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A TRANSFORMATION TECHNIQUE FOR OPTIMAL CONTROL PROBLEMS WITH PARTIALLY LINEAR STATE INEQUALITY CONSTRAINTS

by

A. MIELE, A.K. WU, and C.T. LIU

RICE UNIVERSITY

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This paper considers optimal control problems involving the minimization of a functional subject to differential constraints, terminal constraints, and a state inequality constraint. The state inequality constraint is of a special type, namely, it is linear in some or all of the components of the state vector.

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A Transformation Technique for Optimal Control Problems with Partially Linear State Inequality Constraints[1]

by

A. MIELE[2], A.K. WU[3], and C.T. LIU[4]

Abstract. This paper considers optimal control problems involving the minimization of a functional subject to differential constraints, terminal constraints, and a state inequality constraint. The state inequality constraint is of a special type, namely, it is linear in some or all of the components of the state vector.

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A transformation technique is introduced, by means of which the inequality constrained problem is converted into an equality constrained problem involving differential constraints, terminal constraints, and a control equality constraint. The transformation technique takes advantage of the partial linearity of the state inequality constraint so as to yield a transformed problem characterized by a new state vector of minimal size. This concept is important computationally, in that the computer time per iteration increases with the square of the dimension of the state vector.

In order to illustrate the advantages of the new transformation technique, several numerical examples are solved by means of the sequential gradient-restoration algorithm for optimal control problems involving nondifferential constraints. The examples show the substantial savings in computer time for convergence, which are associated with the new transformation technique.

Key Words. Optimal control, numerical methods, computing methods, transformation techniques, sequential gradient-restoration algorithm, nondifferential constraints, state inequality constraints, linear state inequality constraints, partially linear state inequality constraints.

1. Introduction

In recent years, considerable attention has been devoted to the study of optimal control problems with bounded state. The approaches employed are of two types: (i) the direct approach, in which a predetermined number and sequence of subarcs are assumed, and (ii) the indirect approach in which no predetermined number and sequence of subarcs are assumed.

Direct Approach. In the direct approach, the extremal arc is viewed as being composed of several subarcs, some internal to the state boundary and some lying on the state boundary. At the points of junction of different subarcs, discontinuities in the control and the multipliers are allowed. In practice, this view of problems with state inequality constraints can be implemented through a variety of techniques, all of them requiring satisfaction of the state inequality constraint, with this understanding: strict inequality must be enforced for the subarcs internal to the state boundary, and strict equality must be enforced for the subarcs lying on the state boundary.

Indirect Approach. In the indirect approach, the extremal arc is viewed as a single subarc, even though a portion of it may lie extremely close to the state boundary. Along this single subarc, the control and the multipliers are regarded to be continuous functions of the time. In practice, this view of problems with state inequality constraints can be implemented through either penalty function techniques or transformation techniques. The latter techniques (see, for instance, Refs. 1-5) constitute the key concern of this paper.

Transformation Techniques. In a transformation scheme, a Valentine-type representation is employed for the state inequality constraint $L_0(x,\theta) \geq 0$ (Ref. 4). If k is the order of the state inequality constraint[5], the dimension of the state vector is increased by k. That is, the n-vector state x is augmented by the k-vector y, which is defined in such a way that the state inequality constraint $L_0(x,\theta) \geq 0$ is satisfied automatically at each point of the trajectory.

Concerning the m-vector control u, one has two options: (i) to leave its dimension unchanged through a substitution technique (Ref. 4); or (ii) to increase its dimension by one. In the second option, the m-vector control u is augmented by the scalar control w. In turn, the augmented control (u,w) must satisfy a nondifferential constraint everywhere along the trajectory (Ref. 5).

Research Objectives. A drawback of the transformation schemes of Refs. 4 and 5 is that they cause an increase in the

---

[5]A state inequality constraint $L_0(x,\theta) \geq 0$ is defined to be of order k if the kth time derivative of the function $L_0(x,\theta)$ is the first to contain the control explicitly.

dimension of the state vector and, consequently, an increase in the computer time per iteration. As an example, if one uses the sequential gradient-restoration algorithm for optimal control problems with nondifferential constraints (Ref. 5), the computer time per iteration is proportioned to the square of the dimension of the state vector. Clearly, in devising transformation techniques converting an inequality constrained problem into an equivalent equality constrained problem, it is important to prevent or limit the increase in the dimension of the state vector; in other words, it is important that the dimension of the new state vector be as small as possible.

In this paper, we pursue this point of view with reference to a special category of optimal control problems, namely, problems where the state inequality constraint $L_0(x,\theta) \geq 0$ is linear in some or all of the components of the state vector. We consider the k-vector function $L(x,\theta)$, composed of $L_0(x,\theta)$ and its first k-1 derivatives. We make the following assumptions: (i) the n-vector state x can be partitioned into vectors x_A and x_B having dimensions a and b, respectively; (ii) the k-vector function $L(x_A,x_B,\theta)$ can be partitioned into two vector functions $\tilde{L}(x_A,x_B,\theta)$ and $\hat{L}(x_A,x_B,\theta)$ having dimensions a and c, respectively; and (iii) the a-vector function $\tilde{L}(x_A,x_B,\theta)$ is linear in x_A. With this understanding, the elimination of x_A becomes possible and one can formulate a new optimal control problem involving a state vector of reduced size. In other words, the augmented state vector (x,y) having dimension $n+k$ is replaced by the new state vector (x_B,y) having dimension $b+k$, with the following implication: if the algorithm of Ref. 5 is employed, the computer time per iteration is scaled according to the square of the ratio $(b+k)/(n+k)$.

In theory, the transformation technique proposed here appears to have a very particular nature. In practice, a large number of technical problems fall within the scheme described here. With this in mind, the advantages of the new transformation technique are illustrated through several examples, which are solved by means of the sequential gradient-restoration algorithm for optimal control problems with nondifferential constraints (Ref. 5). The examples show the substantial savings in computer time for convergence, which are associated with the new transformation technique.

2. Statement of the Problem

2.1. Notation. Let θ denote the independent variable, and let $x(\theta)$, $u(\theta)$ denote the dependent variables. The time θ is a scalar, the state $x(\theta)$ is an n-vector, and the control $u(\theta)$ is an m-vector.[6] The initial time is $\theta = 0$ and the final time is $\theta = \tau$. The final time is either given or free.[7]

2.2. Optimization Problem. With the above definitions, the optimization problem can be stated as follows.

Problem P1. Minimize the functional

$$I = \int_0^\tau f(x,u,\theta)d\theta + [g(x,\theta)]_\tau \tag{1}$$

with respect to the state $x(\theta)$, the control $u(\theta)$, and the parameter τ which satisfy the differential constraints

$$dx/d\theta = \phi(x,u,\theta), \qquad 0 \leq \theta \leq \tau, \tag{2}$$

the state inequality constraint

$$L_0(x,\theta) \geq 0, \qquad 0 \leq \theta \leq \tau, \tag{3}$$

and the boundary conditions

---

[6] All vectors are column vectors.

[7] In the latter case, τ is a parameter to be optimized.

$$x(0) = \text{given}, \tag{4}$$

$$[\psi(x,\theta)]_\tau = 0 \; . \tag{5}$$

In the above equations, the quantities I, f, g, L_0 are scalar, the function ϕ is an n-vector, and the function ψ is a q-vector.

3. Transformation of the Problem

In this section, we convert the inequality constrained problem into an equality constrained problem. We employ the Valentine-type transformation proposed by Jacobson in Ref. 4. We assume that the state inequality constraint (3) has order k. This means that the kth time derivative of the function $L_0(x,\theta)$ is the first to contain the control u explicitly.

3.1. Valentine-Type Transformation. We introduce the auxiliary state vector $y(\theta)$ and the auxiliary control variable $w(\theta)$, where y is a k-vector and w is a scalar. We require the k-vector y and the scalar w to satisfy the differential equations

$$
\begin{aligned}
dy_1/d\theta &= y_2 ,\\
dy_2/d\theta &= y_3 ,\\
\dots\dots\dots\dots &,\\
dy_{k-1}/d\theta &= y_k ,\\
dy_k/d\theta &= w .
\end{aligned}
\tag{6}
$$

Next, we discard the state inequality constraint (3) and replace it with the state equality constraint

$$
L_0(x,\theta) - A_0(y) = 0, \qquad A_0(y) = y_1^2, \tag{7}
$$

where $y_1(\theta)$ denotes the first component of the auxiliary state vector $y(\theta)$. Then, we execute k time derivatives of both sides of Eq. (7). This yields the relations

$$
\begin{aligned}
&L_1(x,\theta) - A_1(y) = 0 , \\
&L_2(x,\theta) - A_2(y) = 0 , \\
&L_3(x,\theta) - A_3(y) = 0 , \\
&L_4(x,\theta) - A_4(y) = 0 , \\
&\ldots\ldots\ldots\ldots\ldots\ldots , \\
&L_{k-1}(x,\theta) - A_{k-1}(y) = 0 ,
\end{aligned}
\tag{8}
$$

and

$$M_k(x,u,\theta) - B_k(y,w) = 0. \tag{9}$$

In Eqs. (8)-(9), the symbols L_i and M_k denote the successive total derivatives of $L_0(x,\theta)$, and the symbols A_i and B_k denote the successive total derivatives of $A_0(y)$. Specifically, the derivatives $A_i(y), i = 0,1,\ldots,k-1$, are given by

$$
\begin{aligned}
A_0(y) &= y_1^2 , \\
A_1(y) &= 2y_1y_2 , \\
A_2(y) &= 2y_2^2 + 2y_1y_3 , \\
A_3(y) &= 6y_2y_3 + 2y_1y_4 , \\
A_4(y) &= 6y_3^2 + 8y_2y_4 + 2y_1y_5 , \\
&\ldots\ldots\ldots\ldots ,
\end{aligned} \tag{10}
$$

and the derivative $B_k(y,w)$ is given by

$$
\begin{aligned}
B_1(y,w) &= 2y_1w, && \text{if} \quad k=1, \\
B_2(y,w) &= 2y_2^2 + 2y_1w , && \text{if} \quad k=2, \\
B_3(y,w) &= 6y_2y_3 + 2y_1w , && \text{if} \quad k=3, \\
B_4(y,w) &= 6y_3^2 + 8y_2y_4 + 2y_1w , && \text{if} \quad k=4, \\
B_5(y,w) &= 20y_3y_4 + 10y_2y_5 + 2y_1w, && \text{if} \quad k=5, \\
&\ldots\ldots\ldots\ldots .
\end{aligned} \tag{11}
$$

Note that Eqs. (7) and (8) are successive first integrals of Eq. (9). Also note that Eqs. (7)-(8), when applied at the initial point, yield the initial conditions for the auxiliary state vector. Since x(0) is given, Eqs. (7)-(8) imply that

$$
y(0) = \text{given} . \tag{12}
$$

Summarizing, the state inequality constraint (3) can be replaced by the state equality constraint (7). In turn, (7) can be replaced by the control equality constraint (9), which is to be employed in combination with the differential equations (6) and the initial conditions (12).

3.2. Notation. In more compact notation, the differential equations (6) can be rewritten in the form

$$dy/d\theta = \omega(y,w) , \qquad 0 \leq \theta \leq \tau , \tag{13}$$

where y and ω denote the k-vectors

$$y = [y_1, y_2, \ldots\ldots, y_{k-1}, y_k]^T ,$$
$$\omega(y,w) = [y_2, y_3, \ldots\ldots, y_k, w]^T . \tag{14}$$

In turn, the relations (7)-(8) can be rewritten as

$$L(x,\theta) - A(y) = 0 , \tag{15}$$

where $L(x,\theta)$ and $A(y)$ denote the k-vectors

$$L(x,\theta) = [L_0(x,\theta), L_1(x,\theta), \ldots\ldots, L_{k-1}(x,\theta)]^T ,$$
$$A(y) = [A_0(y), A_1(y), \ldots\ldots, A_{k-1}(y)]^T . \tag{16}$$

Finally, the relation (9) can be rewritten as

$$S(x,y,u,w,\theta) = 0 , \tag{17}$$

where

$$S(x,y,u,w,\theta) = M_k(x,u,\theta) - B_k(y,w) . \tag{18}$$

With the aid of this notation, Problem P1 can be reformulated as follows.

