AD-A062 877   PURDUE UNIV LAFAYETTE IND DEPT OF MATHEMATICS   F/G 12/1
THE PRINCIPAL EIGENVALUE FOR LINEAR SECOND ORDER ELLIPTIC EQUAT--ETC(U)
AUG 78 C J HOLLAND   AFOSR-77-3286
UNCLASSIFIED   AFOSR-TR-78-1524   NL

| OF |
ADA
062877

END
DATE
FILMED
3-79
DDC

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 78-1524
2. GOVT ACCESSION NO.:
3. RECIPIENT'S CATALOG NUMBER:
4. TITLE (and Subtitle): THE PRINCIPAL EIGENVALUE FOR LINEAR SECOND ORDER ELLIPTIC EQUATIONS WITH NATURAL BOUNDARY CONDITIONS
5. TYPE OF REPORT & PERIOD COVERED: Interim rept.
6. PERFORMING ORG. REPORT NUMBER:
7. AUTHOR(s): Charles J. Holland
8. CONTRACT OR GRANT NUMBER(s): AFOSR-77-3286
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Purdue University, Department of Mathematics, West Lafayette, IN 47907
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332
12. REPORT DATE: Aug[redacted] 1978
13. NUMBER OF PAGES: 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 18p.
15. SECURITY CLASS. (of this report): UNCLASSIFIED
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE:
16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

We study the properties of the principal eigenvalue λ^* for the eigenvalue equation

$$-\nabla\circ(u_x a) + u_x b + cu = \lambda u \text{ in } \Omega, \quad (1)$$

DD FORM 1 JAN 73 1473

402451

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract continued.

with the natural boundary condition $u_x an = 0$ on $\partial\Omega$. Above a is a positive definite matrix for each $x \in \bar{\Omega}$, b is a vector, and n is the exterior normal. Throughout we assume that the functions a_{ij}, b_i, c are of class $C^2(\bar{\Omega})$ and the bounded domain is of class $C^{2+\alpha}$. Then the principal eigenvalue λ^* exists and is real and all other eigenvalues λ satisfy $\text{Re}\,\lambda \geq \lambda^*$, see [16]. Here a characterization of the principal eigenvalue is derived motivated by ideas from stochastic control theory. This characterization was derived earlier by Donsker-Varadhan [6] using different methods. The stochastic control theory interpretation suggests a possible numerical scheme for determining the principal eigenvalue and eigenfunction for (1), and this interpretation also gives a new characterization of the smallest eigenvalue for Schrodinger's equation. Finally we study the behavior of the principal eigenvalue λ^ε as $\varepsilon \downarrow 0$ when $a = \varepsilon I$ both for the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ and the mixed condition $\frac{\partial u}{\partial n} + \alpha^2 u = 0$. The above results are only outlined here. Some of these results have appeared in [10], [12].

AFOSR-TR- 78-1524

The Principal Eigenvalue for Linear Second Order Elliptic Equations with Natural Boundary Conditions

Charles J. Holland
Purdue University

August 1978

Accompanying Statement

This paper is a summary of an invited talk presented at the International Conference on Stochastic Analysis, April 10-14, 1978, at Northwestern University. This paper will appear in the proceedings of that conference. The talk consists of two main parts. First, we presents our recent work on the characterization of the principal eigenvalue. We have been able to derive a new characterization of the principal eigenvalue for second order linear elliptic partial differential equations, *not necessarily self-adjoint*, with both natural and Dirichlet boundary conditions, and also give a new alternative *numerical* method for calculating both the principal eigenvalue and corresponding eigenvector in the case of natural boundary conditions. The principal eigenvalue, if appropriate sign changes are made, determines the *stability* of equilibrium solutions to certain second order nonlinear partial differential equations. The corresponding eigenvector enables one to determine the first approximation of the solution of the nonlinear equation to variations of the initial condition from the equilibrium solution. These nonlinear equations are important in the applications. For these reasons it is important to have these characterizations of the principal eigenvalue and eigenvector.

