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EXTERIOR-INTERIOR APERTURE COUPLING OF A RECTANGULAR CAVITY WITH WIRE OBSTACLE PHASE II

for

Department of the Air Force
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
Bolling Air Force Base
Washington, D. C. 20332

by

William Arthur Johnson

30 November 1978

ENGINEERING EXPERIMENT STATION
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TUCSON, ARIZONA

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ABSTRACT

In this work, the exterior-interior coupling problem of a cavity backed aperture in a perfectly conducting infinite sheet is considered. The cavity is assumed to be rectangular and contains a perfectly conducting obstacle. For numerical considerations the obstacle is taken to be a straight, thin wire, oriented perpendicular to one of the cavity walls. The problem is formulated in the frequency domain with an $e^{i\omega\tau}$ time dependence.

The dyadic formulation of this vector electromagnetic boundary value problem is given. The controversy over the longitudinal wave functions and their contribution to field dyads is resolved. Specific attention is given to the singularities and completeness of the Green's dyads' eigenfunction expansions.

Numerically tractable integral equations for the aperture electric fields are obtained. A summation method is developed for evaluation of otherwise slowly converging eigenfunction expansions of potential dyads when source and observation points become close. Suggestions for future extensions of this work are discussed.

CHAPTER 1

INTRODUCTION

The problem considered (Figure 1) is the exterior-interior coupling of a cavity backed aperture in a perfectly conducting infinite sheet. This paper extends the work of Seidel [1977, 1978] eliminating the need for his dipole approximations for the aperture fields. The cavity considered (Figure 1) is rectangular with "a" less than or equal to "b", has perfectly conducting cavity walls, and contains a perfectly conducting wire obstacle. In the general formulation (Chapter 4) the obstacle may be of general shape. For numerical considerations however it is taken to be a thin wire parallel to two of the cavity walls. The problem is formulated in the frequency domain with an $e^{i\omega\tau}$ time dependence. In order to obtain the summation formulas of Chapter 6, it is assumed that the frequency of operation is such that the wave number k times the cavity dimension "a" is less than π.

Chapter 2 deals with the dyadic formulation of three dimensional electromagnetic boundary value problems. In particular eigenfunction expansions are given for rectangular cavity dyads. Completeness of the cavity expansions is considered as well as their singularities. Finally the contribution of the longitudinal vector wave functions for free space, cavity, and waveguide dyadic expansions is examined.

In Chapter 3, attention is given to preliminary two dimensional slot diffraction problems. Sample numerical results, illustrating the

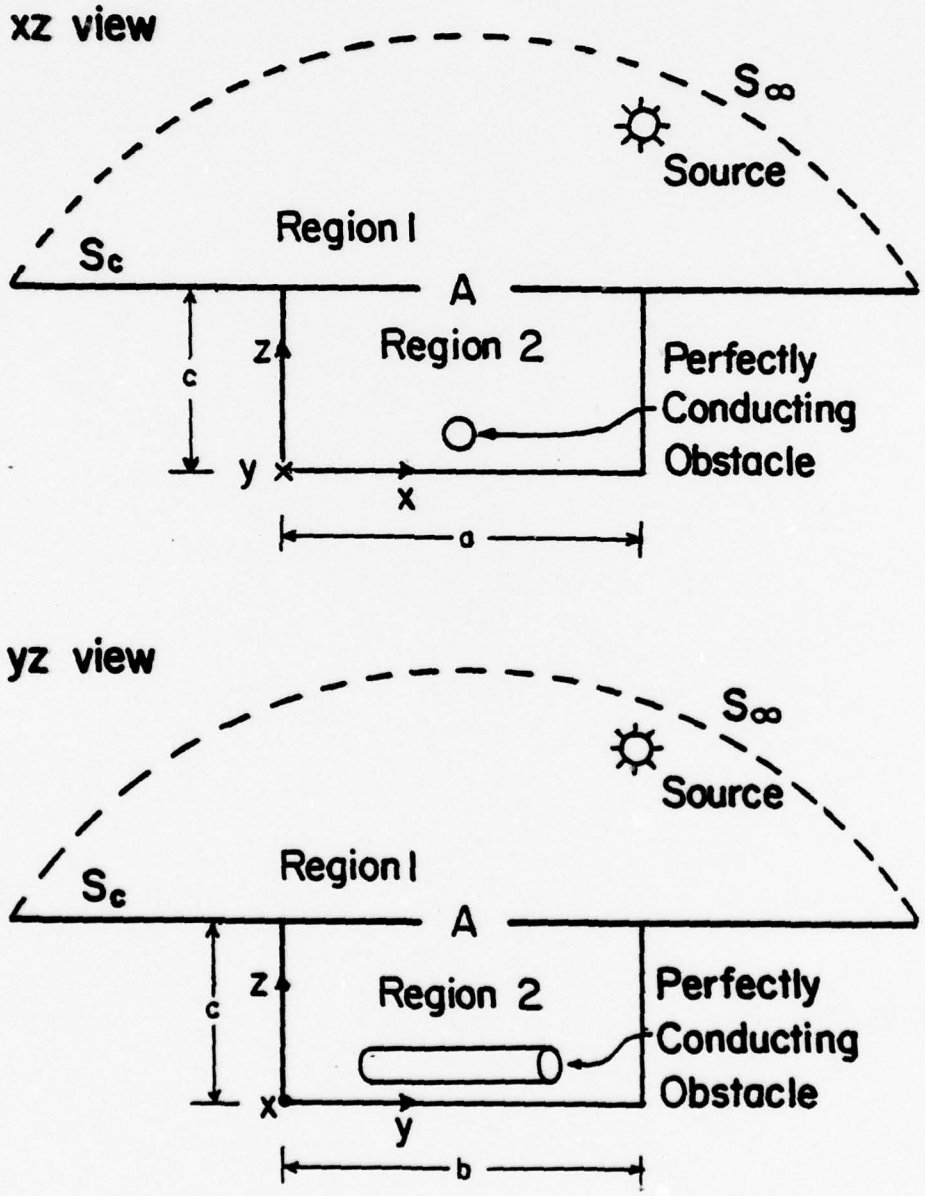


Figure 1. Geometry of Problem

expected field behavior for the three dimensional problem (Figure 1) are given. For one incident field polarization the field's edge behavior has been built into the numerical solution. This illustrates the use of the edge behavior to reduce the matrix size needed to solve the discretized problem. The incorporation of field edge behavior in the three dimensional problem still needs to be studied.

The remaining chapters deal with the numerical formulation of the three dimensional coupling problem of Figure 1. In Chapter 4 techniques of Wilton and Dunaway [1975] are extended to obtain numerically tractable integral equations for aperture fields. In Chapter 5, the method of moments [Harrington 1968] is used to put the problem in matrix form. In the analysis, the wire obstacle has been removed since it has been treated by Seidel [1977, 1978]. Chapter 6 deals with summing otherwise slowly convergent eigenfunction expansions of potential dyads needed for carrying out the numerical procedure of Chapter 5. Finally, Chapter 7 gives conclusions as well as several suggestions for extension of this work.

Background

In recent years problems for which a wire couples to an aperture in an infinite perfectly conducting sheet have been considered [Butler, and Umashankar 1976; Seidel and Butler 1976]. However the image theory needed to apply the techniques put forth in the above papers to the cavity backed aperture problem with obstacle yields a three dimensional array of images [Seidel 1977, p. 22]. Butler, Rahmat-Samii and Mittra [1978] provide a good review of aperture literature.

An alternate method for formulation of the problem utilizes dyadic Green's functions. An expansion for the electric field cavity dyad was found by Weyl [1913, 1915]. Teichmann [1952] found that this expansion was incomplete and provided a correction [Teichmann and Wigner 1953]. The dyad for the magnetic vector potential for the rectangular cavity is formulated in Morse and Feshback [1953, pp. 1849-1851]. More recently these dyads have been formulated by Tai and Rozenfeld [1976] and Rahmat-Samii [1975]. However there has been some controversy over the effect of the longitudinal $\bar{L}$ vector wave function contribution to these field dyads. This point is covered in detail in Chapter 2 and the difficulties resolved. (A review of the appropriate literature is given in Chapter 2.)