<u>Problem P2</u>. Minimize the functional

$$I = \int_0^\tau f(x,u,\theta)d\theta + [g(x,\theta)]_\tau \tag{19}$$

with respect to the state $x(\theta)$, $y(\theta)$, the control $u(\theta), w(\theta)$, and the parameter τ which satisfy the differential constraints

$$dx/d\theta = \phi(x,u,\theta) , \qquad 0 \leq \theta \leq \tau , \tag{20}$$

$$dy/d\theta = \omega(y,w) , \qquad 0 \leq \theta \leq \tau , \tag{21}$$

the nondifferential constraint

$$S(x,y,u,w,\theta) = 0, \qquad 0 \leq \theta \leq \tau, \tag{22}$$

and the boundary conditions

$$x(0) = \text{given}, \qquad y(0) = \text{given}, \tag{23}$$

$$[\psi(x,\theta)]_\tau = 0 . \tag{24}$$

In the above equations, the quantities I,f,g,S are scalar, the function ϕ is an n-vector, the function ω is a k-vector, and the function ψ is a q-vector.

Note that the vectors $x(\theta)$ and $y(\theta)$ are automatically consistent with the relation

$$L(x,\theta) - A(y) = 0, \qquad 0 \leq \theta \leq \tau, \tag{25}$$

at every point of the trajectory, providing they are consistent with (25) at the initial point. Therefore, the initial conditions (23) must be such that

$$L(x(0),0) - A(y(0)) = 0 . \tag{26}$$

3.3. <u>Partitioning of the State Vector</u>. Assume now that the n-vector state $x(\theta)$ is partitioned into vectors $x_A(\theta)$ and $x_B(\theta)$ having dimensions a and b, respectively, such that

$$a + b = n, \qquad 0 \leq a \leq k , \qquad n-k \leq b \leq n . \tag{27}$$

Under these conditions, Problem P2 can be reformulated as follows.

<u>Problem P2</u>. Minimize the functional

$$I = \int_0^\tau f(x_A, x_B, u, \theta) d\theta + [g(x_A, x_B, \theta)]_\tau \tag{28}$$

with respect to the state $x_A(\theta)$, $x_B(\theta)$, $y(\theta)$, the control $u(\theta)$, $w(\theta)$, and the parameter τ which satisfy the differential constraints

$$dx_A/d\theta = \phi_A(x_A, x_B, u, \theta), \qquad 0 \le \theta \le \tau, \tag{29}$$

$$dx_B/d\theta = \phi_B(x_A, x_B, u, \theta), \qquad 0 \le \theta \le \tau, \tag{30}$$

$$dy/d\theta = \omega(y, w), \qquad 0 \le \theta \le \tau, \tag{31}$$

the nondifferential constraint

$$S(x_A, x_B, y, u, w, \theta) = 0, \qquad 0 \le \theta \le \tau, \tag{32}$$

and the boundary conditions

$$x_A(0) = \text{given}, \qquad x_B(0) = \text{given}, \qquad y(0) = \text{given}, \tag{33}$$

$$[\psi(x_A, x_B, \theta)]_\tau = 0. \tag{34}$$

In the above equations, the quantities I, f, g, S are scalar, the function ϕ_A is an a-vector, the function ϕ_B is a b-vector, the function ω is a k-vector, and the function ψ is a q-vector.

Note that the vectors $x_A(\theta)$, $x_B(\theta)$, $y(\theta)$ are automatically consistent with the relation

$$L(x_A, x_B, \theta) - A(y) = 0, \qquad 0 \le \theta \le \tau, \tag{35}$$

at any point of the trajectory, providing they are consistent with (35) at the initial point. Therefore, the initial conditions (33) must be such that

$$L(x_A(0), x_B(0), 0) - A(y(0)) = 0. \tag{36}$$

Full advantage of the partitioning of the state vector x into vectors x_A and x_B is taken in the following section.

4. Further Transformation

In many technical applications, the left-hand side of the state inequality constraint $L_0(x,\theta)$ is linear in some or all of the components of the state vector. In addition, depending on the form of the differential system, some of the successive total derivatives $L_i(x,\theta)$ might be linear in some or all of the components of the state vector. In other words, it is plausible that situations exist such that the following properties hold.

(i) The n-vector state $x(\theta)$ can be partitioned into vectors $x_A(\theta)$ and $x_B(\theta)$ having dimensions a and b, respectively, such that

$$a+b=n, \qquad 0 \le a \le k, \qquad n-k \le b \le n . \tag{37}$$

(ii) The k-vector function $L(x_A,x_B,\theta)$ can be partitioned into two vector functions $\tilde{L}(x_A,x_B,\theta)$ and $\hat{L}(x_A,x_B,\theta)$ having dimensions a and c, respectively, such that

$$a+c=k, \qquad 0 \le a \le k, \qquad 0 \le c \le k . \tag{38}$$

(iii) The a-vector function $\tilde{L}(x_A,x_B,\theta)$ is linear in x_A, that is,

$$\tilde{L}(x_A,x_B,\theta) = M(\theta)x_A + N(x_B,\theta), \tag{39}$$

where $M(\theta)$ is an $a \times a$ matrix and $N(x_B,\theta)$ is an a-vector.

Under the above conditions, important simplifications are possible. If the square matrix $M(\theta)$ is nonsingular, Eq. (39)

admits the formal solution

$$x_A = M^{-1}(\theta)[\tilde{L}(x_A, x_B, \theta) - N(x_B, \theta)] , \tag{40}$$

which in the light of (35) can be rewritten as

$$x_A = M^{-1}(\theta)[\tilde{A}(y) - N(x_B, \theta)] . \tag{41}$$

Here, $\tilde{A}(y)$ is obtained by partitioning the k-vector function $A(y)$ into two vector functions $\tilde{A}(y)$ and $\hat{A}(y)$ having dimensions a and c, respectively.

Under the above premises, the elimination of x_A becomes possible, and one can formulate a new optimal control problem involving the reduced state vector x_B as well as the auxiliary state vector y.

Problem P3. Minimize the functional

$$I = \int_0^\tau f(x_B, y, u, \theta) d\theta + [g(x_B, y, \theta)]_\tau \tag{42}$$

with respect to the state $x_B(\theta), y(\theta)$, the control $u(\theta), w(\theta)$, and the parameter τ which satisfy the differential constraints

$$dx_B/d\theta = \phi_B(x_B, y, u, \theta), \qquad 0 \leq \theta \leq \tau , \tag{43}$$

$$dy/d\theta = \omega(y, w), \qquad 0 \leq \theta \leq \tau , \tag{44}$$

the nondifferential constraint

$$S(x_B, y, u, w, \theta) = 0, \qquad 0 \leq \theta \leq \tau, \tag{45}$$

and the boundary conditions

$$x_B(0) = \text{given}, \qquad y(0) = \text{given}, \tag{46}$$

$$[\psi(x_B, y, \theta)]_\tau = 0 . \tag{47}$$

In the above equations, the quantities I,f,g,S are scalar, the function ϕ_B is a b-vector, the function ω is a k-vector, and the function ψ is a q-vector.[8]

4.1. Remark. Comparison of Problem P2 with Problem P3 shows that the augmented state vector (x,y) having dimension $n + k$ has been replaced by the new state vector (x_B, y) having dimension $b + k$, with the following implication: if the algorithm of Ref. 5 is employed, the computer time per iteration is scaled according to the square of the ratio $(b + k)/(n + k)$.

In theory, the transformation technique proposed here appears to have a very particular nature. In practice, a large number of technical problems fall within the scheme described here. With this in mind, the advantages of the new transformation technique are illustrated in the following sections through several examples, which are solved by means of the sequential gradient-restoration algorithm for optimal control problems with nondifferential constraints (Ref. 5). The examples show the substantial savings in computer time for convergence, which are associated with the new transformation technique.

[8] After Problem P3 is solved and the functions $x_B(\theta)$, $y(\theta)$ are known, the function $x_A(\theta)$ can be computed a posteriori with (41).

5. Description of the Algorithm

With the sequential ordinary gradient-restoration algorithm for optimal control problems involving nondifferential constraints (SOGRA, Refs. 5-6), numerical solutions can be obtained using either the formulation of Problem P2 or the formulation of Problem P3.

5.1. Time Normalization. The sequential ordinary gradient-restoration algorithm (SOGRA) requires that the actual time θ be replaced by the normalized time

$$t = \theta/\tau , \qquad 0 \leq t \leq 1 , \tag{48}$$

defined in such a way that $t = 0$ at the initial point and $t = 1$ at the final point. The actual final time τ, if it is free, becomes a parameter to be optimized.[9] In this way, an optimal control problem with variable final time is converted into an optimal control problem with fixed final time. Even though this replacement of independent variable is necessary computationally, it was not employed in the description of Problems P1 through P3, for the sake of brevity. However, this

[9]The dimension of the parameter is $p = 0$ if τ is fixed and $p = 1$ if τ is free.

replacement of independent variable is employed in the description of the numerical examples of Section 7.

5.2. Auxiliary Functionals. In addition to the functional I, several auxiliary functionals are of interest in the implementation of SOGRA. These auxiliary functionals are denoted by the symbols J, P, Q and have the following meaning: J is the augmented functional, that is, the functional I augmented linearly by the constraints through Lagrange multipliers; P is the constraint error, that is, the norm squared of the error in the feasibility equations; and Q is the optimality condition error, that is, the norm squared of the error in the optimality conditions. In the definitions of J, P, Q, it is tacitly assumed that the given initial conditions are satisfied at every iteration of SOGRA.

5.3. Convergence Conditions. The auxiliary functionals P and Q are defined in such a way that

$$P = Q = 0 \tag{49}$$

for the optimal solution. For an approximation to the optimal solution, Eqs. (49) are replaced by the inequalities

$$P \leq \varepsilon_1, \qquad Q \leq \varepsilon_2, \tag{50}$$

where ε_1 and ε_2 are small, preselected numbers.

5.4. <u>Sequential Ordinary Gradient-Restoration Algorithm</u>. The sequential ordinary gradient-restoration algorithm (SOGRA, Refs. 5-6) is an iterative technique which includes a sequence of cycles having the following properties: (i) the functions available both at the beginning and at the end of each cycle are feasible; that is, they are consistent with the feasibility equations within the preselected accuracy (50-1); and (ii) the functions produced at the end of each cycle are characterized by a value of the the functional I which is smaller than that associated with the functions available at the beginning of the cycle.

To achieve the above properties, each cycle is made of two phases, a gradient phase and a restoration phase. These phases are now described.

The gradient phase is started only when the constraint error P satisfies Ineq. (50-1). It involves a single iteration, which is designed to decrease the value of the functional I or the augmented functional J, while satisfying the constraints to first order. During this iteration, the first variation of the functional I is minimized, subject to the linearized constraints and a quadratic constraint on the variations of the control and the parameter.

The restoration phase is started only when the constraint error P violates Ineq. (50-1). The restoration phase involves one or more iterations. In each restorative iteration, the objective is to reduce the constraint error P, while the constraints are satisfied to first order and the norm squared of the variations of the control and the parameter is minimized. The restoration phase is terminated whenever Ineq. (50-1) is satisfied.

In summary, the main properties of the sequential ordinary gradient-restoration algorithm can be written as follows:

$$\tilde{J} < J, \qquad P \leq \varepsilon_1, \tag{51}$$

$$\tilde{P} < P, \qquad P > \varepsilon_1, \tag{52}$$

$$\hat{I} < I, \qquad P \leq \varepsilon_1, \qquad \hat{P} \leq \varepsilon_1, \tag{53}$$

where (51) hold for a gradient iteration, (52) hold for a restorative iteration, and (53) hold for a complete gradient-restoration cycle. In the above relations, I, J, P denote quantities at the beginning of an iteration or a cycle; $\tilde{I}$, $\tilde{J}$, $\tilde{P}$ denote quantities at the end of an iteration; and $\hat{I}$, $\hat{J}$, $\hat{P}$ denote quantities at the end of a cycle.

Each iteration, gradient or restorative, involves two distinct operations: (i) the computation of the displacements per unit stepsize and (ii) the computation of the stepsize α. Here, the terminology of Refs. 5-6 is employed. Operation (i) entails the solution of a linear, two-point boundary-value problem by means of the method of particular solutions (Refs. 7-9). Operation (ii) entails the execution of a one-dimensional search (Refs. 10-11).

