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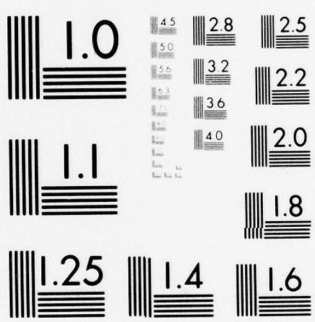

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OPTIMAL CONTROL FOR ANTI-TANK WEAPON.

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and Computer Science

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A severe sensitivity problem existed with the previous optimal control law implementation. It was decided that the control problem had to be reformulated with hard constraints on the controller. A review of the pertinent control literature is given in Chapter III. Our control problem is formulated in Chapter IV as a minimum time, minimum fuel, and minimum energy problem. The solution as a minimum time problem is discussed. The computer implementation in a simple four state simulation is given in Chapter V with conclusions and recommendations for future study given in Chapter VI.

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September 30, 1978

FINAL REPORT

for

Western Kentucky University Contract # DAAK 40-78-M-0102
(Redstone Arsenal Contract # PAN TG-37)

OPTIMAL CONTROL FOR AN ANTI-TANK WEAPON

to

Dr. Harold Pastrick, Contract Supervisor
Guidance and Control Directorate
Research and Development Laboratory
U. S. Army Missile Command
Redstone Arsenal, Alabama 35809

from

Dr. Randy J. York
Department of Mathematics and Computer Science
Western Kentucky University
Bowling Green, Kentucky 42101

Peter Sisler, research assistant

ABSTRACT

A three state optimal control law had been implemented in the missile simulation under the last contract. Before a Kalman filter could be designed and implemented on the 6DOF simulation to reduce the effect on performance of random disturbances, a serious problem of performance sensitivity had to be overcome. To this end, it was decided that the control problem had to be reformulated with hard constraints on the controller. The previous control problem only assumed a 'soft constraint' in the functional J,

$$J = \{\text{other terms at } t_f\} + \beta\int_0^{t_f} u^2(t)dt ,$$

to be minimized.

To solve this control problem, a review of the pertinent control literature was conducted which is given in Chapter III. Our control problem is formulated in Chapter IV as a minimum time, minimum fuel, and minimum energy problem. The solution as a minimum time problem is discussed. The computer implementation in a simple four state simulation is given in Chapter V with conclusions and recommendations for future study given in Chapter VI.

ACKNOWLEDGEMENTS

Dr. Harold Pastrick, contract supervisor, of the U.S. Army Missile Command, has been most helpful with his suggestions and demands. We would also like to acknowledge the assistance of Dr. Thomas Madron and his Research and Computer Services group at Western who have patiently answered our many questions and requests. This contract, with its security requirement, has posed special problems, and we would like to thank Dr. Glenn Crumb and his assistants at the Office of Grants and Contracts for their aid.

This report could not have been prepared without the help of Debra Wheeler, who not only was most careful in the typing of this manuscript, but who was able to spot and correct oversights on our part.

Someone has to be last, but we hope that no affront is taken. We would like to especially thank our wives for their patience and understanding they showed during this contractual period and the preparation of this final report.

CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

Chapter I

INTRODUCTION

Our contractual task was to investigate and reduce the effect of random disturbances on missile performance. A guidance filter was to be designed to produce smoothed estimates of various states which in turn are used to control the missile. Primarily, the line-of-sight angle λ and angular rate $\dot{\lambda}$ along with the attitude angle θ and angular rate output $\dot{\theta}$ are needed in both pitch and yaw channels. Random disturbances which are inherent in measuring the states are to be filtered out to produce the desired smoothed estimate.

Although the three-state control law had been implemented and did produce an acceptable miss distance, the implementation was of little practical value. The states used (Y_d and $\dot{Y}_d$) must be replaced with others (λ and $\dot{\lambda}$). Even if the implementation were altered to use these states along the lines suggested in the contract report [6], a more serious problem needed to be overcome. Small changes in initial conditions or in control parameter values produced unacceptable changes in miss distance and attitude angle at impact. The control law was extremely sensitive to such changes and behaved in a highly discontinuous manner.

Before work could begin on the design of the Kalmer filter, it was essential to resolve the sensitivity problem. To this end, two methods seemed to offer some promise. The first was to try to fly the missile with a four-state controller which was based on previous work done by Pastrick and York [9]. Basically, the four-state controller assumes a first order lag model for the autopilot; whereas the three-state controller assumes instantaneous autopilot response. Chapter II discusses this work.

The second method used to overcome the sensitivity problem was to investigate reformulating the control problem with hard constraints on the controller of the form:

$$|u(t)| \leq M \quad . \tag{1.1}$$

The previous work only assumed a soft constraint with a weighted integral of the square of the control u in the functional J to be minimized, i.e.

$$J = \{\text{other terms}\} + \beta\int_0^{t_f} u^2(t)\, dt$$

To accomplish this goal of reformulating the control problem as one with hard constraints, a review of the control literature was undertaken. Chapter III describes the major types of control problems: minimum time, minimum fuel, and minimum energy. A simple two-state system is used to compare and contrast the three different types of control laws.

Chapter IV gives three formulations of our missile control problem: one for each of the three categories. Its solution as a minimum time problem is given. Chapter V discusses the computer implementation of the resulting control law.

Chapter II

THE SENSITIVITY PROBLEM

2.1 Performance Sensitivity

The three-state control law had been implemented in the 6DOF missile simulation under the assumption of perfect knowledge of the states—Y_d, $\dot{Y}_d$, and θ (see Figure 4.1 and Table 4.1 for a detailed description of these states). In the actual system, however, the line-of-sight angle λ and angular rate are available, not Y_d and $\dot{Y}_d$. This meant that the controller had to be reformulated in these variables along with θ. One method for doing this was outlined in the contract final report.

A far more serious problem existed with the control implementation than the above, and that was a problem of performance sensitivity. Small changes in initial conditions or in the control parameters would produce unacceptable changes in miss distance, as shown in Table 2.1.

Table 2.1. 6DOF Simulation Runs for Various Initial Conditions

$Y_d(t_o)$	$H_t(t_o)$	$\theta(t_o)$	$V_M(t_o)$	RXE	RZE	$\theta(t_f)$
-5000.	-1500.	-5.	1090.856	.3	-10.4	-38.1
-4500.	-1500.	-5.	1090.856	1.6	-68.9	-44.
-5500.	-1500.	-5.	1090.856	-1065.3	1.2	-77.5
-5000.	-1400.	-5.	1090.856	-690.7	.5	-47.2
-5000.	-1600.	-5.	1090.856	-116.7	1.5	-89.7
-5000.	-1500.	-4.	1090.856	.7	-150.4	-32.4
-5000.	-1500.	-6.	1090.856	-495.1	.9	-41.8
-5000.	-1500.	-5.	1085.856	.9	-127.1	-35.5
-5000.	-1500.	-5.	1095.856	-448.7	1.2	-43.4

RXE and RZE are the coordinates of the point on the trajectory where the missile is closest to the target.

2.2 Four-State Controller

To overcome the sensitivity problem, it was decided to investigate using a four-state controller previously developed by Pastrick and York [9]. The pertinent details of this work appear as Appendix A in the contract report [16]. Basically, our mathematical model consists of four states Y_d, $\dot{Y}_d$, θ, and A_L to describe the missile - target geometry in a plane. The differential equations describing the dynamics are

$$\dot{Y}_d = \dot{Y}_d \tag{2.1}$$

$$\ddot{Y}_d = -A_L \cos\theta \tag{2.2}$$

$$\dot{A}_L = -w_1 A_L + K_1 u \tag{2.3}$$

$$\dot{\theta} = K_a u \ . \tag{2.4}$$

The contract work made the assumption that the lag in the autopilot could be ignored and employed only a 'soft constraint' on the controller u in the form of an integral of u^2 in the cost functional to be minimized. By ignoring the autopilot lag, the system could then be described by just three-state variables. It is difficult to say with much precision which assumption causes the sensitively problem, but it is probably the latter.

In the hope, however, of reducing performance sensitivity, it was decided to replace the three-state controller

$$u = c_1(t)Y_d + c_2(t)\dot{Y}_d + c_3(t)\theta \tag{2.5}$$

with the four-state controller

$$u = d_1(t)Y_d + d_2(t)\dot{Y}_d + d_3(t)\theta + d_4(t)A_L \ . \tag{2.6}$$

This four-state controller is considerably more complex than the three-state one, but it gives better performance over a wider range of autopilot lag values. See [9], [16] for comparisons in performance using the two controllers.

2.3 Computer Simulation Results

The four state simulation with the four state controller has not been made to work. At first, there was a coding error, but this has since been found and corrected. The main problem has been a lack of access to the computer facility at the University of Kentucky. During the first part of this contract, the telephone lines used for transmission were out of order. With the telephone maintenance men on strike, this situation was not remedied for over a month. By that time, the decision had already been made to pursue the reformulation of the control problem. A successful four state implementation would have been preferred, but the author does feel that the final result would have been negative, that is, the sensitivity problem would be present with the four state controller as it was with the three state one. The extreme sensitivity shown in Figure 2.1 makes this conclusion almost a certainty.

Chapter III

OPTIMAL CONTROL THEORY REVIEW FOR BOUNDED CONTROLLERS

3.1 The Unconstrained Control Problem

The unconstrained optimal control problem is to find an admissible control u* that causes the system

$$\dot{\bar{x}}(t) = \bar{a}(\bar{x}(t),\bar{u}(t),t) \tag{3.1}$$

to follow an admissible trajectory $\bar{x}^*$ that minimizes the performance measure

$$J(\bar{u}) = h(\bar{x}(t_f),t_f) + \int_{t_o}^{t_f} g(\bar{x}(t),\bar{u}(t),t)dt, \tag{3.2}$$

where $\bar{x}$ is the n × 1 state vector and $\bar{u}$ is the m × 1 vector of control inputs. The problem assumes no bounds on either the state or control regions.