Seidel [1977, 1978] has solved the problem (Figure 1) under the small aperture approximation by the use of dipole moments. The major contributions of the present work are the removal of these dipole approximations, resolution of the controversy over the longitudinal wave functions, the derivation of numerically tractable integral equations for the aperture fields, and the summation of the potential dyads' expansions (Chapter 6) when the distance between source and observation points is small.

CHAPTER 2

VECTOR FIELDS AND DYADIC FORMULATION

The problem considered is the computation of steady state ($e^{i\omega\tau}$) electromagnetic fields in a region with known sources. The region's boundary (Figure 1) is assumed to be one of the following: the sphere at infinity, or a perfectly conducting surface S_c with apertures, or part of the sphere at infinity plus a perfectly conducting surface S_c with apertures.

Maxwell's equations for steady state electric sources in a homogeneous, linear, isotropic medium may be written as:

$$\nabla \times \bar{\hat{E}} = \bar{H} \tag{2.1a}$$

$$\nabla \times \bar{H} = k^2 \bar{\hat{E}} + \bar{J} \tag{2.1b}$$

$$\bar{\hat{E}} = \frac{\bar{E}}{-i\omega\mu} \tag{2.1c}$$

$$\nabla \cdot \bar{\hat{E}} = \frac{\rho}{-i\omega\varepsilon\mu} \tag{2.1d}$$

$$\nabla \cdot \bar{H} = 0 \tag{2.1e}$$

The magnetic source analogues are

$$\nabla \times \bar{\hat{H}} = \bar{\hat{E}} \tag{2.2a}$$

$$\nabla \times \bar{\hat{E}} = k^2 \bar{\hat{H}} + \bar{M} \tag{2.2b}$$

$$\bar{\hat{E}} = - \bar{E} \tag{2.2c}$$

$$\bar{\hat{H}} = \frac{\bar{H}}{-i\omega\varepsilon} \tag{2.2d}$$

$$\nabla \cdot \bar{\hat{E}} = 0 \tag{2.2e}$$

$$\nabla \cdot \bar{\hat{H}} = \frac{\rho_m}{-i\omega\varepsilon\mu} \tag{2.2f}$$

The above equations yield the vector wave equations

$$\nabla \times \nabla \times \bar{\hat{E}} - k^2 \bar{\hat{E}} = \bar{J} \tag{2.3a}$$

$$\nabla \times \nabla \times \bar{H} - k^2 \bar{H} = \nabla \times \bar{J} \tag{2.3b}$$

and

$$\nabla \times \nabla \times \bar{\hat{E}} - k^2 \bar{\hat{E}} = \nabla \times \bar{M} \tag{2.4a}$$

$$\nabla \times \nabla \times \bar{\hat{H}} - k^2 \bar{\hat{H}} = \bar{M} \tag{2.4b}$$

for electric and magnetic source excitations, respectively.

A vector analogue of Green's second identity [Stratton 1941, p. 250] will be used to invert these wave equations, viz:

$$\iiint_V (\bar{Q} \cdot \nabla \times \nabla \times \bar{P} - \bar{P} \cdot \nabla \times \nabla \times \bar{Q}) dV$$

$$= \iint_S (\bar{P} \times \nabla \times \bar{Q} - \bar{Q} \times \nabla \times \bar{P}) \cdot \bar{n} \, dS \, , \tag{2.5}$$

where the volume V is bounded by the surface S and $\bar{n}$ is the outward pointing unit surface normal. $\bar{Q}$ will be identified with a field quantity while $\bar{P}$ will be related to a Green's function.

Since the fields are vectors, the Green's function must be a dyadic vector operator. The basic operations and properties associated with dyads are given by Tai [1971, pp. 46-48]. The dyadic version of Green's second identity is

$$\iiint_V [\bar{P} \cdot \nabla \times \nabla \times \bar{\bar{Q}} - (\nabla \times \nabla \times \bar{P}) \cdot \bar{\bar{Q}}]dv$$

$$= - \iint_S \bar{n} \cdot [\bar{P} \times \nabla \times \bar{\bar{Q}} + (\nabla \times \bar{P}) \times \bar{\bar{Q}}]dS \ . \tag{2.6}$$

This follows from (2.5) by setting $\bar{Q} = \bar{\bar{Q}} \cdot \bar{a}$ where $\bar{a}$ is an arbitrary constant vector.

$\bar{E}$ - Field Formulation

To invert the wave equations (2.3) the following dyadic analogues of (2.1) and (2.3) are needed:

$$\nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}') = \bar{\bar{G}}_h(\bar{r}|\bar{r}') \tag{2.7a}$$

$$\nabla \times \bar{\bar{G}}_h(\bar{r}|\bar{r}') = \bar{\bar{I}}\ \delta(\bar{r}-\bar{r}') + k^2\ \bar{\bar{G}}_e(\bar{r}|\bar{r}') \tag{2.7b}$$

$$\nabla \times \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}') - k^2\ \bar{\bar{G}}_e(\bar{r}|\bar{r}') = \bar{\bar{I}}\ \delta(\bar{r}-\bar{r}') \tag{2.7c}$$

$$\nabla \times \nabla \times \bar{\bar{G}}_h(\bar{r}|\bar{r}') - k^2\ \bar{\bar{G}}_h(\bar{r}|\bar{r}') = \nabla \times (\bar{\bar{I}}\ \delta(\bar{r}-\bar{r}')) \tag{2.7d}$$

Inserting $\bar{\hat{E}}$ and $\bar{\bar{G}}_e$ into (2.6) yields

$$\bar{\hat{E}}(\bar{r}') = \iiint_V \bar{J}(\bar{r}) \cdot \bar{\bar{G}}_e(\bar{r}|\bar{r}')dv$$

$$- \iint_S [\bar{n} \times \bar{\hat{E}}(\bar{r})) \cdot \bar{\bar{G}}_h(\bar{r}|\bar{r}') + \bar{n} \cdot (\bar{H}(\bar{r}) \times \bar{\bar{G}}_e(\bar{r}|\bar{r}'))]dS \tag{2.8}$$

In Appendix A it is shown that

$$\tilde{\bar{\bar{G}}}_e(\bar{r}'|\bar{r}) = \bar{\bar{G}}_e(\bar{r}|\bar{r}') \ , \tag{2.9}$$

where "~" signifies the transpose of the dyad. Interchange of the primed and unprimed coordinates in (2.8) and use of (2.9) give

$$\bar{\hat{E}}(\bar{r}) = \iiint_V \bar{\bar{G}}_e(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}')dV'$$

$$- \iint_S [(\bar{n}' \times \bar{\hat{E}}(\bar{r}')) \cdot \bar{\bar{G}}_h(\bar{r}'|\bar{r}) + (\bar{\bar{G}}_e(\bar{r}|\bar{r}') \times \bar{n}') \cdot \bar{H}(\bar{r}')]dS' \tag{2.10}$$

Since the tangential electric field is zero on the perfectly conducting surface S_c and the radiation condition at infinity holds for the fields and Green's functions, equation (2.10) becomes

$$\bar{\hat{E}}(\bar{r}) = \iiint_V \bar{\bar{G}}_e(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}')dV' - \iint_A (\bar{n}' \times \bar{\hat{E}}(\bar{r}')) \cdot \bar{\bar{G}}_h(\bar{r}'|\bar{r})dS', \tag{2.11a}$$

where

$$\bar{n} \times \bar{\bar{G}}_e(\bar{r}|\bar{r}') = 0 \qquad \text{on } S_c \ , \tag{2.11b}$$

$$\bar{n} \times \bar{\hat{E}}(\bar{r}) = 0 \qquad \text{on } S_c\text{-}A \tag{2.11c}$$

A is the aperture portion of S_c and S_c-A is all of S_c except the aperture surface. Note that the fact that (2.9) and (2.11b) imply

$$\bar{\bar{G}}_e(\bar{r}|\bar{r}') \times \bar{n}' = 0 \qquad \text{on } S_c$$

has been used to obtain (2.11a).