Our method converts the determination of the eigenvalue and eigenvector to determining the solution of a stationary stochastic control problem. This latter problem is solved and from it a numerical scheme arises naturally. This method appears to have applications in solving other problems.

Secondly, we report recent progress on determining the asymptotic behavior of the principal eigenvalue for some singularly perturbed eigenvalue problems as a small nuisance parameter tends to zero. The principal eigenvalue is the optimal value for a singularly perturbed stationary stochastic control problem. We are thus able to determine the asymptotic behavior of the optimal value of certain stationary stochastic control problems.

THE PRINCIPAL EIGENVALUE FOR LINEAR SECOND ORDER ELLIPTIC EQUATIONS WITH NATURAL BOUNDARY CONDITIONS

Charles J. Holland[1,2]

Courant Institute of Mathematical Sciences
New York University
New York City, N.Y.

I. INTRODUCTION

We study the properties of the principal eigenvalue λ^* for the eigenvalue equation

$$-\nabla\cdot(u_x a) + u_x b + cu = \lambda u \text{ in } \Omega, \tag{1}$$

with the natural boundary condition $u_x an = 0$ on $\partial\Omega$. Above a is a positive definite matrix for each $x \in \overline{\Omega}$, b is a vector, and n is the exterior normal. Throughout we assume that the functions a_{ij}, b_i, c are of class $C^2(\overline{\Omega})$ and the bounded domain is of class $C^{2+\alpha}$. Then the principal eigenvalue λ^* exists and is real and all other eigenvalues λ

[1]On leave from Purdue University.

[2]This research was supported by a grant to the Courant Institute from the Sloan Foundation and also by AFOSR 77-3286.

satisfy $\text{Re}\,\lambda \geq \lambda^*$, see [16]. Here a characterization of the principal eigenvalue is derived motivated by ideas from stochastic control theory. This characterization was derived earlier by Donsker-Varadhan [6] using different methods. The stochastic control theory interpretation suggests a possible numerical scheme for determining the principal eigenvalue and eigenfunction for (1), and this interpretation also gives a new characterization of the smallest eigenvalue for Schrodinger's equation. Finally we study the behavior of the principal eigenvalue λ^ε as $\varepsilon \downarrow 0$ when $a = \varepsilon I$ both for the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ and the mixed condition $\frac{\partial u}{\partial n} + \alpha^2 u = 0$. The above results are only outlined here. Some of these results have appeared in [10], [12].

II. CHARACTERIZATION

Define

$$F(\phi,V) = \phi_x a\phi_x^T + c\phi^2 + b^T(4a)^{-1}b\phi^2 - \tfrac{1}{2}\,\text{div}\,b\,\phi^2$$
$$-\tfrac{1}{2}[(2a)^{-1}b - V_x^T]^T(2a)[(2a)^{-1}b - V_x^T]\phi^2. \quad (2)$$

<u>Theorem 1</u>. λ^* satisfies

$$\lambda^* = \min_{\phi\in\Phi}\ \max_{V\in C^1(\Omega)} \left[\int_\Omega F(\phi,V)dx + \int_{\partial\Omega} \tfrac{1}{2}(b\cdot n)\phi^2 ds\right] \quad (3)$$

where Φ is the set of functions ϕ of class $C^2(\Omega) \cap C(\bar{\Omega})$ with $\phi^2 > 0$ in $\bar{\Omega}$ and

$\int \phi^2 = 1$. Note that whenever $(2a)^{-1}b$ is a gradient, then V in (3) is chosen so that $V_x = (2a)^{-1}b$. In particular when $b = 0$ (3) reduces to Rayleigh-Ritz.

As we remarked earlier Donsker-Varadhan [6] utilizing their powerful methods established the analogue of (3) for the Dirichlet problem. Their method can be extended to cover this case. We established the formula (3) using stochastic control theory methods in [12] and outlined how to establish the result also for the Dirichlet problem.

