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A FORTRAN CODE FOR THE CALCULATION OF SOUND PROPAGATION IN A RA--ETC(U)  
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# LEVEL II

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A FORTRAN Code for the  
 Calculation of Sound Propagation  
 in a Range Dependent Ocean I.  
 The Depth Functions.

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Supported by Code 8120, Naval Research Laboratory,  
 Washington, D.C. 20375 under Contract No. N00013-77-C-0008 <sup>173</sup>

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
6. TITLE (and Subtitle) A FORTRAN CODE FOR THE CALCULATION OF SOUND PROPAGATION IN A RANGE DEPENDENT OCEAN. I. THE DEPTH FUNCTIONS.		5. TYPE OF REPORT & PERIOD COVERED Technical Report (final) Oct. 15, 1976-Jan. 15, 1977
7. AUTHOR(s) 10. Anton Nagl, G.L. Zarura and H. Uberall		6. PERFORMING ORG. REPORT NUMBER 8. CONTRACT OR GRANT NUMBER(s) NRL N00013-77-C-0008
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Physics Catholic University of America Washington DC 20064		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program 62759N, Project SF52-552, Task SF52-552-691, Work Unit S01-94.802
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Sea Systems Command Washington DC 20362		12. REPORT DATE June 7, 1978
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Code 8121 Naval Research Laboratory Washington DC 20375		13. NUMBER OF PAGES vi + 81 pp & 21 pp progr. listing 15. SECURITY CLASS. (of this report) Unclassified
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
9. Final rept. 15 Oct 76 - 15 Jan 77		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) 16. F52552 / 17. SF52552691		
18. SUPPLEMENTARY NOTES 15. N000173-77-C-0008		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Sound Propagation; Ocean Acoustics; Range Dependent Ocean; Depth functions; Arbitrary depth dependence; FORTRAN code; Normal Modes; Near-separation; Local depth functions; Eigenvalue problem		
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Volume

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the range and depth dependent parts of the problem. The resulting depth functions satisfy locally a one-dimensional wave equation which together with the imposed boundary conditions defines an eigenvalue problem. The computer code described here permits to solve this problem for a wide variety of conditions. Deep ocean profiles can be treated for frequencies up to a few hundred Hz. For shallow water cases the frequency range easily extends to several kHz.

A further volume of this report will deal with the range functions and with the solution of the mode coupling problem.

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↙ This report describes a computer code which solves the wave equation for sound propagation in the ocean with arbitrary depth dependence and arbitrary but gradual range dependence of the sound velocity distribution.

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### Acknowledgements

The support of the Naval Research Laboratory Code 8120 in the completion of this work is appreciated, as well as earlier support of the Office of Naval Research Code 486.

We wish to thank F. S. Chwieroth and F. Ingenito for helpful discussions, and R.H.Ferris for his interest.

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## I. INTRODUCTION

The purpose of the computer code described in this volume is to find the solutions of the eigenvalue problem given by the differential equation

$$\frac{\partial^2 u_p(z, \vec{\rho})}{\partial z^2} + \left[ \frac{\omega^2}{c^2(z, \vec{\rho})} - k_p^2(\vec{\rho}) \right] u_p(z, \vec{\rho}) = 0 \quad (1.1a)$$

and the boundary conditions

$$u_p(0, \vec{\rho}) = 0 \quad (1.1b)$$

$$\lim_{z \rightarrow \infty} u_p(z, \vec{\rho}) = 0 \quad (1.1c)$$

The variable  $\vec{\rho}$  is considered a parameter. The solution consists of sets of eigenfunctions

$$u_p(z, \vec{\rho}) \quad p = 1, 2, \dots, N(\vec{\rho})$$

and their associated eigenvalues  $k_p(\vec{\rho})$  for each desired value  $\vec{\rho}$ .

Inputs are the sound frequency  $f = \frac{\omega}{2\pi}$  and a function  $c(z, \vec{\rho})$  corresponding in this case to the measured or simulated sound velocity distribution in some part of the ocean.

In the program it is assumed that  $c(z, \vec{\rho})$  is given in the form of a two-dimensional grid of values:

$$c(z_1, \vec{\rho}_1), c(z_1, \vec{\rho}_2), \dots, c(z_1, \vec{\rho}_n) \\ c(z_k, \vec{\rho}_1), \dots, c(z_k, \vec{\rho}_n),$$

and that one can interpolate linearly between grid points. The set of values

$$c(z_i, \vec{\rho}_j) \quad i = 1, \dots, k$$

will be referred to as the sound velocity profile at range point  $\vec{\rho}_j$ . In practice series of sound profiles are usually taken along a straight line course so the vectors  $\vec{\rho}_j$  can be replaced by the scalar range variable  $\rho_j$ .

The problem as outlined above arises as part of the attempt of solving the wave equation

$$\rho(\vec{r}) \vec{\nabla} \cdot [\rho^{-1}(\vec{r}) \vec{\nabla} p(\vec{r})] + k^2(\vec{r}) p(\vec{r}) = \delta(\vec{r} - \vec{r}_0)$$

describing the pressure field of a unit point source (located at  $\vec{r}_0$ ) in an inhomogeneous ocean medium of density  $\rho(\vec{r})$  and a propagation constant  $k(\vec{r}) = \omega/c(\vec{r})$  where  $c(\vec{r})$  is assumed to depend arbitrarily on the depth  $z$  and arbitrarily, but gradually, on the horizontal range coordinate  $\vec{\rho} = (x, y)$ .

Assuming the density  $\rho(\vec{r})$  to be constant throughout horizontal layers and defining a velocity potential  $\phi(\vec{r})$  such that

$$p(\vec{r}) = \rho(\vec{r}) \partial \phi(\vec{r}) / \partial t, \quad (1.2)$$

one obtains for each layer a Helmholtz equation

$$\nabla^2 \phi(\vec{r}) + k^2(\vec{r}) \phi(\vec{r}) = \delta(\vec{r} - \vec{r}_0). \quad (1.3)$$

For a range independent sound velocity distribution, i.e. for  $c(\vec{r}) = c(z)$  the solution of (1.3) can be written as a normal mode sum of the form

$$\phi(\vec{r}) = \sum_p \psi_p(\vec{\rho}) u_p(z). \quad (1.4)$$

The "range functions"  $\psi_p(\vec{\rho})$  then satisfy

$$[\nabla_{\vec{\rho}}^2 + k_p^2(\vec{\rho})] \psi_p(\vec{\rho}) = \delta(\vec{\rho} - \vec{\rho}_0) u_p(z_0), \quad (1.5)$$

which is easily solved, while the "depth functions"  $u_p(z)$  are obtained by solving (1.1) with  $\vec{\rho} = \text{const}$ . For arbitrary but gradual range dependence one can attempt a solution in the form of an almost separated

normal mode sum which takes the form

$$\phi(\vec{r}) = \sum_p \psi_p(\vec{\rho}) u_p(z, \vec{\rho}) \quad (1.5)$$

The equation satisfied by the range functions becomes now much more complicated involving terms which couple all the modes together, whereas the depth functions can still be obtained from (1.1) except that they are now no longer range independent, but gradually range-dependent "local" depth functions.

The problem of arbitrary z-dependence, and arbitrary but gradual range dependence of  $c(\vec{r})$  is the case for which the programs described in this publication have been developed.

Vol. 1 is restricted to the discussion of the depth functions  $u_p(z, \vec{\rho})$  and a method for calculating them. A discussion of the general range equation and a method for obtaining the modal range functions  $\psi_p(\vec{\rho})$  together with a description of the associated computer code may be found in Vol. 2.

The method for solving (1.1) which was used in the computer code described here is based upon the assumption of piecewise linear sound velocity profiles. This assumption does not restrict the range of applicability of the code in any way. On the contrary, it enables one to obtain analytic solutions for the depth functions and thus makes it possible to obtain numerical solutions much more efficiently than by the method of numerical integration which is necessary for arbitrary profiles. In this way, normal mode theory becomes accessible to a much larger class of problems than otherwise possible. This is of particular

importance in view of the fact that the normal mode treatment is the most exact method of investigation sound propagation problems, which still works in cases where other methods (ray tracing, PE-method, etc.) fail. On the other hand, by inserting additional segments any velocity profile can, at least in principle, be approximated to any desired degree of accuracy by the present method. This would, of course, increase the computational effort again to unrealistic magnitudes. However, in practice, a very large number of segments is neither necessary nor is it in general possible, since for any given range point there is usually only a relatively small number of data points available to define the local sound velocity distribution.

## II. DEPTH FUNCTIONS FOR A PIECEWISE LINEAR SOUND VELOCITY PROFILE

Since the boundary conditions are given at  $z = 0$  and  $z = \infty$ , the depth functions  $u(z, \vec{p})$  have to be found within these limits. The vertical sound velocity distribution at some particular range point  $\vec{p}$  which determines the general solution of the depth function equation at this point is, as mentioned earlier, usually approximated by a piecewise linear function:

$$c(z) = c(z_i) + s_i^c (z - z_i) \quad z_i \leq z \leq z_{i+1} \quad (2.1a)$$

with

$$s_i^c = [c(z_{i+1}) - c(z_i)] / (z_{i+1} - z_i) \quad (i = 1, 2, \dots, K+1) \quad (2.1b)$$

The range  $0 \leq z \leq \infty$  is thus subdivided into horizontal layers with boundaries at  $z = z_i, z_{i+1}$  for the  $i$ th layer. The method to proceed with is therefore to find the solutions of the depth function equation within each layer and match the solutions at the layer boundaries. The conditions to be satisfied are that

$$\rho(z) u_p(z, \vec{p}) \quad \text{and} \quad \partial u_p(z, \vec{p}) / \partial z \quad (2.2)$$

are continuous across each boundary. These follow from the requirement that the pressure

$$p = \rho \partial \phi / \partial t$$

and the velocity

$$\vec{v} = -\vec{\nabla} \phi$$

be continuous everywhere.

The water extends from  $z = 0$  to  $z = z_B$ . The water density is assumed to be constant everywhere i.e.

$$\rho(z) = \rho_w \quad 0 \leq z \leq z_B \quad (2.3)$$

The ocean floor is assumed to consist of isovelocity layers

with the density constant within each layer. At the present time only one layer is considered, hence the ocean floor is characterized by 2 numbers

$$\rho(z) = \rho_0 \quad \tau(z) = \tau_B \quad z > z_B$$

However, if the need arises these assumptions can easily be generalized to several layers with linearly varying profiles.

Within the water  $c(z, \vec{\rho})$  at some particular  $\vec{\rho}$  varies between  $c_{\min}(\vec{\rho})$  and  $c_{\max}(\vec{\rho})$ , where always  $c_{\max}(\vec{\rho}) \leq c_B(\vec{\rho})$ .

A solution of the eigenvalue problem (1.1) is obtained only if the local vertical wave number  $K_p(z, \vec{\rho})$ ,

$$K_p(z, \vec{\rho}) = \left\{ \left[ \omega / c(z, \vec{\rho}) \right]^2 - k_p^2(\vec{\rho}) \right\}^{1/2} \quad (2.4)$$

is real for at least part of the range  $0 < z < z_B(\vec{\rho})$ . This limits the range of possible eigenvalues to

$$k_p(\vec{\rho}) < \omega / c_{\min}(\vec{\rho}). \quad (2.5a)$$

Discrete eigenvalues are obtained for

$$k_p(\vec{\rho}) > \omega / c_B(\vec{\rho}). \quad (2.5b)$$

For

$$k_p(\vec{\rho}) < \omega / c_B(\vec{\rho}),$$

any value of  $k_p$  will provide a solution to eigenvalue problem (1.1).

This report is concerned only with the calculation of discrete eigensolutions. With the assumptions made above the solution within the ocean floor, i.e. for the range

$$z_B(\vec{\rho}) \leq z < \infty$$

which satisfies the boundary conditions at  $\infty$  is simply

$$u_p^B(z, \vec{\rho}) = a_p^B(\vec{\rho}) \exp \{ -\gamma_p(\vec{\rho}) z \} \quad (2.6a)$$

with

$$\gamma_p(\vec{\rho}) = \left\{ k_p^2 - \omega^2 / c_B^2(\vec{\rho}) \right\}^{1/2}. \quad (2.6b)$$

This leaves only the solutions for the range

$$0 \leq z \leq z_B(\vec{\rho}),$$

i.e.  $\mu_p^W(z, \vec{\rho})$ , the solutions within the water, to be determined.

They are obtained by solving (1.1a) with  $c(z)$  as given by (2.1) or more precisely with piecewise linear distribution of  $\{\omega/c(z)\}^2$ :

$$k_o^2(z) = \{\omega/c(z)\}^2 = \{\omega/c(z_i)\}^2 + s_i(z-z_i) \quad z_i \leq z \leq z_{i+1} \quad (2.7a)$$

with

$$s_i = \{[\omega/c(z_{i+1})]^2 - [\omega/c(z_i)]^2\} / (z_{i+1} - z_i). \quad i=1, 2, \dots, k \quad (2.7b)$$

Assuming a linear dependence of  $k_o^2$  on  $z$  is compatible to first order with the assumption of a linear  $z$ -dependence of  $c$  if the variations in  $c(z)$  are small compared to the magnitude of  $c(z)$ :

$$k_o^2(z) = \{\omega/c(z)\}^2 = \omega^2 / \{c(z_i) + s_i^c(z-z_i)\}^2,$$

which with

$$\Delta c_i = |c(z_{i+1}) - c(z_i)| \ll c(z_i) \quad (2.8)$$

can be written as

$$k_o^2(z) \cong \{\omega/c(z_i)\}^2 - 2\{\omega/c(z_i)\}^2 \{s_i^c/c(z_i)\}(z-z_i)$$

Hence, if  $c(z)$  is a linear function with slope  $s_i^c$  then  $k_o^2(z)$  is also approximately a linear function with slope  $s_i = -2\omega^2 s_i^c / \{c(z_i)\}^3$  and vice versa, if (2.8) holds. Condition (2.8) is satisfied for sound velocity profiles since typically

$$\Delta c < 50 \text{ m/sec}, \quad c \cong 1500 \text{ m/sec}.$$

Proceeding with the assumption that  $k_o^2(z)$  can be linearized according to (2.8) the depth function equation for the  $i$ th layer

at some particular  $\vec{\rho}$  becomes

$$d^2 u_p(z, \vec{\rho}) / dz^2 + \{ [\omega/c(z_i, \vec{\rho})]^2 + s_i(\vec{\rho})(z - z_i) - k_p^2(\vec{\rho}) \} u_p(z, \vec{\rho}) = 0$$

or

$$d^2 u_p(z, \vec{\rho}) / dz^2 + \{ dp_i(\vec{\rho}) + s_i(\vec{\rho})z \} u_p(z, \vec{\rho}) = 0$$

with

$$dp_i(\vec{\rho}) = \{ \omega/c(z_i, \vec{\rho}) \}^2 - s_i(\vec{\rho})z_i - k_p^2(\vec{\rho}). \quad (2.9)$$

The substitution

$$\zeta_{pi}(z, \vec{\rho}) = -\{ [s_i(\vec{\rho})]^2 \}^{-1/3} \{ dp_i(\vec{\rho}) + s_i(\vec{\rho})z \} \equiv \alpha_i(\vec{\rho}) [z + \beta_i(\vec{\rho})] \quad (2.10)$$

transforms this equation into the Airy equation

$$d^2 u_p / d\zeta^2 = \zeta u_p.$$

Therefore, the solution for the  $p$ th mode in the  $i$ th layer in the water becomes

$$u_{pi}^w(z, \vec{\rho}) = a_{pi} A_i(\zeta_{pi}) + b_{pi} B_i(\zeta_{pi}), \quad z_i \leq z \leq z_{i+1} \quad (2.11)$$

There are 2 constants to be determined for each layer. By watching the solutions and their derivatives at the layer boundaries the coefficients of one layer can be expressed in terms of quantities from the preceding layer.

$$\text{Abbreviating } \begin{cases} u_{pi}(z) = u_{pi}^w(z, \vec{\rho}), & u_{pi}'(z) = u_{pi}^w'(z), \\ A_i = A_i(\zeta_{pi})|_{z=z_i}, & A_{i-1} = A_i(\zeta_{pi-1})|_{z=z_i}, & \alpha_i = d\zeta_i/dz, \end{cases}$$

one can write the matching conditions at the boundary between the  $i$ th and the  $(i-1)$ st layer as

$$u_{pi-1}(z_i) \equiv a_{pi-1} A_{i-1} + b_{pi-1} B_{i-1} = a_{pi} A_i + b_{pi} B_i \equiv u_{pi}(z_i) \quad (2.12)$$

$$u_{pi-1}'(z_i) \equiv -\alpha_{i-1} [a_{pi-1} A_{i-1}' + b_{pi-1} B_{i-1}'] = -\alpha_i [a_{pi} A_i' + b_{pi} B_i'] \equiv u_{pi}'(z_i).$$

Solving these two equations for  $a_i$  and  $b_i$  in terms of  $u_{i-1}(z_i)$

and  $u'_{i-1}(z_i)$  yields

$$a_{pi} = \pi \left\{ u_{pi-1}(z_i) B'_i + \alpha_i^{-1} u'_{pi-1}(z_i) B_i \right\} \quad (2.13)$$

$$b_{pi} = \pi \left\{ -u_{pi-1}(z_i) A'_i - \alpha_i^{-1} u'_{pi-1}(z_i) B_i \right\}.$$

The first derivatives of the depth functions are obtained from (2.11) as

$$-\alpha_i^{-1} u_{pi}^{w'}(z_i, \vec{\rho}) = a_{pi} A'_i(\zeta_{pi}) + b_{pi} B'_i(\zeta_{pi}), \quad (2.14)$$

keeping in mind that the derivatives of the Airy functions are with respect to  $\zeta$ . When matching derivatives at the layer boundaries one has of course to use derivatives with respect to  $z$ , as was done in (2.12). However, in order to propagate the solutions for each mode from one layer to the next, one can, as suggested by eqs. (2.13) and (2.14), use just as well  $\zeta$ -derivatives. This approach was taken in the program described here. Defining

$$u_{pi}(z_i)_\zeta = -\alpha_i^{-1} u'_{pi}(z_i),$$

the coefficients (2.13) then become:

$$\begin{aligned} a_{pi} &= \pi \left[ u_{pi-1}(z_i) B'_i - T u_{pi-1}(z_i)_\zeta B_i \right] \\ b_{pi} &= \pi \left[ -u_{pi-1}(z_i) A'_i + T u'_{pi-1}(z_i)_\zeta A_i \right] \end{aligned} \quad (2.13')$$

with

$$T = \alpha_{i-1} / \alpha_i.$$

An alternate form for propagating the solutions, also used in the program, is

$$\begin{aligned} u_{pi}^{w'}(z_i, \vec{\rho}) &= c_{11} u_{pi-1}(z_i) + c_{12} u_{pi-1}(z_i)_\zeta \\ u_{pi}^{w'}(z_i, \vec{\rho})_\zeta &= c_{21} u_{pi-1}(z_i) + c_{22} u'_{pi-1}(z_i)_\zeta \end{aligned} \quad (2.15)$$

where

$$\begin{aligned}
\tau_{11} &= \pi \{ A_i'(\zeta_{pi}) B_i - B_i(\zeta_{pi}) A_i' \} \\
\tau_{12} &= \pi T \{ B_i(\zeta_{pi}) A_i - A_i(\zeta_{pi}) B_i \} \\
\tau_{21} &= \pi \{ A_i'(\zeta_{pi}) B_i' - B_i(\zeta_{pi}) A_i' \} \\
\tau_{22} &= \pi T \{ B_i'(\zeta_{pi}) A_i - A_i'(\zeta_{pi}) B_i \}.
\end{aligned}$$

Using eqs. (2.11), (2.13) and (2.14), the full solution  $u^W(\bar{z}, \vec{\rho})$  in the water can be determined in terms of just two unknown constants, which can be e.g.  $a_{p1}$  and  $b_{p1}$ , i.e. the coefficients in the layer nearest the surface. One constraint for these coefficients comes from the boundary conditions at the surface:

$$u_p^W(0, \vec{\rho}) = 0 \quad (2.16)$$

Hence from (2.11):

$$a_{p1} A_1 + b_{p1} B_1 = 0. \quad (2.17)$$

If one chooses for  $a_{p1}$  the arbitrary but convenient value

$$a_{p1} = B_1, \quad (2.18)$$

then  $b_{p1}$  is determined:

$$b_{p1} = -A_1, \quad (2.19)$$

and one obtains for the slope of the solution just below the surface from (2.14)

$$u_{p1}^{W'}(0, \vec{\rho}) = -\alpha_1 (B_1 A_1' - A_1 B_1') = \alpha_1(\vec{\rho}) / \pi. \quad (2.20)$$

This choice of coefficients ensures that all solutions start out with a positive derivative near the surface:

$$u_{p1}^{W'}(0, \vec{\rho}) > 0, \quad (2.21)$$

which makes the  $\zeta$  derivative

$$u_{p1}^{W'}(0, \vec{\rho})_{\zeta} = -[\alpha_1(\vec{\rho})]^{-1} u_{p1}^{W'}(0, \vec{\rho}) = -\pi^{-1} \operatorname{sgn}[\alpha_1(\vec{\rho})]. \quad (2.22)$$