$\bar{H}$ - Field Formulation

The inversion of the magnetic source wave equations (2.4) is accomplished similarly. The dyadic analogues of (2.2a-b) and (2.4) are

$$\nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}') = \bar{\bar{g}}_e(\bar{r}|\bar{r}') \tag{2.12a}$$

$$\nabla \times \bar{\bar{g}}_e(\bar{r}|\bar{r}') = k^2\, \bar{\bar{g}}_h(\bar{r}|\bar{r}') + \bar{\bar{I}}\, \delta(\bar{r}-\bar{r}') \tag{2.12b}$$

$$\nabla \times \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}') - k^2\, \bar{\bar{g}}_h(\bar{r}|\bar{r}') = \bar{\bar{I}}\, \delta(\bar{r}-\bar{r}) \tag{2.12c}$$

$$\nabla \times \nabla \times \bar{\bar{g}}_e(\bar{r}|\bar{r}') - k^2\, \bar{\bar{g}}_e(\bar{r}|\bar{r}') = \nabla \times (\bar{\bar{I}}\, \delta(\bar{r}-\bar{r}')) \tag{2.12d}$$

Using duality between the electric and magnetic source driven equations one obtains

$$\bar{\hat{H}}(\bar{r}) = \iiint_V \bar{\bar{g}}_h(\bar{r}|\bar{r}') \cdot \bar{M}(\bar{r}')dV' - \iint_S [(\bar{n}' \times \bar{\hat{H}}(\bar{r}')) \cdot \bar{\bar{g}}_e(\bar{r}'|\bar{r})$$

$$+ (\bar{\bar{g}}_h(\bar{r}|\bar{r}') \times \bar{n}') \cdot \bar{\hat{E}}(\bar{r}')]dS' \,. \tag{2.13}$$

To obtain the above one needs

$$\tilde{\bar{\bar{g}}}_h(\bar{r}'|\bar{r}) = \bar{\bar{g}}_h(\bar{r}|\bar{r}') \tag{2.14}$$

which is shown in Appendix A. The radiation condition at infinity plus

$$\bar{n} \times \bar{\bar{g}}_e(\bar{r}|\bar{r}') = 0 \qquad \text{on } S_c \,, \tag{2.15a}$$

and

$$\bar{n} \times \bar{\hat{E}}(\bar{r}) = 0 \qquad \text{on } S_c\text{-}A \tag{2.15b}$$

imply

$$\bar{\hat{H}}(\bar{r}) = \iiint_V \bar{\bar{g}}_h(\bar{r}|\bar{r}') \cdot \bar{M}(\bar{r}')dv' - \iint_A \bar{\bar{g}}_h(\bar{r}|\bar{r}') \cdot (\bar{n}' \times \bar{\hat{E}}(\bar{r}'))dS' \tag{2.15c}$$

where A is the aperture portion of S_c.

Dyadic Green's Functions for a Rectangular Cavity, Electric Sources

Green's functions for electric sources in a rectangular cavity are constructed, first by use of a dyadic vector potential and then from the $\bar{L}$, $\bar{N}$, $\bar{M}$ vector wave functions. The cavity dimensions are a, b, and c in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions respectively. By (2.7) $\bar{\bar{G}}_h$ is divergence free. Thus it can be expressed as the curl of a vector potential $\bar{\bar{G}}_A$. Maxwell's equations yield [Jackson 1962]

$$-(\nabla^2 + k^2)\, \bar{\bar{G}}_A(\bar{r}|\bar{r}') = \bar{\bar{I}}\, \delta(\bar{r}-\bar{r}') \tag{2.16a}$$

$$-(\nabla^2 + k^2)\, \bar{\Phi} = -\frac{1}{k^2} \nabla\, \delta(\bar{r}-\bar{r}') \tag{2.16b}$$

$$\nabla \cdot \bar{\bar{G}}_A(\bar{r}|\bar{r}') = - k^2 \bar{\Phi} \tag{2.16c}$$

where

$$\bar{\bar{G}}_h(\bar{r}|\bar{r}') = \nabla \times \bar{\bar{G}}_A(\bar{r}|\bar{r}') \tag{2.16d}$$

and

$$\bar{\bar{G}}_e(\bar{r}|\bar{r}') = \bar{\bar{G}}_A - \nabla \Phi = (\bar{\bar{I}} + \frac{\nabla\nabla\cdot}{k^2})\bar{\bar{G}}_A \tag{2.16e}$$

Since the dyadic Green's functions are field responses to sources at $\bar{r}'$, they satisfy electromagnetic field boundary conditions on the perfectly conducting surface S_c, provided $\bar{r}'$ is not on S_c. In all considerations of boundary conditions in this section it is assumed that $\bar{r}'$ is not on S_c. Jones [1974] has shown that

$$\nabla_o \cdot (\bar{n} \times \bar{E}) = i\omega\mu\bar{n}\cdot\bar{H} \tag{2.17}$$

on a surface away from sources; ∇_o is the surface divergence. Thus

$$\bar{n} \times \bar{\bar{G}}_e(\bar{r}|\bar{r}') = 0 \qquad \text{on } S_c \tag{2.18}$$

implies

$$\bar{n} \cdot (\nabla \times \bar{\bar{G}}_A(\bar{r}|\bar{r}')) = 0 \qquad \text{on } S_c \ . \tag{2.19}$$

Note that (2.16e) and (2.18) imply

$$\bar{n} \times \bar{\bar{G}}_A(\bar{r}|\bar{r}') = \bar{n} \times \nabla \bar{\Phi}(\bar{r}|\bar{r}') \qquad \text{on } S_c \ . \tag{2.20}$$

If Dirichlet boundary conditions are imposed on $\bar{\Phi}$, one obtains

$$- (\nabla^2 + k^2)\, \bar{\bar{G}}_A(\bar{r}|\bar{r}') = \bar{\bar{I}}\, \delta(\bar{r}-\bar{r}') \tag{2.21a}$$

$$\left.\begin{aligned} \bar{n} \times \bar{\bar{G}}_A(\bar{r}|\bar{r}') &= 0 \\ \nabla \cdot \bar{\bar{G}}_A(\bar{r}|\bar{r}') &= 0 \end{aligned}\right\} \quad \text{on } S_c \tag{2.21b}$$

From a vector Green's theorem [Morse and Feshbach 1953, p. 1768] one may show that the operator determined by (2.21) is self-adjoint. Thus each component $\bar{\bar{G}}_A \cdot \hat{x}$, $\bar{\bar{G}}_A \cdot \hat{Y}$, and $\bar{\bar{G}}_A \cdot \hat{z}$ may be given in terms of eigenfunction expansions. Seidel [1977] has shown that for $\bar{r}$ not equal to $\bar{r}'$ the operations indicated by (2.16d) and 2.16e) on $\bar{\bar{G}}_A$ may be performed term by term to yield $\bar{\bar{G}}_h$ and $\bar{\bar{G}}_e$. The dyads $\bar{\bar{G}}_A$ and $\bar{\bar{G}}_e$ are given in Table 1.

Not only is $\bar{\bar{G}}_e$ correct away from the source; it is distributionally correct [Stakgold 1968, p. 11]. This fact can be readily demonstrated by reconstructing $\bar{\bar{G}}_e$ in terms of $\bar{L}$, $\bar{M}$, and $\bar{N}$ vector wave functions [Morse and Feshbach 1953, p. 1766]. $\bar{L}$, $\bar{M}$, and N are eigenfunctions of the vector Helmholtz equation (2.21). Although (2.21) has a complete set of eigenfunctions, we will check a completeness relation to insure that all eigenfunctions have been obtained.

