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THE SECOND LAW OF THERMODYNAMICS AND STABILITY, (U)  
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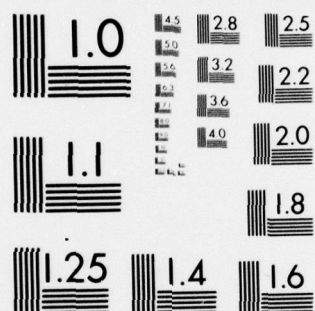
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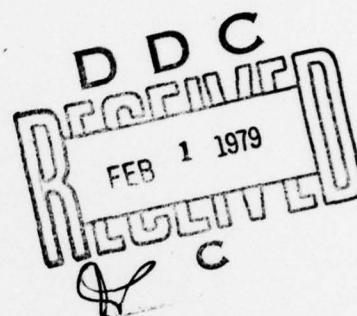
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THE SECOND LAW OF THERMODYNAMICS AND STABILITY<sup>+</sup>

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by



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## 1. Introduction

There are several indications pointing to an intimate relationship between the second law of thermodynamics and "stability". Most notably, the work of Ericksen [1] on thermoelasticity and subsequent developments by Coleman and Dill [2] and Gurtin [3] have demonstrated that the Clausius-Duhem inequality induces Liapunov stability of equilibrium processes. Here I attempt to establish a different connection between stability and the second law, in that "stability" will be interpreted as continuous dependence of thermodynamic processes upon initial state and supply terms.

The ideas will be presented within the context of thermoelasticity theory, without heat conduction. The hyperbolic character of the field equations causes the breakdown of smooth solutions and the development of shock waves so the class of smooth functions is far too narrow to encompass all processes of physical interest. The natural framework is the class of proper processes, characterized by the property that velocity, deformation gradient, specific entropy, stress, temperature and internal energy are functions of bounded variation in the sense of Tonelli-Cesari.<sup>(\*)</sup> The balance laws, in integral as well as in local form, are meaningful in the class of proper processes and the classical geometric theory of wave propagation can be transplanted into this class. One expects that

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<sup>(\*)</sup> that is, measurable functions whose first derivatives are Borel measures. This is essentially the broadest function class in which the Gauss-Green theorem holds. There is an analogy between the geometric structures of functions of bounded variation and functions that are piecewise smooth. For a survey see [4].

existence of solutions to the field equations will eventually be established within the class of proper processes but so far this has only been accomplished in the one-dimensional case [5].

The interpretation of the role of the second law of thermodynamics in the theory of proper processes poses an interesting problem. The idea of restricting constitutive relations so that the Clausius-Duhem inequality be automatically satisfied by smooth processes originated in the work of Coleman and Noll [6] and has by now become standard practice in continuum thermomechanics. On the other hand, within the class of processes with shock waves, the Clausius-Duhem inequality has traditionally been viewed as an admissibility criterion. It turns out that this criterion generally rules out some but not all extraneous processes so that one must either impose shock admissibility restrictions [7] or else strengthen the second law [8], in order to single out those processes that are physically relevant.

Despite the above remarks, it will be shown here that, whenever they exist, smooth processes are stable within the class of (not necessarily smooth) proper processes that satisfy the Clausius-Duhem inequality. In other words, for as long as one is dealing with smooth processes, the second law in its traditional form is a satisfactory admissibility criterion.

The above result is established under certain assumptions on material response, relating to convexity of internal energy. With the exception of one-dimensional bodies, global convexity of the internal energy function is contrary to experience (it is incompatible, in particular, with the principle of frame indifference).

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Even so, internal energy will in general be locally convex on a certain region in state space. In Section 4 we prove uniqueness and stability for smooth processes residing in the convexity region of internal energy. For the case where the body is smooth and bounded and the motion of its boundary is prescribed, stability is established, in Section 5, under the weaker assumption that the smooth process resides in the strong ellipticity region.

The general strategy of the proof was inspired by the important paper [9] of DiPerna on uniqueness of solutions to the initial value problem for quasilinear hyperbolic systems. The heart of the proof is an inequality, derived in Section 3, which estimates the evolution in time of the "distance" between the states of two processes originating at neighboring states. The time rate of increase of this distance can be controlled, with the help of the Clausius-Duhem inequality, provided that at least one of the processes is smooth and resides in the convexity (or the strong ellipticity) region of internal energy.

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## 2. Adiabatic Processes in Thermoelastic Materials

We consider a thermoelastic body with reference configuration  $\mathcal{B}$  in  $R^n$  ( $n = 1, 2$  or  $3$ ). The reference mass density  $\rho(\underline{X})$ , defined on  $\overline{\mathcal{B}}$ , is smooth and strictly positive,

$$(2.1) \quad \rho(\underline{X}) \geq \rho_0 > 0, \quad \underline{X} \in \overline{\mathcal{B}}.$$

A motion  $\underline{x} = \underline{x}(\underline{X}, t)$  determines the velocity field  $\underline{v} = \dot{\underline{x}}$  and the deformation gradient field  $\underline{F} = \text{grad}_{\underline{X}} \underline{x}$ . The internal energy  $\epsilon$ , the Piola-Kirchhoff stress  $\underline{T}$  and the temperature  $\theta$  are determined by the deformation gradient  $\underline{F}$  and the specific entropy  $\eta$  via constitutive relations

$$(2.2) \quad \epsilon = \epsilon^*(\underline{F}, \eta; \underline{X}), \quad \underline{T} = \underline{T}^*(\underline{F}, \eta; \underline{X}), \quad \theta = \theta^*(\underline{F}, \eta; \underline{X}),$$

where  $\epsilon^*, \underline{T}^*$  and  $\theta^*$  are smooth functions defined for  $\underline{F}$  in the set  $M^+$  of  $n \times n$  matrices with positive determinant,  $\eta$  in  $R$  and  $\underline{X}$  in  $\overline{\mathcal{B}}$ .<sup>(\*)</sup> We also require that

$$(2.3) \quad \theta^*(\underline{F}, \eta; \underline{X}) > 0, \quad \underline{F} \in M^+, \quad \eta \in R, \quad \underline{X} \in \overline{\mathcal{B}}.$$

We shall be assuming that the material is a non-conductor of heat so that heat flux and entropy flux vanish.

