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A UNIFIED PARAMETRIC QUADRATIC PROGRAMMING SOLUTION TO SOME STO--ETC(U)  
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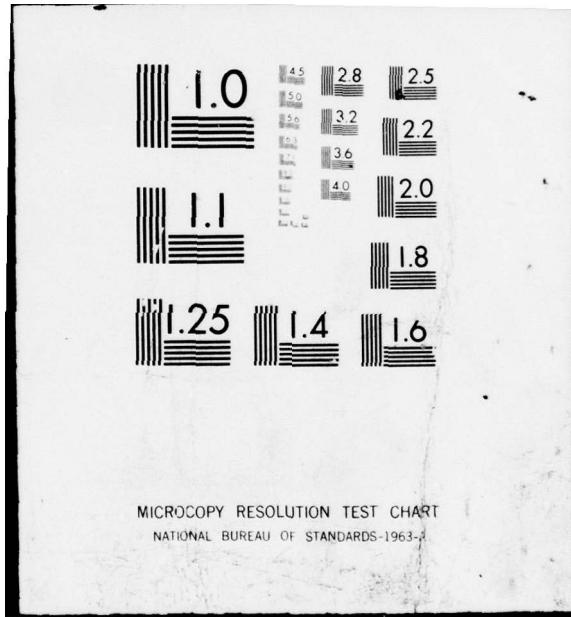
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A UNIFIED PARAMETRIC QUADRATIC  
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STOCHASTIC LINEAR PROGRAMMING MODELS

Masakazu Kojima and Ikuyo Kaneko

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11 ABSTRACT

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In this paper, we consider several deterministic models for a stochastic linear program with a constant feasible region and stochastic cost coefficients having multi-variate normal distribution. Relationships among the solutions of these models are examined and it is shown that solving a parametric quadratic program associated with Markowitz's mean-variance model yields solutions to all other models considered for all relevant values of parameters.

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A UNIFIED PARAMETRIC QUADRATIC PROGRAMMING SOLUTION  
TO SOME STOCHASTIC LINEAR PROGRAMMING MODELS

Masakazu Kojima<sup>\*</sup> and Ikuyo Kaneko<sup>\*\*</sup>

1. Introduction.

The problem we shall consider in this paper is:

$$P0 \quad \text{maximize } \xi^T x \text{ subject to } x \in X,$$

where  $\xi$  is an  $n$ -dimensional multi-variate normal random vector with mean  $c$  and variance-covariance matrix  $V$ , and

$$X = \{x \in R^n : Ax \leq b, x \geq 0\}$$

with an  $m$  by  $n$  constant matrix  $A$  and an  $m$ -dimensional constant vector  $b$ . All through the paper we assume:

- (1)  $V$  is positive definite; and
- (2) the following linear program has a finite optimum<sup>1</sup>:

$$P1 \quad \text{maximize } c^T x \text{ subject to } x \in X.$$

One approach to  $P0$  is to consider a deterministic programming model which produces an "optimal" solution to  $P0$  in a certain sense: various approaches in this direction have been proposed (see below). These models differ from one another in the concept of optimality in the stochastic program  $P0$ . In this paper, we shall show that the mean-variance (parametric quadratic programming) approach (of Markowitz [7]) yields solutions to several other deterministic models of  $P0$ , in a unified way.

The significance of the result is two-fold. It clarifies the relationships among the solutions of these models; it also provides an efficient, unified computational method to

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<sup>1</sup>We could relax this assumption somewhat by allowing the case where  $X$  is empty: the assumed feasibility of  $P1$  (or  $P0$ ) is merely for convenience.

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solve PO based on these various models.

The deterministic models for PO we shall deal with in this paper are summarized as follows:

P2 Find efficient solutions<sup>2</sup> of

$$\text{maximize } f(x) \text{ subject to } x \in X ,$$

where  $f(x) = (c^T x, -x^T V x)^T$ .

$$P3(\alpha) \quad \text{maximize } \alpha c^T x - \frac{1}{2} x^T V x \text{ subject to } x \in X ,$$

where  $\alpha \geq 0$ .

$$P4(\beta) \quad \text{maximize } \Pr\{\xi^T x \geq \beta\} \text{ subject to } x \in X ,$$

where  $\beta \in (-\infty, \infty)$ ; and

$$P5(\gamma) \quad \text{maximize } \theta \text{ subject to } \Pr\{\xi^T x \geq \theta\} \geq \gamma \text{ and } x \in X ,$$

where  $\gamma \in (0, 1)$ .

The problem P2 is the so-called mean-variance model due to Markowitz [7]: it can be shown, under the assumptions stated above, that all efficient solutions of P2 are generated by P3( $\alpha$ ) for all  $\alpha \geq 0$  (see Theorem 2 below). Solving P3( $\alpha$ ), parametrically for all  $\alpha \geq 0$ , can be done in an efficient manner, by using one of several existing methods. For our purpose, the parametric version of Graves' principal pivoting algorithm (Graves [3]; see Cottle [1] for the parametric version) will be most convenient. It is also noted that the model due to Freund [2] based on a risk averse, exponential utility function amounts to solving P3( $\alpha$ ) for  $\alpha > 0$ .

The model P4( $\beta$ ) is often referred to as Charnes-Cooper's P-model. It is not difficult to show that under the condition (1) above and that  $X$  does not contain the origin, P4( $\beta$ ) is equivalent to the fractional program:

$$P4'(\beta) \quad \text{maximize } \frac{c^T x - \beta}{\sqrt{x^T V x}} \text{ subject to } x \in X .$$

<sup>2</sup>A point  $\bar{x} \in X$  is efficient if and only if there is no  $x \in X$  such that  $f(x) \geq f(\bar{x})$  and  $f(x) \neq f(\bar{x})$ .

It is worthwhile, and interesting, to observe that a certain portfolio selection problem based on Tobin's risk-free assets assumption (see [9]) is formulated (Lintner [6]) as the same fractional program stated above.

The problem  $P4(\beta)$  for fixed  $\beta$  may be solved by using some nonlinear programming algorithm applied to  $P4'(\beta)$ , or alternatively to a certain (strictly concave) nonlinear program  $(P4''(\beta, \lambda))$ , as stated in Section 2) equivalent to  $P4'(\beta)$ . The problem  $P4'(\beta)$  arising in the risk-free portfolio selection problem can be transformed into a linear complementarity problem (see Ziemba et al. [10] and also Pang [8]) with a positive definite matrix; and thus it can be solved efficiently by some complementary pivoting algorithm. The validity of this transformation, however, depends critically on the assumption that the problem contains the constraint  $e^T x = 1$ , where  $e$  is the summation vector (which is a reasonable assumption in the portfolio setting).