Since $\bar{M}$ and $\bar{N}$ are divergence free they are also eigenfunctions of the operator

$$\zeta\bar{E} = \nabla \times \nabla \times \bar{E} \tag{2.22a}$$

Table 1. Electric Source Dyadic Green's Functions*

$$\bar{\bar{G}}_A(\bar{r}|\bar{r}') = \frac{1}{abc} \sum_{\ell,m,n=0}^{\infty} \frac{\varepsilon_m \varepsilon_n \varepsilon_\ell}{K_{\ell mn}^2 - k^2} \begin{Bmatrix} (cc)_x (ss)_y (ss)_z & 0 & 0 \\ 0 & (ss)_x (cc)_y (ss)_z & 0 \\ 0 & 0 & (ss)_x (ss)_y (cc)_z \end{Bmatrix}$$

$$\bar{\bar{G}}_e(\bar{r}|\bar{r}') = \frac{1}{abc} \sum_{\ell,m,n=0}^{\infty} \frac{\varepsilon_m \varepsilon_n \varepsilon_\ell}{K_{\ell mn}^2 - k^2} \begin{Bmatrix} (1 - \frac{k_x^2}{k^2})(cc)_x (ss)_y (ss)_z - \frac{k_x k_y}{k^2} (cs)_x (sc)_y (ss)_z - \frac{k_x k_z}{k^2} (cs)_x (ss)_y (sc)_z \\ - \frac{k_x k_y}{k^2} (sc)_x (cs)_y (ss)_z (1 - \frac{k_y^2}{k^2})(ss)_x (cc)_y (ss)_z - \frac{k_y k_z}{k^2} (ss)_x (cs)_y (sc)_z \\ - \frac{k_x k_z}{k^2} (sc)_x (ss)_y (cs)_z - \frac{k_y k_z}{k^2} (ss)_x (sc)_y (cs)_z \quad (1 - \frac{k_z^2}{k^2})(ss)_x (ss)_y (cc)_z \end{Bmatrix}$$

where $(cc)_x (ss)_y (ss)_z = \cos(k_x x) \cos(k_x x') \sin(k_y y) \sin(k_y y') \sin(k_z z) \sin(k_z z')$, etc.

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}, \quad k_z = \frac{\ell\pi}{c}, \quad K_{\ell mn}^2 = k_x^2 + k_y^2 + k_z^2$$

and

$$\varepsilon_i = \begin{cases} 1, & i = 0 \\ 2, & i \neq 0 \end{cases}$$

*The cavity has lengths a, b, and c in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions respectively.

with boundary condition

$$\bar{n} \times \bar{E} = 0 \qquad \text{on } S_c \,. \tag{2.22b}$$

$\bar{L}$ being curl free is in the null space of ζ. Therefore the vector wave functions are also the eigenfunctions of ζ and may be used to construct $\bar{\bar{G}}_e$.

The vector wave functions are given in Table 2. One may check that over the cavity volume these functions are mutually orthogonal and satisfy

$$\iiint_V |\bar{L}_{\ell mn}(\bar{r})|^2 dV = K^2_{\ell mn} \frac{abc}{\varepsilon_n \varepsilon_m \varepsilon_\ell} \tag{2.23a}$$

$$\iiint_V |\bar{N}_{\ell mn}(\bar{r})|^2 dV = \iiint_V |\bar{M}_\ell(\bar{r})|^2 dV = \frac{(k_x{}^2 + k_y{}^2)abc}{\varepsilon_\ell \varepsilon_n \varepsilon_m} \tag{2.23b}$$

for all non-zero wave functions. In the above

$$\varepsilon_i = \begin{cases} 1 \,, & i = 0 \\ 2 \,, & i \neq 0 \end{cases} .$$

Following Tai and Rozenfeld [1976], we write

$$\bar{\bar{I}}\, \delta(\bar{r}-\bar{r}') = \sum_{\ell,m,n}^{\infty} \frac{\varepsilon_\ell \varepsilon_m \varepsilon_n}{abc} \Big[\frac{\bar{L}_{\ell mn}(\bar{r})\, \bar{L}_{\ell mn}(\bar{r}')}{K^2_{\ell mn}}$$

$$+ \frac{1}{(k_x^2 + k_y^2)} (\bar{M}_{\ell mn}(\bar{r})\, \bar{M}_{\ell mn}(\bar{r}') + \bar{N}_{\ell mn}(\bar{r})\, \bar{N}_{\ell mn}(\bar{r}'))\Big] \,. \tag{2.24}$$

Table 2. Vector Wave Functions for Electric Sources

$$\bar{L}_{\ell mn} = \nabla\,(s_x s_y s_z)$$

$$= k_x c_x s_y s_z \hat{x} + k_y s_x c_y s_z \hat{y} + k_z s_x s_y c_z \hat{z}$$

$$\bar{M}_{\ell mn} = \nabla \times (c_x c_y s_z \hat{z}) = -\,k_y c_x s_y s_z \hat{x} + k_x s_x c_y s_z \hat{y}$$

$$\bar{N}_{\ell mn} = \frac{1}{K_{\ell mn}} \nabla \times \nabla \times (s_x s_y c_z \hat{z})$$

$$= -\,\frac{k_x k_z}{K_{\ell mn}} c_x s_y s_z \hat{x} - \frac{k_y k_z}{K_{\ell mn}} s_x c_y s_z \hat{y} + \left(\frac{k_x^2 + k_y^2}{K_{\ell mn}}\right) s_x s_y c_z \hat{z}$$

where

$$c_x s_y s_z = \cos(k_x x)\,\sin(k_y y)\,\sin(k_z z)\ ,\ \text{etc.}$$

$$k_x = \frac{m\pi}{a}\ ,\ k_y = \frac{n\pi}{b}\ ,\ k_z = \frac{\ell\pi}{c}\ ,\ K_{\ell mn}^2 = k_x^2 + k_y^2 + k_z^2$$

$$m = 0,1,2,\ldots\ ,\ n = 0,1,2,\ldots\ ,\ \text{and}\ \ell = 0,1,2,\ldots\ .$$

Inserting the vector wave functions from Table 2 into (2.24) gives

$$\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}') = \sum_{\ell,m,n=0}^{\infty} \frac{\varepsilon_\ell \varepsilon_m \varepsilon_n}{abc} [(cc)_x (ss)_y (ss)_z\, \hat{x}\hat{x}$$

$$+ (ss)_x (cc)_y (ss)_z\, \hat{y}\hat{y} + (ss)_x (ss)_y (cc)_z\, \hat{z}\hat{z}] \,, \qquad (2.25)$$

where $(cc)_x (ss)_y (ss)_z$

$$= \cos(k_x x)\, \cos(k_x x')\, \sin(k_y y)\, \sin(k_y y')\, \sin(k_z z)\, \sin(k_z z') \,,$$

etc.

The above equation is known to be true by scalar spectral theory. Note that dividing each term of (2.25) by $K^2_{\ell mn} - k^2$ yields $\bar{\bar{G}}_A$ in Table 1. Since $\bar{L}$, $\bar{M}$, and $\bar{N}$ are also eigenfunctions of the operator ζ in (2.22)

$$\bar{\bar{G}}_e(\bar{r}|\bar{r}') = \sum_{\ell,m,n}^{\infty} \frac{\varepsilon_\ell \varepsilon_m \varepsilon_n}{abc} [\frac{\bar{L}_{\ell mn}(\bar{r})\, \bar{L}_{\ell mn}(\bar{r}')}{-k^2\, K^2_{\ell mn}}$$

$$+ \frac{\bar{M}_{\ell mn}(\bar{r})\, \bar{M}_{\ell mn}(\bar{r}') + \bar{N}_{\ell mn}(\bar{r})\, \bar{N}_{\ell mn}(\bar{r}')}{(K_x^2 + K_y^2)\,(K^2_{\ell mn} - k^2)}] \,. \qquad (2.26)$$

Inserting the vector wave functions yields the same $\bar{\bar{G}}_e$ as in Table 1. Thus the vector potential approach gives the correct answer even at the source point. The sum over the $\bar{L}$ functions yields a term proportional to the longitudinal δ-function [Morse and Feshbach 1953, p. 1781]. The second term determines a dyad whose entries are square integrable

over the cavity volume. This characterization of the singularities of $\bar{\bar{G}}_e$ may prove useful in later numerical considerations.

