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DIVISION OF ENGINEERING AND WEAPONS

Report EW-16-78

Synthesis of Optimal Ladder Networks

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November 1978

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This project treats synthesis procedures for optimal ladder networks. The first part of the project deals with a chain-matrix decomposition technique for realizing two-port LC ladder networks. Two chain-matrix decomposition theorems are proved. These theorems state the necessary and sufficient conditions that must be satisfied by a (2x2) matrix $S_n(s)$ that is to be decomposed into a product of simple matrices, i.e., $S_n(s) = K_1K_2 \cdots K_n$. It is found that the zeroes of the adjacent elements

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of $S_n(s)$ alternate pairwise along the $j\omega$ axis in the s-plane. It is also found that if the matrix $S_n(s)$ is the overall chain matrix of an LC ladder network, then each of the simple matrices $K_i(s)$ for $i = 1,2,\cdots n$ represents a simple LC ladder section.

These decomposition techniques are then applied to the design of filters. Of particular interest are Butterworth, Chebyshev, and Bessel filters with single and double terminations. These filters are designed by the decomposition of chain matrices whose elements are predetermined by the orthogonal polynomials that approximate the ideal filter characteristics.

The synthesis of inhomogeneous ladder networks is also investigated. By means of the mapping $p = y(s)z(s)$, the optimal process for RC ladders is generalized to synthesize optimal inhomogeneous ladders.

The last part of this project treats the synthesis of double-terminated optimal RC ladders whose eigenvalues follow certain patterns. Several formulas are found to facilitate the calculation of element values for optimal ladders whose eigenvalues are $p_i = ip_1$ as well as for those ladders whose eigenvalues are $p_i = i^2p_1$. These formulas reduce the tedious process of optimal synthesis to simple formula substitutions.

ABSTRACT

This project treats synthesis procedures for optimal ladder networks. The first part of the project deals with a chain-matrix decomposition technique for realizing two-port LC ladder networks. Two chain-matrix decomposition theorems are proved. These theorems state the necessary and sufficient conditions that must be satisfied by a [2x2] matrix $S_n(s)$ that is to be decomposed into a product of simple matrices, i.e., $S_n(s) = K_1 K_2 \cdots K_n$. It is found that the zeroes of the adjacent elements of $S_n(s)$ alternate pairwise along the $j\omega$ axis in the s-plane. It is also found that if the matrix $S_n(s)$ is the overall chain matrix of an LC ladder network, then each of the simple matrices $K_i(s)$ for $i = 1, 2, \cdots n$ represents a simple LC ladder section.

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CHAPTER 1

REVIEW OF THE LITERATURE AND SCOPE OF THE THESIS

The study of ladder network synthesis began in the 1920s. The first paper dealing explicitly with the realization of a one-port whose impedance is a prescribed function of frequency was written by Cauer [6] in 1926. Cauer's method is based on continuous fraction expansions. Cauer also introduced one of the most direct methods for synthesizing two-port networks resulting in ladder structures [7]. Since then, a number of synthesis procedures for two-port networks have been discovered [24]. However, very few of these methods made use of chain parameters.

One of the earlier works on the synthesis of two-element-kind two-port networks using chain matrix factorization was published by Hunt [10]. In Hunt's method, a given chain matrix can be factorized as a product of elementary matrices, each representing one branch of a section of a ladder. However, Hunt did not give any realizability condition on the overall chain matrix. Therefore, at the end of each step of Hunt's procedure, one cannot be sure of the realizability of the remaining matrix. The procedure tends to be one of trial and error. Lee and Brown [15] were the first to prove the necessary and sufficient conditions of the overall chain parameters for the realization of two-element-kind ladder networks using chain matrix decomposition techniques.

The motivation of this thesis is due in part to the resurgence of interest in the optimal realization of two-element-kind two-port networks with specified transmission zeros. Optimal synthesis of RC ladder networks with given terminations was first introduced by Kuh [12] in 1958. He realized a grounded RC two-port from a given transfer function with distinct poles on the negative real axis. He used the Lagrange multipliers method to obtain the optimization and the resulting optimal network has maximum gain. Integrated circuit technology spurred a revival of interest in optimal synthesis of RC ladders nearly two decades after Kuh's discovery. Optimal synthesis of RC ladder networks using chain parameters in the s-domain was discovered by Protonotarios [21] in 1974. He formulated the realization technique of RC ladders with n shunt capacitors and n+1 series resistors. His realization is obtained from a given chain parameter, B(s), resulting in a network having minimum total shunt capacitance. Protonotarios also formulated the synthesis of RC ladders with n shunt capacitors and n series resistors. This realization is derived from a given chain parameter, A(s), resulting in a network having minimum total resistance - total capacitance product. Almost simultaneously, Stein and Salama [23] developed optimal synthesis procedures for the same RC ladders and, furthermore, showed that the optimal RC ladder with n+1 resistors is symmetrical and the optimal RC ladder with n resistors is antimetrical.

Chapter 2 of this thesis deals with lossless two-port synthesis of ladder networks, using the chain matrix decomposition method which is an extension of the work done by Lee and Brown. Two chain matrix decomposition theorems are proved. These theorems give the necessary and sufficient conditions of the chain parameters for the existence of cascaded lossless ladder networks. The results of these decomposition theorems and synthesis algorithms are applied to the design of filters in Chapter 3. The first half of Chapter 3 is concerned with filter networks that are terminated in a one-ohm load resistor [17]. The second half of Chapter 3 deals with double-terminated filter design using chain parameter technique [18]. Classical filters such as Butterworth, Chebyshev, and Bessel are synthesized through the use of the decomposition of chain matrices. It was found that with suitable terminations these filters are optimal.

Chapters 4 and 5 extend the work done by Lee and Brown on inhomogeneous ladder networks [16] as well as the work done by Kuh, Protonotarios, and Stein on optimal RC ladders. Chapter 4 deals with optimal synthesis of the inhomogeneous ladder networks as shown in Figure 4.1 and Figure 4.2. By means of the mapping of $p = y(s)z(s)$, the optimal process was generalized to synthesize inhomogeneous ladders. Chapter 5 is concerned with synthesis procedures of certain classes of optimal RC ladder networks as shown in Figure 5.1 with $R_1 = R_{n+1}$. The state equation is $[\dot{V}] = [A][V]+[B]u$

where the state variable [V] represents the voltages across the capacitors, u is the forcing function, and [A] and [B] are constant matrices of a network of given topology and components. It was found that if the tridiagonal matrix [A] has eigenvalues $p_i = ip_1$, then its diagonal elements are equal and each is equal to the sum of eigenvalues divided by the number of eigenvalues. Several formulas have been found and their applications demonstrated. These formulas reduce the tedious process of optimal synthesis to simple substitution in formulas. Impulse-response and step-response curves as well as tables of residues are given.

CHAPTER 2

SYNTHESIS OF LADDER NETWORKS BY CHAIN MATRIX DECOMPOSITION

Section 2.1 Introduction

The object of this chapter is to derive and demonstrate the synthesis procedures of the two-element-kind lossless ladder networks by chain matrix decomposition method. Cascade synthesis procedures by Darlington and Youla are found useful for immittance function realization but their methods are different approaches than the chain matrix techniques.

The properties of entries of a two-by-two matrix to be synthesized are derived as lemmas. Theorems of decomposition of this matrix into a product of simple matrices are then followed. When the ladder network is shown as in Figure 2.6, the zero distribution of each overall chain matrix entry has a definite pattern. Therefore, suppose there comes a two-by-two matrix with each entry having this same zero distribution pattern, then this matrix can be used as a chain matrix of a realizable, two-element-kind lossless ladder network.

The necessary and sufficient conditions of a matrix $S_n(s)$ which is to be decomposed into a product of simple matrices are obtained as shown in the theorems, and these simple matrices $K_i(s)$ for $i = 1, 2, \ldots, N$ are the chain matrices each of which represents a simple ladder network.

A simple decomposition algorithms is developed in

Section 2.4. Using this algorithms implemented by digital computer programs, synthesis of two-element-kind lossless ladder networks can be accomplished with ease and with little expenditure of time.

Section 2.2 Distribution of Zeros of Chain Parameters of Cascaded Ladders

The following five lemmas provide the properties of the A, B, C, D entries of the overall chain matrix. These lemmas will be used for the proof of the decomposition theorems in the next section.

Lemma 1 Let a matrix $S_n(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$. If

(1) Det $S_n(s) = 1$,

(2) A(s), B(s), C(s) and D(s) are polynomials with positive coefficients, A(s) and D(s) are even functions of s and of degree 2n and 2(n-1) respectively,

B(s) and C(s) are odd functions of s and of degree 2n-1,

(3) Zeros of A(s), $[z_{a_i}]$ and zeros of B(s), $[z_{b_i}]$ interlace on the jω axis,

$0 = z_{b_0} < |z_{a_1}| < |z_{b_1}| < \cdots < |z_{b_{n-1}}| < |z_{a_n}|$,

then

(a) zeros of A(s) and C(s) interlace on the jω axis

$0 = z_{c_0} < |z_{a_1}| < |z_{c_1}| < \cdots < |z_{c_{n-1}}| < |z_{a_n}|$ and

(b) zeros of B(s) and D(s) interlace on the jω axis

$$0 = z_{b_0} < |z_{d_1}| < |z_{b_1}| < \cdots < |z_{d_{n-1}}| < |z_{b_{n-1}}| \text{ and}$$

(c) zeros of C(s) and D(s) interlace on the jω axis

$$0 = z_{c_0} < |z_{d_1}| < |z_{c_1}| < \cdots < |z_{d_{n-1}}| < |z_{c_{n-1}}|$$

Proof of (a): It follows from (1) that

$$A(s)D(s) - B(s)C(s) = 1 \quad \text{for all } s,$$

which yields

$$A(s)D(s) - s^2\bar{B}(s)\bar{C}(s) = 1,$$

where $s\bar{B}(s) = B(s)$ and $s\bar{C}(s) = C(s)$.

Substituting $z = s^2$ gives

$$A(z)D(z) - z\,\bar{B}(z)\bar{C}(z) = 1 \tag{2-1}$$

where zeros of A(z), $\bar{z}_{a_i}$ and $\bar{B}(z)$, $\bar{z}_{b_i}$ interlace on the negative real axis as shown in Figure 2.1, with

$$0 > \bar{z}_{a_1} > \bar{z}_{b_1} > \cdots > \bar{z}_{b_{n-1}} > \bar{z}_{a_n}, \text{ where } \bar{z}_{a_i} = -|z_{a_i}|^2.$$

From Equation (2-1), it follows that

$$A(\bar{z}_{a_i})D(\bar{z}_{a_i}) - \bar{z}_{a_i}\bar{B}(\bar{z}_{a_i})\bar{C}(\bar{z}_{a_i}) = -\bar{z}_{a_i}\bar{B}(\bar{z}_{a_i})\bar{C}(\bar{z}_{a_i}) = 1$$

and similarly,

$$-\bar{z}_{a_{i+1}}\bar{B}(\bar{z}_{a_{i+1}})\bar{C}(\bar{z}_{a_{i+1}}) = 1$$

Therefore,

$$\bar{z}_{a_i}\bar{B}(\bar{z}_{a_i})\bar{C}(\bar{z}_{a_i}) = \bar{z}_{a_{i+1}}\bar{B}(\bar{z}_{a_{i+1}})\bar{C}(\bar{z}_{a_{i+1}}) \tag{2-2}$$

$\bar{B}(\bar{z}_{a_i})$ and $\bar{B}(\bar{z}_{a_{i+1}})$ are of opposite signs because of the zero distributions of A(z) and $\bar{B}(z)$ as shown in Figure 2.2. Therefore, from Equation (2-2), $\bar{C}(\bar{z}_{a_i})$ and $\bar{C}(\bar{z}_{a_{i+1}})$ are also of opposite signs. This implies that the interval

$(\bar{z}_{a_i}, \bar{z}_{a_{i+1}})$ contains an odd number of zeros of $\bar{C}(z)$. There are n-1 of these intervals each of which contains one and only one zero of $\bar{C}(z)$. Therefore, zeros of A(z) and $\bar{C}(z)$ interlace on the negative real axis of the z-plane with

$$0 > \bar{z}_{a_1} > \bar{z}_{c_1} > \cdots > \bar{z}_{c_{n-1}} > \bar{z}_{a_n}$$

This implies that zeros of A(s) and $\bar{C}(s)$ interlace on the jω axis with

$$0<|z_{a_1}|<|z_{c_1}|<\cdots<|z_{c_{n-1}}|<|z_{a_n}|$$

Since $s\bar{C}(s) = C(s)$, this completes the proof of (a) of Lemma 1. Proof of (b) and (c) follow similarly.

Q.E.D.

Lemma 2, 3, 4, and 5 are summarized in Table 2.1 without proofs. The following example illustrates the zero interlacing property of a matrix which satisfies Lemma 1.

Example 2.1. Let

$$S_n(s) = \begin{bmatrix} 6s^4+6s^2+1 & 6s^3+3s \\ C(s) & D(s) \end{bmatrix} \text{ and } \det S_n(s) = 1.$$

As can be seen A(s) and B(s) are of degree four and three respectively. The zeros of A(s) are at ±j 0.459, ±j 0.888 and ther zeros of B(s) are at 0, ±j 0.707. It follows from $A(s)D(s) - B(s)C(s) = 1$ that $C(s) = 6s^3 + 4s$ and $D(s) = 6s^2 + 1$. The zeros of C(s) are at 0, ±j 0.816 and the zeros of D(s) are at ±j 0.409. Indeed the zeros of A(s) and B(s), A(s) and C(s), B(s) and D(s), C(s) and D(s) alternate on the jω axis as stipulated in Lemma 1.

Table 2.1 Summary of Lemma 2, 3, 4, and 5

	Lemma 2	Lemma 3
Hypothesis	(1) Det $S_n(s)=1$ (2) A(s) is of degree 2(n-1) B(s) is of degree 2n-1 C(s) is of degee 2n-3 D(s) is of degee 2(n-1) (3) Zeros of B(s) and D(s) interlace on the jω axis such that $0=z_{b_0}<\|z_{d_1}\|<\|z_{b_1}\|<\cdots<\|z_{d_{n-1}}\|<\|z_{b_{n-1}}\|$	(1) Det $S_n(s)=1$ (2) A(s) is of degree 2(n-1) B(s) is of degree 2n-3 C(s) is of degee 2n-3 D(s) is of degee 2(n-2) (3) Zeros of A(s) and C(s) interlace on the jω axis such that $0=z_{c_0}<\|z_{a_1}\|<\|z_{c_1}\|<\cdots<\|z_{c_{n-2}}\|<\|z_{a_{n-1}}\|$
Implication	Zeros of A(s) and B(s), A(s) and C(s) and C(s) and D(s) interlace on the jω axis such that $0=z_{b_0}<\|z_{a_1}\|<\|z_{b_1}\|<\cdots<\|z_{a_{n-1}}\|<\|z_{b_{n-1}}\|$ $0=z_{c_0}<\|z_{a_1}\|<\|z_{c_1}\|<\cdots<\|z_{c_{n-2}}\|<\|z_{a_{n-1}}\|$ $0=z_{c_0}<\|z_{d_1}\|<\|z_{c_1}\|<\cdots<\|z_{c_{n-2}}\|<\|z_{d_{n-1}}\|$	Zeros of A(s) and B(s) interlace on the jω axis such that $0=z_{b_0}<\|z_{a_1}\|<\|z_{b_1}\|<\cdots<\|z_{b_{n-2}}\|<\|z_{a_{n-1}}\|$

Table 2.1 (Continued)

	Lemma 4	Lemma 5
Hypothesis	(1) Det $S_n(s)=1$ (2) A(s) is of degree 2n B(s) is of degree 2n+1 C(s) is of degree 2n-1 D(s) is of degree 2n (3) Zeros of A(s) and C(s) interlace on the jω axis such that $0=z_{c_0}<\|z_{a_1}\|<\|z_{c_1}\|<\cdots<\|z_{c_{n-1}}\|<\|z_{a_n}\|$	(1) Det $S_n(s)=1$ (2) A(s) is of degree 2(n+1) B(s) is of degree 2n+1 C(s) is of degree 2n+1 D(s) is of degree 2n (3) Zeros of B(s) and D(s) interlace on the jω axis such that $0=z_{b_0}<\|z_{d_1}\|<\|z_{b_1}\|<\cdots<\|z_{d_n}\|<\|z_{b_n}\|$
Implication	Zeros of B(s) and D(s) interlace on the jω axis such that $0=z_{b_0}<\|z_{d_1}\|<\|z_{b_1}\|<\cdots<\|z_{d_n}\|<\|z_{b_n}\|$	Zeros of A(s) and B(s), A(s) and C(s), C(s) and D(s) interlace on the jω axis such that $0=z_{b_0}<\|z_{a_1}\|<\|z_{b_1}\|<\cdots<\|z_{b_n}\|<\|z_{a_{n+1}}\|$ $0=z_{c_0}<\|z_{a_1}\|<\|z_{c_1}\|<\cdots<\|z_{c_n}\|<\|z_{a_{n+1}}\|$ $0=z_{c_0}<\|z_{d_1}\|<\|z_{c_1}\|<\cdots<\|z_{d_n}\|<\|z_{c_n}\|$

Section 2.3 Decomposition Theorems

The following theorems and corollaries give the necessary and sufficient conditions for realizing a cascade of ladder networks.

Theorem 2.1 Let the matrix $S_n(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$.

$S_n(s) = K_1 K_2 \cdots K_{n-1} K_n$ where

$$K_i(s) = \begin{bmatrix} 1+s^2\ell_i c_i & s\ell_i \\ sc_i & 1 \end{bmatrix}, \quad \ell_i, c_i > 0 \text{ for } i = 1,2,\cdots n \qquad (2\text{-}3)$$

if and only if

(a) Det $S_n(s) = 1$,

(b) $A(s)$, $B(s)$, $C(s)$ and $D(s)$ are polynomials with positive coefficients, $A(s)$ and $D(s)$ are even functions of s and of degree $2n$ and $2(n-1)$ respectively, $B(s)$ and $C(s)$ are odd functions of s and of degree $2n-1$,

(c) Zeros of $A(s)$, $[z_{a_i}]$ and zeros of $B(s)$, $[z_{b_i}]$ interlace on the $j\omega$ axis,

$$0 = z_{b_0} < |z_{a_1}| < |z_{b_1}| < \cdots < |z_{b_{n-1}}| < |z_{a_n}|$$

(d) $A(0) = 1$.

Proof of Sufficiency: Suppose $S_n(s)$ is as described in the hypothesis.

$$S_n(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$$

where

$$A(s) = a_{2n}s^{2n} + a_{2(n-1)}s^{2(n-1)} + \ldots + 1$$

$$B(s) = b_{2n-1}s^{2n-1} + b_{2n-3}s^{2n-3} + \ldots + b_1 s$$

$$C(s) = c_{2n-1}s^{2n-1} + c_{2n-3}s^{2n-3} + \ldots + c_1 s$$

$$D(s) = d_{2(n-1)}s^{2(n-1)} + d_{2(n-2)}s^{2(n-2)} + \ldots + 1$$

Define $$K_n(s) = \begin{bmatrix} 1+s^2\ell_n c_n & s\ell_n \\ sc_n & 1 \end{bmatrix} \qquad (2\text{-}4)$$

Therefore $$K_n^{-1}(s) = \begin{bmatrix} 1 & -s\ell_n \\ -sc_n & 1+s^2\ell_n c_n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -sc_n & 1 \end{bmatrix}\begin{bmatrix} 1 & -s\ell_n \\ 0 & 1 \end{bmatrix}.$$

Let $$S_{n-1}(s) = \begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix} = S_n(s)K_n^{-1}(s)$$

$$= \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -sc_n & 1 \end{bmatrix}\begin{bmatrix} 1 & -s\ell_n \\ 0 & 1 \end{bmatrix} \qquad (2\text{-}5)$$

It is next shown that $\begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix}$ satisfies conditions (a), (b), (c), (d) and each of its entries is two degrees lower than those of $S_n(s)$'s.

Equation (2-5) can be written as

$$\begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix} = \begin{bmatrix} A(s) - sc_nB(s) & B(s) \\ C(s) - sc_nD(s) & D(s) \end{bmatrix} \begin{bmatrix} 1 & -s\ell_n \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{A}(s) & B(s) \\ \tilde{C}(s) & D(s) \end{bmatrix} \begin{bmatrix} 1 & -s\ell_n \\ 0 & 1 \end{bmatrix}, \quad (2\text{-}6)$$

where $\tilde{A}(s) = A(s) - sc_nB(s),$ (2-7)

$\tilde{C}(s) = C(s) - sc_nD(s).$ (2-8)

Equation (2-7) can be written as

$$\tilde{A}(s) = (a_{2n} - c_nb_{2n-1})s^{2n} + (a_{2n-2} - c_nb_{2n-3})s^{2n-2}$$

$$+ (a_{2n-4} - c_nb_{2n-5})s^{2n-4} + \cdots + 1. \quad (2\text{-}9)$$

Choose $c_n = \dfrac{a_{2n}}{b_{2n-1}}.$ (2-10)

This causes the coefficient of s^{2n} in Equation (2-9) to vanish. Next it is intended to show that the coefficient of s^{2n-2} in Equation (2-9) is positive, i.e.,

$$a_{2n-2} - c_nb_{2n-3} = \frac{a_{2n-2}b_{2n-1} - a_{2n}b_{2n-3}}{b_{2n-1}} > 0. \quad (2\text{-}11)$$

It follows from hypothesis that

$$A(s) = a_{2n}s^{2n} + a_{2n-2}s^{2n-2} + \cdots + a_2s^2 + 1$$

$$B(s) = b_{2n-1}s^{2n-1} + b_{2n-3}s^{2n-3} + \cdots + b_3s^3 + b_1s$$

$$= s(b_{2n-1}s^{2n-2} + b_{2n-3}s^{2n-4} + \cdots + b_3s^2 + b_1)$$

$$= s\,\bar{B}(s).$$

Substituting $z = s^2$ yields

$A(z) = a_{2n}z^{n} + a_{2n-2}z^{n-1} + \cdots + a_2 z + 1$

$\overline{B}(z) = b_{2n-1}z^{n-1} + b_{2n-3}z^{n-2} + \cdots + b_3 z + b_1.$

By hypothesis, it follows that zeros of $A(z)$, $\overline{z}_{a_i}$ and $\overline{B}(z)$, $\overline{z}_{b_i}$ interlace on the negative real axis, as shown in Figure 2.1, with

$0 > \overline{z}_{a_1} > \overline{z}_{b_1} > \cdots > \overline{z}_{b_{n-1}} > \overline{z}_{a_n}$, where $\overline{z}_{a_i} = -|z_{a_i}|^2$.

From the zero distributions of $A(z)$ and $\overline{B}(z)$, it follows that the sum of the zeros of $A(z)$ is less than the sum of the zeros of $\overline{B}(z)$. Therefore, the negative sum of the zeros of $A(z)$ is larger than the negative sum of the zeros of $\overline{B}(z)$,

$$\frac{a_{2n-2}}{a_{2n}} > \frac{b_{2n-3}}{b_{2n-1}},$$

which implies $a_{2n-2}b_{2n-1} - a_{2n}b_{2n-3} > 0.$ (2-12)

The coefficient of s^{2n-4} of $A(s)$ is positive because the negative product of the zeros of $A(z)$ is larger than the negative product of the zeros of $\overline{B}(z)$. Similarly, all other terms of $A(s)$ are positive. Therefore $A(s)$ is an even function of s and of degree $2(n-1)$.

Next, it is shown that $C(s)$ is an odd function of s and of degree $2n-3$. Equation (2-8) can be written as

$$C(s) = (c_{2n-1}-c_n d_{2n-2})s^{2n-1} + (c_{2n-3}-c_n d_{2n-4})s^{2n-3} + (c_{2n-5}-c_n d_{2n-6})s^{2n-5} + \cdots + 1. \quad (2\text{-}13)$$

Since $A(s)D(s) - B(s)C(s) = 1$, the coefficient of s^{4n-2} must be zero, i.e.,

$$a_{2n}d_{2n-2} - b_{2n-1}c_{2n-1} = 0$$

which implies that $c_n = \frac{a_{2n}}{b_{2n-1}} = \frac{c_{2n-1}}{d_{2n-2}}$. (2-14)

From Equation (2-14) it follows that the coefficient of s^{2n-1} in Equation (2-13) vanishes.

The coefficient of s^{2n-3} of C(s) is

$$c_{2n-3} - c_n d_{2n-4} = c_{2n-3} - \frac{c_{2n-1}}{d_{2n-2}} \cdot d_{2n-4}$$

$$= \frac{c_{2n-3}d_{2n-2} - c_{2n-1}d_{2n-4}}{d_{2n-2}} > 0,$$

which follows from the zero distributions of C(s) and D(s). Similarly, all other terms of C(s) are positive. Therefore C(s) is an odd function of s and of degree 2n-3.

Next it is shown that

B(s) is an odd function of s and of degree 2n-3.

D(s) is an even function of s and of degree 2n-4.

The matrix $\begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$ now satisfies Lemma 2; zeros of A(s) and C(s) interlace on the $j\omega$ axis in a pattern given in Lemma 2.

It follows from Equation (2-6) that

$$S_{n-1}(s) = \begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix} = \begin{bmatrix} \tilde{A}(s) & B(s) - s\ell_n\tilde{A}(s) \\ \tilde{C}(s) & D(s) - s\ell_n\tilde{C}(s) \end{bmatrix}.$$

where $\tilde{B}(s) = B(s) - s\ell_n\tilde{A}(s)$ (2-15)

$\tilde{D}(s) = D(s) - s\ell_n\tilde{C}(s)$ (2-16)

Equation (2-15) can be written as

$$\tilde{B}(s) = (\ell_n c_n b_{2n-1} - \ell_n a_{2n})s^{2n+1} + (\ell_n c_n b_{2n-3} + b_{2n-1} - \ell_n a_{2n-2})s^{2n-1}$$
$$+ (\ell_n c_n b_{2n-5} + b_{2n-3} - \ell_n a_{2n-4})s^{2n-3} + \dots + (b_1 - \ell_n)s \qquad (2\text{-}17)$$

The coefficient of s^{2n+1} of $\tilde{B}(s)$ is

$$\ell_n c_n b_{2n-1} - \ell_n a_{2n} = \ell_n \frac{a_{2n}}{b_{2n-1}} \cdot b_{2n-1} - \ell_n a_{2n} = 0.$$

The coefficient of s^{2n-1} of $\tilde{B}(s)$ is

$$\ell_n c_n b_{2n-3} + b_{2n-1} - \ell_n a_{2n-2}$$

$$= \ell_n \left[\frac{a_{2n}}{b_{2n-1}} \cdot b_{2n-3} - a_{2n-2}\right] + b_{2n-1}$$

$$= \ell_n \frac{a_{2n}b_{2n-3} - a_{2n-2}b_{2n-1}}{b_{2n-1}} + b_{2n-1}. \qquad (2\text{-}18)$$

$$\text{Choosing } \ell_n = \frac{b^2_{2n-1}}{a_{2n-2}b_{2n-1} - a_{2n}b_{2n-3}} > 0, \qquad (2\text{-}19)$$

results in Equation (2-18) to vanish. Therefore, the degree of $\tilde{B}(s)$ is $\leq$ 2n-3.