The solution to the problem is given by first defining the Hamiltonian (H),

$$H(\bar{x}(t),\bar{u}(t),\bar{p}(t),t) \equiv g(\bar{x}(t),\bar{u}(t),t) + \bar{p}^T(t)[\bar{a}(\bar{x}(t),\bar{u}(t),t)]. \tag{3.3}$$

The n × 1 vector $\bar{p}(t)$ is the vector consisting of the set of Lagrange multipliers. The following system of differential equations are then solved for $\bar{x}$, $\bar{p}$, and $\bar{u}$:

$$\dot{\bar{x}}(t) = \frac{\partial H}{\partial \bar{p}}(\bar{x}(t),\bar{u}(t),\bar{p}(t),t) \tag{3.4}$$

$$\dot{\bar{p}}(t) = -\frac{\partial H}{\partial \bar{x}}(\bar{x}(t),\bar{u}(t),\bar{p}(t),t) \tag{3.5}$$

$$\bar{0} = \frac{\partial H}{\partial \bar{u}}(\bar{x}(t),\bar{u}(t),\bar{p}(t),t) \tag{3.6}$$

subject to the two point boundary conditions

$$\bar{x}(t_o) = \bar{x}_o \ , \ \bar{x}(t_f) = \bar{x}_f \text{ (if given)} \tag{3.7}$$

and

$$\left[\frac{\partial h}{\partial \bar{x}}(\bar{x}(t_f),t_f) - \bar{p}(t_f)\right]^T \delta\bar{x}_f + \left[H(\bar{x}(t_f),\bar{u}(t_f),\bar{p}(t_f),t_f) + \frac{\partial h}{\partial t}(\bar{x}(t_f),t_f)\right]\delta t_f = 0. \tag{3.8}$$

Equations (3.4), (3.5), and (3.8) are referred to as the state equations, costate equations, and the transversality condition, respectively. Equations (3.4), (3.5), and (3.6) are necessary (not sufficient) conditions for an optimal controller.

When constraints are placed on the controller, then the controller u* where Equation (3.6) is satisfied may very well lie outside the admissible region; in which case, the partial derivative is undefined. Another approach is needed, and this is provided by Pontryagin's minimum principle.

3.2 The Constrained Optimal Control Problem

When a constraint on the control region is imposed, for example,

$$\alpha_i \leq |u_i(t)| \leq \beta_i \ , \ i = 1,\ldots m \tag{3.9}$$

then Pontryagin's minimum principle must be used to solve the control problem.

Pontryagin's minimum principle: A necessary condition for u* to minimize the functional J is that

$$H(\bar{x}^*(t),\bar{u}^*(t),\bar{p}^*(t),t) \leq H(\bar{x}^*(t),\bar{u}(t),\bar{p}^*(t),t), \tag{3.10}$$

for all $t \in [t_0,t_f]$ and for all admissible controls $\bar{u}$. In essence, Pontryagin's minimum principle states that an optimal control must minimize the Hamiltonian H globally, not locally.

The solution to the constrained control problem then is to solve Equations (3.4), (3.5), and (3.10) instead of (3.6). The boundary conditions remain unchanged.

3.3 Three Major Types of Control Problems

For the solution of our missile control problem, we will be interested primarily in three variations of the functional J: the minimum time, minimum fuel, and minimum energy problems. The solutions of these three problems will be stated for a linear, time-varying system. The example used to illustrate these three types, as well as the missile control problem in which we are interested, is time-invariant however.

A linear, time-varying state system is represented by

$$\dot{\bar{x}}(t) = \bar{a}(\bar{x}(t),t) + \bar{B}(\bar{x}(t),t)\bar{u}(t) \tag{3.11}$$

where $\bar{x}$ is n x 1, $\bar{a}$ is n x 1, $\bar{u}$ is m x 1, and $\bar{B}$ is n x m. We assume that the control restraints are normalized, that is,

$$|u_i(t)| \leq 1 \ , \ i = 1,\ldots m \ . \tag{3.12}$$

Throughout, the object will be to take the system to the origin, whether the final time t_f is specified or left free.

We define the three types of problems.

Definition 3.1. If the functional J is given by

$$J = \int_0^{t_f} 1 \, dt \, , \tag{3.13}$$

then the control problem is referred to as the <u>minimum time</u> problem.

Definition 3.2. If the functional J is given by

$$J = \int_0^{t_f} \left[\sum_{i=1}^{m} \beta_i |u_i(t)| \right] dt \, , \tag{3.14}$$

then the control problem is referred to as the <u>minimum fuel</u> problem.

Definition 3.3. If the functional J is given by

$$J = \int_0^{t_f} \left[\sum_{i=1}^{n} \alpha_i u_i^2(t) \right] dt \, , \tag{3.15}$$

then the control problem is referred to as the <u>minimum energy</u> problem.

The next three sections state the solution to each type of control problem and use the two-state system,

$$\dot{x}_1(t) = x_2(t) \tag{3.16}$$

$$\dot{x}_2(t) = u(t) \tag{3.17}$$

$$|u(t)| \leq 1 \, , \; t_f \text{ free} \, , \tag{3.18}$$

as an example. Note that u has only one component, which will also be the case in our missile problem.

3.3.1 The Minimum Time Problem

The form of the optimal controller u* is given for i = 1,2,...m by

$$u^*_i(t) = \begin{cases} 1 & \text{for } \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) < 0 \\ -1 & \text{for } \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) > 0 \\ \text{undetermined} & \text{for } \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) = 0 \end{cases} \tag{3.19}$$

where $\bar{b}_i(\bar{x}(t),t) \equiv i^{th}$ column of the matrix $\bar{B}$. Note that the minimum time controller is of the 'bang-bang' variety (maximum effort throughout the interval).

When the system

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t), \tag{3.20}$$

is time invariant and controllable, several theorems apply, such as:

Theorem 3.1. (Existence and Uniquences). If all of the eigenvalues of A have non-positive real parts, then a unique optimal controller exists that transfers any initial state x_o to the origin.

Theorem 3.2. (Number of Switchings). If the eigenvalues of A are all real, and a (unique) time-optimal control exists, then each control component can switch at most n - 1 times.

In our example, since the eigenvalues of A are both zero, then a unique optimal controller exists, and it can change sign at most once.

For our example, the Hamiltonian H is given by

$$H = 1 + p_1(t)x_2(t) + p_2(t)u(t) \tag{3.21}$$

and the optimal controller u by

$$u^*(t) = \begin{cases} -1 & \text{for } p_2(t) > 0 \\ +1 & \text{for } p_2(t) < 0. \end{cases} \tag{3.22}$$

The solution to the costate equation is

$$p_1(t) = \pi_1 \tag{3.23}$$

$$p_2(t) = -\pi_1 t + \pi_2. \tag{3.24}$$

As can be seen from Equation (3.24), the control u can change sign at most once. The possible optimal control sequences are {+1}, {-1}, {+1,-1}, {-1,+1}.

To find the switching curve in the state space, we solve the state equations with

$$u = \Delta = \pm 1$$

to obtain

$$x_2(t) = \Delta t + \S_2 \tag{3.25}$$

$$x_1(t) = \frac{\Delta}{2}t^2 + \S_2 t + \S_1 . \tag{3.26}$$

Eliminating the parameter t, we obtain

$$x\ (t) = \frac{\Delta}{2}x_2^2(t) + \tau \ , \tag{3.27}$$

where $\tau = \tau(\Delta, \S_1, \S_2)$ is a constant. The trajectories are given in Figure 3.1 with the solid line corresponding to the (u = +1) command, and the dashed line corresponding to the (u = -1) command.

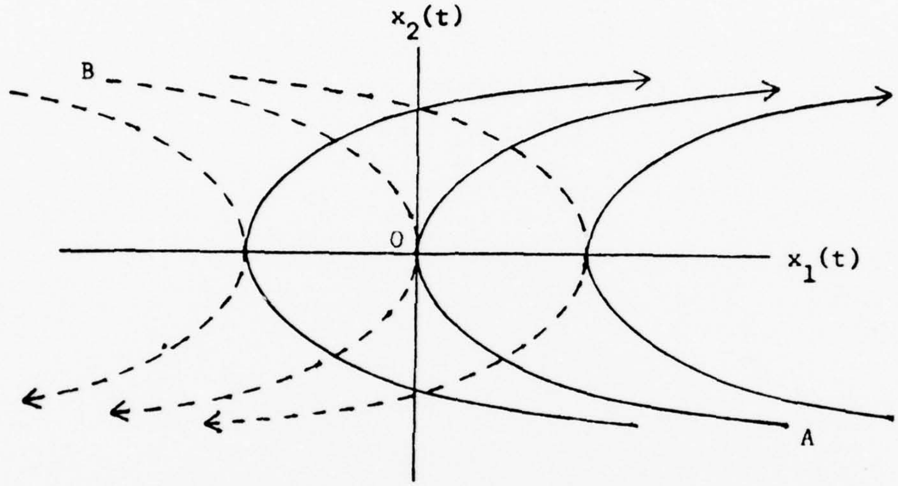

Figure 3.1. Trajectories for u = ± 1

The switching curve AOB is composed of the two portions of the parabolas passing through the origin; its equation is given by

$$x_1(t) = -\mathrm{sgn}(x_2(t))x_2^2(t)/2 \tag{3.28}$$

or

$$s(\bar{x}(t)) \equiv x_1(t) + \mathrm{sgn}(x_2(t))x_2^2(t)/2 = 0 \tag{3.29}$$

and is depicted in Figure 3.2.