5.5. Computational Effort. At each iteration of the gradient phase or the restoration phase, a linear, two-point boundary-value problem must be solved. If the method of particular solutions is employed (Refs. 7-9), one must execute $n_i + p + 1$ independent sweeps of the linearized system governing the variations associated with the gradient phase or the restoration phase, where n_i is the dimension of the state vector and p is the dimension of the parameter.[10] Note that

$$n_2 = n + k \quad \text{for Problem P2,} \tag{54-1}$$

$$n_3 = b + k \quad \text{for Problem P3.} \tag{54-2}$$

Hence, the following factor is indicative of the algorithmic time per iteration:

$$W_i = 2n_i(n_i + p + 1), \qquad i = 2 \text{ or } 3, \tag{55}$$

with the implication that

$$W_2 = 2(n + k)(n + k + p + 1) \quad \text{for Problem P2,} \tag{56}$$

$$W_3 = 2(b + k)(b + k + p + 1) \quad \text{for Problem P3.} \tag{57}$$

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[10] The subscript $i = 2$ is employed for Problem P2, and the subscript $i = 3$ is employed for Problem P3.

5.6. <u>Gradient Stepsize</u>. The computation of the gradient stepsize requires the consideration of the functions

$$\tilde{J} = \tilde{J}(\alpha), \qquad \tilde{P} = \tilde{P}(\alpha). \tag{58}$$

Then, a one-dimensional search scheme is applied to (58-1), and a value of the stepsize α is selected for which the following relations are satisfied:

$$\tilde{J}(\alpha) \leq \tilde{J}(0), \qquad \tilde{P}(\alpha) \leq P_*, \qquad \tilde{\tau}(\alpha) \geq 0, \tag{59}$$

where τ is the final time and P_* is a preselected number, not necessarily small. Satisfaction of Ineq. (59-1) is possible because of the descent property of the gradient phase. Ineq. (59-2) is introduced to prevent excessive constraint violation. And Ineq. (59-3) is required for problems with free final time.

Prior to the satisfaction of (59), a scanning process is employed, leading to the bracketing of the minimum point for $\tilde{J}(\alpha)$. This operation is then followed by a Hermitian cubic interpolation process (Refs. 10-11), which is stopped whenever the following relation is satisfied:[11]

$$|\tilde{J}_\alpha(\alpha)| \leq \varepsilon_3 \quad \text{or} \quad |\tilde{J}_\alpha(\alpha)/\tilde{J}_\alpha(0)| \leq \varepsilon_4, \tag{60}$$

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[11] The symbols ε_3 and ε_4 denote small, preselected numbers.

subject to an upper limit for the number of search steps N_s. Once a stepsize α_o has been selected consistently with either (60) or the prescribed upper limit for the number of search steps, Ineqs. (59) must be checked. If satisfaction occurs, then the stepsize α_o is accepted. If any violation occurs, then the stepsize α_o must be bisected progressively until satisfaction of (59) is finally achieved.

5.7. Restoration Stepsize. The computation of the restoration stepsize requires the consideration of the function

$$\tilde{P} = \tilde{P}(\alpha) . \tag{61}$$

Then, the stepsize α must be selected so that the following relations are satisfied:

$$\tilde{P}(\alpha) < \tilde{P}(0), \qquad \tilde{\tau}(\alpha) \geq 0 . \tag{62}$$

Satisfaction of Ineq. (62-1) is possible because of the descent property of the restoration phase. Ineq. (62-2) is required for problems with free final time.

In order to achieve satisfaction of (62), a bisection process is applied to the restoration stepsize α, starting from the reference stepsize $\alpha_o = 1$. This reference stepsize has the property of yielding one-step restoration for the special case where all the constraints are linear.

5.8. <u>Iterative Procedure for the Restoration Phase</u>. The descent property (52-1) of the restoration phase guarantees satisfaction of Ineq. (62-1) at the end of any iteration, but not satisfaction of Ineq. (50-1). Therefore, the restoration algorithm must be employed iteratively until Ineq. (50-1) is satisfied. At this point, the restoration phase is terminated.

5.9. <u>Descent Property of a Cycle</u>. A descent property exists for a complete gradient-restoration cycle under the assumption of small stepsizes. More specifically, let I, $\tilde{I}$, $\hat{I}$ denote the values of the functional under consideration at the beginning of the gradient phase, at the end of the gradient phase, and at the end of the subsequent restoration phase. Note that I and $\tilde{I}$ are not comparable, since the constraints are not satisfied to the same accuracy. On the other hand, I and $\hat{I}$ are comparable, and the gradient stepsize can be selected so that

$$\hat{I} < I . \tag{63}$$

This inequality constitutes the descent property of a complete gradient-restoration cycle. In order to enforce it, one proceeds as follows. At the end of the restoration phase, one must verify Ineq. (63). If it is satisfied, the next gradient phase is started; otherwise, the previous gradient stepsize is bisected as many times as needed until, after restoration, Ineq. (63) is satisfied.

6. Experimental Conditions

In order to evaluate the transformation techniques discussed in Sections 3-4, several numerical examples were solved. The sequential gradient-restoration algorithm of Ref. 5 was programmed in FORTRAN IV, and the numerical results were obtained in double-precision arithmetic.

Computations were performed at Rice University using an IBM 370/155 computer. For each example, the interval of integration was divided into 50 steps. The differential equations were integrated using Hamming's modified predictor-corrector method with a special Runge-Kutta starting procedure (Ref. 12). The definite integrals I, J, P, Q were computed using a modified Simpson's rule. The method of particular solutions (Refs. 7-9) was used to solve the linear, two-point boundary-value problems associated with both the gradient phase and the restoration phase.

6.1. Convergence Conditions. The parameters ε_1, ε_2, ε_4 appearing in Ineqs. (50) and (60) were set at the levels[12]

$$\varepsilon_1 = E-08, \qquad \varepsilon_2 = E-04, \qquad \varepsilon_4 = E-03 . \tag{64}$$

The tolerance level (64-1) characterizes the restoration phase;

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[12] The symbol $E \pm ab$ stands for $10^{\pm ab}$.

the tolerance levels (64-1) and (64-2), employed in combination, characterize the algorithm as a whole; and the tolerance level (64-3) characterizes the one-dimensional search for the gradient stepsize.

6.2. <u>Safeguards</u>. For the gradient phase, the parameter P_* appearing in Ineq. (59-2) was set at the level

$$P_* = 10 \,. \tag{65}$$

The tolerance level (65) limits the constraint violation which is permissible during the gradient phase. Also for the gradient phase, the number of Hermitian search steps N_s required to satisfy Ineq. (60) was subject to the upper bound

$$N_s \leq 5 \,. \tag{66}$$

6.3. <u>Nonconvergence Conditions</u>. The sequential gradient-restoration algorithm was programmed to stop whenever violation of any of the following inequalities occurred:[13]

$$N_c \leq 30 \,, \qquad N \leq 100, \qquad N_r \leq 10, \tag{67}$$

$$N_{bg} \leq 10, \qquad N_{br} \leq 10, \qquad N_{bc} \leq 5, \tag{68}$$

$$M \leq 0.83\,E+75 \,. \tag{69}$$

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[13]Inequality (69) is characteristic of the IBM 370/155 computer.

Here, N_c is the number of cycles, N is the total number of iterations, N_r is the number of restorative iterations per cycle, N_{bg} is the number of bisections of the gradient stepsize required to satisfy Ineqs. (59), N_{br} is the number of bisections of the restoration stepsize required to satisfy Ineqs. (62), N_{bc} is the number of bisections of the gradient stepsize required to satisfy Ineq. (63), and M is the modulus of any of the quantities employed in the algorithm.

7. Numerical Examples

In this section, eight numerical examples are described employing scalar notation. In particular, the symbols $x_i(\theta), i = 1, \ldots, n$, denote the components of the original state vector; the symbols $y_i(\theta), i = 1, \ldots, k$, denote the components of the auxiliary state vector; the symbol $u(\theta)$ denotes the original control variable (a scalar in the examples reported here); the symbol $w(\theta)$ denotes the auxiliary control variable (a scalar); and the symbol τ denotes the final time.

For all of the examples, a time normalization is used in order to simplify the numerical computations. Specifically, the actual time θ is replaced by the normalized time

$$t = \theta/\tau , \tag{70}$$

which is defined in such a way that $t = 0$ at the initial point and $t = 1$ at the final point. The actual final time τ, if it is free, is regarded as a parameter to be optimized. Thus, the dimension of the parameter is $p = 0$ if τ is fixed and $p = 1$ if τ is free.

The dot denotes derivative with respect to the normalized time, i.e.,

$$\dot{x}_1 = dx_1/dt, \qquad \dot{x}_2 = dx_2/dt, \ldots, \qquad \dot{x}_n = dx_n/dt, \tag{71}$$

$$\dot{y}_1 = dy_1/dt, \qquad \dot{y}_2 = dy_2/dt, \ldots, \qquad \dot{y}_k = dy_k/dt . \tag{72}$$

Concerning the convergence history, the terminology is as follows. N_c denotes the cycle number, N_g is the number of gradient iterations per cycle, N_r is the number of restorative iterations per cycle, N is the total number of iterations, P is the constraint error, Q is the error in the optimality conditions, and I is the value of the functional being optimized.

Example 7.1. This example involves (i) boundary conditions of the fixed endpoint type, (ii) fixed final time $\tau = 1$, and (iii) a linear state inequality constraint of the first order. Problem P1 is as follows:

$$I = \int_0^1 (x_1^2 + u^2)\,dt\,, \tag{73-1}$$

$$\dot{x}_1 = x_1^2 - u\,, \tag{73-2}$$

$$x_1 - 0.9 \geq 0\,, \tag{73-3}$$

$$x_1(0) = 1, \tag{73-4}$$

$$x_1(1) = 1\,. \tag{73-5}$$

In this formulation, the unknowns are the state variable $x_1(t)$ and the control variable u(t).

We introduce the auxiliary state variable $y_1(t)$ and the auxiliary control variable w(t) defined by

$$x_1 - 0.9 = y_1^2 \quad , \quad \dot{y}_1 = w . \tag{74}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 (x_1^2 + u^2)dt , \tag{75-1}$$

$$\dot{x}_1 = x_1^2 - u , \qquad \dot{y}_1 = w , \tag{75-2}$$

$$x_1^2 - u - 2y_1 w = 0 , \tag{75-3}$$

$$x_1(0) = 1 , \qquad y_1(0) = \sqrt{(0.1)} , \tag{75-4}$$

$$x_1(1) = 1 . \tag{75-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $y_1(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0 is linear in x_1, the elimination of the state variable x_1 is possible through the relation

$$x_1 = 0.9 + y_1^2 . \tag{76}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 [(0.9 + y_1^2)^2 + u^2]dt , \tag{77-1}$$

$$\dot{y}_1 = w, \tag{77-2}$$

$$(0.9 + y_1^2)^2 - u - 2y_1 w = 0 , \qquad (77\text{-}3)$$

$$y_1(0) = \sqrt{(0.1)}, \qquad (77\text{-}4)$$

$$y_1(1) = \sqrt{(0.1)} . \qquad (77\text{-}5)$$

In this formulation, the unknowns are the state variable $y_1(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the function $x_1(t)$ can be computed a posteriori with (76).

In the computer runs, the following nominal functions are employed for Problem P2:

$$x_1(t) = 1, \qquad y_1(t) = \sqrt{(0.1)}, \qquad (78\text{-}1)$$

$$u(t) = 1, \qquad w(t) = 1 , \qquad (78\text{-}2)$$

and Problem P3:

$$y_1(t) = \sqrt{(0.1)}, \qquad (79\text{-}1)$$

$$u(t) = 1, \qquad w(t) = 1 . \qquad (79\text{-}2)$$

The numerical results are given in Tables 1-4. Convergence to the desired stopping condition occurs in $N = 12$ iterations for Problem P2 and $N = 10$ iterations for Problem P3. The CPU time for convergence is $T = 21.2$ sec for Problem P2 and $T = 13.3$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 37.3%.

Example 7.2. This example is a minimum time problem and involves (i) boundary conditions of the fixed endpoint type, (ii) variable final time τ, and (iii) a linear state inequality constraint of the first order. After introducing the normalized time (70), Problem P1 is as follows:

$$I = \tau , \tag{80-1}$$

$$\dot{x}_1 = \tau u , \qquad \dot{x}_2 = \tau (u^2 - x_1^2) , \tag{80-2}$$

$$0.4 - x_2 \geq 0 , \tag{80-3}$$

$$x_1(0) = 0, \qquad x_2(0) = 0 , \tag{80-4}$$

$$x_1(1) = 1, \qquad x_2(1) = 0 . \tag{80-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, the control variables $u(t)$, and the parameter τ.

