55090-01	2.1790-02	9.400-01
31	6.54800-01	-1.85360-01	-2.29330 00	-7.80820 00	-1.58600-01	2.0490-02	9.650-01
32	6.75290-01	-2.33780-01	-2.42430 00	-4.92790 00	-1.62890-01	1.9780-02	9.880-01
33	6.95060-01	-2.82500-01	-2.49180 00	-1.84730 00	-1.67990-01	1.9540-02	1.010 00
34	7.14600-01	-3.71330-01	-7.45580 00	1.48330 00	-1.73990-01	1.9750-02	1.040 00
35	7.34350-01	-3.80090-01	-2.43090 00	5.14380 00	-1.81020-01	2.0490-02	1.070 00
36	7.54840-01	-4.28550-01	-2.28380 00	9.26680 00	-1.89310-01	2.2010-02	1.150 00
37	7.76860-01	-4.76200-01	-2.02760 00	1.40920 01	-1.99280-01	2.5230-02	1.330 00
38	8.02080-01	-5.22240-01	-1.59680 00	2.01930 01	-2.11890-01	3.3560-02	1.390 00
39	8.35650-01	-5.67820-01	-7.70430-01	2.94620 01	-2.30180-01	4.6630-02	5.700-01
40	8.82270-11	-5.61280-01	9.61830-01	4.53870 01	-2.56700-01	2.6590-02	6.540-01
41	9.08670-01	-5.18330-01	2.31800 00	5.67820 01	-2.71130-01	1.7400-02	7.970-01
42	9.26270-01	-4.68960-01	3.38090 00	6.54930 01	-2.79750-01	1.3860-02	7.280-01
43	9.40140-11	-4.15540-01	4.34240 00	7.33090 01	-2.85900-01	1.0090-02	9.920-01
44	9.50230-01	-3.67890-01	5.11330 00	7.95670 01	-2.89860-01	1.0010-02	7.590-01
45	9.60240-01	-3.12600-01	5.94770 00	8.63140 01	-2.93270-01	7.6010-03	1.000 00
46	9.67840-01	-2.64870-01	6.62020 00	9.18330 01	-2.95470-01	7.6010-03	9.090-01
47	9.75440-01	-2.11640-01	7.34040 00	9.77260 01	-2.97280-01	6.9130-03	8.510-01
48	9.82350-01	-1.59710-01	8.03560 00	1.03440 02	-2.98570-01	5.8820-03	1.000 00
49	9.88240-01	-1.09630-01	8.65910 00	1.08590 02	-2.99360-01	5.8820-03	1.000 00
50	9.94120-01	-5.67880-02	9.31360 00	1.14010 02	-2.99850-01	5.8820-03	
51	1.00000 00	0.0	1.00010 01	1.19740 02	-3.00020-01		

DC TODD CDPS
EXAMPLE 6

AR000384 19:16 MON AUG 07, 1978

K = 15.0

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	2	51	30	20	1	1

I1	TOL	RSC
0.0	5.00000-04	1.00000-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	7	1	51	6.990-01	1.500 01	3.670-01
NEW SPACING COMPUTED							
UPDATE	2	12	1	26	1.310-03	6.860-04	2.690-04
UPDATE	3	0	0	0	0.0	0.0	0.0