Using (2.11) and (2.14) with (2.18) and (2.19) one can determine the solution and the derivative at the lower boundary of the first layer. In terms of these quantities the coefficients for the second layer can be found using (2.13). In this way the solution  $u^W(z, \vec{\rho})$  can be propagated to  $z = z_B$ , the ocean floor. There the functions  $u_p^W(z, \vec{\rho})$  and  $u_p^B(z, \vec{\rho})$  have to be matched. This leads to the conditions

$$\rho_W u_p^W(z_B, \vec{\rho}) = \rho_B u_p^B(z_B, \vec{\rho}) \quad (2.23)$$

$$\left. \frac{\partial u_p^W(z, \vec{\rho})}{\partial z} \right|_{z=z_B} = \left. \frac{\partial u_p^B(z, \vec{\rho})}{\partial z} \right|_{z=z_B} = -\gamma_p(\vec{\rho}) u_p^B(z_B, \vec{\rho}).$$

(2.24)

As indicated in (2-7a),  $u_p^B$  contains one undetermined coefficient. It can be found by using (2.23). The choice (2.18) was made because it provides a convenient starting point for numerical calculations, as will become apparent below. But it was arbitrary in the sense that solutions calculated with this choice do in general not satisfy the normalization condition for normal mode solutions, in this case:

$$\int_0^{\infty} \frac{\rho(z)}{\rho_W} |u_p(z, \vec{\rho})|^2 dz = 1.$$

(2.25)

To remedy this situation the integral over the square of the unnormalized solution is evaluated to find the necessary normalization factor

$$A_p = \left\{ \int_0^{z_B} |u_p^W(z, \vec{\rho})|^2 dz + (\rho_B/\rho_W) \int_{z_B}^{\infty} |u_p^B(z, \vec{\rho})|^2 dz \right\}^{-1/2} \quad (2.26)$$

where

$$\int_0^{z_B} |u_p^W(z, \vec{\rho})|^2 dz = \sum_{i=1}^N \frac{1}{\alpha_{pi}(\vec{\rho})} \int_{\zeta_{pi}(z_{i-1}, \vec{\rho})}^{\zeta_{pi}(z_i, \vec{\rho})} [a_{pi} Ai(\zeta_{pi}) + b_{pi} Bi(\zeta_{pi})]^2 d\zeta_{pi} \quad (2.27a)$$

$$\begin{aligned} \frac{\rho_B}{\rho_W} \int_{z_B}^{\infty} |u_p^B(z, \vec{\rho})|^2 dz &= \\ &= \frac{\rho_B}{\rho_W} \int_{z_B}^{\infty} [a_{pN}^B(\vec{\rho})]^2 e^{-2\gamma_p(\vec{\rho})z} dz = \frac{\rho_B}{\rho_W} \frac{[a_{pN}^B(\vec{\rho})]^2}{2\gamma_p(\vec{\rho})} e^{-2\gamma_p(\vec{\rho})z_B} \end{aligned} \quad (2.27b)$$

The first integral can also be carried out analytically using

$$\int_{\zeta_1}^{\zeta_2} Ai(\zeta)^2 d\zeta = \left[ \zeta (Ai(\zeta))^2 - (Ai'(\zeta))^2 \right]_{\zeta_1}^{\zeta_2} \quad (2.28a)$$

Similarly

$$\int_{\zeta_1}^{\zeta_2} Ai(\zeta) Bi(\zeta) d\zeta = \left[ \zeta Ai(\zeta) Bi(\zeta) - Ai'(\zeta) Bi'(\zeta) \right]_{\zeta_1}^{\zeta_2},$$

and

$$\int_{\zeta_1}^{\zeta_2} (Bi(\zeta))^2 d\zeta = \left[ \zeta (Bi(\zeta))^2 - (Bi'(\zeta))^2 \right]_{\zeta_1}^{\zeta_2} \quad (2.28b)$$

$$(2.28c)$$

These expressions follow from

$$\int C_1(\zeta) C_2(\zeta) d\zeta = \zeta C_1'(\zeta) C_2(\zeta) - C_1'(\zeta) C_2'(\zeta)$$

by differentiating with respect to  $\zeta$ :

$$C_1 C_2' = C_1' C_2 + \zeta (C_1' C_2 + C_1 C_2') - C_1'' C_2 - C_1' C_2''.$$

Upon rewriting this equation as

$$C_2' (C_1'' - \zeta C_1') + C_1' (C_2'' - \zeta C_2') = 0$$

it is apparent that it is satisfied for  $C_1$  and  $C_2$  being any solution of the Airy equation.

The functions  $u_p(z, \vec{\rho})$  as discussed so far still do not satisfy eq. (2.24). The coefficients of the two partial solutions  $u_p^W(z, \vec{\rho})$  and  $u_p^B(z, \vec{\rho})$  were chosen such that the boundary conditions at  $z = 0$  and  $z = \infty$  were satisfied and that  $u_p(z, \vec{\rho})$  is continuous everywhere. The derivative  $u_p'(z, \vec{\rho})$  is at this point continuous everywhere inside the water and inside the ocean floor. However, so far no provisions have been made to make it also continuous at  $z = z_B$ . In order to achieve this,  $u_p^W$  and  $u_p^B$  have to be adjusted such that (2.24) is satisfied. This can be done by choosing the proper values for  $k_p(\vec{\rho})$  which enter  $u_p^W$  and  $u_p^B$  through (2.9) and (2.6b). In this sense eq. (2.24) provides the characteristic equation for <sup>the</sup> eigenvalue problem at hand. A more convenient form of the characteristic equation

is found by dividing (2.24) by (2.23), which corresponds to matching logarithmic derivatives at the ocean floor:

$$\left. \frac{\partial u_p^W(z, \vec{\rho})}{\partial z} \right|_{z=z_B} = - \frac{\rho_W}{\rho_B} \gamma_p(\vec{\rho}) u_p^W(z_B, \vec{\rho}). \quad (2.29)$$

The characteristic equation defining the eigenvalues  $k_p(\vec{\rho})$  can, of course, be established at the boundary between any two layers. It turns out that sometimes it is advantageous to propagate the solution from infinity up to the boundary between two water layers. The characteristic equation is then

$$u_{p\downarrow}(z_M, \vec{\rho}) u'_{p\uparrow}(z_M, \vec{\rho}) = u'_{p\downarrow}(z_M, \vec{\rho}) u_{p\uparrow}(z_M, \vec{\rho}), \quad (2.30)$$

assuming the match occurs at the boundary between the (M-1)st and the Mth layer. Here  $u_{p\downarrow}$  is the solution started at  $z = 0$ ,  $u_{p\uparrow}$  the solution started at  $\infty$ .

Both (2.23) and (2.30) are used in the paper and will be referred to later on.

### III. ARGUMENTS OF THE AIRY FUNCTIONS AND GENERAL PROPERTIES OF THE EIGENVALUE SPECTRUM

Since the Airy functions play a central role in the problem discussed in this report it is useful to point out some general properties of these functions and to try to get an idea of how the various parameters which enter the problem affect the regions in which the Airy functions are needed. It turns out that the arguments of the Airy functions within a particular layer can be expressed to a good approximation in a relatively simple way in terms of the frequency, the slope of the velocity profile and a quantity with the dimension of a sound velocity which indicates how far the eigenvalue is above or below the wave number corresponding to the local sound speed. This formula can not only be used to determine the range over which the arguments of the Airy functions vary for a particular profile but also to estimate the total number of modes possible and the average separation between successive eigenvalues as a function of mode number.

The Airy functions are characterized by the observation that they behave like an oscillating function for negative arguments and like an exponentially rising or falling function for positive arguments. The characteristics are clearly brought out by the asymptotic expressions for large arguments ( $|\zeta| \gg 1$ ):

$$\begin{aligned}
 \text{Ai}(\zeta) &\sim \frac{e^{-\eta}}{2\sqrt{\pi}\zeta^{1/4}} \left\{ 1 - \frac{15}{216} \frac{1}{\eta} + \dots \right\} \\
 \text{Bi}(\zeta) &\sim \frac{e^{\eta}}{\sqrt{\pi}\zeta^{1/4}} \left\{ 1 + \frac{15}{216} \frac{1}{\eta} + \dots \right\}
 \end{aligned}
 \quad (\eta = \frac{2}{3} \zeta^{3/2})$$

$$\text{Ai}(-\zeta) \sim \frac{1}{\sqrt{\pi} \zeta^{1/4}} \left\{ \left( 1 - \frac{35 \times 99}{2 \times 216^2} \frac{1}{\eta^2} + \dots \right) \sin \left( \eta + \frac{\pi}{4} \right) - \left( \frac{15}{216} \frac{1}{\eta} - \dots \right) \cos \left( \eta + \frac{\pi}{4} \right) \right\} \quad (3.1)$$

$$\text{Bi}(-\zeta) \sim \frac{1}{\sqrt{\pi} \zeta^{1/4}} \left\{ \left( 1 - \frac{35 \times 99}{2 \times 216^2} \frac{1}{\eta^2} + \dots \right) \cos \left( \eta + \frac{\pi}{4} \right) + \left( \frac{15}{216} \frac{1}{\eta} - \dots \right) \sin \left( \eta + \frac{\pi}{4} \right) \right\}.$$

The argument of the Airy functions is given by (2.10) with  $d_{pi}(\vec{\phi})$  and  $s_i(\vec{\phi})$  defined by (2.9) and (2.7b). It follows that (ignoring the  $\vec{\phi}$ -dependence and writing  $c(z_i)$  as  $c_i$  etc.):

$$\zeta_{pi}(z) = - (s_i)^{-2/3} [k_o^2(z) - k_p^2] \quad (3.2)$$

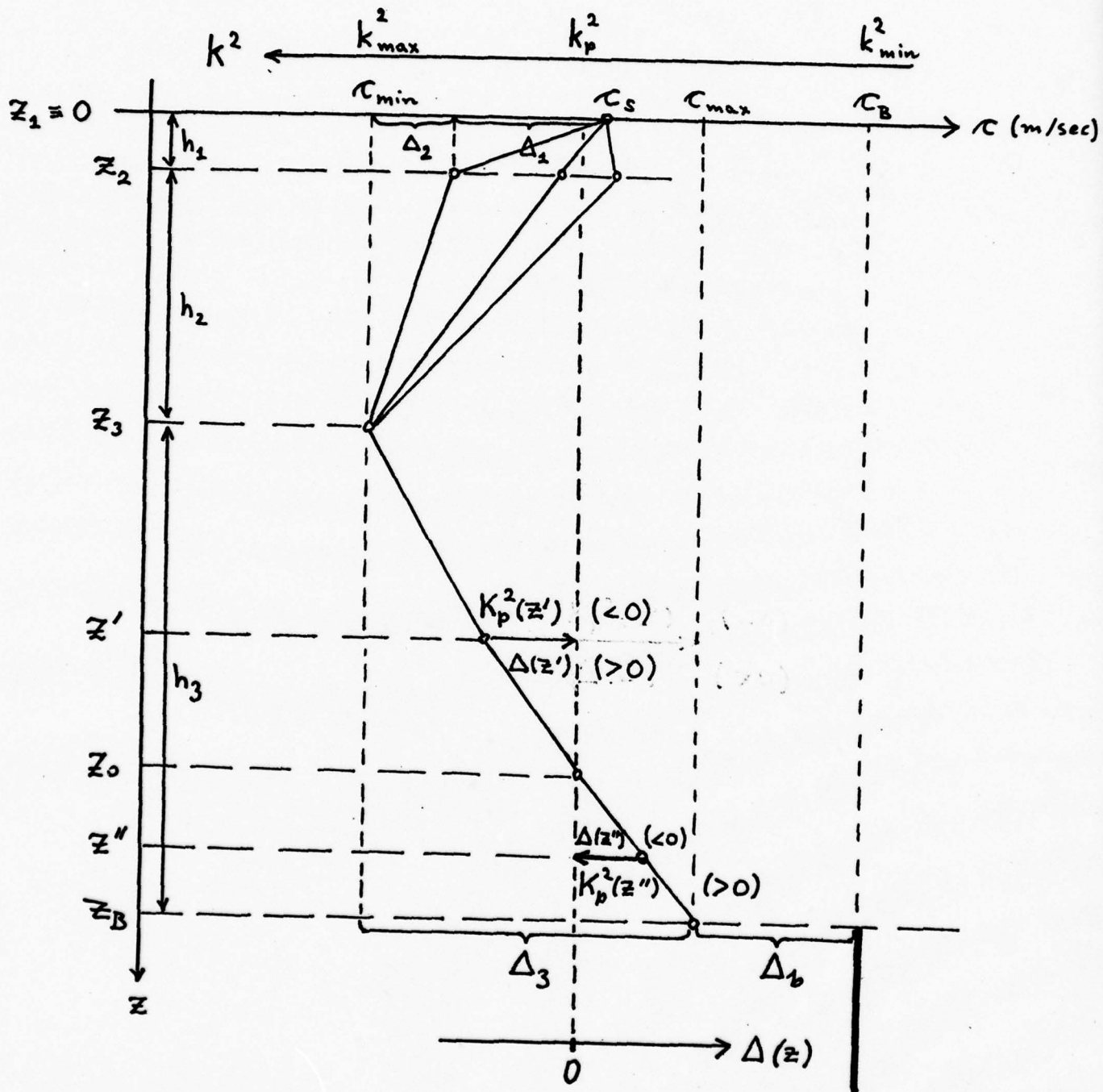
$$k_o^2(z) = \frac{\omega^2}{c_i^2} + s_i (z - z_i)$$

denoting the  $k^2$  - value as a function of  $z$  along the profile. As indicated by (2.5a) and (2.5b) the range of discrete eigenvalues is restricted to

$$\frac{\omega}{c_B} = k_{\min} < k_p < k_{\max} = \frac{\omega}{c_{\min}}.$$

This is indicated in Fig. 1 where schematically a typical profile is shown.

Fig. 1



The quantity which essentially determines the argument of the Airy functions as a function of  $z$  for a particular mode  $p$  is the local vertical wave number  $K_p(z)$  defined by

$$K_p^2(z) = k_0^2(z) - k_p^2 \quad (3.3)$$

Defining new variables

$$\mathcal{K}_0^2(z) = -k_0^2(z) + \omega^2/c_{\min}^2$$

$$\mathcal{K}_p^2 = -k_p^2 + \omega^2/c_{\min}^2$$

with the ranges

$$0 \leq \mathcal{K}_0^2(z) \leq \frac{\omega^2}{c_{\min}^2} - \frac{\omega^2}{c_{\max}^2} = \frac{\omega^2}{c_{\min}^2} - \frac{\omega^2}{(c_{\min} + \Delta_3)^2} \approx \frac{2\omega^2}{c_{\min}^3} \Delta_3$$

and

$$\mathcal{K}_p^2 = \begin{cases} 0 & (p=1) \\ \frac{\omega^2}{c_{\min}^2} - \frac{\omega^2}{c_B^2} = \frac{\omega^2}{c_{\min}^2} - \frac{\omega^2}{(c_{\min} + \Delta_3 + \Delta_b)^2} \approx \frac{2\omega^2(\Delta_3 + \Delta_b)}{c_{\min}^3} & (p=N) \end{cases}$$

one can see that

$$K_p^2(z) = \mathcal{K}_p^2 - \mathcal{K}_0^2(z)$$

varies between

$$- 2\omega^2 \Delta_3 / c_{\min}^3$$

for  $p = 1$  (i.e. for the eigenvalue with the lowest mode number)

and at  $z = z_B$  (the  $z$ -value with the highest sound velocity), and

$$2\omega^2 (\Delta_3 + \Delta_b) / c_{\min}^3$$

for  $p = N$  (for the eigenvalue with the highest mode number and at  $z = z_3$  (the depth with the minimum sound speed), with values

$$2\omega^2 \Delta(z) / c_{\min}^3$$

in between, where

$$-\Delta_3 \leq \Delta(z) \leq \Delta_3 + \Delta_b. \quad (3.4)$$

Hence the coordinate axis of  $\Delta(z)$  has the same units as  $c(z)$  but its origin is not fixed in Fig. 1. Instead, it shifts around and its position corresponds to the location of the square of the eigenvalue ( $k_p^2$ ) on the  $k^2$  - axis.

The slopes  $s_i$  can be written as

$$s_i = \frac{1}{h_i} \left( \frac{\omega^2}{c_{i+1}^2} - \frac{\omega^2}{c_i^2} \right) \quad (3.5)$$

$$\cong 2s_i^c \omega^2 / \omega_{\min}^2 = 2(\Delta_i/h_i)(\omega^2/c_{\min}^3).$$

With these definitions and approximations the argument of the Airy functions in the  $i$ th layer for the  $p$ th mode can be written as

$$\zeta_{pi}(z) = -\frac{2^{1/3}}{c_{\min}} \left( \frac{\omega}{\Delta_i/h_i} \right)^{2/3} \Delta(z). \quad (3.6)$$

Hence, it turns out that positive arguments are generally obtained for low wave numbers, negative arguments for high wave numbers. The slope of the profile in a particular layer does not affect the sign of the argument. However, the absolute value of the argument may grow very large as the slope approaches zero. Layers with small or vanishing vertical sound speed gradients are frequently encountered near the surface and of course near sound speed minima. Some care is necessary to maintain the accuracy of the solutions in these cases as will be shown below.

In order to get some numerical values the following parameters were chosen:

$$f = 250 \text{ Hz}$$

$$s_1^c = \Delta_1 / h_1 = 0.1, 5, 100 \text{ m/sec per } 100 \text{ m}$$

$$s_2^c = \Delta_2 / h_2 = 5 \text{ m/sec per } 100 \text{ m}$$

$$s_3^c = \Delta_3 / h_3 = 1.5 \text{ m/sec per } 100 \text{ m}$$

$$c_{\min} \approx 1500 \text{ m/sec.}$$

The three values chosen for  $s_1^c$  correspond to low, average and high sound velocity gradients as usually encountered at small or intermediate depths. The value chosen for  $s_3^c$  is typical for greater depths. With these numbers are obtained for the Airy function arguments:

$$\zeta_{p1}(z) = -11.4 \Delta(z), \quad -0.84 \Delta(z), \quad -0.114 \Delta(z)$$

$$\zeta_{p2}(z) = -0.84 \Delta(z)$$

$$\zeta_{p3}(z) = -1.87 \Delta(z).$$

For  $f = 25$  Hz all coefficients are smaller by a factor of  $10^{2/3} = 4.64$ . For the lowest eigenvalue:

$$\begin{aligned}\Delta(0) &\cong -(\Delta_1 + \Delta_2) \\ \Delta(z_2) &\cong -\Delta_2 \\ \Delta(z_3) &\cong 0 \\ \Delta(z_B) &\cong -\Delta_3.\end{aligned}\tag{3.7a}$$

For the highest eigenvalue:

$$\begin{aligned}\Delta(0) &\cong \Delta_3 + \Delta_b - \Delta_1 - \Delta_2 \\ \Delta(z_2) &\cong \Delta_3 + \Delta_b - \Delta_2 \\ \Delta(z_3) &\cong \Delta_3 + \Delta_b \\ \Delta(z_B) &\cong \Delta_b.\end{aligned}\tag{3.7b}$$

The parameters encountered in a deep ocean case may be e.g.

$$\begin{aligned}h_1 &= 400 \text{ m}, \quad h_2 = 600 \text{ m}, \quad h_3 = 3000 \text{ m} \\ \Delta_1 &= 0.4 \text{ m/sec}, \quad \Delta_2 = 30 \text{ m/sec}, \quad \Delta_3 = 45 \text{ m/sec}, \quad \Delta_b = 300 \text{ m/sec}.\end{aligned}\tag{3.8}$$

This leads to the following Airy function arguments for the lowest mode:

- $\sim 346$  at the surface ( $z = 0$ )
- $\sim 340$  at the end of the first layer ( $z = z_2$ )
- $\sim 25$  at the beginning of the second layer ( $z = z_2$ )
- $\sim 0$  at the end of the second layer ( $z = z_3$ )

$\sim 0$  at the beginning of the third layer ( $z = z_3$ )

$\sim 84$  at the ocean floor ( $z = z_B$ )

The corresponding numbers for the highest mode are

$\sim -3570, -3575, -263, -288, -644, -560$  for the quoted value  
of  $\Delta_b$

$\sim -165, -170, -4, -38, -84, 0$  for  $\Delta_b = 0$ .