We introduce the auxiliary state varible $y_1(t)$ and the auxiliary control variable $w(t)$ defined by

$$0.4 - x_2 = y_1^2, \qquad \dot{y}_1 = \tau w . \tag{81}$$

With this understanding, Problem P2 is represented by

$$I = \tau , \tag{82-1}$$

$$\dot{x}_1 = \tau u, \qquad \dot{x}_2 = \tau (u^2 - x_1^2) , \qquad \dot{y}_1 = \tau w, \tag{82-2}$$

$$x_1^2 - u^2 - 2y_1 w = 0, \tag{82-3}$$

$$x_1(0) = 0, \qquad x_2(0) = 0, \qquad y_1(0) = \surd(0.4), \tag{82-4}$$

$$x_1(1) = 1, \qquad x_2(1) = 0\,. \tag{82-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, $y_1(t)$, the control variables $u(t)$, $w(t)$, and the parameter τ.

Since L_0 is linear in x_2, the elimination of the state variable x_2 is possible through the relation

$$\dot{x}_2 = 0.4 - y_1^2\,. \tag{83}$$

As a consequence, Problem P3 is represented by

$$I = \tau\,, \tag{84-1}$$

$$\dot{x}_1 = \tau u\,, \qquad \dot{y}_1 = \tau w\,, \tag{84-2}$$

$$x_1^2 - u^2 - 2y_1 w = 0\,, \tag{84-3}$$

$$x_1(0) = 0, \qquad y_1(0) = \surd(0.4), \tag{84-4}$$

$$x_1(1) = 1, \qquad y_1(1) = \surd(0.4). \tag{84-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $y_1(t)$, the control variables $u(t)$, $w(t)$, and the parameter τ. After Problem P3 is solved, the function $x_2(t)$ can be computed a posteriori with (83).

In the computer runs, the following nominal functions are

employed for Problem P2:

$$x_1(t) = t\,, \qquad x_2(t) = 0\,, \qquad y_1(t) = \surd(0.4), \tag{85-1}$$

$$u(t) = 1\,, \qquad w(t) = 1\,, \qquad \tau = 1\,, \tag{85-2}$$

and Problem P3:

$$x_1(t) = t\,, \qquad y_1(t) = \surd(0.4)\,, \tag{86-1}$$

$$u(t) = 1, \qquad w(t) = 1, \qquad \tau = 1\,. \tag{86-2}$$

The numerical results are given in Tables 5-8. Convergence to the desired stopping conditions occurs in $N = 34$ iterations for Problem P2 and $N = 35$ iterations for Problem P3. The CPU time is $T = 103.3$ sec for Problem P2 and $T = 75.9$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 26.6%.

<u>Example 7.3</u>. This example involves (i) boundary conditions of the fixed endpoint type, (ii) fixed final time $\tau = 1$, and (iii) a linear state inequality constraint of the first order. Problem P1 is as follows:

$$I = \int_0^1 (x_1^2 + u^2)\,dt\,, \tag{87-1}$$

$$\dot{x}_1 = x_1^2 - u\,, \tag{87-2}$$

$$x_1 - 0.8 - t + t^2 \geq 0\,, \tag{87-3}$$

$$x_1(0) = 1 , \tag{87-4}$$

$$x_1(1) = 1 . \tag{87-5}$$

In this formulation, the unknowns are the state variable $x_1(t)$ and the control variable $u(t)$.

We introduce the auxiliary state variable $y_1(t)$ and the auxiliary control variable $w(t)$ defined by

$$x_1 - 0.8 - t + t^2 = y_1^2 , \qquad \dot{y}_1 = w . \tag{88}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 (x_1^2 + u^2)dt , \tag{89-1}$$

$$\dot{x}_1 = x_1^2 - u , \qquad \dot{y}_1 = w , \tag{89-2}$$

$$x_1^2 - u - 1 + 2t - 2y_1 w = 0 , \tag{89-3}$$

$$x_1(0) = 1, \qquad y_1(0) = \sqrt{(0.2)} , \tag{89-4}$$

$$x_1(1) = 1 . \tag{89-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $y_1(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0 is linear in x_1, the elimination of the state variable x_1 is possible through the relation

$$x_1 = 0.8 + t - t^2 + y_1^2 . \tag{90}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 [(0.8 + t - t^2 + y_1^2)^2 + u^2]dt , \tag{91-1}$$

$$\dot{y}_1 = w , \tag{91-2}$$

$$(0.8 + t - t^2 + y_1^2)^2 - u - 1 + 2t - 2y_1 w = 0 , \tag{91-3}$$

$$y_1(0) = \surd(0.2) , \tag{91-4}$$

$$y_1(1) = \surd(0.2) . \tag{91-5}$$

In this formulation, the unknowns are the state variable $y_1(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the function $x_1(t)$ can be computed a posteriori with (90).

In the computer runs, the following nominal functions are employed for Problem P2:

$$x_1(t) = 1 , \qquad y_1(t) = \surd(0.2), \tag{92-1}$$

$$u(t) = 1, \qquad w(t) = 1 , \tag{92-2}$$

and Problem P3:

$$y_1(t) = \surd(0.2) , \tag{93-1}$$

$$u(t) = 1 , \qquad w(t) = 1 . \tag{93-2}$$

The numerical results are given in Tables 9-12. Convergence to the desired stopping condition occurs in $N = 24$ iterations for Problem P2 and $N = 23$ iterations for Problem P3. The CPU time is $T = 43.9$ sec for Problem P2 and $T = 29.4$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 33.2%.

Example 7.4. This example involves (i) boundary conditions of the fixed endpoint type, (ii) fixed final time $\tau = \pi/2$, and (iii) a linear state inequality constraint of the second order. Problem P1 is as follows:

$$I = \int_0^1 \tau(u^2 - x_1^2 + \tau t)dt \,, \tag{94-1}$$

$$\dot{x}_1 = \tau u \,, \qquad \dot{x}_2 = \tau(2 - 4x_1^2) \,, \tag{94-2}$$

$$1 - x_2 \geq 0 \,, \tag{94-3}$$

$$x_1(0) = 0, \qquad x_2(0) = 0 \,, \tag{94-4}$$

$$x_1(1) = 1, \qquad x_2(1) = 0 \,. \tag{94-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$ and the control variable $u(t)$.

We introduce the auxiliary state variables $y_1(t)$, $y_2(t)$ and the auxiliary control variable $w(t)$ defined by

$$1 - x_2 = y_1^2 , \qquad \dot{y}_1 = \tau y_2 , \qquad \dot{y}_2 = \tau w . \tag{95}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 \tau(u^2 - x_1^2 + \tau t)dt , \tag{96-1}$$

$$\dot{x}_1 = \tau u, \quad \dot{x}_2 = \tau(2 - 4x_1^2), \quad \dot{y}_1 = \tau y_2, \quad \dot{y}_2 = \tau w, \tag{96-2}$$

$$4x_1 u - y_2^2 - y_1 w = 0, \tag{96-3}$$

$$x_1(0) = 0, \quad x_2(0) = 0, \quad y_1(0) = 1, \quad y_2(0) = -1, \tag{96-4}$$

$$x_1(1) = 1, \quad x_2(1) = 0 . \tag{96-5}$$

In this formulation, the *unknowns* are the state variables $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0 is linear in x_2, the elimination of the state variable x_2 is possible through the relation

$$x_2 = 1 - y_1^2 . \tag{97}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 \tau(u^2 - x_1^2 + \tau t)dt , \tag{98-1}$$

$$\dot{x}_1 = \tau u , \qquad \dot{y}_1 = \tau y_2 , \qquad \dot{y}_2 = \tau w , \tag{98-2}$$

$$4x_1 u - y_2^2 - y_1 w = 0 , \tag{98-3}$$

$$x_1(0) = 0\,, \qquad y_1(0) = 1\,, \qquad y_2(0) = -1\,, \tag{98-4}$$

$$x_1(1) = 1\,, \qquad y_1(1) = -1\,. \tag{98-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $y_1(t)$, $y_2(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the function $x_2(t)$ can be computed a posteriori with (97).

In the computer runs, the following nominal functions are employed for Problem P2:

$$x_1(t) = t, \quad x_2(t) = 0, \quad y_1(t) = 1-2t, \quad y_2(t) = -1, \tag{99-1}$$

$$u(t) = 1/\tau, \quad w(t) = 0, \tag{99-2}$$

and Problem P3:

$$x_1(t) = t, \quad y_1(t) = 1 - 2t, \quad y_2(t) = -1, \tag{100-1}$$

$$u(t) = 1/\tau, \quad w(t) = 0\,. \tag{100-2}$$

The numerical results are given in Tables 13-16. Convergence to the desired stopping condition occurs in $N = 17$ iterations for Problem P2 and $N = 6$ iterations for Problem P3. The CPU time is $T = 66.7$ sec for Problem P2 and $T = 20.0$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 70.1%.

Example 7.5. This example involves (i) boundary conditions of the fixed endpoint type, (ii) fixed final time $\tau = 1$, and (iii) a linear state inequality constraint of the second order. Problem P1 is as follows:

$$I = \int_0^1 u^2 dt , \tag{101-1}$$

$$\dot{x}_1 = x_2 , \qquad \dot{x}_2 = u , \tag{101-2}$$

$$0.15 - x_1 \geq 0 , \tag{101-3}$$

$$x_1(0) = 0, \qquad x_2(0) = 1, \tag{101-4}$$

$$x_1(1) = 0, \qquad x_2(1) = -1. \tag{101-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$ and the control variable $u(t)$.

We introduce the auxiliary state variables $y_1(t)$, $y_2(t)$ and the auxiliary control variable $w(t)$ defined by

$$0.15 - x_1 = y_1^2 , \qquad \dot{y}_1 = y_2, \qquad \dot{y}_2 = w . \tag{102}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 u^2 dt , \tag{103-1}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \qquad \dot{y}_1 = y_2, \qquad \dot{y}_2 = w , \tag{103-2}$$

$$u + 2y_2^2 + 2y_1 w = 0 , \tag{103-3}$$

$$x_1(0)=0, \quad x_2(0)=1, \quad y_1(0)=\surd(0.15), \quad y_2(0)=-1/\surd(0.60), \tag{103-4}$$

$$x_1(1)=0, \quad x_2(1)=-1. \tag{103-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0 and L_1 are linear in x_1 and x_2, the elimination of the state variables x_1 and x_2 is possible through the relations

$$x_1 = 0.15 - y_1^2, \qquad x_2 = -2y_1y_2. \tag{104}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 u^2 dt, \tag{105-1}$$

$$\dot{y}_1 = y_2, \qquad \dot{y}_2 = w, \tag{105-2}$$

$$u + 2y_2^2 + 2y_1 w = 0, \tag{105-3}$$

$$y_1(0) = \surd(0.15), \qquad y_2(0)=-1/\surd(0.60), \tag{105-4}$$

$$y_1(1) = \surd(0.15), \qquad y_2(1)= 1/\surd(0.60). \tag{105-5}$$

In this formulation, the unknowns are the state variable $y_1(t)$, $y_2(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the functions $x_1(t)$ and $x_2(t)$ can be computed a posteriori with (104).

In the computer runs, the following nominal functions are

employed for Problem P2:

$$x_1(t)=0, \quad x_2(t)=1-2t, \quad y_1(t)=\surd(0.15)(2t-1)^2, \quad y_2(t)=(2t-1)/\surd(0.60), \tag{106-1}$$

$$u(t) = -(2t-1)^2, \qquad w(t) = 1, \tag{106-2}$$

and Problem P3:

$$y_1(t) = \surd(0.15)(2t-1)^2, \qquad y_2(t) = (2t-1)/\surd(0.60), \tag{107-1}$$

$$u(t) = -(2t-1)^2, \qquad w(t) = 1 . \tag{107-2}$$

The numerical results are given in Tables 17-20. Convergence to the desired stopping condition occurs in $N = 16$ iterations for Problem P2 and $N = 17$ iterations for Problem P3. The CPU time is $T = 63.4$ sec for Problem P2 and $T = 34.7$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 45.3%.