Dyadic Green's Functions for a Rectangular Cavity, Magnetic Sources

Dyadic Green's functions for magnetic sources in a rectangular cavity are now constructed. The cavity dimensions are the same as in the last section. By (2.12) $\bar{\bar{g}}_e$ is divergence free and may be expressed as the curl of an electric vector potential. As before Maxwell's equations yield

$$- (\nabla^2 + k^2)\, \bar{\bar{g}}_F(\bar{r}|\bar{r}') = \bar{\bar{I}}\, \delta(\bar{r}-\bar{r}') \tag{2.27a}$$

$$(\nabla^2 + k^2)\, \bar{\Phi}(\bar{r}|\bar{r}') = \frac{\nabla\delta(\bar{r}-\bar{r}')}{k^2} \tag{2.27b}$$

$$\nabla \cdot \bar{\bar{g}}_F(\bar{r}|\bar{r}') = - k^2\, \bar{\Phi} \tag{2.27c}$$

where

$$\bar{\bar{g}}_e(\bar{r}|\bar{r}') = \nabla \times \bar{\bar{g}}_F(\bar{r}|\bar{r}') \tag{2.27d}$$

and

$$\bar{\bar{g}}_h(\bar{r}|\bar{r}') = \bar{\bar{g}}_F(\bar{r}|\bar{r}') - \nabla\, \bar{\Phi}(\bar{r}|\bar{r}') = (\bar{\bar{I}} + \frac{\nabla\nabla\cdot}{k^2})\, \bar{\bar{g}}_F(\bar{r}|\bar{r}') \ . \tag{2.27e}$$

For the δ-source away from the perfectly conducting surface we have

$$\bar{n} \times (\nabla \times \bar{\bar{g}}_F) = 0 \qquad \text{on } S_c \ . \tag{2.28}$$

Thus, (2.27e), and (2.17) yield

$$\bar{n} \cdot \bar{\bar{g}}_F - \frac{\partial \bar{\Phi}}{\partial n} = \bar{n} \cdot \bar{\bar{g}}_h = 0 \quad \text{on } S_c \ . \tag{2.29}$$

If homogeneous Neumann boundary conditions are imposed on $\bar{\Phi}$ one obtains

$$- (\nabla^2 + k^2)\ \bar{\bar{g}}_F(\bar{r}|\bar{r}') = \bar{\bar{I}}\ \delta(\bar{r}-\bar{r}') \tag{2.30a}$$

$$\left.\begin{aligned} \bar{n} \times (\nabla \times \bar{\bar{g}}_F(\bar{r}|\bar{r}')) &= 0 \\ \bar{n} \cdot \bar{\bar{g}}_F(\bar{r}|\bar{r}') &= 0 \end{aligned}\right\} \quad \text{on } S_c \ . \tag{2.30b}$$

Using a vector Green's theorem [Morse and Feshbach 1953, p. 1768] one may show that a self-adjoint operator is associated with (2.30). Thus $\bar{\bar{g}}_F$ may be expanded in terms of a complete set of eigenfunctions. This result is given in Table 3. Again Seidel [1977, p. 12] has shown that $\bar{\bar{g}}_e$ and $\bar{\bar{g}}_h$ may be obtained by differentiating $\bar{\bar{g}}_F$ term by term provided $\bar{r}$ is not equal to $\bar{r}'$.

We now show that a complete set of eigenfunctions of (2.30) are the vector wave functions $\bar{L}$, $\bar{M}$, and $\bar{N}$. The vector wave functions are given in Table 4. Since $\bar{M}$ and $\bar{N}$ are divergence free, they are also eigenfunctions of the operator

$$\theta\bar{H} = \nabla \times \nabla \times \bar{H} \tag{2.31a}$$

with boundary condition

$$\bar{n} \times (\nabla \times \bar{H}) = 0 \qquad \text{on } S_c \ . \tag{2.31b}$$

Table 3. Magnetic Source Dyadic Green's Functions*

$$\bar{\bar{g}}_F(\bar{r}|\bar{r}') = \frac{1}{abc} \sum_{\ell,m,n=0}^{\infty} \frac{\varepsilon_n \varepsilon_m \varepsilon_\ell}{K_{\ell mn}^2 - k^2} \begin{Bmatrix} (ss)_x(cc)_y(cc)_z & 0 & 0 \\ 0 & (cc)_x(ss)_y(cc)_z & 0 \\ 0 & 0 & (cc)_x(cc)_y(ss)_z \end{Bmatrix}$$

$$\bar{\bar{g}}_h(\bar{r}|\bar{r}') = \frac{1}{abc} \sum_{\ell,m,n=0}^{\infty} \frac{\varepsilon_n \varepsilon_m \varepsilon_\ell}{K_{\ell mn}^2 - k^2} \begin{Bmatrix} (1 - \frac{k_x^2}{k^2})(ss)_x(cc)_y(cc)_z & -\frac{k_x k_y}{k^2}(sc)_x(cs)_y(cc)_z & -\frac{k_x k_z}{k^2}(sc)_x(cc)_y(cs)_z \\ -\frac{k_x k_y}{k^2}(cs)_x(sc)_y(cc)_z & (1 - \frac{k_y^2}{k^2})(cc)_x(ss)_y(cc)_z & -\frac{k_y k_x}{k^2}(cc)_x(sc)_y(cs)_z \\ -\frac{k_x k_z}{k^2}(cs)_x(cc)_y(sc)_z & -\frac{k_z k_x}{k^2}(cs)_x(cc)_y(sc)_z & (1 - \frac{k_z^2}{k^2})(cc)_x(cc)_y(ss)_z \end{Bmatrix}$$

*The notation is explained in Table 1.

Table 4. Vector Wave Functions for Magnetic Sources*

$$\bar{L}_{\ell mn} = \nabla(c_x c_y c_z)$$

$$= - k_x s_x c_y c_z \hat{x} - k_y c_x s_y c_z \hat{y} - k_z c_x c_y s_z \hat{z}$$

$$\bar{M}_{\ell mn} = \nabla \times (s_x s_y c_z \hat{z})$$

$$= k_y s_x c_y c_z \hat{x} - k_x c_x s_y c_z \hat{y}$$

$$\bar{N}_{\ell mn} = \frac{1}{K_{\ell mn}} \nabla \times \nabla \times (c_x c_y s_z \hat{z})$$

$$= - \frac{k_x k_z}{K_{\ell mn}} s_x c_y c_z \hat{x} - \frac{k_y k_z}{K_{\ell mn}} c_x s_y c_z \hat{y} + \left(\frac{k_y^2 + k_x^2}{K_{\ell mn}}\right) c_x c_y s_z \hat{z}$$

*The notation is the same as in Table 2.

The curl free $\bar{L}$ vectors are in the null space of θ. It may be checked that the $\bar{L}$, $\bar{M}$, and $\bar{N}$ vectors are mutually orthogonal and satisfy

$$\iiint_V |\bar{L}|^2 dv = K^2_{\ell mn} \frac{abc}{\varepsilon_m \varepsilon_n \varepsilon_\ell} \tag{2.32a}$$

$$\iiint_V |\bar{M}|^2 dv = \iiint_V |\bar{N}|^2 dv = (k_x^2 + k_y^2) \frac{abc}{\varepsilon_m \varepsilon_n \varepsilon_\ell} \tag{2.32b}$$

for all non-zero vectors. Following Tai and Rozenfeld [1976] the completeness relation may be expressed as

$$\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}') = \frac{1}{abc} \sum_{\ell,m,n} \varepsilon_\ell \varepsilon_n \varepsilon_m \left[\frac{\bar{L}_{\ell mn}(\bar{r})\bar{L}_{\ell mn}(\bar{r}')}{K^2_{\ell mn}} + \frac{\bar{M}_{\ell mn}(\bar{r})\,\bar{M}_{\ell mn}(\bar{r}') + \bar{N}_{\ell mn}(\bar{r})\,N_{\ell mn}(\bar{r}')}{k_x^2 + k_y^2} \right] . \tag{2.33}$$

This may be verified by inserting the vector wave functions from Table 4.

$$\bar{\bar{I}}\,\delta(\bar{r}-\bar{r}') = \frac{1}{abc} \sum_{\ell,m,n=0}^{\infty} \varepsilon_\ell \varepsilon_m \varepsilon_n \begin{bmatrix} (ss)_x (cc)_y (cc)_z & 0 & 0 \\ 0 & (cc)_x (ss)_y (cc)_z & 0 \\ 0 & 0 & (cc)_x (cc)_y (ss)_z \end{bmatrix} \tag{2.34}$$

Since we have shown the vector wave functions to be eigenfunctions of the operator θ in (2.31),

$$\bar{\bar{g}}_h(\bar{r}|\bar{r}') = \frac{1}{abc} \sum_{\ell mn} \varepsilon_\ell \varepsilon_m \varepsilon_n \left[\frac{- \bar{L}_{\ell mn}(\bar{r})\ \bar{L}_{\ell mn}(\bar{r}')}{k^2\ K^2_{\ell mn}} \right.$$

$$\left. + \frac{\bar{M}_{\ell mn}(\bar{r})\ \bar{M}_{\ell mn}(\bar{r}') + \bar{N}_{\ell mn}(\bar{r})\ \bar{N}_{\ell mn}(\bar{r}')}{(k_x^2 + k_y^2)\ \ (K^2_{\ell mn} - k^2)} \right] \tag{2.35}$$

The first term is proportional to the longitudinal dyadic delta function, while the second part determines square integrable functions over the cavity volume. Substitution of the vector wave functions from Table 4 in (2.35) yields $\bar{\bar{g}}_h$ in Table 3. Thus the expression for $\bar{\bar{g}}_h$ is also valid at $\bar{r}$ equals $\bar{r}'$.