It is apparent that a combination of high frequency, small slopes in the profile, very low and very high mode numbers can result in very large positive or negative arguments for the Airy functions. From the asymptotic expressions for the Airy functions one can see that large positive arguments are particularly serious since they can lead to exponents of the order of several thousand. For the lowest mode e.g., the Airy functions are of the order of unity near  $z = z_2$ , but in the example given, of the order

$$e^{1/3 \cdot 84^{3/2}} \cong 10^{223}$$

near  $z = z_b$ , i.e. the eigenfunctions for the lowest mode grow over more than 200 orders of magnitude between the ocean floor and the location of the sound speed minimum. For very small slopes which may occur near the surface or near a sound speed minimum the exponents may be even higher. As the small slopes

do not usually persist over large distances the growths of the Airy functions between the beginning and the end of a layer are relatively small, but the coefficients obtained when matching the solutions to the other layers become small to the same order as the Airy function grows large, and near-cancellation of very large numbers occurs. Special precautions have to be taken to avoid overflows and underflows in the computer and to ensure that the accuracy of the results does not deteriorate due to the near cancellation of large numbers. This problem will be addressed in the next section.

Eq. (3.6) can be used to estimate the number of discrete modes available for a particular profile and frequency.

The highest mode is an oscillatory function all the way from the surface to the ocean floor since the arguments of the Airy functions are negative throughout. Also since the arguments are large in most of the region the Airy functions can be approximated by

$$Ai(-\zeta) \propto \sin\left(\frac{2}{3}\zeta^{3/2} + \frac{\pi}{4}\right)$$

$$Bi(-\zeta) \propto \sin\left(\frac{2}{3}\zeta^{3/2} + \frac{3\pi}{4}\right).$$

By keeping track of the phase differences of the sines between the two boundaries of each layer and adding them up over the whole water depth one can approximately determine the number

of half-waves accommodated in the water column and thus deduce the wave number for the highest mode and at the same time the total number of modes supported by the profile:

$$N \cong \frac{2}{3\pi} \sum_{j=1}^M \left| \left[ -\zeta_{N_j}(z_{j+1}) \right]^{3/2} - \left[ -\zeta_{N_j}(z_j) \right]^{3/2} \right|, \quad (3.9)$$

or using (3.6)

$$N \cong \frac{2}{3\pi} \frac{\sqrt{2}\omega}{c_{\min}^{3/2}} \sum_{j=1}^M \frac{h_j}{\Delta_j} \left| \Delta^{3/2}(z_{j+1}) - \Delta^{3/2}(z_j) \right|. \quad (3.9b)$$

Usually a good estimate can be obtained by approximating the profile by two or three straight lines.

With (3.7b) one obtains

$$N \cong \frac{4\sqrt{2}}{3} \frac{f}{c_{\min}^{3/2}} \left\{ \frac{h_1}{\Delta_1} \left| (\Delta_3 + \Delta_b - \Delta_2)^{3/2} - (\Delta_3 + \Delta_b - \Delta_1 - \Delta_2)^{3/2} \right| \right. \\ \left. + \frac{h_2}{\Delta_2} \left| (\Delta_3 + \Delta_b)^{3/2} - (\Delta_3 + \Delta_b - \Delta_2)^{3/2} \right| + \frac{h_3}{\Delta_3} \left| \Delta_b^{3/2} - (\Delta_3 + \Delta_b)^{3/2} \right| \right\}. \quad (3.10)$$

For  $\Delta_b \gg \Delta_1, \Delta_2, \Delta_3$ :

$$N \cong \frac{2\sqrt{2}f}{c_{\min}^{3/2}} \left\{ h_1 (\Delta_3 + \Delta_b - \Delta_2)^{1/2} + h_2 (\Delta_3 + \Delta_b)^{1/2} \right. \\ \left. + h_3 \Delta_b^{1/2} \right\} \quad (3.11)$$

$$\cong \frac{2\sqrt{2}f}{c_{\min}^{3/2}} H \sqrt{\Delta_b},$$

where  $H$  is the total water depth. For the parameters chosen above one obtains 843 modes. For  $\Delta_b = 0$  the result would be ( $\Delta_1 \ll \Delta_3$ ):

$$N \cong \frac{4\sqrt{2}}{3} \frac{f}{c_{\min}^{3/2}} \left\{ h_1 (\Delta_3 - \Delta_2)^{1/2} + \frac{h_2}{\Delta_2} \left[ \Delta_3^{3/2} - (\Delta_3 - \Delta_2)^{3/2} \right] + h_3 \Delta_3^{1/2} \right\} \cong 215 \text{ modes.} \quad (3.12)$$

Two approximations of (3.12) are of special interest:

$$\Delta_3 = \Delta_2: \quad N \cong \frac{4\sqrt{2}}{3} \frac{f}{c_{\min}^{3/2}} (h_2 + h_3) \sqrt{\Delta_3}$$

and

$$\Delta_2 \ll \Delta_3: \quad N \cong \frac{4\sqrt{2}}{3} \frac{f}{c_{\min}^{3/2}} \left( \frac{3}{2} h_1 + \frac{3}{2} h_2 + h_3 \right) \sqrt{\Delta_3}.$$

Hence it turns out that the total number of modes lies approximately in the range

$$N = \left( 1 \dots \frac{3}{2} \right) \frac{4\sqrt{2}}{3} \frac{f}{c_{\min}^{3/2}} H \sqrt{\Delta} \quad (3.13)$$

with  $\Delta$  = maximum sound speed difference,  $H$  = total water depth.

To determine the average spacing between eigenvalues one only has to divide the difference

$$k_{\max} - k_{\min} = \frac{\omega}{c_{\min}} - \frac{\omega}{c_{\max}} \cong \frac{\omega}{c_{\min}^2} (\Delta_2 + \Delta_b) \quad (3.14)$$

by the total number of eigenvalues as determined above. However, some care is necessary to obtain a useful result. It turns out that in the two regions

$$\omega^2/c_{\max}^2 < k^2 < \omega^2/c_{\min}^2, \quad (3.15)$$

$$\omega^2/c_b^2 < k^2 < \omega^2/c_{\max}^2, \quad (3.16)$$

the eigenvalues are spaced very differently. For the first region to which the eigenvalues are confined when  $c_b = c_{\max}$  (i.e.  $\Delta_b = 0$ ) the approximate structure of the eigenvalue spectrum can be found by calculating the number of modes as a function of the wave number for a two-layer profile (i.e.  $k_1 = 0$ ). Then the total phase accumulated in the profile is given by the two Airy function arguments at  $z = z_3$ , i.e.

$$\Delta(z) = \Delta(z_3).$$

If one takes  $\Delta(z_3)$  as a function of  $k$  varying between 0 and  $\Delta_3$  one can obtain the number of modes as a function of  $k$  as

$$n = \frac{4\sqrt{2}}{3} \frac{f}{c_{\min}^{3/2}} \left\{ \frac{h_2}{\Delta_2} + \frac{h_3}{\Delta_3} \right\} [\Delta(k)]^{3/2}$$

with  $\Delta(k)$  determined by

$$\frac{\omega^2}{c_{\min}^2} - \frac{\omega^2}{[c_{\min} + \Delta(k)]^2} = k_{\max}^2 - (k_{\max} - \delta k)^2$$

$$\text{or } \Delta(k) \cong (\epsilon_{\min}^2 / \omega) \delta k = (\epsilon_{\min}^2 / \omega) (k_{\max} - k). \quad (3.17)$$

$$\text{Hence } n \cong \frac{2\epsilon_{\min}^{3/2}}{3\pi^{3/2} f^{1/2}} \left\{ \frac{h_2}{\Delta_2} + \frac{h_3}{\Delta_3} \right\} (\delta k)^{3/2} \quad (3.18)$$

or

$$k_{\max} - k(n) = \left(\frac{3}{2}\right)^{2/3} \frac{\pi f^{1/3}}{\epsilon_{\min}} \left(\frac{h_2}{\Delta_2} + \frac{h_3}{\Delta_3}\right)^{-2/3} n^{2/3} = \alpha n^{2/3},$$

i.e. the wave number changes with the two-thirds power of the mode number. Differentiating with respect to the mode number yields

$$dk/dn = -\frac{2}{3} \alpha n^{-1/3} \cong \Delta k / \Delta n.$$

For  $\Delta n = 1$  one obtains the spacing between eigenvalues as a function of the mode number:

$$\Delta k(n) = \left(\frac{2}{3} f\right)^{1/3} \frac{\pi}{\epsilon_{\min}} \left(\frac{h_2}{\Delta_2} + \frac{h_3}{\Delta_3}\right)^{-2/3} n^{-1/3}. \quad (3.19)$$

For the region (3.16) the eigenvalue distribution is quite different as one can see by inverting (3.11) and differentiating it with respect to  $n$ , after replacing  $\Delta_0$  by (3.17):

$$n(k) = \frac{2\sqrt{2} f}{\epsilon_{\min}^{3/2}} H \left[ (\epsilon_{\min}^2 / \omega) (k_{\max} - k) \right]^{1/2}, \quad (3.20)$$

$$k_{\max} - k = (\pi \epsilon_{\min} / 4 f H^2) n^2,$$

with the result:

$$\Delta k(n) = -(\pi \epsilon_{\min} / 2 f H^2) n, \quad (3.21)$$

i.e. in this region the spacing between eigenvalues increases linearly with mode number, whereas in the region  $k > \omega/c_{\max}$  it decreases slowly. Hence, the average eigenvalue spacing  $\langle \Delta k \rangle$  is a useful quantity only in the latter region, since there the spacing remains relatively constant except for the first few modes.  $\langle \Delta k \rangle$  can be defined for this case by dividing (3.14) with  $\Delta_2 = 0$  by (3.12):

$$\langle \Delta k \rangle = \frac{3\pi}{2\sqrt{2}} \left( \frac{\Delta_3}{c_{\min}} \right)^{1/2} \left\{ h_1 \left( 1 - \frac{\Delta_2}{\Delta_3} \right)^{1/2} + h_2 \frac{\Delta_2}{\Delta_3} \left[ 1 - \left( 1 - \frac{\Delta_2}{\Delta_3} \right)^{1/2} \right] + h_3 \right\}^{-1} \quad (3.22)$$

Taking as a special case  $\Delta_2 = \Delta_3$  and  $\Delta_2 \ll \Delta_3$ , one gets

$$\langle \Delta k \rangle = \frac{3\pi}{2\sqrt{2}} \left( \frac{\Delta_3}{c_{\min}} \right)^{1/2} (h_2 + h_3)^{-1} \quad (\Delta_2 = \Delta_3)$$

$$\langle \Delta k \rangle = \frac{3\pi}{2\sqrt{2}} \left( \frac{\Delta_3}{c_{\min}} \right)^{1/2} \left( h_1 + \frac{3}{2} h_2 + h_3 \right)^{-1} \quad (\Delta_2 \ll \Delta_3).$$

Hence a good approximate guess is

$$\langle \Delta k \rangle = \frac{3\pi}{2\sqrt{2}} \left( \frac{\Delta_3}{c_{\min}} \right)^{1/2} \frac{1}{H} \quad (3.23)$$

where H is the total water depth.

This expression is useful for establishing the step size for the wave number in the search for the eigenvalues. Upon some further manipulation one can obtain the expression

$$\frac{\Delta k(n)}{\langle \Delta k \rangle} \approx \frac{2}{3} \left( \frac{n}{N} \right)^{-1/3} \quad (3.24)$$

From this one can see that near the highest mode ( $n = N$ ) the actual spacing is  $2/3$  of the average, for  $n = N/8$  i.e. for a very low mode the actual spacing is  $4/3$  of the average. Hence by using  $6 \times \langle \Delta k \rangle$  as the step size one has approximately four steps between the highest modes, 16 steps between modes  $N/64$  and  $N/64 + 1$  which is usually sufficient to catch all modes except when nearly degenerate modes are present.

#### IV. SOLUTION OF THE CHARACTERISTIC EQUATION

As outlined in section 2, the eigenvalue problem (1.1) can be recast into the form (2.29) or (2.30) with the provision that the functions  $u_p^w(z, \vec{\rho})$  and  $u_p^b(z, \vec{\rho})$  or  $u_{p\downarrow, \uparrow}(z, \vec{\rho})$  satisfy the wave equation (1.1a) and the boundary conditions (1.1b) and (1.1c). This requirement has been met by starting solutions of (1.1a) at the boundaries  $z = 0$  and  $z = \infty$ , matching the boundary conditions there and propagating them layer by layer towards the joining point  $z = z_B$  (for 2.29) or  $z = z_M$  (for (2.30)).

The subscripts  $p$  on the solutions indicates that the particular functions describe the  $p$ th eigenmode corresponding to the eigenvalue  $k_p$ . Hence all the solutions  $u_p^w$ ,  $u_p^b$  and  $u_{p\uparrow}$ ,  $u_{p\downarrow}$  can be considered implicit functions of a variable  $k$  which for the  $p$ th eigenfunction assumes the value  $k = k_p$ .

The task of solving the eigenvalue problem (1.1) is thus reduced to finding those values of  $k$  for which (2.29) or (2.30) are satisfied, more precisely to finding the zeros of the functions

$$W(k) = \left. \frac{\partial u^w(z, \vec{\rho}, k)}{\partial z} \right|_{z=z_B} + \frac{\rho_w}{\rho_B} \gamma(\vec{\rho}, k) u^w(z_B, \vec{\rho}, k) \quad (4.1)$$

or

$$W(k) = u_{\downarrow}(z_M, \vec{p}, k) u'_{\uparrow}(z_M, \vec{p}, k) - u'_{\downarrow}(z_M, \vec{p}, k) u_{\uparrow}(z_M, \vec{p}, k). \quad (4.2)$$

As eq. (4.2) suggests, finding the eigenvalues corresponds to finding the zeros of the Wronskian between the down and up solutions at the joining point. Eq. (4.1) is essentially equivalent to (4.2) except that in this case use has been made of the fact that the up-solution is a simple exponential. This made it possible to factor out the up-solution and in fact drop it since it is nonzero everywhere.

An important property of the function  $W(k)$  is that it alternates signs between successive eigenvalues. For (4.1) this fact can be deduced in a straightforward way if this equation is interpreted as a matching condition of the derivatives of the down- and up-solutions:

$$W(k) = u^W(z_B, \vec{p}, k)' - u^B(z_B, \vec{p}, k)'$$

where subsequently the relation

$$u^B(z_B, \vec{p}, k)' = \gamma(\vec{p}, k) u^B(z_B, \vec{p}, k)$$

and the matching condition of the functions themselves

$$\rho_B u^B(z_B, \vec{p}, k) = \rho_W u^W(z_B, \vec{p}, k)$$

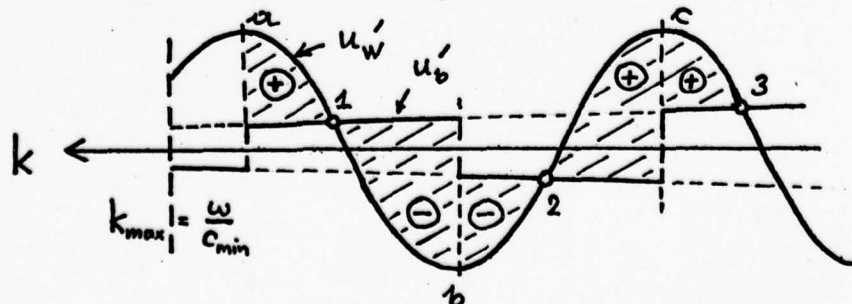
have been used.

It is well known that eigenfunctions of the type  $u_p(z, \vec{\varphi})$ , regardless of their specific form, always have a standing wave type pattern of some sort with  $p$  nodes for the  $p$ th eigenmode. Hence if the functions  $u_p^W(z, \vec{\varphi})$  are always made to start out with the same slope at one boundary; e.g. the surface, they will have alternately positive and negative values at the other boundary. Therefore, since there are no modes within the ocean floor,  $u^W(z_b, \vec{\varphi}, k)$ , and because of (2.23) also  $u^b(z_b, \vec{\varphi}, k)$ , have different signs between successive eigenvalues. Hence the derivatives of  $u^W(z_b, \vec{\varphi}, k)$  as a function of  $k$  oscillates continuously between positive and negative values whereas the derivatives of  $u^b(z_b, \vec{\varphi}, k)$  is positive (negative) as long as  $u^W(z_b, \vec{\varphi}, k)$  is negative (positive) and it shows the discontinuous behavior indicated in Fig. 2, switching signs whenever  $u^W(z_b, \vec{\varphi}, k)$  changes sign (which is assumed to take place at points a, b, c etc. in Fig. 2). The plus and minus signs in Fig. 2 mark the areas where

$$u^W(z_B, \vec{\varphi}, k)' \gtrless u^b(z_B, \vec{\varphi}, k)'$$

The numbered circles indicate the locations where  $W(k)$  changes sign, i.e. the locations of the eigenvalues.

Fig. 2



For eq. (4.2) an argument similar to the one just given applies if  $z_M$  was chosen such that the up solution is of the exponential type throughout. If both the down- and up-functions have oscillatory behavior at  $z = z_M$  one can simulate their behavior there schematically by the functions

$$u_{\downarrow}(z, k) = A \sin \frac{k\pi}{z_B} z \quad 0 < z < z_M$$

$$u_{\uparrow}(z, k) = B \sin \frac{k\pi}{z_B} (z_B - z) \quad z_M < z < z_B$$

which have matching logarithmic derivatives at  $z_M$  whenever  $k$  is an integer. Then

$$[W(k)]_{z=z_M} = \frac{k\pi}{z_B} AB \sin k\pi,$$

i.e., the Wronskian is zero for eigenvalues and alternates sign only there. This simple example can also be used to show that had one taken the condition of matching the logarithmic derivatives directly,

$$V(k) = \frac{u_{\downarrow}'}{u_{\downarrow}} - \frac{u_{\uparrow}'}{u_{\uparrow}},$$

instead of in the form of the Wronskian (4.2), the desirable property of the signs alternately only at the eigenvalues

would be lost:

$$V(k) \propto \frac{\sin k\pi}{\sin \frac{k\pi}{z_B} z_M \sin k\pi \left(1 - \frac{z_M}{z_B}\right)}$$

While  $V(k)$  still has zeros only for  $k = \text{integer}$  the sign of  $V(k)$  now switches not only when an eigenvalue is passed (i.e. for  $k = \text{integer}$ ) but also when one of the sines in the denominator has a zero.

Even though the arguments given above were not rigorous the qualitative conclusions apply directly to the general case of arbitrary eigenfunctions as long as these eigenfunctions are all non-degenerate. If this restriction is lifted and degenerate eigenmodes are possible the situation may become rather complicated and the simple method of finding the eigenvalues outlined in this section would certainly no longer work. However, it is very unlikely, as long as one is dealing with empirical sound velocity distributions, that one should encounter cases with truly degenerate eigenmodes. On the other hand, almost degenerate eigenmodes may occur occasionally in cases where the sound velocity distribution has pronounced local minima. The method of this section may then still be applied. However, a little care may be necessary to catch the sign change of  $W(k)$  between the almost degenerate eigenvalues.

For the purpose of this report it will be assumed that  $W(k)$  changes sign at every eigenvalue and only there. Then, in order to find the eigenvalues, i.e. those values of the  $k$  for which

$$W(k) = 0$$

is satisfied, one may simply use the following procedure.  $W(k)$  is calculated as a function of  $k$  with  $k$  changing in small enough steps and, whenever the result changes sign between two successive steps, one switches over to an iteration routine which locates the exact position of the zero. The range of discrete eigenvalues is restricted to

$$k_{\max} > k > k_{\min} \quad (4.3)$$

The lowest eigenmode is found for  $k \approx k_{\max}$ . Hence the search is started at  $k = k_{\max}$  and proceeds by decreasing  $k$  steps of  $XKD$ ; the quantity  $XKD$  has to be chosen properly. If it is too small, excessive amounts of computer time will be required to complete the search. If it is too large the search will locate only part of the eigenvalues. A good choice is usually

$$XKD \cong \langle \Delta k \rangle / 6, \quad (4.4)$$

where  $\langle \Delta k \rangle$  is the average eigenvalue spacing as defined in section 3. If the bottom velocity  $c_B$  is much larger than  $c_{\max}$ , the maximum sound velocity in the water, one may save consider-

able computation time by lettering XKD increase linearly with mode number for wave numbers

$$k < k_{\min} = \omega / c_{\max} .$$

Occasionally there are almost degenerate eigenmodes which are separated by only

$$\Delta k \cong \langle \Delta k \rangle / 100$$

or even less. In those cases it is usually still best to use (4.4) and catch the missed eigenvalues by a more detailed local search.

Formally eq. (4.1) and the method outlined for finding  $u^w(z, \vec{\xi}, k)$  is sufficient to find all eigenvalues. In practice, however, several difficulties arise which have to be dealt with. Usually no problems arise for intermediate or high mode numbers since in these cases the function  $W(k)$  is sufficiently well-behaved near the sea floor  $z = z_b$ . The problems which do appear are basically caused by the fact that for low mode numbers  $u^w(z, \vec{\xi}, k)$  has only a small oscillatory part near the sound speed minimum and two long exponential sections between there and the surface and the ocean floor, within which the function drops sometimes over several hundred orders of magnitude (see section 3). However, despite the exponentially falling appearance, both exponentially falling and exponentially rising components have to be present in the solutions to allow matching of the logarithmic derivatives at the boundaries. Very near the eigen-

values, the coefficients for the exponentially rising parts are of course extremely small so that the required exponentially falling appearance can be achieved. However for wavenumbers away from the eigenvalues matching at the boundary between the oscillating and the exponential part of  $u^W$  require large coefficients for the exponentially rising part. This way  $W(k)$  can assume very large values near the sea floor which may easily exceed the capacity of the computer. And more seriously, since a zero of  $W(k)$  has to be achieved essentially by cancellation between  $u^W$  and its derivative the loss of accuracy due to very near cancellation of very large numbers does not permit to obtain a solution. Even the use of double precision arithmetic is not sufficient in many cases. Four steps were taken to permit the calculation of the eigenvalues despite these difficulties:

(1) Instead of always matching the down- and up-solutions at the sea floor (i.e. using eq. (4.1)) the matching is sometimes carried out at some appropriate boundary between two layers in the water (i.e. eq. (4.2) is used). This makes  $W(k)$  much more well-behaved for the low mode numbers.