<u>Example 7.6</u>. This example involves (i) boundary conditions of the free endpoint type, (ii) fixed final time $\tau = 1$, and (iii) a linear state inequality constraint of the first order. Problem P1 is as follows:

$$I = \int_0^1 (x_1^2 + x_2^2 + u^2/200)\,dt , \tag{108-1}$$

$$\dot{x}_1 = x_2 , \qquad \dot{x}_2 = u - x_2, \tag{108-2}$$

$$8t^2 - 8t + 1.5 - x_2 \geq 0 , \tag{108-3}$$

$$x_1(0) = 0, \qquad x_2(0) = -1 . \tag{108-4}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$ and the control variable $u(t)$.

We introduce the auxiliary state variable $y_1(t)$ and the auxiliary control variable $w(t)$ defined by

$$8t^2 - 8t + 1.5 - x_2 = y_1^2 , \qquad \dot{y}_1 = w . \tag{109}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 (x_1^2 + x_2^2 + u^2/200)\,dt , \tag{110-1}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u - x_2, \qquad \dot{y}_1 = w , \tag{110-2}$$

$$16t - 8 - u + x_2 - 2y_1 w = 0 , \tag{110-3}$$

$$x_1(0) = 0, \qquad x_2(0) = -1, \qquad y_1(0) = \sqrt{(2.5)} . \tag{110-4}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, $y_1(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0 is linear in x_2, the elimination of the state variable x_2 is possible through the relation

$$x_2 = 8t^2 - 8t + 1.5 - y_1^2 . \tag{111}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 [x_1^2 + (8t^2 - 8t + 1.5 - y_1^2)^2 + u^2/200]dt\ , \tag{112-1}$$

$$\dot{x}_1 = 8t^2 - 8t + 1.5 - y_1^2\ , \qquad \dot{y}_1 = w, \tag{112-2}$$

$$8t^2 + 8t - 6.5 - u - y_1^2 - 2y_1 w = 0\ , \tag{112-3}$$

$$x_1(0) = 0, \qquad y_1(0) = \surd(2.5). \tag{112-4}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $y_1(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the function $x_2(t)$ can be computed a posteriori with (111).

In the computer runs, the following nominal functions are employed for Problem P2:

$$x_1(t) = 0, \qquad x_2(t) = -1, \qquad y_1(t) = \surd(2.5), \tag{113-1}$$

$$u(t) = 1, \qquad w(t) = 1\ , \tag{113-2}$$

and Problem P3:

$$x_1(t) = 0, \qquad y_1(t) = \surd(2.5), \tag{114-1}$$

$$u(t) = 1, \qquad w(t) = 1. \tag{114-2}$$

The numerical results are given in Tables 21-24. Convergence to the desired stopping condition occurs in $N = 19$ iterations for Problem P2 and $N = 19$ iterations for Problem P3. The

CPU time is T = 50.2 sec for Problem P2 and T = 36.5 sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 27.4%.

Example 7.7. This example involves (i) boundary conditions of the free endpoint type, (ii) fixed final time $\tau = 1$, and (iii) a linear state inequality constraint of the second order. Problem P1 is as follows:

$$I = \int_0^1 (x_1^2 + x_2^2 + u^2/200)dt, \tag{115-1}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u - x_2, \tag{115-2}$$

$$8t^2 - 8t + 1.5 - x_1 \geq 0, \tag{115-3}$$

$$x_1(0) = 0, \qquad x_2(0) = -1. \tag{115-4}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$ and the control variable $u(t)$.

We introduce the auxiliary state variables $y_1(t)$, $y_2(t)$ and the auxiliary control variable $w(t)$ defined by

$$8t^2 - 8t + 1.5 - x_1 = y_1^2, \qquad \dot{y}_1 = y_2, \qquad \dot{y}_2 = w. \tag{116}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 (x_1^2 + x_2^2 + u^2/200)dt, \tag{117-1}$$

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u - x_2, \qquad \dot{y}_1 = y_2, \qquad \dot{y}_2 = w, \tag{117-2}$$

$$16 - u + x_2 - 2y_2^2 - 2y_1 w = 0 , \tag{117-3}$$

$$x_1(0) = 0, \quad x_2(0) = -1, \quad y_1(0) = \sqrt{(1.5)}, \quad y_2(0) = -7/\sqrt{6}. \tag{117-4}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, $y_1(t)$, $y_2(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0 and L_1 are linear in x_1 and x_2, the elimination of the state variables x_1 and x_2 is possible through the relations

$$x_1 = 8t^2 - 8t + 1.5 - y_1^2 , \qquad x_2 = 16t - 8 - 2y_1 y_2 . \tag{118}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 [(8t^2-8t+1.5-y_1^2)^2 + (16t-8-2y_1y_2)^2 + u^2/200]dt, \tag{119-1}$$

$$\dot{y}_1 = y_2 , \qquad \dot{y}_2 = w , \tag{119-2}$$

$$16t + 8 - u - 2y_1y_2 - 2y_2^2 - 2y_1 w = 0 , \tag{119-3}$$

$$y_1(0) = \sqrt{(1.5)}, \qquad y_2(0) = -7/\sqrt{6} . \tag{119-4}$$

In this formulation, the unknowns are the state variables $y_1(t)$, $y_2(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the functions $x_1(t)$ and $x_2(t)$ can be computed a posteriori with (118).

In the computer runs, the following nominal functions are

employed for Problem P2:

$$x_1(t)=0, \quad x_2(t)=-1, \quad y_1(t)=\sqrt{(1.5)}, \quad y_2(t)=-7/\sqrt{6}\ , \tag{120-1}$$

$$u(t)=1, \qquad w(t)=1, \tag{120-2}$$

and Problem P3:

$$y_1(t)=\sqrt{(1.5)}, \qquad y_2(t)=-7/\sqrt{6}\ , \tag{121-1}$$

$$u(t)=1, \qquad w(t)=1\ . \tag{121-2}$$

The numerical results are given in Tables 25-28. Convergence to the desired stopping condition occurs in $N=22$ iterations for Problem P2 and $N=26$ iterations for Problem P3. The CPU time is $T=81.9$ sec for Problem P2 and $T=55.6$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 32.2%.

Example 7.8. This example involves (i) boundary conditions of the fixed endpoint type, (ii) fixed final time $\tau=1$, and (iii) a linear state inequality constraint of the third order. Problem P1 is as follows:

$$I=\int_0^1 u^2 dt\ , \tag{122-1}$$

$$\dot{x}_1=x_2\ , \qquad \dot{x}_2=x_3\ , \qquad \dot{x}_3=u\ , \tag{122-2}$$

$$0.5-x_1 \geq 0\ , \tag{122-3}$$

$$x_1(0) = 0\,, \qquad x_2(0) = 1\,, \qquad x_3(0) = 2\,, \tag{122-4}$$

$$x_1(1) = 0\,, \qquad x_2(1) = -1\,, \qquad x_3(1) = 2\,. \tag{122-5}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$ and the control variable $u(t)$.

We introduce the auxiliary state variables $y_1(t)$, $y_2(t)$, $y_3(t)$ and the auxiliary control variable $w(t)$ defined by

$$0.5 - x_1 = y_1^2\,, \qquad \dot{y}_1 = y_2, \qquad \dot{y}_2 = y_3, \qquad \dot{y}_3 = w\,. \tag{123}$$

With this understanding, Problem P2 is represented by

$$I = \int_0^1 u^2 dt\,, \tag{124-1}$$

$$\dot{x}_1 = x_2\,, \qquad \dot{x}_2 = x_3\,, \qquad \dot{x}_3 = u\,, \tag{124-2}$$

$$\dot{y}_1 = y_2\,, \qquad \dot{y}_2 = y_3\,, \qquad \dot{y}_3 = w\,, \tag{124-3}$$

$$u + 6y_2y_3 + 2y_1w = 0\,, \tag{124-4}$$

$$x_1(0) = 0\,, \qquad x_2(0) = 1\,, \qquad x_3(0) = 2\,, \tag{124-5}$$

$$y_1(0) = 1/\sqrt{2}\,, \qquad y_2(0) = -1/\sqrt{2}\,, \qquad y_3(0) = -3/\sqrt{2}, \tag{124-6}$$

$$x_1(1) = 0\,, \qquad x_2(1) = -1\,, \qquad x_3(1) = 2\,. \tag{124-7}$$

In this formulation, the unknowns are the state variables $x_1(t)$, $x_2(t)$, $x_3(t)$, $y_1(t)$, $y_2(t)$, $y_3(t)$ and the control variables $u(t)$, $w(t)$.

Since L_0, L_1, L_2 are linear in x_1, x_2, x_3, the elimination of the state variables x_1, x_2, x_3 is possible through the relations

$$x_1 = 0.5 - y_1^2, \qquad x_2 = -2y_1y_2, \qquad x_3 = -2y_2^2 - 2y_1y_3 . \tag{125}$$

As a consequence, Problem P3 is represented by

$$I = \int_0^1 u^2 dt , \tag{126-1}$$

$$\dot{y}_1 = y_2 , \qquad \dot{y}_2 = y_3 , \qquad \dot{y}_3 = w , \tag{126-2}$$

$$u + 6y_2y_3 + 2y_1w = 0 , \tag{126-3}$$

$$y_1(0) = 1/\sqrt{2}, \qquad y_2(0) = -1/\sqrt{2}, \qquad y_3(0) = -3/\sqrt{2}, \tag{126-4}$$

$$y_1(1) = 1/\sqrt{2}, \qquad y_2(1) = 1/\sqrt{2}, \qquad y_3(1) = -3/\sqrt{2}. \tag{126-5}$$

In this formulation, the unknowns are the state variables $y_1(t)$, $y_2(t)$, $y_3(t)$ and the control variables $u(t)$, $w(t)$. After Problem P3 is solved, the functions $x_1(t)$, $x_2(t)$, $x_3(t)$ can be computed a posteriori with (125).

In the computer runs, the following nominal functions are employed for Problem P2:

$$x_1(t) = 0, \qquad x_2(t) = 1 - 2t, \qquad x_3(t) = 2 , \tag{127-1}$$

$$y_1(t) = 1/\sqrt{2}, \qquad y_2(t) = (1/\sqrt{2})(2t-1), \qquad y_3(t) = -3/\sqrt{2}, \tag{127-2}$$

$$u(t) = 0, \qquad w(t) = 0 , \tag{127-3}$$

and Problem P3:

$$y_1(t) = 1/\sqrt{2}, \qquad y_2(t) = (1/\sqrt{2})(2t-1), \qquad y_3(t) = -3/\sqrt{2}, \qquad (128\text{-}1)$$

$$u(t) = 0, \qquad w(t) = 0 \,. \qquad (128\text{-}2)$$

The numerical results are given in Tables 29-32. Convergence to the desired stopping condition occurs in $N = 26$ iterations for Problem P2 and $N = 21$ iterations for Problem P3. The CPU time is $T = 171.0$ sec for Problem P2 and $T = 60.0$ sec for Problem P3. The reduction in CPU time due to the use of the new formulation is 65.0%.