Equation (2-16) can be written as

$$\tilde{D}(s) = (\ell_n c_n d_{2n-2} - \ell_n c_{2n-1}) s^{2n} + (\ell_n c_n d_{2n-4} + d_{2n-2} - \ell_n c_{2n-3}) s^{2n-2}$$

$$+ (\ell_n c_n d_{2n-6} + d_{2n-4} - \ell_n c_{2n-5}) s^{2n-4} + \ldots + 1 \qquad (2\text{-}20)$$

The coefficient of s^{2n} of $\tilde{D}(s)$ is

$$\ell_n c_n d_{2n-2} - \ell_n c_{2n-1} = \ell_n \cdot \frac{c_{2n-1}}{d_{2n-2}} \cdot d_{2n-2} - \ell_n c_{2n-1} = 0.$$

The coefficient of s^{2n-2} of $\tilde{D}(s)$ is

$$\ell_n c_n d_{2n-4} + d_{2n-2} - \ell_n c_{2n-3}$$

$$= \ell_n \left[\frac{a_{2n}}{b_{2n-1}} \cdot d_{2n-4} - c_{2n-3} \right] + d_{2n-2}$$

$$= \frac{b^2_{2n-1}}{a_{2n-2} b_{2n-1} - a_{2n} b_{2n-3}} \cdot \frac{a_{2n} d_{2n-4} - b_{2n-1} c_{2n-3}}{b_{2n-1}} + d_{2n-2}$$

$$= \frac{b_{2n-1}(a_{2n} d_{2n-4} - b_{2n-1} c_{2n-3}) + d_{2n-2}(a_{2n-2} b_{2n-1} - a_{2n} b_{2n-3})}{a_{2n-2} b_{2n-1} - a_{2n} b_{2n-3}}$$

$$(2\text{-}21)$$

Now, again since $A(s)D(s) - B(s)C(s) = 1$, the coefficient of s^{4n-4} must vanish which implies that

$$a_{2n} d_{2n-4} - b_{2n-1} c_{2n-3} = b_{2n-3} c_{2n-1} - a_{2n-2} d_{2n-2}. \qquad (2\text{-}22)$$

Substituting Equations (2-14) and (2-22) into (2-21) causes the coefficient of s^{2n-2} of $\tilde{D}(s)$ to vanish. Therefore, the degree of $\tilde{D}(s)$ is $\leq 2n-4$.

Now, consider the matrix $S_{n-1}(s) = \begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix}$

where

$\tilde{A}(s)$ is an even function of s and of degree 2n-2

$\tilde{B}(s)$ is an odd function of s and of degree $\leq$ 2n-3

$\tilde{C}(s)$ is an odd function of s and of degree 2n-3

$\tilde{D}(s)$ is an even function of s and of degree $\leq$ 2n-4.

Since

$$\tilde{A}(s)\tilde{D}(s) - \tilde{B}(s)\tilde{C}(s) = 1,$$

$$\tilde{A}(s)\tilde{D}(s) - s^2\tilde{\tilde{B}}(s)\tilde{\tilde{C}}(s) = 1,$$

where both $\tilde{\tilde{B}}(s)$ and $\tilde{\tilde{C}}(s)$ are even functions of s.

Let $z = s^2$. Then,

$$\tilde{A}(z)\tilde{D}(z) - z\,\tilde{\tilde{B}}(z)\tilde{\tilde{C}}(z) = 1,$$

and

$$z_{a_i}\tilde{\tilde{B}}(\bar{z}_{a_i})\tilde{\tilde{C}}(\bar{z}_{a_i}) = z_{a_{i+1}}\tilde{\tilde{B}}(\bar{z}_{a_{i+1}})\tilde{\tilde{C}}(\bar{z}_{a_{i+1}}). \qquad (2\text{-}23)$$

$\tilde{\tilde{C}}(\bar{z}_{a_i})$ and $\tilde{\tilde{C}}(\bar{z}_{a_{i+1}})$ are of opposite signs because of the zero distributions of $\tilde{A}(z)$ and $\tilde{\tilde{C}}(z)$. Therefore, $\tilde{\tilde{B}}(\bar{z}_{a_i})$ and $\tilde{\tilde{B}}(\bar{z}_{a_{i+1}})$ are also of opposite signs. This implies that the interval $(\bar{z}_{a_i}, \bar{z}_{a_{i+1}})$ contains an odd number of zeros of $\tilde{\tilde{B}}(z)$. There are n-2 of these intervals each of which contains one and only one zero of $\tilde{\tilde{B}}(z)$. Therefore $\tilde{\tilde{B}}(z)$ is of degree $\geq$ n-2 and $\tilde{B}(s)$ is of degree

$\geq$ 2n-4 and $\tilde{B}(s) = s\tilde{\tilde{B}}(s)$ is of degree $\geq$ 2n-3. Since the degree of $\tilde{B}(s)$ has been previously shown to be $\leq$ 2n-3, therefore $\tilde{B}(s)$ is of degree 2n-3. Similarly, $\tilde{D}(s)$ is of degree 2n-4. This proves the $S_{n-1}(s)$. This decomposition process is continued until n=1. This results in

$$S_1(s) = \begin{bmatrix} a_2s^2+1 & b_1s \\ c_1s & 1 \end{bmatrix}.$$

Define

$$K_1(s) = \begin{bmatrix} 1+s^2\ell_1c_1 & s\ell_1 \\ sc_1 & 1 \end{bmatrix},$$

where $c_1 = \frac{a_2}{b_1} = c_1$ and $\ell_1 = b_1$.

The product

$$S_1(s)K_1^{-1}(s) = \begin{bmatrix} a_2s^2+1 & b_1s \\ c_1s & 1 \end{bmatrix}\begin{bmatrix} 1 & -\ell_1s \\ -c_1s & \ell_1c_1s^2+1 \end{bmatrix}$$

$$= \begin{bmatrix} (a_2-b_1c_1)s^2+1 & (b_1\ell_1c_1-a_2\ell_1)s^3+(b_1-\ell_1)s \\ (c_1-c_1)s & (\ell_1c_1-c_1\ell_1)s^2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, the above process results in

$$S_n(s) = K_1(s)\cdot K_2(s)\cdots K_n(s)$$

as described in the theorem. This completes the proof of the sufficient part of the theorem.

Necessity: It is intended to prove by induction that if there is $K_1(s) \cdot K_2(s) \cdots K_n(s)$ where each $K_i(s)$ is described by Equation (2-3), then

$$S_n(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$$

possesses the properties (a), (b), (c), and (d) of the theorem.

Obviously $S_1(s)$, $S_2(s)$ $\cdots$ satisfy conditions (a), (b), (c), and (d). If $S_n(s)$ has properties (a), (b), (c), and (d), then

$$S_n(s)K_{n+1}(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} 1+s^2\ell_{n+1}c_{n+1} & s\ell_{n+1} \\ sc_{n+1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} 1 & s\ell_{n+1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sc_{n+1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A(s) & \tilde{B}(s) \\ C(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sc_{n+1} & 1 \end{bmatrix}. \qquad (2\text{-}24)$$

Then, $\tilde{B}(s) = s\ell_{n+1}A(s) + B(s)$ and of degree $2n+1$,

$\tilde{D}(s) = s\ell_{n+1}C(s) + D(s)$ and of degree $2n$.

Since $\det \begin{bmatrix} A(s) & \tilde{B}(s) \\ C(s) & \tilde{D}(s) \end{bmatrix} = 1$ and zeros of $A(s)$ and $C(s)$ interlace on the $j\omega$ axis, it follows that, by Lemma 4, zeros of $\tilde{B}(s)$ and $\tilde{D}(s)$ interlace on the $j\omega$ axis.

Equation (2-24) is now written as

$$S_n(s)K_{n+1}(s) = \begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix} = S_{n+1}(s),$$

where $\tilde{A}(s) = A(s) + sc_{n+1}\tilde{B}(s)$ and is of degree $2(n+1)$,

$\tilde{C}(s) = C(s) + sc_{n+1}\tilde{D}(s)$ and is of degree $2n+1$.

Since $\det \begin{bmatrix} \tilde{A}(s) & \tilde{B}(s) \\ \tilde{C}(s) & \tilde{D}(s) \end{bmatrix} = 1$ and zeros of $\tilde{B}(s)$ and $\tilde{D}(s)$ interlace on the $j\omega$ axis, it follows that, by Lemma 5, zeros of $\tilde{A}(s)$ and $\tilde{B}(s)$, $\tilde{A}(s)$ and $\tilde{C}(s)$, $\tilde{C}(s)$ and $\tilde{D}(s)$ interlace on the $j\omega$ axis. This shows that the matrix $S_{n+1}(s)$ also has properties (a), (b), (c), and (d) of the theorem.

Q.E.D.

Corollaries to this theorem are summarized in Table 2.2 without proofs. Theorem 2.1 and its corollaries formulate the method of decomposition from the right. The following theorem shows decomposition from the left.

Theorem 2.2 Let the matrix $S_n(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$.

$$S_n(s) = K_n(s)K_{n-1}(s)\cdots K_2K_1$$

where

$$K_i(s) = \begin{bmatrix} 1+s^2\ell_i c_i & s\ell_i \\ sc_i & 1 \end{bmatrix}, \quad \ell_i,\ c_i > 0 \text{ for } i = 1,2,\cdots n$$

Table 2.2 Summary of Decomposition Corollaries

Corollary 2.1	Corollary 2.2
$S_n(s) = K_1(s) \cdot K_2(s) \cdots K_n(s)$	$S_n(s) = K_1(s) \cdot K_2(s) \cdots K_{n+1}(s)$
$K_i = \begin{bmatrix} 1 & s\ell_i \\ sc_i & 1+s^2\ell_i c_i \end{bmatrix}$	$K_i = \begin{bmatrix} 1 & s\ell_i \\ sc_i & 1+s^2\ell_i c_i \end{bmatrix}$ $K_1 = \begin{bmatrix} 1 & s\ell_1 \\ 0 & 1 \end{bmatrix}$
if and only if	
1. Det $S_n(s)=1$; $A(0)=1$	
2. A(s) is of degree 2(n-1) B(s) is of degree 2n-1 C(s) is of degree 2n-1 D(s) is of degree 2n	A(s) is of degree 2(n-1) B(s) is of degree 2n-1 C(s) is of degree 2n-3 D(s) is of degree 2(n-1)
3. Zeros of C(s) and D(s) interlace on the $j\omega$ axis with $0 = z_{c_0} < \lvert z_{d_1} \rvert < \lvert z_{c_1} \rvert < \cdots < \lvert z_{c_{n-1}} \rvert < \lvert z_{d_n} \rvert$	Zeros of A(s) and B(s) interlace on the $j\omega$ axis with $0 = z_{b_0} < \lvert z_{a_1} \rvert < \lvert z_{b_1} \rvert < \cdots < \lvert z_{a_{n-1}} \rvert < \lvert z_{b_{n-1}} \rvert$

Table 2.2 (Continued)

Corollary 2.3
$S_n(s) = K_1(s) \cdot K_2(s) \cdots K_{n+1}(s)$
$K_i = \begin{bmatrix} 1+s^2\ell_i c_i & s\ell_i \\ sc_i & 1 \end{bmatrix}$ $K_1 = \begin{bmatrix} 1 & 0 \\ sc_1 & 1 \end{bmatrix}$
if and only if
1. Det $S_n(s)=1$; $A(0)=1$
2. $A(s)$ is of degree $2(n-1)$ $B(s)$ is of degree $2n-3$ $C(s)$ is of degree $2n-1$ $D(s)$ is of degree $2(n-1)$ Zeros of $C(s)$ and $D(s)$ interlace on the $j\omega$ axis with $0 = z_{c_0} < \lvert z_{d_1} \rvert < \lvert z_{c_1} \rvert < \cdots < \lvert z_{d_{n-1}} \rvert < \lvert z_{c_{n-1}} \rvert$

if and only if

(a) Det $S_n(s) = 1$,

(b) A(s), B(s), C(s) and D(s) are polynomials with positive coefficients A(s) and D(s) are even functions of s and of degree 2n and 2(n-1) respectively.
B(s) and C(s) are odd functions of s and of degree 2n-1,

(c) Zeros of A(s), $[z_{a_i}]$ and zeros of C(s), $[z_{c_i}]$ interlace on the $j\omega$ axis with

$$0 = z_{c_0} < |z_{a_1}| < |z_{c_1}| < \cdots < |z_{c_{n-1}}| < |z_{c_n}|,$$

(d) A(0) = 1.

Proof of this theorem is similar to that of Theorem 2.1 and therefore omitted.

Q.E.D.

Consider a passive, reciprocal 2-port network with the polarities of I_1, I_2, V_1, and V_2 as shown in Figure 2.3. The chain matrix expression of this 2-port is then given as

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V_2 \\ I_2 \end{bmatrix}.$$

If this 2-port is of simple ladder structure of Figure 2.4 then the chain matrix of this ladder is

$$K = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} = \begin{bmatrix} 1+YZ & Z \\ Y & 1 \end{bmatrix},$$

If all series arms and shunt arms are inductive and capacitive elements, respectively, then $Z(s) = \ell s$, $Y(s) = cs$, and therefore K_i matrix in the foregoing theorem is realizable by this ladder. It follows that a cascade of n of these ladders as shown in Figure 2.5 has an overall chain matrix of $K_1 \cdot K_2 \cdot \cdots K_n = S_n$ which is the overall matrix stated in the theorem.

Section 2.4 The Decomposition Algorithm

The algorithm for determining the values of ladder elements will now be established. The decomposition theorem is used to develop a computer-supported procedure based on the two known entries in the overall chain matrix.

$$A(s) = a_{2n}s^{2n} + a_{2(n-1)}s^{2(n-1)} + \cdots + a_2 s^2 + 1$$

$$B(s) = b_{2n-1}s^{2n-1} + b_{2n-3}s^{2n-3} + \cdots + b_3 s^3 + b_1 s$$

A Routh array is formed;

$$
\begin{array}{lllllll}
\dfrac{a_{2n}}{b_{2n-1}} & a_{2n} & a_{2(n-1)}\cdots & a_2 & 1 & & s^{2n} \\[2ex]
\dfrac{b_{2n-1}}{\alpha_{2(n-1)}} & b_{2n-1} & b_{2n-3}\cdots & b_1 & & & s^{2n-1} \\[2ex]
\dfrac{\alpha_{2(n-1)}}{\gamma_{2n-3}} & \alpha_{2(n-1)} & \alpha_{2(n-2)}\cdots & \alpha_0 & & & s^{2(n-1)} \\[2ex]
 & \gamma_{2n-3} & \gamma_{2n-5}\cdots & b_1 & & & s^{2n-3} \\
 & \cdot & & & & & \cdot \\
 & \cdot & & & & & \cdot \\
 & \cdot & & & & & \cdot \\
\dfrac{\gamma_1}{\alpha_0} & \gamma_1 & & & & & \\
 & \alpha_0 & & & & & s^0
\end{array}
$$

where each element starting from the second column to the right is evaluated by Routh criterion method, i.e.,

$$\alpha_{2(n-1)} = \frac{b_{2n-1}a_{2(n-1)} - a_{2n}b_{2n-3}}{b_{2n-1}},$$

$$\alpha_{2(n-2)} = \frac{b_{2n-1}a_{2(n-2)} - a_{2n}b_{2n-5}}{b_{2n-1}},$$

$$\gamma_{2n-3} = \frac{\alpha_{2(n-1)}b_{2n-3} - b_{2n-1}\alpha_{2(n-2)}}{\alpha_{2(n-1)}}$$

$$\gamma_{2n-5} = \frac{\alpha_{2(n-1)}b_{2n-5} - b_{2n-1}\alpha_{2(n-3)}}{\alpha_{2(n-1)}}, \text{ etc.}$$

In the first column, (k,1) entry gives the value of ℓ_k if k is odd and the value of c_k if k is even. All the elements in the array and the values of ℓ's and c's can be computed by a simple digital computer program.

The above algorithm applies to the circuit as shown in Figure 2.6 where the number of reactive elements is even (Theorem 2.1). If the number of reactive elements is odd as shown in Figure 2.7 (Corollary 2.2), then the algorithm should be modified as follows:

(1) In the Routh array, the coefficients of B(s) should be located in the first row, and those of A(s) in the second row.

(2) In the first column, (k,1) entry gives the values of ℓ_k if k is odd and the values of c_k if k is even.

In both Figure 2.6 and Figure 2.7, m denotes total number of elements.

Example 2.2

$$A(s) = (s^2+1)(10s^2+1)(20s^2+1)(36s^2+1)$$

$$= 7200s^8+8460s^6+1346s^4+67s^2+1$$

$$B(s) = (6s^2+1)(12s^2+1)(25s^2+1)s$$

$$= 1800s^7+522s^5+43s^3+s$$

$c_1 = 4$	7200	8460	1346	67	1	s^8
$\ell_2 = 0.28$	1800	522	43	1	0	s^7
$c_3 = 33.47$	6372	1174	63	1	0	s^6
$\ell_4 = 0.58$	190.36	25.20	0.72	0	0	s^5
$c_5 = 120.53$	330.36	38.98	1	0	0	s^4
$\ell_6 = 0.12$	2.74	0.14	0	0	0	s^3
$c_7 = 1335.54$	21.95	1	0	0	0	s^2
$\ell_8 = 0.016$	0.016	0	0	0	0	s^1
	1	0	0	0	0	s^0

The completed circuit is shown in Figure 2.8.

Example 2.3 (Corollary 2.2)

Given: $A(s) = 240s^6 + 234s^4 + 48s^2 + 1$

$B(s) = 1440s^7 + 1452s^5 + 330s^3 + 12s$

$\ell_1 = 6$	1440	1452	330	12	s^7
$c_2 = 5$	240	234	48	1	s^6
$\ell_3 = 2$	48	42	6		s^5
$c_4 = 4$	24	18	1		s^4
$\ell_5 = 3$	6	4			s^3
$c_6 = 2$	2	1			s^2
$\ell_7 = 1$	1				s^1
	1				s^0

The completed circuit is shown in Figure 2.9.

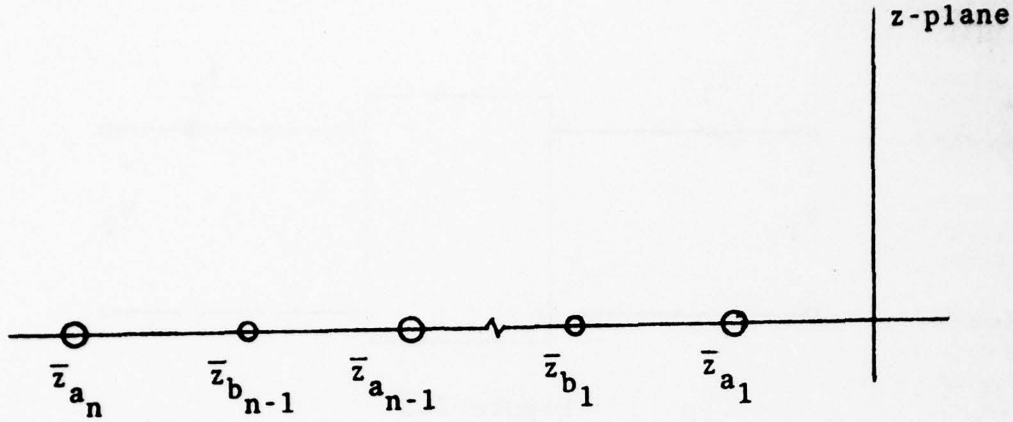

Figure 2.1

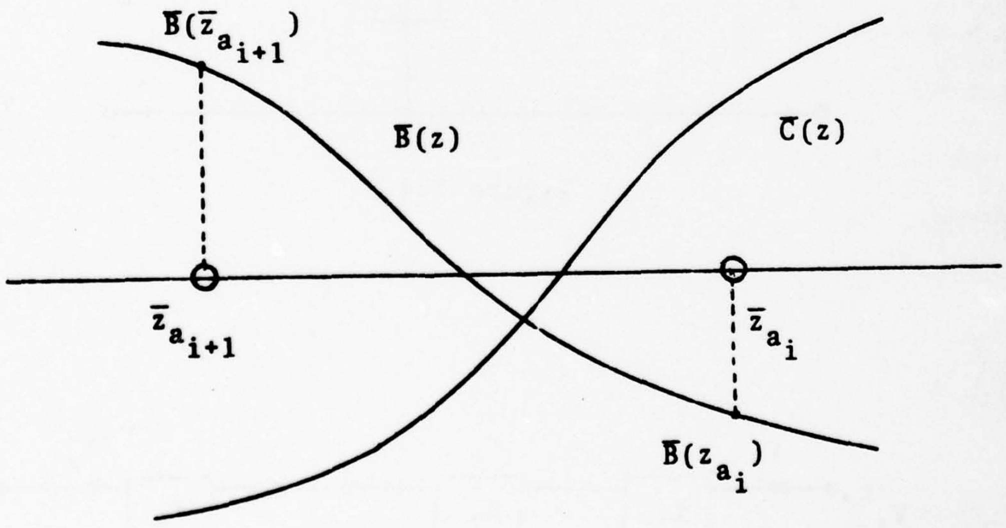

Figure 2

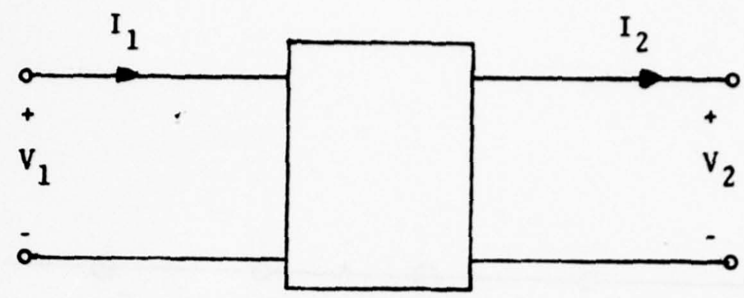

Figure 2.3

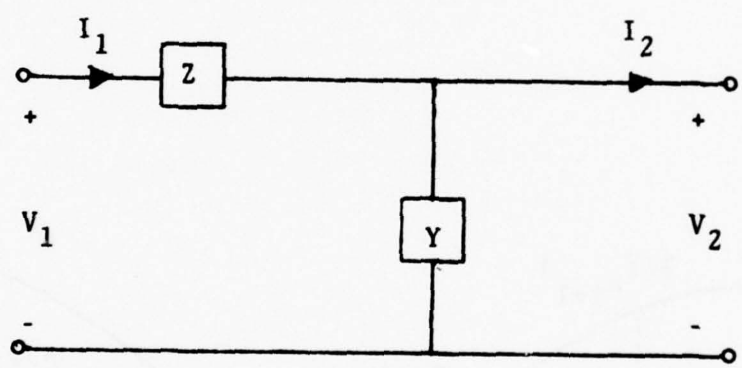

Figure 2.4

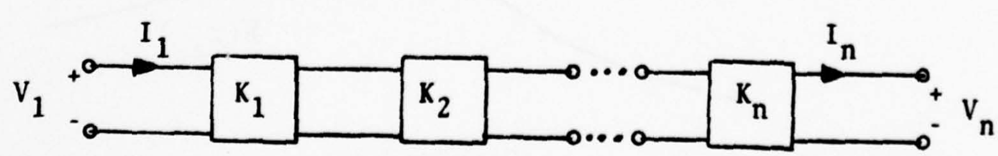

Figure 2.5

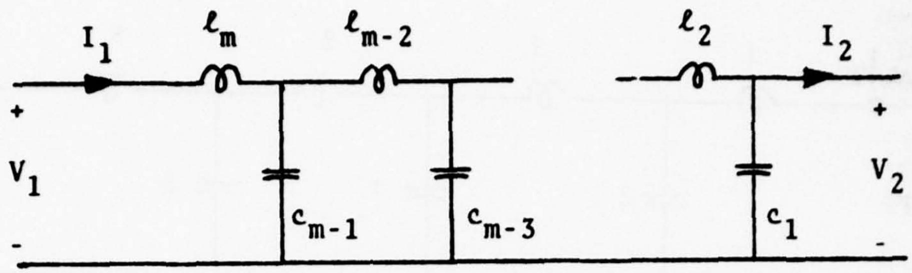

Figure 2.6

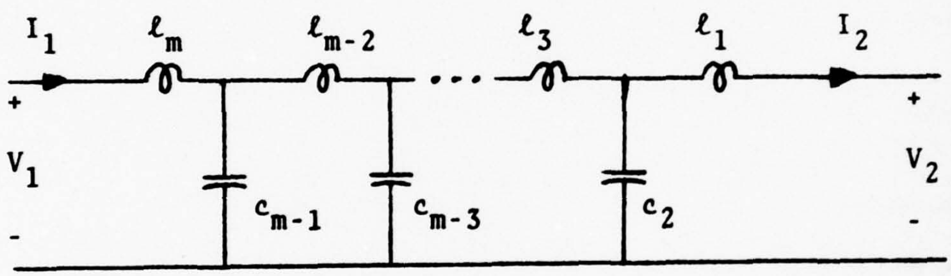

Figure 2.7

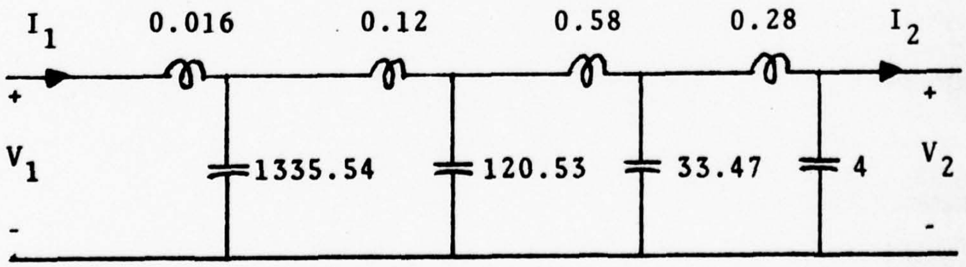

Figure 2.8

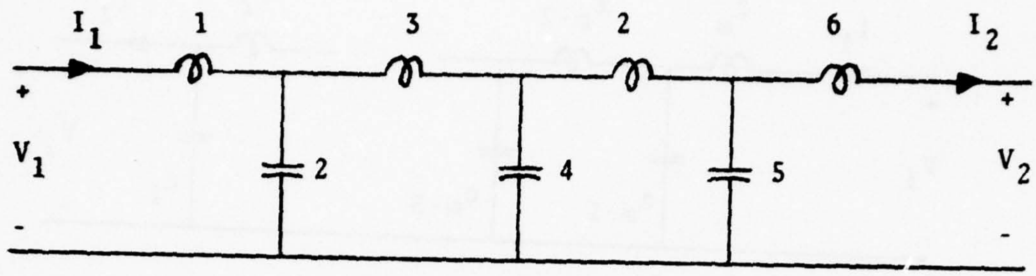

Figure 2.9

CHAPTER 3

APPLICATION OF CHAIN MATRIX DECOMPOSITION TECHNIQUES

Section 3.1 Introduction

In this chapter the techniques which were developed earlier will now be used in several practical situations which are of interest. Fortunately, the literature [25] is rich in characteristics for filter networks in wide variety and degrees of sophistication. For example such filters as Butterworth, Chebyshev, and Bessel are well known and completely characterized. These will now be synthesized through use of the decomposition of chain matrices. The elements of these matrices are predetermined by the orthogonal polynomials used to approximate the ideal filter characteristics. The chain-matrix method is shown to be simple and straightforward. Results of the above filters are compiled into a table for both single and double-terminated cases.