(2) In the cases which up-layers are introduced the solutions increase very rapidly between the sea floor and the matching point. To prevent exceeding the capacity of the computer the function  $u_p(z, \vec{\rho}, k)$  and its derivative  $u'_p(z, \vec{\rho}, k)$  are scaled down at each layer boundary in steps of  $10^{10}$  until  $u_p < 10^{-5}$ .

(3) If the solution is of the exponential type over the whole extent of a wide layer the propagation of the solution between the two boundaries becomes inaccurate if it is carried out in one step. In these cases the layers are divided into sublayers and the solutions are only propagated from one sublayer boundary to the next. This way the chances for errors due to near cancellation of large numbers is very much reduced.

(4) If the Airy function arguments are large and positive the corresponding values for the Airy functions can easily exceed the capacity of the computer. In these cases the Airy functions can be approximated by

$$Ai(\zeta) \propto f(\zeta) e^{-\frac{2}{3} \zeta^{3/2}}$$

$$Bi(\zeta) \propto g(\zeta) e^{\frac{2}{3} \zeta^{3/2}}$$

with corresponding expressions for the derivatives, where  $\zeta$  may be of the order of 100 or even more.

Fortunately it turns out that one can find a formulation of the problem (see eqs. (2-15) and (2-16) where these very large or very small numbers are never needed individually but only in pairs in such a way that each pair contains one positive and one negative exponent. Hence, by carrying  $f(\zeta)$ ,  $g(\zeta)$  etc. and the exponents along separately one can avoid

overflows and underflows since the exponents appearing in the results are always only the small differences of two large numbers.

The computer code described in this report uses the method for solving the characteristic equation outlined earlier in this section. With the additional features just discussed it can be used to solve a wide range of practical problems. Propagation problems in the deep ocean ( $H \sim 4000$  m) can be solved for frequencies up to several hundred Hz with high precision. For shallow water cases (water depth = a few hundred meters) the frequency range in which the program can be used extends up to several kHz. If a few precautionary measures are taken in preparing the profile data (see Section 7) then the range of application is basically only limited by the amount of computer time one is willing to spend.

The computer code consists of the main routine SEARCH and a set of subroutines. A detailed description of these programs is given in the next two sections. It should be noted that use has been made of this program in the following publication: A. Nagl, H. Überall, A. Haug and G. L. Zarur, "Adiabatic Mode Theory of Underwater Sound Propagation in a Range Dependent Environment", J. Acoust. Soc. Am. 63, 739 (1978).

## V. DESCRIPTION OF THE COMPUTER CODE (MAIN PROGRAM)

Purpose of this section and the next is to discuss the set of computer codes written to solve the problem outlined in section 2 using the method described in section 4. The set consists of the main program SEARCH to which this section is devoted and the subroutines

PROP, LARGE, START, PR, LG, DAIRY, EXTREM, SINCOS, PARAM and SELAUT

which will be described in the next section.

The main program obtains input data from the subroutine PARAM, which provides mainly control parameters, and the file NFILE(1) in which the sound velocity profiles are stored. The inputs consist of the quantities (indices I refer to ith layer or ith layer boundary, indices J refer to jth profile):

NUMBER = number of boundaries for selected profile;  
 (X(I), Y(I), I = 1, NUMBER) = depths  $z_i$  and sound velocities  $c(z_i, \bar{\rho})$  for profile selected;

CBOTM = sound velocity within the sea floor. These data come from the file NFILE(1), the following ones from the subroutine PARAM.

FREQ = sound frequency  $f$   
RH01 = water density  $\rho_w$   
RH02 = density of sea floor material  $\rho_B$   
XKD = step size of wave number  $k$   
IPRMIN = first profile to be investigated  
IPRMAX = last profile to be investigated  
XKST(J) = starting value of  $k$  for eigenvalue search  
for  $j$ th profile  
XINCR1, = depth increment and maximum depth at which  
XMAX1  
depth functions are to be calculated  
NOPT = parameter to indicate whether or not the depth  
functions are to be calculated at a second  
set of depths. (NOPT = 1, NO; NOPT = 2, YES)  
XINCR2, = depth increment and maximum depth for second  
XMAX2  
set of depth function values (needed only  
if NOPT = 2).  
NFILE(K) = Names of data files  
NFILE(1) = Profile data file  
NFILE(2) = Intermediate file  
NFILE(3) = Depth function values at source depth  
NFILE(4) = Depth function values at receiver  
depth  
NFILE(5) = Eigenvalues  
NFILE(6) = Wave functions and derivatives at  
layer boundaries  
NFILE(7) = Intermediate file

JMIND = maximum number of up-layers allowed (once the down-solution has reached this layer a switch to the up-solution part may occur.)

ICT = maximum number of eigenvalues allowed for this run.

NPPL(I) = number of depth function values in ith layer.

ISLAY = number of points at which the depth functions are calculated (dimension of first depth function array).

ISOU = index of element in depth function array for source depth.

IREC1 = index element in depth function array for receiver depth for those modes for which there are no up-layers (if there are up-layers the index is called IREC and is computed in the program).

IRLAY = dimension of second depth function array for those modes for which there are up-layers (needed only if NOPT = 2).

IRLAY1 = dimension of second depth function array for modes without up-layers (needed only if NOPT = 2).

In executing the main program a number of essential steps can be distinguished which are described below. In order to facilitate associating variables used in the previous section with the corresponding symbols in the computer code both labels are sometimes given, connected by an arrow. In the program listing provided in the appendix the instruction, or set of instructions, each of the following steps is concerned with, are marked by arrows or brackets and are labeled by the appropriate step numbers.

- (1) Control parameters are read in from PARAM
- (2) A profile is selected and the data describing it are read in.
- (3) The wave number  $k \rightarrow XK$  for the first iteration is determined. Generally the starting value is  $\omega/c_{min}$ .  
However, if for a particular profile a value  $XKST < \omega/c_{min}$  is provided the iterations will start with this value.
- (4) This step determines the slopes  $S(I)$  and the internal derivations for each layer. All quantities are multiplied with powers of  $H$  such that they come out dimensionless ( $H =$  total water depth).
- (5) The search for the first eigenvalue is started at deterrent number 890. The program returns to this point each time the calculations for a particular eigenvalue have been completed in order to initiate the search for the next eigenvalue.
- (6) At statement number 990 the wave number  $k \rightarrow XK$  is reduced in steps of  $\Delta k \rightarrow XKD$ . For each value of  $k$  thus selected the program proceeds to calculate the Wronskian  $W(k)$ . This process continues until  $W(k)$  has different signs for two subsequent values of  $k$ . Then the value of  $k$  is determined by a different part of the program as described below. Control of the value of  $k$  remains there until  $k$  has converged to an eigenvalue.
- (7) Calculation of  $W(k)$ , whatever the value of  $k$ , is commenced at the statement number 994. First  $\gamma \rightarrow UK$  is calculated. This is up to a minus sign the logarithmic derivative of

the function  $u_p^B(z)$  inside the sea floor.

- (8) Then in a DO-loop which extends to statement 3 the solution in the water is calculated by starting at the surface with the boundary conditions  $u_p^W(0) = 0$  and  $u_p^W(b)' = |a|/\pi$  and propagating the function layer by layer down towards the sea floor. The solution and its derivative at the end of a layer (UOUT, UPOUT) in terms of the solution and its derivative at the beginning of the layer (UIN, UPIN) are calculated in one of the three subroutines START, LARGE, PROP. PROP is chosen if the argument  $\zeta \rightarrow DX$  of the Airy functions is smaller or equal DM, which is chosen to be 11. LARGE is called if the argument is larger than DM. START is used for the first layer regardless of the argument. However, START can call LARGE if the argument is larger than 15 in at least part of the first layer. Similarly PROP will <sup>call</sup>/LARGE if the argument is  $\leq 11$  at the beginning of the layer but grows  $> 11$  somewhere within the layer. Conversely, if the argument drops from  $> 11$  to  $\leq 11$  somewhere within a layer, LARGE calls PROP at this point to complete the layer.

This process is carried through all the way to the sea floor unless at the start of the calculations for some layer it is found that the argument of the Airy functions at the end of the layer is larger than 7, and that in addition the layer number J is larger than JD,

the number of completed down layers, which can be set as a parameter. In this case control of the program is transferred to another section of the program which is described in the next two paragraphs.

If this transfer does not occur and the propagation of the solution through all the layers is completed the program proceeds to calculate  $u_p^B(z_B) \rightarrow \text{SCALE}$  and  $W(k) \rightarrow \text{FUNC}$  which in effect corresponds to the Wronskian of the sea floor as outlined in section 4, and then skips to statement 45 in order to calculate the overall normalization constant (see paragraph 12.) The program actually assumes  $u_p^B(z_B) = 1$  and  $u_p^B(z_B)' = -\gamma$  with  $\gamma \rightarrow UK$ . In this sense SCALE is the mismatch between  $u_p^W(z_B)$  and  $u_p^B(z_B)$ . Since  $u_p^W(z)'$  is calculated as the derivative with respect to the argument of the Airy function (see Section 2) and not with respect to  $z$ ,  $u_p^B(z_B)' \rightarrow \text{UPBOTM}$  has to be divided by the internal derivative at  $z = z_B (= -AL)$  before it can be used to calculate the Wronskian. The additional factor is contained in the definition of UPBOTM to keep the derivative dimensionless. This is also necessary since  $u_p^W(z)'$  is dimensionless by virtue of the fact that the argument of the Airy functions is defined to be dimensionless.

- (9) The section of the program described in this and the next paragraph are entered if the arguments of the Airy fns.

grow too large in the lower layers of the profile for a reliable solution of the eigenvalue problem to be possible with the procedure outlined in the previous paragraph. If this part of the program is entered the result of the downward section of the calculation (UIN, UPIN) is kept for later use and a new solution is started at  $z = \infty$ . The coefficient of the solution within the sea floor is chosen such that the value of the solution at the sea floor is

$$u_p^w(z_B) = XFACT * 10^{-25}.$$

The quantity XFACT is at first set equal to one. Later it is adjusted to provide a match of the upward part of the solution with downward part. Since  $u_p(z)$  for  $z \geq z_B$  is an exponential the derivative at the sea floor is simply:

$$u_p^w(z_B)' = -UK * XFACT * 10^{-25}.$$

In the program this result is divided by (-AL) in order to give the derivative with respect to the argument of the Airy functions rather than with respect to  $z$ . The reason for this was explained in section 2. The factor H (=total water depth) which also appears is included to keep the derivative dimensionless.

Hence, because of conditions (2.23) and (2.24), the boundary values for the upward solution at the sea floor are

$$UDIN = RHO2 / RHO1 * XFACT * 10^{-25}$$

$$UPDIN = (-UK) * H / (-AL) * XFACT * 10^{-25}$$

Starting with these values the solution in the water,  $u_w(z)$ , is then propagated upward layer by layer in a DO-loop which extends to statement #4 whereby the solution and its derivative at the end of a layer, i.e. the upper boundary of the layer (UDOUT, UPDOUT) are calculated in terms of the solution and its derivative at the beginning, i.e. the lower boundary, of the layer (UDIN, UPDIN). To carry out this calculation the subroutines PR and LG are called if the arguments of the Airy function are  $\leq 11$  and  $> 11$  respectively. Transfers between PR and LG within a layer can occur analogous to the procedure outlined in paragraph 8.

- (10) Since for the lower modes the upward solution may grow very rapidly over many orders of magnitude ( $\propto \exp 2/3 \zeta^{3/2}$  sometimes  $\zeta \sim 50$  or even more) it was found necessary to provide the possibility of scaling down the solutions and their derivatives at the end of each upward layer. Generally, the downscaling consists of reducing  $u$  and  $u'$  in steps of  $10^{10}$  until  $|u| < 10^{-5}$ . No scaling is performed in the uppermost up-layer. To keep track of the number of scalings at the end of each layer, this number is stored in the array ISC(J). This array is updated for each new value of  $k$ . However, once the Wronskian has changed signs the number of scalings at each boundary is kept constant in order to prevent the scaling pro-

cedure from disrupting the iteration process which finds the  $k$  for which

$$W(k) = 0.$$

If the number of scalings in two successive iterations were allowed to be different the convergence of the iteration process would be uncertain.

The fact that the quantity XFACT may change its value adds another complication. XFACT corresponds to the mismatch between the down-and up-solutions at their joining point. Until  $k$  has converged to some eigenvalue  $k_p$ , XFACT is actually kept equal to one. This does not interfere with finding the eigenvalue. However, in order to get a continuous solution and to find the correct value for the normalization integral it is necessary at the end to recalculate the solution with the eigenvalue just found and with XFACT assuming its proper value.

In this calculation the number of scalings in each layer may be different than before. Hence in order to keep track of the number of scalings in the final iteration another array has to be introduced: MSC(J). If the solutions are required at two different set of depths (NOPT = 2) an additional iteration is carried out where the number of scalings to be performed in each layer is taken from MSC(J).

(11) After the calculation for all the up-layers has been completed the Wronskian at the layer boundary where the up- and down-solutions join, is determined. If the program is in the phase where an eigenvalue has been found in the previous iteration (MFLAG = 1) two additional calculations are carried out in this section. First, the array NSC(ICOUNT, K) is determined which gives the total number of scalings which have been carried out on the solution with mode number ICOUNT between the sea floor and the Kth layer. In other words, in order to obtain the correct values for the solution for the mode  $p \rightarrow$  ICOUNT in the Jth layer, the solution as stored in the file at this point has to be multiplied by

$$10^{10 * NSC(ICOUNT, K)} .$$

The second operation carried out in this section concerns the contributions of the up-layer to the normalization integral. They are also affected by the scalings and have to be corrected. XS(J) and XT(J) are contributions to the normalization integral accumulated in layer J while the solution was calculated in subroutines PR and LG, respectively. It was assumed that only the uppermost up-layer ( $J=JX$ ) contributes significantly to the normalization integral. The solution can be assumed to be exponentially decaying for at least a considerable part of this layer so that in the lower layers the square of the solution will be

insignificantly small. Hence, the contribution of all the up-layers to the normalization integral was taken to be

$$XNORM2 = (XS(JX) + XT(JX)) * 10^{-20 * NS}$$

where NS is the difference

$$ISC(J) - MSC(J)$$

assumed over all up-layers.

- (12) The contributions of the down-layers to the normalization integral are accumulated through the variable XNORM, each layer adding the quantity

$$\int_{z_n}^{z_{n+1}} |u_p^W|^2 dz.$$

The contributions of the up-layer are calculated similarly, added up to yield XNORM as outlined in the previous paragraph. The calculation of these integrals is described in the section on the subroutines. The contribution of the solution inside the sea floor,

$$\int_{z_B}^{\infty} |u_p^B|^2 dz,$$

is called XNOR1. It is obtained by observing that  $u_p^B(z)$  is an exponential with the exponent  $-\gamma \rightarrow -XK$ . Hence the overall normalization constant is obtained as

$$SQN = \left( \frac{XNOR1}{SCALE^2} + XNORM + XNORM2 \right)^{-1/2}$$

where the fact was taken into consideration that SCALE is the mismatch between  $\phi_w u_p^W(z_B)$  and  $\phi_B u_p^B(z_B)$ . SQN is

stored in the file NTYPE(2) with FORMAT (D).

- (13) If MFLAG = 1 (= eigenvalue found in previous iteration) and NOPT = 2 (= second set of depth functions requested) the program will return to statement #994 (see paragraph 7) to calculate the depth functions in the range

$$0 \leq z \leq XMAX 2$$

in steps of XINCR2. These values are also stored in file NTYPE(2). The format in this case is FORMAT (D 20.8).

- (14) If MFLAG = 1 then all calculations concerning a particular eigenvalue are finished when statement #892 is reached and the program returns from this point to statement number 890 (see paragraph 5) to start the search for next eigenvalue.

- (15) The program enters this section only once for each eigenvalue, namely if it is found that the wave numbers for two successive iterations differ by less than  $10^{-15}$ . This condition is used as the criterion that the search for the eigenvalue has converged. The value  $k \rightarrow XK$  obtained is stored in file NTYPE(2) with FORMAT(D). If there are up-layers the mismatch between the down- and the up-solutions (XFACT) is calculated here and the program returns to statement number 994 (see paragraph 7) with MFLAG = 1 for one (NOPT = 1) or two (NOPT = 2) additional iterations in which the following quantities are calculated and stored (as already partially indicated above):

- (i) contributions from each layer to the normalization integral

(ii) the values of the solution  $u_p^W(z)$  at one or two different sets of depths.

(iii) the values of the solutions and their derivatives at each layer boundary.

- (16) This section of the program determines the wave number  $k$  for the next iteration. If the Wronskian  $W(k)$  has not changed sign since the last eigenvalue was found ( $IFLAG = 0$ ) the program returns to statement #990 where  $k \rightarrow XK$  is reduced by  $\Delta k \rightarrow XKD$ . If the Wronskian has changed sign between this iteration and the previous one,  $JFLAG$  is set equal to 1 and the wave number for the next iteration is calculated by quadratic interpolation:

$$k_{n+1} = k_n - \left[ \frac{W(k_n)}{W(k_n) - W(k_{n-1})} \right]^{1/2} (k_n - k_{n-1}).$$

If  $JFLAG = 1$  already,  $k$  for the next iteration is calculated by linear interpolation:

$$k_{n+1} = k_n - \frac{W(k_n)}{W(k_n) - W(k_{n-1})} (k_n - k_{n-1}),$$

and the program also returns to statement number 994.

- (17) This section of the program is entered after all eigenvalues for a particular profile have been found. It essentially reads back all the data written into file

NFILE(1), pickes out the depth function values at the source and receiver depths and normalizes them. Then it stores the values at the source depth for all modes into file FILE (3), the corresponding values at the receiver depths into file NFILE (4), the eigenvalues into fileNFILE (5), the wave functions and their derivatives at the larger boundaries into file NFILE (6). In all cases FORMAT (86D) is used.

## VI. DESCRIPTION OF THE COMPUTER CODE (SUBROUTINES).

### Subroutine SELAUT

This routine transfers the data for the selected profile from the profile data file NFILE (1) to the temporary file NFILE (2) converting them in the process into a format which is convenient for reading them to the main program SEARCH. This arrangement is convenient since the main program does not have to be changed if for some reason the arrangement of the data in the profile data field should be altered.

### Subroutine PARAM

Purpose of this program is to provide a convenient input channel for all the input parameters except the profile data. Through the set of input parameters provided it is usually possible to set up the program for specific cases without having to change and recompile the main program and any of the other subroutines every time. The set of parameters provided is listed in the previous section, where the main program is described. Information on how to set up a run is given in section 7.

Subroutines DAIRY

It calculates the Airy functions and their derivatives to 14 significant digits for arguments in the range

$$-1000 < DX < 15 .$$

It is called from START, PROP and PR. The arguments for which the Airy functions are needed in these subroutines are usually smaller than DM, which is 11 in PROP and PR and 15 in START.

If the argument is larger than that value in some part of a layer the program switches over to LARGE or LG.

The only time when the argument can be larger than 11 in PROP and PR is in the case of a layer in which the argument decreases from above 11 to below 11. Then the program starts the layer in LARGER or LG and then switches to PROP or PR at the beginning of the sublayer in which the argument decreases below 11. At this point the argument may be larger than 11 by the amount DA by which the argument may change within a sub-layer. Hence if one makes sure that  $DA < 4$  the arguments for which DAIRY is used always stay below the upper limit for which DAIRY is defined.

For layers with very small slopes in the velocity profile the arguments may fall considerably below -1000 (see section 3). For these cases the

Subroutine SINCOS

is called which uses the asymptotic expansion for large negative arguments given in (3.1).

Subroutine EXTREM

Calculates the Airy functions and their derivatives for large positive arguments. This routine is called from LARGE and LG. The arguments for which the Airy functions are needed there are usually larger than 11. An exception are the calculations in a sublayer in which the arguments rise above or fall below 11, i.e. the sublayers at the beginning (the end) of which the switch from (to) the routines PROP, START or LG occurs. Then the arguments can be below 11 by the amount by which they can change within a sublayer.