Table 1. Convergence history, Example 7.1, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.14E+01		
1	0	3	3	0.52E-09	0.35E+00	1.83569
2	1	2	6	0.15E-16	0.14E-01	1.66599
3	1	1	8	0.10E-09	0.24E-03	1.65742
4	1	1	10	0.61E-17	0.15E-03	1.65697
5	1	1	12	0.92E-18	0.97E-04	1.65678

Table 2. Converged solution, Example 7.1, Problem P2.

t	x_1	y_1	u	w
0.0	1.0000	0.3162	1.7479	-1.1826
0.1	0.9410	0.2025	1.3353	-1.1104
0.2	0.9095	0.0978	1.0097	-0.9323
0.3	0.9006	0.0247	0.8366	-0.5177
0.4	0.9000	-0.0090	0.8067	-0.1866
0.5	0.9003	-0.0176	0.8104	-0.0018
0.6	0.9000	-0.0094	0.8135	0.1818
0.7	0.9005	0.0238	0.7863	0.5158
0.8	0.9094	0.0973	0.6442	0.9396
0.9	0.9410	0.2024	0.4360	1.1098
1.0	1.0000	0.3162	0.2474	1.1898

$\tau = 1.00000$

Table 3. Convergence history, Example 7.1, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.14E+01		
1	0	1	1	0.20E-28	0.76E+00	2.00000
2	1	2	4	0.22E-18	0.26E-01	1.67338
3	1	1	6	0.10E-11	0.24E-03	1.65745
4	1	1	8	0.86E-18	0.14E-03	1.65697
5	1	1	10	0.39E-19	0.91E-04	1.65675

Table 4. Converged solution, Example 7.1, Problem P3.

t	x_1	y_1	u	w
0.0	1.0000	0.3162	1.7627	-1.2060
0.1	0.9410	0.2025	1.3324	-1.1035
0.2	0.9095	0.0977	1.0106	-0.9379
0.3	0.9006	0.0245	0.8362	-0.5111
0.4	0.9000	-0.0081	0.8072	-0.1759
0.5	0.9002	-0.0161	0.8104	0.0000
0.6	0.9000	-0.0081	0.8129	0.1763
0.7	0.9006	0.0246	0.7858	0.5116
0.8	0.9095	0.0978	0.6439	0.9374
0.9	0.9410	0.2025	0.4383	1.1037
1.0	1.0000	0.3162	0.2384	1.2040

$\tau = 1.00000$

Table 5. Convergence history, Example 7.2, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.53E+01		
1	0	4	4	0.69E-10	0.10E+00	1.63945
2	1	2	7	0.70E-11	0.15E-01	1.59570
3	1	2	10	0.28E-16	0.74E-02	1.59032
4	1	1	12	0.30E-09	0.51E-02	1.58767
5	1	1	14	0.14E-09	0.22E-02	1.58626
6	1	1	16	0.74E-11	0.25E-02	1.58530
7	1	1	18	0.75E-11	0.89E-03	1.58468
8	1	1	20	0.56E-12	0.14E-02	1.58422
9	1	1	22	0.71E-12	0.40E-03	1.58390
10	1	1	24	0.72E-13	0.87E-03	1.58364
11	1	1	26	0.10E-12	0.19E-03	1.58346
12	1	1	28	0.13E-13	0.55E-03	1.58330
13	1	1	30	0.25E-13	0.10E-03	1.58319
14	1	1	32	0.19E-14	0.33E-03	1.58311
15	1	1	34	0.73E-14	0.62E-04	1.58304

Table 6. Converged solution, Example 7.2, Problem P2.

t	x_1	x_2	y_1	u	w
0.0	0.0000	0.0000	0.6324	0.8971	-0.6363
0.1	0.1414	0.1253	0.5241	0.8860	-0.7298
0.2	0.2793	0.2382	0.4022	0.8527	-0.8070
0.3	0.4103	0.3274	0.2693	0.7982	-0.8704
0.4	0.5303	0.3832	0.1294	0.7098	-0.8595
0.5	0.6355	0.3990	0.0302	0.6482	-0.2694
0.6	0.7428	0.3966	0.0582	0.6926	0.6178
0.7	0.8425	0.3595	0.2010	0.5483	1.0172
0.8	0.9183	0.2728	0.3565	0.4084	0.9487
0.9	0.9712	0.1488	0.5011	0.2581	0.8747
1.0	1.0000	0.0000	0.6324	0.1144	0.7802

$\tau = 1.58304$

Table 7. Convergence history, Example 7.2, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.48E+01		
1	0	5	5	0.41E-16	0.14E+00	1.65592
2	1	2	8	0.17E-10	0.16E-01	1.59799
3	1	2	11	0.11E-15	0.97E-02	1.59082
4	1	1	13	0.39E-09	0.47E-02	1.58787
5	1	1	15	0.55E-10	0.26E-02	1.58641
6	1	1	17	0.55E-11	0.23E-02	1.58544
7	1	1	19	0.26E-11	0.10E-02	1.58481
8	1	1	21	0.36E-12	0.13E-02	1.58434
9	1	1	23	0.27E-12	0.49E-03	1.58401
10	1	1	25	0.42E-13	0.87E-03	1.58374
11	1	1	27	0.44E-13	0.25E-03	1.58355
12	1	1	29	0.73E-14	0.57E-03	1.58339
13	1	1	31	0.10E-13	0.13E-03	1.58327
14	1	1	33	0.13E-14	0.36E-03	1.58317
15	1	1	35	0.33E-14	0.81E-04	1.58310

Table 8. Converged solution, Example 7.2, Problem P3.

t	x_1	x_2	y_1	u	w
0.0	0.0000	0.0000	0.6324	0.8974	-0.6366
0.1	0.1414	0.1253	0.5240	0.8862	-0.7303
0.2	0.2794	0.2383	0.4020	0.8530	-0.8077
0.3	0.4104	0.3276	0.2690	0.7980	-0.8706
0.4	0.5303	0.3831	0.1299	0.7078	-0.8455
0.5	0.6354	0.3988	0.0332	0.6490	-0.2621
0.6	0.7426	0.3963	0.0604	0.6920	0.6008
0.7	0.8425	0.3596	0.2008	0.5494	1.0156
0.8	0.9183	0.2729	0.3564	0.4081	0.9492
0.9	0.9712	0.1489	0.5010	0.2577	0.8749
1.0	1.0000	0.0000	0.6324	0.1158	0.7799

$\tau = 1.58310$

Table 9. Convergence history, Example 7.3, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.21E+01		
1	0	4	4	0.48E-14	0.11E+01	2.43959
2	1	2	7	0.74E-16	0.90E-02	2.10181
3	1	1	9	0.79E-13	0.15E-02	2.09857
4	1	1	11	0.23E-13	0.25E-02	2.09694
5	1	1	13	0.59E-15	0.62E-03	2.09599
6	1	1	15	0.99E-15	0.11E-02	2.09535
7	1	1	17	0.34E-16	0.30E-03	2.09492
8	1	1	19	0.49E-16	0.59E-03	2.09460
9	1	1	21	0.26E-17	0.16E-03	2.09437
10	1	1	23	0.43E-17	0.34E-03	2.09419
11	1	0	24	0.66E-08	0.98E-04	2.09396

Table 10. Converged solution, Example 7.3, Problem P2.

t	x_1	y_1	u	w
0.0	1.0000	0.4472	1.3993	-1.5645
0.1	0.9748	0.2912	1.0571	-1.5565
0.2	0.9786	0.1367	0.7664	-1.4938
0.3	1.0104	0.0222	0.6505	-0.6597
0.4	1.0400	-0.0064	0.8809	-0.0527
0.5	1.0500	-0.0070	1.1025	-0.0009
0.6	1.0400	-0.0065	1.2823	0.0540
0.7	1.0104	0.0224	1.3912	0.6614
0.8	0.9786	0.1367	1.1495	1.4920
0.9	0.9748	0.2912	0.8434	1.5566
1.0	1.0000	0.4472	0.6007	1.5644

$\tau = 1.00000$

Table 11. Convergence history, Example 7.3, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.16E+01		
1	0	3	3	0.18E-14	0.25E+01	2.85114
2	1	2	6	0.62E-17	0.18E-01	2.10447
3	1	1	8	0.58E-14	0.14E-02	2.09848
4	1	1	10	0.21E-15	0.24E-02	2.09655
5	1	1	12	0.25E-17	0.52E-03	2.09564
6	1	1	14	0.28E-17	0.91E-03	2.09511
7	1	1	16	0.30E-18	0.26E-03	2.09475
8	1	1	18	0.17E-19	0.49E-03	2.09449
9	1	1	20	0.32E-19	0.15E-03	2.09429
10	1	1	22	0.30E-21	0.29E-03	2.09414
11	1	0	23	0.63E-08	0.93E-04	2.09396

Table 12. Converged solution, Example 7.3, Problem P3.

t	x_1	y_1	u	w
0.0	1.0000	0.4472	1.3996	-1.5648
0.1	0.9748	0.2912	1.0572	-1.5571
0.2	0.9786	0.1364	0.7671	-1.5001
0.3	1.0104	0.0217	0.6495	-0.6543
0.4	1.0400	-0.0063	0.8809	-0.0475
0.5	1.0500	-0.0066	1.1025	-0.0011
0.6	1.0400	-0.0064	1.2823	0.0488
0.7	1.0104	0.0219	1.3921	0.6566
0.8	0.9786	0.1365	1.1485	1.4981
0.9	0.9748	0.2912	0.8432	1.5572
1.0	1.0000	0.4472	0.6005	1.5646

$\tau = 1.00000$

Table 13. Convergence history, Example 7.4, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.54E+01		
1	0	3	3	0.85E-08	0.18E-01	1.24356
2	1	1	5	0.19E-11	0.55E-02	1.23749
3	1	1	7	0.98E-16	0.20E-02	1.23593
4	1	1	9	0.13E-16	0.15E-02	1.23520
5	1	1	11	0.98E-19	0.69E-03	1.23481
6	1	1	13	0.20E-18	0.11E-02	1.23438
7	1	0	14	0.14E-08	0.24E-03	1.23427
8	1	1	16	0.10E-18	0.63E-03	1.23405
9	1	0	17	0.27E-08	0.97E-04	1.23391

Table 14. Converged solution, Example 7.4, Problem P2.

t	x_1	x_2	y_1	y_2	u	w
0.0	0.0000	0.0000	1.0000	-1.0000	0.9911	-1.0000
0.1	0.1568	0.3090	0.8312	-1.1438	0.9914	-0.8258
0.2	0.3099	0.5875	0.6421	-1.2580	0.9555	-0.6201
0.3	0.4558	0.8081	0.4380	-1.3342	0.8903	-0.3579
0.4	0.5875	0.9494	0.2247	-1.3778	0.7875	-0.2128
0.5	0.7040	0.9999	0.0062	-1.4010	0.6968	-0.0726
0.6	0.8059	0.9541	-0.2139	-1.3972	0.5969	0.1308
0.7	0.8902	0.8143	-0.4308	-1.3576	0.4717	0.3785
0.8	0.9527	0.5924	-0.6383	-1.2776	0.3211	0.6398
0.9	0.9901	0.3108	-0.8301	-1.1572	0.1517	0.8893
1.0	1.0000	0.0000	-0.9999	-1.0000	-0.0262	1.1049

$\tau = 1.00000$

Table 15. Convergence history, Example 7.4, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.79E+00		
1	0	3	3	0.33E-09	0.14E-02	1.23414
2	1	1	5	0.27E-16	0.16E-03	1.23377
3	1	0	6	0.91E-10	0.44E-04	1.23373

Table 16. Converged solution, Example 7.4, Problem P3.

t	x_1	x_2	y_1	y_2	u	w
0.0	0.0000	0.0000	1.0000	-1.0000	0.9874	-1.0000
0.1	0.1555	0.3090	0.8312	-1.1448	0.9846	-0.8395
0.2	0.3076	0.5882	0.6417	-1.2633	0.9542	-0.6571
0.3	0.4538	0.8098	0.4361	-1.3482	0.8996	-0.4231
0.4	0.5884	0.9515	0.2200	-1.3969	0.8104	-0.1994
0.5	0.7075	0.9999	-0.0009	-1.4115	0.7039	0.0126
0.6	0.8089	0.9508	-0.2216	-1.3932	0.5848	0.2199
0.7	0.8906	0.8090	-0.4369	-1.3425	0.4537	0.4251
0.8	0.9509	0.5880	-0.6417	-1.2597	0.3112	0.6282
0.9	0.9879	0.3092	-0.8311	-1.1453	0.1578	0.8279
1.0	1.0000	0.0000	-1.0000	-0.9999	-0.0053	1.0214

$\tau = 1.57079$

Table 17. Convergence history, Example 7.5, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.76E+01		
1	0	4	4	0.93E-08	0.15E-01	5.94696
2	1	1	6	0.88E-16	0.33E-02	5.93350
3	1	1	8	0.10E-15	0.15E-02	5.92983
4	1	1	10	0.18E-19	0.62E-03	5.92838
5	1	1	12	0.15E-18	0.42E-03	5.92763
6	1	0	13	0.40E-08	0.23E-03	5.92713
7	1	0	14	0.34E-08	0.13E-03	5.92683
8	1	0	15	0.70E-08	0.16E-03	5.92658
9	1	0	16	0.76E-08	0.42E-04	5.92646

Table 18. Converged solution, Example 7.5, Problem P2.

t	x_1	x_2	y_1	y_2	u	w
0.0	0.0000	1.0000	0.3872	-1.2909	-4.4309	1.4169
0.1	0.0794	0.6055	0.2655	-1.1399	-3.4551	1.6124
0.2	0.1243	0.3091	0.1600	-0.9654	-2.4738	1.9045
0.3	0.1445	0.1096	0.0741	-0.7394	-1.5065	2.7869
0.4	0.1497	0.0129	0.0162	-0.4002	-0.4453	3.8483
0.5	0.1499	0.0000	-0.0040	-0.0012	0.0323	4.0484
0.6	0.1497	-0.0128	0.0160	0.3994	-0.4435	3.8777
0.7	0.1445	-0.1096	0.0739	0.7407	-1.5101	2.7892
0.8	0.1243	-0.3092	0.1600	0.9659	-2.4733	1.8971
0.9	0.0794	-0.6055	0.2655	1.1400	-3.4547	1.6104
1.0	0.0000	-1.0000	0.3872	1.2909	-4.4300	1.4157