Finally, the model  $P5(\gamma)$  was proposed by Kataoka [5]<sup>3</sup>, who outlined two algorithms which are potentially capable of solving  $P5(\gamma)$  for fixed  $\gamma$ . These algorithms are iterative schema solving a (concave maximization) quadratic program in each iteration. In [5] it was assumed (in addition to (1) and (2) above) that  $0 \notin X$  and  $\gamma \in (\frac{1}{2}, 1)$ .

In the next section we shall derive some properties of the solutions of  $P3(\alpha)$  and also relate  $P3(\alpha)$  to  $P2$ . We shall then demonstrate that solving  $P3(\alpha)$ ,  $\alpha \geq 0$ , will simultaneously solve  $P4(\beta)$  (treated in Section 3) and  $P5(\gamma)$  (Section 4), for all relevant values of  $\beta$  and  $\gamma$  respectively. It will also be shown that solving  $P4(\beta)$  (or  $P5(\gamma)$ , respectively) for fixed value of  $\beta$  (or  $\gamma$ ) can be done efficiently without having to solve  $P3(\alpha)$  parametrically for all  $\alpha \geq 0$ . Certain relationships between the values of the parameters ( $\beta$  and  $\gamma$ ) and the corresponding optimal objective values of the problems  $P4(\beta)$  and  $P5(\gamma)$  will be expounded in the fifth and final section.

## 2. Some Properties of the Solutions of the Parametric Quadratic Program $P3(\alpha)$ and Their Relationship to Efficient Solutions of $P2$ .

We first note that it follows from (1) and (2) that for every  $\alpha \geq 0$ ,  $P3(\alpha)$  has a

<sup>3</sup>Kataoka's model has  $P\{\xi^T x \geq \theta\} = \gamma$  rather than  $\geq$  as in  $P5(\gamma)$ ; but these two are equivalent under the condition  $0 \notin X$  assumed in [5].

unique solution. In fact, under these conditions,  $X$  is nonempty and the objective value of  $P3(\alpha)$  is bounded from above by the optimal objective value of  $P1$ ; uniqueness is an obvious consequence of the assumed positive definiteness of  $V$ .

For each  $\alpha \geq 0$ , we shall let  $\varphi(\alpha)$  denote the unique solution of  $P3(\alpha)$ . It is noted that  $\varphi(\alpha)$  for all  $\alpha \geq 0$  may be computed by the parametric principal pivoting algorithm; in particular,  $\varphi(\alpha)$  is continuous and piecewise-linear in  $\alpha \geq 0$ .

In the remainder of the paper we shall use the following notation:

$$X^1 = \{x : x \text{ solves } P1\} ,$$

$$X^2 = \{x : x \text{ is an efficient solution of } P2\} ;$$

$$X^3 = \{\varphi(\alpha) : \alpha \geq 0\} ,$$

$$x^* = \operatorname{argmin}\{x^T V x : x \in X^1\} ,$$

$$\alpha^* = \text{the final "critical value" generated in the parametric Graves' algorithm: if no pivot is necessary, set } \alpha^* = 0.$$

$$r(\alpha) = c^T \varphi(\alpha) - \frac{1}{\alpha} \varphi(\alpha)^T V \varphi(\alpha) \quad (\text{used in Section 3}),$$

$$s(\alpha) = \frac{-1}{\alpha} (\varphi(\alpha)^T V \varphi(\alpha))^{\frac{1}{2}} \quad (\text{used in Section 4}) .$$

Note that  $x^*$  exists and is unique, and that  $X^1 \cap X^2 = \{x^*\}$ .

We shall prove:

Theorem 1.

For  $\alpha \geq \alpha^*$ ,  $\varphi(\alpha) = x^*$  and (hence)  $X^3 = \{\varphi(\alpha) : 0 \leq \alpha \leq \alpha^*\}$ .

Proof: We first show that  $X^3$  is bounded. For every  $\alpha \geq 0$ , we have:

$$ac^T \varphi(\alpha) - \frac{1}{2} \varphi(\alpha)^T V \varphi(\alpha) \geq ac^T x^* - \frac{1}{2} (x^*)^T V x^* ,$$

and

$$ac^T \varphi(\alpha) \leq ac^T x^* .$$

Thus, for each  $\alpha \geq 0$ ,

$$(3) \quad \varphi(\alpha)^T V \varphi(\alpha) \leq (x^*)^T V x^* .$$

Since  $V$  is positive definite, the boundedness follows.

Now, by definition  $\alpha^*$  is the last critical value generated in the parametric version of Graves' principal pivoting algorithm (see Cottle [1]) applied to the parametric linear complementarity problem representing the Kuhn-Tucker conditions of the parametric quadratic program  $P3(\alpha)$ ,  $\alpha \geq 0$ . That is, the same complementary basis of  $(I, -M)$  is feasible for every  $\alpha \geq \alpha^*$ , where

$$M = \begin{bmatrix} V & A^T \\ -A & 0 \end{bmatrix} .$$

It then follows that,

$$(4) \quad \varphi(\alpha) = \varphi(\alpha^*) \quad , \quad \alpha \geq \alpha^* \quad ,$$

i.e.,  $\varphi(\alpha^*)$  solves  $P3(\alpha)$  for  $\alpha \geq \alpha^*$ .

For  $\alpha \geq \alpha^*$ ,  $\alpha > 0$ , we then have that

$$(5) \quad c^T \varphi(\alpha^*) - \frac{1}{2\alpha} \varphi(\alpha^*)^T V \varphi(\alpha^*) \geq c^T x^* - \frac{1}{2\alpha} (x^*)^T V x^* .$$

Letting  $\alpha \rightarrow \infty$  in (5) we obtain  $c^T \varphi(\alpha^*) \geq c^T x^*$ . But since  $x^*$  solves  $P1$  and  $\varphi(\alpha^*) \in X$ , the inequality holds in the other direction also, implying  $c^T \varphi(\alpha^*) = c^T x^*$ , or  $\varphi(\alpha^*) \in X^1$ .