This means that the ranges of arguments for which EXTREM is called extends from about 7 to possibly several hundred. To avoid overflows resulting from the corresponding large values for the Airy functions the results are given in 2 parts, one being the exponent of the asymptotic form (= ZETA), the other one the value of the polynomial

multiplying the exponential (see (3.1)). In order to take advantage of the high precision of the subroutine DAIRY for the smaller arguments, this routine is called by EXTREM to calculate the Airy functions for arguments  $\leq 15$ .

Purpose of the remaining five subroutines (PROP, PR, START, LARGE, LG) is to carry out the following functions:

- (1) To provide the depth functions and derivatives at the end of an interval in terms of the depth function and its derivative at the beginning of the interval. START, PROP and LARGE are used for down-layers, and PR and LG for up-layers.
- (2) If MFLAG = 1, to calculate the integral of the square of the depth function over the range of the layer.
- (3) If MFLAG = 1, to calculate the depth function at integer multiples of XINCR. The inputs to all 5 of these routines are with a few exceptions:

AL, BE quantities in terms of which the arguments of the Airy functions are defined. According to

(2.10) the arguments can be written as

$$\zeta_{pi}(z, \vec{\rho}) = -\alpha_i(\vec{\rho})[z + \beta_i(\vec{\rho})] \quad \text{or} \quad -AL * (BE + X),$$

where AL corresponds to the internal derivative  $d\zeta/dz$

and BE to  $d_{pi}(\vec{\rho})/s_i(\vec{\rho})$ , with  $d_{pi}(\vec{\rho})$  being

defined by (2.9). AL and BE can always be assumed

to be constant within the range (X, XJ) of the interval. Hence the arguments of the Airy functions are always either a monotonically increasing or a monotonically decreasing function of the depth.

- X, XJ Depth variables defining the beginning and the end of the depth interval for which the calculations are performed.
- U, UP Depth function and its derivative at beginning of the interval.
- XNEW Depth variable which is changed in increments of XINCR inside the subroutines. It specifies the depth at which depth functions may be calculated and it is used to define the boundaries of sublayers. The input value of this variable is always  $\leq X$  for START, PROP, LARGE and  $\geq X$  for PR, LG.
- XINCR Step size with which the depth variable XNEW may be changed.
- TT Ratio of internal derivatives at boundary between two intervals as defined in (2.13') (internal derivative for previous interval divided by internal derivative for this interval).

**MFLAG** Flag that is set in the main program SEARCH and transmitted to the subroutines through a COMMON statement. It indicates that an eigenvalue has just been found. In the subroutines it sets up the conditions for calculating the contribution to the normalization integral from the layer under consideration and, if in addition  $MFLAG_1 = 1$ , for calculating the depth functions at prescribed depths.

**MFLAG<sub>1</sub>** This flag is also transmitted to the subroutines through a COMMON statement. It is set to 1 in the main program SEARCH if the maximum depths ( $XMAX_1$  or  $XMAX_2$ ), down to which the values of the depth functions are requested, are within or below the layer under consideration. If the flag has the value 1 it causes the subroutines to calculate the depth functions for all values of  $XNEW$  which fall inside the interval, provided MFLAG also has the value 1.

The output from the subroutines consists of:

**UJ, UPJ** Depth function and its derivative at end of interval.

During execution of the subroutines this variable has been increased (START, PROP, LARGE) or decreased (PR, LG) by an integer multiple of XINCR from its input value. Its output value is always  $\leq$  XJ for START, PROP, LARGE, and  $\geq$  XJ for PR and LG.

**XNORM** This is the variable through which the contribution to the normalization integral of all the down-layers is accumulated, i.e. whenever any of the subroutines START, PROP, or LARGE is called and the condition MFLAG = 1 exists the integral of the square of the depth function within the layer boundaries is calculated and the result added to XNORM. The variable is transmitted to and from the main program via a COMMON statement.

**X1SQ, X2SQ** These are the contributions to the normalization integral calculated in the subroutines PR (=X1SQ) and PG (=X2SQ). PR and LG have both quantities as arguments since a transfer between the 2 subroutines can occur within a layer. The contribution of all the up-layers are added up in the main program.

Subroutines PROP and PR have a similar structure and are therefore discussed together. They consist essentially of 3 sections:

Section (1): Calculations at the beginning of the layer. The coefficients AJ, BJ of the Airy functions defining the depth function in this layer are determined here following eqs. (2-13'). If MFLAG = 1 the value of the normalization integral at the start of the integration interval ( =DEG) is also calculated at this point.

Section (2): Calculations at sublayer boundaries. This section is entered in two cases:

(a) If the argument of the Airy function at XJ, the end of the layer, is  $> 11$ .

(b) If MFLAG1 = 1, i.e. if the calculation of the depth function at points within this layer is requested. In both cases sublayer boundaries are established within the layer at regular intervals, starting with the input value of XNEW increased for PROP (decreased for PR) by XINCR and then proceeding in steps of XINCR. At each boundary the values of depth function (WF) and its derivative (WP) are calculated. In case (2) the depth function at each sublayer boundary is stored in file NFILE (2) with FORMAT (D20.8). In case (1) the program proceeds from one sublayer to the next until at some sublayer boundary the argument of the Airy function exceeds

11. Then subroutine LARGE is called using as inputs

X = XNEW

U = WF

UP = WP

TT = 1 (since the internal derivatives do not change across  
the sublayer boundaries),

completing there the remainder of the calculation for this layer. After completion of the CALL statement the program immediately returns to the main program. If MFLAG = 1 the value of the normalization integral at the latest XNEW is calculated (-CEN) and the quantity (-CEN -BEG) is added to XNORM before returning to the main program, i.e. the contribution to the normalization integral of the region between the beginning of the layer and the XNEW at which the transfer to LARGE occurs as added to the total.

Section (3): Unless a transfer to subroutines LARGE occurred in Section 2 the calculations in PROP and PR are wound up by calculating the depth function and its derivative at the end of the layer (i.e. UJ, UPJ) and if MFLAG = 1 by calculating the normalization integral at the end of the layer (-CEN) and adding the difference to XNORM. If Section 2 was skipped (i.e. if the argument of the Airy functions was smaller than 11 for the whole layer and if MFLAG1 was 0, so that the depth functions did not

have to be calculated inside this layer), then XNEW still has its initial value and it has to be updated. If the argument of the Airy functions is  $\leq 11$  at the beginning of the layer but  $> 11$  at the end, only the solution for the first part of the layer is calculated in PROP ( or PR) up to the last sublayer boundary at which the argument is less than 11. The solution for the rest of the layer is then calculated in LARGE (or LG) as indicated above. The argument at which the transfer occurs may be from very slightly less than 11 to considerably less than 11. The latter is the case if the slope is large and the argument at the next sublayer boundary is just above 11. The reverse case where the argument is  $> 11$  at the beginning of the layer and  $\leq 11$  at the end, i.e. when the calculations for the first part of the layer are made in LARGE (or LG) and then the transfer is made to PROP (or PR) is discussed in the section on LARGE and LG.

Subroutine START is called to perform the calculations in the layer nearest the surface. It contains 4 sections: the first three are used if the argument of the Airy functions remain below 15 for the whole layer, the fourth one if the argument exceeds 15 for at least part of the layer.

Section 1: calculations at the surface. Calculation of the coefficients to the Airy functions according to assumptions (2-18) and (2-19). If MFLAG = 1 the value of the normalization integral at the surface is also determined here.

Section 2: calculations at the end of the layer. This section determines the depth function and its derivative at the end of the layer and, if MFLAG = 1, the contribution ZNORM of the layer to the normalization integral XNORM.

Section 3: updates XNEW to the largest value  $\leq$  XJ and for MFLAG1 = 1 calculates the depth function inside the first layer at intervals XINCR and writes them into file (NFILE (2)).

Section 4: This section is used to transfer the calculations for the entire first layer to LARGE. This is done if anywhere inside the layer the argument of the Airy function exceeds 15. For the part of the layer for which the argument is  $\leq$  15 the calculation should actually be done in START but it is assumed that if the argument is  $>$  15 somewhere in the layer it is not much less anywhere else. This is a reasonable assumption since surface layers with large arguments are never very wide. So the argument may be large but it cannot vary much over the extent of the layer.

Subroutines LARGE and LG again have similar structure and may therefore be discussed together. These subroutines are used for those intervals in which the Airy function arguments are large ( $>$  DM). Such an interval may extend over a whole layer or over

part of a layer. Since the Airy function arguments are monotonically increasing or decreasing with<sup>in</sup> a layer, it is clear that of the arguments at the beginning ( $= DX$ ) and at the end ( $= DX1$ ) at least one must be larger than  $DM$ , if the routines LARGE and LG are involved in the calculation for a particular layer, and one can distinguish 3 cases:

- (1)  $DX > DM$  and  $DX1 > DM$ . The layer is calculated entirely by LARGE or LG (interval extends between the layer boundaries  $X$  and  $XJ$ .)
- (2)  $DX > DM$  but  $DX1 \leq DM$ . First part of layer is calculated in LARGE or LG (interval extends from  $X$  to value of  $XNEW$  at switch. For calculations of remainder of layer PROP or PR are called.
- (3)  $DX \leq DM$ ,  $DX1 > DM$ . First part of layer is calculated in PROP or PR. For calculations of remainder of layer LARGE or LG are called by PROP or PR (interval extends from value of  $XNEW$  at switch to  $XJ$ ).

All intervals for which LARGE and LG are used are divided up into sublayers of width  $XINCR$ , and the depth functions are propagated from one sublayer boundary to the next using eqs. (2-15). The first sublayer boundary extends from  $X$  ( $=$  interval boundary, which is the layer boundary in cases (1) and (2) and the value of  $XNEW$  at the switch in case (3)) to  $XNEW_{input} + XINCR$ , the first

sublayer boundary within the interval. Subsequent sublayers are established by changing the depth variable XNEW in steps of XINCR and using these values as boundaries. The last sublayer extends from the highest (lowest) value for XNEW which is  $\leq XJ$  ( $\geq XJ$ ), or in case (3) to the first value of XNEW for which the Airy function argument is  $< 11$ .

Both LARGE and LG essentially consist of 3 sections:

Section (1): At the beginning of the interval only the Airy functions for the argument at this point are calculated.

Section (2): Calculations for each sublayer. The solutions are propagated from one sublayer to the next by expressing the depth functions and their derivatives at the end of each sublayer ( $U_1$ ,  $U_1P$ ) in terms of the corresponding values at the end of the previous sublayer ( $U$ ,  $UP$ ) and the coefficients  $C_{ij}$  following eqs. (2-16).

If MFLAG = 1 the depth function value is written into the file NFILE (2) at every value of XNEW inside the interval.

If MFLAG = 1 the contribution of the sublayer to the normalization integral is calculated.

If at some sublayer boundary it is found that within the following sublayer the Airy function argument goes below DM, the calculations

are transferred to PROP or PR, in the case MFLAG = 1 after the contribution of all the sublayers up to this point to the normalization integral is calculated and added to XNORM.

The argument at which the transfer occurs may vary from slightly larger than 11 to considerably larger than 11. The latter case may happen if the argument changes very much over the extent of the sublayer and it is only slightly less than 11 at the next sublayer boundary. This means that the Airy function argument could be larger than 11. This means that the Airy function argument could be larger than 15 for calculations in PROP or PR. This could cause problems since the subroutine DAIRY which is used in PROP and PR for calculating the Airy functions is accurate only for arguments below 15, and more seriously, the value of the Airy functions may exceed the capacity of the computer if the argument is allowed to go too high. In order to avoid these problems the width of the sublayers should be chosen small enough to keep the change in the Airy function arguments within a sublayer below 4.

According to (3.6) the difference in the Airy function arguments can be written as

$$\Delta \xi = - (2^{1/3} / \epsilon_{\min}) \left( \frac{\omega}{\Delta_i / h_i} \right)^{2/3} [\Delta(z_{n+1}) - \Delta(z_n)]$$

where

$$\Delta(z_{n+1}) - \Delta(z_n) = (\Delta_i / h_i) (z_{n+1} - z_n) = (\Delta_i / h_i) \times INCR$$

i.e. the difference in the  $\Delta$ 's is just the sound velocity slope in the layer times the thickness of the sublayer. Hence

$$\Delta \zeta_{SL} = - \frac{2^{1/3}}{c_{min}} \omega^{2/3} \left( \frac{\Delta_i}{h_i} \right)^{2/3} XINCR.$$

So in order to keep the Airy function arguments from exceeding 15 for calculations in PROP and PR, XINCR should be chosen such that

$$\frac{2^{1/3}}{c_{min}} \omega^{2/3} \left| \frac{\Delta_i}{h_i} \right|^{2/3} XINCR < 4 \quad (6.4)$$

for all layers.

Section (3): Calculations at the end of a layer consist of propagating the solution from the last sublayer boundary to the end of the layer and if MFLAG = 1 of calculating the contribution of the whole interval to the normalization integral and adding it to XNORM.

## VII. INSTRUCTIONS FOR USING THE COMPUTER CODE

This section outlines the steps necessary to obtain the solutions to the eigenvalue problem (1.1) using the computer code described in the previous two sections.

### (1) Preparation of input parameters (entered into PARAM).

(NFILE(J), J = 1.5)

Naming of files:

J = 1 profile file

J = 2 temporary file

J = 3 depth functions at source depth

J = 4 depth functions at receiver depth

J = 5 eigenvalue file

J = 6 file for depth functions and

derivatives at layer boundaries

FREQ, RH01, RH02

Sound frequency, water density, ~~sea~~ floor density

IPRMIN, IPRMAX

First and last profile to be investigated

XKST(J)

Starting value of k for eigenvalue search for jth profile. Ordinarily the values of this array are all set equal to 1000. This way the first mode to be found will be the one with mode number one. If desired a run can be started with a higher mode number by specifying the appropriate

ICT

values of the wave numbers XKST for each of the selected profiles.

This parameter specifies the number of modes to be calculated. If a number  $>N$  ( $N$  = maximum number of modes for a profile) is selected all modes are calculated.

JMIND

Maximum number of up-layers allowed. Usually a good choice is to take one less than the number of layers below the location of the sound speed minimum.

XKD

Step size of wave number during search for sign change of  $W(k)$ . This quantity determines to a large extent the amount of computer time required for a particular profile. A good value to start with is  $XKD = \langle \Delta k \rangle / 6$  with  $\langle \Delta k \rangle$  as defined by (3-23). For the region

$$k < \frac{\omega}{c_{\max}} \quad c_B - c_{\max} \gg c_{\max} - c_{\min}$$

the spacing between eigenvalues increases linearly with mode number according to (3.21)

Hence, if

$$c_B - c_{\max} \gg c_{\max} - c_{\min}$$

with, e.g.,

$$c_{\min} \cong 1490 \text{ m/s}, \quad c_{\max} \cong 1540 \text{ m/s}, \\ c_B \cong 1800 \text{ m/s},$$

the spacing between eigenvalues can grow very large and one can save a considerable amount of computer time if the step size XKD is made to increase linearly with mode number when searching for eigenvalues in the region  $k < \frac{\omega}{c_{\max}}$ . This can be done by making use of (3-21), increasing XKD by

$$\frac{\pi c_{\min}}{2 f H^2}$$

each time an eigenvalue has been found. These quantities specify the depth at which the depth functions are calculated. In addition XINCR1 and XINCR2 determine the widths of the sublayers.

If the source and receiver depths are related by an integer or if they have a convenient common factor the depth functions at both locations can be calculated with one pass through the program after a particular eigenvalue has been found. In this case one sets

XINCR1, XMAX1

XINCR2, XMAX2

NOPT

NOPT = 1

and XINCR2, XMAX2 are not needed.

XINCR1 is chosen such that both the source and receiver depth are an integer multiple of it and XMAX1 is taken to be the source or receiver depth, whichever is larger (e.g., source depth = 200m, receiver depth = 500m, XINCR1 = 100m, XMAX1 = 500m).

If the source and receiver depth have no convenient common factor (e.g. source depth = 29m, receiver depth = 100m) the depth functions at these depths are calculated at two different passes through the program.

In this case

NOPT = 2

XINCR1 = XMAX1 = Source depth

XINCR2 = XMAX2 = Receiver depth.

The option NOPT = 1 must also be used if the receiver position falls into an up-layer. This case occurs when the water depth is very large and the receiver is located near the ocean floor. At the end of section 6 it was mentioned that there is a constraint on XINCR, which took the form (6.1). Accordingly XINCR1 and XINCR2 have to satisfy the condition

$$XINCR1, XINCR2 < 1400 f^{-2/3} (\Delta/h)_{\max}^{-1/3}$$

where  $f$  is the frequency and  $(\Delta/h)_{\max}$  the maximum sound speed gradient of profile.

NPPL(I), ISLAY,

ISOU, IRECL,

IRLAY, IRLAY]

are the parameters needed to extract the desired output data (eigenvalues, normalized depth function at source and receiver and the depth function and their derivatives at each layer boundary) from the temporary output file NFILE(2).

If  $\text{NOPT} = 1$  only the first 4 parameters are needed.  $\text{NPPL}(I)$  gives the number of data points in each layer (the surface is not counted as a data point, data points at layer boundaries belong to the preceding layer), i.e. if  $\text{XINCRL} = 50$  and the widths of the first and second layer are 100m and 240m,  $\text{NPPL}(1) = 2$  and  $\text{NPPL}(2) = 5$ , etc.

Data are taken at intervals of  $\text{XINCRL}$  to the largest multiple of  $\text{XINCRL}$  which falls into the layer in which  $\text{XMAX1}$  lies.  $\text{ISLAY}$  is the total number of data points taken,  $\text{ISOU}$  and  $\text{IREC1}$  specify the data points which correspond to the source and receiver depths respectively. If e.g. the source depth is 100m, the receiver depth 150m, in the example given above, then

$$\text{ISLAY} = 7 \quad (= \text{NPPL}(1) + \text{NPPL}(2))$$

$$\text{ISOU} = 2$$

$$\text{IREC1} = 3$$

If  $\text{NOPT} = 2$  the depth functions at the receiver depth are obtained in a second

loop through the programs as explained above. If the receiver position does not fall in an up-layer for any of the modes one sets

$$\text{IRLAY} = \text{IRLAX1} = \text{total number of data points taken.}$$

If the receiver depth does fall into an up-layer for some modes then it is necessarily very large and one sets

$$\text{XMAX2} = \text{total water depth } H.$$

Then

$$\text{IRLAY1} = \text{total number of data points taken } (\leq H/\text{XINCR2})$$

and

$$\text{IRLAY} = \text{IRLAY1 if the ocean bottom is not a data point,}$$

$$\text{IRLAY} = \text{IRLAY1}-1 \text{ if the ocean bottom is a data point.}$$

- (2) Preparation of profile data file. The data essentially consist of the array (1.2) plus the corresponding depth values  $Z_i$ . Sometimes the sound velocities are not given as the same depth for all profiles. Therefore the depth values are also taken as a function of range.

The profile data file is read by the subroutine SELAUT. At present the following data structure is assumed:

1st line:  $N$  = Number of layer boundaries for first profile,  
FORMAT (I3).

2nd- $j$ th lines:  $N + 1$  velocity values, 5 to a line, FORMAT (5D14.7)  
= sound velocity at each layer boundary + bottom velocity  
for first profile.

( $j + 1$ )st- $k$ th line:  $N + 1$  depth values, 5 to a line, FORMAT  
(5D14.7) = depth values of layer boundaries + bottom depth  
for first profile.

1st- $k$ th line repeated for each additional profile.

If this arrangement is found inconvenient it can easily be changed. The only program affected by this change would be the short subroutine SELAUT.

The program in its general form does not accept isovelocity layers. From (3.6) one can see that for those cases the Airy function arguments become  $\infty$  and (3.1) shows that the Airy functions are not defined there. Isovelocity layers do rarely occur in empirical profiles and if they do they can easily be avoided by replacing the vanishing slopes by very small slopes (e.g. .01 m/sec per 100m). If desired the program can

easily be generalized to include the (essentially trivial case) of isovelocity layers by adding 2 subroutines which propagate the down- and up- solutions for these cases between layer boundaries and which are called by the main program in place of PROP, LARGE or START and PR or LG.