$\tau = 1.00000$

Table 19. Convergence history, Example 7.5, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.44E+01		
1	0	2	2	0.69E-10	0.25E+00	6.14402
2	1	1	4	0.10E-09	0.16E-01	5.95065
3	1	1	6	0.28E-16	0.41E-02	5.93466
4	1	1	8	0.12E-15	0.16E-02	5.93041
5	1	1	10	0.42E-19	0.76E-03	5.92872
6	1	1	12	0.21E-18	0.46E-03	5.92786
7	1	0	13	0.64E-08	0.29E-03	5.92727
8	1	0	14	0.44E-08	0.13E-03	5.92693
9	1	1	16	0.30E-21	0.23E-03	5.92676
10	1	0	17	0.38E-09	0.41E-04	5.92661

Table 20. Converged solution, Example 7.5, Problem P3.

t	x_1	x_2	y_1	y_2	u	w
0.0	0.0000	1.0000	0.3872	-1.2909	-4.4377	1.4258
0.1	0.0794	0.6054	0.2656	-1.1397	-3.4540	1.6117
0.2	0.1243	0.3091	0.1601	-0.9655	-2.4720	1.8973
0.3	0.1445	0.1096	0.0740	-0.7400	-1.5095	2.7963
0.4	0.1497	0.0129	0.0162	-0.3987	-0.4435	3.8649
0.5	0.1499	0.0000	-0.0038	0.0000	0.0308	4.0305
0.6	0.1497	-0.0129	0.0162	0.3987	-0.4435	3.8647
0.7	0.1445	-0.1096	0.0740	0.7400	-1.5096	2.7964
0.8	0.1243	-0.3091	0.1601	0.9655	-2.4720	1.8973
0.9	0.0794	-0.6054	0.2656	1.1397	-3.4540	1.6117
1.0	0.0000	-1.0000	0.3872	1.2909	-4.4377	1.4258

$\tau = 1.00000$

Table 21. Convergence history, Example 7.6, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.53E+02		
1	0	4	4	0.97E-10	0.20E+00	0.61522
2	1	2	7	0.19E-11	0.22E-01	0.33462
3	1	2	10	0.43E-15	0.46E-02	0.23701
4	1	2	13	0.14E-14	0.13E-02	0.18428
5	1	1	15	0.85E-10	0.37E-03	0.17757
6	1	1	17	0.20E-08	0.41E-03	0.17444
7	1	1	19	0.15E-11	0.80E-04	0.17283

Table 22. Converged solution, Example 7.6, Problem P2.

t	x_1	x_2	y_1	u	w
0.0	0.0000	-1.0000	1.5811	11.8479	-6.5926
0.1	-0.0529	-0.2025	0.9902	4.0080	-5.3566
0.2	-0.0603	-0.0227	0.4927	-0.2990	-4.5906
0.3	-0.0691	-0.1922	0.1105	-2.7927	-2.7103
0.4	-0.1006	-0.4212	-0.0347	-2.0552	-0.4898
0.5	-0.1482	-0.5025	-0.0509	-0.4993	0.0316
0.6	-0.1957	-0.4203	-0.0186	1.2095	0.8015
0.7	-0.2275	-0.2047	0.1572	2.1037	2.8354
0.8	-0.2384	-0.0355	0.5055	0.9462	3.7765
0.9	-0.2387	0.0178	0.8730	0.2981	3.5047
1.0	-0.2352	0.0587	1.2005	0.7978	3.0240

$\tau = 1.00000$

Table 23. Convergence history, Example 7.6, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.70E+02		
1	0	4	4	0.26E-10	0.20E+00	0.62152
2	1	2	7	0.79E-09	0.12E-01	0.29797
3	1	2	10	0.31E-16	0.31E-02	0.20454
4	1	2	13	0.28E-14	0.23E-02	0.18697
5	1	1	15	0.13E-09	0.32E-03	0.17853
6	1	1	17	0.99E-09	0.40E-03	0.17288
7	1	1	19	0.33E-11	0.30E-04	0.17157

Table 24. Converged solution, Example 7.6, Problem P3.

t	x_1	x_2	y_1	u	w
0.0	0.0000	-1.0000	1.5811	12.8747	-6.9174
0.1	-0.0503	-0.1664	0.9728	3.9858	-5.4232
0.2	-0.0550	-0.0038	0.4730	-0.4751	-4.5748
0.3	-0.0628	-0.1908	0.1043	-2.8834	-2.4311
0.4	-0.0943	-0.4203	-0.0195	-2.0359	-0.3992
0.5	-0.1417	-0.5007	-0.0275	-0.4961	0.0838
0.6	-0.1891	-0.4200	-0.0034	1.1839	0.5831
0.7	-0.2209	-0.2022	0.1492	2.1949	2.6900
0.8	-0.2311	-0.0261	0.4961	0.9679	3.8354
0.9	-0.2306	0.0234	0.8697	0.2259	3.5626
1.0	-0.2271	0.0534	1.2027	0.6597	3.0736

$\tau = 1.00000$

Table 25. Convergence history, Example 7.7, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.37E+02		
1	0	4	4	0.30E-15	0.85E-01	0.97910
2	1	1	6	0.19E-08	0.32E-02	0.79284
3	1	1	8	0.18E-09	0.11E-02	0.75471
4	1	1	10	0.13E-14	0.36E-03	0.75132
5	1	1	12	0.24E-14	0.57E-03	0.74892
6	1	1	14	0.17E-15	0.19E-03	0.74719
7	1	1	16	0.18E-15	0.34E-03	0.74585
8	1	1	18	0.28E-16	0.11E-03	0.74482
9	1	1	20	0.21E-16	0.22E-03	0.74399
10	1	1	22	0.55E-17	0.77E-04	0.74332

Table 26. Converged solution, Example 7.7, Problem P2.

t	x_1	x_2	y_1	y_2	u	w
0.0	0.0000	-1.0000	1.2247	-2.8577	-3.4340	0.8576
0.1	-0.1085	-1.1435	0.9426	-2.7882	-1.9189	0.6509
0.2	-0.2256	-1.1872	0.6675	-2.7060	-1.2952	1.0958
0.3	-0.3430	-1.1382	0.4038	-2.5527	0.2239	1.9870
0.4	-0.4453	-0.8539	0.1591	-2.3435	3.6035	1.7528
0.5	-0.5049	-0.3254	-0.0706	-2.3029	4.9089	-1.1205
0.6	-0.5159	0.0519	-0.3098	-2.4983	2.1866	-2.2304
0.7	-0.5044	0.1432	-0.5696	-2.6831	0.1680	-1.3840
0.8	-0.4914	0.1087	-0.8434	-2.7809	-0.4350	-0.6379
0.9	-0.4832	0.0585	-1.1239	-2.8211	-0.2957	-0.1943
1.0	-0.4778	0.0670	-1.4063	-2.8204	0.8164	0.2342

$\tau = 1.00000$

Table 27. Convergence history, Example 7.7, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.40E+02		
1	0	4	4	0.49E-18	0.89E-01	0.98869
2	1	1	6	0.44E-08	0.40E-02	0.79388
3	1	1	8	0.11E-11	0.36E-02	0.76835
4	1	1	10	0.24E-12	0.81E-03	0.75838
5	1	1	12	0.19E-14	0.10E-02	0.75313
6	1	1	14	0.31E-14	0.32E-03	0.75009
7	1	1	16	0.17E-15	0.49E-03	0.74799
8	1	1	18	0.19E-15	0.17E-03	0.74647
9	1	1	20	0.36E-16	0.29E-03	0.74531
10	1	1	22	0.24E-16	0.11E-03	0.74441
11	1	1	24	0.62E-17	0.18E-03	0.74368
12	1	1	26	0.45E-17	0.76E-04	0.74309

Table 28. Converged solution, Example 7.7, Problem P3.

t	x_1	x_2	y_1	y_2	u	w
0.0	0.0000	-1.0000	1.2247	-2.8577	-3.4315	0.8565
0.1	-0.1084	-1.1428	0.9426	-2.7886	-1.9086	0.6433
0.2	-0.2254	-1.1858	0.6674	-2.7075	-1.2929	1.0830
0.3	-0.3428	-1.1381	0.4035	-2.5547	0.1944	2.0000
0.4	-0.4452	-0.8562	0.1588	-2.3413	3.6041	1.8130
0.5	-0.5049	-0.3244	-0.0705	-2.2985	4.9466	-1.1524
0.6	-0.5158	0.0532	-0.3095	-2.4985	2.1647	-2.2655
0.7	-0.5042	0.1427	-0.5694	-2.6843	0.1643	-1.3756
0.8	-0.4912	0.1093	-0.8433	-2.7809	-0.4146	-0.6262
0.9	-0.4828	0.0607	-1.1237	-2.8205	-0.2915	-0.1963
1.0	-0.4773	0.0654	-1.4061	-2.8212	0.7183	0.2035

$\tau = 1.00000$

Table 29. Convergence history, Example 7.8, Problem P2.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.56E+02		
1	0	7	7	0.15E-17	0.15E+03	276.22402
2	1	2	10	0.92E-21	0.10E+02	201.10661
3	1	1	12	0.52E-11	0.26E+01	193.87600
4	1	1	14	0.47E-14	0.51E+00	192.48949
5	1	1	16	0.16E-16	0.18E+00	192.13689
6	1	1	18	0.13E-18	0.41E-01	192.03952
7	1	1	20	0.79E-21	0.15E-01	192.01184
8	1	0	21	0.92E-08	0.37E-02	192.00340
9	1	1	23	0.19E-22	0.13E-02	192.00105
10	1	0	24	0.11E-09	0.41E-03	192.00023
11	1	0	25	0.12E-09	0.10E-03	192.00002
12	1	0	26	0.17E-09	0.55E-04	191.99993

Table 30. Converged solution, Example 7.8, Problem P2.

t	x_1	x_2	x_3	u
0.0	0.0000	1.0000	2.0000	-23.9758
0.1	0.1062	1.0880	-0.1601	-19.2074
0.2	0.2112	0.9839	-1.8405	-14.3999
0.3	0.2981	0.7359	-3.0402	-9.5955
0.4	0.3551	0.3919	-3.7596	-4.7917
0.5	0.3749	0.0000	-3.9990	0.0000
0.6	0.3551	-0.3919	-3.7596	4.7918
0.7	0.2981	-0.7359	-3.0402	9.5958
0.8	0.2111	-0.9839	-1.8404	14.4002
0.9	0.1062	-1.0880	-0.1601	19.2058
1.0	0.0000	-1.0000	2.0000	23.9878

t	y_1	y_2	y_3	w
0.0	0.7071	-0.7071	-2.1213	10.5894
0.1	0.6275	-0.8668	-1.0699	10.8698
0.2	0.5374	-0.9154	0.1528	14.1789
0.3	0.4492	-0.8191	1.8903	21.0205
0.4	0.3805	-0.5149	4.2429	23.5208
0.5	0.3535	0.0000	5.6552	0.0000
0.6	0.3805	0.5149	4.2428	-23.5209
0.7	0.4492	0.8190	1.8904	-21.0209
0.8	0.5374	0.9154	0.1528	-14.1789
0.9	0.6275	0.8668	-1.0699	-10.8685
1.0	0.7071	0.7071	-2.1213	-10.5979

$\tau = 1.00000$

Table 31. Convergence history, Example 7.8, Problem P3.