With this equality, (5) yields

$$\varphi(\alpha^*)^T V \varphi(\alpha^*) \leq (x^*)^T V x^* .$$

Thus by the definition of  $x^*$ , we have  $\varphi(\alpha^*) = x^*$ .

□

Using Theorem 1 we shall give a proof of the well-known fact that all efficient solutions of  $P2$  are generated by solving  $P3(\alpha)$  for  $\alpha \geq 0$ .

Theorem 2.

$$X^2 = \{ \varphi(\alpha) : 0 \leq \alpha \leq \alpha^* \} .$$

Proof: In view of Theorem 1, we only need to show  $X^2 = X^3$ . The relation  $X^2 \supset X^3$  is obvious. To show the other implication let  $\hat{x} \in X^2$ . Define the closed, convex set  $K$  by

$$K = \left\{ \begin{array}{l} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbb{R}^2 : \\ y_1 \leq c^T x - c^T \hat{x} \\ y_2 \leq \frac{1}{2} (-x^T V x + \hat{x}^T V \hat{x}) \end{array} \quad \begin{array}{l} \text{for some} \\ x \in X \end{array} \right\} .$$

It follows from  $\hat{x} \in X^2$  that  $K$  has the empty intersection with the interior of the non-negative orthant of  $\mathbb{R}^2$ . Then by the separation theorem, there exists  $p = (p_1, p_2)^T \in \mathbb{R}^2$ ,  $p \geq 0$ ,  $p \neq 0$  for which  $p^T y \leq 0$  for every  $y \in K$ . Since for each  $x \in X$  the point

$$\begin{pmatrix} c^T x - c^T \hat{x} \\ \frac{1}{2} [-x^T V x + \hat{x}^T V \hat{x}] \end{pmatrix}$$

belongs to  $K$ , it follows that for every  $x \in X$ ,

$$p_1 c^T \hat{x} - \frac{1}{2} p_2 \hat{x}^T V \hat{x} \geq p_1 c^T x - \frac{1}{2} p_2 x^T V x .$$

If  $p_2 > 0$ , then  $\hat{x} = \varphi(p_1/p_2)$ ; otherwise ( $p_1 > 0$ , necessarily)  $\hat{x} \in X^1$ . Since  $\hat{x} \in X^2$  by hypothesis  $\hat{x} = x^*$  and hence  $\hat{x} \in X^3$ .

□

Finally, we shall present a couple of properties of  $\varphi(\alpha)$  which will be useful in the subsequent sections.

Property 3.

- (i)  $\varphi(\alpha)^T V \varphi(\alpha)$  is monotone nondecreasing in  $\alpha \geq 0$ .
- (ii) Assume  $0 \in X$ . Then either  $\varphi(\alpha) = 0$  for each  $\alpha \geq 0$ , or  $\varphi(\alpha) \neq 0$  for each  $\alpha > 0$ .

Proof: The proof of (i) is contained in Kataoka [5]. To show (ii) assume  $0 \in X$

(so  $\varphi(0) = 0$ ) and suppose  $\varphi(\tilde{\alpha}) \neq 0$  for some  $\tilde{\alpha} > 0$ . From (i) and Theorem 1 it follows that  $x^* \neq 0$ , which implies that

$$\max\{c^T x : x \in X\} > 0 ;$$

thus  $c^T x^* > 0$ . Since both  $0$  and  $x^*$  belong to  $X$ , so does  $\lambda x^*$  for every  $\lambda \in [0,1]$ .

For  $\lambda \in (0,1]$ , we have that the objective value of  $P3(\alpha)$  for each  $\alpha > 0$  is

$$\lambda a c^T x^* - \frac{\lambda^2}{2} (x^*)^T V x^* .$$

Since  $c^T x^* > 0$ , this value is strictly positive for sufficiently small  $\lambda > 0$ . Thus,  $x = 0$  can not be the solution of  $P3(\alpha)$  for  $\alpha > 0$ .

□

### 3. Relationships between the Parametric Quadratic Program $P3(\alpha)$ and P-Model $P4(\beta)$ .

To simplify the argument, we shall first assume that  $X$  does not contain the origin and derive the relationships between  $P3(\alpha)$  and  $P4(\beta)$ . We shall then show how to modify the argument to handle the case  $0 \in X$ .

As was mentioned in Section 1, the problem  $P4(\beta)$  for fixed  $\beta$  is equivalent to  $P4'(\beta)$  (under  $0 \notin X$ ). By using a standard result in fractional programming (see e.g. Jagannathan [4]),  $P4'(\beta)$  is further transformed into:

$$P4''(\beta, \lambda) \quad \text{maximize} \quad c^T x - \beta - \lambda (x^T V x)^{\frac{1}{2}} \quad \text{subject to} \quad x \in X .$$

More specifically, for fixed  $\beta \in \mathbb{R}$   $x \in X$  solves  $P4'(\beta)$  if and only if there exists  $\lambda \in \mathbb{R}$  such that  $x$  solves  $P4''(\beta, \lambda)$  and  $c^T x - \beta - \lambda (x^T V x)^{\frac{1}{2}} = 0$ .

We prove:

#### Theorem 4.

Assume  $0 \notin X$ .

(i) For every  $\alpha > 0$ ,  $\varphi(\alpha)$  solves  $P4(\beta)$  with  $\beta = r(\alpha)$ , where  $r(\alpha) = c^T \varphi(\alpha) - \varphi(\alpha)^T V \varphi(\alpha) / \alpha$ .

(ii) For every  $\beta \in (-\infty, \beta^*)$ , where  $\beta^* = c^T x^*$ , there exists  $\alpha > 0$  such that  $\beta = r(\alpha)$  and  $\varphi(\alpha)$  solves  $P4(\beta)$ .

(iii) For every  $x$  solving  $P4(\beta)$  with  $\beta \in (-\infty, \beta^*)$ , it holds that  $x = \varphi(\alpha)$ , where  $\alpha = (x^T V x) / (c^T x - \beta)$ .