Before setting up the profile data file it is necessary to check the widths of the vertical layers and to make sure they do not exceed a certain maximum value which is given by the requirement that the Airy functions should not grow or decay within one layer by more than the maximum number the computer can handle. The program in its present form was written for the PDP10 in which the exponents are limited to  $\approx \pm 35$ . At the end of each up-layer the depth functions are scaled down to  $\leq 10^{-5}$ . Hence the maximum allowable growth within one up-layer is  $10^{40}$  (or  $10^{60}$  for the bottom layer where the depth function is made to start out with  $\sim 10^{-25}$ ). Layers which do not satisfy this requirement have to be subdivided. An estimate for the maximum allowable layer width can be obtained by using the formulas derived in section 3. From there it follows that the Airy function arguments within the lowest layer (which is usually the one causing trouble) is

$$\zeta(z) = (2\omega^2 \Delta_i/h_i)^{1/3} (z_B - z)/c_{\min}$$

The ratio of the Airy function values at 2 locations within that layer is

$$\begin{aligned} \frac{Ai(\zeta_2)}{Ai(\zeta_1)} &\sim e^{\frac{2}{3} [\zeta(z_1)^{3/2} - \zeta(z_2)^{3/2}]} \\ &= e^{\frac{4\pi}{3} \sqrt{2} f (\Delta_n/h_n)^{1/2} (h_n/c_{\min})^{3/2} \left[ \left( \frac{z_B - z_1}{h_n} \right)^{3/2} - \left( \frac{z_B - z_2}{h_n} \right)^{3/2} \right]}, \end{aligned}$$

where  $\Delta_n/h_n$  is the sound speed gradient and  $h$  the width of the layer. For the example chosen in section 3 ( $f = 250 \text{ Hz}$ ,

$$\Delta_n/h_n = .015 \text{ 1/3} \quad \text{and} \quad h_n = 3000 \text{ m):}$$

$$\frac{Ai(\zeta_2)}{Ai(\zeta_1)} \sim 10^{223} \left[ \left( \frac{z_B - z_1}{h_n} \right)^{3/2} - \left( \frac{z_B - z_2}{h_n} \right)^{3/2} \right].$$

With  $z_2 = z_B = 4000 \text{ m}$ ,  $z_1 = 1000 \text{ m}$  as in the example one obtains

$$\frac{Ai(\zeta_2)}{Ai(\zeta_1)} \sim 10^{223}$$

as before (see section 3). Hence in this case the 3000 m wide bottom layer would have to be divided up into about 6 separate layers of  $\sim 500 \text{ m}$  each.

The example presented is actually something like an extreme case since  $250 \text{ Hz}$  for a deep ocean case is about the upper limit

of application for the normal mode theory and furthermore the layer thicknesses as they are usually given rarely exceed 1000 m. Hence only in rare cases do layers have to be divided up. The program presented in this report was written for the PDP10 in which the exponents are limited to  $\sim \pm 35$ . For a machine with a larger exponent range the restrictions on the layer widths can of course be relaxed correspondingly.

(3) Adjustment of array dimensions.

The maximum number of layers in any of the profiles determines the dimension of the arrays.

ISC, MSC, S, B, X, Y, T, TD, C, P, ALPH, XS, XT, NPPL,  
and the second dimension of

NSC.

The dimensions of these quantities should be chosen as the number of layers + 2.

The dimension of the arrays

JUPA, IRECA,

the first dimension of

NSC

and the second dimensions of

UO, UF, XKA

are determined by the maximum number of modes obtained for any particular profile.

The array

XKST

has to be dimensioned according to the number of profiles.

Dimensioned arrays are limited to the main program, except for

XKST and NFILE

which appear both in the main program and in the subroutines

PARAM and

X and Y

which are used in SELAUT.

(4) Execution of program.

After the preparations described in (1)-(3) are taken care of the program can be executed. The following routines and files are involved:

Main program: SEARCH

Subroutines: PARAM, START, PROP, LARGE, PR, LG, DAIRY,  
EXTREM, SELAUT

Profile data file: NFILE (1)

The results of the calculations is output through the files

NFILE (3): depth functions at source depth

NFILE (4): depth functions at receiver depth

NFILE (5): eigenvalues

NFILE (6): depth functions and derivatives at layer boundaries

(5) Mode number test.

If the eigenvalue spectrum contains almost degenerate modes it is possible that the search procedure contained in the program skipped over some modes. Therefore after execution of the program one has to carry out a check to see whether the eigenvalue spectrum for each of the profiles is complete. This consists of rerunning the program for the highest mode for each profile (using the appropriate starting values XKST for the eigenvalue search) with a sufficiently small XINCRI to obtain a complete picture of the modes and plotting the modes as a function of depth. The depth function values can be taken from the temporary file NFILE(2). If one finds that for a particular profile the number of modes corresponds to the mode number then one can conclude that the solution of the eigenvalue problem was successfully completed for this profile. If the number of modes is larger than the mode number this means that some modes are missing. The missing modes can be found by repeating the procedure described above for some lower modes (e.g. starting with a mode number  $\cong 1/4$  of the highest mode number), until the mode number of the missing modes has been established. The exact location of the missing modes can then be found by running the program between the neighboring two modes with a sufficiently small XKD.

```

00000 C
00000 C
00000 C SEARCH
00000 C (MAIN PROGRAM)
00000 C
00010 IMPLICIT DOUBLE PRECISION(A-H,O-Z)
00020 DIMENSION FU(100),EIG(100),ISC(60),MSC(60),JUPA(80)
00030 DIMENSION S(60),B(60),X(60),Y(60),T(60),TD(60),C(60)
00040 DIMENSION P(60),ALPH(60),UO(6,80),UF(6,80),XKA(6,80)
00050 DIMENSION XS(60),XT(60),NPPL(60),NSC(80,25),IRECA(80)
00060 DIMENSION XKST(10),NFILE(7)
00070 COMMON /IFLAGS/ MFLAG,MFLAG1,MTYPE /JC/JCOUNT
00080 COMMON /SQNORM/ XNORM
00090 COMMON /DATCOM/XKST, FREQ,RHO1,RHO2,XKD,XMAX1,XMAX2,
00100 1 XINCR1,XINCR2,NPPL,JMIND,NFILE,ISLAY,IRLAY,
00110 2 ISOU,IREC,NOPT,IPRMIN,IPRMAX,ICT,IRLAY1,IREC1
00120 CALL PARAM ← (1)
00130 DM=11,000
00140 XKDS=XKD
00150 MTYPE=NFILE(2)
00160 NTYPE=NFILE(2)
00170 DO 15 IPR=IPRMIN,IPRMAX
00180 CLOSE (UNIT=NFILE(1))
00190 CLOSE (UNIT=NFILE(2))
00200 CALL SELAUT(IPR,NFILE(1),NFILE(2))
00210 CLOSE (UNIT=NFILE(2))
00220 XNQR1=0,000
00230 SCALE=1,000
00240 SCALE2=1,000
00250 DSCALE=1,0020
00260 ICGUNT=0
00270 READ(NFILE(2),26) NUMBER
00280 26 FORMAT(I3)
00290 READ(NFILE(2),1) (X(I),Y(I),I=1,NUMBER)
00300 READ(NFILE(2),1) CBOTM
00310 CLOSE(UNIT=NFILE(2))
00320 CMIN=Y(1)
00330 DO 199 I=1,NUMBER
00340 IF(Y(I),LT,CMIN) CMIN=Y(I)
00350 199 CONTINUE
00360 NMIN=NUMBER-1
00370 H=X(NUMBER)
00380 JD=NMIN-JMIND
00390 H2=H*H
00400 PI=3,141592653589783238400
00410 EDIF=1,000
00420 1 FORMAT(20)
00430 OM=2,000*FREQ*PI
00440 OM2=OM**2
00450 XK=OM/CMIN
00460 IF(XK,GT,XKST(IPR)) XK=XKST(IPR)
00470 P(1)=H2*OM2/(Y(1)**2)
00480 DO 43 I=1,NMIN
00490 X(I+1)=X(I+1)/H
00500 43 P(I+1)=H2*OM2/(Y(I+1)**2)
00510 DO 44 I=1,NMIN
00520 S(I)=(P(I+1)-P(I))/(X(I+1)-X(I))
00530 C(I)=P(I)-S(I)*X(I)
00540 44 CONTINUE
00550 ITER=0

```

(2)

(3)

(4)

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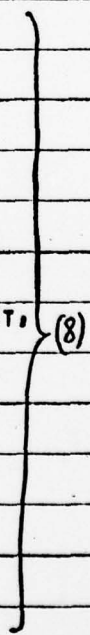
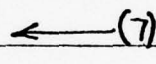
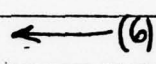
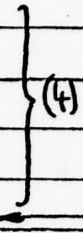
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00560      TH=1.000/3.000
00570      DO 66 I=1,NMIN
00580      SN=S(I)/DABS(S(I))
00590      66      ALPH(I)=SN*(S(I)**TH)
00600      T(1)=S(1)/DABS(S(1))
00610      DO 67 I=2,NMIN
00620      T(I)=ALPH(I-1)/ALPH(I)
00630      67      CONTINUE
00640      890      IFLAG=0
00650      ICAL=0
00660      XMAX=XMAX1/H
00670      XINCR=XINCR1/H
00680      IF(ICOUNT.EQ.1CT) GO TO 982
00690      MFLAG=0
00700      MFLAG1=0
00710      XNORM=0.000
00720      JTER=0
00730      KFLAG=0
00740      XFACT=1.000
00750      990      XK=XK-XKD
00760      UBQTM=RHQ2/RHO1
00770      IF(XK.GT.(OM/CBOTM)) GO TO 991
00780      IF (XKD,NE,XKDS) GO TO 980
00790      XK=XK+XKD
00800      XKD=XKDS/10
00810      GO TO 990
00820      991      CONTINUE
00830      DO 20 I=1,NMIN
00840      ISC(I)=0
00850      20      MSC(I)=0
00860      994      CONTINUE
00870      UK=DSQRT((XK**2)-((OM/CBOTM)**2))
00880      IF(MFLAG.EQ.1.AND,ICAL,NE,1) JUPA(ICOUNT)=JUP
00890      JTER=JTER+1
00900      XN=H2*(XK**2)
00910      DO 3      J=1,NMIN
00920      JUP=0
00930      TT=T(J)
00940      IF(X(J),GT,XMAX) MFLAG1=0
00950      IF(J.EQ.1) XNEW=-XINCR
00960      SN=S(J)/DABS(S(J))
00970      AL=SN*(S(J)**TH)
00980      B(J)=C(J)-XN
00990      BE=B(J)/S(J)
01000      DX=-AL*(X(J)+BE)
01010      DX1=-AL*(X(J+1)+BE)
01020      SW=1.00
01030      IF(J.EQ.1) CALL START(AL,BE,X(J),X(J+1),UOUT,UPOUT,
01040      1 XNEW,XINCR,SW,TT)
01050      IF(J.EQ.1) GO TO 812
01060      CONTINUE
12 01070      IF(J.LE.JD) GO TO 605
11 01080      IF(KFLAG.EQ.1) GO TO 600
10 01090      IF(DX1.LE.7) GO TO 605
9 01100      KFLAG=1
8 01110      JD=J-1
7 01120      GO TO 600
6 01130      605      CONTINUE
5 01140      IF(DX.GT,DM) CALL LARGE(AL,BE,X(J),X(J+1),UIN,
4 01150      1 UOUT,UPIN,UPOUT,XNEW,XINCR,TT)
3

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01160      IF(DX,LE,DM) CALL PROP(AL,BE,X(J),X(J+1),UIN,
01170      1  UDOUT,UPIN,UPOUT,XNEW,XINCR,TT)
01180      812  UIN=UOUT
01190      UPIN=UPOUT
01200      IF (MFLAG,NE,1,OR,ICAL,EQ,1) GO TO 115
01210      IF(J,NE,1) GO TO 120
01220      CALL DAIRY(-AL*BE,AI,AIP,BI,BIP)
01230      UOUT=0,
01240      UPOUT=0DABS(BI*AIP-AI*BIP)
01250      JPL=1
01260      WRITE(NFILE(7),130) JPL,KFLAG,X(1),AL,BE,UOUT,UPOUT
01270      120  CONTINUE
01280      JPL=J+1
01290      WRITE(NFILE(7),130) JPL,KFLAG,X(J+1),AL,BE,UOUT,UPOUT
01300      130  FORMAT(2I4,5D)
01310      115  CONTINUE
01320      3    CONTINUE
01330      SCALE=UBOTM/UIN
01340      SCALE2=(SCALE**2)
01350      UPBOTM=-H*UK/-AL
01360      FUNC=UIN*UPBOTM=UPIN*UBOTM
01370      XNORM2=0,000
01380      GO TO 45
01390      600  CONTINUE
01400      KFLAG=1
01410      ISW=ICOUNT+1
01420      JX=J
01430      DO 4 J=NMIN,JX,-1
01440      JUP=NMIN=JX+1
01450      FCT=1,000
01460      MS=0
01470      IF(MFLAG1,EQ,1) MS=1
01480      IF(X(J),GT,XMAX,AND,MS,EQ,1) MFLAG1=0
01490      IF(J,EQ,NMIN) XNEW=1,000
01500      SN=S(J)/DABS(S(J))
01510      AL=SN*(S(J)**TH)
01520      TD(J)=ALPH(J+1)/ALPH(J)
01530      IF(J,LT,NMIN) GO TO 606
01540      TD(J)=1,000/AL
01550      UDIN=UBOTM*XFACT/1,0025
01560      UPDIN=-H*UK/-AL
01570      UPDIN=UPDIN*XFACT/1,0025
01580      606  CONTINUE
01590      B(J)=C(J)-XN
01600      BE=B(J)/S(J)
01610      DX1=-AL*(X(J)+BE)
01620      DX=-AL*(X(J+1)+BE)
01630      IF(DX,GT,DM) CALL LG(AL,BE,X(J+1),X(J),UDIN,
01640      1  UDOUT,UPDIN,UPDOUT,XNEW,XINCR,TD(J),XS(J),XT(J))
01650      IF(DX,LE,DM) CALL PR(AL,BE,X(J+1),X(J),UDIN,
01660      1  UDOUT,UPDIN,UPDOUT,XNEW,XINCR,TD(J),XS(J),XT(J))
01670      IF(J,EQ,JX) GO TO 300
01680      IF(MFLAG,EQ,1) GO TO 110
01690      IF(IFLAG,EQ,1) GO TO 608
01700      603  IF(IFLAG,NE,1,AND,DABS(UDOUT),GT,1,D-5) ISC(J)=ISC(J)+1
01710      IF(ISC(J),EQ,0) GO TO 300
01720      UDOUT=UDOUT/1,010
01730      UPDOUT=UPDOUT/1,010
01740      IF(DABS(UDOUT),GT,1,D-5) GO TO 603
01750      GO TO 300

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01760 608 UDOUT=UDOUT/10.00**((ISC(J)*10)
01770 UPDOUT=UPDOUT/10.00**((ISC(J)*10)
01780 GO TO 300
01790 110 CONTINUE
01800 IF(ICAL,EQ,1) GO TO 301
01810 604 IF(DABS(UDOUT).GT,1,0-5)MSC(J)=MSC(J)+1
01820 IF(MSC(J),EQ,0) GO TO 300
01830 UDOUT=UDOUT/1,010
01840 UPDOUT=UPDOUT/1,010
01850 IF(DABS(UDOUT).GT,1,0-5) GO TO 604
01860 GO TO 300
01870 301 UDOUT=UDOUT/10.00**((MSC(J)*10)
01880 UPDOUT=UPDOUT/10.00**((MSC(J)*10)
01890 300 CONTINUE
01900 JPL=J+1
01910 UX=0
01920 PX=0.
01930 IF(J,EQ,JX.AND,JX,NE,NMIN) UX=UDIN
01940 IF(J,EQ,JX.AND,JX,NE,NMIN) PX=UPDIN
01950 IF(ICAL,NE,1.AND,MFLAG,EQ,1)WRITE(NFILE(7),130)
01960 1 JPL,KFLAG,X(J),AL,BE,UX,PX
01970 UDIN=UDOUT
01980 UPDIN=UPDOUT
01990 IF(MS,EQ,1) MFLAG1=1
02000 4 CONTINUE
02010 UDOUT=UDOUT/1,0015
02020 UPDOUT=UPDOUT/1,0015
02030 FUNC=UIN*UPDOUT-UPIN*UDOUT
02040 IF(MFLAG,NE,1)GO TO 45
02050 NS=0
02060 IF(JX,EQ,NMIN) GO TO 22
02070 DO 28 I=JX+1,NMIN
02080 28 NS=NS-MSC(I)+ISC(I)
02090 DO 47 K=JX,NMIN
02100 NSC(ICOUNT,K)=NS
02110 IF(K,LE,JX) GO TO 47
02120 DO 46 N=JX,K
02130 46 NSC(ICOUNT,K)=NSC(ICOUNT,K)+MSC(N)
02140 47 CONTINUE
02150 22 XNORM2=0,00
02160 XST=X(JX)*XT(JX)
02170 IF(NS,EQ,0) XNORM2=XST
02180 IF(NS,NE,0.AND,DLOG10(XST)-20,00*NS.GT,-30,00)
02190 1 XNORM2=10.00**((DLOG10(XST)-NS*20,00)
02200 45 CONTINUE
02210 IF(ICAL,EQ,1) GO TO 892
02220 C TYPE 1,XK,FUNC
02230 XNOR1=0,000
02240 IF(KFLAG,EQ,0) XNOR1=RH02/((2,000*H+UK)
02250 XNR=XNOR1+(XNORM2+XNORM)*SCALE2
02260 IF(XNR,GT,0,000) SQN=(1,000/DSORT(XNR))*DABS(SCALE)
02270 IF(MFLAG,EQ,1) WRITE(NTYPE,1) SQN
02280 IF(MFLAG,NE,1) GO TO 165
02290 C STORE U,UP AT LAYER BOUNDARIES FOR MODE JCOUNT
02300 WRITE(21,170) NUMBER
02310 170 FORMAT(I4)
02320 CLOSE(UNIT=29)
02330 DO 160 IN=1,NUMBER
02340 READ(NFILE(7),130) JC,KC,XC,ALC,BEC,UXC,UPXC
02350 UXC=UXC*SQN

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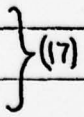
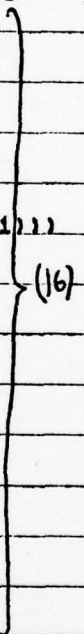
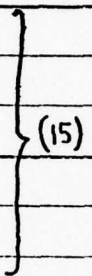
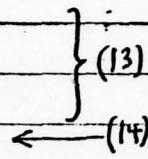
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02360      UPXC=UPXC*SQN
02370      IF(KC.EQ.1) UXC=UXC*10**(-10*NSC(ICOUNT,JX))
02380      IF(KC.EQ.1) UPXC=UPXC*10**(-10*NSC(ICOUNT,JX))
02390      160      WRITE(21,132) XC,ALC,BEC,UXC,UPXC
02400      132      FORMAT(50)
02410      165      CONTINUE
02420      CLOSE(UNIT=NFILE(7))
02430      IF(MFLAG.NE.1.OR.NOPT.EQ.1) GO TO 892
02440      ICAL=1
02450      XMAX=XMAX2/H
02460      XINCR=XINCR2/H
02470      MFLAG1=1
02480      GO TO 994
02490      892      IF(MFLAG.EQ.1) GO TO 890
02500      IF(DABS(EDIF).LT.1.0D-15) ITER=1
02510      IF(ITER.EQ.0) GO TO 444
02520      ICOUNT=ICOUNT+1
02530      EDIF=1.0D0
02540      MFLAG1=1
02550      MFLAG=1
02560      IF(KFLAG.EQ.1) XFACT=UIN/(UDOUT*1.0D15)
02570      WRITE(NTYPE,1) XK
02580      TYPE 770,XK,ICOUNT
02590      ITER=0
02600      GO TO 994
02610      444      CONTINUE
02620      FU(JTER)=FUNC
02630      EIG(JTER)=XK
02640      IF(IFLAG.EQ.1) GO TO 880
02650      IF(JTER.EQ.1) GO TO 990
02660      PROD=FU(JTER)/DABS(FU(JTER))
02670      PROD=PROD*FU(JTER-1)/DABS(FU(JTER-1))
02680      IF(PROD.GT.0.0D0) GO TO 990
02690      IFLAG=1
02700      XK=EIG(JTER)-DSQRT(FU(JTER)/(FU(JTER)-FU(JTER-1)))
02710      1      *(EIG(JTER)-EIG(JTER-1))
02720      GO TO 994
02730      880      JMIN=JTER-1
02740      K=JMIN
02750      GO TO 882
02760      DO 881 K=JMIN,1,-1
02770      PROD=FU(JTER)/DABS(FU(JTER))
02780      PROD=PROD*FU(K)/DABS(FU(K))
02790      IF(PROD.LT.0.0D0) GO TO 882
02800      881      CONTINUE
02810      882      SE=(FU(JTER)-FU(K))/(EIG(JTER)-EIG(K))
02820      SIN=FU(JTER)-(SE*EIG(JTER))
02830      XK=-SIN/SE
02840      EDIF=XK-EIG(JTER)
02850      770      FORMAT(10,15)
02860      GO TO 994
02870      980      CONTINUE
02880      TYPE 31,(JUPA(I),I=1,ICOUNT)
02890      31      FORMAT(' UP=LAYERS ='/10I4)
02900      CLOSE (UNIT=NTYPE)
02910      NTT=0
02920      DO 41 I=1,NMIN
02930      41      NTT=NTT+NPPL(I)
02940      DO 16 I=1,ICOUNT
02950      IF(I.GE.ISW.OR.NOPT.EQ.1) IRLAY=IRLAY1