The Longitudinal Component of the Electric Green's Dyad

This section is concerned with two intimately related topics, the characterization of the electric Green's dyad's singularity and the effect of neglecting the longitudinal vector wave functions in the dyad's eigenfunction expansion.

Hansen [1935] first used spherical coordinate vector wave functions to expand a free space potential Green's function. The expansion given was an eigenfunction expansion in the angular variables and closed form in the radial variable. Hansen found that the corresponding electric field expansion, exterior to a finite sphere containing all sources, contained no longitudinal vector wave functions.

Morse and Feshbach [1953] use vector wave functions to represent a potential dyadic Green's function as a full eigenfunction expansion. The unit delta dyadic is broken into transverse and longitudinal components. The transverse component is divergence free and the longitudinal component is curl free. It is implied [Morse and Feshbach 1953, p. 1781] that away from the source the transverse and longitudinal components of the unit delta dyadic vanish identically.

Tai [1971] has no longitudinal vector wave functions in the full eigenfunction expansion of the electric Green's dyad. Subsequently he concludes [Tai 1973] that there is a missing longitudinal vector wave contribution, specifically, $-\hat{z}\hat{z}\,\delta(\bar{r}-\bar{r}')/k^2$.

Collin [1973] takes a different approach, correcting for the incompleteness of the $\bar{E}$ and $\bar{H}$ model expansions in a waveguide. His correction term is the same as Tai's $\hat{z}\hat{z}$ delta function term. In his conclusion Collin alludes to a private communication with Tai and implies that this $\hat{z}\hat{z}$ delta correction term must be added to the transverse vector wave functions to obtain the complete dyadic Green's function.

Howard [1974] uses a cylindrical vector wave function expansion (the sin $m\Phi$ terms should be added for completeness) to produce a closed form expression for the longitudinal vector wave function contribution to the full Green's dyadic eigenfunction expansion. This term does not identically vanish away from the source, apparently contradicting the previously cited literature.

Tai and Rozenfeld [1976] correctly state that the longitudinal vector wave functions in the full dyadic expansion contribute not only

at the source point but away from the source as well. But then later they seem to imply, as Tai [1973], that the longitudinal vector wave function contribution is a $-\hat{z}\hat{z}\,\delta(\bar{r}-\bar{r}')/k^2$ term. These two statements are in contradiction.

Several papers deal with the singular nature of the electric Green's dyadic. Van Bladel [1961] expresses the free space electric Green's dyadic in terms of a spherical principal value plus a delta function type singularity. Rahmat-Samii [1975] extracts different singularities from cavity and waveguide dyadics. Yaghjian [1978] used "principal volumes" and delta function type singularities to characterize the electric Green's dyadic's singularity. He notes that different "principal volumes" and their associated different delta type singularities have a sum total singularity which is the same irrespective of the "principal volume" chosen.

Some of the confusion in the literature may be rectified by clearly stating which longitudinal vector wave functions are being used. However, the papers of Howard [1974], Tai [1973], and Tai and Rozenfeld [1973] still remain in contradiction. Second it is physically surprising that Rahmat-Samii [1975] has extracted different singularities for cavity and waveguide problems.

It is rigorously shown that the dominant singularity of the electric Green's dyad is proportional to the longitudinal component of the unit delta dyad. Contrary to Morse and Feshbach [1953], this term does not vanish identically away from the source. The longitudinal vector wave functions in the full eigenfunction expansion of the Green's

dyad not only contribute at source points but away from the source as well. The full eigenfunction expansion is useful for two reasons. First spectral theory will guarantee the completeness of the eigenfunction expansion. Second if one desires to compute the Green's dyad at different spacial locations, the full expansion allows one the flexibility to close in a manner insuring the greatest exponential convergence. Howard's central result for free space [Howard 1974, eq. 13] will be mathematically verified and shown to be unique. The apparent controversy on the effect of the longitudinal vector wave function contribution will be resolved.

The free space electric Green's dyadic is studied first. Results are later generalized to cavity and waveguide problems. The free space electric Green's dyad, $\bar{\bar{G}}_e$, satisfies (2.7c) plus the Sommerfeld radiation condition. $\bar{\bar{G}}_e$ is obtained by vector wave function expansions. Expansion of the unit delta dyadic yields [Morse and Feshbach 1953]

$$\bar{\bar{I}}\delta(\bar{r}-\bar{r}') = \sum_m \frac{\bar{L}_m(\bar{r})\ \bar{L}_m^*(\bar{r}')}{\Lambda_m} + \sum\left[\frac{\bar{M}_m(\bar{r})\ \bar{M}_m^*(\bar{r}')}{\Lambda_m'} + \frac{\bar{N}_m(\bar{r})\ \bar{N}_m^*(\bar{r}')}{\Lambda_m''}\right]$$

$$= \bar{\bar{D}}_L(\bar{r}-\bar{r}') + \bar{\bar{D}}_T(\bar{r}-\bar{r}') \tag{2.36}$$

where Λ_m, Λ_m' and Λ_m'' are normalization constants. The above sum may be over a continuous or discrete set of m. The first term is the longitudinal component of the delta dyad; the second is transverse. By spectral theory

$$\bar{\bar{G}}_e(\bar{r}|\bar{r}') = \frac{-\bar{\bar{D}}_L(\bar{r}-\bar{r}')}{k^2}$$

$$+ \sum \left[\frac{\bar{M}_m(\bar{r})\ \bar{M}_m^*(\bar{r}')}{\Lambda_m'} + \frac{\bar{N}_m(\bar{r})\ \bar{N}_m^*(\bar{r}')}{\Lambda_m''} \right] \frac{1}{k_m^2 - k^2} \tag{2.37}$$

One may show that the second term is square integrable over finite volumes by noting the corresponding cavity sum is square integrable and that the difference of the cavity and free space sum is analytic. Thus the dominant singularity of $\bar{\bar{G}}_e$ is proportional to $\bar{\bar{D}}_L(\bar{r}-\bar{r}')$, the longitudinal component of the unit delta dyadic. Morse and Feshbach [1953] indicate that $\bar{\bar{D}}_L(\bar{r}-\bar{r}')$ is identically zero for $\bar{r}$ not equal to $\bar{r}'$. This implies that the $\bar{L}$ functions may be neglected away from the source. However we proceed to show that $\bar{D}_L(\bar{r}-\bar{r}')$ does not vanish identically away from the source. This is done by two independent proofs.

First a proof by Helmholtz's theorem is given. Following Howard [1974]

$$\bar{\bar{I}}\ \delta(\bar{r}-\bar{r}') = \nabla^2 \frac{\bar{\bar{I}}}{4\pi|\bar{r}-\bar{r}'|} = \nabla \times \nabla \times \frac{\bar{\bar{I}}}{4\pi|\bar{r}-\bar{r}'|} - \nabla\nabla \frac{1}{4\pi|\bar{r}-\bar{r}'|} \tag{2.38}$$

The first term is transverse; the second longitudinal. One expects that

$$\bar{\bar{D}}_L(\bar{r}-\bar{r}') = -\nabla\nabla \frac{1}{4\pi|\bar{r}-\bar{r}'|} \tag{2.39}$$

A mathematical proof of (2.39) is now given. It is also shown that this dyadic distribution is unique.