Proof: (i) Let  $\alpha > 0$  be given. Since  $\varphi(\alpha)$  solves the quadratic program  $P3(\alpha)$ , the following Kuhn-Tucker conditions hold:

$$(6.1) \quad z = -\alpha c + V \varphi(\alpha) + A^T u \geq 0, \quad \varphi(\alpha) \geq 0, \quad z^T \varphi(\alpha) = 0 ,$$

$$(6.2) \quad y = b - A\varphi(\alpha) \geq 0, \quad u \geq 0, \quad y^T u = 0.$$

Letting  $\lambda = -s(\alpha) = (\varphi(\alpha)^T V \varphi(\alpha))^{-\frac{1}{2}} / \alpha > 0$ ,  $z' = z/\alpha$  and  $u' = u/\alpha$ , we have:

$$z' = -c + \lambda V \varphi(\alpha) / (\varphi(\alpha)^T V \varphi(\alpha))^{-\frac{1}{2}} + A^T u' \geq 0,$$

$$\varphi(\alpha) \geq 0, \quad (z')^T \varphi(\alpha) = 0,$$

$$y = b - A\varphi(\alpha) \geq 0, \quad u' \geq 0, \quad y^T u' = 0,$$

which implies that  $\varphi(\alpha)$  satisfies the Kuhn-Tucker conditions for  $P4''(\beta, \lambda)$ . Since  $\lambda > 0$ , the objective function of  $P4''(\beta, \lambda)$  is strictly concave and hence  $\varphi(\alpha)$  solves  $P4''(\beta, \lambda)$ .

From  $\beta = c^T \varphi(\alpha) - (\varphi(\alpha)^T V \varphi(\alpha)) / \alpha$  it follows that

$$c^T \varphi(\alpha) - \beta - \lambda (\varphi(\alpha)^T V \varphi(\alpha))^{-\frac{1}{2}} = 0$$

and so  $\varphi(\alpha)$  solves  $P4'(\beta)$ , and hence  $P4(\beta)$ . This proves the first part.

(ii) The second part is proved by examining some properties of  $r(\alpha)$ . By Theorem 1,  $\varphi(\alpha) = x^*$  for  $\alpha \geq \alpha^*$ ; and so

$$\lim_{\alpha \rightarrow \infty} r(\alpha) = \beta^*.$$

Since  $0 \notin X$ ,  $\varphi(0) \neq 0$ ; and thus

$$\lim_{\alpha \rightarrow 0^+} r(\alpha) = -\infty.$$

Since  $r(\alpha)$  is clearly continuous we have that<sup>4</sup>

$$(7) \quad \{r(\alpha) : \alpha > 0\} = (-\infty, \beta^*).$$

Thus, for every  $\beta \in (-\infty, \beta^*)$  there exists  $\alpha > 0$  such that  $\beta = r(\alpha)$ : by part (i)  $\varphi(\alpha)$  solves  $P4(\beta)$ . This completes the proof of part (ii).

(iii) The proof of part (iii) can be done by tracing the steps of the proof for part (i) in the opposite way. That is, if  $x$  solves  $P4(\beta)$ , then it solves  $P4''(\beta, \lambda)$  with

<sup>4</sup> Furthermore, it can be shown (see Section 5) that  $r(\alpha)$  is monotone nondecreasing in  $\alpha \geq 0$ .

$\lambda = (c^T x - \beta) / (x^T V x)^{\frac{1}{2}} > 0$ . Then by setting  $\alpha = (x^T V x)^{\frac{1}{2}} / \lambda = (x^T V x) / (c^T x - \beta)$  it follows that  $x$  solves  $P3(\alpha)$  since  $P3(\alpha)$  has a unique solution.

□

The above theorem implies that the set of solutions of  $P3(\alpha)$ ,  $\alpha > 0$ , is exactly the same as that of  $P4(\beta)$ ,  $\beta \in (-\infty, \beta^*)$ . (Note that from definition it follows that when  $\beta = \beta^*$ ,  $x^* = \varphi(\alpha^*)$  solves  $P4(\beta)$ .) In view of Theorem 2, it may be argued that solving  $P4(\beta)$  for  $\beta > \beta^*$  is unnecessary, at least from the standpoint of efficient solutions.

The following result is proved directly from the above theorem.

Corollary.

Assume  $0 \notin X$ . Let  $\bar{\beta} \in (-\infty, \beta^*)$  and  $\bar{\alpha} > 0$  be given. If  $\bar{\beta} = r(\bar{\alpha})$ , then  $\varphi(\bar{\alpha})$  solves  $P4(\bar{\beta})$ . If  $\bar{\beta} > r(\bar{\alpha})$ , then there exists  $\hat{\alpha} \in (\bar{\alpha}, \infty)$  such that  $\bar{\beta} = r(\hat{\alpha})$  and  $\varphi(\hat{\alpha})$  solves  $P4(\bar{\beta})$ . If  $\bar{\beta} < r(\bar{\alpha})$ , then there exists  $\hat{\alpha} \in (0, \bar{\alpha})$  for which  $\bar{\beta} = r(\hat{\alpha})$  and  $\varphi(\hat{\alpha})$  solves  $P4(\bar{\beta})$ .

This result is useful when one wants to solve  $P4(\beta)$  for fixed value  $\beta = \bar{\beta}$ ; it suggests that one does not have to solve  $P3(\alpha)$  for all  $\alpha \geq 0$ . Instead, first make a guess on a likely value of a solution of the problem  $P4(\bar{\beta})$ , say  $\bar{x}$ . In many practical cases, a reasonably good estimate of an optimal solution is available. Let  $\bar{\alpha}$  be defined by

$$\bar{\alpha} = \frac{-\bar{x}^T V \bar{x}}{c^T \bar{x} - \bar{\beta}} > 0,$$

and solve  $P3(\bar{\alpha})$ . Depending on the relative magnitude of  $r(\bar{\alpha})$  and  $\bar{\beta}$ , one then solves  $P3(\alpha)$  parametrically for  $\alpha \geq \bar{\alpha}$ , or  $\alpha \leq \bar{\alpha}$ . If the initial guess  $\bar{x}$  is close to the correct solution, this second stage of the computation would generate  $\hat{\alpha}$  such that  $\bar{\beta} = r(\hat{\alpha})$  quickly.

It is worth noting that the value of  $r(\alpha)$  for each  $\alpha > 0$  may be determined without evaluating the quadratic form of  $\varphi(\alpha)$ . In fact, it is well-known that if  $(x, u)$  is a Kuhn-Tucker point of the quadratic program  $P3(\alpha)$ , then it follows that  $x^T V x = c^T x - b^T u$ . Thus if we let  $u(\alpha)$  be the dual variable in  $P3(\alpha)$  corresponding to  $\varphi(\alpha)$ , we have that

$$r(\alpha) = \frac{1}{\alpha} b^T u(\alpha) .$$

In the remainder of this section we deal with the case where the origin is contained in  $X$ ; i.e.  $b \geq 0$ . In this case, some of the arguments employed above are no longer valid: for instance, the problem  $P4'(\beta)$  is not well-defined at  $x = 0$ , and  $r(\alpha)$  does not approach  $-\infty$  as  $\alpha$  approaches to zero. However, it will be shown that essentially the same results as those in Theorem 4 and its corollary hold in this case also.