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02960      IREC=NTI-IREC1+1
02970      DO 42 J=1,NMIN-JUPA(I)
02980      42      IREC=IREC+NPPL(J)
02990      95      IF(I.GE,ISW) IREC=IREC1
03000      IRECA(I)=IREC
03010      READ(NTYPE,1) XKA(IPR,I)
03020      DO 70 K=1,ISLAY
03030      READ(NTYPE,2) BXTMP
03040      IF(K.EQ,ISOU) BX=BXTMP
03050      70      IF(K.EQ,IREC.AND.NOPT.EQ.1) CX=BXTMP
03060      READ(NTYPE,1) XN
03070      IF(NOPT,EQ.1) GO TO 30
03080      DO 71 K=1,IRLAY
03090      READ(NTYPE,2) CXTMP
03100      71      IF(K.EQ,IREC) CX=CXTMP
03110      30      CONTINUE
03120      NSPPL=0
03130      DO 39 L=1,NMIN
03140      NSPPL=NSPPL+NPPL(L)
03150      KL=L
03160      IF(NSPPL.GT,IREC1) GO TO 40
03170      39      CONTINUE
03180      40      CONTINUE
03190      UO(IPR,I)=BX*XN
03200      16      UF(IPR,I)=CX*XN*10.D0**(-10*NSC(I,KL))
03210      CLOSE (UNIT=NTYPE)
03220      19      XKD=XKDS
03230      TYPE 48,(IRECA(I),I=1,ICOUNT)
03240      48      FORMAT('      # OF REC IN WFCT FILE ='/10I4)
03250      TYPE 18,IPR,ICOUNT,ISW
03260      18      FORMAT('      END OF PROFILE #',I2,'      # OF EV IN THIS P
03270      18      1ROFILE =',I2,'      ISW=',I2)
03280      15      CONTINUE
03290      DO 17 J=IPRMIN,IPRMAX
03300      WRITE(NFILE(3),7) (UO(J,I),I=1,IM)
03310      WRITE(NFILE(4),7) (UF(J,I),I=1,IM)
03320      WRITE(NFILE(5),7) (XKA(J,I),I=1,IM)
03330      17      CONTINUE
03340      2      FORMAT(D20.0)
03350      7      FORMAT(86D)
03360      996     STOP
03370      END
    
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00010      SUBROUTINE SELAUT(IP,NFIL,NTEMP)
00020      DOUBLE PRECISION X(55),Y(55)
00030      DO 10 I=1,IP
00040      READ(NFIL,2) N
00050      READ(NFIL,1) (Y(J),J=1,N+1)
00060      10 READ(NFIL,1) (X(J),J=1,N+1)
00070      WRITE(NTEMP,2) N
00080      DO 20 I=1,N
00090      20 WRITE(NTEMP,3) X(I),Y(I)
00100      WRITE(NTEMP,3) Y(N+1)
00110      1 FORMAT(5D14,7)
00120      2 FORMAT(1,3)
00130      3 FORMAT(2D)
00140      RETURN
00150      END
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00010      SUBROUTINE PARAM
00020      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00030      DIMENSION NPPL(60),XKST(10),NFILE(7)
00040      COMMON /DATCOM/XKST,FREQ,RHO1,RHO2,XKD,XMAX1,XMAX2,
00050      1 XINCR1,XINCR2,NPPL,JMIND,NFILE,ISLAY,IRLAY,
00060      2 ISOU,IREC,NOPT,IPRMIN,IPRMAX,ICT,IRLAY1,IREC1
00070      DATA NPPL/10,14,24,57*0/
00080      DATA XKST/10*1000,00/
00090      DATA NFILE/38,24,25,27,28,21,29/
00100      CON=-30487804
00110      FREQ=5.00
00120      RHO1=1.00
00130      RHO2=1.00
00140      XMAX1=100,
00150      XINCR1=50,
00160      XMAX2=100,
00170      ISLAY=10
00180      IRLAY=51
00190      ISOU=1
00200      IREC=2
00210      IRLAY1=51
00220      IREC1=2
00230      XINCR2=100.
00240      XKD=,40=4
00250      ICT=20
00260      JMIND=0
00270      IPRMIN=1
00280      IPRMAX=4
00290      NOPT=1
00300      RETURN
00310      END

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CATHOLIC UNIV OF AMERICA WASHINGTON D C DEPT OF PHYSICS F/G 9/2  
A FORTRAN CODE FOR THE CALCULATION OF SOUND PROPAGATION IN A RA--ETC(U)  
JUN 78 A NAGL, G L ZARUR, H UEBERALL N00173-77-C-0008

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	00010		SUBROUTINE DAIRY(DX,AI,AIP,BI,BIP)	26230
	00020		IMPLICIT DOUBLE PRECISION (A-W,0-Z)	
	00030	C	FOR DOUBLE PRECISION ARGUMENTS, THIS ROUTINE CALCULATES THE AIRY	26240
	00040	C	FUNCTION AI(X) AND ITS DERIVATIVE AIP(X), IT ALSO FINDS	26250
	00050	C	THE OTHER REAL LINEARLY INDEPENDENT SOLUTION BI(X) AND	26260
	00060	C	ITS DERIVATIVE BIP(X).	26270
	00070	C	THE DEFINITIONS AND NORMALIZATIONS ARE AS IN NBS HANDBOOK	26280
	00080	C	OF MATHEMATICAL FUNCTIONS,P,446	26290
	00090	C	THE METHODS USED ARE POWER SERIES EXPANSION FOR SMALL X	26300
	00100	C	AND GAUSSIAN INTEGRATION FOR LARGE X	26310
	00110		DIMENSION X(16),W(16),XSQ(16)	26320
	00120	C	DOUBLE PRECISION DX,AI,AIP,BI,BIP	26330
	00130	C	DOUBLE PRECISION XS ,XCUBE,AISUM,AIPSUM	26340
	00140	C	DOUBLE PRECISION DF,DFP,DG,DGP	26350
	00150	C	DOUBLE PRECISION FJM2,EJM1,FJ,FJP1,FJP2,FACTOR	26360
	00160	C	DOUBLE PRECISION C1,C2,ROOT3	26370
	00170	C	DOUBLE PRECISION DZETA,DARG,DROOTX	26380
	00180	C	DOUBLE PRECISION ROOT4X,S,CO,RATIO,EFAC,ZETASQ	26390
	00190	C	DOUBLE PRECISION SUMR,SUMI,SUMRP,SUMIP,TERMR,TERMI	26400
	00200	C	DOUBLE PRECISION DZERO,DA,DB,DEN,ONE	26410
	00210	C	DOUBLE PRECISION X,W,XSQ	26420
	00220	C	DOUBLE PRECISION RSQ,TEMP, RTP1,RTP12	26430
	00230	C	DOUBLE PRECISION TERMA,TERMB	26440
	00240		LOGICAL NEEDBI	26450
	00250		DATA DZERO,ONE /0,000,1,000/	26460
	00260		DATA ROOT3/1.73205080756887700/	26470
	00270		DATA C1,C2 /,35502805388781700, .25881940379280700/	26480
	00280		DATA RTP1 /,282094791773878100/	26490
	00290		DATA RTP12/,564189583547756200/	26500
	00300	C	POSITIONS AND WEIGHTS FOR 10-TERM SUM FOR AIRY FUNCTIONS	26510
	00310		DATA W( 1) / 3.15425157629647870-14/	26520
	00320		DATA W( 2) / 6.63942108195849210-11/	26530
	00330		DATA W( 3) / 1.75838890613456690-08/	26540
	00340		DATA W( 4) / 1.37123923704358150-06/	26550
	00350		DATA W( 5) / 4.43509666392843500-05/	26560
	00360		DATA W( 6) / 7.15550109177182550-04/	26570
	00370		DATA W( 7) / 6.48895661033353810-03/	26580
	00380		DATA W( 8) / 3.64404158757732820-02/	26590
	00390		DATA W( 9) / 1.43997924185909990-01/	26600
	00400		DATA W(10) / 8.12311413362614860-01/	26610
	00410		DATA X( 1) / 1.40830810721809640+01/	26620
	00420		DATA X( 2) / 1.02148854791973310+01/	26630
	00430		DATA X( 3) / 7.44160184504509300+00/	26640
	00440		DATA X( 4) / 5.30709430617819270+00/	26650
	00450		DATA X( 5) / 3.63401350291324620+00/	26660
	00460		DATA X( 6) / 2.33106523030524500+00/	26670
	00470		DATA X( 7) / 1.34479708246092680+00/	26680
	00480		DATA X( 8) / 6.41888583695672960-01/	26690
	00490		DATA X( 9) / 2.01003459981210460-01/	26700
	00500		DATA X(10) / 8.05943591720528330-03/	26710
	00510		DATA XSQ( 1) /0.198333172485621700 03/	26720
12	00520		DATA XSQ( 2) /0.104343885353116500 03/	26730
11	00530		DATA XSQ( 3) /0.553774380201781700 02/	26740
10	00540		DATA XSQ( 4) /0.281652499746689900 02/	26750
9	00550		DATA XSQ( 5) /0.132060541393558000 02/	26760
8	00560		DATA XSQ( 6) /0.543386510793804400 01/	26770
7	00570		DATA XSQ( 7) /0.180847919299542000 01/	26780
6	00580		DATA XSQ( 8) /0.412020953878836900 00/	26790
5	00590		DATA XSQ( 9) /0.404023909244180700-01/	26800
4	00600		DATA XSQ(10) /0.649545073035383900-04/	26810
3				

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	00610	C	POSITIONS AND WEIGHTS FOR 4-TERM SUM FOR AIRY FUNCTIONS	26820
	00620		DATA W(11) / 4.77639030575772630-05/	26830
	00630		DATA W(12) / 4.99143064329109590-03/	26840
	00640		DATA W(13) / 8.61698469938403120-02/	26850
	00650		DATA W(14) / 9.08790958459811020-01/	26860
	00660		DATA X(11) / 3.91983295544550910+00/	26870
	00670		DATA X(12) / 1.69156190048235040+00/	26880
	00680		DATA X(13) / 5.02755324672630180-01/	26890
	00690		DATA X(14) / 1.92470605620156920-02/	26900
	00700		DATA XSQ(11) / 0.153650903985966700 02/	26910
	00710		DATA XSQ(12) / 0.286138166316346100 01/	26920
	00720		DATA XSQ(13) / 0.252762916486681800 00/	26930
	00730		DATA XSQ(14) / 0.370449340277899800-03/	26940
	00740	C	POSITIONS AND WEIGHTS FOR 2-TERM SUM FOR AIRY FUNCTIONS	26950
	00750		DATA W(15) / 9.68072805957736040-01/	26960
	00760		DATA W(16) / 3.19271940421639580-02/	26970
	00770		DATA X(15) / 3.68006018661530440-02/	26980
	00780		DATA X(16) / 1.05924693821123780+00/	26990
	00790		DATA XSQ(15) / 0.135428429771110700-02/	27000
	00800		DATA XSQ(16) / 0.112200407610988100 01/	27010
	00802		IF(DX.GT.-1000.00) GO TO 991	
	00804		CALL SINCOS(DX, AI, AIP, BI, BIP)	
	00806		RETURN	
	00808	991	CONTINUE	
	00810		IF(DX.LT.-5.000) GO TO 100	27020
	00820		NEEDBI=,FALSE.	27030
	00830		IF(DX.GT.3.700) GO TO 200	27040
	00840	C	THIS ROUTE FOR SMALLX, USING POWER SERIES.	27050
	00850	C	INITIALIZE	27060
	00860	10	XS = DX*DX	27070
	00870		XCUBE = XS *DX	27080
	00880		XS = XS *0.500	27090
	00890		DF = C1	27100
	00900		DFP = C1*XS	27110
	00910		DG = C2*DX	27120
	00920		DGP = C2	27130
	00930		AISUM = DF - DG	27140
	00940		AIPSUM = DFP - DGP	27150
	00950		BI = DF + DG	27160
	00960		BIP = DFP + DGP	27170
	00970		FJM2=-2.000	27180
	00980	20	FJM2=FJM2+3.000	27190
	00990		FJM1=FJM2+ONE	27200
	01000		FJ=FJM1+ONE	27210
	01010		FJP1=FJ+ONE	27220
	01020		FJP2=FJP1+ONE	27230
	01030		RATIO = XCUBE/FJ	27240
	01040		DF = DF*RATIO/FJM1	27250
	01050		DFP = DFP*RATIO/FJP2	27260
	01060		DG = DG*RATIO/FJP1	27270
	01070		DGP = DGP*RATIO/FJM2	27280
12	01080		BI = BI + (DF+DG)	27290
11	01090		BIP = BIP + (DFP+DGP)	27300
10	01100		IF(NEEDEDI) GO TO 80	27310
9	01110		AISUM = AISUM + (DF-DG)	27320
8	01120		AIPSUM = AIPSUM + (DFP-DGP)	27330
7	01130	C	CONVERGENCE TEST	27340
6	01140	80	IF(DABS(DF).GT.1.00-16) GO TO 20	27350
5	01150	C	CONVERGENCE, COMPUTE FUNCTIONS	27360
4	01160	99	BI = ROOT3*BI	27370
3				

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	01170		BIP = ROOT3*BIP	27380
	01180	C	THIS RETURNS IF X IS BETWEEN 3.7 AND 8.0, SINCE IN SUCH CASES MORE	27390
	01190	C	ACCURATE VALUES OF AI AND AIP HAVE ALREADY BEEN FOUND BY GAUSSIAN	27400
	01200	C	INTEGRATION	27410
	01210		IF(NEEDBI)RETURN	27420
	01220		AI = AISUM	27430
	01230		AIP = AIPSUM	27440
	01240		RETURN	27450
	01250	C	GAUSSIAN INTEGRATION FOR LARGE NEGATIVE X	27460
	01260	100	DROOTX = DSQRT(-DX)	27470
	01270		ROOT4X = DSQRT(DROOTX)	27480
	01280		DZETA = -.6666666666666667*DX*DROOTX	27490
	01290		DARG = DZETA - .7853981633974483	27500
	01300		SUMR = DZERO	27510
	01310		SUMI = DZERO	27520
	01320		SUMRP = DZERO	27530
	01330		SUMIP = DZERO	27540
	01340	C	TEST TO SEE HOW MANY TERMS ARE NEEDED IN GAUSSIAN INTEGRATION	27550
	01350		IF(DX.LT.(-200.D0)) GO TO 140	27560
	01360		IF(DX.LT.(-15.D0)) GO TO 130	27570
	01370	C	THIS CASE FOR DX BETWEEN -5.0 AND -15.0	27580
	01380		LIMLO=1	27590
	01390		LIMHI=10	27600
	01400		GO TO 149	27610
	01410	C	THIS CASE FOR DX BETWEEN -15.0 AND -200.	27620
	01420	130	LIMLO=11	27630
	01430		LIMHI=14	27640
	01440		GO TO 149	27650
	01450	C	THIS CASE FOR DX.LT.-200.	27660
	01460	140	LIMLO=15	27670
	01470		LIMHI=16	27680
	01480	149	ZETASQ=DZETA**2	27690
	01490		DO 150 K=LIMLO,LIMHI	27700
	01500		TERMR=W(K)/((ZETASQ+XSQ(K))**2)	27710
	01510		SUMR = SUMR + TERMR	27720
	01520		TERMR=TERMR*X(K)	27730
	01530		SUMI=SUMI+TERMR	27740
	01540		TERMR=TERMR*X(K)	27750
	01550		SUMRP=SUMRP+TERMR	27760
	01560	150	SUMIP=SUMIP+TERMR*X(K)	27770
	01570		SUMR=(SUMR*ZETASQ+SUMRP)*ZETASQ	27780
	01580		TEMP=SUMI*ZETASQ	27790
	01590		SUMI=(TEMP+SUMIP)*DZETA	27800
	01600		SUMRP=SUMRP*DZETA	27810
	01610		SUMIP=SUMIP-TEMP	27820
	01620	C	FORM AIRY FUNCTIONS	27830
	01630	196	S = DSIN(DARG)	27840
	01640		CO = DCOS(DARG)	27850
	01650		RATIO = RTP12/ROOT4X	27860
	01660		AI = RATIO*(CO*SUMR + S*SUMI)	27870
	01670		BI = RATIO*(CO*SUMI - S*SUMR)	27880
12	01680		SUMRP=SUMRP+SUMRP	27890
11	01690		RATIO = -.2500/DX	27900
10	01700		FACTOR = -RTP12*ROOT4X	27910
9	01710		AIP = RATIO*AI - DROOTX*BI + FACTOR*(CO*SUMRP+S*SUMIP)	27920
8	01720		BIP = RATIO*BI + DROOTX*AI + FACTOR*(CO*SUMIP-S*SUMRP)	27930
7	01730		RETURN	27940
6	01740	C	GAUSSIAN INTEGRATION FOR LARGE POSITIVE X	27950
5	01750	200	DROOTX = DSQRT(DX)	27960
4	01760		DZETA = .6666666666666667*DX*DROOTX	27970
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01770      EFAC = DEXP(-DZETA)                                27980
01780      ROOT4X = DSQRT(DROOTX)                            27990
01790      AI = DZERO                                         28000
01800      BI = DZERO                                         28010
01810      AIP = DZERO                                       28020
01820      BIP = DZERO                                       28030
01830      IF(DX.LT.8.000) NEEDBI=.TRUE.                    28040
01840      C TEST TO SEE HOW MANY TERMS ARE NEEDED IN GAUSSIAN INTEGRATION 28050
01850      IF(DX.GT.15.000) GO TO 230                        28060
01860      C THIS CASE FOR DX BETWEEN 3.7 AND 15.          28070
01870      LIMLO=1                                           28080
01880      LIMHI=10                                          28090
01890      GO TO 249                                         28100
01900      C THIS CASE FOR DX GREATER THAN 15,             28110
01910      230 LIMLO=11                                      28120
01920      LIMHI=14                                          28130
01930      249 DO 250 K=LIMLO,LIMHI                          28140
01940      DA=DZETA*X(K)                                     28150
01950      TERMA = W(K)/DA                                   28160
01960      AI = AI + TERMA                                    28170
01970      AIP=AIP+TERMA*X(K)/DA                            28180
01980      IF(NEEDBI) GO TO 250                              28190
01990      DB=DZETA-X(K)                                     28200
02000      TERMB = W(K)/DB                                   28210
02010      BI = BI + TERMB                                    28220
02020      BIP=BIP+TERMB*X(K)/DB                            28230
02030      250 CONTINUE                                     28240
02040      C FORM FUNCTIONS                                  28250
02050      FACTOR=RTP1*DZETA/ROOT4X                          28260
02060      RATIO = 0.2500/DX                                 28270
02070      AI=AI*EFAC*FACTOR                                 28280
02080      AIP=- (DROOTX+RATIO)*AI+RTP1*ROOT4X*EFAC*AIP    28290
02090      C THIS IS SATISFIED ONLY FOR X BETWEEN 3.7 AND 8.0 IN THESE CASES 28300
02100      C THE BI AND BIP ABOUT TO BE COMPUTED ARE NOT SUFFICIENTLY ACCURATE, 28310
02110      C THUS RETURN TO POWER SERIES FOR BI AND BIP.  28320
02120      IF(NEEDBI) GO TO 10                               28330
02130      FACTOR=FACTOR+FACTOR                              28340
02140      BI=BI*FACTOR/EFAC                                 28350
02150      BIP=(DROOTX-RATIO)*BI-RTP12*ROOT4X*BIP/EFAC    28360
02160      RETURN                                           28370
02170      END                                              28380

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00100      SUBROUTINE SINCOS(Z, AI, AIP, BI, BIP)
00200      IMPLICIT DOUBLE PRECISION (A-H, O-Z)
00300      PI=3.1415926359000
00400      AR=2.*(-Z)**(1.5)/3.
00500      ARG=AR+PI/4
00600      Z1=(-Z)**(-0.25)
00700      Z2=1./Z1
00800      PI2=1./DSQRT(PI)
00900      AI=PI2*Z1*(DSIN(ARG)-5./72./AR*DCOS(ARG))
01000      BI=PI2*Z1*(DCOS(ARG)+5./72./AR*DSIN(ARG))
01100      AIP=-PI2*Z2*(DCOS(ARG)+7./72./AR*DSIN(ARG))
01200      BIP=PI2*Z2*(DSIN(ARG)-7./72./AR*DCOS(ARG))
01300      RETURN
01400      END
01500

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00010      SUBROUTINE EXTREM(X,A,AP,B,BP,ZETA)
00020      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00030      DIMENSION XI(4),W(4)
00032      IF (X.GT.15.00) GO TO 100
00034      CALL DAIRY(X,A,AP,B,BP)
00036      ZETA=1.00
00038      RETURN
00039      100  CONTINUE
00040      XI(1)=3.919832955445509100
00050      XI(2)=1.691561900482350400
00060      XI(3)=5.02755324672630180-01
00070      XI(4)=1.92470605620156920-02
00080      W(1)=4.7763903057572630-05
00090      W(2)=4.99143064329109590-03
00100      W(3)=8.61698469938403120-02
00110      W(4)=9.08790958459811020-01
00120      DROOTX=DSQRT(X)
00130      ZETA=0.66666666666666700*X*DROOTX
00140      ROOT4X=DSQRT(DROOTX)
00150      A=0.000
00160      AP=0.000
00170      B=0.000
00180      BP=0.000
00190      DO 1 K=1,4
00200      DA=ZETA*XI(K)
00210      TERMA=W(K)/DA
00220      A=A+TERMA
00230      AP=AP+TERMA*XI(K)/DA
00240      OB=ZETA*XI(K)
00250      TERMB=W(K)/OB
00260      B=B+TERMB
00270      BP=BP+TERMB*XI(K)/OB
00280      1  CONTINUE
00290      RTP1=0.282094791773878100
00300      RTP12=0.564189583547756200
00310      FAC=RTP1*ZETA/ROOT4X
00320      RAT=0.25000/X
00330      A=A+FAC
00340      AP=-((DROOTX+RAT)*A+RTP1*ROOT4X*AP
00350      FAC=FAC+FAC
00360      B=B+FAC
00370      BP=(DROOTX-RAT)*B-RTP12*ROOT4X*BP
00380      RETURN
00390      END