First note that any dyadic distribution is uniquely determined by its operation on infinitely differentiable testing vectors $\bar{F}(\bar{r})$ with compact support. This may be seen by putting the dyad in matrix form and $\bar{F}$ in vector form and the use of scalar distribution theory. The Helmholtz theorem [Morse and Feshbach 1953] shows

$$\bar{F}(\bar{r}) = \nabla \Phi + \nabla \times \bar{A} \; ; \quad \nabla \cdot \bar{A} = 0 \tag{2.40a}$$

where

$$\Phi(\bar{r}) = - \iiint \frac{\nabla \cdot \bar{F}(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|} \, dv' \tag{2.40b}$$

and

$$\bar{A}(\bar{r}) = \iiint \frac{\nabla \times \bar{F}(r')}{4\pi|\bar{r}-\bar{r}'|} \, dv' \tag{2.40c}$$

Note that surface terms in (2.40a) and (2.40b) are not present since $\bar{F}$ has compact support. Thus the longitudinal part of the test vector $\bar{F}$ is

$$\bar{F}_L(\bar{r}) = - \nabla \iiint \frac{\nabla' \cdot \bar{F}(\bar{r}')}{4\pi|\bar{r}-\bar{r}'|} \, dv' \tag{2.41a}$$

Kellogg [1953] shows the gradient may be brought inside the integral; thus

$$\bar{F}_L(\bar{r}) = - \iiint [\nabla' \cdot \bar{F}(\bar{r}')] \nabla \frac{1}{4\pi|\bar{r}-\bar{r}'|} dv' \tag{2.41b}$$

From distribution theory

$$\bar{F}_L(\bar{r}) = - \iiint \bar{F}(\bar{r}') \cdot \nabla\nabla \frac{1}{4\pi|\bar{r}-\bar{r}'|} dv' \tag{2.41c}$$

Since a distribution is determined uniquely by its operation on testing vectors $\bar{F}(\bar{r})$,

$$\bar{D}_L(\bar{r}-\bar{r}') = - \nabla\nabla \frac{1}{4\pi|\bar{r}-\bar{r}'|} \tag{2.42}$$

is uniquely determined. Direct computation shows that $\bar{D}_L(\bar{r}-\bar{r}')$ does not vanish identically away from the source.

An alternate constructive proof of the above result (2.42) is now given. Expansion of the free space delta dyadic in terms of cartesian coordinate vector wave functions, (2.36), and orthogonality yields

$$\bar{\bar{I}}\ \delta(\bar{r}-\bar{r}') = \bar{\bar{D}}_L(\bar{r}-\bar{r}') + \bar{\bar{D}}_T(\bar{r}-\bar{r}') \tag{2.43a}$$

where

$$\bar{\bar{D}}_L(\bar{r}-\bar{r}') = \frac{1}{(2\pi)^3} \iiint \frac{\bar{L}(\bar{r})\ \bar{L}^*(\bar{r})}{k_x^2 + k_y + k_z} dk_x\ dk_y\ dk_z \tag{2.43b}$$

$$\bar{\bar{D}}_T(\bar{r}-\bar{r}') = \frac{1}{(2\pi)^3} \iiint \frac{\bar{M}(\bar{r})\ \bar{M}^*(\bar{r}') + \bar{N}(\bar{r})\ \bar{N}^*(\bar{r}')}{k_x + k_y} dk_x\ dk_y\ dk_z \tag{2.43c}$$

$$\bar{L}(\bar{r}) = \nabla \psi \ , \ \bar{M}(\bar{r}) = \nabla \times (\hat{z}\ \psi), \ \bar{N}(\bar{r}) = \frac{\nabla \times \bar{M}(\bar{r})}{\sqrt{k_x^2 + k_y^2 + k_z^2}} \tag{2.43d}$$

and

$$\psi = e^{-i\bar{K}\cdot\bar{r}} \ , \ \bar{K} = k_x\ \hat{x} + k_y\ \hat{y} + k_z\ \hat{z}$$

By direct computation

$$\bar{\bar{D}}_L(\bar{r}-\bar{r}') = \frac{-1}{(2\pi)^3} \iiint \begin{bmatrix} k_x^2 & k_x k_y & k_x k_z \\ k_x k_y & k_y^2 & k_y k_z \\ k_x k_z & k_y k_z & k_z^2 \end{bmatrix} \frac{e^{-i\bar{K}\cdot(\bar{r}-\bar{r}')}}{k_x^2 + k_y^2 + k_z^2}\, dk_x\, dk_y\, dk_z \tag{2.44a}$$

$$= \frac{-1}{(2\pi)^3} \nabla\nabla \iiint \frac{e^{-i\bar{K}\cdot(\bar{r}-\bar{r}')}}{k_x^2 + k_y^2 + k_z^2}\, dk_x\, dk_y\, dk_z \tag{2.44b}$$

$$= -\ \nabla\nabla \frac{1}{4\pi|\bar{r}-\bar{r}'|} \tag{2.44c}$$

which is identical to (2.39).

Although the previous discussion was restricted to free space one may suspect analogous results to hold for cavity and waveguide problems. This is indeed the case as is now shown. The electric field cavity dyadic $\bar{\bar{G}}_e^c$ is given by (2.16e) where $\bar{\bar{G}}_A^c$ satisfies (2.21). The superscript "c" has been added to cavity dyads to avoid confusion with free space dyads. The boundary value problem (2.21) is self-adjoint

and has a complete set of associated eigenfunctions. These same wave functions may be used to expand the delta dyad and $\bar{\bar{G}}_e^c$ as in (2.36) and (2.37). Although Rahmat-Samii [1975] extracted a delta singularity from $\bar{\bar{G}}_e^c$, its true dominant singularity is $\bar{\bar{D}}_L^c(\bar{r}-\bar{r}')$ defined analogously to (2.36).

As a device to study $\bar{\bar{D}}_L^c(\bar{r}-\bar{r}')$ we consider $\bar{\bar{G}}_A^c$ in (2.16e) with k set equal to zero, $\bar{\bar{G}}_{A_o}^c$. Expanding in vector wave functions

$$\bar{\bar{G}}_{A_o}^c(r\ r') = \sum_m \frac{\bar{L}_m^c(\bar{r})\ \bar{L}_m^c(\bar{r}')^*}{\Lambda_m\ k_m^2} + \sum_m \left[\frac{\bar{M}_m^c(\bar{r})\ \bar{M}_m^c(\bar{r}')^*}{\Lambda_m'\ k_m^2} + \frac{\bar{N}_m^c(\bar{r})\ \bar{N}_m^c(\bar{r}')^*}{\Lambda_m\ k_m^2}\right] \tag{2.45}$$

From (2.16e)

$$\nabla^2 \left(\bar{\bar{G}}_A^c(\bar{r}|\bar{r}') - \frac{\bar{\bar{I}}}{4\pi|\bar{r}-\bar{r}'|}\right) \equiv 0 \tag{2.46}$$

interior to the cavity or waveguide. From the ellipticity of the above operator it follows that the difference of the two potentials is analytic [Treves, 1975]. However

$$-\nabla\nabla \cdot \left(\bar{\bar{G}}_A^c(\bar{r}|\bar{r}') - \frac{\bar{\bar{I}}}{4\pi|r-r'|}\right) = \sum_m \frac{\bar{L}_m^c(\bar{r})\ \bar{L}_m^c(\bar{r}')^*}{\Lambda_m} - \bar{\bar{D}}_I(\bar{r}-\bar{r}') \tag{2.47a}$$

$$= \bar{\bar{D}}_L^c(\bar{r}-\bar{r}') - \bar{\bar{D}}_L(\bar{r}-\bar{r}') \tag{2.47b}$$

To obtain (2.47a) term by term differentiation was used which is distributionally correct [Stakgold 1968]. Since the left side of (2.47a) is analytic it follows that $\bar{\bar{D}}_L^c(\bar{r}-\bar{r}')$ has the same singularity as $\bar{\bar{D}}_L(\bar{r}-\bar{r}')$. Thus $\bar{\bar{D}}_L^c(\bar{r}-\bar{r}')$ does not vanish identically for $\bar{r}\neq\bar{r}'$. Although the above results are for $\bar{r},\bar{r}'$ in the cavity or waveguide interior, by use of an image term one can obtain similar results for $\bar{r}'$ on a wall.

Different representations of the electric Green's dyadic are now considered. The electric Green's dyad may be expressed as an eigenfunction expansion in two variables and closed form in the third. As previously stated, this is the form in which Hansen [1935] first introduced the vector wave function expansions and it was found that no longitudinal vector wave functions were needed outside of a sphere containing all sources. It will now be shown that the effect of neglecting the longitudinal wave functions in this form is precisely the $\hat{z}\hat{z}$ delta function correction term [Tai, 1973; Tai and Rozenfeld 1976; Collin, 1973]. Although free space is considered, results for waveguides follow analogously.