CONVERGED SOLUTION

CHEKDE	18	2.280-15	0.0	0.0	0.0		
CHEKBM	12	39	12	26	9.040-04	1.500-02	2.030-01 9.550-05

SOLUTION WRITTEN ON UNIT 20

2

OC TODD CDP5
EXAMPLE 6

AR000384

19:16

MON AUG 07, 1978

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-1.49910 01	2.44740 02	0.0	4.8180-03	1.000 00
2	4.81760-03	-6.94470-02	-1.38500 01	2.29050 02	-1.65480-04	4.8180-03	1.000 00
3	9.63510-03	-1.33570-01	-1.27820 01	2.14440 02	-6.60550-04	4.8180-03	1.000 00
4	1.44530-02	-1.92710-01	-1.17830 01	2.00830 02	-1.44840-03	4.8180-03	1.350 00
5	1.92700-02	-2.47190-01	-1.08460 01	1.88150 02	-2.50990-03	6.5020-03	1.050 00
6	2.57730-02	-3.13860-01	-9.67470 00	1.72390 02	-4.33810-03	6.8030-03	1.000 00
7	3.25760-02	-3.75800-01	-8.55370 00	1.57420 02	-6.68830-03	6.8030-03	1.420 00
8	3.97790-02	-4.30460-01	-7.52990 00	1.43860 02	-9.43490-03	9.6290-03	1.030 00
9	4.90080-02	-4.96570-01	-6.22890 00	1.26790 02	-1.39080-02	9.9130-03	1.480 00
10	5.89200-02	-5.52350-01	-5.05040 00	1.11500 02	-1.91160-02	1.4630-02	1.440 00
11	7.25480-02	-6.15010-01	-3.56370 00	9.24910 01	-2.76810-02	2.1030-02	2.210 00
12	9.45780-02	-6.71340-01	-1.87130 00	7.10170 01	-4.12690-02	4.6500-02	8.170-01
13	1.41070-01	-6.95280-01	6.18410-01	3.96640 01	-7.34860-02	3.7970-02	6.970-01
14	1.79040-01	-6.47320-01	1.80960 00	2.38780 01	-9.91160-02	2.6470-02	8.700-01
15	2.95510-01	-5.92040-01	2.33110 00	1.57760 01	-1.15560-01	2.3020-02	9.290-01
16	2.28530-01	-5.34730-01	2.62580 00	9.96930 00	-1.28530-01	2.1380-02	9.640-01
17	2.49910-01	-4.76670-01	2.78820 00	5.32070 00	-1.39350-01	2.0620-02	9.910-01
18	2.79530-01	-4.18350-01	2.85630 00	1.36140 00	-1.48580-01	2.0420-02	1.010 00
19	2.90950-01	-3.59980-01	2.84760 00	-2.14840 00	-1.56530-01	2.0700-02	1.040 00
20	3.11650-01	-3.01730-01	2.76940 00	-5.34410 00	-1.63370-01	2.1450-02	1.060 00
21	3.33100-01	-2.43790-01	2.62240 00	-8.30630 00	-1.69220-01	2.2770-02	1.100 00
22	3.55870-01	-1.86480-01	2.40090 00	-1.10800 01	-1.74110-01	2.4950-02	1.170 00
23	3.80820-01	-1.30300-01	2.09090 00	-1.36840 01	-1.78040-01	2.9270-02	1.260 00
24	4.10100-01	-7.53450-02	1.65240 00	-1.61480 01	-1.81020-01	3.6770-02	1.450 00
25	4.46860-01	-2.60770-02	1.01410 00	-1.83040 01	-1.82810-01	5.3140-02	1.000 00
26	5.00000-01	1.09130-03	-1.58400-03	-1.94940 01	-1.83240-01	5.3140-02	6.920-01
27	5.53140-01	-2.62030-02	-1.01480 00	-1.83330 01	-1.83670-01	3.6770-02	7.960-01
28	5.89900-01	-7.54570-02	-1.65150 00	-1.61730 01	-1.85460-01	2.9270-02	8.520-01
29	6.15180-01	-1.30390-01	-2.09010 00	-1.37040 01	-1.88440-01	2.4950-02	9.130-01
30	6.44130-01	-1.86550-01	-2.40030 00	-1.10960 01	-1.92380-01	2.2770-02	9.420-01
31	6.66900-01	-2.43840-01	-2.62200 00	-8.31930 00	-1.97270-01	2.1450-02	9.650-01
32	6.88350-01	-3.01780-01	-2.76920 00	-5.35560 00	-2.03120-01	2.0700-02	9.870-01
33	7.09050-01	-3.60030-01	-2.84760 00	-2.15950 00	-2.09970-01	2.0420-02	1.010 00
34	7.29470-01	-4.18400-01	-2.85650 00	1.34970 00	-2.17920-01	2.0620-02	1.040 00
35	7.50090-01	-4.76730-01	-2.78870 00	5.30730 00	-2.27140-01	2.1380-02	1.080 00
36	7.71470-01	-5.34800-01	-2.62680 00	9.95240 00	-2.37960-01	2.3020-02	1.150 00
37	7.94490-01	-5.92150-01	-2.33290 00	1.57530 01	-2.50950-01	2.6470-02	1.430 00
38	8.20960-01	-6.47500-01	-1.81450 00	2.38360 01	-2.67390-01	3.7970-02	1.220 00
39	8.58930-01	-6.95940-01	-6.41700-01	3.95170 01	-2.93030-01	4.6500-02	4.520-01
40	9.05420-01	-6.72970-01	1.87610 00	7.06510 01	-3.25310-01	2.1030-02	6.960-01
41	9.26450-01	-6.16300-01	3.58460 00	9.22000 01	-3.38930-01	1.4630-02	6.780-01
42	9.41080-01	-5.53350-01	5.06990 00	1.11280 02	-3.47510-01	9.9130-03	9.710-01
43	9.50990-01	-4.97380-01	6.24740 00	1.26610 02	-3.52720-01	9.6290-03	7.070-01
44	9.60620-01	-4.31100-01	7.54730 00	1.43710 02	-3.57200-01	6.8030-03	1.000 00
45	9.67420-01	-3.76320-01	8.57060 00	1.57310 02	-3.59950-01	6.8030-03	9.560-01
46	9.74230-01	-3.14260-01	9.69100 00	1.72300 02	-3.62310-01	6.5020-03	7.410-01
47	9.80730-01	-2.47500-01	1.08620 01	1.88080 02	-3.64140-01	4.8180-03	1.000 00
48	9.85550-01	-1.92940-01	1.17980 01	2.00780 02	-3.65200-01	4.8180-03	1.000 00
49	9.90360-01	-1.33720-01	1.27980 01	2.14410 02	-3.65990-01	4.8180-03	1.000 00
50	9.95180-01	-6.95170-02	1.38660 01	2.29040 02	-3.66480-01	4.8180-03	
51	1.00000 00	0.0	1.50070 01	2.44740 02	-3.66650-01		