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<sup>(\*)</sup> In particular, we assume that the partial derivatives of  $\epsilon^*, \theta^*$  and  $\underline{T}^*$ , at any fixed  $\underline{F} \in M^+$  and  $\eta \in R$ , are bounded functions of  $\underline{X}$  on  $\overline{\mathcal{B}}$ .

By a proper thermodynamic process we mean fields  $(\underline{x}, \eta)(\underline{X}, t)$  such that  $(\underline{y}, \underline{F}, \eta)(\underline{X}, t)$  are functions of locally bounded variation, in the sense of Tonelli-Cesari, satisfying the balance laws of momentum and energy, viz.,

$$(2.4) \quad \rho \dot{\underline{v}}_i = T_{i\alpha, \alpha} + \rho b_i,$$

$$(2.5) \quad \rho \left( \epsilon + \frac{1}{2} v_i v_i \right) = (T_{i\alpha} v_i)_{, \alpha} + \rho b_i v_i + \rho r,$$

where  $b(\underline{X}, t)$  is the body force and  $r(\underline{X}, t)$  is the energy supply. The process will be called admissible if it also satisfies the Clausius-Duhem inequality

$$(2.6) \quad \dot{\eta} - \frac{r}{\theta} \geq 0.$$

The reader should keep in mind that for the class of proper processes (2.4)-(2.6) only hold in the sense of measures (or distributions) and thus one cannot simplify (2.5) in the standard fashion, since the usual product differentiation rule does not apply to products of functions of bounded variation.

A process  $(\bar{\underline{x}}, \bar{\eta})(\underline{X}, t)$  will be called smooth if the functions  $(\bar{\underline{v}}, \bar{\underline{F}}, \bar{\eta})(\underline{X}, t)$  are Lipschitz continuous, uniformly on bounded subsets of their domain. Thus a smooth process may contain weak waves but not shock waves. For smooth processes, one may write the balance laws in reduced form,

$$(2.7) \quad \rho \dot{\bar{\underline{v}}}_i = \bar{T}_{i\alpha, \alpha} + \rho \bar{b}_i,$$

$$(2.8) \quad \rho \dot{\bar{\epsilon}} = \bar{T}_{i\alpha} \bar{v}_{i,\alpha} + \rho \bar{r}.$$

As it is known, the standard requirement that every smooth process be admissible will be satisfied if and only if

$$(2.9) \quad \bar{T}^* = \rho \frac{\partial \epsilon^*}{\partial \bar{F}}, \quad \theta^* = \frac{\partial \epsilon^*}{\partial \eta}.$$

Indeed, upon using (2.9), (2.8) yields

$$(2.10) \quad \dot{\bar{\eta}} - \frac{\bar{r}}{\theta} = 0,$$

which shows that the Clausius-Duhem inequality is automatically satisfied by smooth processes, as an equality.

### 3. The Evolutionary Inequality

We establish here the estimate that will be the basis of our stability analysis. We define

$$(3.1) \quad H^*(v, F, \eta; \bar{v}, \bar{F}, \bar{\eta}; X) = \frac{1}{2} \rho(v_i - \bar{v}_i)(v_i - \bar{v}_i) + \rho \epsilon^*(F, \eta; X) \\ - \rho \epsilon^*(\bar{F}, \bar{\eta}; X) - T_{i\alpha}^*(\bar{F}, \bar{\eta}; X)(F_{i\alpha} - \bar{F}_{i\alpha}) - \rho \theta^*(\bar{F}, \bar{\eta}; X)(\eta - \bar{\eta}),$$

$$(3.2) \quad G_\alpha^*(v, F, \eta; \bar{v}, \bar{F}, \bar{\eta}; X) = -(T_{i\alpha}^*(F, \eta; X) - T_{i\alpha}^*(\bar{F}, \bar{\eta}; X))(v_i - \bar{v}_i).$$

On account of (2.9),  $H^*$  and  $G^*$  are of quadratic order in  $(v - \bar{v}, F - \bar{F}, \eta - \bar{\eta})$ .

Consider now a smooth process  $(\bar{x}, \bar{\eta})(X, t)$ , with supply terms  $(\bar{b}, \bar{r})(X, t)$ , and let  $(x, \eta)(X, t)$  be any admissible process with supply terms  $(b, r)(X, t)$ . We set

$$(3.3) \quad H(X, t) = H^*(v(X, t), F(X, t), \eta(X, t); \bar{v}(X, t), \bar{F}(X, t), \bar{\eta}(X, t); X),$$

$$(3.4) \quad G(X, t) = G^*(v(X, t), F(X, t), \eta(X, t); \bar{v}(X, t), \bar{F}(X, t), \bar{\eta}(X, t); X).$$

We shall be viewing  $H$  as a measure of the "distance" between the two processes. In order to see how  $H$  evolves in time, we compute below  $\dot{H} + \text{div}_X G$ , keeping in mind that the usual product differentiation rule applies (in the sense of measures) to the product of a Lipschitz continuous function with a function of bounded variation. We have

$$\begin{aligned}
(3.5) \quad \dot{H} + G_{\alpha, \alpha} &= \rho \left( \epsilon + \frac{1}{2} v_i v_i \right) - \rho v_i \dot{\bar{v}}_i - \rho \bar{v}_i \dot{v}_i + \rho \bar{v}_i \dot{\bar{v}}_i - \rho \dot{\epsilon} \\
&- \dot{\bar{T}}_{i\alpha} (F_{i\alpha} - \bar{F}_{i\alpha}) - \bar{T}_{i\alpha} \dot{F}_{i\alpha} + \bar{T}_{i\alpha} \dot{\bar{F}}_{i\alpha} - \rho \dot{\theta} (\eta - \bar{\eta}) \\
&- \rho \bar{\theta} (\dot{\eta} - \dot{\bar{\eta}}) - (T_{i\alpha} v_i)_{, \alpha} + (T_{i\alpha} - \bar{T}_{i\alpha}) \dot{\bar{F}}_{i\alpha} + \bar{T}_{i\alpha} \dot{F}_{i\alpha} \\
&+ v_i \bar{T}_{i\alpha, \alpha} + \bar{v}_i T_{i\alpha, \alpha} - \bar{v}_i \bar{T}_{i\alpha, \alpha}.
\end{aligned}$$