Let us outline our derivation of (3). If λ^*, $u^* > 0$ represent the principal eigenvalue and normalized eigenfunction, then (1) can be rewritten as

$$-\text{tr } a u^*_{xx} + u^*_x(b+d) + cu^* = \lambda^* u^* \qquad (4)$$

for an appropriate vector function d. Now $u^* > 0$ in $\bar{\Omega}$. Define $u^* = \exp(-\psi)$, then ψ satisfies the equation

$$\text{tr } a\psi_{xx} - \psi_x a \psi_x^T - \psi_x(b+d) + c = \lambda^* \qquad (5)$$

with the boundary condition $\psi_x an = 0$. We recognize that (5) is the Hamilton-Jacobi equation for a stationary stochastic control problem and that it can be rewritten as

$$\text{tr } a\psi_{xx} + \min_{v \in R^n} [\psi_x v + (b+d+v)(4a)^{-1}(b+d+v) + c] = \lambda^* \qquad (6)$$

A similar stochastic control theory interpretation has been derived independently by Fleming [8] to obtain some Ventsel-Freidlin estimates.

The stochastic control problem is the following. For each Lipschitz function w, let p_w be the invariant measure associated with the Ito stochastic differential equation

$$d\xi = w(\xi)dt + \sigma(\xi)dz(t),\ \sigma\sigma^T = 2a, \tag{7}$$

with reflection in the direction $-an$ at the boundary. With each control w associate the cost

$$C(w) = \int_\Omega [(b+d+w)(4a)^{-1}(b+d+w)+c]p_w\, dx. \tag{8}$$

Then $\lambda^* = \operatorname*{Min}_w C(w)$. Since the min in (6) is obtained when $w = -b - d + 2(u^*)^{-1}(u^*_x a)^T$, then the Min $C(w)$ can be restricted to $\mathscr{W} = \{w\colon w = -b-d+2u^{-1}(u_x a)^T,$ $\int u^2 = 1,\ u \in C^2(\Omega) \cap C(\bar{\Omega})\}$.

Now the representation (3) is obtained by discovering the relationship between w and p_w for $w \in \mathscr{W}$. For any w, p_w satisfies the Fokker-Planck equation and boundary condition

$$\nabla\cdot[(p_x a)^T - p(d+w)] = 0 \text{ in } \Omega$$

$$((p_x a) - p(d+w)^T)\cdot n = 0 \text{ on } \partial\Omega. \tag{9}$$

A little work shows that every $w \in \mathscr{W}$ can be written as

$$w = -d + 2u^{-1}(u_x a)^T + (-b+2(W_x a)^T) \tag{10}$$

where $p_w = u^2$ and u, W satisfy

$$\nabla\cdot(u^2(-b + 2(W_x a)^T) = 0 \quad \text{in} \quad \Omega$$

$$(-b^T + 2W_x a)\cdot n = 0 \quad \text{on} \quad \partial\Omega. \tag{11}$$

Now W minimizes

$$I(V) = \int_\Omega ((2a)^{-1}b - V_x^T)^T (2a)((2a)^{-1}b - V_x^T)u^2 dx. \tag{12}$$

Substituting (10) into (8) and using the fact that W minimizes (12), one obtains the representation (3). This concludes the outline of the proof.

Corollary 1. $\underset{\phi}{\text{Min}}\ \underset{C^1(\bar\Omega)}{\text{Max}} = \underset{C^1(\bar\Omega)}{\text{Max}}\ \underset{\phi}{\text{Min}}$ in (3).

We shall outline a proof of this fact below. Note that the interchange of Min and Max gives the representation

$$\lambda^* = \underset{V\in C^1(\bar\Omega)}{\text{Min}} \quad \mu(V)$$

where $\mu(V)$ is the smallest eigenvalue to

$$-\text{tr}\ a\phi_{xx} + M(V)\phi = \mu\phi \text{ in } \Omega, \tag{14}$$

$$M(V) = c + b^T(4a)^{-1}b - \tfrac{1}{2}\,\text{div}\ b$$
$$-\tfrac{1}{2}[(2a)^{-1}b - V_x^T]^T(2a)[(2a)^{-1}b - V_x^T],$$