Theorem 5.

Assume  $0 \in X$ .

- (i) For every  $\alpha > 0$  such that  $r(\alpha) \neq 0$ ,  $\varphi(\alpha)$  solves  $P4(\beta)$  with  $\beta = r(\alpha)$ .
- (ii) For every  $\beta \in (-\infty, 0]$ ,  $x = 0$  solves  $P4(\beta)$ . For every  $\beta \in (0, \beta^*)$ , there exists  $\alpha > 0$  such that  $\beta = r(\alpha)$  and  $\varphi(\alpha)$  solves  $P4(\beta)$ .
- (iii) For every  $x$  solving  $P4(\beta)$  for  $\beta \in (0, \beta^*)$ , it holds that  $x = \varphi(\alpha)$ , where  $\alpha = (x^T Vx) / (c^T x - \beta)$ .

Proof: (i) Since  $r(\alpha) \neq 0$ , we have  $\varphi(\alpha) \neq 0$ . We note that since  $r(\alpha) = b^T u(\alpha) / \alpha$  and  $b \geq 0$ ,  $r(\alpha) \neq 0$  implies  $r(\alpha) > 0$ . Let  $\epsilon^* = e^T \varphi(\alpha) > 0$ , where  $e$  is the summation vector. Then  $\varphi(\alpha)$  solves  $P3(\alpha)$  with the additional constraint

$$(8) \quad e^T x \geq \epsilon ,$$

for every  $\epsilon \in [0, \epsilon^*]$ . From Theorem 4 it follows that  $\varphi(\alpha)$  solves  $P4(r(\alpha))$  with the additional constraint (8), for each  $\epsilon \in (0, \epsilon^*]$ . This implies that no  $x \in X$ ,  $x \neq 0$  is a "better" solution than  $\varphi(\alpha)$  in  $P4(r(\alpha))$ . Since  $r(\alpha) > 0$ , the objective value in  $P4(r(\alpha))$  for  $x = 0$  is obviously zero but that corresponding to  $\varphi(\alpha)$  is positive since  $\varphi(\alpha) \neq 0$ ; thus  $\varphi(\alpha)$  is a "better" solution than  $x = 0$  in  $P4(r(\alpha))$ . Therefore,  $\varphi(\alpha)$  solves  $P4(r(\alpha))$ .

(ii) First note that whenever  $\beta \leq 0$ ,  $x = 0$  solves  $P4(\beta)$  by definition of the problem. In particular, if  $x^* = 0$ , then  $\beta^* = c^T x^* = 0$  and so  $x = 0$  is the trivial solution. Henceforth we shall assume  $x^* \neq 0$ . To prove the assertion in (ii) for  $\beta \in (0, \beta^*]$ , we shall examine some properties of  $r(\alpha)$  under the assumption  $0 \in X$ . First, the continuity

of  $r(\alpha)$  and that  $r(\alpha)$  approaches  $\beta^*$  as  $\alpha \rightarrow \infty$  are true whether or not  $0 \in X$ . We have, however,

$$(9) \quad \lim_{\alpha \rightarrow 0^+} r(\alpha) = 0 .$$

To see this, recall that  $\varphi(\alpha)$  is continuous, piecewise-linear in  $\alpha$ ; and thus for small  $\alpha$ ,  $\varphi(\alpha)$  takes on the form  $\varphi(\alpha) = \alpha d$  for some  $n$ -vector  $d$ . Then we have

$$r(\alpha) = \alpha c^T d - \frac{1}{\alpha} (\alpha^2 d^T V d) = \alpha (c^T d - d^T V d) ;$$

hence (9) holds. Thus for every  $\beta \in (0, \beta^*)$ , there exists  $\alpha > 0$  for which  $\beta = r(\alpha)$ ; by part (i)  $\varphi(\alpha)$  solves  $P4(\beta)$ .

(iii) The third part is proved in a similar way to that of the proof of part (iii) in Theorem 4 for the  $0 \notin X$  case. We note, however, that unless  $X = \{0\}$ ,  $x = 0$  cannot be a solution to  $P4(\beta)$  for  $\beta \in (0, \beta^*)$ : if  $X = \{0\}$ , then  $\beta^* = 0$  and so the statement is true by default. For  $x \neq 0$ , solving  $P4(\beta)$ , we can "cut off" the origin to make the argument based on the Kuhn-Tucker conditions valid.

□

The following is direct from Theorem 5.

Corollary.

Assume  $0 \in X$ . Let  $\bar{\beta} \in (0, \beta^*)$  and  $\bar{\alpha} > 0$  be given. If  $\bar{\beta} = r(\bar{\alpha})$ , then  $\varphi(\bar{\alpha})$  solves  $P4(\bar{\beta})$ . If  $\bar{\beta} > r(\bar{\alpha})$ , then there exists  $\hat{\alpha} \in (\bar{\alpha}, \infty)$  such that  $\bar{\beta} = r(\hat{\alpha})$  and  $\varphi(\hat{\alpha})$  solves  $P4(\bar{\beta})$ . If  $\bar{\beta} < r(\bar{\alpha})$ , then there exists  $\hat{\alpha} \in (0, \bar{\alpha})$  such that  $\bar{\beta} = r(\hat{\alpha})$  and  $\varphi(\hat{\alpha})$  solves  $P4(\bar{\beta})$ .

#### 4. Relationships between the Parametric Quadratic Program $P3(\alpha)$ and Kataoka Model $P5(\gamma)$ .

Assuming (1)-(2) and  $0 \notin X$ , Kataoka [5] showed that  $P5(\gamma)$  for fixed  $\gamma \in (0, 1)$  is transformed into the following problem:

$$P5'(\delta) \quad \text{maximize } c^T x + \delta (x^T V x)^{\frac{1}{2}} \text{ subject to } x \in X .$$

Here  $\delta = I^{-1}(\gamma)$ , where for  $s \in \mathbb{R}$

$$I(s) = \frac{1}{\sqrt{2\pi}} \int_s^{\infty} e^{-\frac{y^2}{2}} dy ,$$

and  $I^{-1}$  is the inverse map of  $I$ . In view of this equivalence<sup>5</sup>, it is clear that  $\theta$  in  $P5(\gamma)$  need not be treated explicitly and hence we shall say  $x$  (rather than  $(x, \theta)$ ) solves the problem  $P5(\gamma)$ .