Observing that  $v_{i, \alpha} = \dot{F}_{i\alpha}$ ,  $\bar{v}_{i, \alpha} = \dot{\bar{F}}_{i\alpha}$  and using the balance laws (2.4), (2.5), (2.7), (2.8), we may rewrite (3.5) in the form

$$\begin{aligned}
(3.6) \quad \dot{H} + G_{\alpha, \alpha} &= \rho (v_i - \bar{v}_i) (b_i - \bar{b}_i) + \rho (r - \bar{r}) - \dot{\bar{T}}_{j\beta} (F_{j\beta} - \bar{F}_{j\beta}) \\
&+ \dot{\bar{F}}_{i\alpha} (T_{i\alpha} - \bar{T}_{i\alpha}) - \rho \dot{\theta} (\eta - \bar{\eta}) - \rho \bar{\theta} (\dot{\eta} - \dot{\bar{\eta}}) \\
&= \rho (v_i - \bar{v}_i) (b_i - \bar{b}_i) + \rho (r - \bar{r}) - \overline{\frac{\partial T_{j\beta}^*}{\partial F_{i\alpha}}} \dot{\bar{F}}_{i\alpha} (F_{j\beta} - \bar{F}_{j\beta}) \\
&- \overline{\frac{\partial T_{j\beta}^*}{\partial \eta}} \dot{\bar{\eta}} (F_{j\beta} - \bar{F}_{j\beta}) + \dot{\bar{F}}_{i\alpha} (T_{i\alpha} - \bar{T}_{i\alpha}) - \rho \overline{\frac{\partial \theta^*}{\partial F_{i\alpha}}} \dot{\bar{F}}_{i\alpha} (\eta - \bar{\eta}) \\
&- \rho \overline{\frac{\partial \theta^*}{\partial \eta}} \dot{\bar{\eta}} (\eta - \bar{\eta}) - \rho \bar{\theta} (\dot{\eta} - \dot{\bar{\eta}}).
\end{aligned}$$

By virtue of (2.9),

$$(3.7) \quad \rho \frac{\partial \theta^*}{\partial F_{i\alpha}} = \frac{\partial T_{i\alpha}^*}{\partial \eta}, \quad \frac{\partial T_{j\beta}^*}{\partial F_{i\alpha}} = \frac{\partial T_{i\alpha}^*}{\partial F_{j\beta}},$$

so that (3.6) yields

$$\begin{aligned}
(3.8) \quad \dot{H} + G_{\alpha, \alpha} = & \rho(v_i - \bar{v}_i)(b_i - \bar{b}_i) + \rho(r - \bar{r}) \\
& + \dot{\bar{F}}_{i\alpha} \{T_{i\alpha} - \bar{T}_{i\alpha} - \frac{\partial T_{i\alpha}^*}{\partial F_{j\beta}} (F_{j\beta} - \bar{F}_{j\beta}) - \frac{\partial T_{i\alpha}^*}{\partial \eta} (\eta - \bar{\eta})\} \\
& + \rho \dot{\bar{\eta}} \{ \theta - \bar{\theta} - \frac{\partial \theta^*}{\partial F_{j\beta}} (F_{j\beta} - \bar{F}_{j\beta}) - \frac{\partial \theta^*}{\partial \eta} (\eta - \bar{\eta}) \} \\
& - \rho \dot{\bar{\eta}} (\theta - \bar{\theta}) - \rho \bar{\theta} (\dot{\eta} - \dot{\bar{\eta}}).
\end{aligned}$$

From the Clausius-Duhem inequality (2.6) and (2.10), we obtain

$$(3.9) \quad \rho(r - \bar{r}) - \rho \dot{\bar{\eta}} (\theta - \bar{\theta}) - \rho \bar{\theta} (\dot{\eta} - \dot{\bar{\eta}}) \leq \frac{\rho}{\bar{\theta}} (\theta - \bar{\theta})(r - \bar{r}) - \frac{\rho \bar{r}}{\theta \bar{\theta}} (\theta - \bar{\theta})^2$$

hence (3.8) takes the form

$$\begin{aligned}
(3.10) \quad \dot{H} + G_{\alpha, \alpha} \leq & \rho(v_i - \bar{v}_i)(b_i - \bar{b}_i) + \frac{\rho}{\bar{\theta}} (\theta - \bar{\theta})(r - \bar{r}) - \frac{\rho \bar{r}}{\theta \bar{\theta}} (\theta - \bar{\theta})^2 \\
& + \dot{\bar{F}}_{i\alpha} \{T_{i\alpha} - \bar{T}_{i\alpha} - \frac{\partial T_{i\alpha}^*}{\partial F_{j\beta}} (F_{j\beta} - \bar{F}_{j\beta}) - \frac{\partial T_{i\alpha}^*}{\partial \eta} (\eta - \bar{\eta})\} \\
& + \frac{\rho \bar{r}}{\bar{\theta}} \{ \theta - \bar{\theta} - \frac{\partial \theta^*}{\partial F_{j\beta}} (F_{j\beta} - \bar{F}_{j\beta}) - \frac{\partial \theta^*}{\partial \eta} (\eta - \bar{\eta}) \}.
\end{aligned}$$

Inequality (3.10) will be the starting point of our stability analysis. The crucial observation is that its right hand side is of quadratic order in  $(v - \bar{v}, F - \bar{F}, \eta - \bar{\eta}, b - \bar{b}, r - \bar{r})$  so that one may establish continuous dependence of processes upon initial data and supply terms by applying Gronwall type inequalities, provided that  $H$  be positive definite in an appropriate sense. This program will be implemented in the following two sections.