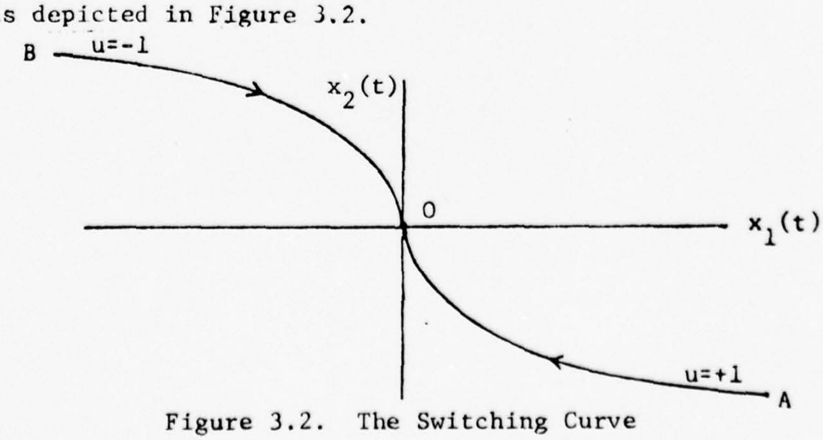

Figure 3.2. The Switching Curve

To summarize, the solution to the control problem is to apply a (u = -1) control for those initial states which lie above $(s(\bar{x}(t)) > 0)$ the switching curve. The resulting trajectory will then intersect the switching curve AOB at which time the control switches to +1. In a similar fashion, for those points below $(s(\bar{x}(t)) < 0)$ the switching curve, the reversed sequence will bring the system to the origin (see Figure 3.3).

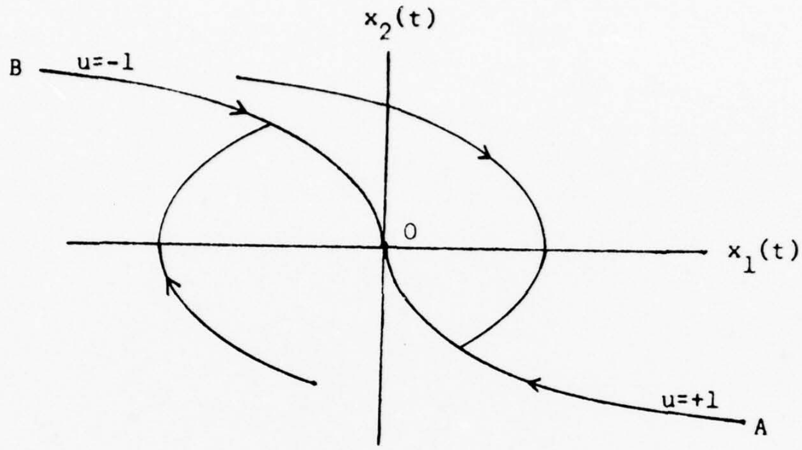

Figure 3.3 Optimal Trajectories for Various Initial States

3.3.2 The Minimum Fuel Problem

The form of the optimal controller is given for i = 1,...m by

$$u_i^*(t) = \begin{cases} 1.0 & \text{for} \quad \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) < -1.0 \\ 0.0 & \text{for} \quad -1.0 < \bar{p}^T(t)b_i(\bar{x}(t),t) < 1.0 \\ -1.0 & \text{for} \quad 1.0 < \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) \\ v & \text{if} \quad \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) = -1.0 \\ -w & \text{if} \quad \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) = 1.0 \end{cases} \tag{3.30}$$

where v and w are undetermined constants in the interval [0,1]. Note that the minimum fuel controller is of the bang-off-bang variety.

For some initial state values in our two-state example (Equations (3.16) and (3.17)), an optimal minimal fuel controller does not exist. Let us examine the situation more closely. Our controller is given by

$$u^*(t) = \begin{cases} +1.0 & \text{if} \quad p_2(t) < -1.0 \\ 0.0 & \text{if} \quad -1.0 < p_2(t) < 1.0 \\ -1.0 & \text{if} \quad p_2(t) > 1.0\ , \end{cases} \tag{3.31}$$

if we ignore the possibility of singular controls for the moment. The Hamiltonian H is given by

$$H = |u(t)| + p_1(t)x_2(t) + p_2(t)u(t) \ . \qquad (3.32)$$

Note that the state and costate equations remain unchanged. The possible control sequences are {+1}, {-1}, {0}, {+1,0}, {-1,0}, {0,+1}, {0,-1}, {1,0,-1}, {-1,0,+1}. The locus of points that can be driven to the origin with control (u = +1) or (u = -1) is the same as before (see Figure 3.2).

With control (u = 0), the state trajectories are

$$\begin{aligned} x_2(t) &= §_2 \\ x_1(t) &= §_2 t + §_1 \end{aligned} \qquad (3.33)$$

and are sketched in Figure 3.4.

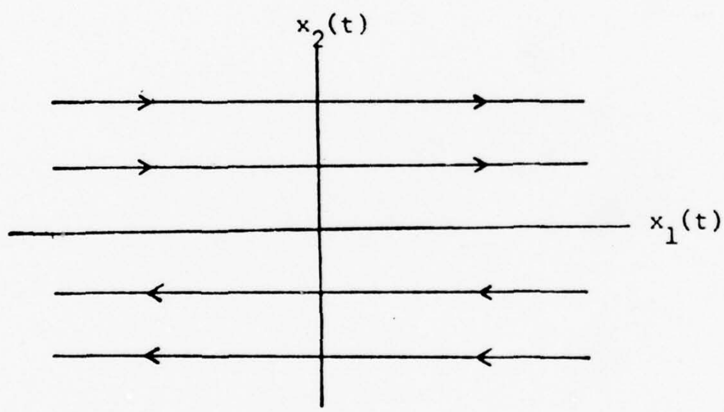

Figure 3.4. State Trajectories for u = 0

Note that if $§_2 = 0$, then x_1 does not change until u becomes non-zero. Hence, unless the initial state is the origin, the control (u = 0) cannot bring the system to the origin; consequently, any control sequence candidate ending in '0' should be dismissed from consideration.

We can now give some idea of how to steer any initial state to the origin. If the initial state $(§_1, §_2)$ is in the set

$$R_1 \equiv \{(x_1,x_2) \mid (x_1,x_2) \text{ lies above the switching curve } (s(\bar{x}(t)) = 0) \text{ and } x_2 \geq 0\} \ , \qquad (3.34)$$

then apply the control u = -1 until x_2 becomes negative. Next, allow the system to coast (u = 0), until the switching curve is intersected at which time, the control u = +1 will drive the system to the origin (see Figure 3.5).

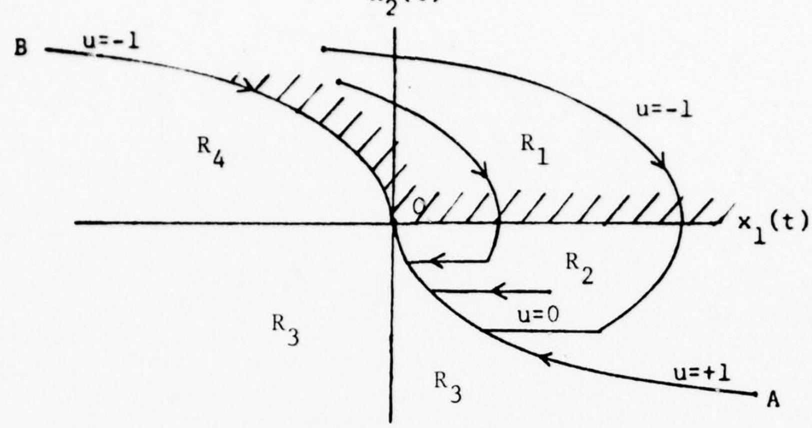

Figure 3.5. Trajectories for Initial Values in R_1 and R_2

As the drawing would indicate, an optimal controller does not exist since if u is any {-1,0,+1} control sequence which drives the system to the origin, a more optimal one can be obtained by switching to '0' earlier. The switching cannot be made at x_2 equal to zero, but must wait until x_2 becomes negative.

If the initial state is in R_2 defined by

$$R_2 \equiv \{(x_1,x_2) \mid (x_1,x_2) \text{ lies above the switching curve and } x_2 < 0\}, \tag{3.35}$$

then the control sequence {0,+1} will bring the system to the origin. The controller is optimal, but, interestingly, it is not unique (see Figure 3.5). The reason for this is that the control problem is singular in this region. The control

$$u(t) = v \ , \ 0 < v < 1 \tag{3.36}$$

can be applied throughout the interval of operation. The transversality condition, Equation (3.8), is satisfied if

$$H(t_f) = |v| + 0 + p_2(t_f)v = 0. \tag{3.37}$$

Equations (3.37), (3.30), (3.23), (3.24) imply that

$$p_1(t) = 0 \text{ and } p_2(t) = -1 \ , \ t \in [0,t_f]. \tag{3.38}$$

The resulting trajectory is parabolic, running through the origin and has been drawn in Figure 3.5. With Equation (3.36) holding for u, then

$$x_2(t) = vt + \S_2 \tag{3.39}$$

and, so at $t = t_f$,

$$vt_f + \xi_2 = 0 \ . \tag{3.40}$$

The time of impact depends on v. An infinite number of such controllers is available to steer the system to the origin. Similar remarks would apply to the other two complementary regions which lie below the switching curve AOB.

3.3.3 The Minimum Energy Problem

The form of the optimal controller is given for $i = 1,\ldots m$ by (see Appendix A for a derivation)

$$u^*_i(t) = \begin{cases} -1.0 & \text{if } \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) \geq 2r_{ii} \\ \dfrac{-\bar{p}^T(t)\bar{b}_i(\bar{x}(t),t)}{2r_{ii}} & \text{if } -2r_{ii} < \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) < 2r_{ii} \\ +1.0 & \text{if } \bar{p}^T(t)\bar{b}_i(\bar{x}(t),t) \leq -2r_{ii} \ . \end{cases} \tag{3.41}$$

Note that the minimum energy controller can assume all values between -1 and +1; that is, it is a continuous controller whereas the minimum fuel and minimum time controllers were discontinuous.

For some initial states in our two state example, an optimal controller will exist; but, for others it will not. It will be shown, however, that all initial states can be steered to the origin.