Section 3.2 Synthesis of Single-Terminated Ladder Network by Chain Matrix Decomposition

An ideal low-pass filter characteristic of Figure 3.1 can be approximated in various ways, such as

$$\left|\frac{V_2(s)}{V_1(s)}\right|^2_{s=j\omega} = \frac{1}{1+\omega^{2n}} \tag{3-1}$$

$$\left|\frac{V_2(s)}{V_1(s)}\right|^2_{s=j\omega} = \frac{1}{1+\varepsilon^2 C_n^{\,2}} \qquad (3\text{-}2)$$

$$\left|\frac{V_2(s)}{V_1(s)}\right|^2_{s=j\omega} = e^{-sT} \qquad (3\text{-}3)$$

where $V_1(s)$ and $V_2(s)$ are the input and output voltages of the filter networks respectively.

As is well known, Equations (3-1), (3-2), and (3-3) result in Butterworth, Chebyshev, and Bessel polynomials and hence the corresponding filters. Synthesis of these filters have been extensively investigated and widely documented. They are discussed and illustrated in most of the textbooks on network synthesis. In this section, synthesis of these filter networks are accomplished through the application of the decomposition of chain parameter matrices whose elements are predetermined by the orthogonal polynomials used to approximate the ideal characteristics. These polynomials are well known in the literature and can be found in many mathematical handbooks.

Consider the circuit shown in Figure 3.2 in which m number of reactive elements are connected to a voltage source and terminated in a 1-Ω resistor. The chain parameter matrix is

$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix} \qquad (3\text{-}4)$$

Since $V_2(s) = I_2(s)$, it follows from Equation (3-4) that

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{A(s)+B(s)} \qquad (3\text{-}5)$$

Therefore,

$$\left|\frac{V_2(s)}{V_1(s)}\right|^2_{s=j\omega} = \left[\frac{1}{A(s)+B(s)}\right]\left[\frac{1}{A(-s)+B(-s)}\right]_{s=j\omega}$$

$$= \left.\frac{1}{B_m(s)}\,\frac{1}{B_m(-s)}\right|_{s=j\omega} \qquad (3\text{-}6)$$

where $B_m(s)$ are orthogonal polynomials of mth order used for the approximation of the ideal filter characteristics. It follows from Equation (3-6) that

$$\frac{1}{A(s)+B(s)} = \frac{1}{B_m(s)}.$$

Since A(s) and B(s) are the elements of the chain matrix of Theorem 2.1, A(s) is an even polynomial and B(s) is an odd polynomial and hence they are uniquely determined. $A(s) = Ev[B_m(s)]$, $B(s) = Od[B_m(s)]$. In other words, if

$$B_m(s) = s^m+a_{m-1}S^{m-1}+a_{m-2}s^{m-2}+\cdots+a_1s+a_0 \qquad (3\text{-}7)$$

where m is even, then

$$A(s) = s^m + a_{m-2}s^{m-2} + \cdots + a_2s^2 + a_0 \tag{3-8}$$

$$B(s) = a_{m-1}s^{m-1} + a_{m-3}s^{m-3} + \cdots + a_3s^3 + a_1s \tag{3-9}$$

The following examples demonstrate the application of decomposition method to filter designs.

Example 3.1. (Application of Theorem 2.1)

Given 6th order Bessel polynomial

$$B_6(s) = s^6+21s^5+210s^4+1{,}260s^3+4{,}725s^2+10{,}395s+10{,}395$$

Solution:

$$A(s) = \mathrm{Ev}[B_6(s)] = s^6+210s^4+4{,}725s^2+10{,}395$$

$$B(s) = \mathrm{Od}[B_6(s)] = 21s^5+1{,}260s^3+10{,}395s.$$

A Routh array is formed and circuit elements computed in accordance with the deomposition algorithm developed in Chapter 2:

c_6=0.048	1	210	4725	10395	s^6
ℓ_5=0.14	21	1260	10395	0	s^5
c_4=0.225	150	4230	10395	0	s^4
ℓ_3=0.301	667.8	8939.7	0	0	s^3
c_2=0.382	2221.98	10395	0	0	s^2
ℓ_1=0.559	5815.56	0	0	0	s^1
	10395	0	0	0	s^0

The completed 6th order Bessel filter circuit is shown in Figure 3.3.

Example 3.2. (Application of Corollary 2.2)

Given 5th order Butterworth polynomial

$$B_5(s) = s^5+3.236s^4+5.236s^3+5.236s^2+3.236s+1$$

Solution:

$$A(s) = Ev[B_5(s)] = 3.236s^4+5.236s^2+1$$

$$B(s) = Od[B_5(s)] = s^5+5.236s^3+3.236s.$$

$\ell_5=0.309$	1	5.236	3.236	s^5
$c_4=0.894$	3.236	5.236	1	s^4
$\ell_3=1.382$	3.618	2.927		s^3
$c_2=1.694$	2.618	1		s^2
$\ell_1=1.545$	1.545			s^1
	1			s^0

The completed 5th order Butterworth filter circuit is shown in Figure 3.4.

Tables 3.1, 3.2, and 3.3 show the normalized element values for a single-terminated Butterworth, Chebyshev, and Bessel filters respectively.

Table 3.1 Normalized Element Values for a Single-Terminated Butterworth Filter

m	L_1	C_2	L_3	C_4	L_5	C_6	L_7	C_8
1	1.000							
2	0.707	1.414						
3	0.500	1.333	1.500					
4	0.383	1.082	1.577	1.531				
5	0.309	0.894	1.382	1.694	1.545			
6	0.259	0.758	1.202	1.553	1.759	1.553		
7	0.223	0.656	1.055	1.397	1.659	1.799	1.558	
8	0.195	0.578	0.937	1.259	1.528	1.729	1.825	1.561

Table 3.2 Normalized Element Values for a Single-Terminated Chebyshev Filter with $\frac{1}{2}$-Decibel Ripple

m	L_1	C_2	L_3	C_4	L_5	C_6	L_7	C_8
1	0.349							
2	0.701	0.940						
3	0.798	1.300	1.347					
4	0.835	1.392	1.728	1.314				
5	0.853	1.429	1.814	1.643	1.539			
6	0.863	1.448	1.849	1.710	1.902	1.404		
7	0.869	1.460	1.868	1.737	1.971	1.725	1.598	
8	0.873	1.467	1.875	1.751	1.998	1.784	1.957	1.438

Table 3.3 Normalized Element Values for a Single-Terminated Bessel Filter

m	L_1	C_2	L_3	C_4	L_5	C_6	L_7	C_8
1	1.000							
2	0.333	1.000						
3	0.167	0.480	0.833					
4	0.100	0.290	0.463	0.710				
5	0.067	0.195	0.310	0.422	0.623			
6	0.048	0.140	0.225	0.301	0.382	0.560		
7	0.036	0.106	0.170	0.229	0.283	0.349	0.511	
8	0.028	0.082	0.134	0.181	0.223	0.264	0.321	0.473

Section 3.3 Synthesis of Double-Terminated Ladder Network by Chain Matrix Decomposition

Consider the circuit shown in Figure 3.5. This lossless ladder network is described by chain parameters A(s), B(s), C(s), and D(s) and it is double-terminated by one-ohm resistors. Its chain matrix is

$$\begin{bmatrix} V(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix} \qquad (3\text{-}10)$$

or

$$\begin{bmatrix} V_1(s) \\ I_1(s) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} \begin{bmatrix} V_2(s) \\ I_2(s) \end{bmatrix}. \qquad (3\text{-}11)$$

It follows from Equation (3-11) that

$$V_1(s) = [A(s)+C(s)]V_2(s)+[B(s)+D(s)]I_2(s). \qquad (3\text{-}12)$$

Since $V_2(s) = I_2(s)$, Equation (3-12) results in

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{A(s)+B(s)+C(s)+D(s)}. \qquad (3\text{-}13)$$

The voltage transfer function of the ladder network of Figure 3.5 can be written as

$$\frac{V_2(s)}{V_1(s)} = \frac{H}{B_m(s)}, \qquad (3\text{-}14)$$

where $B_m(s)$ is the normalized orthogonal polynomial of mth order that is used to approximate ideal low-pass filter characteristics and H is the constant multiplier. At s=0, the network is purely resistive and therefore,

$$\left.\frac{V_2(s)}{V_1(s)}\right|_{s=0} = \frac{H}{B_m(0)} = \frac{1}{2}.$$

Since normalized $B_m(0) = 1$, the constant multiplier H is one half. Thus

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{A(s)+B(s)+C(s)+D(s)} = \frac{1}{2B_m(s)}. \qquad (3\text{-}15)$$

In Chapter 2, it is shown that for an LC ladder network, A(s) and D(s) are even functions of s and B(s) and C(s) are odd functions of s. Therefore, it follows from Equation (3-15) that

$$A(s)+D(s) = Ev[2B_m(s)] \qquad (3\text{-}16)$$

$$B(s)+C(s) = Od[2B_m(s)]. \qquad (3\text{-}17)$$

The objective of this section is to apply the chain matrix decomposition method to synthesize double-terminated Butterworth filters. The terminations are assumed to be one ohm as shown in Figure 3.5. It has been shown [25] that these filters are symmetrical if m is odd. Furthermore, if m is even, they are antimetrical. Therefore,

$$A(s) = D(s) \text{ for m odd, and} \qquad (3\text{-}18)$$

$$B(s) = C(s) \text{ for m even.} \qquad (3\text{-}19)$$

Equation (3-15) to (3-19) are useful for synthesizing double-terminated filters by chain matrix decomposition.

The following example illustrates the procedure.

Example 3.3: Synthesize the double-terminated 6th order Butterworth filter given the Butterworth polynomial.

$$B_6(s) = s^6+3.8637s^5+7.4641s^4+9.1416s^3+$$
$$7.4641s^2+3.8637s+1$$

Solution: From Equations (3-16) and (3-17), it follows that

$$A(s)+D(s) = Ev[2B_6(s)] = 2(s^6+7.4641s^4+7.4641s^2+1) \quad (3\text{-}20)$$

and

$$B(s)+C(s) = Od[2B_6(s)] = 2(3.8637s^5+9.1416s^3+3.8637s).$$

Since m=6, it follows from Equation (3-19) that

$$B(s) = C(s) = 3.8637s^5+9.1416s^3+3.8637s. \quad (3\text{-}21)$$

For passive networks, $A(s)D(s)-B(s)C(s) = 1$, thus,

$$A(s)D(s) = B(s)C(s)+1$$
$$= (3.8637s^5+9.1416s^3+3.8637s)^2+1.$$

But,

$$[A(s)-D(s)]^2 = [A(s)+D(s)]^2-4A(s)D(s)$$
$$= 4(s^6+7.4641s^4+7.4641s^2+1)^2-$$
$$4(3.8637s^5+9.1416s^3+3.8637s)^2-4$$
$$= 4s^{12}.$$

Therefore,

$$A(s)-D(s) = 2s^6. \quad (3\text{-}22)$$

Adding Equations (3-20) and (3-22) results in

$$A(s) = 2s^6+7.4641s^4+7.4641s^2+1. \quad (3\text{-}23)$$

The chain parameters A(s) [Equation (3-23)] and B(s) [Equation (3-21)] can now be used to realize the filter. The computation of element values is shown in Table 3.4. The corresponding filter network is shown in Figure 3.6. Table 3.5 shows the normalized element values for a double-terminated Butterworth filter.

Table 3.4 Computation of Element Values of Example 3.3

	2	7.4641	7.4641	1
C_6 = 0.5176				
	3.8637	9.1416	3.8637	
L_5 = 1.4142				
	2.7321	5.4641	1	
C_4 = 1.9319				
	1.4142	2.4495		
L_3 = 1.9319				
	0.7320	1		
C_2 = 1.4142				
	0.5176			
L_1 = 0.5176				
	1			

Table 3.5 Normalized Element Values for a Double-Terminated Butterworth Filter (Equal Terminations)

m	L_1	C_2	L_3	C_4	L_5	C_6	L_7	C_8
1	2.000							
2	1.414	1.414						
3	1.000	2.000	1.000					
4	0.765	1.848	1.848	0.765				
5	0.618	1.618	2.000	1.618	0.618			
6	0.518	1.414	1.932	1.932	1.414	0.518		
7	0.445	1.248	1.802	2.000	1.802	1.248	0.445	
8	0.390	1.111	1.663	1.962	1.962	1.663	1.111	0.390

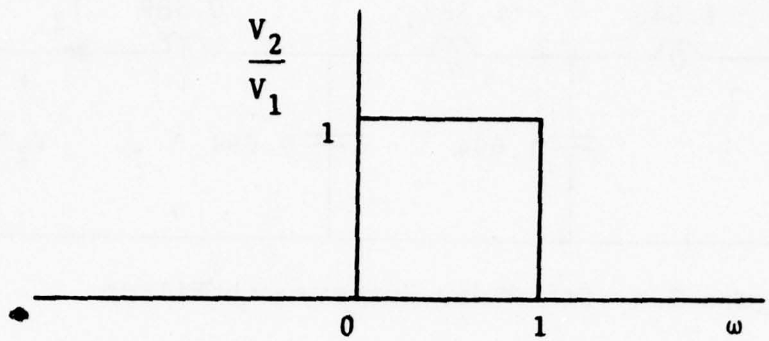

Figure 3.1

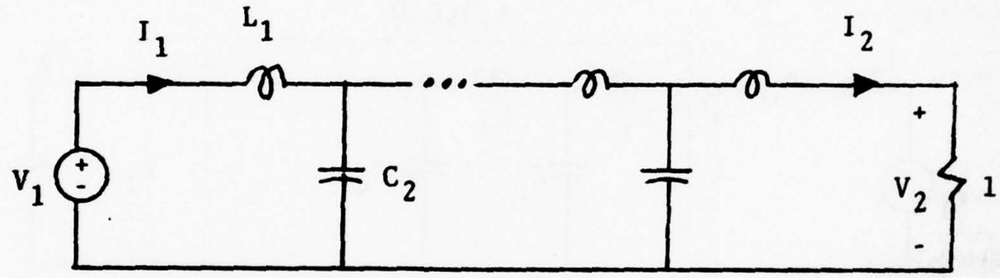

Figure 3.2 General Form of Filters in Tables 3.1, 3.2, and 3.3

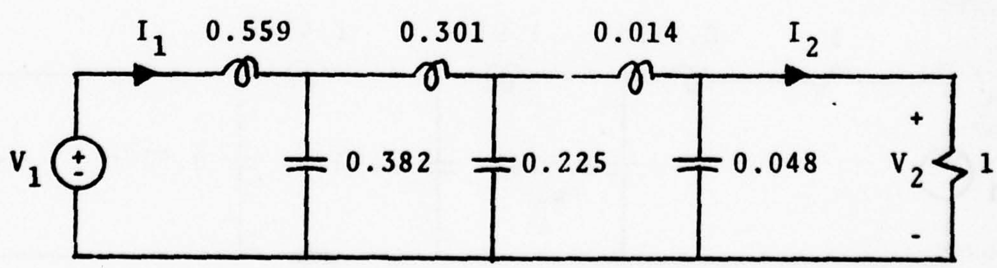

Figure 3.3 6th Order Bessel Filter

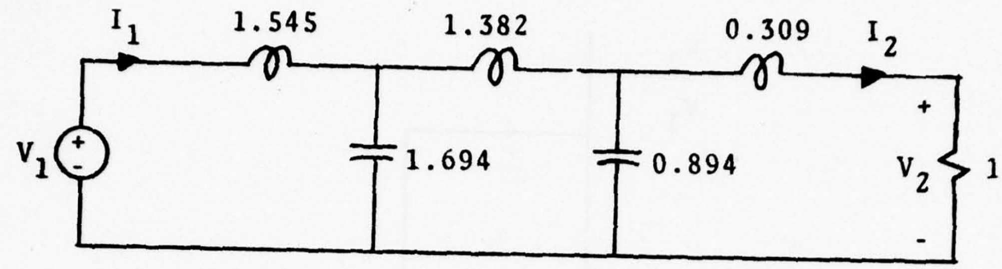

Figure 3.4 5th Order Butterworth Filter

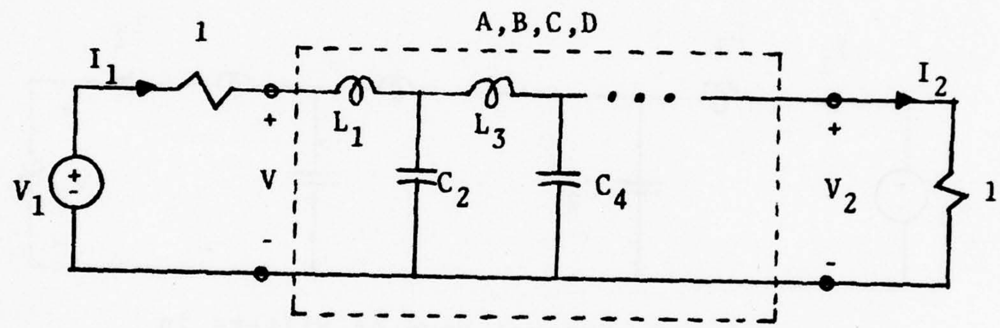

Figure 3.5 General Form of Double-Terminated Filter

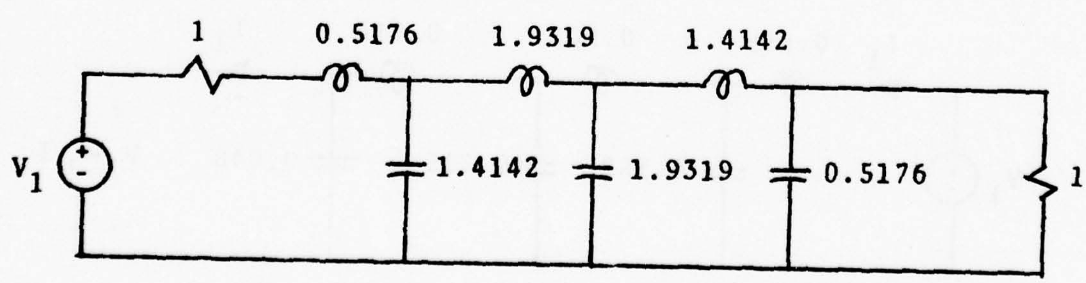

Figure 3.6 6th Order Butterworth Filter

CHAPTER 4

OPTIMAL SYNTHESIS OF INHOMOGENEOUS LADDER NETWORK USING CHAIN MATRIX

Section 4.1 Introduction

The synthesis of inhomogeneous ladder networks using chain matrix techniques has not been investigated to any significant extent. Very little material has appeared in the literature on this subject. A recent paper by Lee and Brown sheds some light on the pole-zero distribution pattern of the immittance function for inhomogeneous ladder networks. Much more work is necessary in order to clearly characterize such systems. In this chapter several synthesis procedures of optimal inhomogeneous ladder networks are developed through use of the pole-zero distribution pattern developed by Lee and Brown and a simple complex variable transformation.

Section 4.2 Optimal Synthesis of Inhomogeneous Ladder Networks

Fundamental definitions concerning inhomogeneous ladder networks and optimal synthesis are given below in order to facilitate proofs of Lemmas which will establish the synthesis procedures.

Definition 4.1: A ladder network is said to be inhomogenous if all its series arm impedances are $f_i z(s)$ and all its shunt arm admittances are $g_i y(s)$, where $i = 1, 2, \cdots, n$ and

$z(s) \neq y(s)$. Such a network appears in Figure 4.1.

Definition 4.2 An arc is denoted by $[G(s), k]$, if it satisfies the root locus equation,

$$1 + \frac{k}{G(s)} = 0,$$

for $k \geq 0$ in the s-plane.

Definition 4.3 Suppose,

$$P(s) = k_p \prod_{i=1}^{N} [G(s)+p_i]$$

and

$$Q(s) = k_q \prod_{i=1}^{N(\text{or } N-1)} [G(s)+q_i].$$

If $0 < p_i < q_i < p_{i+1}$ for all $i = 1, 2, \cdots, N-1$, then zeros of $P(s)$ and $Q(s)$ are said to interlace with respect to $[G(s), k]$. [Observation is made here that the zeros of $P(p)$ and $Q(p)$ interlace on the negative $p = G(s)$ axis of the plane.]

Definition 4.4 Synthesis of an inhomogeneous ladder network from a given transfer function is said to be optimal if $(\Sigma f_i)(\Sigma g_i)$ is minimized. The quantities f_i, g_i are the impedance and admittance level for the ith section of the network of Figure 4.1.

Let $S_n(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix} = \prod_{i=1}^{N} K_i$ be the chain matrix of

Figure 4.1. If

$$K_i = \begin{bmatrix} 1+f_i g_i y(s)z(s) & f_i z(s) \\ g_i y(s) & 1 \end{bmatrix}$$

where $z(s) = \frac{n_a(s)}{d_a(s)}$, $y(s) = \frac{n_b(s)}{d_b(s)}$, n = degree of $[n_a(s)n_b(s)]$

or degree of $[d_a(s)d_b(s)]$ whichever is greater. Then

$$A(s) = K_a \frac{\prod_{i=1}^{Nn} (s+z_{a_i})}{[d_a(s)d_b(s)]^N}, \tag{4-1}$$

$$A(s) \Big|_{[s|n_a(s)n_b(s) = 0]} = 1$$

$$z^{-1}(s)B(s) = K_b \frac{\prod_{i=1}^{(N-1)n} (s+z_{b_i})}{[d_a(s)d_b(s)]^{N-1}}, \tag{4-2}$$

$$z^{-1}(s)B(s) \Big|_{[s|n_a(s)n_b(s) = 0]} = \Sigma f_i = f_T$$

$$y^{-1}(s)C(s) = K_c \frac{\prod_{i=1}^{(N-1)n} (s+z_{c_i})}{[d_a(s)d_b(s)]^{N-1}}, \tag{4-3}$$

$$y^{-1}(s)C(s) \Bigg|_{[s \mid n_a(s)n_b(s) = 0]} = \Sigma g_i = g_T$$

$$D(s) = K_d \frac{\prod_{i=1}^{(N-1)n} (s+z_{d_i})}{[d_a(s)d_b(s)]^{N-1}}, \tag{4-4}$$

$$D(s) \Bigg|_{[s \mid n_a(s)n_b(s) = 0]} = 1$$

where zeros $-z_{a_i}$ and $-z_{b_i}$, $-z_{a_i}$ and $-z_{c_i}$, $-z_{b_i}$ and $-z_{d_i}$, $-z_{c_i}$ and $-z_{d_i}$, interlace with respect to [y(s)z(s), k]. Therefore, it follows from observation that there exists a transformation of y(s)z(s) = p such that

$$A(p) = \prod_{i=1}^{N} (1+\frac{p}{\alpha_i}),$$

$$[z^{-1}B](p) = f_T \prod_{i=1}^{N-1} (1+\frac{p}{\beta_i}),$$

$$[y^{-1}C](p) = g_T \prod_{i=1}^{N-1} (1+\frac{p}{\gamma_i}),$$

$$D(p) = \prod_{i=1}^{N-1} (1+\frac{p}{\delta_i}),$$

where

$$0 < \alpha_i < \beta_i < \alpha_{i+1},$$

$$0 < \alpha_i < \gamma_i < \alpha_{i+1},$$

$$0 < \beta_i < \delta_i < \beta_{i+1},$$

$$0 < \gamma_i < \delta_i < \gamma_{i+1}.$$

If an inhomogeneous ladder network ends in a series impedance as shown in Figure 4.2, then its chain parameters have the following forms:

$$A(s) = K_a \frac{\prod_{i=1}^{Nn} (s+z_{a_i})}{[d_a(s)d_b(s)]^N}, \tag{4-5}$$

$$A(s)\Big|_{[s|n_a(s)n_b(s) = 0]} = 1,$$

$$z^{-1}B(s) = K_b \frac{\prod_{i=1}^{Nn} (s+z_{b_i})}{[d_a(s)d_b(s)]^N}, \tag{4-6}$$

$$z^{-1}(s)B(s)\Big|_{[s|n_a(s)n_b(s) = 0]} = \Sigma f_i = f_T,$$

$$y^{-1}(s)C(s) = K_c \frac{\prod_{i=1}^{(N-1)n} (s+z_{c_i})}{[d_a(s)d_b(s)]^{N-1}}, \tag{4-7}$$

$$y^{-1}(s)C(s) \Big|_{[s|n_a(s)n_b(s) = 0]} = g_i = g_T,$$

$$D(s) = K_d \frac{\prod_{i=1}^{Nn} (s+z_{d_i})}{[d_a(s)d_b(s)]^{N-1}}, \tag{4-8}$$

$$D(s) \Big|_{[s|n_a(s)n_b(s) = 0]} = 1,$$

where zeros $-z_{a_i}$ and $-z_{b_i}$, $-z_{a_i}$ and $-z_{c_i}$, $-z_{b_i}$ and $-z_{d_i}$, $-z_{c_i}$ and $-z_{d_i}$, interlace with respect to $[y(s)z(s),k]$. Therefore, there exists a transformation of $y(s)z(s) = p$ such that

$$A(p) = \prod_{i=1}^{N} (1+\frac{p}{\alpha_i}),$$

$$[z^{-1}B](p) = f_T \prod_{i=1}^{N} (1+\frac{p}{\beta_i}),$$

$$[y^{-1}C](p) = g_T \prod_{i=1}^{N-1} (1+\frac{p}{\gamma_i}),$$

$$D(p) = \prod_{i=1}^{N} (1+\frac{p}{\delta_i}),$$

where

$$0 < \alpha_i < \beta_i < \alpha_{i+1},$$

$$0 < \alpha_i < \gamma_i < \alpha_{i+1},$$

$$0 < \beta_i < \delta_i < \beta_{i+1},$$

$$0 < \gamma_i < \delta_i < \gamma_{i+1}.$$

Lemma 4.1: Let A(s), B(s), C(s), and D(s) be the chain parameters of any 2-port network. If

$$A(s) = K_a \frac{\prod_{i=1}^{Nn} (s+z_{a_i})}{[g(s)]^N},$$

$$A(s) \Big|_{[s|f(s) = 0]} = 1,$$

$$\frac{g_1(s)}{f_1(s)} B(s) = K_b \frac{\prod_{i=1}^{Nn} (s+z_{b_i})}{[g(s)]^N},$$

$$\frac{g_1(s)}{f_1(s)} B(s) \Big|_{[s|f(s) = 0]} = f_T,$$

where $f_1(s) \cdot f_2(s) = f(s)$, $g_1(s) \cdot g_2(s) = g(s)$, $-z_{a_i}$ and $-z_{b_i}$ interlace with respect to $[\frac{f(s)}{g(s)}, k]$, then

$$\frac{f_1(s)}{g_1(s)} C(s) = pA(p) \sum_{i=1}^{N} \frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)(p+\alpha_i)} \Bigg|_{p=\frac{f(s)}{g(s)}} \tag{4-9}$$

and

$$D(s) = \frac{[z^{-1}B](p)}{f_T} \left[1 - f_T \sum_{i=1}^{N} \frac{p}{\beta_i [z^{-1}B]'(-\beta_i) A(-\beta_i)(p+\beta_i)} \right] \Bigg|_{p=\frac{f(s)}{g(s)}} \tag{4-10}$$

where $-\alpha_i$ and $-\beta_i$ are the zeros of $A(p)$ and $[z^{-1}B](p)$ in p-plane.