#### 4. Stability of Smooth Processes in the Convexity Region of Internal Energy

In this section we shall be dealing with pairs of processes that render  $H(\underline{X}, t)$  pointwise positive definite. On account of (3.1) and (2.9), the sign of  $H^*$  is dependent upon the convexity properties of the function  $\varepsilon^*$ .

For fixed  $\underline{X} \in \mathcal{D}$ , we let  $J^*(\underline{F}, \underline{\eta}; \underline{X})$  denote the Hessian matrix of  $\varepsilon^*(\underline{F}, \underline{\eta}; \underline{X})$ . We will say that a process  $(\bar{\underline{x}}, \bar{\underline{\eta}})(\underline{X}, t)$  resides in the convexity region of internal energy if  $J^*(\bar{\underline{F}}(\underline{X}, t), \bar{\underline{\eta}}(\underline{X}, t); \underline{X})$  is uniformly positive definite on the closure of the domain of the process. By virtue of (2.9),  $J^*$  is the Jacobian matrix of the transformation  $(\underline{F}, \underline{\eta}) \mapsto (\rho^{-1}\underline{T}, \theta)$ . Thus, by Sylvester's theorem,  $J^*(\bar{\underline{F}}(\underline{X}, t), \bar{\underline{\eta}}(\underline{X}, t); \underline{X})$  will be positive definite if and only if

$$(4.1) \quad \left[ \frac{\partial \underline{T}^* (\bar{\underline{F}}(\underline{X}, t), \bar{\underline{\eta}}(\underline{X}, t); \underline{X})}{\partial \underline{F}} \right] \text{ positive definite}$$

$$(4.2) \quad \det J^*(\bar{\underline{F}}(\underline{X}, t), \bar{\underline{\eta}}(\underline{X}, t); \underline{X}) > 0.$$

In order to see the thermodynamic interpretation of (4.2), we observe that when  $J^*$  is nonsingular at a point one may invert locally the transformation  $(\underline{F}, \underline{\eta}) \mapsto (\rho^{-1}\underline{T}, \theta)$  to get  $\underline{F} = \underline{F}^*(\rho^{-1}\underline{T}, \theta; \underline{X})$ ,  $\underline{\eta} = \underline{\eta}^*(\rho^{-1}\underline{T}, \theta; \underline{X})$  and

$$(4.3) \quad \frac{\partial \underline{\eta}^*}{\partial \theta} = \rho^{-n} (\det J^*)^{-1} \det \left[ \frac{\partial \underline{T}^*}{\partial \underline{F}} \right].$$

It follows that (4.1), (4.2) are equivalent to the following two conditions (in the notation of classical thermodynamics):

$$(4.4) \quad \left( \frac{\partial \tilde{T}}{\partial \tilde{F}} \right)_{\eta} \text{ positive definite,}$$

$$(4.5) \quad \left( \frac{\partial \eta}{\partial \theta} \right)_{\tilde{T}} > 0.$$

It is easily seen that (4.4), (4.5) are in turn equivalent to

$$(4.6) \quad \left( \frac{\partial \tilde{T}}{\partial \tilde{F}} \right)_{\theta} \text{ positive definite,}$$

$$(4.7) \quad \left( \frac{\partial \eta}{\partial \theta} \right)_{\tilde{F}} > 0.$$

The relevance of (4.5), (4.7) to thermodynamic stability was first pointed out by Gibbs [10]. It would be appropriate to impose these conditions on any region in state space that does not contain critical points at which the material undergoes phase transitions. On the contrary, (4.4) and (4.6) are quite restrictive. These conditions may be globally satisfied for one dimensional bodies but are incompatible, in the multidimensional case, with the principle of material frame indifference on a large portion of state space that includes, in particular, natural states. The theory of elastic stability (e.g. [11]) elucidates the role of (4.4), (4.6) and makes it plausible that these conditions will be satisfied on some region in state space where strains are moderate and stresses are predominately tensile.

We now state two stability results, the first one for the mixed initial-boundary value problem and the second for the pure initial value problem.

Theorem 4.1. Assume that  $\mathcal{D}$  is bounded and has finite perimeter [4]. Let  $(\bar{x}, \bar{\eta})(\underline{x}, t)$  be a smooth process defined on  $\mathcal{D} \times [0, t_0]$ , residing in the convexity region of internal energy, with supply terms  $(\bar{b}, \bar{r})(\underline{x}, t) \in L^\infty(\mathcal{D} \times [0, t_0])$ . Then there are positive constants  $\delta, \alpha, M, N$  with the following property: If  $(\underline{x}, \eta)(\underline{x}, t)$  is any admissible process defined on  $\mathcal{D} \times [0, t_0]$ , with supply terms  $(\underline{b}, \underline{r})(\cdot, t) \in L^1([0, t_0]; L^2(\mathcal{D}))$ , and such that<sup>(\*)</sup>

$$(4.8) \quad |\underline{F}(\underline{x}, t) - \bar{F}(\underline{x}, t)| + |\underline{\eta}(\underline{x}, t) - \bar{\eta}(\underline{x}, t)| < \delta, \quad (\underline{x}, t) \in \mathcal{D} \times [0, t_0],$$

$$(4.9) \quad (\underline{v} - \bar{v}) \cdot (\underline{\tau} - \bar{\tau}) \leq 0, \quad \text{on } \partial \mathcal{D},$$

then we have, for any  $s \in [0, t_0]$ ,

$$(4.10) \quad \begin{aligned} & ||(\underline{v} - \bar{v}, \underline{F} - \bar{F}, \underline{\eta} - \bar{\eta})(\cdot, s)||_{L^2(\mathcal{D})} \\ & \leq Me^{\alpha s} ||(\underline{v} - \bar{v}, \underline{F} - \bar{F}, \underline{\eta} - \bar{\eta})(\cdot, 0)||_{L^2(\mathcal{D})} \\ & + Ne^{\alpha s} \int_0^s ||(\underline{b} - \bar{b}, \underline{r} - \bar{r})(\cdot, t)||_{L^2(\mathcal{D})} dt. \end{aligned}$$

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<sup>(\*)</sup> In Equation (4.9)  $\underline{\tau}, \bar{\tau}$  denote the stress vectors.