We have:

Theorem 6.

Assume  $0 \notin X$ .

- (i) For every  $\alpha > 0$ ,  $\varphi(\alpha)$  solves  $P5(\gamma)$  with  $\gamma = I(s(\alpha))$ , where  $s(\alpha) = -(\varphi(\alpha)^T V \varphi(\alpha))/\alpha$ .
- (ii) For every  $\gamma \in (\frac{1}{2}, 1)$  there exists  $\alpha > 0$  such that  $\gamma = I(s(\alpha))$  and  $\varphi(\alpha)$  solves  $P5(\gamma)$ .
- (iii) For every  $x$  solving  $P5(\gamma)$ ,  $\gamma \in (\frac{1}{2}, 1)$ , it holds that  $x = \varphi(\alpha)$ , where  $\alpha = (x^T V x)^{\frac{1}{2}} / I^{-1}(\gamma)$ .

Proof: (i) By definition,  $\varphi(\alpha)$  satisfies the Kuhn-Tucker conditions (6) for  $P3(\alpha)$ .

Letting  $u' = u/\alpha$ , and  $z' = z/\alpha$  we have

$$z' = -c - s(\alpha) V \varphi(\alpha) / (\varphi(\alpha)^T V \varphi(\alpha))^{\frac{1}{2}} + A^T u' \geq 0 ,$$

$$\varphi(\alpha) \geq 0, \quad (z')^T \varphi(\alpha) = 0 ,$$

$$y = b - A \varphi(\alpha) \geq 0, \quad u' \geq 0, \quad y^T u' = 0 .$$

That is,  $\varphi(\alpha)$  satisfies the Kuhn-Tucker conditions for  $P5'(\delta)$  for  $\delta = s(\alpha)$ . Since  $s(\alpha) < 0$  the conditions are sufficient for optimality. Thus  $\varphi(\alpha)$  solves  $P5'(\delta)$  and hence it solves  $P5(\gamma)$ , where  $\gamma = I(s(\alpha))$ . This completes the proof of (i).

(ii) To prove (ii), we shall examine some properties of  $s(\alpha)$ . By Theorem 1, we have

$$\varphi(\alpha) = x^* \quad \text{for } \alpha \geq \alpha^* ; \text{ so,}$$

$$\lim_{\alpha \rightarrow \infty} s(\alpha) = 0 .$$

<sup>5</sup>If  $X$  contains the origin, then  $P5(\gamma)$  may be restated as: Find  $\bar{x} \in X$ ,  $\bar{x} \neq 0$  such that  $c^T \bar{x} + I^{-1}(\gamma) \sqrt{\bar{x}^T V \bar{x}} = \max\{c^T x + I^{-1}(\gamma) \sqrt{x^T V x} : x \in X\} > 0$ : if no such  $x$  exists, then  $x = 0$  solves the problem.

Since  $\varphi(0) \neq 0$ ,

$$\lim_{\alpha \rightarrow 0^+} s(\alpha) = -\infty .$$

Furthermore,  $s(\alpha)$  is clearly continuous; thus

$$\{s(\alpha) : \alpha > 0\} = (-\infty, 0) ,$$

or,

$$\{I(s(\alpha)) : \alpha > 0\} = (\frac{1}{2}, 1) .$$

Thus for every  $\gamma \in (\frac{1}{2}, 1)$ , there exists  $\alpha > 0$  for which  $\gamma = I(s(\alpha))$ : by part (i),  $\varphi(\alpha)$  solves P5( $\gamma$ ).

□

Again, it may be argued that solving P5( $\gamma$ ) for  $\gamma < \frac{1}{2}$  is unnecessary from the standpoint of efficient solutions. We note that for  $\gamma = \frac{1}{2}$ , any solution to P1 solves P5( $\gamma$ ). We note that letting  $u(\alpha)$  denote the dual variable corresponding to  $\varphi(\alpha)$  in P3( $\alpha$ ), then we have

$$(10) \quad s(\alpha) = -\frac{1}{2} (c^T \varphi(\alpha) - b^T u(\alpha)) \frac{1}{2} .$$

For computational convenience (see the remarks immediately after Corollary to Theorem 4 in the preceding section) we state:

Corollary.

Assume  $0 \notin X$ . Let  $\bar{\gamma} \in (\frac{1}{2}, 1)$  and  $\bar{\alpha} > 0$  be given. If  $\bar{\gamma} = I(s(\bar{\alpha}))$ , then  $\varphi(\bar{\alpha})$  solves P5( $\bar{\gamma}$ ). If  $\bar{\gamma} < I(s(\bar{\alpha}))$ , then there exists  $\hat{\alpha} \in (\bar{\alpha}, \infty)$  such that  $\bar{\gamma} = I(s(\hat{\alpha}))$  and  $\varphi(\hat{\alpha})$  solves P5( $\bar{\gamma}$ ). If  $\bar{\gamma} > I(s(\bar{\alpha}))$ , then there exists  $\hat{\alpha} \in (0, \bar{\alpha})$  for which  $\bar{\gamma} = I(s(\hat{\alpha}))$  and  $\varphi(\hat{\alpha})$  solves P5( $\bar{\gamma}$ ).

Now we shall consider the case  $0 \in X$ .

Recall that  $\varphi(\alpha)$  is continuous and piecewise-linear in  $\alpha > 0$ . If  $0 \in X$ , then  $\varphi(0) = 0$  and so for small  $\alpha \geq 0$ ,  $\varphi(\alpha) = \alpha d$  holds for some  $n$ -vector  $d$ . In view of Property 3,  $d = 0$  if and only if  $x^* = 0$ . Let

$$(11) \quad s^* = \lim_{\alpha \rightarrow 0^+} s(\alpha) = -\frac{1}{2} (d^T v d)$$

$$(12) \quad \gamma^* = \begin{cases} I(s^*) & \text{if } d \neq 0 \\ \frac{1}{2} & \text{if } d = 0 \end{cases} .$$

We have:

Theorem 7.

Assume  $0 \in X$ .