Our controller u* is given by

$$u^*(t) = \begin{cases} -1.0 & \text{if } p_2(t) \geq 2 \\ -\dfrac{p_2(t)}{2} & \text{if } -2 < p_2(t) < 2 \\ +1.0 & \text{if } p_2(t) \leq -2 \ . \end{cases} \tag{3.42}$$

The Hamiltonian is given by

$$H = u^2(t) + p_1(t)x_2(t) + p_2(t)u(t) \ . \tag{3.43}$$

Note that the state and costate equations remain unchanged. Hence,

$$p_1(t) = \pi_1 \text{ and } p_2(t) = -\pi_1 t + \pi_2. \tag{3.44}$$

Rather than list all possible control sequences, let us examine the transversality condition (Equation (3.8)) and its consequences to narrow the list. The transversality condition is that

$$H(t_f) = 0 = u^2(t_f) + p_1(t_f)\cancelto{0}{x_2(t_f)} + p_2(t_f)u(t_f) \; . \qquad (3.45)$$

Thus

$$u(t_f) = 0 \quad \text{or} \quad u(t_f) = -p_2(t_f) \; . \qquad (3.46)$$

In the latter case, since the control constraint

$$|u(t)| \leq 1 \qquad (3.47)$$

must be met, it follows that

$$|p_2(t_f)| \leq 1 \qquad (3.48)$$

From the form of the optimal control, Equation (3.42), we also have that

$$u(t_f) = -p_2(t_f)/2 \; . \qquad (3.49)$$

Hence, from Equations (3.49) and (3.46), it follows that

$$p_2(t_f) = 0 \; , \qquad (3.50)$$

which implies that

$$u(t_f) = 0 \; . \qquad (3.51)$$

Hence, in either case of Equation (3.46), we have that Equation (3.51) holds and that consequently

$$p_2(t_f) = 0 \; . \qquad (3.52)$$

Using Equation (3.44), combined with Equation (3.52), we conclude that π_1 and π_2 have the same sign. This last fact when combined with Equation (3.52) limits the possible graphs of $p_2(t)$ vs. t to those shown in Figure 3.6.

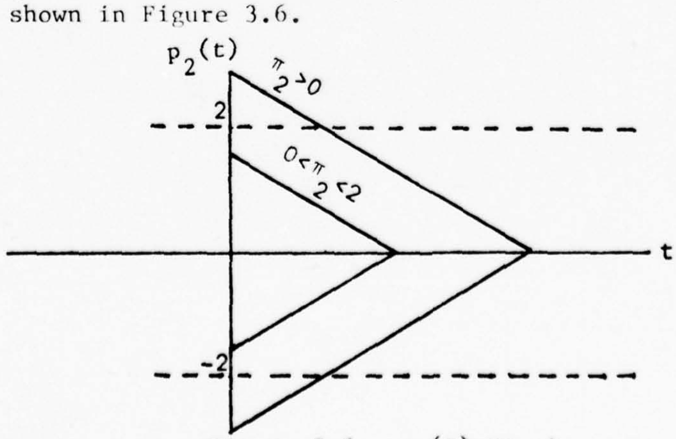

Figure 3.6. $p_2(t)$ vs. t

Figure 3.6 in conjunction with the form of the optimal control obtained using Pontryagin's minimum principle (Equation 3.42) guarantees that there are only three possible optional control sequences:

$$\{-\frac{p_2(t)}{2}\},\ \{\Delta, -\frac{p_2(t)}{2}\},\ \text{where } \Delta = \pm 1\ . \qquad (3.53)$$

<u>Case</u> <u>I</u>: The control sequence $\{-\frac{p_2(t)}{2}\}$

We will obtain the locus of all points that can be steered to the origin with control $\{-p_2(t)/2\}$. Since the final time t_f is free, the Hamiltonian will be identically zero along an extremal trajectory. Several equations that will be needed are:

$$H(0) = 0 \rightarrow \pi_2^2 = 4\pi_1 x_2(0) \qquad (3.54)$$

$$H(t_f) = 0 \rightarrow \pi_1 t_f = \pi_2 \qquad (3.55)$$

$$|p_2(t)| < 2 \rightarrow |\pi_2| \leq 2 \qquad (3.56)$$

$$x_2(t_f) = 0 \rightarrow 0 = \frac{\pi_1}{4} t_f^2 - \frac{\pi_2}{2} t_f + x_2(0) \qquad (3.57)$$

$$x_1(t_f) = 0 \rightarrow 0 = \frac{\pi_1}{12} t_f^2 - \frac{\pi_2}{4} t_f^2 + x_2(0) t_f + x_1(0). \qquad (3.58)$$

Several substitutions will be required to obtain inequalities involving only $x_1(0)$, $x_2(0)$. From Equations (3.57) and (3.55), we obtain

$$\pi_2 t_f = 4x_2(0)\ . \qquad (3.59)$$

Using Equations (3.58) and (3.55), we have that

$$\frac{\pi_2 t_f^2}{6} = x_2(0) t_f + x_1(0)\ . \qquad (3.60)$$

Substituting Equation (3.59) into (3.60) yields

$$\frac{2}{3} x_2(0) t_f = x_2(0) t_f + x_1(0) \qquad (3.61)$$

or

$$t_f = -\frac{3x_1(0)}{x_2(0)}\ . \qquad (3.62)$$

Substituting Equation (3.62) into (3.59) yields

$$\pi_2 = -\frac{4}{3}\frac{x_2^2(0)}{x_1(0)}\ . \qquad (3.63)$$

Placing this result and Equation (3.62) into Equation (3.55), we obtain

$$\pi_1 = \frac{4}{9}\frac{x_2^3(0)}{x_1^2(0)} \quad . \tag{3.64}$$

Finally, using Equation (3.63) with Equation (3.56) gives us the following inequalities:

$$x_2 \geq -\sqrt{3x_1/2} \quad \text{or} \quad x_2 \leq \sqrt{-3x_1/2} \quad . \tag{3.65}$$

<u>Sign considerations for Case I</u>:

Equation (3.55) $\rightarrow$ $\text{sgn}(\pi_1) = \text{sgn}(\pi_2)$

Equation (3.63) $\rightarrow$ $\text{sgn}(\pi_2) = -\text{sgn}(x_1(0))$

Equation (3.64) $\rightarrow$ $\text{sgn}(\pi_1) = \text{sgn}(x_2(0))$.

Consequently,

$$\text{sgn}(x_2(0)) = -\text{sgn}(x_1(0)) \quad , \tag{3.66}$$

and we conclude that we are confined to quadrants II and IV.

<u>Shape of optimal control trajectories for Case I</u>:

It can be shown that $x_2(t)/\pi_1$ is a monotone, decreasing function bounded below by zero. Also, $x_1(t)/\pi_1$ is a monotone, increasing function bounded above by zero. From the implied relationship

$$x_2 = f(x_1) \quad , \tag{3.67}$$

it can be established that $f(x_1)$ is concave upward in quadrant II, and concave downward in quadrant IV. In summary, any initial starting point $(x_1(0), x_2(0))$ in the region $R_0^- \cup R_0^+$ where

$$R_0^- \equiv \{(x_1,x_2) \mid (x_1,x_2) \text{ lies below the curve } x_2^2 = \text{sgn}(x_1)\tfrac{3}{2}x_1 \text{ and } x_2 > 0\} \tag{3.68}$$

and

$$R_0^+ \equiv \{(x_1,x_2) \mid (x_1,x_2) \text{ lies above the curve } x_2^2 = \text{sgn}(x_1)\tfrac{3}{2}x_1 \text{ and } x_2 < 0\} \quad , \tag{3.69}$$

can be driven to the origin in an optimal fashion with

$$u(t) = -\frac{p_2(t)}{2} \quad . \tag{3.70}$$

Equations (3.63) and (3.64) give the value of the constants π_1 and π_2 appearing in $p_2(t)$ in terms of the initial state values (see Figure 3.7).

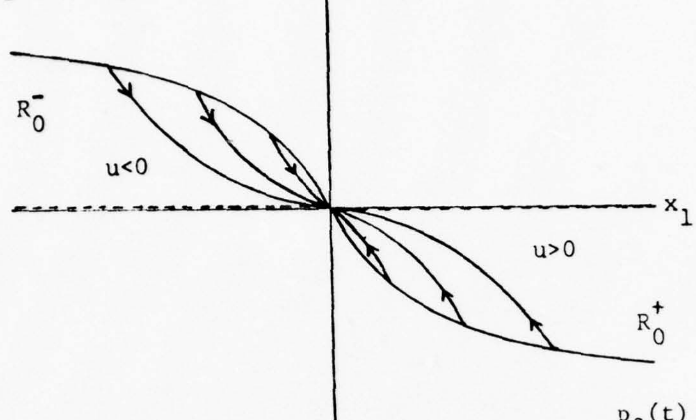

Figure 3.7. Optimal Trajectories for $u = \frac{-p_2(t)}{2}$

<u>Case II</u>: The control sequence $\{\Delta, \frac{-p_2(t)}{2}\}$, $\Delta = \pm 1$

Let t_1 denote the time when the control changes from Δ to $-p_2(t)/2$. For $0 \leq t \leq t_1$, we integrate the state equations to obtain

$$x_2(t) = \Delta t + x_2(0) \tag{3.71}$$

$$x_1(t) = \Delta\frac{t^2}{2} + x_2(0)t + x_1(0) \; . \tag{3.72}$$

Consequently, we can eliminate t from these two equations and obtain

$$x_1 = \frac{\Delta}{2}(\Delta x_2 - \Delta x_2(0))^2 + x_2(0)(\Delta x_2 - \Delta x_2(0)) + x_1(0) \tag{3.73}$$

or

$$x_1 - x_1(0) = \frac{\Delta}{2}(x_2^2 - x_2^2(0)) \; . \tag{3.74}$$

Equation (3.74) is a parabola in the state space with vertex

$$x_1 = x_1(0) - \frac{\Delta}{2}x_2^2(0) \tag{3.75}$$

$$x_2 = 0 \; . \tag{3.76}$$

From Equation (3.74), the parabolas passing through the origin are

$$x_1 = \pm\frac{x_2^2}{2} \; . \tag{3.77}$$

When $\Delta = +1$, we can see from Equations (3.71), (3.72) that

$$x_2(t) \uparrow x_2(t_1) \text{ and so } x_2(0) < 0 \quad . \tag{3.78}$$

Similarly, when $\Delta = -1$, then

$$x_2(t) \downarrow x_2(t_1) \text{ and so } x_2(0) > 0 \quad . \tag{3.79}$$

To summarize, the state space consists of three major regions: those points that can be brought to the origin with $u = \{-p_2(t)/2\}(R_0)$; those points that can be brought to the origin with the control switching sequence $u = \{\Delta, -p_2(t)/2\}(R_1)$; and all other points for which there does not exist an optimal control (R_2). Figure 3.8 shows various trajectories for these regions. The definitions of these regions follows the figure.