Proof of Lemma 4.1: It follows from the hypothesis, there exists a transformation $p = y(s)z(s)$, such that

$$A(p) = \prod_{i=1}^{N} (1+\frac{p}{\alpha_i}),$$

$$[z^{-1}B](p) = f_T \prod_{i=1}^{N} (1+\frac{p}{\beta_i}),$$

where $0 < \alpha_i < \beta_i < \alpha_{i+1}$.

The open circuit input admittance is given by

$$\frac{1}{pz_{11}} = \frac{[zC](p)}{pA(p)}$$

$$= \frac{g_T p \prod_{i=1}^{N-1} (1+\frac{p}{\gamma_i})}{p \prod_{i=1}^{N} (1+\frac{p}{\alpha_i})}$$

$$= \sum_{i=1}^{N} \frac{\Lambda i}{p+\alpha_i}, \tag{4-11}$$

where

$$\Lambda_i = \left. \frac{[zC](p)}{pA'(p)} \right|_{p=-\alpha_i}$$

$$= \frac{[zC](-\alpha_i)}{-\alpha_i A'(-\alpha_i)}.$$

$$\therefore \frac{[zC](p)}{A(p)} = p \sum_{i=1}^{N} \frac{[zC](-\alpha_i)}{-\alpha_i A'(-\alpha_i)(p+\alpha_i)}.$$

Since $A(p)D(p)-[z^{-1}B](p)[zC](p) = 1$,

then

$$[zC](-\alpha_i) = \frac{-1}{[z^{-1}B](-\alpha_i)}.$$

Thus Equation (4-11) becomes

$$\frac{[zC](p)}{A(p)} = p \sum_{i=1}^{N} \frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)(p+\alpha_i)},$$

or

$$[zC](p) = pA(p) \sum_{i=1}^{N} \frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)(p+\alpha_i)}.$$

This completes the proof of Equation (4-9).

To prove Equation (4-10), consider the short-circuit input admittance

$$\frac{y_{11}}{p} = \frac{D(p)}{p|z^{-1}B|(p)}$$

$$= \frac{1}{f_T p} + \sum_{i=1}^{N} \frac{D(-\beta_i)}{-\beta_i [z^{-1}B]'(-\beta_i)(p+\beta_i)}$$

$$= \frac{1}{f_T p} - \sum_{i=1}^{N} \frac{1}{\beta_i A(-\beta_i)[z^{-1}B]'(-\beta_i)(p+\beta_i)}.$$

Rearranging the expression for $\frac{y_{11}}{p}$ yields:

$$\frac{D(p)}{[z^{-1}B](p)} = \frac{1}{f_T} - p\sum_{i=1}^{N} \frac{1}{\beta_i A(-\beta_i)[z^{-1}B]'(-\beta_i)(p+\beta_i)}.$$

It follows therefore that

$$D(p) = \frac{[z^{-1}B](p)}{f_T}\left[1 - f_T \sum_{i=1}^{N} \frac{p}{\beta_i [z^{-1}B]'(-\beta_i)A(-\beta_i)(p+\beta_i)}\right].$$

This completes the proof of Equation (4-10).

Q.E.D.

Lemma 4.2: Let A(s), B(s), C(s), and D(s) be the chain parameters of the network of Figure 4.2. If

$$A(s) = K_a \frac{\prod_{i=1}^{Nn} (s+z_{a_i})}{[d_a(s)d_b(s)]^N},$$

$$A(s)\Big|_{[s|n_a(s)n_b(s) = 0]} = 1,$$

$$z^{-1}(s)B(s) = K_b \frac{\prod_{i=1}^{Nn} (s+z_{b_i})}{[d_a(s)d_b(s)]^N},$$

$$z^{-1}(s)B(s)\Big|_{[s|n_a(s)n_b(s) = 0]} = f_T,$$

where $-z_{a_i}$ and $-z_{b_i}$ interlace with respect to

$$\left[z(s)y(s) = \frac{n_a(s)n_b(s)}{d_a(s)d_b(s)},\ k\right],$$

then

$$g_T = \sum_{i=1}^{N} \frac{1}{\alpha_i^2 A'(-\alpha_i)[z^{-1}B](-\alpha_i)}\Bigg|_{p=y(s)z(s)}, \qquad (4\text{-}12)$$

and

$$g_T = -\sum_{i=1}^{N} \frac{1}{\beta_i^2 [z^{-1}B]'(-\beta_i)A(-\beta_i)} + \frac{1}{f_T}\left(a_1 + \frac{b_1}{f_T}\right)\Bigg|_{p=y(s)z(s)} \qquad (4\text{-}13)$$

Proof of Lemma 4.2: Again by observation, as in the proof of Lemma 4.1, there exists a transformation $p = y(s)z(s)$, and $A(s)$ and $z^{-1}(s)B(s)$ become

$$A(p) = \prod_{i=1}^{N} (1+\frac{p}{\alpha_i}),$$

$$[z^{-1}B](p) = f_T \prod_{i=1}^{N} (1+\frac{p}{\beta_i}),$$

where $0 < \alpha_i < \beta_i < \alpha_{i+1}$.

The open-circuit input admittance is given by

$$\frac{1}{pz_{11}(p)} = \frac{[zC](p)}{pA(p)}$$

$$= \frac{g_T p \prod_{i=1}^{N-1} (1+\frac{p}{\gamma_i})}{p \prod_{i=1}^{N} (1+\frac{p}{\alpha_i})}$$

$$= \sum_{i=1}^{N} \frac{\Lambda i}{p+\alpha_i}, \tag{4-14}$$

where

$$\Lambda_i = \left.\frac{[zC](p)}{pA'(p)}\right|_{p=-\alpha_i}$$

$$= \frac{[zC](-\alpha_i)}{-\alpha_i A'(-\alpha_i)}.$$

$$\therefore \frac{1}{z_{11}(p)} = \frac{[zC](p)}{A(p)}$$

$$= p \sum_{i=1}^{N} \frac{[zC](-\alpha_i)}{-\alpha_i A'(-\alpha_i)(p+\alpha_i)}.$$

Since $A(p)D(p) - [z^{-1}B](p)[zC](p) = 1$,

then

$$[zC](-\alpha_i) = \frac{-1}{[z^{-1}B](-\alpha_i)}.$$

$$\therefore \frac{[zC](p)}{A(p)} = \sum_{i=1}^{N} \frac{p}{\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)(p+\alpha_i)}$$

From Equation (4-14) it follows that

$$g_T = \lim_{p\to 0} \frac{[zC](p)}{pA(p)}$$

$$= \sum_{i=1}^{N} \frac{1}{\alpha_i^2 A'(-\alpha_i)[z^{-1}B](-\alpha_i)} \qquad (4\text{-}15)$$

This completes the proof of Equation (4-12).

To prove Equation (4-13), consider the following fraction expansion:

$$\frac{1}{pA(p)[z^{-1}B](p)} = \frac{1}{p \prod_{i=1}^{N}(1+\frac{p}{\alpha_i}) f_T \prod_{i=1}^{N}(1+\frac{p}{\beta_i})}$$

$$= \frac{1}{f_T p} + \sum_{i=1}^{N} \frac{K_i}{p+\alpha_i} + \sum_{i=1}^{N} \frac{K'_i}{p+\beta_i}$$

where

$$K_i = \left. \frac{1}{pA'(p)[z^{-1}B](p)} \right|_{p=-\alpha_i}$$

$$= \frac{1}{-\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)}$$

$$K'_i = \left. \frac{1}{pA(p)[z^{-1}B]'(p)} \right|_{p=-\beta_i}$$

$$= \frac{1}{-\beta_i A(-\beta_i)[z^{-1}B]'(-\beta_i)}$$

$$\therefore \frac{1}{A(p)[z^{-1}B](p)} = \frac{1}{f_T} + p\sum_{i=1}^{N} \frac{1}{-\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)(p+\alpha_i)}$$

$$+ \sum_{i=1}^{N} \frac{p}{-\beta_i A(-\beta_i)[z^{-1}B]'(-\beta_i)(p+\beta_i)}$$

$$\frac{1}{p}\left[\frac{1}{A(p)[z^{-1}B](p)} - \frac{1}{f_T}\right] = -\sum_{i=1}^{N} \frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B](-\alpha_i)(p+\alpha_i)}$$

$$- \sum_{i=1}^{N} \frac{1}{\beta_i A(-\beta_i)[z^{-1}B]'(-\beta_i)(p+\beta_i)}$$

$$\lim_{p\to 0} \frac{1}{p}\left[\frac{1}{\Lambda(p)[z^{-1}B](p)} - \frac{1}{f_T}\right] = -\sum_{i=1}^{N} \frac{1}{\alpha_i^{2}\Lambda'(-\alpha_i)[z^{-1}B](-\alpha_i)}$$

$$-\sum_{i=1}^{N} \frac{1}{\beta_i^{2}\Lambda(-\beta_i)[z^{-1}B]'(-\beta_i)}$$

$$= -g_T - \sum_{i=1}^{N} \frac{1}{\beta_i^{2}\Lambda(-\beta_i)[z^{-1}B]'(-\beta_i)}$$

(4-16)

On the other hand:

$$\frac{1}{\Lambda(p)[z^{-1}B](p)} = \frac{1}{\pi(1+\frac{p}{\alpha_i})\, f_T\, \pi(1+\frac{p}{\beta_i})}$$

$$= \frac{1}{(1+a_1p+\cdots)(f_T+b_1p+\cdots)}$$

$$= \frac{1}{f_T+(a_1f_T+b_1)p+(a_1b_1+\cdots)p^2+\cdots}$$

$$= \frac{1}{f_T} - \frac{1}{f_T}(a_1+\frac{b_1}{f_T})p+\cdots$$

$$\frac{1}{\Lambda(p)[z^{-1}B](p)} - \frac{1}{f_T} = \frac{-1}{f_T}(a_1+\frac{b_1}{f_T})p+\cdots$$

$$\lim_{p\to 0} \frac{1}{p}\left[\frac{1}{\Lambda(p)[z^{-1}B](p)} - \frac{1}{f_T}\right] = -\frac{1}{f_T}(a_1+\frac{b_1}{f_T}) \qquad (4\text{-}17)$$

Equating the right hand sides of (4-16) and (4-17) results in

$$g_T = -\sum_{i=1}^{N} \frac{1}{\beta_i^2 A(-\beta_i)[z^{-1}B]'(-\beta_i)} + \frac{1}{f_T}(a_1 + \frac{b_1}{f_T}). \qquad (4\text{-}18)$$

This completes the proof of Equation (4-13).

Q.E.D.

Lemma 4.3: Let the chain matrix parameter A(s) of the network of Figure 4.1 be

$$A(s) = \frac{K_a \prod_{i=1}^{Nn} (s+z_{a_i})}{[d_a(s)d_b(s)]^N},$$

and

$$A(s)\Big|_{\{s|n_a(s)n_b(s)=0\}} = 1$$

where $-z_{a_i} \in [y(s)z(s),k]$. Then the minimum value for the product $f_T g_T$ is given by

$$(f_T g_T)_{min} = [\sum_{i=1}^{N} \frac{1}{\alpha_i^{3/2}|A'(-\alpha_i)|}]^2, \qquad (4\text{-}19)$$

where α_i are the zeros of $A(s)\Big|_{y(s)z(s)=p}$.

The minimum of $f_T g_T$ is achieved for the network of Figure 4.1 whose short-circuit output impedance is

$$z_{2,s}(p) = \frac{1}{y_{22}} = \frac{f_T}{\sqrt{(f_T g_T)_{min}}} \sum_{i=1}^{N} \frac{1}{\sqrt{\alpha_i}|A'(-\alpha_i)|(p+\alpha_i)}. \qquad (4\text{-}20)$$

Proof of Lemma 4.3: Let $y(s)z(s) = p$, then

$$A(p) = \sum_{i=1}^{N} [1+\frac{p}{\alpha_i}].$$

The network is completely determined by specifying

$$[z^{-1}B](p) = f_T \sum_{i=1}^{N} [1+\frac{p}{\beta_i}],$$

where

$$\alpha_i < \beta_i < \alpha_{i+1}. \tag{4-21}$$

According to Equation (4-12), it is possible to write

$$f_T g_T = \sum_{i=1}^{N} \frac{1}{\alpha_i^2 A'(-\alpha_i)[z^{-1}B_1](-\alpha_i)}$$

$$= f(\beta_1, \beta_2, \cdots, \beta_n), \tag{4-22}$$

where

$$[z^{-1}B_1](p) = \frac{1}{f_T}[z^{-1}B](p)$$

$$= \sum_{i=1}^{N} [1+\frac{p}{\beta_i}]. \tag{4-23}$$

Consider $\beta_1 < \beta_2 < \cdots < \beta_n$ as adjustable parameters of the network, varying in the intervals established in Equation (4-21).

The problem is to find for what values of β_1, β_2, $\cdots$, β_n in the open intervals such that the product $f_T g_T$ has a minimum and also to determine this minimum value.

First, observe that for inhomogeneous ladders ending in a shunt arm admittance, $\beta_n = \infty$, i.e., let

$$f_T g_T = f(\beta_1,\ \beta_2,\ \cdots,\ \beta_{n-1}). \tag{4-24}$$

At the point where the minimum occurs,

$$\frac{\partial}{\partial \beta_K}(f_T g_T) = \frac{\partial f}{\partial \beta_K} = 0 \tag{4-25}$$

Since

$$f_T g_T = \sum_{i=1}^{N} \frac{1}{\alpha_i^{\,2} A'(-\alpha_i)[z^{-1}B_1](-\alpha_i)} \tag{4-26}$$

$$= f(\beta_1,\ \beta_2,\ \cdots,\ \beta_{n-1}),$$

where

$$[z^{-1}B_1](p) = \prod_{i=1}^{N-1} [1+\frac{p}{\beta_i}], \tag{4-27}$$

expanding Equation (4-26) gives

$$f_T g_T = \sum_{i=1}^{N} \frac{1}{\alpha_i^{\,2}\Lambda'(-\alpha_i)\left[\dfrac{[z^{-1}B_1](-\alpha_i)}{(1-\frac{\alpha_i}{\beta_K})}\right]} \cdot \frac{1}{(1-\frac{\alpha_i}{\beta_K})},$$

where the bracketed term is independent of β_K. Thus

$$\frac{\partial}{\partial\beta_K}(f_T g_T) = \sum_{i=1}^{N} \frac{1}{\alpha_i^{\,2}\Lambda'(-\alpha_i)\left[\dfrac{[z^{-1}B_1](-\alpha_i)}{(1-\frac{\alpha_i}{\beta_K})}\right]} \cdot \frac{\frac{\alpha_i}{\beta_K^{\,2}}}{(1-\frac{\alpha_i}{\beta_K})^2}.$$

Simplifying,

$$\frac{\partial}{\partial\beta_K}(f_T g_T) = \frac{1}{\beta_K}\sum_{i=1}^{N} \frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B_1](-\alpha_i)(-\beta_K+\alpha_i)} = 0 \tag{4-28}$$

Therefore, the numbers $-\beta_K(k = 1, 2, \cdots, n-1)$ which minimize $f_T g_T = f(\beta_1, \beta_2, \cdots, \beta_{n-1})$ satisfy the zeros of the following rational function:

$$L(p) = \sum_{i=1}^{N} \frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B_1](-\alpha_i)(p+\alpha_i)}. \tag{4-29}$$

The poles of the rational function L(P) are obviously $-\alpha_1$, $-\alpha_2, \cdots, -\alpha_n$. Expansion of

$$\frac{[z^{-1}B_1](p)}{A(p)} = \sum_{i=1}^{N} \frac{[z^{-1}B_1](-\alpha_i)}{A'(-\alpha_i)(p+\alpha_i)} \tag{4-30}$$

can be used to estimate L(p) within a constant multiplier. Therefore,

$$\frac{1}{\alpha_i A'(-\alpha_i)[z^{-1}B_1](-\alpha_i)(p+\alpha_i)} = \lambda \frac{[z^{-1}B_1](-\alpha_i)}{A'(-\alpha_i)(p+\alpha_i)}.$$

$$\therefore \left|[z^{-1}B_1](-\alpha_i)\right| = \frac{1}{\sqrt{\lambda\alpha_i}}. \tag{4-31}$$

Since $A'(-\alpha_i)[z^{-1}B_1](-\alpha_i) > 0$ for $i = 1, 2, \cdots, n$, it follows from Equation (4-30) that

$$\frac{[z^{-1}B_1](p)}{A(p)} = \sum_{i=1}^{N} \frac{[z^{-1}B_1](-\alpha_i)}{A'(-\alpha_i)(p+\alpha_i)}$$

$$= \sum_{i=1}^{N} \frac{1}{\sqrt{\lambda\alpha_i}\, A'(-\alpha_i)(p+\alpha_i)}$$

$$= \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{N} \frac{1}{\sqrt{\alpha_i}\,|A'(-\alpha_i)|(p+\alpha_i)}. \tag{4-32}$$

Solving for λ at p=0 results in

$$\frac{[z^{-1}B_1](0)}{A(0)} = \frac{1}{\sqrt{\lambda}} \sum_{i=1}^{N} \frac{1}{\alpha_i^{3/2}|A'(-\alpha_i)|}.$$

Since $[z^{-1}B_1](0) = 1$, $A(0) = 1$, then

$$\sqrt{\lambda} = \sum_{i=1}^{N} \frac{1}{\alpha_i^{3/2}|A'(-\alpha_i)|}. \qquad (4\text{-}33)$$

The short-circuit output impedance can now be written as

$$y_{22}^{-1}(p) = \frac{[z^{-1}B](p)}{A(p)}$$

$$= f_T \frac{[z^{-1}B_1](p)}{A(p)}$$

$$= \frac{f_T}{\sqrt{\lambda}} \sum_{i=1}^{N} \frac{1}{\sqrt{\alpha_i}\ A'(-\alpha_i)\ (p+\alpha_i)}. \qquad (4\text{-}34)$$

Substituting Equation (4-31) into Equation (4-22) yields

$$(f_T g_T)_{min} = \sum_{i=1}^{N} \frac{1}{\alpha_i^{2} A'(-\alpha_i) \frac{1}{\sqrt{\lambda\alpha_i}}}$$

$$= \sum_{i=1}^{N} \frac{\sqrt{\lambda}}{\alpha_i^{3/2}|A'(-\alpha_i)|}$$

$$= \left[\sum_{i=1}^{N} \frac{1}{\alpha_i^{3/2}|A'(-\alpha_i)|}\right]^2 = \lambda$$

This completes the proof of Lemma 4.3.

Q.E.D.

Lemma 4.4: Let the chain matrix parameter B(s) of the network of Figure 4.2 be

$$z^{-1}(s)B(s) = \frac{K_b \prod_{i=1}^{Nn} (s+z_{b_i})}{[d_a(s)d_b(s)]^N},$$

and

$$z^{-1}(s)B(s)\Big|_{[s|n_a(s)n_b(s) = 0]} = f_T$$

where $-z_{b_i} \in [y(s)z(s), k]$. Then the quantity

$$g_T = 4 \sum_{j=1}^{(n+1)/2} \frac{1}{\beta^2_{2j-1}\left|[z^{-1}B]'(-\beta_{2j-1})\right|}, \qquad (4\text{-}35)$$

where β_i are the zeros of $z^{-1}(s)B(s)\Big|_{y(s)z(s) = p}$,

gives the smallest possible value for the total admittance g_T.

Proof of Lemma 4.4: Let $y(s)z(s) = p$, then

$$\frac{1}{[z^{-1}B](p)} = \frac{1}{f_T \prod_{i=1}^{N} (1+\frac{p}{\beta_i})}.$$

The parameter $D(p) = \prod_{i=1}^{N} (1+\frac{p}{\delta_i})$ is introduced with

$\delta_i (i = 1, 2, \cdots, n)$ arbitrarily chosen in the intervals

$$\beta_{i-1} < \delta_i < \beta_i \quad i = 1, 2, \cdots, n \tag{4-36}$$

Consider $\delta_1, \delta_2, \cdots, \delta_n$ as parameters of the system. Use Equation (4-13) which is repeated here as Equation (4-37),

$$g_T = -\sum_{i=1}^{N} \frac{1}{\beta_i^2 [z^{-1}B]'(-\beta_i)D(-\beta_i)} + \frac{1}{f_T}(a_1 + \frac{b_1}{f_T}). \tag{4-37}$$

This identity shows that the total admittance g_T can be expressed as a rational function of $\delta_1, \delta_2, \cdots, \delta_n$:

$$g_T = F(\delta_1, \delta_2, \cdots, \delta_n). \tag{4-38}$$

This function has at least one minimum inside the intervals defined in (4-36). At the point where the minimum occurs,

$$\frac{\partial F}{\partial \delta_K} = -\frac{1}{\delta_K} \sum_{i=1}^{N} \frac{1}{\beta_i [z^{-1}B]'(-\beta_i)D(-\beta_i)(-\delta_K+\beta_i)} - \frac{1}{f_T \delta_K^2} \tag{4-39}$$

for k = 1, 2, $\cdots$, n. The derivation of (4-39) is similar to that of Equation (4-28). From Equation (4-39), it follows that the numbers $-\delta_K$ (k = 1, 2,$\cdots$, n), which implies minimizing g_T, satisfy the zeros of

$$L(p) = \sum_{i=1}^{N} \frac{1}{\beta_i [z^{-1}B]'(-\beta_i) D(-\beta_i)(p+\beta_i)} + \frac{1}{f_T p}. \quad (4\text{-}40)$$

By observation it follows that the poles of L(p) are the numbers $-\beta_1, -\beta_2, \cdots, -\beta_n$. Expansion of

$$\frac{D(p)}{p[z^{-1}B](p)}$$

yields

$$\frac{D(p)}{p[z^{-1}B](p)} = \sum_{i=1}^{n} \frac{D(-\beta_i)}{\beta_i [z^{-1}B]'(-\beta_i)(p+\beta_i)} + \frac{1}{f_T p}. \quad (4\text{-}41)$$

The foregoing expression should be able to estimate L(p) within a constant multiplier which implies

$$D^2(-\beta_i) = 1, \quad i = 1, 2, \cdots, n. \quad (4\text{-}42)$$

Therefore, it follows from Equation (4-41) that for g_T to be a minimum,

$$\frac{D(p)}{[z^{-1}B](p)} = \frac{1}{f_T} + p \sum_{i=1}^{n} \frac{1}{\beta_i \left|[z^{-1}B]'(-\beta_i)\right| (p+\beta_i)}. \quad (4\text{-}43)$$

If A(p) is used instead of D(p), an identical expression is obtained for $\frac{A(p)}{[z^{-1}B](p)}$. Hence, for g_T to be a minimum, A(p) = D(p), so the ladder is symmetrical as shown in Figure 4.3. The parameters $A^*(p)$ and $[z^{-1}B]^*(p)$ of the first half of the ladder network can be obtained as follows.

$$\begin{bmatrix} A(p) & [z^{-1}B](p) \\ [zC](p) & D(p) \end{bmatrix} =$$

$$\begin{bmatrix} A^*(p) & [z^{-1}B]^*(p) \\ [zC]^*(p) & D^*(p) \end{bmatrix} \begin{bmatrix} D^*(p) & [z^{-1}B]^*(p) \\ [zC]^*(p) & A^*(p) \end{bmatrix},$$

$$[z^{-1}B](p) = 2A^*(p)[z^{-1}B]^*(p)$$

$$= f_T \prod_{i=1}^{n} (1+\frac{p}{\beta_i}). \qquad (4\text{-}44)$$

Since $A^*(p)\Big|_{p=0} = 1$, zeros of $A^*(p)$ and $[z^{-1}B]^*(p)$ interlace in the intervals defined in (4-36). Therefore,

$$A^*(p) = \prod_{j=1}^{(n+1)/2} (1+\frac{p}{\beta_{2j-1}}), \qquad (4\text{-}45)$$

and

$$[z^{-1}B]^*(p) = \frac{f_T}{2} \prod_{j=1}^{n/2} (1+\frac{p}{\beta_{2j}}). \qquad (4\text{-}46)$$

Then applying the identity

$$g_T = \prod_{i=1}^{N} \frac{1}{\alpha_i^2 A'(-\alpha_i)[z^{-1}B](-\alpha_i)} \qquad (4\text{-}47)$$

for the first half of the ladder, yields the following expression:

$$\text{minimum } g_T = 4 \sum_{j=1}^{(n+1)/2} \frac{1}{\beta^2_{2j-1} \left| [z^{-1}B]'(-\beta_{2j-1}) \right|} . \quad (4\text{-}48)$$

This completes the proof of Lemma 4.4.

Q.E.D.

For the inhomogeneous ladder network of Figure 4.1, the open-circuit voltage transfer function is given by

$$\frac{V_2(s)}{V_1(s)} = \frac{1}{A(s)} \quad (4\text{-}49)$$

where

$$A(s) = \frac{K_a \prod_{i=1}^{Nn} (s+a_{a_i})}{[d_a(s)d_b(s)]^N},$$

and

$$A(s) \Big|_{[s|n_a(s)n_b(s) = 0]} = 1,$$

and $-z_{a_i} \in [y(s)z(s), k]$. If $y(s)z(s) = p$, Equation (4-49) becomes

$$A^{-1}(p) = \frac{H}{\prod_{i=1}^{N} (p+\alpha_i)}, \quad (4\text{-}50)$$

where

$$H = \prod_{i=1}^{N} \alpha_i .$$

To synthesize the ladder network one makes use of the following relationship between z parameters and chain parameters:

$$\frac{z_{12}(p)}{z_{11}(p)} = \frac{1}{[zC](p)} \Bigg/ \frac{A(p)}{[zC](p)} = \frac{1}{A(p)}.$$

It is possible to construct $z_{11}(p)$ as follows.

$$z_{11}(p) = \frac{A(p)}{[zC](p)} = \frac{ka(p)}{[zC](p)} \tag{4-51}$$

where $a(p) = \prod_{i=1}^{N} (p+\alpha_i)$,

and

$$[zC](p) = p \prod_{i=1}^{N-1} (p+\gamma_i),$$

and $0 \le \gamma_1 < \alpha_1 < \gamma_2 < \cdots < \gamma_n < \alpha_n$, (4-52)

where γ_i are to be determined. Once they are found then the continued fraction expansion of $z_{11}(p)$ about $p = \infty$ yields the inhomogeneous ladder network shown in Figure 4.1. For this purpose let the objective function be defined as

$$F = f_T + w_F g_T. \tag{4-53}$$

Definition 4.5: Synthesis of an inhomogeneous ladder network from a given transfer function is said to be optimal if the realization of Equation (4-51) is obtained with minimum value of F.

Lemma 4.5: If an inhomogeneous ladder network is optimal, then (a) $f_T = w_F g_T$ and (b) the solution for $\gamma_1, \gamma_2, \cdots, \gamma_n$

is independent of w_F.