DC TODD CDPS
EXAMPLE 6

AR000384 19:16 MON AUG 07, 1978

K = 20.0

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	3	51	30	20	1	1

II	TOL	RSC
0.0	5.00000-04	1.00000-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	7	1	51	7.740-01	2.000 01	4.000-01
NEW SPACING COMPUTED							
UPDATE	2	12	1	26	2.840-03	1.450-03	5.010-04
UPDATE	3	0	0	0	0.0	0.0	0.0

CONVERGED SOLUTION

CHEKDE	33	8.020-15	0.0	0.0	0.0		
CHEKBM	12	11	12	26	2.160-03	3.830-02	8.650-01 1.930-04

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 6

ARD00384 19:16 MON AUG 07, 1978

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-1.99790 01	4.19740 02	0.0	4.1260-03	1.000 00
2	4.12620-03	-7.89530-02	-1.83120 01	3.88080 02	-1.65250-04	4.1260-03	1.000 00
3	8.25240-03	-1.51290-01	-1.67720 01	3.58930 02	-6.42450-04	4.1260-03	1.000 00
4	1.23790-02	-2.17520-01	-1.52470 01	3.32070 02	-1.40540-03	4.1260-03	1.080 00
5	1.65050-02	-2.78090-01	-1.40290 01	3.07320 02	-2.42970-03	4.4760-03	1.430 00
6	2.09930-02	-3.37850-01	-1.27100 01	2.82680 02	-3.81040-03	6.4110-03	1.000 00
7	2.73920-02	-4.13790-01	-1.10010 01	2.50980 02	-6.22580-03	6.4110-03	1.020 00
8	3.38030-02	-4.79360-01	-9.48360 00	2.23060 02	-9.09410-03	6.5420-03	1.580 00
9	4.03450-02	-5.36820-01	-6.10910 00	1.97990 02	-1.24230-02	1.0340-02	1.070 00
10	5.06880-02	-6.10730-01	-6.24210 00	1.64400 02	-1.83740-02	1.1090-02	1.610 00
11	6.17830-02	-6.70500-01	-4.58540 00	1.35190 02	-2.54990-02	1.7900-02	2.560 00
12	7.96840-02	-7.33290-01	-2.54330 00	9.94530 01	-3.81170-02	4.5910-02	8.650-01
13	1.25590-01	-7.70300-01	5.85860-01	4.56800 01	-7.31730-02	3.9730-02	6.860-01
14	1.65320-01	-7.16770-01	1.96860 00	2.47660 01	-1.02890-01	2.7250-02	8.710-01
15	1.92570-01	-6.55150-01	2.51170 00	1.55190 01	-1.21620-01	2.3740-02	9.300-01
16	2.16320-01	-5.91740-01	2.80570 00	9.46430 00	-1.36440-01	2.2070-02	9.660-01
17	2.38380-01	-5.27910-01	2.56230 00	4.85980 00	-1.48800-01	2.1320-02	9.930-01
18	2.59700-01	-4.63950-01	3.02420 00	1.03140 00	-1.59370-01	2.1160-02	1.020 00
19	2.80860-01	-3.99980-01	3.00970 00	-2.33940 00	-1.68510-01	2.1500-02	1.040 00
20	3.02370-01	-3.26050-01	2.92570 00	-5.42320 00	-1.76420-01	2.2330-02	1.060 00
21	3.24700-01	-2.72320-01	2.77170 00	-8.31360 00	-1.83210-01	2.3760-02	1.100 00
22	3.48460-01	-2.09070-01	2.54080 00	-1.10600 01	-1.88920-01	2.6040-02	1.160 00
23	3.74500-01	-1.46970-01	2.21770 00	-1.36760 01	-1.93540-01	3.0150-02	1.270 00
24	4.04650-01	-8.67190-02	1.76610 00	-1.61580 01	-1.97720-01	3.8260-02	1.490 00