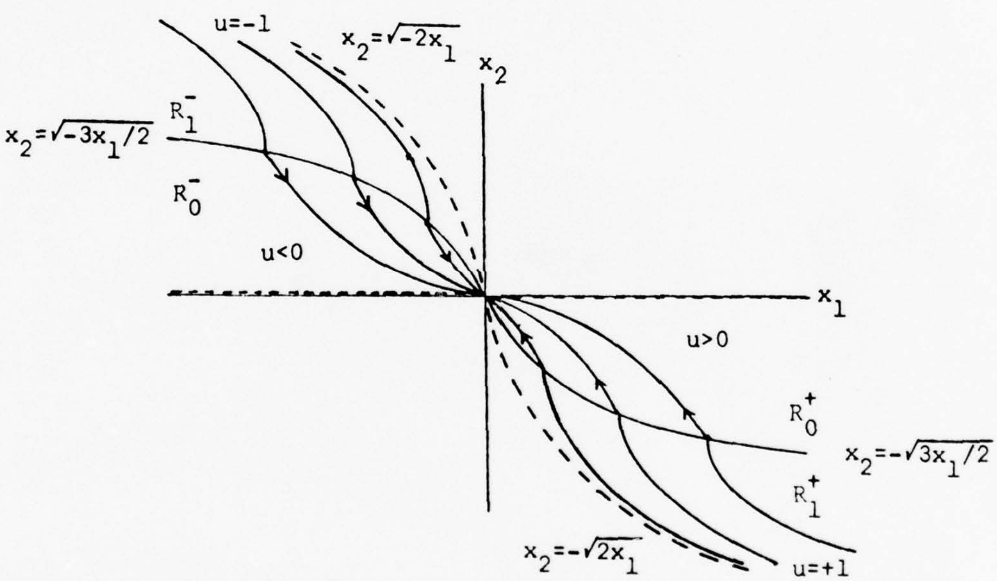

Figure 3.8. Various Optimal Control Trajectories

Although regions R_0^- and R_0^+ were defined previously, their definitions are repeated with those of the new regions for the sake of completeness.

$$R_0^- \equiv \{(x_1,x_2) \mid 0 < x_2 \leq \sqrt{-3x_1/2}\} \tag{3.80}$$

$$R_0^+ \equiv \{(x_1,x_2) \mid 0 > x_2 \geq -\sqrt{3x_1/2}\} \tag{3.81}$$

$$R_0 \equiv R_0^- \cup R_0^+$$

$$R_1^- \equiv \{(x_1,x_2) \mid \sqrt{-3x_1/2} < x_2 < \sqrt{-2x_1}\} \quad (3.82)$$

$$R_1^+ \equiv \{(x_1,x_2) \mid -\sqrt{-3x_1/2} > x_2 > -\sqrt{2x_1}\} \quad (3.83)$$

$$R_1 \equiv R_1^- \cup R_1^+$$

$$R_2^- = \{(x_1,x_2) \mid x_2 \geq 0 \text{ and } (x_1,x_2) \notin R_0 \cup R_1\} \quad (3.84)$$

$$R_2^+ = \{(x_1,x_2) \mid x_2 \leq 0 \text{ and } (x_1,x_2) \notin R_0 \cup R_1\} \quad (3.85)$$

$$\hat{R}^- \equiv \{(x_1,x_2) \mid x_2 = \sqrt{-2x_1}\} \quad (3.86)$$

$$\hat{R}^+ \equiv \{(x_1,x_2) \mid x_2 = -\sqrt{2x_1}\} \ . \quad (3.87)$$

The optimal solutions to the minimum energy problem are restricted to $R_0 \cup R_1$. Sub-optimal solutions may be used in the rest of the state space: $\{\Delta\}$ on $\hat{R}^- \cup \hat{R}^+$ and $\{-\Delta, -\frac{1}{2}p_2(t)\}$ on $R_2^- \cup R_2^+$. Note that the switching of the control u from $-\Delta$ to $-p_2(t)/2$ must occur some arbitrarily small distance inside the region R_0 and not on the x_1 axis.

3.4 Some Remarks about the Three Types of Control Problems

First, it should be noted that there are other types of control problems than the three mentioned in section 3.3. These are of interest to us because we can formulate our missile control problem in any of the three forms (see Chapter IV).

Some observations are in order about the three control problems. In terms of difficulty, the minimum time problem is the easiest to solve, the minimum fuel would be next, and the minimum energy last. This is due mainly to the general form of the controller for each case. More important, however, is the form of the controller as it pertains to the problem of implementation.

The minimum time controller is a bang-bang type (± 1), and the minimum fuel is a bang-off-bang controller ($0, \pm 1$). The minimum energy one can take on all values between +1 and -1. This controller is continuous, whereas the other two are not. From an implementation point of view, the minimum energy controller would be preferable since the input to the system would be changing in a less drastic manner. Furthermore, in a system that identifies the control with tail fin deflection, instantaneous changes are physically unrealizable. Because of the continuity of the controller, one would expect the implemented version of this control to closer approximate the idealized version than would be the case with the other two. The tradeoff, of course, is the additional mathematical complexity in obtaining the solution to the control problem in the first place.

Chapter IV

THE MISSILE CONTROL PROBLEM WITH CONTROL CONSTRAINTS

4.1 Missile - Target Geometry and Dynamics

Our objective is to control a missile in such a way as to impact on the target with the center of the missile being vertical; that is, we want not only to insure that we hit the target, but we want also to impose conditions of the missile's attitude at impact. Figure 4.1 depicts the geometry of the terminal guidance phase. and Table 4.1 defines the variables introduced.

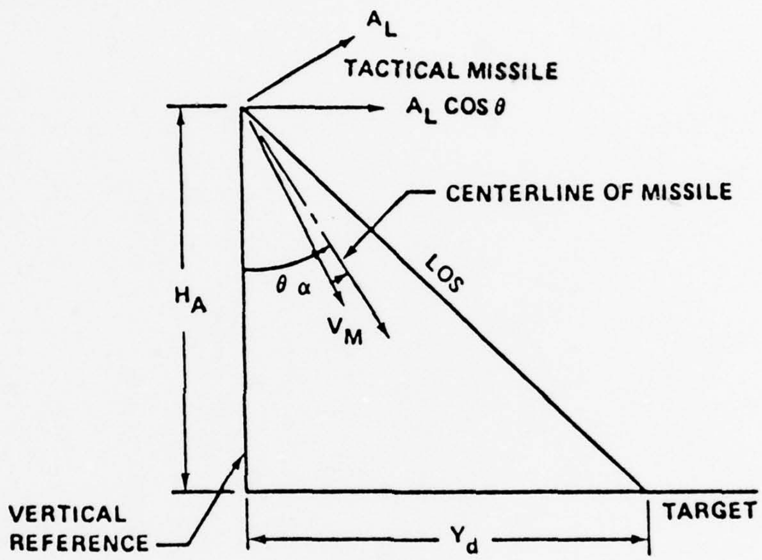

Figure 4.1. Geometry of Tactical Missile - Target Positions

Table 4.1. Definition of Variables

Variable	Definition
Y_m	Missile position variable projected on the ground (ft)
Y_t	Target position variable (ft)
Y_d	Position variable from missile to target projected on the ground ($Y_d = Y_t - Y_m$)
$\dot{Y}_d$	Time derivative of Y_d (ft/sec)
A_L	Lateral acceleration of the missile (ft/sec^2)
θ	Body attitude angle of the missile (deg)

The following set of state variables was chosen for the modeling process.

$$\overline{X} = \begin{bmatrix} Y_d \\ \dot{Y}_d \\ A_L \\ \theta \end{bmatrix} . \tag{4.1}$$

Under the assumption that the angle of attack α is small and can thus be neglected, the state variables are subject to the following dynamics:

$$\dot{Y}_d = \dot{Y}_d \tag{4.2}$$

$$\ddot{Y}_d = -A_L \cos\theta \tag{4.3}$$

$$\dot{A}_L = -w_1 A_L + K_1 u \tag{4.4}$$

$$\dot{\theta} = K_a u \quad . \tag{4.5}$$

Performance is considered to be acceptable when the following constraints are satisfied:

$$|Y_d(t_f)| \leq 5 \text{ feet} \tag{4.6}$$

$$|\theta(t_f)| \leq 5 \text{ degrees} \quad . \tag{4.7}$$

4.2 Formulations of the Missile Control Problem

We will impose the normalized control constraints

$$|u(t)| \leq 1 \tag{4.8}$$

on our problem. In order to facilitate our goal of obtaining a closed form solution, we will add an additional assumption that will reduce the number of state variables to three. To this end, we assume that the autopilot has zero lag. This means that Equation (4.4) can be omitted, and that we can rewrite Equation (4.3) by using the relationship

$$A_L = \frac{K_1}{w_1} u \ . \tag{4.9}$$

We linearize our differential equation (4.3) about the impact conditions,

$$b = \cos\theta \ , \tag{4.10}$$

and obtain the following three state system dynamics:

$$\dot{Y}_d = Y_d \tag{4.11}$$

$$\ddot{Y}_d = \frac{-bK_1}{w_1} u \tag{4.12}$$

$$\dot{\theta} = K_a u \quad . \tag{4.13}$$

4.2.1 Minimum Time Problem

Find a controller u, satisfying

$$|u(t)| \leq M \quad , \tag{4.14}$$

that will drive the system (4.11), (4.12), (4.13) to the origin and will minimize

$$J = \int_0^{t_f} 1 \, dt \quad . \tag{4.15}$$

At first glance, one might think that our control problem as outlined in section 4.1 cannot be viewed as a minimum time problem, but that is in fact incorrect. The key lies in the selection of state variables. Minimum time in this context means far more than getting to the target in the shortest possible time. The end condition

$$Y_d(t_f) = 0 \tag{4.16}$$

means that the miss distance at impact is to be zero. The end condition for the second state variable

$$\dot{Y}_d(t_f) = 0 \tag{4.17}$$

implies that the velocity vector is to be directed straight downward. Finally, the last state's end condition

$$\theta(t_f) = 0 \tag{4.18}$$

insures that the center line of the missile is perpendicular to the horizon upon impact.