Proof: The variable k in Equation (4-51) is merely an impedance scaling factor. Therefore, the objective function $F = f_T + w_F g_T$ can be written as

$$F = kf_{T1} + \frac{1}{k} w_F g_{T1}, \tag{4-54}$$

where $kf_{T1} = f_T$ and $\frac{1}{k} g_{T1} = g_T$. For the network to be optimal, the derivative of F must vanish. Thus

$$\frac{\partial F}{\partial k} = f_{T1} - k^{-2} w_F g_{T1} = 0.$$

Solving for k yields

$$k = \sqrt{\frac{w_F g_{T1}}{f_{T1}}}. \tag{4-55}$$

Substituting the optimal value for k gives

$$f_T = kf_{T1} = \sqrt{\frac{w_F g_{T1}}{f_{T1}}}\, f_{T1} = \sqrt{w_F f_{T1} g_{T1}}, \text{ and}$$

$$w_F g_T = w_F \frac{1}{k} g_{T1} = w_F \sqrt{\frac{f_{T1}}{w_F g_{T1}}}\, g_{T1} = \sqrt{w_F f_{T1} g_{T1}}.$$

Therefore $f_T = w_F g_T = \sqrt{w_F f_{T1} g_{T1}}$ which proves (a).

To prove (b), note that for the network to be optimal the derivative of F with respect to γ_i must vanish. Hence,

$$\frac{\partial F}{\partial \gamma_i} = k \frac{\partial f_{T1}}{\partial \gamma_i} + \frac{1}{k} w_F \frac{g_{T1}}{\partial \gamma_i} = 0, \quad i = 1, 2, \cdots, n. \quad (4\text{-}56)$$

Substituting Equation (4-55) into Equation (4-56) yields

$$\frac{\partial}{\partial \gamma_i}(f_T g_T) = 0, \quad i = 1, 2, \cdots, n. \quad (4\text{-}57)$$

The solution of Equation (4-57) for $\gamma_1, \gamma_2, \cdots, \gamma_n$ is independent of both k and w_F. This completes the proof.

Q.E.D.

Lemma 4.5 makes it possible to let $w_F = 1$ without any loss of generality. Consequently, it is only necessary to find the network which minimizes the function, $F = f_T + g_T$.

Lemma 4.6: For the inhomogeneous ladder network of Figure 4.2, the minimum value of the function $F = f_T + w_F g_T$ is obtained if and only if

(a) $$f_i = f_{n+2-i} \begin{cases} i=1,2,\cdots,\ \frac{1}{2}n; & n \text{ even} \\ i=1,2,\cdots,\ \frac{1}{2}(n+1); & n \text{ odd} \end{cases}$$

$$g_i = g_{n+1-i} \begin{cases} i=1,2,\cdots\ \frac{1}{2}n; & n \text{ even} \\ i=1,2,\cdots\ \frac{1}{2}(n-1); & n \text{ odd} \end{cases}$$

(b) $f_T = w_F g_T$.

Proof of Necessity:

(a) The open-circuit admittance parameter $\frac{1}{z_{11}(p)}$ of the network is given by Equation (4-51):

$$y^*_{11}(p) = \frac{1}{z_{11}(p)} = \frac{[zC](p)}{ka(p)}$$

where $a(p) = \prod_{i=1}^{N} (p+\alpha_i)$. Thus an equivalent optimization problem is to determine k and [zC](p) so that the realization of $y^*_{11}(p)$ yields the optimum network.

Since $y^*_{22}(p)$ must be similar in form to $y^*_{11}(p)$, and in particular, must have the same poles, let

$$y^*_{22}(p) = \frac{[zC]^*(p)}{k^*a(p)}$$

where k* and [zC]*(p) are to be determined so that the realization of $y^*_{22}(p)$ yields the optimum network. The two optimization problems in terms of $y^*_{11}(p)$ and $y^*_{22}(p)$ are identical and must lead to identical solutions. Therefore,

$$y^*_{11}(p) = y^*_{22}(p) \tag{4-58}$$

is a necessary condition for the optimum network. The element values are obtained by the continued fraction expansion of $y^*_{11}(p)$ [or $y^*_{22}(p)$ or $z_{11}(p)$]. Since the continued fraction expansion is a unique process, Equation

(4-58) implies physical symmetry, that is, condition (a). The necessity of condition (b) has already been established by Lemma 4.5.

Proof of Sufficiency: To prove sufficiency, it must be shown that the network which satisfies both conditions (a) and (b) and has the prescribed transfer function is unique. This is established by using the fact that Figure 4.2 is a compact network in p-domain. Thus,

$$\frac{y_{11}}{p} = \sum_{i=1}^{N} \frac{k_{11}^{(i)}}{p+\alpha_i} \quad , \quad \frac{y_{22}}{p} = \sum_{i=1}^{N} \frac{k_{22}^{(i)}}{p+\alpha_i}$$

where $k_{11}^{(i)} k_{22}^{(i)} = [k_{12}^{(i)}]^2$, $i = 0, 1, 2, \cdots, n$ and $k_{11} = k_{22} > 0$. This implies that $k_{11} = k_{22} = |k_{12}|$.

$\frac{y_{11}(p)}{p}$ can be obtained (to within a constant multiplier) by expanding

$$\frac{1}{pk[z^{-1}B](p)}$$

in partial fractions and changing the sign of any negative residues. The continued fraction of the function so obtained and impedance scaling to adjust $f_T = w_F g_T$ yields the element values. The synthesis process is unique and can lead to only one network for a given A(p). This completes the proof.

Q.E.D.

From the method described in the proof, it follows that

$$\frac{1}{pz_{11}(p)} = \frac{1}{p[z^{-1}B](p)}$$

$$= \frac{1}{p\prod_{i=1}^{N}(1+\frac{p}{\beta_i})}$$

$$= \frac{k_0}{p} + \sum_{i=1}^{N}\frac{k_i}{p+\beta_i}$$

where the residues

$$k_0 = p[\frac{1}{p[z^{-1}B](p)}]\bigg|_{p=0} = \frac{1}{[z^{-1}B](0)}$$

$$k_i = \frac{1}{p[z^{-1}B]'(p)}\bigg|_{p=-\beta_i} = \frac{1}{-\beta_i[z^{-1}B]'(-\beta_i)}$$

$$\therefore \frac{1}{pz_{11}(p)} = \frac{1}{p[z^{-1}B](0)} + \sum_{i=1}^{N}\frac{1}{-\beta_i[z^{-1}B]'(-\beta_i)(p+\beta_i)}$$

$$\frac{1}{z_{11}(p)} = \frac{1}{[z^{-1}B](0)} + \sum_{i=1}^{N}\frac{p}{-\beta_i[z^{-1}B]'(-\beta_i)(p+\beta_i)}$$

$$= \frac{1}{[z^{-1}B](0)}\left[1 + \sum_{i=1}^{N}\frac{[z^{-1}B](0)p}{\beta_i|[z^{-1}B]'(-\beta_i)|(p+\beta_i)}\right]. \quad (4\text{-}59)$$

Lemma 4.7: If the inhomogeneous ladder network of Figure 4.1 is optimal, then

$$f_i = w_F g_{n+1-i};\; i = 1, 2, \cdots, n. \quad (4\text{-}60)$$

Proof: Without loss of generality let $w_F = 1$. Then Equation (4-60) becomes

$$f_i = g_{n+1-i};\ i = 1, 2, \cdots, n. \tag{4-61}$$

Equation (4-61) exhibits the characteristic of an antimetrical network which can be defined by the following equation.

$$z_{11}(p) = \frac{1}{p} y_{22}(p). \tag{4-62}$$

For the inhomogeneous ladder network of Figure 4.1, it is well known that

$$z_{11}(p) = \frac{ka(p)}{p[zC_1](p)}, \tag{4-63}$$

where $[zC_1](p) = \sum_{i=1}^{N-1} (p+\gamma_i)$. The continued fraction expansion of $z_{11}(p)$ yields the network. In order to derive Equation (4-62), an alternate approach using y parameters is employed. One can write

$$y_{22}(p) = \frac{A(p)}{[z^{-1}B](p)}$$

$$= \frac{ka(p)}{[z^{-1}B](p)}$$

$$= \frac{ha(p)}{[z^{-1}B]^*(p)} \tag{4-64}$$

where

$$[z^{-1}B](p) = \prod_{i=1}^{N-1} (1+\frac{p}{\beta_i}),$$

and

$$[z^{-1}B]^*(p) = \prod_{i=1}^{N-1} (p+\beta_i).$$

The equivalent optimization problem is to determine h and $[z^{-1}B]^*(p)$ so that the realization of $y_{22}(p)$ yields the optimal network. Note that the form of $y_{22}(p)$ in Equation (4-64) differs from that of $z_{11}(p)$ in Equation (4-63) only in that $y_{22}(p)$ has no pole at the origin. Let

$$y^*_{22}(p) = \frac{y_{22}(p)}{p} = \frac{ha(p)}{p[z^{-1}B]^*(p)}. \qquad (4\text{-}65)$$

The continued fraction expansion of $y^*_{22}(p)$ about $p = \infty$ yields the ladder network shown in Figure 4.4. Let $F^* = f_T^* + g_T^*$, where f_T^* and g_T^* are the sum of impedances and admittances, respectively, in Figure 4.4. Since Figure 4.4 is the dual of Figure 4.1, minimization of F^* in Figure 4.4 is equivalent to minimizing F in Figure 4.1. Finally, the problem of minimizing F in terms of $z_{11}(p)$ given by Equation (4-63) is identical to that of minimizing F^* in terms of $y^*_{22}(p)$ given by Equation (4-65). Therefore, identical solutions will be obtained, that is, h=k, and $[z^{-1}B]^*(p) = [zC_{\cdot}](p)$. Thus Equation (4-62) is established for the optimal network in the case of $w_F = 1$. This completes the proof.

Q.E.D.

Example 4.1: This example illustrates the optimal synthesis procedure of the inhomogeneous ladder network that is shown in Figure 4.1. Let the chain parameter, A(s), of the ladder be given as

$$A(s) = \frac{s^6+23s^5+236s^4+1338s^3+4477s^2+8287s+7238}{36(s^3+15s^2+75s+125)}.$$

It is desired to realize a ladder with minimum total impedance-total admittance product.

Solution: The chain parameter A(s) can be factored to display its poles and zeros,

$$A(s) = \frac{(s+2\pm j1.732)(s+3.5\pm j3.122)(s+6\pm j3.317)}{36(s+5)^3}.$$

Some algebraic manipulations yield

$$A(s) = \left[1+\frac{(s+1)(s+2)}{s+5}\right]\left[1+\frac{(s+1)(s+2)}{4(s+5)}\right]\left[1+\frac{(s+1)(s+2)}{9(s+5)}\right]$$

As can be seen from Figure 4.5, the zeros of A(s) interlace with respect to

$$\left[\frac{(s+1)(s+2)}{s+5},\ k\right]$$

for k = 1,4,9.

The next step in the synthesis process is to pick arbitrarily

$$z(s) = \frac{s+1}{s+5} \quad \text{and} \quad y(s) = s+2.$$

Hence,

$$A(s) = [1+y(s)z(s)][1+\tfrac{1}{4}y(s)z(s)][1+\tfrac{1}{9}y(s)z(s)].$$

By use of the mapping $p = y(s)z(s)$, $A(s)$ is transformed into p-plane as

$$A(p) = (1+p)(1+\tfrac{p}{4})(1+\tfrac{p}{9}).$$

The minimum value of the total impedance-total admittance product is obtained from Equation (4-19):

$$f_Tg_T\Big|_{min} = \left[\sum_{i=1}^{3} \frac{1}{\alpha_i^{3/2}|A'(-\alpha_i)|}\right]^2,$$

where

$$A'(-\alpha_1) = A'(-1) = \frac{2}{3},$$

$$A'(-\alpha_2) = A'(-4) = -\frac{5}{12},$$

$$A'(-\alpha_3) = A'(-9) = \frac{10}{9},$$

and

$$f_Tg_T\Big|_{min} = \left[\frac{1}{\alpha_1^{3/2}|A'(-\alpha_1)|} + \frac{1}{\alpha_2^{3/2}|A'(-\alpha_2)|} + \frac{1}{\alpha_3^{3/2}|A'(-\alpha_3)|}\right]^2$$

$$= (\frac{11}{6})^2.$$

Therefore, for optimal ladder, $f_T = g_T = \frac{11}{6}$.

This minimum value is achieved by the ladder with the short-circuit output impedance given by Equation (4-20).

$$\frac{1}{y_{22}(p)} = \frac{f_T}{\sqrt{(f_T g_T)}_{min}} \sum_{i=1}^{3} \frac{1}{\sqrt{\alpha_i}\,|\Lambda'(-\alpha_i)|\,(p+\alpha_i)}$$

$$= \frac{1}{\frac{2}{3}(p+1)} + \frac{1}{2(\frac{5}{12})(p+4)} + \frac{1}{3(\frac{10}{9})(p+9)}$$

$$= \frac{\frac{3}{2}(p^2+13p+36) + \frac{6}{5}(p^2+10p+9) + \frac{3}{10}(p^2+5p+4)}{(p+1)(p+4)(p+9)}$$

$$= \frac{3p^2+33p+66}{p^3+14p^2+49p+36}.$$

Thus,

$$y_{22}(p) = \frac{p^3+14p^2+49p+36}{3p^2+33p+66}.$$

The optimum values of impedance level and admittance level are shown in Table 4.1. The corresponding ladder network is shown in Figure 4.6.

Example 4.2: This example demonstrates the synthesis procedure of the optimal inhomogeneous ladder network of Figure 4.2 with minimum total admittance level, g_T. Let the chain parameter, B(s), of the network be given as

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Table 4.1 Optimal Network of Example 4.1

	1	14	49	36
$g_3 = \frac{1}{3}$				
	3	33	66	
$f_3 = 1$				
	3	27	36	
$g_2 = \frac{1}{2}$				
	6	30		
$f_2 = \frac{1}{2}$				
	12	36		
$g_1 = 1$				
	12			
$f_1 = \frac{1}{3}$				
	36			

$$B(s) = \frac{4(s^7+19s^6+158s^5+738s^4+2074s^3+3470s^2+3149s+1155)}{3(s^3+12s^2+48s+64)}$$

Solution: The chain parameter B(s) can be factored to display its poles and zeros,

$$B(s) = \frac{8(s+1)(s+2.5\pm j0.866)(s+3\pm j1.414)(s+3.5\pm j1.658)}{6(s+4)^3}$$

Some algebraic manipulations yield

$$B(s) = 8(s+1)\left[1+\frac{(s+1)(s+3)}{s+4}\right]\left[1+\frac{(s+1)(s+3)}{2(s+4)}\right]\left[1+\frac{(s+1)(s+3)}{3(s+4)}\right].$$

Arbitrarily picking $z(s) = s+1$ and $y(s) = \frac{s+3}{s+4}$, results in

$$z^{-1}(s)B(s) = 8[1+y(s)z(s)][1+\tfrac{1}{2}y(s)z(s)][1+\tfrac{1}{3}y(s)z(s)].$$

As can be seen from Figure 4.7, the zeros of $z^{-1}(s)B(s)$ interlace with respect to

$$\left[\frac{(s+1)(s+3)}{s+4}, k\right]$$

for $k = 1,2,3$. The mapping $p = y(s)z(s)$ yields

$$[z^{-1}B](p) = 8(1+p)(1+\tfrac{p}{2})(1+\tfrac{p}{3}).$$

The minimum total admittance level, g_T, is obtained from Equation (4-35):

$$g_T\Big|_{min} = 4\sum_{j=1}^{2} \frac{1}{\beta^2_{2j-1}\left|[z^{-1}B]'(-\beta_{2j-1})\right|}$$

$$= 4\left[\frac{1}{\beta_1{}^2\ [z^{-1}B]\ (-\beta_1)} + \frac{1}{\beta_3{}^2\left|[z^{-1}B]'(-\beta_3)\right|}\right]$$

where $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 3$, $[z^{-1}B]'(-\beta_1) = \frac{8}{3}$, $[z^{-1}B]'(-\beta_3) = -\frac{8}{3}$, and $g_T = 4[\frac{1}{8/3} + \frac{1}{9(8/3)}] = \frac{5}{3}$.

The ladder with the specified $[z^{-1}B](p)$ that achieves this minimum total admittance level is a symmetrical ladder as shown in Figure 4.3. Let $A^*(p)$ and $[z^{-1}B]^*(p)$ be the corresponding parameters of the first half of the symmetrical ladder. Then it follows from Equation (4-44) that

$$[z^{-1}B](p) = 2A^*(p)[z^{-1}B]^*(p)$$

$$= 8(1+p)(1+\tfrac{p}{2})(1+\tfrac{p}{3}).$$

The short-circuit output impedance $\frac{1}{y^*(p)}$ of the first half of the two-port network in Figure 4.3 is

$$\frac{1}{y^*(p)} = Z^*(p) = \frac{[z^{-1}B]^*(p)}{A^*(p)} = \frac{4(1+\frac{p}{2})}{(1+p)(1+\frac{p}{3})}.$$

Thus

$$y^*(p) = \frac{p^2+4p+3}{6p+12}.$$

The values of impedance level and admittance level are listed in Table 4.2. The corresponding ladder network is shown in Figure 4.8.

Table 4.2 Half of Symmetrical Ladder of Example 4.2

	1	4	3
$\frac{1}{2}g_2 = \frac{1}{6}$			
	6	12	
$f_2 = 3$			
	2	3	
$g_1 = \frac{2}{3}$			
	3		
$f_1 = 1$			
	3		

Table 4.3 Element Values of Example 4.1

R_{1a_1}	0.333	R_{2a_1}	0.500	R_{3a_1}	1.000
R_{1a_2}	0.067	R_{2a_2}	0.100	R_{3a_2}	0.200
L_{1a}	0.067	L_{2a}	0.100	L_{3a}	0.200
C_{1a}	3.000	C_{2a}	2.000	C_{3a}	1.000
R_{1b}	0.500	R_{2b}	1.000	R_{3b}	1.500
C_{1b}	1.000	C_{2b}	0.500	C_{3b}	0.333

$$g_T\Big|_{min} = f_T\Big|_{min} = \frac{11}{6}$$

Table 4.4 Element Values of Example 4.2

R_{1a}	1	R_{2a}	3	R_{3a}	3	R_{4a}	1
C_{1a}	1	C_{2a}	3	C_{3a}	0.333	C_{4a}	1
R_{1b_1}	1.5	R_{2b_1}	3	R_{3b_1}	1.5		
R_{1b_2}	2	R_{2b_2}	4	R_{3b_2}	2		
L_{1b}	0.5	L_{2b}	1	L_{3b}	0.5		
C_{1b}	0.167	C_{2b}	0.083	C_{3b}	0.167		