25	4.42910-01	-3.16110-02	1.05960 00	-1.84000 01	-1.99210-01	5.7090-02	1.000 00
26	5.00000-01	7.67500-05	-1.43280-03	-1.97090 01	-1.99810-01	5.7090-02	6.700-01
27	5.57090-01	-3.17260-02	-1.79990 00	-1.84460 01	-2.00410-01	3.8260-02	7.880-01
28	5.95350-01	-8.68060-02	-1.76500 00	-1.61930 01	-2.02600-01	3.0150-02	8.640-01
29	6.25500-01	-1.47030-01	-2.21690 00	-1.37000 01	-2.06090-01	2.6040-02	9.130-01
30	6.51540-01	-2.09110-01	-2.54030 00	-1.17780 01	-2.10710-01	2.3760-02	9.400-01
31	6.75300-01	-2.72360-01	-2.77150 00	-8.32770 00	-2.16420-01	2.2330-02	9.630-01
32	6.97630-01	-3.36090-01	-2.92570 00	-5.43670 00	-2.23200-01	2.1500-02	9.840-01
33	7.19140-01	-4.00020-01	-3.01010 00	-2.35440 00	-2.31110-01	2.1160-02	1.010 00
34	7.40300-01	-4.64000-01	-3.02490 00	1.01220 00	-2.40260-01	2.1320-02	1.040 00
35	7.61620-01	-5.27980-01	-2.96350 00	4.83290 00	-2.50830-01	2.2070-02	1.080 00
36	7.83680-01	-5.91840-01	-2.80780 00	9.42320 00	-2.63190-01	2.3740-02	1.150 00
37	8.07470-01	-6.55320-01	-2.51550 00	1.54510 01	-2.78010-01	2.7250-02	1.460 00
38	8.34680-01	-7.17120-01	-1.97970 00	2.46260 01	-2.96750-01	3.9730-02	1.160 00
39	8.74410-01	-7.71820-01	-6.43080-01	4.60710 01	-3.26500-01	4.5910-02	3.900-01
40	9.27320-01	-7.36920-01	2.57120 00	9.80010 01	-3.61690-01	1.7900-02	6.200-01
41	9.38220-01	-6.73210-01	4.64620 00	1.34110 02	-3.74270-01	1.1090-02	9.320-01
42	9.49710-01	-6.12820-01	6.29280 00	1.63560 02	-3.81520-01	1.0340-02	6.330-01
43	9.59650-01	-5.38410-01	8.15500 00	1.97750 02	-3.87490-01	6.5420-03	9.800-01
44	9.66200-01	-4.80660-01	9.52710 00	2.22540 02	-3.90830-01	6.4110-03	1.000 00
45	9.72610-01	-4.14820-01	1.10420 01	2.50570 02	-3.93700-01	6.4110-03	6.980-01
46	9.79320-01	-3.38660-01	1.27480 01	2.82370 02	-3.96130-01	4.4760-03	9.220-01
47	9.83500-01	-2.78700-01	1.40670 01	3.07080 02	-3.97510-01	4.1260-03	1.000 00
48	9.87620-01	-2.17970-01	1.53840 01	3.31890 02	-3.98540-01	4.1260-03	1.000 00
49	9.91750-01	-1.51590-01	1.68690 01	3.58810 02	-3.99300-01	4.1260-03	1.000 00
50	9.95870-01	-7.91010-02	1.83490 01	3.88020 02	-3.99780-01	4.1260-03	
51	1.00000 00	0.0	2.00150 01	4.19740 02	-3.99940-01		

DC TODD CDPS
EXAMPLE 6

AR000384 19:16 MON AUG 07, 1978

K = 25.0

NDIM	JCASE	N	NT	JOUT	LPRNT	LNORM
250	4	51	30	20	1	1

11	TOL	RSC
0.0	5.00000-04	1.00000-02

A	B	C	D
1.00000 00	0.0	0.0	0.0
1.00000 00	0.0	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	6	1	51	8.220-01	2.510 01	4.200-01
NEW SPACING COMPUTED							
UPDATE	2	12	1	26	3.680-03	2.330-03	5.930-04
UPDATE	3	0	0	0	0.0	0.0	0.0