with the boundary condition $\phi_x an + \frac{1}{2}(b\cdot n)\phi^2 = 0$ on $\partial\Omega$. It is not necessary that $\frac{1}{2}(b\cdot n) \geq 0$ to guarantee existence of the smallest eigenvalue. Now the fact that Min Max = Inf Sup = Sup Inf follows from the fact that the functional in (3) is strictly concave in V if we require the additional assumption that $V(0) = 0$. Convexity in the argument ϕ^2 follows from the following lemma which is the obvious modification of Lemma 2.2 in Donsker-Varadhan [6]. Note that if $q = \phi^2$, then $\int (q_x a q_x^T) q^{-1} dx = \int 4\phi_x a \phi_x^T$.

Lemma 1. Let $\mathcal{U} = \{p: p \in C^2(\Omega) \cap C^1(\bar{\Omega}),\ p > 0$ in $\bar{\Omega}$, $p_x an = 0$ on $\partial\Omega\}$. If

$q \in \mathcal{U}$, then $\int_\Omega (q_x a q_x^T) q^{-1} dx = -\inf_{p \in \mathcal{U}} \int_\Omega \frac{\nabla \circ (p_x a)}{p} q dx.$

Convexity in ϕ^2, concavity in V is sufficient to guarantee that Inf Sup = Sup Inf, see Theorem 4.1 in Sion [17]. The fact that Sup Inf = Max Min follows from the fact that $\mu(V)$ exists and Remark 2.3, p. 175 in Ekeland-Temam [7].

III. SCHRÖDINGER'S EQUATION

In [10] we considered the Schrödinger equation

$$ih \frac{\partial \psi}{\partial t} = H\psi \tag{15}$$

in bounded domains under the cases of periodic, zero Dirichlet, and zero Neumann boundary conditions. In (15) $H = -(h^2/2m)\Delta_x + K$, h is Planck's constant divided by 2π, and $K \geq 0$ is the potential function. Let us discuss only the Neumann case here. We look for solutions of (15) of the form

$\psi(t,x) = \phi(x) \cdot \exp(-i\lambda h^{-1} t)$ in Ω satisfying $\frac{\partial \phi}{\partial n} = 0$ on $\partial\Omega$. Then ϕ satisfies

$$H\phi = \lambda\phi. \tag{16}$$

Let λ^* denote the smallest eigenvalue of (16) and ϕ^* the corresponding normalized eigenfunction. We now formulate a stochastic control problem for λ^*, ϕ^*. Define $\phi^* = \exp(-Q/h)$. Then Q satisfies the equation

$$\frac{h}{2m} \Delta_x Q + \min_w [Q_x w + \tfrac{1}{2}m|w|^2 + K] = \lambda^* \qquad (17)$$

For each control function v let p_v be the invariant density p_v generated by the diffusion process

$$d\xi = v(\xi)dt + (h/m)^{1/2} dz(t) \qquad (18)$$

with reflection in the interior normal direction. Associate with each control v, the cost

$$C(v) = \int_\Omega [\tfrac{1}{2}m|v|^2 + K] p_v \, dx. \qquad (19)$$

Theorem 2. $\lambda^* = \mathrm{Min}_v C(v)$ where the minimum is taken over Lipschitz functions $v: \bar{\Omega} \to R^n$. The minimum is obtained when $v = (h/m)(\phi^*_x/\phi^*)$ and $p_v = (\phi^*)^2$.

Note that v plays the role of "velocity" of the process (18). Thus λ^* can be thought of as the minimum energy of (19) and $(\phi^*)^2$ as the invariant probability distribution of the process. This interpretation corresponds to the standard physical interpretations. See Nelson [15] for another approach to (15) using stochastic differential equations. Of course, the interpretation (18), (19) for the self-adjoint problem $H\phi = \lambda\phi$ is Rayleigh-Ritz in disguise.