Theorem 4.2. Assume that  $\mathcal{D} = \mathbb{R}^n$ . Let  $(\bar{x}, \bar{\eta})(\underline{x}, t)$  be a smooth process defined on  $\mathcal{D} \times [0, t_0]$ , residing in the convexity region of internal energy, with supply terms  $(\bar{b}, \bar{r})(\underline{x}, t) \in L_{loc}^\infty(\mathcal{D} \times [0, t_0])$  and such that  $(\bar{F}, \bar{\eta})(\underline{x}, t)$  is bounded on  $\mathcal{D} \times [0, t_0]$ . Then there are positive constants  $\delta, k, \alpha, M, N$  with the following property: If  $(x, \eta)(\underline{x}, t)$  is any admissible process defined on  $\mathcal{D} \times [0, t_0]$ , with supply terms  $(b, r)(\cdot, t) \in L^1([0, t_0]; L_{loc}^2(\mathcal{D}))$ , and such that

$$(4.11) \quad |F(\underline{x}, t) - \bar{F}(\underline{x}, t)| + |\eta(\underline{x}, t) - \bar{\eta}(\underline{x}, t)| < \delta, \quad (\underline{x}, t) \in \mathcal{D} \times [0, t_0],$$

then we have for every  $a > 0, s \in [0, t_0]$ ,

$$(4.12) \quad \begin{aligned} & ||(\underline{v} - \bar{\underline{v}}, \underline{F} - \bar{\underline{F}}, \eta - \bar{\eta})(\cdot, s)||_{L^2(|\underline{x}| < a)} \\ & \leq Me^{\alpha s} ||(\underline{v} - \bar{\underline{v}}, \underline{F} - \bar{\underline{F}}, \eta - \bar{\eta})(\cdot, 0)||_{L^2(|\underline{x}| < a + ks)} \\ & + Ne^{\alpha s} \int_0^s ||(\underline{b} - \bar{\underline{b}}, \underline{r} - \bar{\underline{r}})(\cdot, t)||_{L^2(|\underline{x}| < a + k(s-t))} dt. \end{aligned}$$

From the above propositions one draws immediately the following

Corollary 4.1. Let  $\mathcal{D}, (\bar{x}, \bar{\eta})(\underline{x}, t)$  and  $(\underline{x}, \eta)(\underline{x}, t)$  be as in Theorem 4.1 or as in Theorem 4.2. Assume that the corresponding supply terms  $(\bar{b}, \bar{r})(\underline{x}, t)$  and  $(b, r)(\underline{x}, t)$  coincide on  $\mathcal{D} \times [0, t_0]$  and that both processes originate from the same state, that is,

$$(4.13) \quad \underline{x}(\underline{x}, 0) = \bar{x}(\underline{x}, 0), \quad \underline{v}(\underline{x}, 0) = \bar{v}(\underline{x}, 0), \quad \underline{x} \in \mathcal{D}.$$

Then  $(\bar{x}, \bar{\eta})(\bar{x}, t)$  and  $(x, \eta)(x, t)$  coincide on  $\mathcal{D} \times [0, t_0]$ .

The above uniqueness result is only local since the two processes are restricted a priori by (4.8) or (4.11). Roughly speaking, the Corollary states that the only way that an admissible process may bifurcate out of a smooth process, at an instant  $t_1$ , is by the spontaneous generation at  $t_1$  of a shock wave of large ( $\geq \delta$ ) amplitude. In the one dimensional case it is known [12] that this is impossible, i.e., shock waves originating at a smooth state start out with "zero" amplitude and are then gradually amplified. One would need a similar property for the multidimensional case in order to deduce global uniqueness from Corollary 4.1 but a proof does not seem possible at the present stage of development of the theory.

For the proof of Theorems 4.1 and 4.2 we shall need the following Gronwall type inequality.

Lemma 4.1. Assume that the nonnegative functions  $y(t) \in L^\infty[0, s]$  and  $g(t) \in L^1[0, s]$  satisfy the inequality

$$(4.14) \quad y^2(\sigma) \leq M^2 y^2(0) + \int_0^\sigma [2\alpha y^2(t) + 2N g(t) y(t)] dt, \quad \sigma \in [0, s],$$

where  $\alpha, M, N$  are nonnegative constants. Then

$$(4.15) \quad y(s) \leq M e^{\alpha s} y(0) + N e^{\alpha s} \int_0^s g(t) dt.$$

The proof of the above lemma is straightforward and in any case a more general result (Lemma 5.1) will be established in Section 5.

Proof of Theorem 4.1. Since  $(\bar{x}, \bar{\eta})(X, t)$  resides in the convexity region of internal energy, there exist, on account of (2.9), positive constants  $\delta, \lambda$  such that, whenever (4.8) holds,

$$\begin{aligned}
 (4.16) \quad & \varepsilon^*(\underline{F}(X, t), \eta(X, t); X) - \varepsilon^*(\bar{F}(X, t), \bar{\eta}(X, t); X) \\
 & - \rho^{-1} T_{i\alpha}^* (\bar{F}(X, t), \bar{\eta}(X, t); X) [F_{i\alpha}(X, t) - \bar{F}_{i\alpha}(X, t)] \\
 & - \theta^* (\bar{F}(X, t), \bar{\eta}(X, t); X) [\eta(X, t) - \bar{\eta}(X, t)] \\
 & \geq \lambda (|F(X, t) - \bar{F}(X, t)|^2 + |\eta(X, t) - \bar{\eta}(X, t)|^2),
 \end{aligned}$$

for every  $(X, t) \in \mathcal{D} \times [0, t_0]$ .