- (i) For every  $\alpha > 0$  such that  $s(\alpha) \neq 0$ ,  $\varphi(\alpha)$  solves  $P5(\gamma)$  with  $\gamma = I(s(\alpha))$ .
- (ii) For every  $\gamma \in [\gamma^*, 1)$ ,  $x = 0$  solves  $P5(\gamma)$ . For every  $\gamma \in (\frac{1}{2}, \gamma^*)$ , there exists  $\alpha > 0$  for which  $\gamma = I(s(\alpha))$  and  $\varphi(\alpha)$  solves  $P5(\gamma)$ .
- (iii) For every  $x$  solving  $P5(\gamma)$  for  $\gamma \in (\frac{1}{2}, \gamma^*)$ , it holds that  $x = \varphi(\alpha)$ , where  $\alpha = (x^T V x)^{\frac{1}{2}} / I^{-1}(\gamma)$ .

Proof: (i) Since  $s(\alpha) \neq 0$ ,  $\varphi(\alpha) \neq 0$ . Let  $\epsilon^* = e^T \varphi(\alpha) > 0$ . Then  $\varphi(\alpha)$  solves  $P3(\gamma)$  with the additional constraint

$$(13) \quad e^T x \geq \epsilon$$

for every  $\epsilon \in [0, \epsilon^*]$ . By Theorem 6  $\varphi(\alpha)$  solves  $P5(I(s(\alpha)))$  with the additional constraint (13) for every  $\epsilon \in (0, \epsilon^*]$ . This implies that no  $x \in X$ ,  $x \neq 0$  is a "better" solution than  $\varphi(\alpha)$  in  $P5(I(s(\alpha)))$ . The objective value of  $x = 0$  in  $P5(I(s(\alpha)))$  is zero; whereas, that of  $\varphi(\alpha)$  is  $r(\alpha)$  (c.f. Section 5) which is nonnegative (c.f. the proof of Theorem 5). Thus,  $\varphi(\alpha)$  is at least as "good" as  $x = 0$ . Hence,  $\varphi(\alpha)$  solves  $P5(I(s(\alpha)))$ .

(ii) If  $\gamma^* = 1/2$ , then  $x^* = 0$  and  $\beta^* = 0$ . In this case, for any  $x \in X$ ,  $x \neq 0$ , the mean  $c^T x$  of the normal random variable  $\xi^T x$  is nonpositive: thus the objective value in  $P5(\gamma)$  corresponding to  $x \in X$ ,  $x \neq 0$  is nonpositive. Hence  $x = 0$  solves  $P5(\gamma)$  trivially. Assume  $\gamma^* > \frac{1}{2}$ . Then by the definition of  $\gamma^*$ ,  $d \neq 0$ . It follows from Property 3 that  $x^* \neq 0$ ; and so  $\beta^* > 0$ .

Suppose  $x = 0$  does not solve  $P5(\bar{\gamma})$  for some  $\bar{\gamma} \in [\gamma^*, 1)$ . Then there exists  $\bar{x} \in X$ ,  $\bar{x} \neq 0$  and  $\bar{\beta} > 0$  such that  $\Pr\{\xi^T \bar{x} \geq \bar{\beta}\} \geq \bar{\gamma}$ . Since  $\bar{\gamma} > \frac{1}{2}$ ,  $\bar{\beta} < \beta^*$ . Letting  $\hat{\beta} = \bar{\beta}/2$  we have  $0 < \hat{\beta} < \beta^*$  and  $\Pr\{\xi^T \bar{x} \geq \hat{\beta}\} > \bar{\gamma} \geq \gamma^*$ . By Corollary to Theorem 5 we can find

$\hat{\alpha} \in (0, \alpha^*]$  such that  $\hat{\beta} = r(\hat{\alpha})$  and  $\varphi(\hat{\alpha})$  solves  $P4(\hat{\beta})$ . It follows then that  $\Pr\{\xi^T \bar{x} \geq \hat{\beta}\} \leq I(s(\hat{\alpha}))$  and hence  $\gamma^* < I(s(\hat{\alpha}))$ . On the other hand, since  $s(\alpha)$  is nondecreasing in  $\alpha$  (Property 9 in Section 5) and by definition of  $\beta^*$  we have that  $\gamma^* \geq I(s(\alpha))$  for every  $\alpha > 0$ ; thus a contradiction. Hence,  $x = 0$  solves  $P5(\gamma)$  for every  $\gamma \in [\gamma^*, 1)$ . The proof of (ii) is complete if we note (11) - (12), that  $s(\alpha)$  is continuous and that  $s(\alpha) \rightarrow 0$  as  $\alpha$  approaches infinity.

(iii) Part (iii) follows from a similar argument to that proving (iii) in Theorem 6.

□

Corollary.

Assume  $0 \in X$ . Let  $\bar{\gamma} \in (\frac{1}{2}, \gamma^*)$  and  $\bar{\alpha} > 0$  be given. If  $\bar{\gamma} = I(s(\bar{\alpha}))$ , then  $\varphi(\bar{\alpha})$  solves  $P5(\bar{\gamma})$ . If  $\bar{\gamma} < I(s(\bar{\alpha}))$ , then there exists  $\hat{\alpha} \in (\bar{\alpha}, \infty)$  such that  $\bar{\gamma} = I(s(\hat{\alpha}))$  and  $\varphi(\hat{\alpha})$  solves  $P5(\bar{\gamma})$ . If  $\bar{\gamma} > I(s(\bar{\alpha}))$ , then there exists  $\hat{\alpha} \in (0, \bar{\alpha})$  for which  $\bar{\gamma} = I(s(\hat{\alpha}))$  and  $\varphi(\hat{\alpha})$  solves  $P5(\bar{\gamma})$ .

5. Relationships between P-Model  $P4(\beta)$  and Kataoka Model  $P5(\gamma)$ .

Comparing the definitions of  $P4(\beta)$  and  $P5(\gamma)$  one would note that the roles of the parameter and optimal value are reversed in the two problems, suggesting some close relationships between them. In fact, Theorems 4 - 7 imply the following.

Property 8.

For  $\alpha > 0$  let  $r(\alpha) = c^T \varphi(\alpha) - \varphi(\alpha)^T V \varphi(\alpha) / \alpha$  and  $s(\alpha) = -(\varphi(\alpha)^T V \varphi(\alpha)) / \alpha$ . Then for  $\alpha > 0$ ,  $\varphi(\alpha)$  solves  $P4(r(\alpha))$  and the corresponding objective value is  $I(s(\alpha))$ ;  $\varphi(\alpha)$  also solves  $P5(I(s(\alpha)))$  and the corresponding objective value is  $r(\alpha)$ .