IV. NUMERICAL APPROXIMATIONS

The control theory interpretation in Section 2 also suggests an alternative numerical method for approximating the principal eigenvalue and eigenfunction for the Neumann problem (1). The procedure is to discretize the verification equation (6) obtaining the verification equations for stationary control of a Markov chain. The Markov chain problem is the discrete analogue of the stochastic control problem associated with (6). The Markov chain control problem has a special form which can be solved efficiently using an iterative method due to White [19]. See [10], Section 3 for details. We have done some numerical work on simple examples using this method, however, the convergence of this method has not been justified theoretically.

This type of numerical approximation can also be used for the periodic eigenvalue problem. However, the formal approximation suggested in [10] for the Dirichlet problem fails.

V. SINGULARLY PERTURBED PROBLEMS

Here the asymptotic behavior as $\varepsilon \downarrow 0$ of the principal eigenvalue to the singularly perturbed eigenvalue problem

$$-\varepsilon\Delta\phi - \phi_x(b(x)+\varepsilon\tilde{b}(x)) + ((c(x)+\varepsilon\tilde{c}(x))\phi = \lambda\phi \quad (20)$$

in a bounded convex domain Ω with boundary data $\frac{\partial\phi}{\partial n} + \alpha(x)\phi = 0$ on $\partial\Omega$ is studied. The behavior of λ^ε has been extensively studied for the Dirichlet problem with $c = 0$; see the papers [3], [9], [18]. There the behavior can vary from $\lambda^\varepsilon \to 0$ to $\lambda^\varepsilon \to \infty$ as $\varepsilon \downarrow 0$ depending upon the nature of the vector field b. For the zero Neumann problem such a wide range of behavior is not possible since one knows that $\min_{x\in\overline{\Omega}}(c(x)+\varepsilon\tilde{c}(x)) \le \lambda^\varepsilon \le \max_{x\in\overline{\Omega}}(c(x)+\varepsilon\tilde{c}(x))$. However, the behavior of the principal eigenvalue for Neumann and mixed problems is of interest, especially in the stability analysis of equilibrium states of nonlinear diffusion processes occurring in chemical reactors. See H. Keller [14] who treats the eigenvalue problem in one spatial dimension by transforming the problem to the standard self-adjoint Sturmian form.

The problem with the zero Neumann condition $\frac{\partial u}{\partial n} = 0$ is treated first. The mixed problem with $\alpha \not\equiv 0$ will then be solved by transforming the mixed problem into a Neumann problem. The properties of the principal eigenvalue λ^ε depend upon the nature of the vector field $b + \varepsilon\tilde{b}$, as is evident from (7)-(8). We need the following definition. Throughout let $\xi_x^o(t)$ be the solution to the differential equation

$$d\xi_x^0(t) = b(\xi_x^0(t))dt, \quad \xi_x^0(0) = x \in \bar{\Omega} \tag{21}$$

with reflection. Reflection means that if $y \in \partial\Omega$ and $b(y)\cdot n(y) \geq 0$, n the exterior normal as before, then $\xi_y^0(t)$ travels along the boundary with velocity equal to the projection of b on the tangent space to Ω at $\xi_y^0(t)$. Otherwise $\xi_y^0(t)$ is the solution to (21). This prevents the path from escaping from $\bar{\Omega}$. See Bensoussan-Lions [2] or Holland [11] for a more detailed discussion of the path.

First, the self-adjoint case $b = \tilde{b}$ is disposed of.

<u>Lemma 2</u>. If $b = \tilde{b} = 0$, then $\lambda^\varepsilon \to \min_{x\in\bar{\Omega}} c(x)$ as $\varepsilon \downarrow 0$.

Proof: This follows immediately from the Rayleigh-Ritz representation.

<u>Theorem 3</u>. Suppose that the equation (21) with reflection has a globally (for all x in $\bar{\Omega}$) asymptotically stable rest point $\bar{x}$. Then $\lambda^\varepsilon \to c(\bar{x})$ as $\varepsilon \downarrow 0$.