CONVERGED SOLUTION

CHEKDE	18	5.370-14	0.0	0.0	0.0
CHEKBM	12	11	12	26	2.970-03 6.010-02 1.460 00 2.350-04

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 6

AR000384 19:16 MON AUG 07, 1978

	X	Y	YP	YPP	T	H	S
1	0.0	0.0	-2.4969D 01	6.4474D 02	0.0	3.6410-03	1.000 00
2	3.6412D-03	-2.6765D-02	-2.2721D 01	5.9042D 02	-1.6045D-04	3.6410-03	1.000 00
3	7.2824D-03	-1.6570D-01	-2.0663D 01	5.4083D 02	-6.2236D-04	3.6410-03	1.000 00
4	1.0924D-02	-2.3745D-01	-1.8778D 01	4.9555D 02	-1.3584D-03	3.6410-03	1.000 00
5	1.4565D-02	-3.0263D-01	-1.7050D 01	4.5420D 02	-2.3436D-03	3.6410-03	1.360 00
6	1.8206D-02	-3.6175D-01	-1.5466D 01	4.1645D 02	-3.5550D-03	4.9620-03	1.260 00
7	2.3168D-02	-4.3360D-01	-1.3517D 01	3.7023D 02	-5.5323D-03	6.2440-03	1.000 00
8	2.9412D-02	-5.1113D-01	-1.1368D 01	3.1963D 02	-8.4888D-03	6.2440-03	1.240 00
9	3.5656D-02	-5.7617D-01	-9.5110D 00	2.7633D 02	-1.1890D-02	7.7530-03	1.440 00
10	4.3409D-02	-6.4209D-01	-7.5520D 00	2.3115D 02	-1.6622D-02	1.120D-02	1.51D 00
11	5.4611D-02	-7.1336D-01	-5.2657D 00	1.7952D 02	-2.4238D-02	1.687D-02	2.56D 00
12	7.1484D-02	-7.7999D-01	-2.7997D 00	1.2429D 02	-3.6894D-02	4.318D-02	9.30D-01
13	1.1466D-01	-8.1776D-01	6.1986D-01	5.1082D 01	-7.1909D-02	4.014D-02	6.96D-01
14	1.5480D-01	-7.6045D-01	2.0676D 00	2.4300D 01	-1.0378D-01	2.793D-02	8.67D-01
15	1.8273D-01	-6.9461D-01	2.6029D 00	1.4651D 01	-1.2413D-01	2.421D-02	9.39D-01
16	2.0694D-01	-6.2792D-01	2.8842D 00	8.8541D 00	-1.4015D-01	2.273D-02	9.70D-01
17	2.2967D-01	-5.6046D-01	3.0350D 00	4.5422D 00	-1.5366D-01	2.205D-02	9.93D-01
18	2.5171D-01	-4.9275D-01	3.0948D 00	9.5663D-01	-1.6527D-01	2.190D-02	1.01D 00
19	2.7361D-01	-4.2502D-01	3.0801D 00	-2.2401D 00	-1.7532D-01	2.220D-02	1.03D 00
20	2.9581D-01	-3.5744D-01	2.9969D 00	-5.2127D 00	-1.8400D-01	2.296D-02	1.07D 00
21	3.1877D-01	-2.9026D-01	2.8443D 00	-8.0431D 00	-1.9143D-01	2.448D-02	1.10D 00
22	3.4325D-01	-2.2333D-01	2.6131D 00	-1.0789D 01	-1.9771D-01	2.693D-02	1.14D 00
23	3.7018D-01	-1.5721D-01	2.2857D 00	-1.3456D 01	-2.0281D-01	3.076D-02	1.28D 00
24	4.0094D-01	-9.3691D-02	1.8311D 00	-1.5987D 01	-2.0663D-01	3.934D-02	1.52D 00
25	4.4028D-01	-3.4728D-02	1.1505D 00	-1.8326D 01	-2.0907D-01	5.972D-02	1.00D 00
26	5.0000D-01	-5.1352D-06	-1.2607D-03	-1.9742D 01	-2.0977D-01	5.972D-02	6.59D-01
27	5.5972D-01	-3.4830D-02	-1.1505D 00	-1.8390D 01	-2.1046D-01	3.934D-02	7.82D-01
28	5.9906D-01	-9.3757D-02	-1.8300D 00	-1.6029D 01	-2.1290D-01	3.076D-02	8.75D-01
29	6.2982D-01	-1.5725D-01	-2.2850D 00	-1.3479D 01	-2.1673D-01	2.693D-02	9.09D-01
30	6.5675D-01	-2.2336D-01	-2.6128D 00	-1.0804D 01	-2.2183D-01	2.448D-02	9.38D-01
31	6.8127D-01	-2.9028D-01	-2.8441D 00	-8.0538D 00	-2.2811D-01	2.296D-02	9.67D-01
32	7.0419D-01	-3.5746D-01	-2.9970D 00	-5.2224D 00	-2.3554D-01	2.220D-02	9.86D-01
33	7.2639D-01	-4.2504D-01	-3.0804D 00	-2.2514D 00	-2.4422D-01	2.190D-02	1.01D 00
34	7.4829D-01	-4.9278D-01	-3.0953D 00	9.4045D-01	-2.5427D-01	2.205D-02	1.03D 00
35	7.7033D-01	-5.6051D-01	-2.0360D 00	4.5160D 00	-2.6588D-01	2.273D-02	1.06D 00
36	7.9306D-01	-6.2780D-01	-2.8862D 00	8.8076D 00	-2.7946D-01	2.421D-02	1.15D 00
37	8.1727D-01	-6.9476D-01	-2.6069D 00	1.4562D 01	-2.9542D-01	2.793D-02	1.44D 00
38	8.4520D-01	-7.6082D-01	-2.0823D 00	2.4072D 01	-3.1578D-01	4.014D-02	1.08D 00
39	8.8534D-01	-8.1985D-01	-7.0618D-01	4.9776D 01	-3.4768D-01	4.318D-02	3.91D-01
40	9.2852D-01	-7.8494D-01	2.8479D 00	1.2120D 02	-3.8288D-01	1.687D-02	6.64D-01
41	9.4539D-01	-7.1692D-01	5.3588D 00	1.7730D 02	-3.9561D-01	1.120D-02	6.92D-01
42	9.5659D-01	-6.4471D-01	7.6290D 00	2.2951D 02	-4.0326D-01	7.753D-03	8.05D-01
43	9.6434D-01	-5.7822D-01	9.5810D 00	2.7505D 02	-4.0801D-01	6.244D-03	1.00D 00
44	9.7059D-01	-5.1276D-01	1.1431D 01	3.1861D 02	-4.1142D-01	6.244D-03	7.95D-01
45	9.7663D-01	-4.3485D-01	1.3576D 01	3.6944D 02	-4.1438D-01	4.962D-03	7.34D-01
46	9.8179D-01	-3.6276D-01	1.5522D 01	4.1584D 02	-4.1637D-01	3.641D-03	1.00D 00
47	9.8544D-01	-3.0340D-01	1.7104D 01	4.5372D 02	-4.1758D-01	3.641D-03	1.00D 00
48	9.8908D-01	-2.3802D-01	1.8831D 01	4.9519D 02	-4.1857D-01	3.641D-03	1.00D 00
49	9.9272D-01	-1.6607D-01	2.0715D 01	5.4060D 02	-4.1931D-01	3.641D-03	1.00D 00
50	9.9636D-01	-8.6953D-02	2.2773D 01	5.9031D 02	-4.1977D-01	3.641D-03	1.00D 00
51	1.0060D 00	0.0	2.5021D 01	6.4474D 02	-4.1993D-01	3.641D-03	