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00010 SUBROUTINE PROP(AL,BE,X,XJ,U,UJ,UP,UPJ,XNEW,XINCR,TT)
00020 IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00030 COMMON /IFLAGS/ MFLAG,MFLAG1,NTYPE
00040 COMMON /SQNORM/ XNORM
00050 DM=11.0D0
00070 DX=-AL*(X+BE)
00080 DX1=-AL*(XJ+BE)
00090 CALL DAIRY(DX,AI,AIP,BI,BIP)
00100 PI=3.141592653589783238400
00110 AJ=PI*((BIP*U)-(TT*BI*UP))
00120 BJ=PI*((TT*AI*UP)-(AIP*U))
00125 TT=1.0D0
00130 IF(MFLAG,NE,1) GO TO 5
00140 B1=(AJ**2)*(((DX/AL)*(AI**2))-((AIP**2)/AL))
00150 B2=(BJ**2)*(((DX/AL)*(BI**2))-((BIP**2)/AL))
00160 B3=(2.0D0*AJ*BJ)*(((DX/AL)*(AI*BI))-((AIP*BIP)/AL))
00170 BEG=-B1-B2-B3
00180 5 CONTINUE
00190 IF(MFLAG1,EQ,1) GO TO 4
00200 IF((DX,LE,DM),AND,(DX1,LE,DM)) GO TO 3
00210 4 CONTINUE
00220 1 XNEW=XNEW+XINCR
00230 IF(XNEW,GT,XJ) GO TO 2
00240 DX=-AL*(XNEW+BE)
00250 IF(DX,LE,DM,OR,IA,EQ,0) GO TO 6
00255 IF(XNEW-XINCR,LE,X) GO TO 6
00260 XNEW=XNEW-XINCR
00270 IF(MFLAG,NE,1) GO TO 7
00280 DX=-AL*(XNEW+BE)
00290 C1=(AJ**2)*(((DX/AL)*(AI**2))-((AIP**2)/AL))
00300 C2=(BJ**2)*(((DX/AL)*(BI**2))-((BIP**2)/AL))
00310 C3=(2.0D0*AJ*BJ)*(((DX/AL)*(AI*BI))-((AIP*BIP)/AL))
00320 CEN=C1+C2+C3
00330 ZNORM=-CEN-BEG
00340 XNORM=XNORM+ZNORM
00350 7 CONTINUE
00370 CALL LARGE(AL,BE,XNEW,XJ,WF,UJ,WP,UPJ,XNEW,XINCR,TT)
00380 RETURN
00390 6 CONTINUE
00400 CALL DAIRY(DX,AI,AIP,BI,BIP)
00410 IA=IA+1
00420 WF=AJ*AI+BJ*BI
00430 WP=AJ*AIP+BJ*BIP
00440 IF(MFLAG1,EQ,1) WRITE(NTYPE,11) WF
00450 11 FORMAT(2D20,8)
00460 GO TO 1
00470 2 XNEW=XNEW-XINCR
00480 8 DX=-AL*(XJ+BE)
00482 120 CONTINUE
00484 IF(XNEW+XINCR,GT,XJ) GO TO 150
00486 XNEW = XNEW+XINCR
12 00486 GO TO 120
11 00488 150 XNEW=XNEW-XINCR
10 00490 CALL DAIRY(DX,AI,AIP,BI,BIP)
9 00500 UJ=AJ*AI+BJ*BI
8 00510 UPJ=AJ*AIP+BJ*BIP
7 00520 IF(MFLAG,NE,1) RETURN
6 00530 C1=(AJ**2)*(((DX/AL)*(AI**2))-((AIP**2)/AL))
5 00540 C2=(BJ**2)*(((DX/AL)*(BI**2))-((BIP**2)/AL))
4 00550 C3=(2.0D0*AJ*BJ)*(((DX/AL)*(AI*BI))-((AIP*BIP)/AL))
3

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00560 CEN=C1+C2+C3
00570 100 FORMAT(7D14,3)
00580 ZNORM=-CEN-BEG
00590 XNORM=XNORM+ZNORM
00600 RETURN
00610 END

```

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00010 SUBROUTINE PR(AL,BE,X,XJ,U,UJ,UP,UPJ,XNEW,XINCR,TT,
00020 1 X1SQ,X2SQ)
00030 IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00040 COMMON /IFLAGS/ MFLAG,MFLAG1,NTYPE
00045 COMMON /JC/ JCOUNT
00050 X1SQ=0.000
00060 X2SQ=0.000
00070 DM=11.000
00080 IA=0
00090 DX=-AL*(X+BE)
00100 DX1=-AL*(XJ+BE)
00110 CALL DAIRY(DX,AI,AIP,BI,BIP)
00120 PI=3.141592653589783238400
00130 AJ=PI*((BIP*U)-(TT*BI*UP))
00140 BJ=PI*((TT*AI*UP)-(AIP*U))
00150 TT=1.000
00160 IF(MFLAG,NE,1) GO TO 5
00170 B1=(AJ**2)*(((DX/AL)*(A)**2))-((AIP**2)/AL)
00180 B2=(BJ**2)*(((DX/AL)*(B)**2))-((BIP**2)/AL)
00190 B3=(2.000*AJ*BJ)*(((DX/AL)*(A*B)))-((AIP*BIP)/AL)
00200 BEG=-B1-B2-B3
00210 5 CONTINUE
00220 IF(MFLAG1,EQ,1) GO TO 4
00230 IF((DX.LE,DM),AND,(DX1.LE,DM)) GO TO 3
00240 4 CONTINUE
00250 1 XNEW=XNEW-XINCR
00260 IF(XNEW,LT,XJ) GO TO 2
00270 DX=-AL*(XNEW+BE)
00280 IF(DX,LE,DM) GO TO 6
00290 XNEW=XNEW+XINCR
00300 IF(MFLAG,NE,1) GO TO 7
00310 DX=-AL*(XNEW+BE)
00320 C1=(AJ**2)*(((DX/AL)*(A)**2))-((AIP**2)/AL)
00330 C2=(BJ**2)*(((DX/AL)*(B)**2))-((BIP**2)/AL)
00340 C3=(2.000*AJ*BJ)*(((DX/AL)*(A*B)))-((AIP*BIP)/AL)
00350 CEN=C1+C2+C3
00360 X1SQ=CEN*BEG
00370 7 CONTINUE
00390 375 FORMAT(' PR',5D15,3)
00400 CALL LG(AL,BE,XNEW,XJ,WF,UJ,WP,UPJ,XNEW,XINCR,TT,
00410 1 X2SQ,Y2SQ)
00420 RETURN
00430 6 CONTINUE
00440 CALL DAIRY(DX,AI,AIP,BI,BIP)
00450 WF=AJ*AI+BJ*BI
00460 WP=AJ*AIP+BJ*BIP
00470 IF(MFLAG1,EQ,1) WRITE(NTYPE,11) WF
00480 11 FORMAT(10D0,8)
00490 GO TO 1
00500 2 XNEW=XNEW+XINCR
00510 3 DX=-AL*(XJ+BE)
12 00520 CALL DAIRY(DX,AI,AIP,BI,BIP)
11 00530 UJ=AJ*AI+BJ*BI
10 00540 UPJ=AJ*AIP+BJ*BIP
9 00550 IF(MFLAG,NE,1) RETURN
8 00560 C1=(AJ**2)*(((DX/AL)*(A)**2))-((AIP**2)/AL)
7 00570 C2=(BJ**2)*(((DX/AL)*(B)**2))-((BIP**2)/AL)
6 00580 C3=(2.000*AJ*BJ)*(((DX/AL)*(A*B)))-((AIP*BIP)/AL)
5 00590 CEN=C1+C2+C3
4 00600 X1SQ=CEN*BEG
3

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00610 RETURN  
00620 END

```

00010 SUBROUTINE START(AL,BE,X,XJ,U,UP,XNEW,XINCR,SW,TT)
00020 IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00030 COMMON /IFLAGS/ MFLAG,MFLAG1,NITYPE
00040 COMMON /SQNORM/ XNORM
00050 DX=-AL*(X+BE)
00060 DX1=-AL*(XJ+BE)
00070 IF((DX.GT.15.000).OR.(DX1.GT.15.000)) GO TO 4
00080 CALL DAIRY(DX,AI,AIP,BI,BIP)
00090 A=SW*BI*TT
00100 B=-AI*SW*TT
00110 AJ=A
00120 BJ=B
00130 IF(MFLAG,NE,1) GO TO 5
00140 B1=(AJ**2)*(((DX/AL)*(AI**2))-((AIP**2)/AL))
00150 B2=(BJ**2)*(((DX/AL)*(BI**2))-((BIP**2)/AL))
00160 B3=(2.000*AJ*BJ)*(((DX/AL)*(AI*BI))-((AIP*BIP)/AL))
00170 BEG=-B1-B2-B3
00180 5 CONTINUE
00190 DX=-AL*(XJ+BE)
00200 CALL DAIRY(DX,AI,AIP,BI,BIP)
00210 U=A*AI+B*BI
00220 UP=A*AIP+B*BIP
00230 IF(MFLAG,NE,1) RETURN
00240 ZA=AJ**2
00250 ZB=(DX/AL)
00260 ZC=AI**2
00270 ZD=(AIP**2)/AL
00280 C1=ZA*((ZB*ZC)-ZD)
00290 C1=(AJ**2)*(((DX/AL)*(AI**2))-((AIP**2)/AL))
00300 C2=(BJ**2)*(((DX/AL)*(BI**2))-((BIP**2)/AL))
00310 C3=(2.000*AJ*BJ)*(((DX/AL)*(AI*BI))-((AIP*BIP)/AL))
00320 CEN=C1+C2+C3
00330 ZNORM=-CEN-BEG
00340 XNORM=XNORM+ZNORM
00350 1 XNEW=XNEW+XINCR
00360 IF(XNEW.GT.XJ) GO TO 3
00370 DX=-AL*(XNEW+BE)
00380 CALL DAIRY(DX,AI,AIP,BI,BIP)
00390 WF=A*AI+B*BI
00400 IF((MFLAG1.EQ.1).AND.(XNEW.GT.0.000)) WRITE(NITYPE,11)
00410 1 WF
00420 WP=A*AIP+B*BIP
00430 11 FORMAT(2D20.8)
00440 GO TO 1
00450 3 XNEW=XNEW-XINCR
00460 RETURN
00470 4 CONTINUE
00480 C DX=-AL*(X+BE)
00490 G CALL EXTREM(DX,AZ,AZP,BZ,BZP,Z)
00500 XNEW=X
00510 T=1.000
12 00520 C A=BZ*SW
11 00530 C B=-AZ*SW
10 00540 C U1=A*AZ+B*BZ
9 00550 C UP1=A*AZP+B*BZP
8 00555 U1=0
7 00556 UP1=-TT/3,141592653589783200
6 00555 C TYPE 11,X,DX,U1,UP1
6 00580 CALL LARGE(AL,BE,X,XJ,U1,U,UP1,UP,XNEW,XINCR,T)
4 00590 RETURN
3

```

00600 END

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00010 SUBROUTINE LARGE(AL,BE,X,XJ,U,UJ,UP,UPJ,XNEW,XINCR,TT)
00020 IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00030 COMMON /IFLAGS/ MFLAG,MFLAG1,NTYPE
00040 COMMON /SQNORM/ XNORM
00050 DM=11.000
00060 UINIT=U
00070 SUMO=0.000
00080 SUME=0.000
00090 IA=0
00100 PI=3.1415926535889783238400
00110 DX=-AL*(X+BE)
00120 CALL EXTREM(DX,AZ,AZP,BZ,BZP,Z)
00130 1 XNEW=XNEW+XINCR
00135 IF(XNEW,LT,X) GO TO 1
00140 IF(XNEW,GT,XJ) GO TO 2
00150 DX=-AL*(XNEW+BE)
00160 IF(DX,GT,DM) GO TO 4
00170 XNEW=XNEW-XINCR
00180 IF(MFLAG,NE,1) GO TO 5
00190 IF(IDF,EQ,0) SUME=SUME-(4.000*U1*U1)
00200 IF(IDF,NE,0) SUMO=SUMO-(2.000*U1*U1)
00210 SEND=(UINIT**2)+(U1**2)
00220 ZNORM=(XINCR/3.000)*(SUME+SUMO+SEND)
00230 XNORM=XNORM+ZNORM
00240 5 CONTINUE
00250 CALL PROP(AL,BE,XNEW,XJ,U,UJ,UP,UPJ,XNEW,XINCR,TT)
00260 RETURN
00270 4 CONTINUE
00280 CALL EXTREM(DX,AZ2,AZP2,BZ2,BZP2,Z2)
00290 DEX=DEXP(Z-Z2)
00300 C11=PI*((AZ2*BZP+DEX)-((BZ2*AZP)/DEX))
00310 C12=(PI*TT)*(((BZ2*AZ)/DEX)-(AZ2*BZ+DEX))
00320 C21=(PI/TT)*((AZP2*BZP+DEX)-((BZP2*AZP)/DEX))
00330 C22=PI*((BZP2*AZ)/DEX)-(AZP2*BZ+DEX)
00340 U1=C11*U+C12*UP
00350 IF(MFLAG1,EQ,1) WRITE(NTYPE,11) U1
00360 11 FORMAT(2D20,8)
00370 IF(MFLAG,NE,1) GO TO 3
00380 IA=IA+1
00390 IH=IA/2
00400 IDF=(IA-2*IH)
00410 IF(IDF,EQ,0) SUME=SUME+(4.000*U1*U1)
00420 IF(IDF,NE,0) SUMO=SUMO+(2.000*U1*U1)
00430 3 CONTINUE
00440 U1P=C21*U+C22*UP
00450 U1P=U1P*TT
00460 TT=1.000
00470 DX=-AL*(XNEW+BE)
00480 AZ=AZ2
00490 AZP=AZP2
00500 BZ=BZ2
12 00510 BZP=BZP2
11 00520 Z=Z2
10 00530 U=U1
9 00540 UP=U1P
8 00550 100 FORMAT(4D15,6)
7 00560 GO TO 1
6 00570 2 CONTINUE
5 00580 XNEW=XNEW-XINCR
3 00590 DX=-AL*(XJ+BE)

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00600 CALL EXTREM(OX,AZ2,AZP2,BZ2,BZP2,Z2)
00610 DEX=DEXP(Z-Z2)
00620 C11=PI*((AZ2*BZP*DEX)-((BZ2*AZP)/DEX))
00630 C12=(PI*TT)*(((BZ2*AZ)/DEX)-(AZ2*BZ*DEX))
00640 C21=(PI/TT)*((AZP2*BZP*DEX)-((BZP2*AZP)/DEX))
00650 C22=PI*((BZP2*AZ)/DEX)-(AZP2*BZ*DEX)
00660 UJ=C11*U+C12*UP
00670 UPJ=C21*U+C22*UP
00680 UPJ=UPJ*TT
00690 IF(MFLAG,NE,1) RETURN
00700 SEND=(UINIT**2)+(UJ**2)
00710 ZNORM=(XINCR/3.000)*(SUME+SUMO+SEND)
00720 XNORM=XNORM+ZNORM
00730 RETURN
00740 END

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00010 SUBROUTINE LG(AL,BE,X,XJ,U,UJ,UP,UPJ,XNEW,XINCR,TT,
00020 1 X1SQ,X2SQ)
00030 IMPLICIT DOUBLE PRECISION (A-H,O-Z)
00040 COMMON /IFLAGS/ MFLAG,MFLAG1,NTYPE
00050 COMMON /JC/ JCOUNT
00051 DIMENSION W(250),WP(250)
00060 DM=11.000
00080 X1SQ=0.000
00090 X2SQ=0.000
00110 IA=0
00130 UINIT=U
00140 SUM0=0.000
00150 SUME=0.000
00160 PI=3.1415926535889783238400
00170 DX=-AL*(X+BE)
00180 CALL EXTREM(DX,AZ,AZP,BZ,BZP,Z)
00190 1 XNEW=XNEW-XINCR
00200 FCT=1.000
00210 IF(XNEW,LT,XJ) GO TO 2
00220 DX=-AL*(XNEW+BE)
00230 IF(DX,GT,DM,OR,IA,LT,1) GO TO 4
00240 XNEW=XNEW+XINCR
00250 IF(MFLAG,NE,1) GO TO 5
00260 IM=IA-1
00265 IF(IM,LT,1) GO TO 20
00270 DO 14 K=1,IM,2
00280 14 IF(DABS(W(K)),GT,1,D-18) SUM0=SUM0+(2.000*W(K)*W(K))
00285 IF(IM,LT,2) GO TO 20
00290 DO 15 K=2,IM,2
00300 15 IF(DABS(W(K)),GT,1,D-18) SUME=SUME+(4.000*W(K)*W(K))
00305 20 CONTINUE
00310 SEND=(UINIT**2)+W(IA)**2
00320 X1SQ=(XINCR/3.000)*(SUME+SUM0+SEND)
00330 100 FORMAT(5D14,3)
00340 5 CONTINUE
00350 CALL PR(AL,BE,XNEW,XJ,U,UJ,UP,UPJ,XNEW,XINCR,TT,
00360 1 X2SQ,Y2SQ)
00370 RETURN
00380 4 CONTINUE
00390 CALL EXTREM(DX,AZ2,AZP2,BZ2,BZP2,Z2)
00400 DEX=DEXP(Z-Z2)
00410 C11=PI*((AZ2*BZP*DEX)-((BZ2*AZP)/DEX))
00420 C12=(PI*TT)*(((BZ2*AZ)/DEX)-(AZ2*BZ*DEX))
00430 C21=(PI/TT)*((AZP2*BZP*DEX)-((BZP2*AZP)/DEX))
00440 C22=PI*((BZP2*AZ)/DEX)-(AZP2*BZ*DEX)
00450 U1=C11*U+C12*UP
00460 11 FORMAT(1D20,8)
00470 U1P=(C21*U+C22*UP)*TT
00480 TT=1.000
00500 IA=IA+1
00510 502 FORMAT(I3)
12 00520 W(IA)=U1
11 00530 WP(IA)=U1P
10 00610 U=W(IA)
9 00620 UP=WP(IA)
8 00628 30 FORMAT(I3,0)
7 00630 IF(MFLAG1,EQ,1) WRITE(NTYPE,11) W(IA)
6 00640 DX=-AL*(XNEW+BE)
5 00650 AZ=AZ2
4 00660 AZP=AZP2
3

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00670      BZ=BZ2
00680      BZP=BZP2
00690      Z=Z2
00700      GO TO 1
00710      2      CONTINUE
00720      IF(MFLAG,NE,1) GO TO 3
00725      IF(IA,LT,1) GO TO 3
00730      DO 16 K=1,IA,2
00740      16      IF(DABS(W(K)),GT,1,D-18) SUMO=SUMO+2.0D0*W(K)**2
00745      IF(IA,LT,2) GO TO 3
00750      DO 17 K=2,IA,2
00760      17      IF(DABS(W(K)),GT,1,D-18) SUME=SUME+4.0D0*W(K)**2
00770      3      CONTINUE
00780      XNEW=XNEW+XINCR
00790      DX=-AL*(XJ+BE)
00795      IF(DX,LT,5) TYPE 28,DX
00798      28      FORMAT('  WARNING: IN LG ARG OF EXTREM IS ',D)
00800      CALL EXTREM(DX,AZ2,AZP2,BZ2,BZP2,Z2)
00810      DEX=DEXP(Z-Z2)
00820      C11=PI*((AZ2*BZP*DEX)-((BZ2*AZP)/DEX))
00830      C12=(PI*TT)*(((BZ2*AZ)/DEX)-(AZ2*BZ*DEX))
00840      C21=(PI/TT)*((AZP2*BZP*DEX)-((BZP2*AZP)/DEX))
00850      C22=PI*((BZP2*AZ)/DEX)-(AZP2*BZ*DEX)
00860      UJ=C11*U+C12*UP
00870      UPJ=(C21*U+C22*UP)*TT
00880      IF(MFLAG,NE,1) RETURN
00930      SEND=(UINIT**2)*(UJ**2)
00940      X1SQ=(XINCR/3.0D0)*(SUME+SUMO+SEND)
00950      RETURN
00960      END
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