<u>Remark</u>. A rest point $\bar{x}$ is a point in $\bar{\Omega}$ such that $\xi_{\bar{x}}^0(t) = \bar{x}$ for all t. If $\bar{x} \in \Omega$, then $b(\bar{x}) = 0$. If $\bar{x} \in \bar{\Omega}$, then either $b(\bar{x}) = 0$ or else $b(\bar{x})$ is parallel to $n(\bar{x})$.

Outline of proof: Let λ^ε denote the principal eigenvalue and $\phi^\varepsilon > 0$ the corresponding eigenfunction. Define

$$u^{\varepsilon}(x,t) = \phi^{\varepsilon}(x)e^{-\lambda^{\varepsilon}t}. \qquad (22)$$

Then u^{ε} satisfies

$$\varepsilon\Delta u + u_x(b+\varepsilon\tilde{b}) - (c+\varepsilon\tilde{c})u - u_t = 0 \qquad (23)$$

with the boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega x[0,\infty)$.

Then using the Ito differential rule we have that

$$u^{\varepsilon}(x,t) = E\phi(\xi_x^{\varepsilon}(t))\exp\int_0^t -c(\xi_x^{\varepsilon}(s))ds \qquad (24)$$

where $\xi_x^{\varepsilon}(t)$ is the solution to the Ito equation

$$d\xi_x^{\varepsilon}(t) = (b+\varepsilon\tilde{b})(\xi_x^{\varepsilon}(t))dt + (2\varepsilon)^{1/2}dz(t), \xi_x^{\varepsilon}(0) = x, \qquad (25)$$

$z(t)$ Brownian motion as before, with reflection in the interior normal direction at the boundary of $\partial\Omega$. See [2] for a construction of the process.

The proof is then completed using the representations (22) and (24) for sufficiently large t. This type of approach follows ideas of Kac [13]. The proof requires the estimate that in convex domains

$$\Pr_{0\leq t\leq t'} [\|\xi_x^{\varepsilon}(\cdot)-\xi_x^{0}(\cdot)\| > \mu] \downarrow 0$$

as $\varepsilon \downarrow 0$ for any fixed $\mu > 0$. This estimate is derived in [11] (see equation (15) there) and depends upon estimates (3.13), (3.14) in [2] for reflecting processes in convex domains.

<u>Example</u>. If Ω is the unit disc in R^2 and $b = (b_1,b_2) = (1,-x_2^2)$, then $\lambda^{\varepsilon} \rightarrow c(1,0)$ as $\varepsilon \downarrow 0$.

<u>Theorem 4</u>. Suppose that the differential equation (21)

with reflection has a limit cycle Γ whose domain of attraction consists of all of $\bar{\Omega}$ except for isolated critical points in Ω. Further, at a critical point x^* assume that the eigenvalues of $b_x(x^*)$ have only positive real parts. Then

$$\lambda^\varepsilon \to \frac{1}{T^*}\int_0^{T^*} c(\xi_{\bar{x}}^{o}(t))dt \quad \text{as} \quad \varepsilon \downarrow 0$$

where T^* is the period of Γ and $\bar{x}$ is an arbitrary point on Γ.

Proof: The proof is similar to the proof of Theorem 3 except one has the added complication of possible critical points x^*.

We now consider the mixed problem for the cases considered in Theorems 3 and 4.

Theorem 5. Let H be a smooth solution to $\Delta H - H = 0$ in Ω with $\frac{\partial H}{\partial n} = \alpha$ on $\partial\Omega$. Suppose b satisfies the assumptions of Theorem 3. Then $\lambda^\varepsilon \to c(\bar{x}) + \nabla H(\bar{x})\cdot b(\bar{x})$ as $\varepsilon \downarrow 0$. Suppose b satisfies the assumptions of Theorem 4. Then

$$\lambda^\varepsilon \to \frac{1}{T^*}\int_0^{T^*} c(\xi_{\bar{x}}^{o}(t)) + \nabla H(\xi_{\bar{x}}^{o}(t))\cdot b(\xi_{\bar{x}}^{o}(t))dt \quad \text{as} \quad \varepsilon \downarrow 0.$$

Proof: The result follows by considering the eigenvalue problem for $v = e^{H(x)}u$.