STOP

DC TODD CDPS
EXAMPLE 7

AH000384 19:29 MON AUG 07, 1978
Y'' = EXP(Y)

J = 19 LAMBDA = 8.9252240994552D-01 40E47C5941A9C251 40E47C5941A9C251

NDIM JCASE N NT JOUT LPRNT LNORM
250 1 51 30 20 1 1

I1 TOL RSC
0.0 5.0000D-04 1.0000D-32

A	B	C	D
1.0000D 00	0.0	0.0	0.0
1.0000D 00	0.0	0.0	0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE	1	26	51	51	1.13D-01	4.62D-01	7.58D-02
NEW SPACING COMPUTED							
UPDATE	2	26	50	51	4.62D-03	1.64D-03	4.30D-03
UPDATE	3	26	50	51	8.54D-08	2.94D-08	7.86D-08

CONVERGED SOLUTION

CHEKOE	28	2.33D-16	0.0	0.0	0.0
CHEKBM	50	27	40	0	2.17D-07 1.95D-06 4.73D-09 0.0

SOLUTION WRITTEN ON UNIT 20

DC TODD COPS
EXAMPLE 7AR000384 19:20 MON AUG 07, 1978
Y'' = EXP(Y)

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-4.6363D-01	1.0000D 00	0.0	1.883D-02	1.010 00
2	1.8833D-02	-8.5547D-03	-4.4488D-01	9.9148D-01	-8.1108D-05	1.896D-02	1.010 00
3	3.7796D-02	-1.6813D-02	-4.2616D-01	9.8333D-01	-3.2219D-04	1.909D-02	1.010 00
4	5.6885D-02	-2.4770D-02	-4.0746D-01	9.7553D-01	-7.1966D-04	1.921D-02	1.010 00
5	7.6099D-02	-3.2419D-02	-3.8879D-01	9.6810D-01	-1.2696D-03	1.933D-02	1.010 00
6	9.5434D-02	-3.9755D-02	-3.7014D-01	9.6102D-01	-1.9679D-03	1.945D-02	1.010 00
7	1.1488D-01	-4.6774D-02	-3.5151D-01	9.5430D-01	-2.8101D-03	1.956D-02	1.010 00
8	1.3445D-01	-5.3468D-02	-3.3291D-01	9.4794D-01	-3.7912D-03	1.967D-02	1.010 00
9	1.5412D-01	-5.9834D-02	-3.1432D-01	9.4192D-01	-4.9063D-03	1.978D-02	1.010 00
10	1.7390D-01	-6.5867D-02	-2.9575D-01	9.3626D-01	-6.1499D-03	1.988D-02	1.000 00
11	1.9377D-01	-7.1560D-02	-2.7719D-01	9.3094D-01	-7.5162D-03	1.997D-02	1.000 00
12	2.1374D-01	-7.6911D-02	-2.5865D-01	9.2597D-01	-8.9994D-03	2.006D-02	1.000 00
13	2.3380D-01	-8.1913D-02	-2.4012D-01	9.2135D-01	-1.0593D-02	2.014D-02	1.000 00
14	2.5395D-01	-8.6563D-02	-2.2161D-01	9.1708D-01	-1.2290D-02	2.022D-02	1.000 00
15	2.7417D-01	-9.0857D-02	-2.0310D-01	9.1315D-01	-1.4085D-02	2.029D-02	1.000 00
16	2.9446D-01	-9.4791D-02	-1.8461D-01	9.0956D-01	-1.5969D-02	2.036D-02	1.000 00
17	3.1482D-01	-9.8362D-02	-1.6612D-01	9.0632D-01	-1.7936D-02	2.042D-02	1.000 00
18	3.3524D-01	-1.0157D-01	-1.4765D-01	9.0342D-01	-1.9978D-02	2.047D-02	1.000 00
19	3.5572D-01	-1.0440D-01	-1.2918D-01	9.0087D-01	-2.2087D-02	2.052D-02	1.000 00
20	3.7624D-01	-1.0686D-01	-1.1071D-01	8.9865D-01	-2.4256D-02	2.056D-02	1.000 00
21	3.9680D-01	-1.0895D-01	-9.2252D-02	8.9678D-01	-2.6475D-02	2.060D-02	1.000 00