We now fix  $s \in [0, t_0]$ , we integrate (3.10) over  $\mathcal{D} \times [0, \sigma]$ ,  $\sigma \in [0, s]$ , we apply the Gauss-Green theorem and we use (4.9), (4.16) and the observation that the right-hand side of (3.10) is of quadratic order in  $(\underline{v}-\bar{v}, \underline{F}-\bar{F}, \eta-\bar{\eta}, \underline{b}-\bar{b}, r-\bar{r})$  thus arriving at an estimate of the form (4.14) with

$$(4.17) \quad y(t) = ||(\underline{v}-\bar{v}, \underline{F}-\bar{F}, \eta-\bar{\eta})(\cdot, t)||_{L^2(\mathcal{D})},$$

$$(4.18) \quad g(t) = ||(\underline{b}-\bar{b}, r-\bar{r})(\cdot, t)||_{L^2(\mathcal{D})}.$$

An application of Lemma 4.1 completes the proof.

Proof of Theorem 4.2. As in the proof of Theorem 4.1 there are positive constants  $\delta, \lambda$  such that (4.16) is satisfied for all

$(\underline{X}, t) \in \mathcal{D} \times [0, t_0]$ , whenever (4.11) holds. Consequently, since  $\underline{G}^*$  is of quadratic order in  $(\underline{v}-\bar{\underline{v}}, \underline{F}-\bar{\underline{F}}, \underline{\eta}-\bar{\underline{\eta}})$ , one may determine a sufficiently large positive constant  $k$  with the property

$$(4.19) \quad kH(\underline{X}, t) - \underline{G}(\underline{X}, t) \cdot \frac{\underline{X}}{|\underline{X}|} \geq 0, \quad (\underline{X}, t) \in \mathcal{D} \times [0, t_0].$$

We now fix  $s \in [0, t_0]$ ,  $a > 0$ , and for each  $\sigma \in [0, s]$  we integrate (3.10) over the frustum  $\{(\underline{X}, t) \mid t \in [0, \sigma], |\underline{X}| < a + k(s-t)\}$ . Applying the Gauss-Green theorem and using (4.19), (4.16) we arrive, as in the proof of Theorem 4.1, to an estimate of the form (4.14) with

$$(4.20) \quad y(t) = ||(\underline{v}-\bar{\underline{v}}, \underline{F}-\bar{\underline{F}}, \underline{\eta}-\bar{\underline{\eta}})(\cdot, t)||_{L^2(|\underline{X}| < a+k(s-t))}^2,$$

$$(4.21) \quad g(t) = ||(\underline{b}-\bar{\underline{b}}, \underline{r}-\bar{\underline{r}})(\cdot, t)||_{L^2(|\underline{X}| < a+k(s-t))}^2,$$

so that Lemma 4.1 yields (4.12). The proof of the theorem is complete.

From the proof of Theorems 4.1 and 4.2 it becomes clear that in the one-dimensional case, where the internal energy is globally convex, one may select  $\delta$ , in (4.8) or (4.11), arbitrarily large so that the stability and uniqueness results become global. As a matter of fact, in the one-dimensional case and under some additional assumptions (most notably genuine nonlinearity), DiPerna [9] has established uniqueness of piecewise smooth processes in the class of proper admissible processes. DiPerna's analysis relies on a number of novel ideas and the problem of extending it to the multidimensional case is still open.

## 5. Stability of Smooth Processes in the Strong Ellipticity Region

The object of the restrictive convexity assumption, in Section 4, was to ensure that  $H(\underline{X}, t)$  be pointwise positive definite. In reviewing the proof of Theorem 4.1, however, it becomes clear that definiteness of  $\int_{\mathcal{B}} H(\underline{X}, t) d\underline{X}$  would suffice for stability. We prove here that, whenever the body is smooth and the motion of its boundary is prescribed, stability is induced by a mere strong ellipticity condition.

We will say that a process  $(\bar{\underline{x}}, \bar{\eta})(\underline{X}, t)$  resides in the strong ellipticity region if there is a positive constant  $\nu$  with the property that for any vectors  $\xi, \zeta \in R^n$ , every  $\mu \in R$  and all  $(\underline{X}, t)$  in the domain of the process

$$(5.1) \quad \frac{\partial^2 \epsilon^* (\bar{\underline{F}}(\underline{X}, t), \bar{\eta}(\underline{X}, t); \underline{X})}{\partial F_{i\alpha} \partial F_{j\beta}} \xi_i \xi_j \zeta_\alpha \zeta_\beta + \frac{\partial^2 \epsilon^* (\bar{\underline{F}}(\underline{X}, t), \bar{\eta}(\underline{X}, t); \underline{X})}{\partial F_{i\alpha} \partial \eta} \xi_i \zeta_\alpha \mu \\ + \frac{\partial^2 \epsilon^* (\bar{\underline{F}}(\underline{X}, t), \bar{\eta}(\underline{X}, t); \underline{X})}{\partial \eta^2} \mu^2 \geq \nu (|\xi|^2 |\zeta|^2 + \mu^2).$$

Strong ellipticity in nonlinear elasticity has been studied in connection to wave propagation (e.g. [13]) as well as to existence of equilibrium configurations [14]. As an assumption it is weaker than convexity of the internal energy and, in particular, it is not incompatible with frame indifference at a natural state. For a discussion on the extent of the strong ellipticity region in two-dimensional, homogeneous, isotropic, hyperelastic bodies, see [15]. See also [16] for relevant illuminating remarks.