Example. The limits for λ^ε in Theorem 5 can be evaluated in the following case. Suppose $\Omega \subset R^n$ and b satisfies the assumptions of Theorem 3 with

$\bar{x} \in \partial\Omega$. Then $b(\bar{x}) = |b(\bar{x})|n(\bar{x})$ and hence $\lambda^{\varepsilon} \to c(\bar{x}) + \alpha(\bar{x})|b(\bar{x})|$. Thus if $\Omega = (0,1)$, $b > 0$ on $[0,1]$, then $\lambda^{\varepsilon} \to c(1) + \alpha(1)b(1)$ as $\varepsilon \downarrow 0$. This last remark is the content of the limit result (1.7) in [14].

ACKNOWLEDGMENT

The author would like to thank S. R. S. Varadhan for some stimulating discussions.

REFERENCES

1. Anderson, R., Orey, S., Small random perturbations of dynamical systems with reflecting boundary, Proc. Symposium Pure Math. Vol. XXXI, Univ. Ill., 1976, pp. 1-4, American Math Soc.
2. Bensoussan, A., Lions, J., Diffusion processes in bounded domains and singular perturbation problems for variational inequalities with Neumann boundary conditions in Probabilistic Methods in Differential Equations, Springer Verlag Lecture Notes in Mathematics #451(1975), 8-25.
3. Devinatz, A., Ellis, R., Friedman, A., The asymptotic behavior of the first real eigenvalue of second order elliptic operators with a small parameter in the highest derivatives II, Indiana Univ. Math. J. 23(1974) 991-1011.
4. Devinatz, A., Friedman, A., Asymptotic behavior of the principal eigenfunction for a singularly perturbed Dirichlet problem, preprint.
5. Donsker, M., Varadhan, S. R. S., Asymptotic evaluation of certain Wiener integrals for large time, in Proceedings of the International Conference on Integration in function spaces, Clarendon Press, Oxford (1974) 82-88.
6. Donsker, M., Varadhan, S. R. S., On the principal eigenvalue of second order differential operators, Comm. Pure. Appl. Math. 29(1976) 595-621.

7. Ekeland, I., Teman, R., Convex Analysis and Variational Problems. North-Holland (1976).
8. Fleming, W., Exit probabilities and optimal stochastic control, preprint.
9. Friedman, A., The asymptotic behavior of the first real eigenvalue of a second order elliptic operator with a small parameter in the highest derivatives, Indiana Univ. Math. J. 22(1973) 1005-1015.
10. Holland, C., A new energy characterization of the smallest eigenvalue of the Schrödinger equation Comm. Pure Appl. Math. 30(1977) 755-765.
11. Holland, C., The regular expansion in the Neumann problem for elliptic equations Comm. in Partial Differential Equations 1(1976) 191-213.
12. Holland, C., A minimum principle for the smallest eigenvalue for second order linear elliptic equations with natural boundary conditions, Comm. Pure Appl. Math., to appear.
13. Kac, M., On some connections between probability theory and differential and integral equations, Proc. of the 2nd Bekeley Symposium (1950) pp. 189-215.
14. Keller, H., Stability theory for multiple equilibrium states of a nonlinear diffusion process, SIAM J. Math. Anal. 4(1973) 134-140.
15. Nelson, E., Dynamical Theories of Brownian Motion, Princeton University Press, 1967.
16. Protter, M., Weinberger, H., On the spectrum of general second order operators, Bull. Amer. Math. Soc. 72(1966) 251-255.
17. Sion, M., On general minimax theorem, Pac. J. Math. 8(1958) 171-176.
18. Ventsel, A.,Freidlin, M., On small random perturbations of dynamical systems, Russ. Math. Surveys 25(1970) 1-56, Uspekhi Math. Nauk, 28(1970) 3-55.
19. White,D., Dynamic programming, Markov chains, and the method of successive approximations, J. Math. Anal. Appl. 6(1963) 373-376.