22	4.1740D-01	-1.1066D-01	-7.3797D-02	8.9525D-01	-2.8738D-02	2.063D-02	1.000 00
23	4.3803D-01	-1.1199D-01	-5.5345D-02	8.9405D-01	-3.1034D-02	2.065D-02	1.000 00
24	4.5867D-01	-1.1294D-01	-3.6895D-02	8.9320D-01	-3.3357D-02	2.066D-02	1.000 00
25	4.7933D-01	-1.1351D-01	-1.8447D-02	8.9269D-01	-3.5697D-02	2.067D-02	1.000 00
26	5.0000D-01	-1.1370D-01	-1.9327D-02	8.9252D-01	-3.8046D-02	2.067D-02	1.000 00
27	5.2067D-01	-1.1351D-01	1.8447D-02	8.9269D-01	-4.0394D-02	2.066D-02	9.99D-01
28	5.4133D-01	-1.1294D-01	3.6895D-02	8.9320D-01	-4.2734D-02	2.065D-02	9.99D-01
29	5.6197D-01	-1.1199D-01	5.5345D-02	8.9405D-01	-4.5057D-02	2.063D-02	9.99D-01
30	5.8260D-01	-1.1066D-01	7.3797D-02	8.9525D-01	-4.7354D-02	2.060D-02	9.98D-01
31	6.0320D-01	-1.0895D-01	9.2252D-02	8.9678D-01	-4.9616D-02	2.056D-02	9.98D-01
32	6.2376D-01	-1.0686D-01	1.1071D-01	8.9865D-01	-5.1835D-02	2.052D-02	9.98D-01
33	6.4428D-01	-1.0440D-01	1.2918D-01	9.0087D-01	-5.4004D-02	2.047D-02	9.97D-01
34	6.6476D-01	-1.0157D-01	1.4765D-01	9.0342D-01	-5.6113D-02	2.042D-02	9.97D-01
35	6.8518D-01	-9.8362D-02	1.6612D-01	9.0632D-01	-5.8155D-02	2.036D-02	9.97D-01
36	7.0554D-01	-9.4791D-02	1.8461D-01	9.0956D-01	-6.0122D-02	2.029D-02	9.96D-01
37	7.2583D-01	-9.0857D-02	2.0310D-01	9.1315D-01	-6.2006D-02	2.022D-02	9.96D-01
38	7.4615D-01	-8.6563D-02	2.2161D-01	9.1708D-01	-6.3801D-02	2.014D-02	9.96D-01
39	7.6620D-01	-8.1913D-02	2.4012D-01	9.2135D-01	-6.5498D-02	2.006D-02	9.96D-01
40	7.8626D-01	-7.6911D-02	2.5865D-01	9.2597D-01	-6.7092D-02	1.997D-02	9.95D-01
41	8.0623D-01	-7.1560D-02	2.7719D-01	9.3094D-01	-6.8575D-02	1.988D-02	9.95D-01
42	8.2610D-01	-6.5867D-02	2.9575D-01	9.3626D-01	-6.9941D-02	1.978D-02	9.95D-01
43	8.4588D-01	-5.9834D-02	3.1432D-01	9.4192D-01	-7.1185D-02	1.967D-02	9.94D-01
44	8.6555D-01	-5.3468D-02	3.3291D-01	9.4794D-01	-7.2300D-02	1.956D-02	9.94D-01
45	8.8512D-01	-4.6774D-02	3.5151D-01	9.5430D-01	-7.3281D-02	1.945D-02	9.94D-01
46	9.0457D-01	-3.9755D-02	3.7014D-01	9.6102D-01	-7.4127D-02	1.933D-02	9.94D-01
47	9.2390D-01	-3.2419D-02	3.8879D-01	9.6810D-01	-7.4822D-02	1.921D-02	9.94D-01
48	9.4311D-01	-2.4770D-02	4.0746D-01	9.7553D-01	-7.5372D-02	1.909D-02	9.93D-01
49	9.6220D-01	-1.6813D-02	4.2616D-01	9.8333D-01	-7.5769D-02	1.896D-02	9.93D-01
50	9.8117D-01	-8.5547D-03	4.4488D-01	9.9148D-01	-7.6010D-02	1.883D-02	
51	1.0000D 00	0.0	4.6363D-01	1.0000D 00	-7.6091D-02		