$$g_T\Big|_{min} = \frac{5}{3}$$

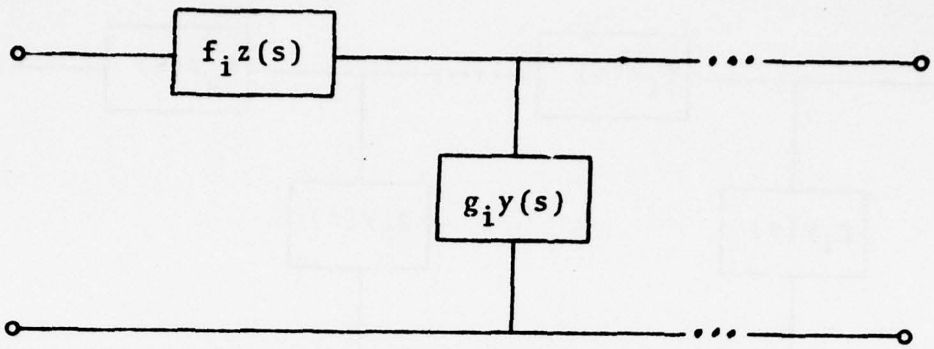

Figure 4.1 Inhomogeneous Ladder Ending in a Shunt Admittance

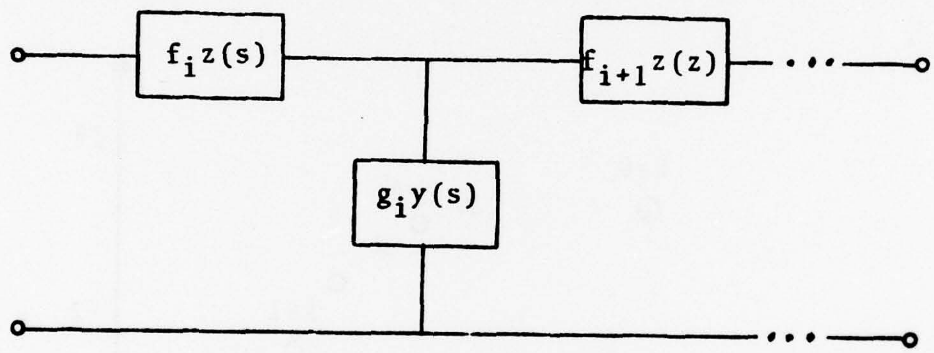

Figure 4.2 Inhomogeneous Ladder Ending in a Series Impedance

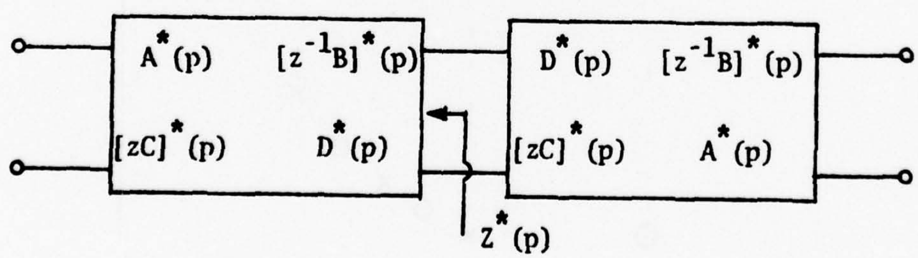

Figure 4.3 A Symmetrical Network

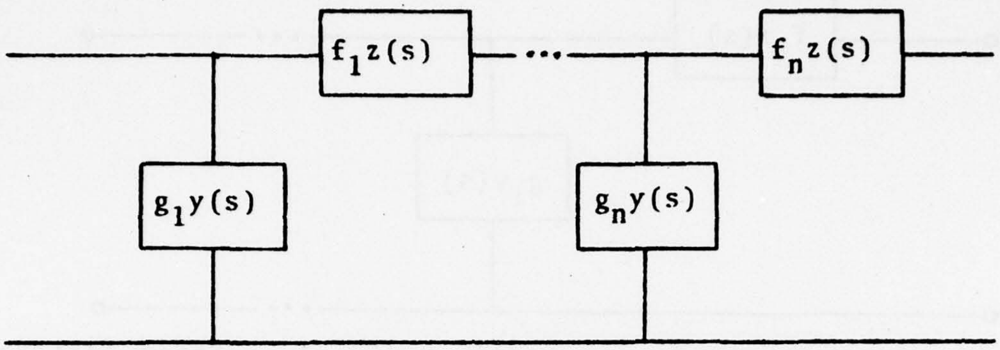

Figure 4.4 Dual of Figure 4.1

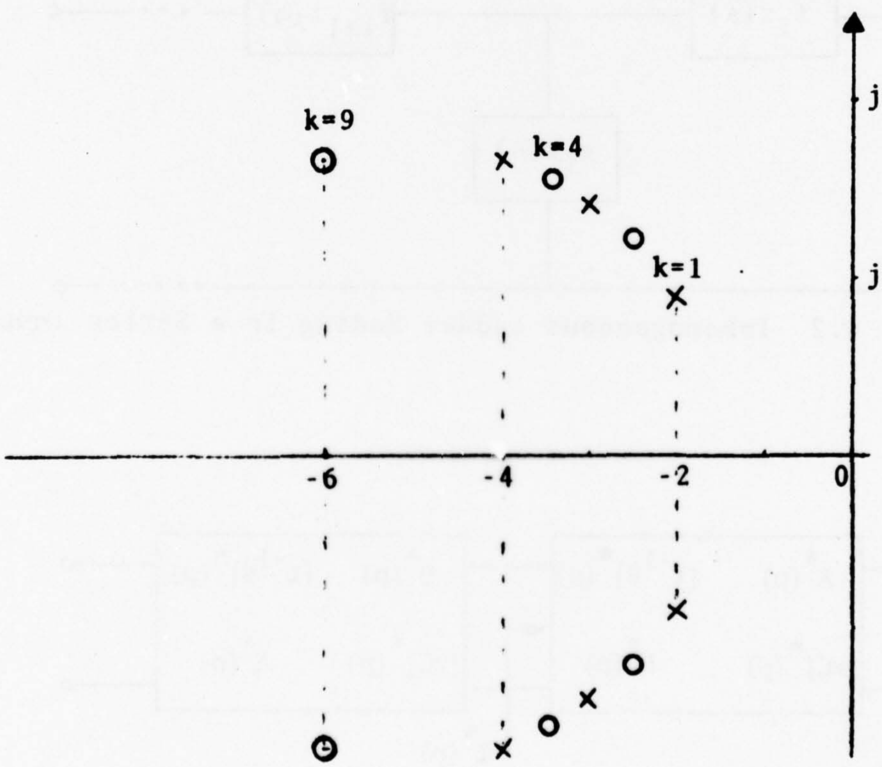

Figure 4.5 Locations of the Zeros of Example 4.1

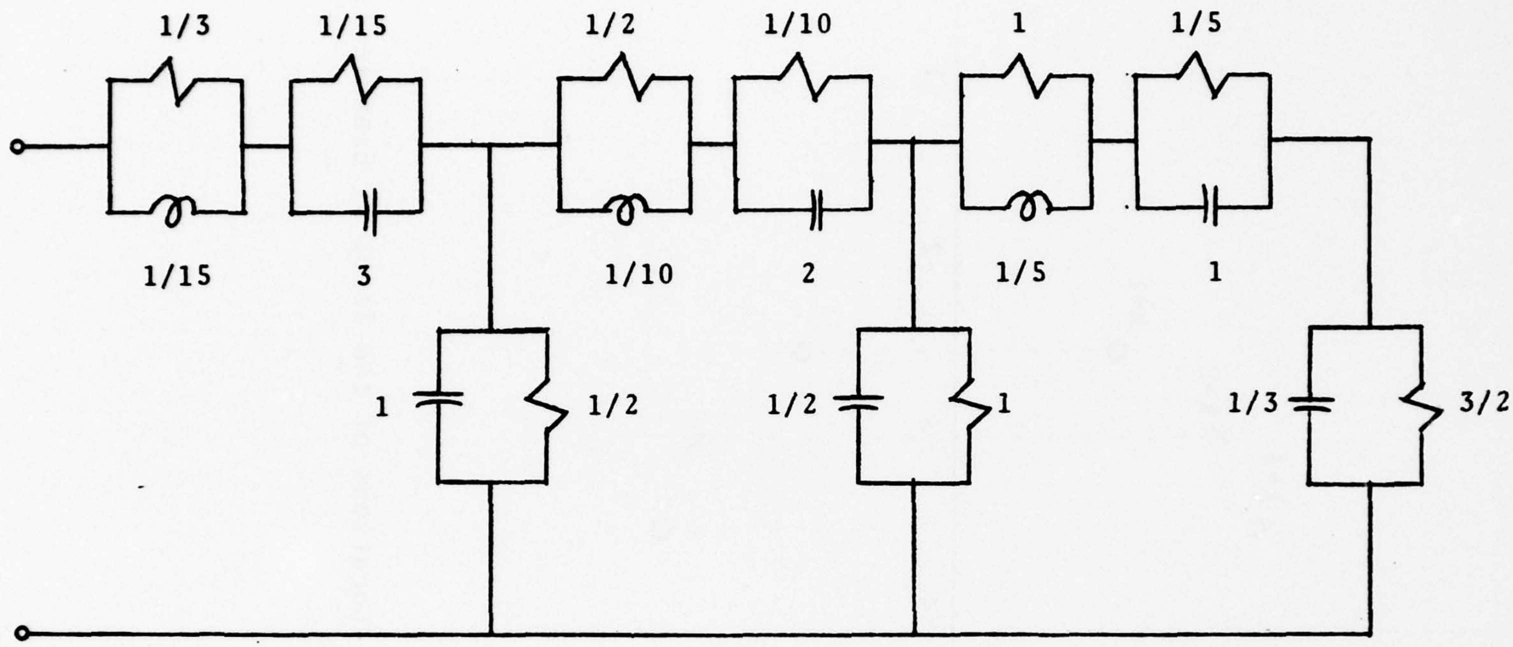

Figure 4.6 Optimal Inhomogeneous Ladder of Example 4.1

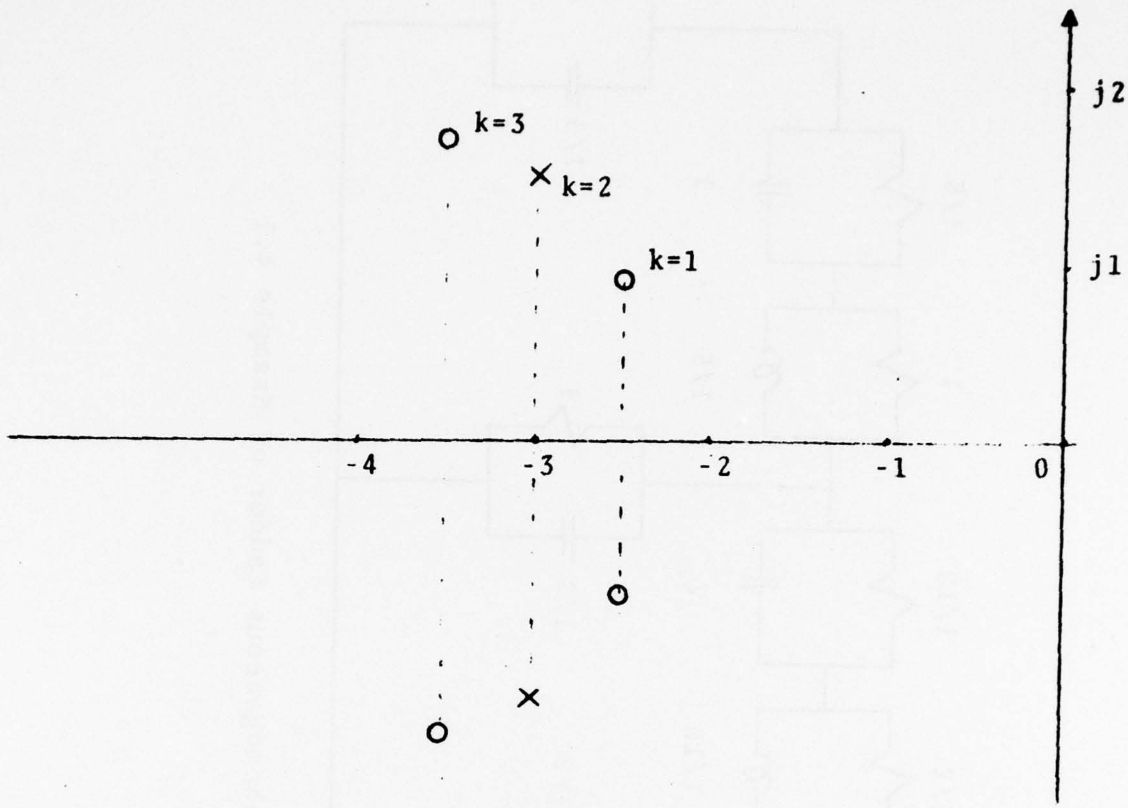

Figure 4.7 Locations of the Zeros of Example 4.2

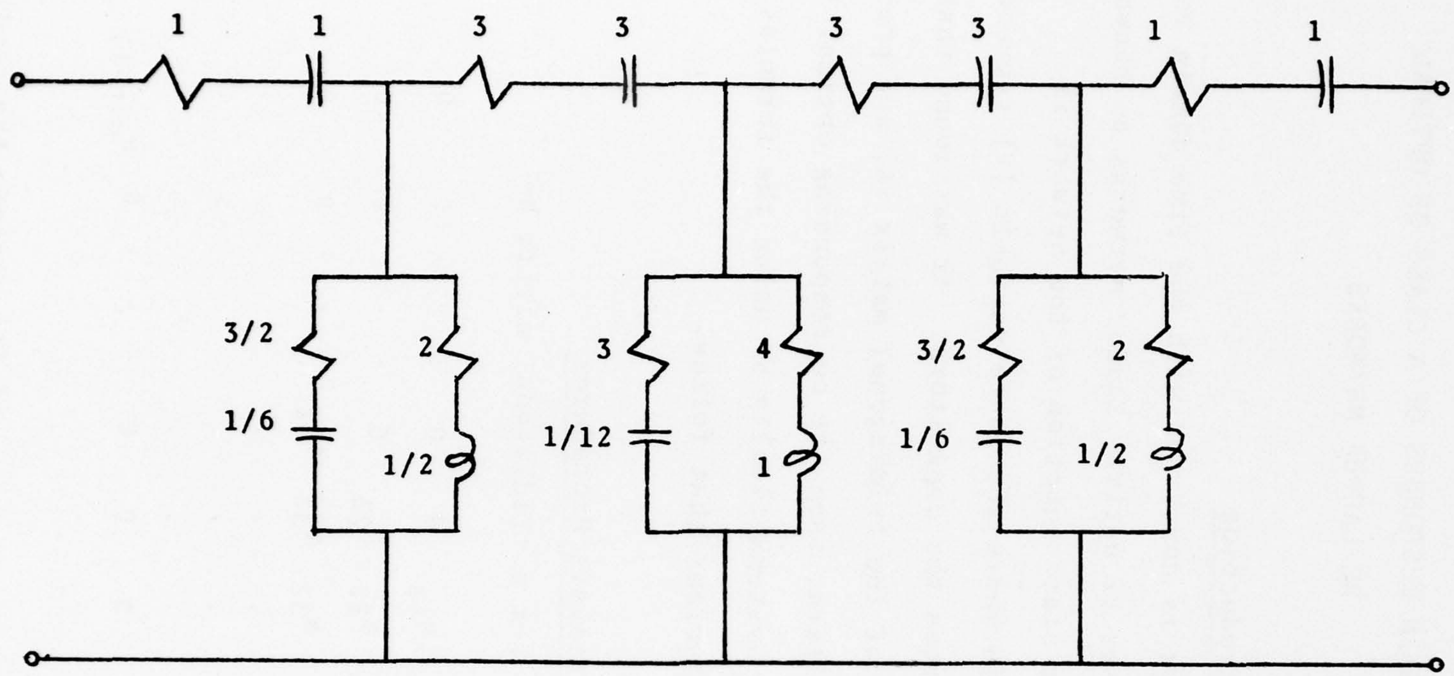

Figure 4.8 Optimal Inhomogeneous Ladder of Example 4.2

CHAPTER 5

TIME DOMAIN RESPONSES OF A CLASS OF OPTIMAL RC LADDER NETWORKS

Section 5.1 Introduction

This chapter is concerned with the time domain responses of certain classes of optimal ladder networks as shown in Figure 5.1. The state equation of the network is $[\dot{V}] = [A][V]+[B]u$ where the state variable [V] represents the voltages across the capacitors. It was found that if the eigenvalues of the tridiagonal matrix [A] are prescribed in a certain pattern, then the corresponding optimal ladder can be obtained systemmatically by using the formulas presented in the theorems that follow.

Section 5.2 Synthesis Procedures

Definition 5.1 Let a tridiagonal matrix be

$$[A] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

If the off-diagonal elements of [A] satisfy the condition that $a_{ij} = a_{(n-i+1)(n-j+1)}$, then the tridiagonal matrix [A] is said to be antimetrical.

Consider the double-terminated RC ladder network of Figure 5.1. Let the state equation be $[\dot{V}] = [A][V]+[B]u$ where the state variable $[V]$ represents the voltages across the capacitors. The $[A]$ matrix is tridiagonal as shown below.

$$[A]=\begin{bmatrix} -\frac{1}{C_1}(\frac{1}{R_1}+\frac{1}{R_2}) & \frac{1}{R_2C_1} & 0 & 0 & \cdot\ \cdot\ \cdot & 0 \\ \frac{1}{R_2C_2} & -\frac{1}{C_2}(\frac{1}{R_2}+\frac{1}{R_3}) & \frac{1}{R_3C_2} & 0 & \cdot\ \cdot\ \cdot & 0 \\ 0 & \frac{1}{R_3C_3} & -\frac{1}{C_3}(\frac{1}{R_3}+\frac{1}{R_4}) & & & \cdot \\ 0 & 0 & & & & \cdot \\ \cdot & \cdot & & & & 0 \\ \cdot & \cdot & & & & \frac{1}{R_nC_{n-1}} \\ \cdot & \cdot & & & & \\ 0 & 0 & \cdot\ \cdot\ \cdot & 0 & \frac{1}{R_nC_n} & -\frac{1}{C_n}(\frac{1}{R_n}+\frac{1}{R_{n+1}}) \end{bmatrix} \tag{5-1}$$

Let an RC ladder network be double-terminated as shown in Figure 5.1 where $R_1 = R_{n+1}$. If

$$R_k = \frac{n(n-1)\cdots(n-k+2)R_1}{(k-1)!}, \tag{5-2}$$

where k = 2,3,$\cdots$,n+1, and

$$C_k = \frac{2(k-1)!}{n(n-1)\cdots(n-k+1)p_1 R_1}, \tag{5-3}$$

where k = 1,2,$\cdots$,n, [Observation: Equations (5-2) and (5-3) imply that $R_k = R_{n-k+2}$ where k = 2,3,$\cdots$,n+1, and $C_k = C_{n-k+1}$ where k = 1,2,$\cdots$,n.] then the impulse response is

$$V_o(t) = \sum_{i=1}^{N} k_i e^{-p_i t}$$

where $p_i = ip_1$ (5-4)

and

$$k_i = (-1)^{i+1} \frac{H}{(i-1)!(n-i)!p_1^{n-1}} \tag{5-5}$$

and

$$H = \frac{R_{n+1}}{\sum_{i=1}^{N+1} R_i} \prod_{i=1}^{N} p_i = \frac{1}{\prod_{i=1}^{N} R_i \prod_{i=1}^{N} C_i}. \tag{5-6}$$

For the n=2 case, the voltage transfer function is

$$\frac{V_o(s)}{V_s(s)} = \frac{\dfrac{1}{2R_1^{\,2}C_1^{\,2}}}{s^2+\dfrac{3}{R_1C_1}s+\dfrac{2}{R_1^{\,2}C_1^{\,2}}}$$

$$= \frac{1}{2R_1^{\,2}C_1^{\,2}}\left[\frac{R_1C_1}{s+\dfrac{1}{R_1C_1}} - \frac{R_1C_1}{s+\dfrac{2}{R_1C_1}}\right]. \qquad (5\text{-}7)$$

Equation (5-7) shows that the impulse response is

$$V_o(t) = k_1e^{-p_1t} + k_2e^{-p_2t},$$

where

$$p_1 = \frac{1}{R_1C_1},\; p_2 = \frac{2}{R_1C_1} = 2p_1,$$

and

$$k_1 = R_1C_1H,\; k_2 = -R_1C_1H,$$

and

$$H = \frac{1}{2R_1^{\,2}C_1^{\,2}}.$$

For the n=3 case, the voltage transfer function is

$$\frac{V_o(s)}{V_s(s)} = \frac{R_1}{4.5R_1^{\,4}C_1^{\,3}s^3+18R_1^{\,3}C_1^{\,2}s^2+22R_1^{\,2}C_1s+8R_1},$$

$$= \frac{\dfrac{2}{9R_1^{\,3}C_1^{\,3}}}{s^3+\dfrac{4}{R_1C_1}s^2+\dfrac{44}{9R_1^{\,2}C_1^{\,2}}s+\dfrac{16}{9R_1^{\,3}C_1^{\,3}}},$$

$$= \frac{\frac{2}{9R_1^3C_1^3}}{(s+\frac{2}{3R_1C_1})(s+\frac{4}{3R_1C_1})(s+\frac{2}{R_1C_1})},$$

$$= \frac{2}{9R_1^3C_1^3}\left[\frac{\frac{9}{8}R_1^2C_1^2}{s+\frac{2}{3R_1C_1}} - \frac{\frac{9}{4}R_1^2C_1^2}{s+\frac{4}{3R_1C_1}} + \frac{\frac{9}{8}R_1^2C_1^2}{s+\frac{2}{R_1C_1}}\right]. \qquad (5\text{-}8)$$

Equation (5-8) shows that the impulse response is

$$V_o(t) = k_1e^{-p_1t} + k_2e^{-p_2t} + k_3e^{-p_3t},$$

where

$$p_1 = \frac{2}{3R_1C_1},$$

$$p_2 = \frac{4}{3R_1C_1} = 2p_1,$$

$$p_3 = \frac{2}{R_1C_1} = 3p_1,$$

and

$$k_1 = \frac{9}{8}R_1^2C_1^2H,$$

$$k_2 = -\frac{9}{4}R_1^2C_1^2H,$$

$$k_3 = \frac{9}{8}R_1^2C_1^2H,$$

and

$$H = \frac{2}{9R_1^3C_1^3}.$$

Values of these two cases agree with the results specified in Equations (5-4), (5-5), and (5-6).

It is noted that the value of H given in (5-6) is exactly the maximum gain of Kuh's optimal ladder [12].

Lemma 5.1: Let an RC ladder network be double-terminated as shown in Figure 5.1 where $R_1 = R_{n+1}$. If

$$R_k = \frac{n(n-1)\cdots(n-k+2)R_1}{(k-1)!}, \tag{5-9}$$

where $k = 2,3,\cdots,n+1$, and

$$C_k = \frac{2(k-1)!}{n(n-1)\cdots(n-k+1)p_1R_1}, \tag{5-10}$$

where $k = 1,2,\cdots,n$, then

(a) The diagonal elements of the [A] matrix of (5-1) are equal and each is equal to the sum of the eigenvalues divided by the number of eigenvalues; i.e.,

$$\frac{1}{C_k}\left(\frac{1}{R_k}+\frac{1}{R_{k+1}}\right) = \frac{(n+1)p_1}{2}. \tag{5-11}$$

(b) The [A] matrix is antimetrical as defined in Definition 5.1.

Proof:

(a) Since $p_i = ip_1$ for $n = 2,3,\cdots,n$, therefore, the sum of p_i is

$$\sum_{i=1}^{n} p_i = \frac{n(n+1)}{2}p_1. \tag{5-12}$$

It follows from Equation (5-9) that

$$\frac{1}{R_k} + \frac{1}{R_{k+1}} =$$

$$\frac{(k-1)!}{n(n-1)(n-2)\cdots(n-k+2)R_1} + \frac{k!}{n(n-1)(n-2)\cdots(n-k+1)R_1} =$$

$$\frac{(k-1)!(n-k+1)+(k-1)!k}{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)R_1} =$$

$$\frac{(k-1)!(n+1)}{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)R_1}. \tag{5-13}$$

From Equation (5-10) and (5-13), it follows that

$$\frac{1}{C_k}\left(\frac{1}{R_k} + \frac{1}{R_{k+1}}\right) =$$

$$\frac{n(n-1)(n-2)\cdots(n-k+1)p_1R_1}{2(k-1)!} \cdot \frac{(k-1)!(n+1)}{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)R_1}.$$

This results in the equation

$$\frac{1}{C_k}\left(\frac{1}{R_k} + \frac{1}{R_{k+1}}\right) = \frac{n+1}{2}p_1,$$

and therefore by Equation (5-12), this value is the sum of p_i divided by n. This completes the proof of (a).

To prove that the [A] matrix is antimetrical, it suffices, according to Definition 5.1 and Equation (5-1), to show that

$$\frac{1}{R_k C_k} = \frac{1}{R_{n-k+2} C_{n-k+1}}. \qquad (5\text{-}14)$$

It follows from Equation (5-9) that

$$R_{n-k+2} = \frac{n(n-1)\cdots(n-k+2)(n-k+1)\cdots kR_1}{(n-k+1)(n-k)\cdots k\cdot(k-1)!}$$

$$= \frac{n(n-1)\cdots(n-k+2)R_1}{(k-1)!},$$

$$= R_k.$$

Similarly, it can be shown that $C_{n-k+1} = C_k$. Therefore, (5-14) follows.

Q.E.D.

Lemma 5.2: Let an optimal RC ladder network with minimum total capacitance and with $R_1 = R_{n+1}$ be double-terminated as shown in Figure 5.1. Let its impulse response be

$$V_o(t) = \sum_{i=1}^{n} k_i e^{-p_i t},$$

where $$p_i = ip_1, \tag{5-15}$$

$$k_i = (-1)^{i+1} \frac{H}{(i-1)!(n-i)!p_1^{n-1}}, \tag{5-16}$$

and

$$H = \frac{R_{n+1}}{\sum_{i=1}^{n+1} R_i} \prod_{i=1}^{n} p_i = \frac{1}{\prod_{i=1}^{n} R_i \prod_{i=1}^{n} C_i}. \tag{5-17}$$

Then

$$R_k = \frac{n(n-1)\cdots(n-k+2)R_1}{(k-1)!}, \tag{5-18}$$

where $k = 2,3,\cdots,n+1$, and

$$C_k = \frac{2(k-1)!}{n(n-1)\cdots(n-k+1)p_1 R_1}, \tag{5-19}$$

where $k = 1,2,\cdots,n$.

From the given impulse response, it follows that the voltage transfer function of the ladder network is

$$\frac{V_o(s)}{V_s(s)} = \frac{H}{\prod_{i=1}^{n} (s+p_i)}.$$

For n=3, the optimal RC ladder has the short circuit input impedance

$$\frac{1}{y_{11}} = \frac{s^3+6p_1s^2+11p_1{}^2s+6p_1{}^3}{s^3+4.5p_1s^2+5p_1{}^2s+0.75p_1{}^3}, \qquad (5\text{-}20)$$

which can be realized by using the decomposition algorithm presented in Chapter 2. This is shown in Table 5.1.

Table 5.1 Three-Section Optimal Ladder

	1	$6p_1$	$11p_1{}^2$	$6p_1{}^3$
$R_1 = 1$				
	1	$\frac{9}{2}p_1$	$5p_1{}^2$	$\frac{3}{4}p_1{}^3$
$C_1 = \frac{2}{3p_1}$				
	$\frac{3}{2}p_1$	$6p_1{}^2$	$\frac{21}{4}p_1{}^3$	
$R_2 = 3$				
	$\frac{1}{2}p_1$	$\frac{3}{2}p_1{}^2$	$\frac{3}{4}p_1{}^3$	
$C_2 = \frac{1}{3p_1}$				
	$\frac{3}{2}p_1{}^2$	$3p_1{}^3$		
$R_3 = 3$				
	$\frac{1}{2}p_1{}^2$	$\frac{3}{4}p_1{}^3$		
$C_3 = \frac{2}{3p_1}$				
	$\frac{3}{4}p_1{}^3$			
$R_4 = 1$				
	$\frac{3}{4}p_1{}^3$			

The values of resistors and capacitors in Table 5.1 satsify Equations (5-18) and (5-19). This three-section optimal ladder is shown in Figure 5.2.

The residues k_i as expressed in Equation (5-16) are tabulated in Table 5.2. A common factor is taken out of the residues to make more apparent their variations. Plots of impulse responses are shown in Figures 5.3 and 5.10.

Table 5.3 shows the residues of the unit step responses of optimal ladders with eigenvalues $p_i = ip_1$. Figure 5.11 shows plots of the responses of these ladders to unit steps. Therefore, a circuit whose desired impulse response or step response has similar variations can be realized as an optimal ladder by the use of Equations (5-2) and (5-3).

Table 5.2 Residues of Impulse Response,

$$V_o(t) = \sum_{i=1}^{n} (-1)^{i+1} k_i e^{-p_i t},$$

of Optimal Ladders where $p_i = ip_1$

n	Common Factor	k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
2	$\frac{1}{2}p_1$	1	-1						
3	$\frac{3}{8}p_1$	1	-2	1					
4	$\frac{1}{4}p_1$	1	-3	3	-1				
5	$\frac{5}{32}p_1$	1	-4	6	-4	1			
6	$\frac{3}{32}p_1$	1	-5	10	-10	5	-1		
7	$\frac{7}{128}p_1$	1	-6	15	-20	15	-6	1	
8	$\frac{1}{32}p_1$	1	-7	21	-35	35	-21	7	-1

Table 5.3 Residues of Unit Step Response,

$$V_o(t) = k_o + \sum_{i=1}^{n} (-1)^i k_i e^{-p_i t},$$

of Optimal Ladders where $p_i = ip_1$

n	Common Factor	k_o	k_1	k_2	k_3	k_4	k_5	k_6
2	$\frac{1}{4}p_1$	1	-2	1				
3	$\frac{1}{8}p_1$	1	-3	3	-1			
4	$\frac{1}{16}p_1$	1	-4	6	-4	1		
5	$\frac{1}{32}p_1$	1	-5	10	-10	5	-1	
6	$\frac{1}{64}p_1$	1	-6	15	-20	15	-6	1

Theorems 5.1 and 5.2 are concerned with optimal ladder networks where the eigenvalues of the [A] matrix are $p_i=ip_1$. If instead the eigenvalues are $p_i=i^2p_1$ with k_i given in Table 5.4 and Table 5.5 for impulse and step response respectively, then the element values of the optimal network can be calculated from the following two formulas.

$$R_k = \frac{1\cdot 3\cdot 5\cdots(2k-3)}{(2n-1)(2n-3)\cdots(2n-2k+3)} \cdot \frac{n(n-1)\cdots(n-k+2)R_1}{(k-1)!}, \tag{5-23}$$

where $k = 2,3,\cdots,n+1$ and

$$C_k = \frac{(2n-1)(2n-3)\cdots(2n-2k+3)}{1\cdot 3\cdot 5\cdots(2k-1)} \cdot \frac{2(k-1)!}{n(n-1)\cdots(n-k+1)p_1R_1}, \tag{5-24}$$

where $k = 1,2,\cdots,n$.

The impulse responses of optimal ladders with eigenvalues $p_i = i^2p_1$ are shown in Figure 5.12. The responses to unit steps are shown in Figure 5.13.

Table 5.4 Residues of Impulse Response,

$$V_o(t) = \sum_{i=1}^{n} (-1)^{i+1} k_i e^{-p_i t},$$

of Optimal Ladders where $p_i = i^2p_1$

n	Common Factor	k_1	k_2	k_3	k_4	k_5	k_6
2	$\frac{1}{2}p_1$	1	-1				
3	$\frac{3}{32}p_1$	5	-8	3			
4	$\frac{1}{16}p_1$	7	-14	9	-2		
5	$\frac{5}{512}p_1$	42	-96	81	-32	5	
6	$\frac{3}{512}p_1$	66	-165	165	-88	25	-3

The formulas derived and presented so far in this chapter provide straightforward methods of synthesizing optimal ladder networks whose eigenvalues follow certain patterns. When the eigenvalues are not arranged in these

Table 5.5 Residues of Unit Step Response,

$$V_o(t) = k_o + \sum_{i=1}^{n} (-1)^i k_i e^{-p_i t},$$

of Optimal Ladders where $p_i = i^2 p_1$

n	Common Factor	k_o	k_1	k_2	k_3	k_4	k_5
2	$\frac{1}{8}p_1$	3	-4	1			
3	$\frac{1}{32}p_1$	10	-15	6	-1		
4	$\frac{1}{128}p_1$	35	-56	28	-8	1	
5	$\frac{1}{512}p_1$	126	-210	120	-45	10	-1

patterns, the existing synthesis methods require a continued fraction expansion. The complexity of such a procedure increases rapidly with the number of ladder sections. However, for a two-section ladder it is possible to derive simple and useful formulas based on the following equations:

$$\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{p_1+p_2}{2}, \qquad (5\text{-}25)$$

and

$$p_1 p_2 = \frac{R_1+R_2+R_3}{R_1 R_2 R_3 C_1 C_2}. \qquad (5\text{-}26)$$

Since $R_1 = R_3 = 1\Omega$, $C_1 = C_2$, solving Equations (5-25) and (5-26) for R_2 and C_1 yields the following two formulas:

$$R_2 = \frac{2p_1}{p_2 - p_1}, \tag{5-27}$$

$$C_1 = \frac{1}{p_1}. \tag{5-28}$$

These formulas are easy to apply and are useful for two-section optimal ladder with any two eigenvalues. Example 5.3 illustrates the process.

For the n=3 case, the short circuit input impedance of the optimal ladder is

$$\frac{1}{y_{11}} = \frac{s^3 + a_2 s^2 + a_1 s + a_o}{s^3 + b_2 s^2 + b_1 s + b_o},$$

where

$$a_o = p_1 p_2 p_3,$$

$$a_1 = p_1 p_2 + p_1 p_3 + p_2 p_3,$$

$$a_2 = p_1 + p_2 + p_3,$$

$$b_o = \frac{p_1 p_2 p_3 p_{21} p_{31} p_{32}}{p_1 p_2 p_{21} + p_1 p_3 p_{31} + p_2 p_3 p_{32} + p_{21} p_{31} p_{32}},$$

$$b_1 = \frac{p_1^2 p_2^2 p_{21} + p_1^2 p_3^2 p_{31} + p_2^2 p_3^2 p_{32} + p_{21} p_{31} p_{32} (p_1 p_2 + p_1 p_3 + p_2 p_3)}{p_1 p_2 p_{21} + p_1 p_3 p_{31} + p_2 p_3 p_{32} + p_{21} p_{31} p_{32}},$$

$$b_2 = \frac{p_1p_2p_{21}(p_1+p_2)+p_1p_3p_{31}(p_1+p_3)+p_2p_3p_{32}(p_2+p_3)}{p_1p_2p_{21}+p_1p_3p_{31}+p_2p_3p_{32}+p_{21}p_{31}p_{32}}$$

$$+ \frac{p_{21}p_{31}p_{32}(p_1+p_2+p_3)}{p_1p_2p_{21}+p_1p_3p_{31}+p_2p_3p_{32}+p_{21}p_{31}p_{32}},$$

$p_{21} = p_2 - p_1$,

$p_{31} = p_3 - p_1$, and

$p_{32} = p_3 - p_2$.