This particular way of viewing our control problem as one with final time t_f unspecified possesses an especially desirable feature from an implementation point of view. It is very difficult to obtain an accurate estimate of final time with our lack of range, range-rate information. A control law in which control changes are made depending only on state measurements and not on time-to-go is preferable.

4.2.2 Minimum Fuel Problem

The dynamics remain unchanged. The control problem then is to drive the system to the origin with a controller that will minimize

$$J = \int_0^{t_f} |u(t)|\, dt \quad . \tag{4.19}$$

This controller would be somewhat more desirable than the minimum time controller since it has the added 'coasting' command. This controller too is time invariant, and the system would be driven by state measurements.

4.2.3 Minimum Energy Problem

Again, the dynamics remain unchanged. The control problem is to drive the system to the origin with a controller that will minimize

$$J = \int_0^{t_f} u^2(t)\, dt \quad . \tag{4.20}$$

This controller would yield the desired type of trajectory and would be the most desirable since for optimal trajectories it changes in a continuous fashion. It too would be time invariant, and state measurements only would drive the system.

4.3 Solution of the Missile Control Problem as a Minimum Time Problem

There are two important concerns to investigate before trying to solve the minimum time control problem with dynamics

$$\dot{Y}_d = \dot{Y}_d \tag{4.21}$$

$$\ddot{Y}_d = \frac{-bK_1}{w_1} u \tag{4.22}$$

$$\dot{\theta} = K_a u \quad , \qquad |u| \leq M \tag{4.23}$$

and state vector

$$X^T = [Y_d, \dot{Y}_d, \theta] \quad . \tag{4.24}$$

First, check to see if the system is controllable, and, secondly, check to see if the matrix A is in Jordan cannonical form where the system is viewed as

$$\dot{X} = AX + Bu \quad . \tag{4.25}$$

4.3.1 Controllability

A system of the form given in Equation (4.25) where X has n states is controllable if and only if

$$\text{rank } [B, AB, A^2B, \ldots A^{n-1}B] = n \quad . \tag{4.26}$$

For our system, we have that

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ -a \\ k \end{bmatrix} \tag{4.27}$$

where

$$a = K_1 b/w_1 \quad \text{and} \quad k = K_a \quad . \tag{4.28}$$

Calculating the matrix $[B, AB, A^2B]$, we obtain

$$M = \begin{bmatrix} 0 & -a & 0 \\ -a & 0 & 0 \\ k & 0 & 0 \end{bmatrix} \tag{4.29}$$

which has rank 2. We conclude then that our particular system is uncontrollable in the sense that it will not be possible to steer an arbitrary point in x_1-x_2-x_3 space to the origin. Since it is more important, however, that x_1 and x_3 be driven to zero than x_2, we will continue with the solution to investigate this possibility.

4.3.2 Jordan Cannonical Form of A

When necessary a change of variables whould be introduced to insure that A is in Jordan cannonical form as this will later simplify the equations of various switching curves. Our matrix A has an eigenvalue 0, of multiplicitly three. The characteristic polynomial $\Delta(t)$ of A is

$$\Delta(t) \equiv \det[tI - A] = t^3 \quad , \tag{4.30}$$

and the minimum polynomial m(t) of A is

$$m(t) = t^2 \quad . \tag{4.31}$$

The Jordan cannonical form is a block diagonal matrix with blocks J_{ij} of the form

$$J_{ij} = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 & 0 \\ 0 & \lambda_i & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_i & 1 \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix} \quad . \tag{4.32}$$

Since there is at least one J_{ij} of order 2 (the power of $(t-\lambda_i)$ in the *minimum polynomial*), and the sum of the orders of J_{ij}'s is equal to 3 (the power of $(t-\lambda_i)$ in the characteristic polynomial), there can be only one possible form for the cannonical form of the matrix A, namely,

$$J = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \quad .$$

Consequently, A is already in Jordan cannonical form.

4.3.3 An Altered Minimum Time Problem

Section 4.3.1 showed that our system was not controllable in the sense that for an arbitrary initial value (x_{10}, x_{20}, x_{30}) we could not drive all three states to zero. Since the rank of the matrix M was 2, we should be able to drive two of our state variables to zero. Returning to the physical meaning of our states, it is clear that it is most important that the miss distance (x_1) and the attitude angle (x_3) be zero at final time. A non-zero value for $x_2(t_f)$ can be tolerated, and in fact, there will be some best possible value c for x_2.

<u>Control Problem</u>: Find a controller v(t) to steer the system

$$\dot{x}_1 = x_2 \tag{4.33}$$

$$\dot{x}_2 = \frac{-K_1}{w_1} bv \tag{4.34}$$

$$\dot{x}_3 = K_a v \tag{4.35}$$

to

$$[\bar{x}(t_f)]^T = [0, c, 0]^T \tag{4.36}$$

in the shortest possible time subject to the constraint

$$|v(t)| \leq M \quad . \tag{4.37}$$

Change of Variable to Normalize the Controller: Define

$$u = \frac{v}{M} \quad , \quad a = \frac{K_1 b\ M}{w_1} \quad , \quad k = K_a M \ . \tag{4.38}$$

Then the system's new state equations are

$$\dot{x}_1 = x_2 \tag{4.39}$$

$$\dot{x}_2 = -au \tag{4.40}$$

$$\dot{x}_3 = ku \tag{4.41}$$

where u must satisfy the constraint

$$|u(t)| \leq 1 \ . \tag{4.42}$$

Determination of c: From Equations (4.40) and (4.41),

$$k\dot{x}_2 + a\dot{x}_3 = -aku + aku = 0 \ . \tag{4.43}$$

Integrating, we get

$$k(x_2(t) - x_2(0)) + a(x_3(t) - x_3(0)) = 0 \ . \tag{4.44}$$

Letting $t = t_f$, we see that when x_3 is driven to zero,

$$x_2(t_f) = \frac{kx_2(0) + ax_3(0)}{k} \equiv c \ . \tag{4.45}$$

Solution of the State Equations: Assuming that $u = \Delta = \pm 1$, the solution of the state equations is

$$x_1 = \frac{-a\Delta t^2}{2} + x_2(0)t + x_1(0) \tag{4.46}$$

$$x_2 = -a\Delta t + x_2(0) \tag{4.47}$$

$$x_3 = k\Delta t + x_3(0) \ . \tag{4.48}$$

Projection of the Switching Curves: Solving Equation (4.47) for t and substituting into Equation (4.46) yields

$$x_1 - x_1(0) = \frac{-\Delta}{2a}(x_2^2 - x_2^2(0)) \ . \tag{4.49}$$

From Equation (4.44), and the definition of c, we have

$$kx_2 + ax_3 = kc \ . \tag{4.50}$$

Since we would like to investigate the projection of the state trajectories in the $x_1 - x_3$ plane, we evaluate Equation (4.49) at $t = t_f$ and make a substitution relating x_2 to x_3. From (4.49), since

$$x_1(t_f) = 0 \text{ and } x_2(t_f) = c \ , \tag{4.51}$$

we have that

$$x_1(0) = \frac{\Delta}{2a}(c^2 - x_2^2(0)) \ . \tag{4.52}$$

From Equation (4.50) at t = 0, we obtain

$$x_2(0) = \frac{kc - ax_3(0)}{k} \ . \tag{4.53}$$

Substituting into Equation (4.52), we have the desired result

$$x_1(0) = \frac{-\Delta}{2ak^2}(a^2x_3^2(0) - 2akcx_3(0)) \ . \tag{4.54}$$

The shape of the locus of initial points in the $x_1 - x_3$ plane that can be driven to the origin is a parabola.

Qualitative Discussion: From Equations (4.47), (4.48), we see that

$$\Delta = +1 \text{ implies that } x_2\downarrow \text{ and } x_3\uparrow \tag{4.55}$$

and

$$\Delta = -1 \text{ implies that } x_2\uparrow \text{ and } x_3\downarrow \ . \tag{4.56}$$

From Equation (4.54), we also note that

$$\Delta = +1 \text{ implies that the parabola opens left} \tag{4.57}$$

and

$$\Delta = -1 \text{ implies that the parabola opens right} \ . \tag{4.58}$$

The x_3-intercepts of the parabola are 0 and $\frac{2kc}{a}$.

Graphs: Figure 4.2 shows trajectories (projections) in the $x_2 - x_3$ plane, while Figures 4.3a and 4.3b show the switching curve in the $x_1 - x_3$ plane.