N_c	N_g	N_r	N	P	Q	I
0	0	0	0	0.39E+02		
1	0	5	5	0.90E-18	0.40E+02	220.61775
2	1	1	7	0.75E-08	0.50E+01	196.13345
3	1	1	9	0.37E-15	0.12E+01	193.00474
4	1	1	11	0.15E-14	0.31E+00	192.27617
5	1	1	13	0.10E-18	0.99E-01	192.08057
6	1	1	15	0.45E-19	0.27E-01	192.02401
7	1	0	16	0.76E-08	0.90E-02	192.00723
8	1	1	18	0.94E-22	0.26E-02	192.00201
9	1	0	19	0.60E-10	0.82E-03	192.00049
10	1	0	20	0.23E-09	0.29E-03	191.99995
11	1	0	21	0.24E-09	0.61E-04	191.99982

Table 32. Converged solution, Example 7.8, Problem P3.

t	x_1	x_2	x_3	u
0.0	0.0000	1.0000	2.0000	-24.0214
0.1	0.1061	1.0879	-0.1607	-19.2006
0.2	0.2111	0.9838	-1.8406	-14.3974
0.3	0.2981	0.7358	-3.0401	-9.5932
0.4	0.3551	0.3918	-3.7591	-4.7858
0.5	0.3749	0.0000	-3.9981	0.0000
0.6	0.3551	-0.3918	-3.7590	4.7858
0.7	0.2981	-0.7358	-3.0402	9.5932
0.8	0.2111	-0.9838	-1.8406	14.3974
0.9	0.1061	-1.0879	-0.1607	19.2006
1.0	0.0000	-1.0000	2.0000	24.0213

t	y_1	y_2	y_3	w
0.0	0.7071	-0.7071	-2.1213	10.6217
0.1	0.6275	-0.8668	-1.0693	10.8672
0.2	0.5374	-0.9153	0.1533	14.1786
0.3	0.4492	-0.8189	1.8907	21.0169
0.4	0.3805	-0.5148	4.2423	23.5029
0.5	0.3536	0.0000	5.6530	0.0001
0.6	0.3805	0.5148	4.2422	-23.5028
0.7	0.4492	0.8189	1.8908	-21.0171
0.8	0.5374	0.9153	0.1533	-14.1786
0.9	0.6275	0.8668	-1.0693	-10.8671
1.0	0.7071	0.7071	-2.1213	-10.6217

$\tau = 1.00000$

8. Discussion and Conclusions

In this paper, we consider optimal control problems involving the minimization of a functional subject to differential constraints, terminal constraints, and a state inequality constraint. The state inequality constraint is of a special type, namely, it is linear in some or all of the components of the state vector.

A transformation technique is introduced, by means of which the inequality constrained problem is converted into an equality constrained problem involving differential constraints, terminal constraints, and a control equality constraint. The transformation technique takes advantage of the partial linearity of the state inequality constraint so as to yield a transformed problem characterized by a new state vector of minimal size. This concept is important computationally, in that the computer time per iteration increases with the square of the dimension of the state vector.

In order to illustrate the advantages of the new transformation technique, eight numerical examples are solved by means of the sequential ordinary gradient-restoration algorithm for optimal control problems involving nondifferential constraints (Refs. 5-6). Two formulations are employed: that associated with Problem P2 and that associated with Problem P3. If n is the number of original state variables and k is the order of the state inequality constraint, the transformed Problem P2 is characterized by $n_2 = n + k$ state variables, while the transformed Problem P3 is characterized by $n_3 = n + b$ state variables, with $b < k$.

For Examples 7.1 through 7.8, the convergence history and the converged solution for both Problem P2 and Problem P3 are given in Tables 1-32. To facilitate the comparison between Problem P2 and Problem P3, summary results are presented in Tables 33-36.

Table 33 shows the number of cycles N_c, the number of gradient iterations ΣN_g, the number of restorative iterations ΣN_r, and the total number of iterations N for both Problem P2 and Problem P3.

Table 34 shows the CPU time for Problem P2, the CPU time for Problem P3, the ratio of CPU times T_3/T_2, and the relative saving in CPU time $(T_2-T_3)/T_2$. From Table 34, it appears that savings in computer time ranging from 26.6% to 70.1% are obtained by employing the formulation associated with Problem P3, rather than the formulation associated with Problem P2.

Table 35 shows the CPU time per iteration for Problem P2, the CPU time per iteration for Problem P3, the ratio of CPU times per iteration τ_3/τ_2, and the relative saving in CPU time per iteration $(\tau_2 - \tau_3)/\tau_2$. From Table 35, it appears that savings in computer time per iteration ranging from 15.1% to

56.6% are obtained by employing the formulation associated with Problem P3, rather than the formulation associated with Problem P2. From Tables 34-35, the beneficial effects due to the use of the new transformation technique are apparent.

Table 36 shows the value of the functional I at convergence for both Problem P2 and Problem P3. It is clear that the value of the functional I obtained with Problem P2 is the same as that obtained with Problem P3 [of course, within the tolerance limits specified by the stopping conditions(64)].

The converged state variables and control variables for Problem P2 and Problem P3 are also the same [of course, within the tolerance limits specified by the stopping conditions (64)]. This can be seen by inspection of Tables 1-32.

Table 33. Summary results.

Example	Problem P2				Problem P3			
	N_c	ΣN_g	ΣN_r	N	N_c	ΣN_g	ΣN_r	N
7.1	5	4	8	12	5	4	6	10
7.2	15	14	20	34	15	14	21	35
7.3	11	10	14	24	11	10	13	23
7.4	9	8	9	17	3	2	4	6
7.5	9	8	8	16	10	9	8	17
7.6	7	6	13	19	7	6	13	19
7.7	10	9	13	22	12	11	15	26
7.8	12	11	15	26	11	10	11	21

Table 34. Summary results.

Example	T_2 (sec)	T_3 (sec)	T_3/T_2	$(T_2-T_3)/T_2$
7.1	21.23	13.32	0.627	0.373
7.2	103.34	75.94	0.734	0.266
7.3	43.99	29.42	0.668	0.332
7.4	66.79	20.03	0.299	0.701
7.5	63.44	34.73	0.547	0.453
7.6	50.24	36.51	0.726	0.274
7.7	81.99	55.61	0.678	0.322
7.8	171.02	60.00	0.350	0.650

Table 35. Summary results.

Example	τ_2 (sec/iter)	τ_3 (sec/iter)	τ_3/τ_2	$(\tau_2-\tau_3)/\tau_2$
7.1	1.769	1.332	0.752	0.248
7.2	3.039	2.169	0.713	0.287
7.3	1.832	1.279	0.698	0.302
7.4	3.928	3.338	0.849	0.151
7.5	3.965	2.043	0.515	0.485
7.6	2.644	1.921	0.726	0.274
7.7	3.726	2.138	0.573	0.427
7.8	6.577	2.857	0.434	0.566

Table 36. Summary results.

Example	Problem P2			Problem P3		
	P	Q	I	P	Q	I
7.1	0.92E-18	0.97E-04	1.65678	0.39E-19	0.91E-04	1.65675
7.2	0.73E-14	0.62E-04	1.58304	0.33E-14	0.81E-04	1.58310
7.3	0.66E-08	0.98E-04	2.09396	0.63E-08	0.93E-04	2.09396
7.4	0.27E-08	0.97E-04	1.23391	0.91E-10	0.44E-04	1.23373
7.5	0.76E-08	0.42E-04	5.92646	0.38E-09	0.41E-04	5.92661
7.6	0.15E-11	0.80E-04	0.17283	0.33E-11	0.30E-04	0.17157
7.7	0.55E-17	0.77E-04	0.74332	0.45E-17	0.76E-04	0.74309
7.8	0.17E-09	0.55E-04	191.99993	0.24E-09	0.61E-04	191.99982

9. Appendix: Computational Effort

At each iteration of the gradient phase or the restoration phase a linear, two-point boundary-value problem must be solved. If the method of particular solutions (Refs. 7-9) is employed in conjunction with the sequential ordinary gradient-restoration algorithm for optimal control problems with nondifferential constraints (Refs. 5-6), one must execute $n_i + p + 1$ independent sweeps of the linearized system governing the variations associated with the gradient phase or the restoration phase, where n_i is the dimension of the state vector and p is the dimension of the parameter.[14] Note that

$$n_2 = n + k \text{ for Problem P2,} \tag{129-1}$$

$$n_3 = b + k \text{ for Problem P3.} \tag{129-2}$$

Hence, the following factor is indicative of the algorithmic time per iteration:

$$W_i = 2n_i(n_i + p + 1), \qquad i = 2 \text{ or } 3, \tag{130}$$

[14] The subscript $i = 2$ is employed for Problem P2, and the subscript $i = 3$ is employed for Problem P3.

with the implication that

$$W_2 = 2(n+k)(n+k+p+1) \text{ for Problem P2,} \qquad (131\text{-}1)$$

$$W_3 = 2(b+k)(b+k+p+1) \text{ for Problem P3.} \qquad (131\text{-}2)$$

In certain applications, the nondifferential constraint can be solved in terms of the original control u, assumed to be a scalar. Then, one can arrive at a new formulation of Problems P2 and P3, where the original control u is absent, while the auxiliary control w is present. This procedure is feasible with Examples 7.1, 7.3, and 7.5 through 7.8, owing to the fact that the nondifferential constraint is linear in u. It is not advisable with Examples 7.2 and 7.4, owing to the presence of square roots (Example 7.2) and singularities (Example 7.4) in the analytical solution for u.

For the sake of argument, assume that the above substitution procedure is employed. Then, one can use the sequential ordinary gradient-restoration algorithm for optimal control problems without nondifferential constraints (Refs. 13-14). Here, one must execute $q+1$ independent sweeps of the linearized system governing the variations associated with the gradient phase or the restoration phase, where q is the number of final conditions. Hence, the following factor is indicative of the algorithmic time per iteration:

$$W_i = 2n_i(q+1), \qquad i = 2 \text{ or } 3, \tag{132}$$

with the implication that

$$W_2 = 2(n+k)(q+1) \text{ for Problem P2}, \tag{133-1}$$

$$W_3 = 2(b+k)(q+1) \text{ for Problem P3}. \tag{133-2}$$

Numerical results illustrating the above analysis are given in Tables 37-39. Table 37 contains the principal data for Example 7.1 through 7.8, more specifically: the number of original state variables n, the order of the state inequality constraint k, the number of parameters p, and the number of final conditions q. The table also contains the dimension a of the vector x_A, the dimension b of the vector x_B, the transformed number of state variables n_2 associated with Problem P2, and the transformed number of state variables n_3 associated with Problem P3.

Table 38 pertains to SOGRA with nondifferential constraints (Refs. 5-6) and contains the work factor W_i, $i = 2$ or 3, which is indicative of the algorithmic time per iteration [see Eqs. (131)]. The table also contains the ratio W_3/W_2 and the relative difference $(W_2 - W_3)/W_2$, which constitutes an upper limit to the savings in CPU time per iteration to be expected by using the new transformation technique in connection with the algorithm of Refs. 5-6.

Finally, Table 39 pertains to SOGRA without nondifferential constraints (Refs. 13-14) and contains the work factor W_i, $i = 2$ or 3, which is indicative of the algorithmic time per iteration [see Eqs. (133)]. The table also contains the ratio W_3/W_2 and the relative difference $(W_2 - W_3)/W_2$, which constitutes an upper limit to the savings in CPU time per iteration to be expected by using the new transformation technique in connection with the algorithm of Refs. 13-14.

Table 37. Principal data for the examples.

Example	n	k	p	q	a	b	n_2	n_3
7.1	1	1	0	1	1	0	2	1
7.2	2	1	1	2	1	1	3	2
7.3	1	1	0	1	1	0	2	1
7.4	2	2	0	2	1	1	4	3
7.5	2	2	0	2	2	0	4	2
7.6	2	1	0	0	1	1	3	2
7.7	2	2	0	0	2	0	4	2
7.8	3	3	0	3	3	0	6	3

Table 38. Computational effort, SOGRA with nondifferential constraints, Refs. 5-6.

Example	W_2	W_3	W_3/W_2	$(W_2-W_3)/W_2$
7.1	12	4	0.333	0.667
7.2	30	16	0.533	0.467
7.3	12	4	0.333	0.667
7.4	40	24	0.600	0.400
7.5	40	12	0.300	0.700
7.6	24	12	0.500	0.500
7.7	40	12	0.300	0.700
7.8	84	24	0.285	0.715

Table 39. Computational effort, SOGRA without nondifferential constraints, Refs. 13-14.

Example	W_2	W_3	W_3/W_2	$(W_2-W_3)/W_2$
7.1	8	4	0.500	0.500
7.2	18	12	0.667	0.333
7.3	8	4	0.500	0.500
7.4	24	18	0.750	0.250
7.5	24	12	0.500	0.500
7.6	6	4	0.667	0.333
7.7	8	4	0.500	0.500
7.8	48	24	0.500	0.500

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