STOP

NOMENCLATURE

a_{1i}	Eq. (104)
a_{2i}	Eq. (105)
$A_{ij}, i = 1 \text{ and } N,$ $j = 1 \text{ and } N$	Eqs. (B-26) thru (B-29)
$A_i, B_i, C_i, D_i,$ $i = 1 \text{ and } N$	End conditions parameters, Eq. (24)
d_{lmjk}	System coefficients, Eqs. (111) thru (133)
D	Differentiation operator, Eq. (40)
D_{mi}	Eq. (110)
E	Eq. (B-4)
E_i	Operator, Eq. (38)
F	Eq. (B-5)
$f(x, y, y', I)$	Eq. (134)
g_i	Integral of quintic spline at x_i , Eqs. (95) and (96)
H	Eq. (B-1)
h_i	Length of the i th interval, Eq. (1)
I	Integral of $y(x)$, Eq. (93)
I_1	Value of I at x_i
$J_1(a,b)$	Integral of the cubic spline, Eq. (27)
$K_1(a,b)$	Integral of the quintic spline, Eq. (89)
k	Parameter in example problems, Eqs. (146), (152), (159), and (169)
M_i	Second derivative of the cubic spline at x_i , Eq. (8)

$\tilde{M}$	Eq. (B-3)
m_i	First derivative of the cubic spline at x_i , Eq. (7)
$\tilde{m}$	Eq. (B-2)
m_i	First derivative of the quintic spline at x_i , Eqs. (74) and (65)
$\mathcal{M}_i$	Second derivative of the quintic spline at x_i , Eqs. (75) and (59)
N	Number of points
$p_i(x)$	Cubic spline polynomial in the i th interval, Eqs. (5) and (11)
Q_i	Eq. (98)
$q_i(x)$	Quintic spline polynomial in the i th interval, Eqs. (73) and (86)
R_i	Eq. (109)
$x(x)$	Cubic spline, Eq. (5)
V_1	Eq. (108)
x	Independent variable
x_1	Given abscissas
y	Dependent variable
y_{A1}, y_{B1}	Used in definition of $p_i(x)$, Eqs. (9) and (10)
$y_{a1}, y_{b1}, y_{c1}, y_{d1}$	Used in the definition of $q_i(x)$, Eqs. (82) thru (85)
y_i	Given ordinates or else $y(x_i)$
$z(x)$	Quintic spline, Eq. (73)
z_i	$z(x_i)$
$a(x), \beta(x), \lambda(x), \epsilon(x)$	Coefficients of linear or linearized differential equation, Eqs. (92) and (136) thru (139)
δ_1	Parameter used in definition of m_i , Eq. (66)

Δ_i	Parameter used in definition of $\mathfrak{M}_i$, Eq. (60)
η_i	Eq. (3)
θ_i	Eq. (4)
λ	Eqs. (153) and (177)
$\lambda_i, i = 1 \text{ and } N$	Lagrange multipliers, Eq. (B-5)
σ_i	Eq. (2)