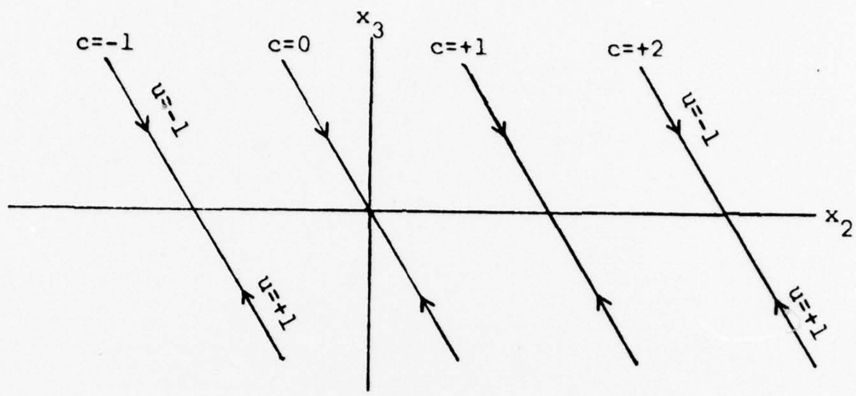

Figure 4.2. Projection of State Trajectories in the x_2 - x_3 Plane

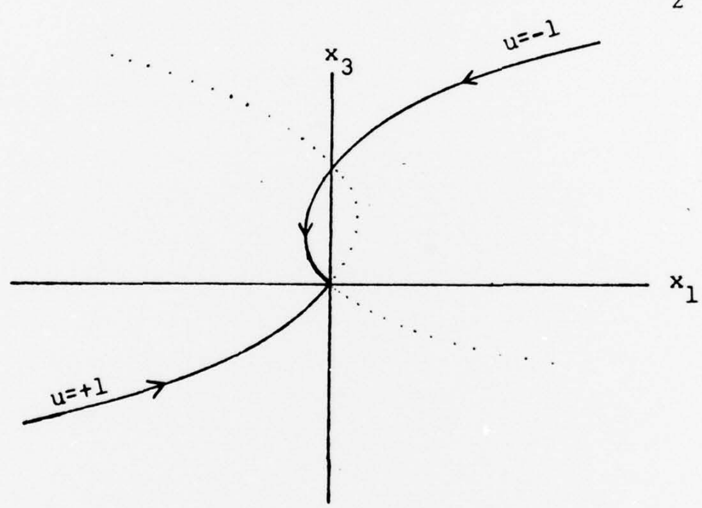

Figure 4.3a. Switching Curve Projection in the x_1 - x_3 Plane ($c \geq 0$)

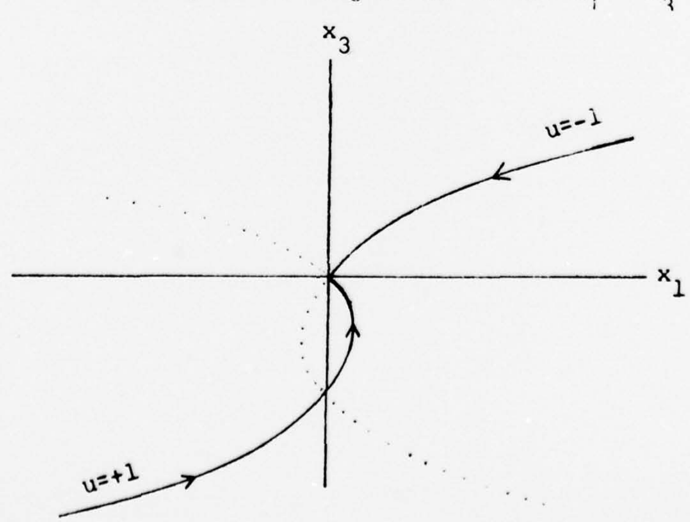

Figure 4.3b. Switching Curve Projection in the x_1 - x_3 Plane ($c < 0$)

The Control Sequence $\{-\Delta,\Delta\}$: First, calculate c, which depends on $x_2(0)$, $x_3(0)$. In Figure 4.3a, if the initial point lies above the switching curve, apply u = -1 until the switching curve is intercepted, and then change to u = +1. The opposite would be true if the initial point lies below the switching curve. The same comments apply to the switching curve for c < 0 given in Figure 4.3b. Several trajectories going to the origin have been shown in Figure 4.4.

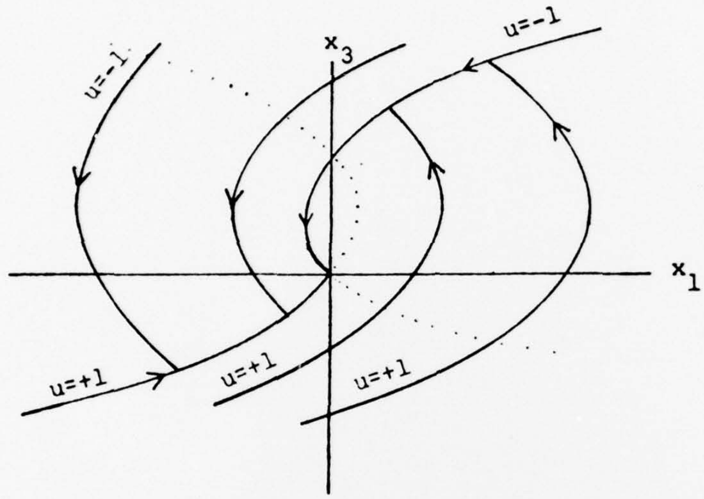

Figure 4.4. Various Optimal Control Trajectories

Conclusions and Remarks: Figure 4.4 shows that it is mathematically possible to take any initial point in x_1-x_2-x_3 space and drive x_1 and x_3 to zero, x_2 to c in an optimal manner. Although mathematically correct, this turns out to be physically impossible since the mathematical model used was not stated as precisely as it should have been. The error is that there are inequality constraints on the state variables which have been overlooked. The switching curves in Figure 4.3 are not actual trajectories since the missile will flip when x_3 changes sign to a negative value. A sign change in x_1 means that the missile has overflown the target.

In order to make the model more closely represent the physics of the situation, these inequality constraints must be incorporated into the model. According to the definition, we must have x_1 and x_3 positive throughout the flight, and x_2 negative.

Chapter V

COMPUTER SIMULATION IMPLEMENTATION

5.1 Effect of the State Inequality Constraints

As was mentioned at the end of Chapter IV, although it was possible to drive miss distance x_1 and attitude angle x_3 to zero in the imaginary world of mathematics, the physical world imposes constraints that will prevent this in practice. Reality imposes the following constraints on our state variables:

$$x_1 \geq 0 \tag{5.1}$$

$$x_2 \leq 0 \tag{5.2}$$

$$x_3 \geq 0 \; . \tag{5.3}$$

Given the initial value of velocity, $x_2(0)$, and missile attitude, $x_3(0)$, a value for the constant c is calculated where c is defined by

$$c = \frac{kx_2(0) + ax_3(0)}{k} \; . \tag{5.4}$$

If c is non-negative, then the switching curve given in Figure 4.3a applies; whereas, if c is negative, then Figure 4.3b applies. Should c be negative, then for certain initial conditions, it will be possible to drive both x_1 and x_3 to zero. However, should c be positive, then we can drive x_1 to zero, but there will be some non-zero best possible value for x_3 at t_f.

5.1.1 Case I. $c < 0$

The constraints, Equations (5.1), (5.2), (5.3) force an admissable trajectory to lie entirely in the first quadrant when the trajectory projection in the $x_1 - x_3$ plane is considered. Should the initial state lie above the switching curve, point A for example, then it is not possible to drive x_3 to zero at impact as there is insufficient time for the missile to react. The best that can be done is to give a hard over tail fin command to bring the nose of the missile down (trajectory AE in Figure 5.1).

Should the initial state lie on the switching curve, most unlikely, a command of {-1} will bring both states to zero. With the initial condition lying below the switching curve, point B for example, then a {+1,-1} sequence will bring x_1 and x_3 to zero most rapidly (trajectory BCO in Figure 5.1). It should be kept in mind, however, that since our math model is a rather simple reflection of reality, perhaps a better control strategy might be the sequence {0,-1}. It too will drive both variables to zero. A computer simulation that is more realistic than the three-state system is needed to decide questions of this type.

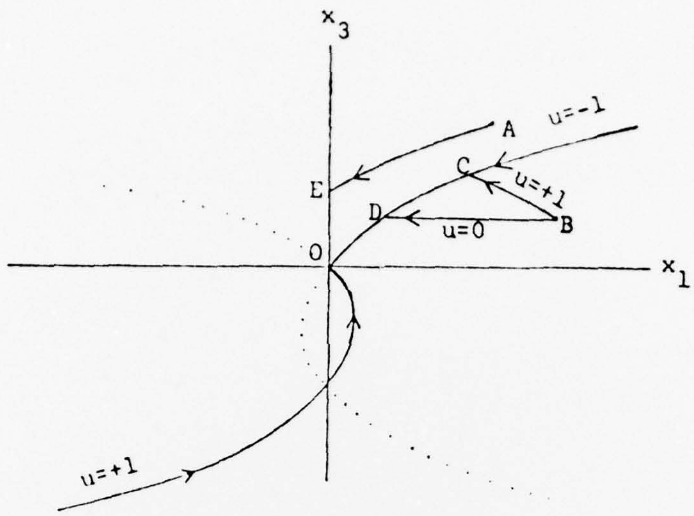

Figure 5.1. Various Trajectories for Case I

5.1.2 Case II. $c \geq 0$

When $c > 0$, it is not possible to drive both x_1 and x_3 to zero at t_f. In a similar fashion, if the initial point lies above the switching curve, a hard over command is the best that can be done. If the initial state lies on the switching curve, then the upper x_3-intercept is the best possible value that can be achieved for x_3 with control {-1}. Similar concerns as before apply to those initial values below the curve. Is the control sequence {+1,-1} or {0,-1} best, or perhaps $u = v$ where v is a negative constant between 0 and -1? Again a simulation is needed to decide.