If we note that the function  $I(s)$  is strictly decreasing in  $s$ , the above result implies the following relationship between  $r(\alpha)$  and  $s(\alpha)$ : for  $\alpha_1, \alpha_2 > 0$ ,

$$(14) \quad \text{sign}\{r(\alpha_1) - r(\alpha_2)\} = \text{sign}\{s(\alpha_1) - s(\alpha_2)\} .$$

In closing the paper we shall present the following result (which was used to prove the second part of Theorem 7).

Property 9.

Both  $r(\alpha)$  and  $s(\alpha)$  are monotone nondecreasing in  $\alpha > 0$ .

Proof: If  $x^* = 0$ , then  $\varphi(\alpha) = 0$  for all  $\alpha > 0$  and so  $r(\alpha) = 0$  and  $s(\alpha) = 0$  for all  $\alpha > 0$ . Henceforth, we assume that  $x^* \neq 0$ ; thus  $\varphi(\alpha) \neq 0$  for  $\alpha > 0$ . We shall establish the monotonicity of  $s(\alpha)$ ; then by (14) the monotonicity of  $r(\alpha)$  follows.

Noting that  $s(\alpha)$  is continuous and is converging to zero as  $\alpha \rightarrow \infty$ , the monotonicity is proved if one can show the following:

$$(15) \quad \begin{array}{ll} 0 < \alpha_1 < \alpha_2 & \text{implies} \\ s(\alpha_1) = s(\alpha_2) & \alpha \in [\alpha_1, \alpha_2] \end{array} \quad \begin{array}{l} s(\alpha) = s(\alpha_1) \\ \alpha \in [\alpha_1, \alpha_2] \end{array}$$

To show this assume  $0 < \alpha_1 < \alpha_2$  and  $s(\alpha_1) = s(\alpha_2)$ . Let  $\delta = s(\alpha_1)$ . Let  $\alpha \in (\alpha_1, \alpha_2)$  be given; define

$$\lambda_1 = \frac{\alpha_2 - \alpha}{\alpha_2 - \alpha_1} \quad \text{and} \quad \lambda_2 = \frac{\alpha - \alpha_1}{\alpha_2 - \alpha_1}.$$

Then  $\lambda_1 \geq 0$ ,  $\lambda_2 \geq 0$ ,  $\lambda_1 + \lambda_2 = 1$  and  $\alpha = \lambda_1 \alpha_1 + \lambda_2 \alpha_2$ . Let  $\epsilon^*$  be the minimum of  $e^T \varphi(\alpha_1)$  and  $e^T \varphi(\alpha_2)$ : since  $x^* \neq 0$  by hypothesis and  $\alpha_1, \alpha_2 > 0$ ,  $\epsilon^*$  is positive. Using the equivalence of P3( $\alpha$ ) and P5'(s( $\alpha$ )) where  $0 \notin X$ , we have, in this case, that  $\varphi(\alpha_1)$  and  $\varphi(\alpha_2)$  solve P5'(s( $\alpha$ )) with the additional constraint

$$(16) \quad e^T x \geq \epsilon$$

for every  $\epsilon \in [0, \epsilon^*]$ .

Since  $\delta < 0$ , P5'( $\delta$ ) is a concave maximization; thus  $\bar{x} = \lambda_1 \varphi(\alpha_1) + \lambda_2 \varphi(\alpha_2)$  also solves P5'( $\delta$ ). Then,

$$\begin{aligned} & c^T \bar{x} + \delta \frac{(x^T V x)^{-1}}{2} \\ &= \sum_{i=1}^2 \lambda_i (c^T \varphi(\alpha_i) + \delta (\varphi(\alpha_i)^T V \varphi(\alpha_i)) \frac{1}{2}), \\ &= c^T \bar{x} + \sum_{i=1}^2 \lambda_i \delta (\varphi(\alpha_i)^T V \varphi(\alpha_i)) \frac{1}{2}. \end{aligned}$$

It follows then

$$(17) \quad (\bar{x}^T V \bar{x})^{\frac{1}{2}} = \sum_{i=1}^2 \lambda_i (\varphi(\alpha_i)^T V \varphi(\alpha_i))^{\frac{1}{2}} .$$

By definition we have

$$\delta = s(\alpha_i) = -\frac{1}{\alpha_i} (\varphi(\alpha_i)^T V \varphi(\alpha_i))^{\frac{1}{2}} ,$$

or

$$(18) \quad (\varphi(\alpha_i)^T V \varphi(\alpha_i))^{\frac{1}{2}} = -\alpha_i \delta, \quad i = 1, 2 .$$

Substituting the quadratic forms in (17) by (18) we obtain

$$(19) \quad \delta = s(\alpha_1) = -\frac{1}{\alpha} (\bar{x}^T V \bar{x})^{\frac{1}{2}} .$$

The implication (15) will be proved, then, by showing that  $\bar{x} = \varphi(\alpha)$ .

Since  $\bar{x}$  solves  $P5'(\delta)$  with (16), the following Kuhn-Tucker conditions hold for some  $u, u_0, y$  and  $y_0$ :

$$(20.1) \quad z = -c + \delta V \bar{x} / (\bar{x}^T V \bar{x})^{\frac{1}{2}} + Au - eu_0 \geq 0$$

$$(20.2) \quad x \geq 0, \quad z^T x = 0 ,$$

$$(20.3) \quad y = b - Ax \geq 0, \quad u \geq 0, \quad y^T u = 0 ,$$

$$(20.4) \quad y_0 = -c^* + e^T x \geq 0, \quad u_0 \geq 0, \quad y_0^T u_0 = 0 .$$

Multiplying (20.1) by  $\alpha (= -(\bar{x}^T V \bar{x})^{\frac{1}{2}} / \delta$  by (19)), and letting  $u' = \alpha u$  and  $u_0' = \alpha u_0$ , we see that  $(\bar{x}, u', u_0')$  satisfies the Kuhn-Tucker conditions for  $P3(\alpha)$  with (16). Letting  $\epsilon \rightarrow 0$  we conclude that  $\bar{x}$  solves  $P3(\alpha)$ ; since  $P3(\alpha)$  has a unique solution  $\bar{x} = \varphi(\alpha)$ , as needed.

□

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