The elements are:

$R_1 = R_4 = 1$

$$R_2 = R_3 = \frac{(a_2-b_2)^2}{b_2(a_2-b_2)-(a_1-b_1)} \tag{5-29}$$

$$C_1 = C_3 = \frac{1}{a_2-b_2} \tag{5-30}$$

$$C_2 = \frac{C_1[b_2(a_2-b_2)-(a_1-b_1)]^2}{(a_1-b_1)[b_2(a_2-b_2)-(a_1-b_1)]-(a_2-b_2)[b_1(a_2-b_2)-(a_o-b_o)]}. \tag{5-31}$$

Example 5.1: This example illustrates the optimal realization technique for the ladder network of Figure 5.1 with the unit impulse response,

$$V_o(t) = \frac{H}{24}(e^{-t}-4e^{-2t}+6e^{-3t}-4e^{-4t}+e^{-5t}).$$

The terminating resistors are $R_1 = R_6 = 1\Omega$. For $n=5$, Equations (5-2) and (5-3) yield the following element values:

$$R_2 = R_5 = 5R_1 = 5\Omega,$$

$$R_3 = R_4 = 10R_1 = 10\Omega,$$

$$C_1 = C_5 = \frac{2}{5p_1R_1} = \frac{2}{5}F,$$

$$C_2 = C_4 = \frac{1}{10p_1R_1} = \frac{1}{10}F, \text{ and}$$

$$C_3 = \frac{1}{15p_1R_1} = \frac{1}{15}F.$$

The optimal network is shown in Figure 5.14. The maximum gain of the optimal ladder network is given by Equation (5-6).

$$H = \frac{R_6}{\sum_{i=1}^{6} R_i} \prod_{i=1}^{5} p_i$$

$$= \frac{R_6p_1p_2p_3p_4p_5}{R_1+R_2+R_3+R_4+R_5+R_6}$$

$$= \frac{15}{4}$$

The optimal network is shown in Figure 5.14.

Example 5.2: This example demonstrates the optimal synthesis procedure for the ladder network of Figure 5.1 with eigenvalues $p_i = i^2 p_1$. The unit impulse response is

$$V_o(t) = \frac{H}{120}(5e^{-t} - 8e^{-4t} + 3e^{-9t}).$$

The terminating resistors are $R_1 = R_4 = 1\Omega$. The problem is to realize the network with maximum possible gain H.

Solution: Application of Equations (5-23) and (5-24) with n=3 yields the following values.

$$R_2 = R_3 = \frac{3}{5}\Omega,$$

$$C_1 = C_3 = \frac{2}{3}F, \text{ and}$$

$$C_2 = \frac{5}{9}F.$$

The maximum gain H is obtained from Equation (5-6),

$$H = \frac{R_4 \; p_1 p_2 p_3}{R_1 + R_2 + R_3 + R_4} = \frac{45}{4}.$$

The optimal network is shown in Figure 5.15.

Example 5.3: This example illustrates the optimal synthesis procedure for the ladder network of Figure 5.1. The unit impulse response is

$$V_o(t) = H(\tfrac{1}{3}e^{-2t} - \tfrac{1}{3}e^{-5t}).$$

The problem is to realize a ladder network with maximum possible H.

Solution: Substitution of the eigenvalues into formulas (5-27) and (5-28) yields the following:

$$R_2 = \frac{2p_1}{p_2-p_1} = \frac{4}{3},$$

$$C_1 = \frac{1}{p_1} = \frac{1}{2}.$$

The value of H is obtained from Equation (5-6).

$$H = \frac{R_3}{R_1+R_2+R_3}p_1p_2 = 3$$

The optimal network is shown in Figure 5.16.

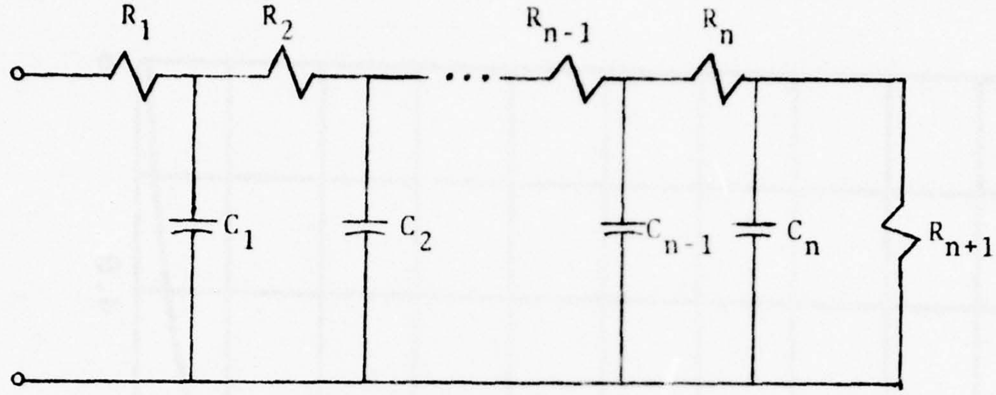

Figure 5.1 Double-Terminated RC Ladder

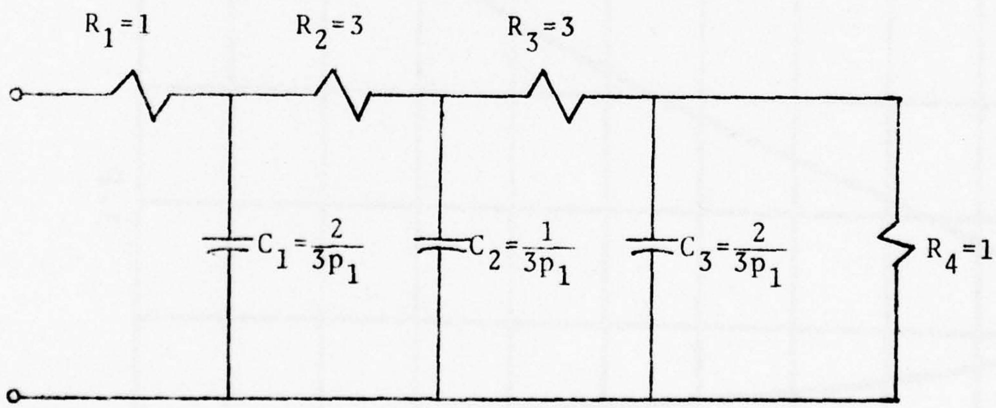

Figure 5.2 Three-Section Optimal Ladder

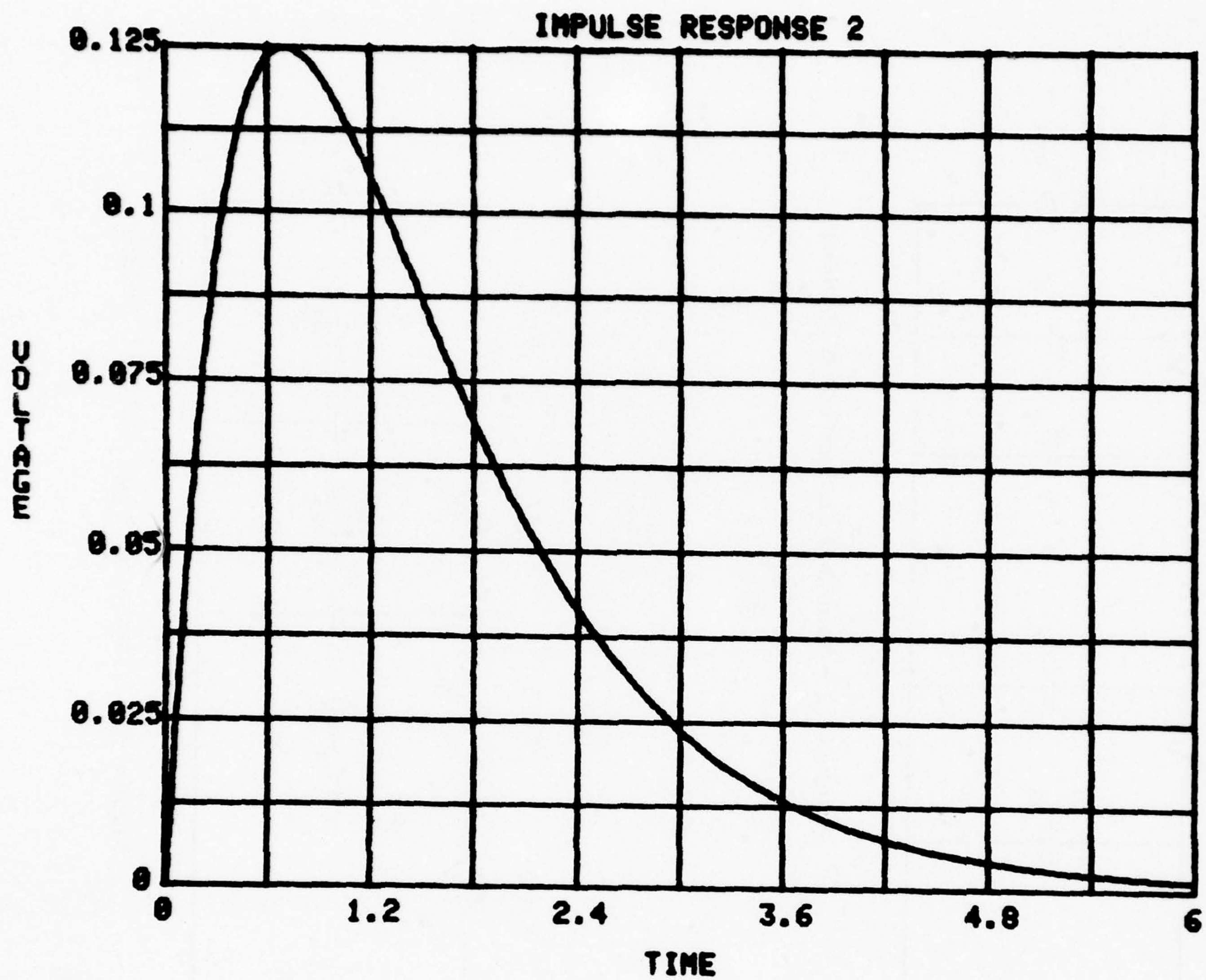
IMPULSE RESPONSE 2
VOLTAGE
0.125
0.1
0.075
0.05
0.025
0
0
1.2
2.4
3.6
4.8
6
TIME

Figure 5.3

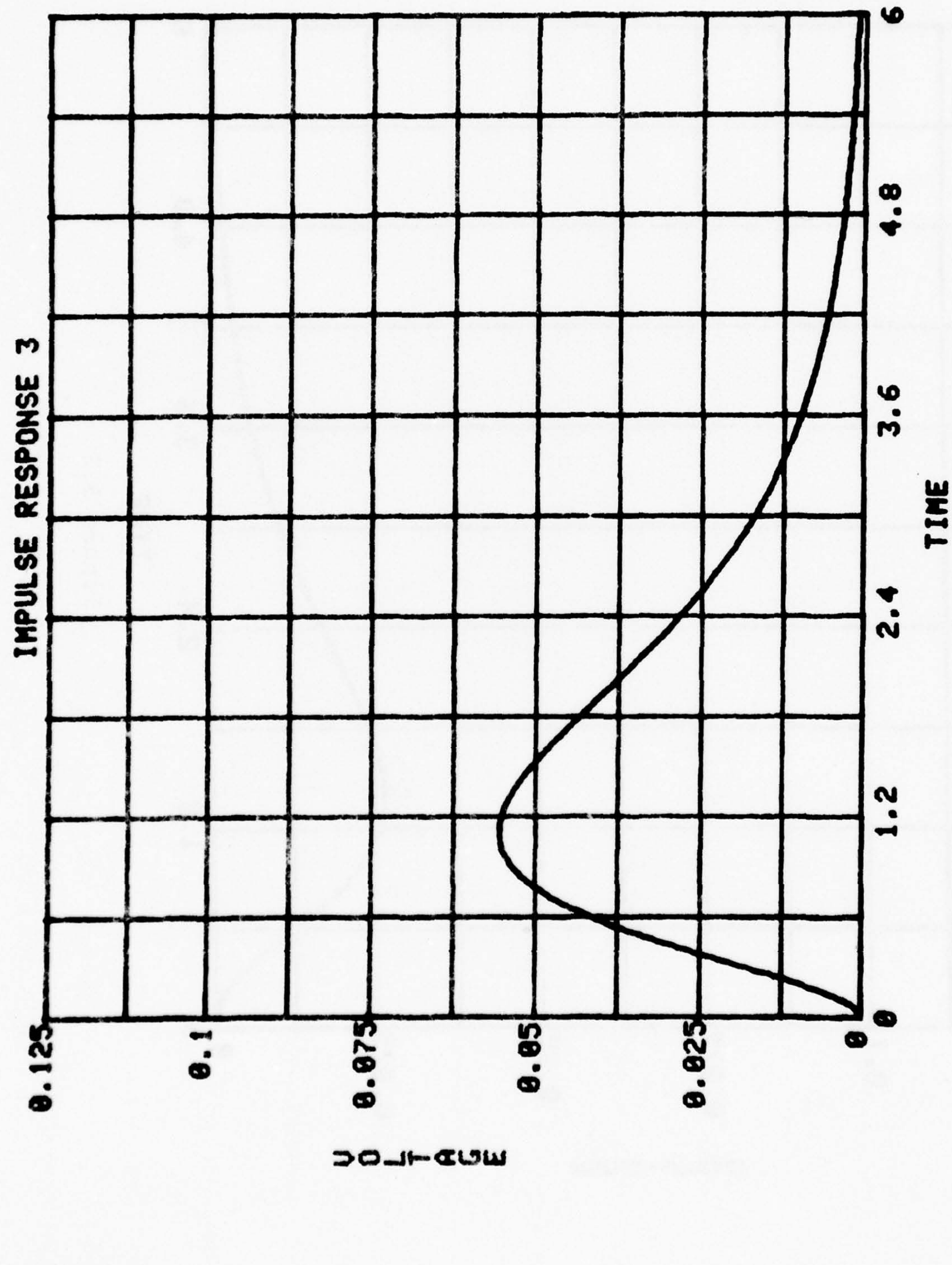
IMPULSE RESPONSE 3
VOLTAGE
TIME
0.125
0.1
0.075
0.05
0.025
0
0
1.2
2.4
3.6
4.8
6

Figure 5.4

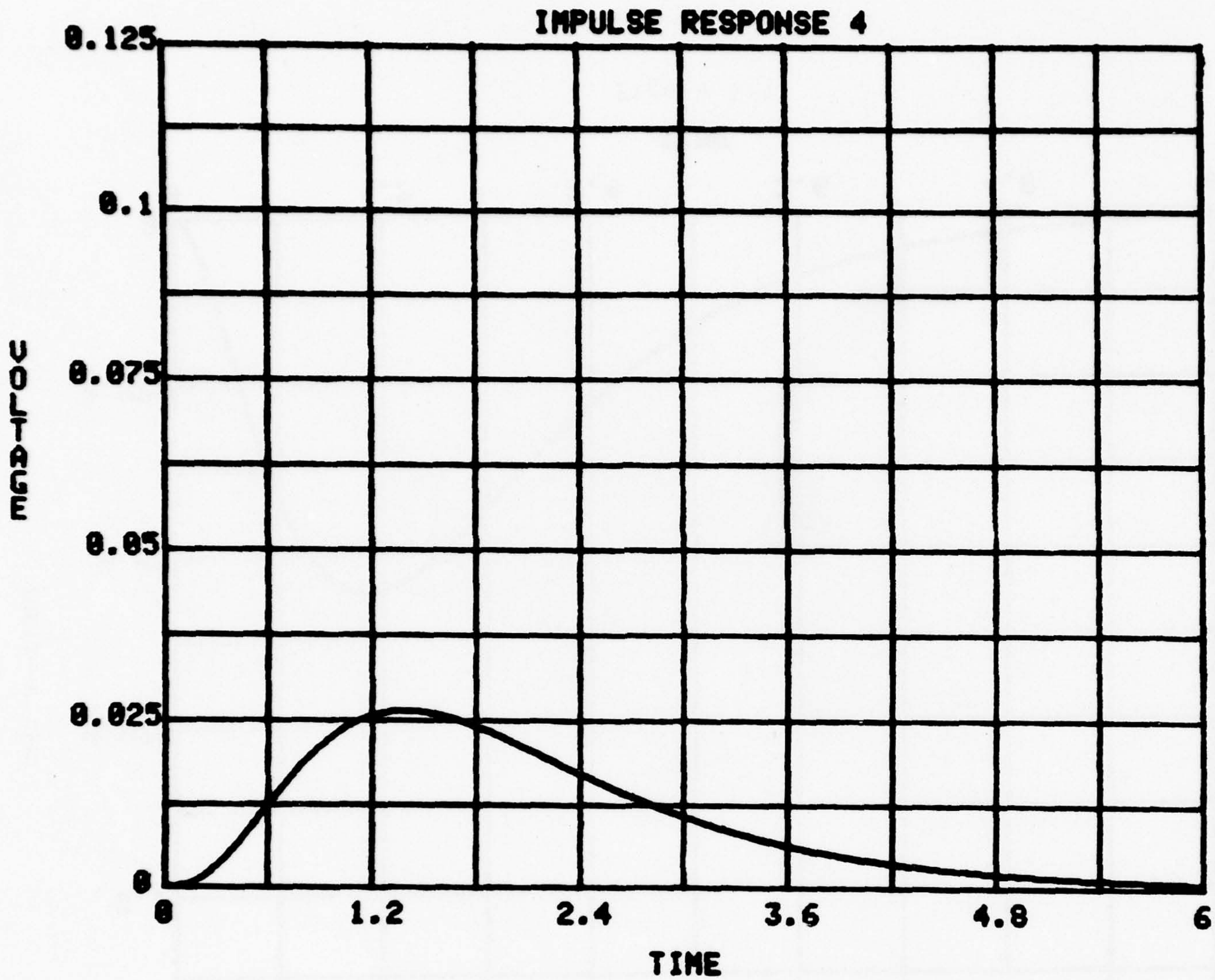
IMPULSE RESPONSE 4
0.125
0.1
0.075
0.05
0.025
0
VOLTAGE
0
1.2
2.4
3.6
4.8
6
TIME

Figure 5.5

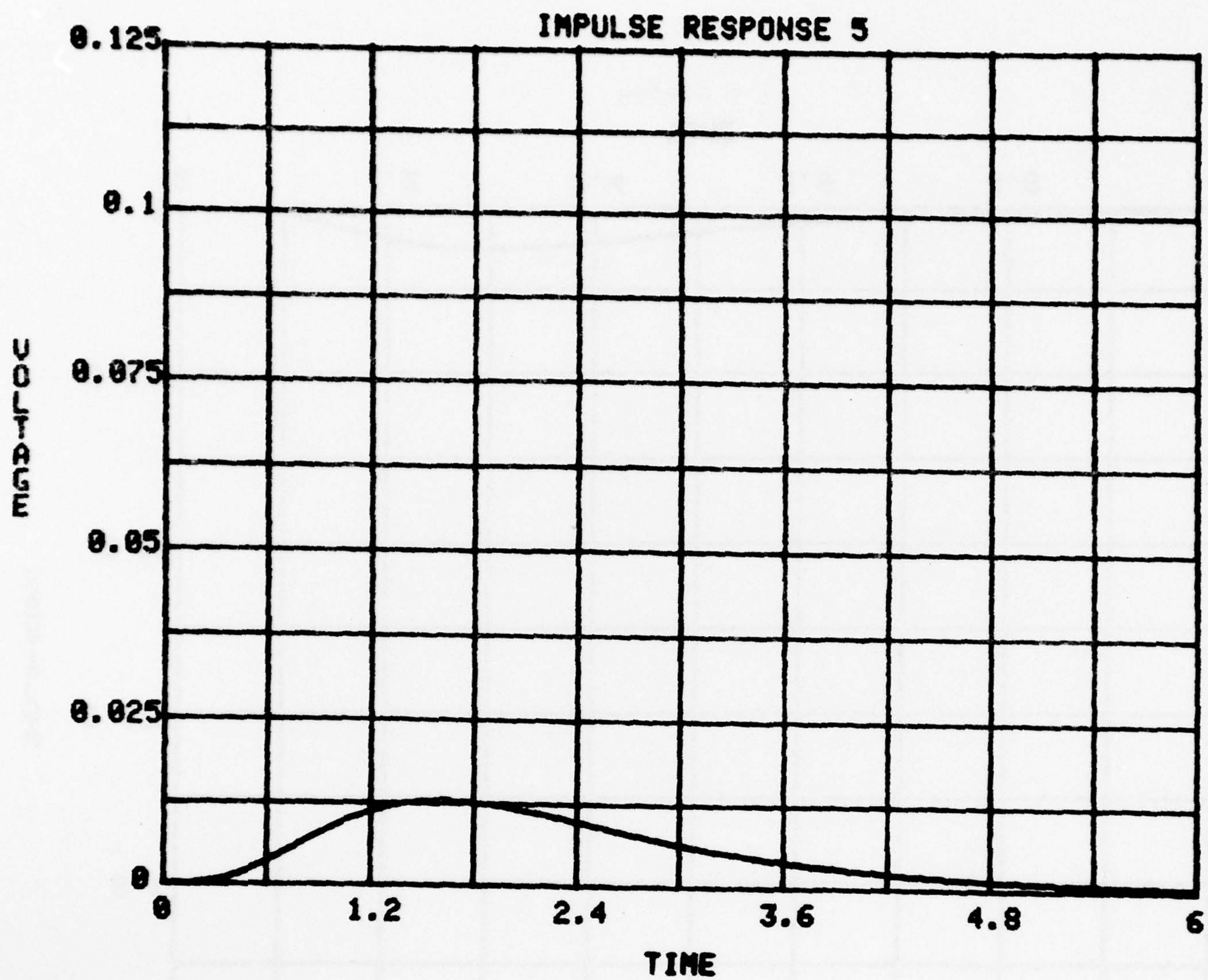
IMPULSE RESPONSE 5
VOLTAGE
0.125
0.1
0.075
0.05
0.025
0
0
1.2
2.4
3.6
4.8
6
TIME

Figure 5.6

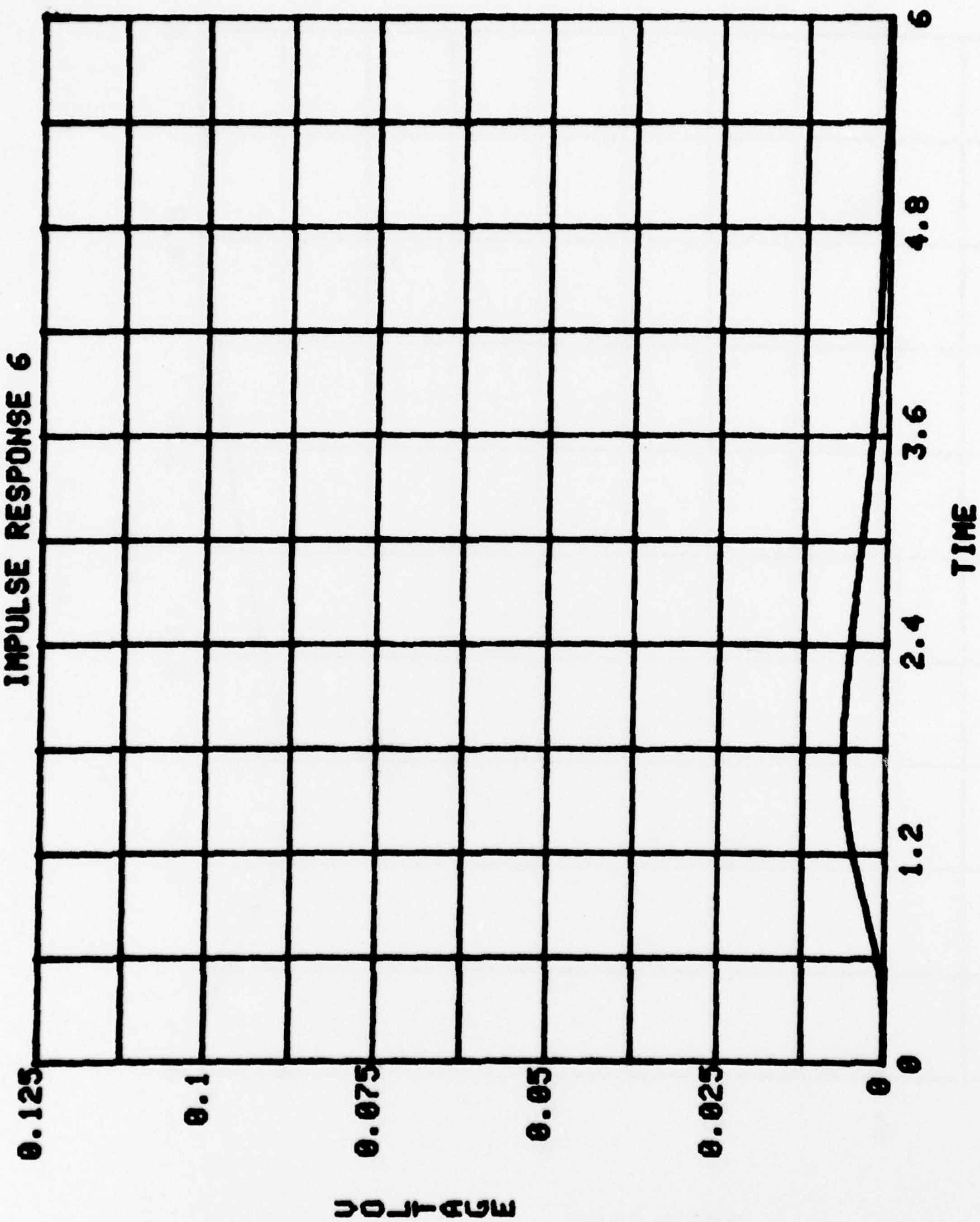
IMPULSE RESPONSE 6
VOLTAGE
0.125
0.1
0.075
0.05
0.025
0
0
1.2
2.4
3.6
4.8
6
TIME

Figure 5.7

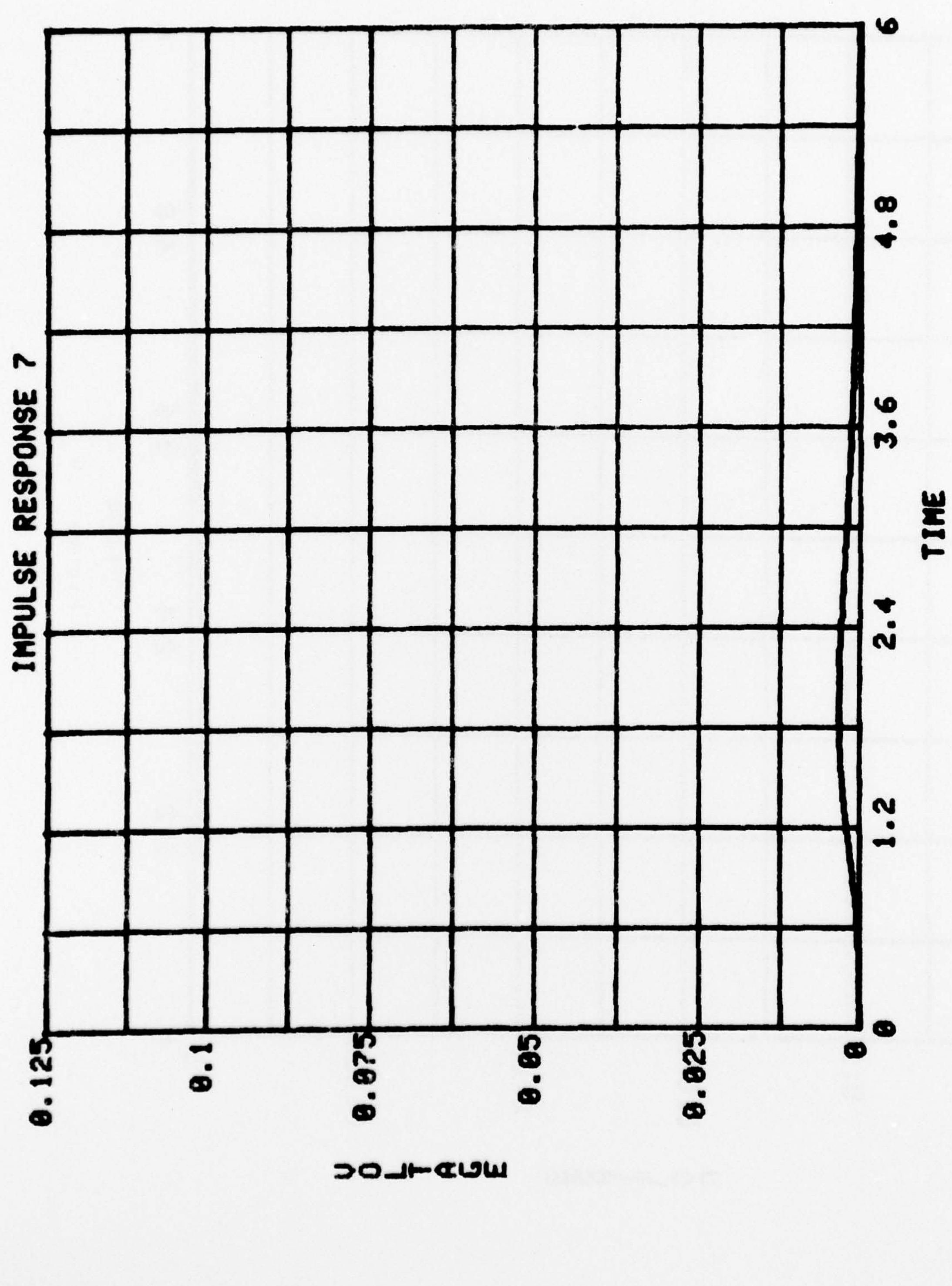
IMPULSE RESPONSE 7
VOLTAGE
TIME
0.125
0.1
0.075
0.05
0.025
0
0
1.2
2.4
3.6
4.8
6