We now state the stability result:

Theorem 5.1. Assume that  $\mathcal{D}$  is smooth and bounded. Let  $(\bar{x}, \bar{n})(X, t)$  be a smooth process defined on  $\mathcal{D} \times [0, t_0]$ , residing in the strong ellipticity region, with supply terms  $(\bar{b}, \bar{r})(X, t) \in L^\infty(\mathcal{D} \times [0, t_0])$ . Then there are positive constants  $\delta, \alpha, \beta, M, N$  with the following property: If  $(x, n)(X, t)$  is any admissible process defined on  $\mathcal{D} \times [0, t_0]$ , with supply terms  $(b, r)(X, t) \in L^1([0, t_0]; L^2(\mathcal{D}))$  and such that

$$(5.2) \quad |F(X, t) - \bar{F}(X, t)| + |n(X, t) - \bar{n}(X, t)| < \delta \quad (X, t) \in \mathcal{D} \times [0, t_0],$$

$$(5.3) \quad x(X, t) = \bar{x}(X, t), \quad X \in \partial\mathcal{D}, \quad t \in [0, t_0],$$

then we have, for any  $s \in [0, t_0]$ ,

$$(5.4) \quad \begin{aligned} & \| (v - \bar{v}, F - \bar{F}, n - \bar{n})(\cdot, s) \|_{L^2(\mathcal{D})} \\ & \leq M \exp(\alpha s + \beta s^2) \| (v - \bar{v}, F - \bar{F}, n - \bar{n})(\cdot, 0) \|_{L^2(\mathcal{D})} \\ & \quad + N \exp(\alpha s + \beta s^2) \int_0^t \| (b - \bar{b}, r - \bar{r})(\cdot, t) \|_{L^2(\mathcal{D})} dt. \end{aligned}$$

From the above proposition one draws the following uniqueness result:

Corollary 5.1. Let  $\mathcal{D}$ ,  $(\bar{x}, \bar{\eta})(X, t)$  and  $(x, \eta)(X, t)$  be as in Theorem 5.1. Assume that the corresponding supply terms  $(\bar{b}, \bar{r})(X, t)$  and  $(b, r)(X, t)$  coincide on  $\mathcal{D} \times [0, t_0]$  and that both processes originate from the same state, i.e.,

$$(5.5) \quad \underline{x}(X, 0) = \bar{x}(X, 0), \quad \underline{v}(X, 0) = \bar{v}(X, 0), \quad X \in \mathcal{D}.$$

Then  $(\bar{x}, \bar{\eta})(X, t)$  and  $(x, \eta)(X, t)$  coincide on  $\mathcal{D} \times [0, t_0]$ .

For the proof of Theorem 5.1 we will employ the following Gronwall type inequality.

Lemma 5.1. Assume that the nonnegative functions  $y(t) \in L^\infty[0, s]$  and  $g(t) \in L^1[0, s]$  satisfy the inequality

$$(5.6) \quad y^2(\sigma) \leq M^2 y^2(0) + \int_0^\sigma [(2\gamma + 4\beta\sigma)y^2(t) + 2Ng(t)y(t)]dt, \quad \sigma \in [0, s],$$

with  $\beta, \gamma, M, N$  nonnegative constants. Then

$$(5.7) \quad y(s) \leq M \exp(\alpha s + \beta s^2) y(0) + N \exp(\alpha s + \beta s^2) \int_0^s g(t)dt,$$

where  $\alpha = \gamma + \beta/\gamma$ .

Proof. We define a nonnegative function  $z(\sigma)$  by

$$(5.8) \quad z^2(\sigma) = M^2 y^2(0) + \int_0^\sigma [(2\gamma + 4\beta\sigma)y^2(t) + 2Ng(t)y(t)]dt, \quad \sigma \in [0, s],$$

and we note that

$$(5.9) \quad \begin{aligned} 2z(\sigma)\dot{z}(\sigma) &= (2\gamma+4\beta\sigma)y^2(\sigma) + 2Ng(\sigma)y(\sigma) + 4\beta \int_0^\sigma y^2(t)dt \\ &\leq (2\alpha+4\beta\sigma)z^2(\sigma) + 2Ng(\sigma)z(\sigma). \end{aligned}$$

Hence

$$(5.10) \quad \dot{z}(\sigma) \leq (\alpha+2\beta\sigma)z(\sigma) + Ng(\sigma).$$

Integrating the differential inequality (5.10) under the initial condition  $z(0) = My(0)$  we arrive at (5.7). The proof is complete.

We note that, when  $\beta = 0$ , Lemma 5.1 reduces to Lemma 4.1.

Lemma 5.2. Let  $\mathcal{D}, (\bar{x}, \bar{\eta})(X, t)$  and  $(x, \eta)(X, t)$  be as in Theorem 5.1. Then there are constants  $\lambda > 0$  and  $\kappa$  with the property that, for any  $\sigma \in [0, t_0]$ ,

$$(5.11) \quad \begin{aligned} \int_{\mathcal{D}} \rho(X) &\left\{ \frac{\partial^2 \epsilon^* (\bar{F}(X, \sigma), \bar{\eta}(X, \sigma); X)}{\partial F_{i\alpha} \partial F_{j\beta}} [F_{i\alpha}(X, \sigma) - \bar{F}_{i\alpha}(X, \sigma)] [F_{j\beta}(X, \sigma) - \bar{F}_{j\beta}(X, \sigma)] \right. \\ &+ \frac{\partial^2 \epsilon^* (\bar{F}(X, \sigma), \bar{\eta}(X, \sigma); X)}{\partial F_{i\alpha} \partial \eta} [F_{i\alpha}(X, \sigma) - \bar{F}_{i\alpha}(X, \sigma)] [\eta(X, \sigma) - \bar{\eta}(X, \sigma)] \\ &\left. + \frac{\partial^2 \epsilon^* (\bar{F}(X, \sigma), \bar{\eta}(X, \sigma); X)}{\partial \eta^2} [\eta(X, \sigma) - \bar{\eta}(X, \sigma)]^2 \right\} dX \\ &\geq 2\lambda \int_{\mathcal{D}} \{ |F(X, \sigma) - \bar{F}(X, \sigma)|^2 + |\eta(X, \sigma) - \bar{\eta}(X, \sigma)|^2 \} dX \\ &- \kappa \int_{\mathcal{D}} |x(X, \sigma) - \bar{x}(X, \sigma)|^2 dX. \end{aligned}$$