5.1.3 Physical Meaning of Case I and II

Combining the definition of c given by Equation (5.4) with the constraints on x_1, x_2, and x_3, we can derive an inequality relating $x_2(0)$ to $x_3(0)$ for both cases. Since k is positive, Equation (5.4) implies that

$$kx_2(0) + ax_3(0) < 0 \tag{5.5}$$

or

$$x_2(0) < -\frac{a}{k}x_3(0) \ . \tag{5.6}$$

Since

$$x_3(t) \geq 0 \quad \text{and} \quad x_2(t) \leq 0 \ , \tag{5.7}$$

it follows that for $c < 0$,

$$x_2(0) < -\frac{a}{k}x_3(0) \leq 0 \ . \tag{5.8}$$

Similarly, for $c > 0$,

$$-\frac{a}{k}x_3(0) < x_2(0) \leq 0 \ . \tag{5.9}$$

Returning to the physical meaning of our state variables, this would say that it is possible to determine whether or not the desired trajectory can be achieved by comparing missile velocity, $x_2(0)$, to a multiple of missile attitude, $-ax_3(0)/k$. If Equation (5.9) holds, the missile velocity is not sufficiently large in magnitude to achieve the stated objective. If Equation (5.8) holds, the missile velocity is sufficiently large to drive both variables to zero.

5.2 Implementation of the Controller for the Three State System

5.2.1 The Linear and Non-linear Three State System

The following procedure was adopted in trying to implement our control strategy with the three state system. First, our system in the computer simulation was kept identical to the normalized linear system given by Equations (4.39), (4.40), (4.41). Then this system was replaced by the non-linear system

$$\dot{x}_1 = x_2 \tag{5.10}$$

$$\dot{x}_2 = -\frac{a}{b}\cos\theta \cdot u \tag{5.11}$$

$$\dot{x}_3 = ku \ , \qquad |u| \leq 1 \tag{5.12}$$

which is a normalized version of the original non-linear three state system given by

$$\dot{Y}_d = \dot{Y}_d \tag{5.13}$$

$$\ddot{Y}_d = -\frac{K_1}{w_1}\cos\Theta \cdot v \tag{5.14}$$

$$\dot{\theta} = K_a v \ , \qquad |v| \leq M \ . \tag{5.15}$$

The control value for u depends on whether or not the point (x_1, x_3) lies below or above the corresponding switching curve ($c \geq 0$ or $c < 0$). The implementation uses an altered form of Equation (4.54). Cancelling the a on the right hand side of (4.54), we obtain for the equation of the switching curve

$$x_1(0) = -\frac{\Delta}{2k^2}(ax_3^2(0) - 2kcx_3(0)) \ . \tag{5.16}$$

Note that this one equation represents both cases since c is included. A point (x_1, x_3) will thus lie below the switching curve (see Figure 5.1) if

$$x_1(0) > -\frac{\Delta}{2k^2}(ax_3^2(0) - 2kcx_3(0)) \tag{5.17}$$

or

$$x_1(0) + \frac{\Delta}{2k^2}(ax_3^2(0) - 2kcx_3(0)) > 0 \; . \tag{5.18}$$

Equation (5.18) was used in the simulation.

5.2.2 Selection of Control Strategy

The problem of selecting a control strategy could not be pursued at this stage for reasons which will shortly become apparent. The computer implementation as mentioned above (see Appendix B for a listing) was executed with the additional feature of updating c every one tenth of a second throughout the flight for greater accuracy. Some difficulty was experienced in evaluating K_a which appears in the differential equation for $\dot{\theta}$. The simulation would drive both θ and Y_d to zero at the same time, but it would have no way of detecting the time of impact. On some runs, the missile would overfly the target with θ achieving zero as it did so.

Chapter VI

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE STUDY

The implemented control law in section 5.2.2 did not accomplish what was intended. Although it did drive θ and Y_d to zero at the same time, this was not always the time at impact. With a little reflection, one realizes that this is all the control law was designed to do. The concept of 'above' or 'below' the ground does not appear in the formulation of the problem. It would seem that this can be overcome by electing to drive Y_d and $\dot{Y}_d$ to zero at t_f. This selection would shape the trajectory so that the missile would have a high attitude angle at impact. This approach is being pursued and results of it will be communicated to the contract supervisor. Had there been more access to the computer, perhaps this difficulty might have been discovered sooner and corrected. We conclude with some recommendations for future study.

The more accurately the mathematical control model reflects the actual system, the better the chances of getting the corresponding control law to work. For our missile problem, the most accurate representation of the actual system is our 6DOF computer simulation which contains at least thirty to forty coupled, first order, ordinary differential equations. To produce a simpler system for which a closed form mathematical solution of the control problem might be feasible, a selection of the more important states must be made. For us, this next simpler model was a four state system using the state variables Y_d, $\dot{Y}_d$, θ, and A_L. We decided that rather than try to solve this problem, we would simplify the system representation even further to a three state one for which we could obtain a control law. The assumption that was made to produce this last model was that the lag in the autopilot could be ignored.

Because of the form of the resulting system, our control problem turned out to be not completely controllable. This fact simplified the mathematics considerably, and a control strategy was devised which would drive the miss distance Y_d to zero along with the attitude angle θ at impact. The other variable, $\dot{Y}_d$, could not be driven to zero at final time except for rather special sets of conditions. Due to this controllability problem, our three state system was effectively reduced to a two state one.

Based upon our experiences, we would recommend that there might be some merit in working with a control for the four state system. Generally, such mathematical solutions are almost impossible to obtain, but if this system has some of the controllability features of the three state one, then the resulting control problem might very well turn out to be like a three state system that is completely controllable. A closed form solution might well be obtained for such a system.

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Remark: Of the three books on optimal control theory, the text by Kirk in the Prentice-Hall's Electrical Engineering Series is the most readable. The book by Athens and Falb and the book by Bryson and Ho are more rigorous than Kirk's, with the Bryson-Ho book being probably the most difficult text to read. It has numerous, interesting applied control examples however. The Athens-Falb book is almost encyclopedic in nature, consisting of about 900 pages.

Appendix A: Derivation of the Minimum Energy Optimal Control Law

<u>Problem</u>: Determine the form of the optimal control for the system

$$\dot{\bar{x}}(t) = \bar{a}(\bar{x}(t),t) + \bar{B}(\bar{x}(t),t)\bar{u}(t) \qquad \text{(A-1)}$$

in order to drive the system to the origin so as to minimize the functional

$$J = \int_{t_o}^{t_f} \bar{u}^T(t)R\bar{u}(t)\, dt \ , \qquad \text{(A-2)}$$

where t_f is free, R a diagonal matrix with positive elements. Assume that

$$|u_i(t)| \leq 1 \ , \ i = 1,\ldots m \ . \qquad \text{(A-3)}$$

<u>Solution</u>: Define the Hamiltonian H to be

$$H = \bar{u}^T(t)R\bar{u}(t) + \bar{p}^T(t)\bar{a}(\bar{x}(t),t) + \bar{p}^T(t)\bar{B}(\bar{x}(t),t)\bar{u}(t). \qquad \text{(A-4)}$$

If the control constraints are not violated, then

$$0 = \frac{\partial H}{\partial \bar{u}} = 2R\bar{u}(t) + \bar{B}^T(\bar{x}(t),t)\bar{p}(t) \ . \qquad \text{(A-5)}$$

Equation (A-5) implies that

$$\bar{u}(t) = R^{-1}\bar{B}^T(\bar{x}(t),t)\bar{p}(t)/2 \ . \qquad \text{(A-6)}$$

To determine the i^{th} component of $\bar{u}(t)$, let $\bar{b}_i$ denote the i^{th} column of the matrix $\bar{B}$. Then

$$\text{the } i^{th} \text{ component of } \bar{B}^T(\bar{x}(t),t)\bar{p}(t) = \bar{b}_i^T\bar{p}(t) = \bar{p}^T(t)\bar{b}_i \ . \qquad \text{(A-7)}$$

The argument has been omitted, but $\bar{b}_i$ is a function of $\bar{x}(t)$,t. Hence, from Equation (A-6), we have that

$$u_i(t) = -\frac{\bar{p}^T(t)\bar{b}_i}{2r_{ii}} \ . \qquad \text{(A-8)}$$

Since Equation (A-3) holds, Equation (A-8) holds for

$$-2r_{ii} < \bar{p}^T(t)\bar{b}_i < 2r_{ii} \ . \qquad \text{(A-9)}$$

Suppose now that the control is saturated. Then Equation (A-5) is no longer valid, and Pontryagin's minimum principle must be applied. The 'u' part of H is

$$\bar{u}^T(t)R\bar{u}(t) + \bar{p}^T(t)B(\bar{x}(t),t)\bar{u}(t) \quad . \tag{A-10}$$

Assuming that the components of the control $\bar{u}$ are linearly independent, $u_i(t)$ is to be chosen so as to minimize the u_i^{th} part of the Hamiltonian:

$$u_i(t)[r_{ii}u_i(t) + \bar{p}^T(t)\bar{b}_i] \tag{A-11}$$

Since r_{ii} is positive and the magnitude of u is 1, then the following choices of $u_i(t)$ will yield the smallest value of the Hamiltonian.

$$u_i(t) = \begin{cases} -1 & \text{if } \bar{p}^T(t)\bar{b}_i \geq 2r_{ii} \\ +1 & \text{if } \bar{p}^T(t)\bar{b}_i \leq -2r_{ii} \quad . \end{cases} \tag{A-12}$$

In summary then, the form of the optimal controller is

$$u_i^*(t) = \begin{cases} -1 & \text{if } \bar{p}^T(t)\bar{b}_i \geq 2r_{ii} \\ -\dfrac{\bar{p}^T(t)\bar{b}_i}{2r_{ii}} & \text{if } -2r_{ii} < \bar{p}^T(t)\bar{b}_i < 2r_{ii} \\ +1 & \text{if } \bar{p}^T(t)\bar{b}_i \leq -2r_{ii} \quad . \end{cases} \tag{A-13}$$

Appendix B

FOUR STATE SIMULATION LISTING

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