Figure 5.8

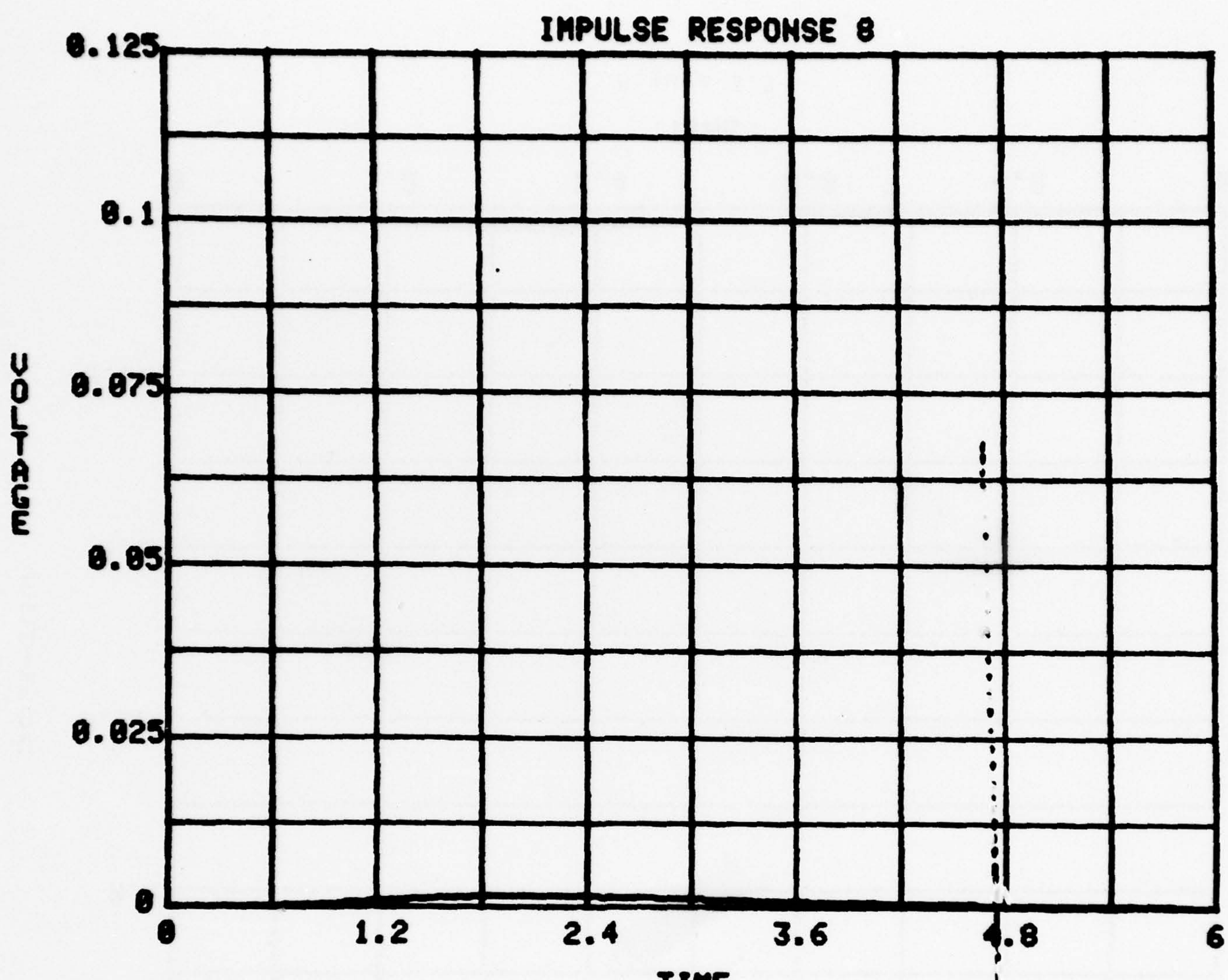
IMPULSE RESPONSE 8
0.125
0.1
0.075
0.05
0.025
0
VOLTAGE
0
1.2
2.4
3.6
4.8
6
TIME

Figure 5.9

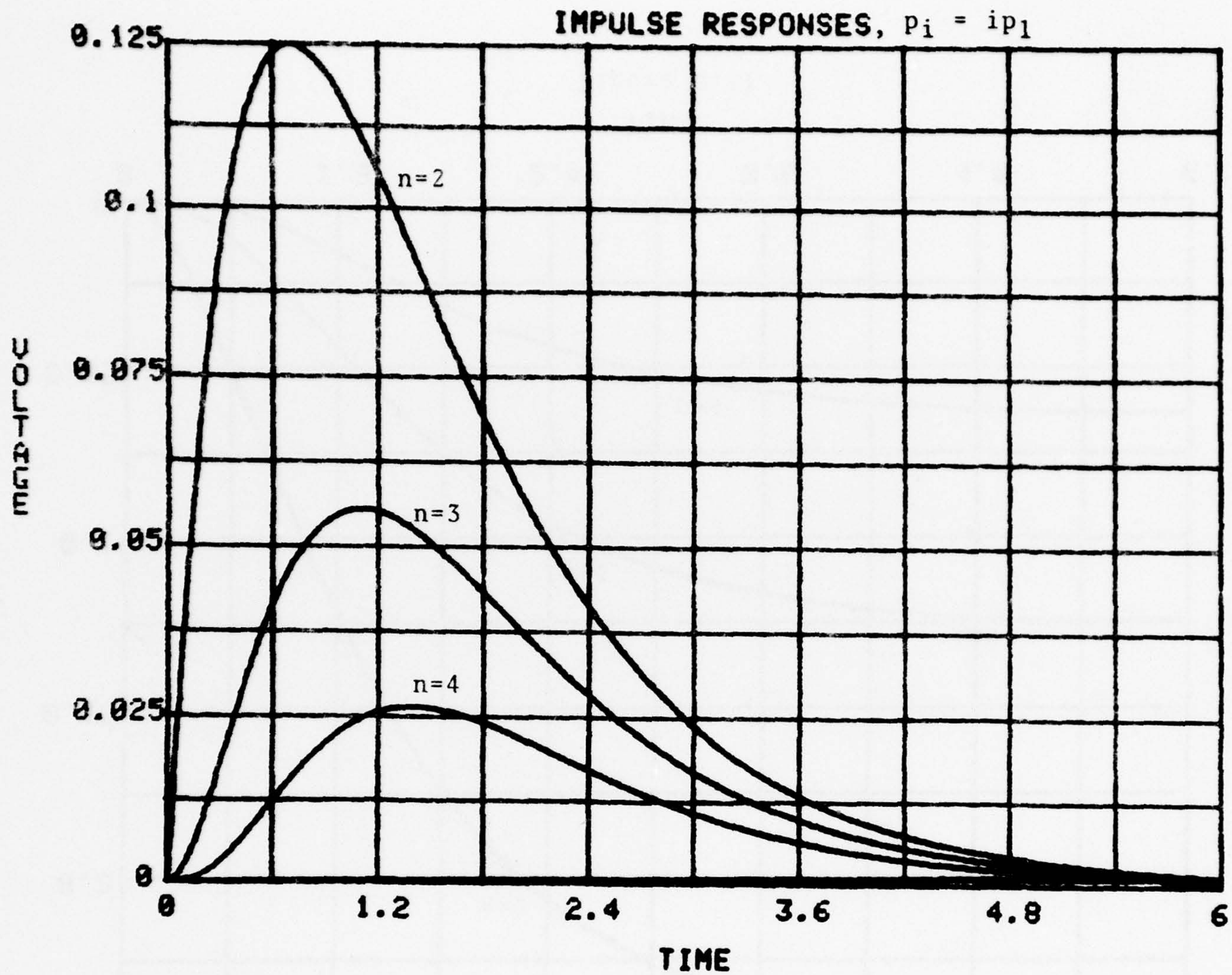
IMPULSE RESPONSES, $p_i = ip_1$
VOLTAGE
0.125
0.1
0.075
0.05
0.025
0
n=2
n=3
n=4
0
1.2
2.4
3.6
4.8
6
TIME

Figure 5.10

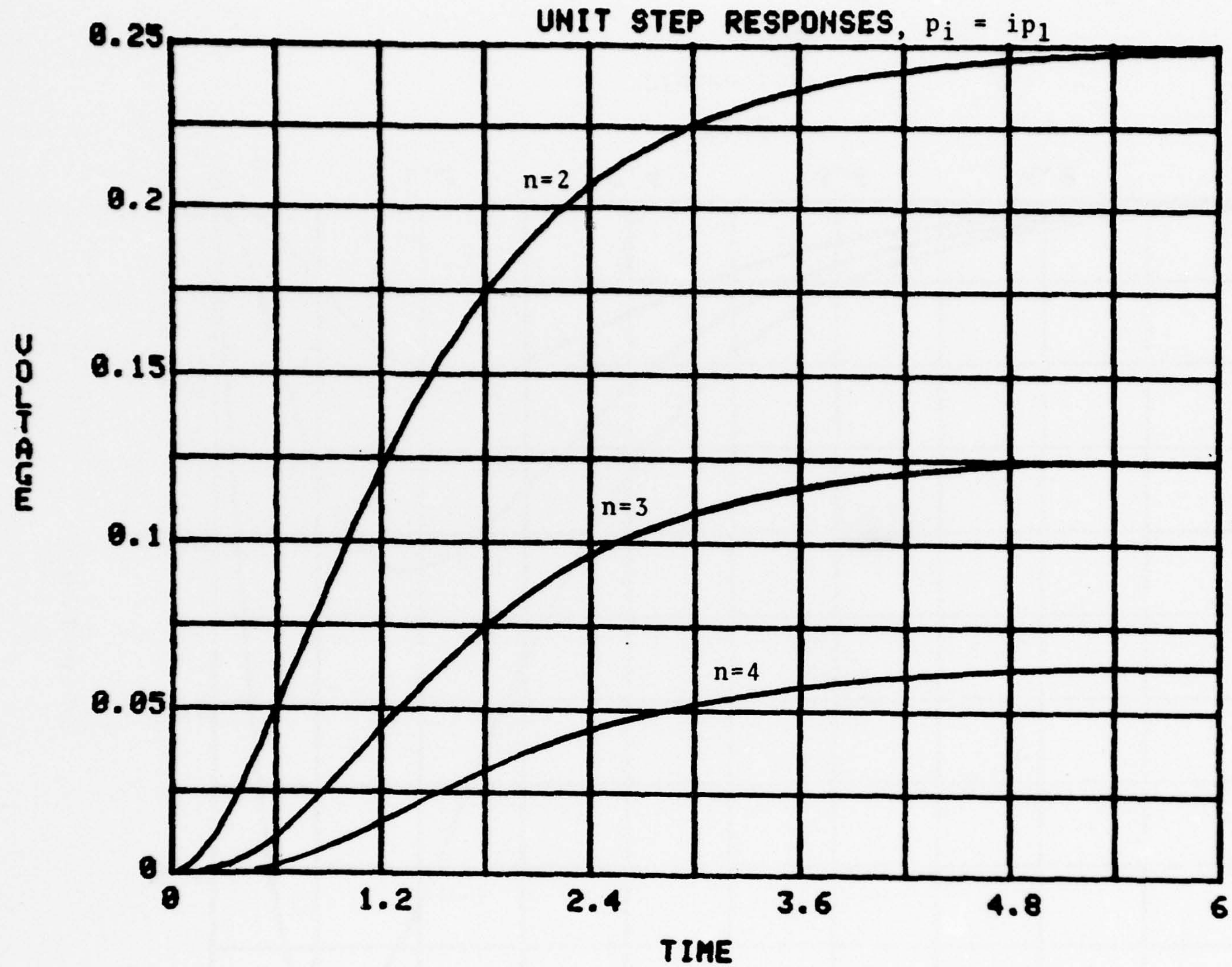
UNIT STEP RESPONSES, $p_i = ip_1$
0.25
0.2
0.15
0.1
0.05
0
VOLTAGE
0
1.2
2.4
3.6
4.8
6
TIME
n=2
n=3
n=4

Figure 5.11

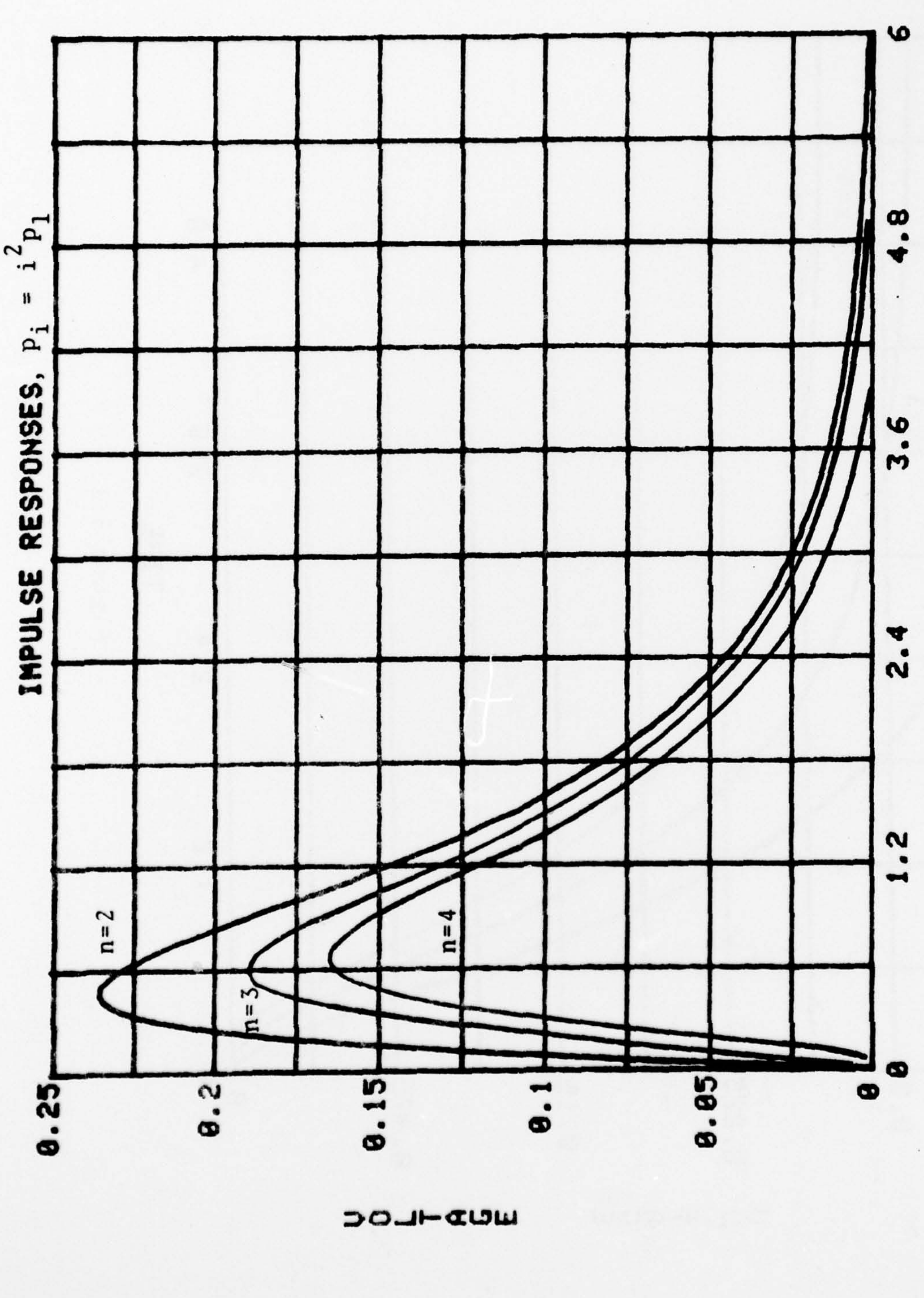
IMPULSE RESPONSES, $p_i = i^2 p_1$
VOLTAGE
TIME
0.25
0.2
0.15
0.1
0.05
0
0
1.2
2.4
3.6
4.8
6
n=2
n=3
n=4

Figure 5.12

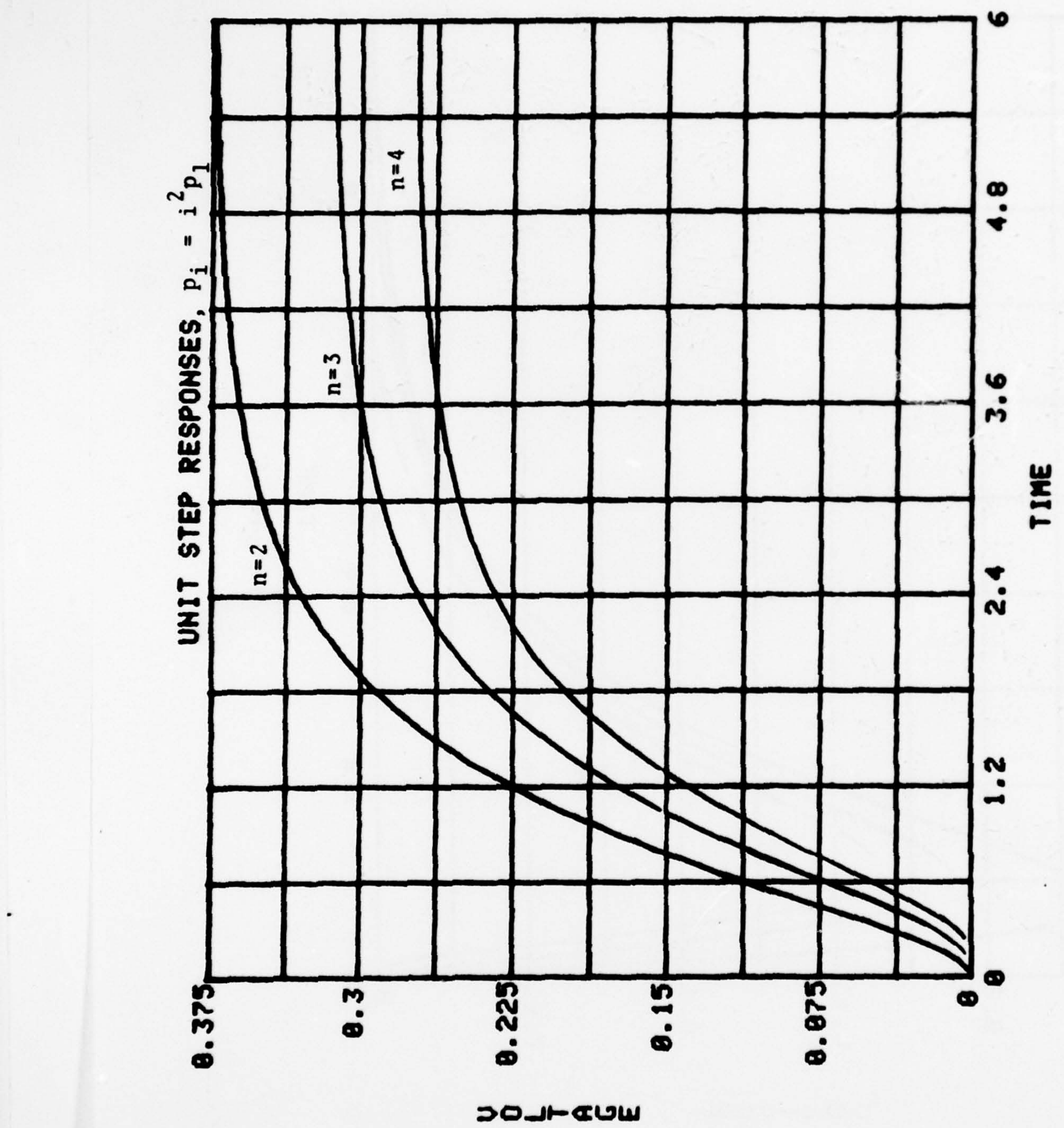
UNIT STEP RESPONSES, $p_i = i^2 p_1$
n=2
n=3
n=4
VOLTAGE
0.375
0.3
0.225
0.15
0.075
0
TIME
0
1.2
2.4
3.6
4.8
6

Figure 5.13

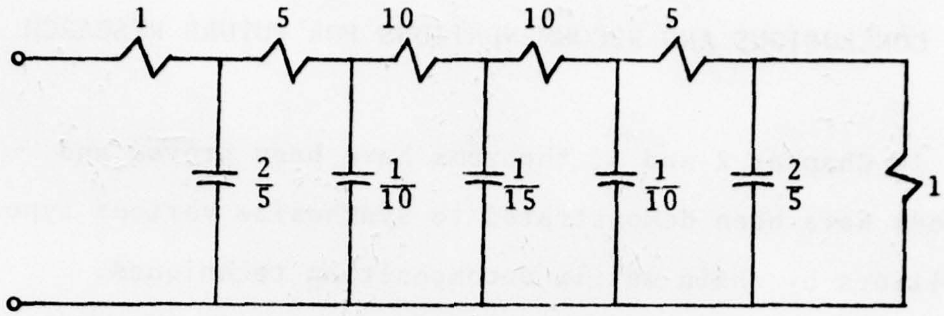

Figure 5.14 Circuit of Example 5.1

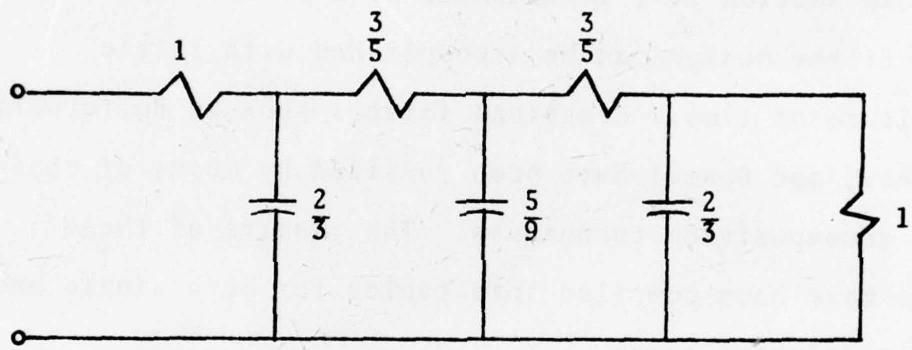

Figure 5.15 Circuit of Example 5.2

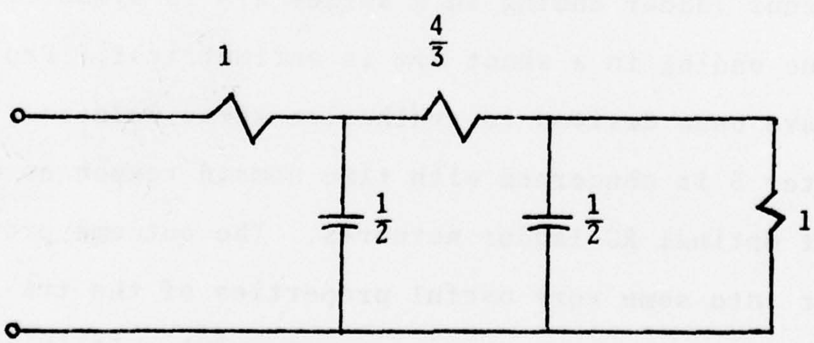

Figure 5.16 Circuit of Example 5.3

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

In Chapter 2 and 3, theorems have been proved and methods have been demonstrated to synthesize various types of filters by chain matrix decomposition techniques. Lossless (LC) ladder networks terminated in a load resistance as well as those resistively terminated at both ends have been investigated. By the use of simple algorithms presented in Section 2.4, implemented by digital computer programs, filter design can be accomplished with little expenditure of time. Classical filters such as Butterworth, Chebyshev, and Bessel have been realized by means of chain matrix decomposition techniques. The results of these filters have been compiled into tables for both single and double-terminated cases.

Chapter 4 deals with optimal synthesis of inhomogeneous ladder networks. It has been found that an optimal inhomogeneous ladder ending in a series arm is symmetrical and the one ending in a shunt arm is antimetrical. Procedures have been derived to synthesize these ladders.

Chapter 5 is concerned with time domain responses of a class of optimal RC ladder networks. The outcome provides an insight into some very useful properties of the tridiagonal matrix [A] of the state equation $[\dot{V}] = [A][V]+[B]u$, which describes a certain type of double-terminated ladder networks. It has been found that the [A] matrix of the optimal ladder is antimetrical as established in Definition 5.1 and the sum of the diagonal elements is equal to the sum of the eigenvalues of [A]. If, in particular, the eigenvalues p_i are distributed in such a way that $p_i = ip_1$, then the diagonal elements of [A] are equal and each is equal to the sum of eigenvalues of [A] divided by the number of eigenvalues. Several formulas have been found to facilitate the calculation of element values of optimal networks of this type as well as those whose eigenvalues have the property of $p_i = i^2 p_1$. Realization of these types of optimal ladders is reduced to mere substitution of the given eigenvalues into these formulas.

While the present thesis has been limited to optimal synthesis of networks whose eigenvalues have certain special types of distribution pattern, future research should include problems whose eigenvalues have arbitrary locations. This thesis has investigated this type of ladder for the n=2 case and formulas have been found that greatly simplify the realization process. It is believed that perhaps formulas can be found for the n=3 case, and for the n=4 case, etc. Sensitivity analysis of these optimal ladders is also an interesting topic for future research.

A worthwhile topic in connection with chain matrix decomposition is the derivation of the necessary and sufficient conditions for the synthesis of general ladder network where $z_i(s) \neq z_j(s)$ and $y_i(s) \neq y_j(s)$.

The ground work of time domain analysis was laid in

Chapter 5. Some distinct properties of the [A] matrix were found; however, no more synthesis methods were found to be of value. Further study may lead to a promising approach to the time domain synthesis.

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