Proof. We recall that  $(\bar{x}, \bar{\eta})(X, t)$  resides in the strong ellipticity region and that  $\bar{x} - \bar{x}$  vanishes on  $\partial\mathcal{D}$ . In the case where  $\eta(X, t) - \bar{\eta}(X, t)$  vanishes identically, (5.11) reduces to the classical Gording inequality. In general, (5.11) is established by imitating the proof of Gording's inequality, as described, for example, in [17].

Proof of Theorem 5.1. On account of (2.9),

$$\begin{aligned}
 (5.12) \quad \epsilon^*(F, \eta; X) - \epsilon^*(\bar{F}, \bar{\eta}; X) &= \frac{1}{\rho(X)} T_{i\alpha}^*(\bar{F}, \bar{\eta}; X) (F_{i\alpha} - \bar{F}_{i\alpha}) - \theta^*(\bar{F}, \bar{\eta}; X) (\eta - \bar{\eta}) \\
 &= \frac{\partial^2 \epsilon^*(\bar{F}, \bar{\eta}; X)}{\partial F_{i\alpha} \partial F_{j\beta}} (F_{i\alpha} - \bar{F}_{i\alpha}) (F_{j\beta} - \bar{F}_{j\beta}) + \frac{\partial^2 \epsilon^*(\bar{F}, \bar{\eta}; X)}{\partial F_{i\alpha} \partial \eta} (F_{i\alpha} - \bar{F}_{i\alpha}) (\eta - \bar{\eta}) \\
 &\quad + \frac{\partial^2 \epsilon^*(\bar{F}, \bar{\eta}; X)}{\partial \eta^2} (\eta - \bar{\eta})^2 + o(|F - \bar{F}|^2 + |\eta - \bar{\eta}|^2).
 \end{aligned}$$

Combining (5.11) with (5.12) we conclude that there is a positive constant  $\delta$  with the property that, when (5.2) is satisfied,

$$\begin{aligned}
 (5.13) \quad \int_{\mathcal{D}} \{ \rho(X) \epsilon^*(F(X, \sigma), \eta(X, \sigma); X) - \rho(X) \epsilon^*(\bar{F}(X, \sigma), \bar{\eta}(X, \sigma); X) \\
 - T_{i\alpha}^*(\bar{F}(X, \sigma), \bar{\eta}(X, \sigma); X) [F_{i\alpha}(X, \sigma) - \bar{F}_{i\alpha}(X, \sigma)] \\
 - \rho(X) \theta^*(\bar{F}(X, \sigma), \bar{\eta}(X, \sigma); X) [\eta(X, \sigma) - \bar{\eta}(X, \sigma)] \} dX \\
 \geq \lambda \int_{\mathcal{D}} \{ |F(X, \sigma) - \bar{F}(X, \sigma)|^2 + |\eta(X, \sigma) - \bar{\eta}(X, \sigma)|^2 \} dX \\
 - \kappa \int_{\mathcal{D}} |x(X, \sigma) - \bar{x}(X, \sigma)|^2 dX.
 \end{aligned}$$

Furthermore, upon using Schwarz's inequality and the Poincaré inequality,

$$\begin{aligned}
 (5.14) \quad & \int_{\mathcal{D}} |\underline{x}(\underline{X}, \sigma) - \bar{\underline{x}}(\underline{X}, \sigma)|^2 d\underline{X} \\
 &= \int_{\mathcal{D}} |\underline{x}(\underline{X}, 0) - \bar{\underline{x}}(\underline{X}, 0) + \int_0^\sigma \{\underline{v}(\underline{X}, t) - \bar{\underline{v}}(\underline{X}, t)\} dt|^2 d\underline{X} \\
 &\leq 2c \int_{\mathcal{D}} |\underline{F}(\underline{X}, 0) - \bar{\underline{F}}(\underline{X}, 0)|^2 d\underline{X} \\
 &+ 2\sigma \int_0^\sigma \int_{\mathcal{D}} |\underline{v}(\underline{X}, t) - \bar{\underline{v}}(\underline{X}, t)|^2 d\underline{X} dt.
 \end{aligned}$$

We now proceed as in the proof of Theorem 4.1. We fix  $s \in [0, t_0]$ , we integrate (3.10) over  $\mathcal{D} \times [0, \sigma]$ ,  $\sigma \in [0, s]$ , and we apply the Gauss-Green theorem. We use the boundary conditions (5.3), (5.13), (5.14) and the observation that the right-hand side of (3.10) is of quadratic order in  $(\underline{v} - \bar{\underline{v}}, \underline{F} - \bar{\underline{F}}, \underline{n} - \bar{\underline{n}}, \underline{b} - \bar{\underline{b}}, \underline{r} - \bar{\underline{r}})$  thus arriving at an estimate of the form (5.6) with

$$(5.15) \quad y(t) = \|(\underline{v} - \bar{\underline{v}}, \underline{F} - \bar{\underline{F}}, \underline{n} - \bar{\underline{n}})(\cdot, t)\|_{L^2(\mathcal{D})},$$

$$(5.16) \quad g(t) = \|(\underline{b} - \bar{\underline{b}}, \underline{r} - \bar{\underline{r}})(\cdot, t)\|_{L^2(\mathcal{D})}.$$

An application of Lemma 5.1 yields (5.4) and completes the proof.

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