Closing (2.44b) in the complex k_z plane and then differentiating yields

$$\bar{\bar{D}}_L(\bar{r}-\bar{r}') = \iint \begin{bmatrix} k_x^2 & k_x k_y & -ik_x k_{cs} \\ k_x k_y & k_y^2 & -ik_y k_{cs} \\ -ik_x k_{cs} & -ik_y k_{cs} & -k_c^2 \end{bmatrix} \cdot$$

$$\cdot \frac{e^{-ik_x(x-x')} \; e^{-ik_y(y-y')} \; e^{-ik_c|z-z'|}}{2k_c} \frac{dk_x \, dk_y}{(2\pi)^2} + \hat{z}\hat{z} \; \delta(\bar{r}-\bar{r}') \tag{2.48}$$

where

$$k_c = \sqrt{k_x^2 + k_y^2} \quad \text{and} \quad k_{cs} = k_c \; \mathrm{sgn}(z-z')$$

Closing the transverse part of $\bar{\bar{G}}_e$ (2.37) in the complex k_z plane yields

$$\bar{\bar{G}}_{e_T}(\bar{r}|\bar{r}') = i \iint \begin{bmatrix} k_y^2 & -k_x k_y & 0 \\ -k_x k_y & k_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

$$\cdot \frac{e^{-ik_x(x-x')} \; e^{-ik_y(y-y')} \; e^{ik_g|z-z'|}}{2k_c^2 \; k_g} \frac{dk_x \, dk_y}{(2\pi)^2}$$

$$+ \iint \begin{bmatrix} k_x^2 & k_x k_y & -ik_x k_{cs} \\ k_x k_y & k_y^2 & -ik_y k_{cs} \\ -ik_x k_{cs} & -ik_y k_{cs} & k_c^2 \end{bmatrix}$$

$$\cdot \frac{e^{-ik_x(x-x')} \; e^{-ik_y(y-y')} \; e^{-k_c|z-z'|}}{2k_c \; k^2} \frac{dk_x \, dk_y}{(2\pi)^2} +$$

$$+ i \iint \begin{bmatrix} k_x^2 k_g^2 & k_x k_y k_g^2 & k_x k_{cs} k_c k_g \\ k_x k_y k_g^2 & k_y^2 k_g^2 & k_y k_{cs} k_c k_g \\ k_x k_{cs} k_c k_g & k_y k_{cs} k_c k_g & k_c^4 \end{bmatrix}$$

$$\cdot \frac{e^{-ik_x(x-x')} \; e^{-ik_y(y-y')} \; e^{ik_g|z-z'|}}{2k^2 \; k_c^2 \; k_g} \frac{dk_x \; dk_y}{(2\pi)^2} \qquad (2.49)$$

where $k_g = \sqrt{k^2 - k_c^2}$. Recalling (2.37), we see that the second term of (2.49), the static pole contribution, cancels with the contribution from the first term of (2.48). The remaining terms of $\bar{\bar{G}}_e$ are just the closed form in z, eigenfunction in x and y, $\bar{M}$ and $\bar{N}$ expantion function contributions plus the famous $- \hat{z}\hat{z}\, \delta(\bar{r}-\bar{r}')/k^2$ term.

In conclusion the true dominant singularity of the electric Green's dyadic is proportional to the longitudinal component of the unit delta dyadic. Contrary to Morse and Feshbach [1953], this term does not identically vanish away from the source. Rahmat-Samii [1975] only extracted part of the singularity from his cavity and waveguide dyads, their true dominant singularities are the same away from the cavity or waveguide walls. At the walls, image terms must be considered. The longitudinal $\bar{L}$ vector wave functions in the full eigenfunction expansion of the electric Green's dyad contribute not only at the source but away from the source point as well. The full eigenfunction expansion is useful for two reasons, one theoretical and one practical. First spectral theory guarantees the completeness of the electric Green's dyadic

expansion. Second if one desires to compute the Green's dyad away from the source, one has the flexibility to close in the variable which gives fastest exponential convergence.

The confusion regarding the longitudinal vector wave function contribution in the literature has been resolved. In Tai's early formulation [Tai, 1971] the missing $\bar{L}$ function contribution is non-zero away from the source. Only when the Green's function is displayed as a two-dimensional eigenfunction expansion with closed form in the third dimension is the $\bar{L}$ function contribution completely given by $-\hat{z}\hat{z}\,\delta(r-r')/k^2$. It is crucial to note that if this partially closed representation is used, the $\hat{z}\hat{z}$ delta function term is not the total singularity since the $\bar{M}$ and $\bar{N}$ function contribution is highly singular. The total dominant singularity is still proportional to the longitudinal component of the unit delta dyadic. Finally similar techniques may be applied to show that the cavity dyad $\bar{g}_h$ (2.35) has the same type of singularity.

CHAPTER 3

A PRELIMINARY PROBLEM

In an effort to gain insight into some of the difficulties that may occur in the three-dimensional vector problem of Chapter 1, a scalar two-dimensional diffraction problem is considered. This problem illustrates how to deal with the field and kernel signularities involved in solution of the aperture integral equations. As expected, the dyadic formulation for this problem reduces to the scalar formulation.

The problem considered is the diffraction of steady state electromagnetic waves through an aperture in a thin, perfectly conducting, infinite sheet (Figure 2). As illustrated, the geometry and the sources are y independent; thus the fields also have no y variation. Maxwell's equations for electric sources (2.1) decouple, viz:

$$- (\nabla^2 + k^2)\, \hat{E}_y = J_y \tag{3.1a}$$

$$H_x = - \frac{\partial \hat{E}_y}{\partial z} \tag{3.1b}$$

$$H_z = \frac{\partial \hat{E}_y}{\partial x} \tag{3.1c}$$

The magnetic source analogues (2.2) becomes

$$- (\nabla^2 + k^2)\, \hat{H}_y = M_y \tag{3.2a}$$

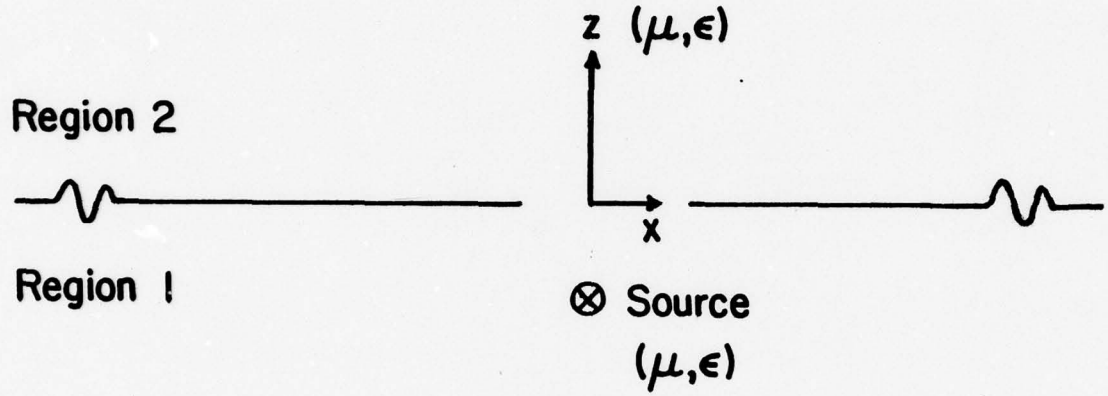


Figure 2. Two Dimensional Slot Diffraction

$$\hat{E}_x = -\frac{\partial \hat{H}_y}{\partial z} \tag{3.2b}$$

$$\hat{E}_z = \frac{\partial \hat{H}_y}{\partial x} \quad . \tag{3.2c}$$

Equations (3.1) and (3.2) have been called the "hard" and "easy" polarizations respectively.

The Easy Polarization

An integral equation for the tangential electric field across the aperture is developed. Once this tangential electric field is known Green's second identity [Kellogg 1953, p. 215] may be used to obtain H_y at any point. The free space Green's function for (3.2a) is given, apart from a complex conjugate needed to account for an $e^{i\omega\tau}$ time dependence, by Morse and Feshbach [1953, p. 811] as

$$g_o(x,x',z,z') = \frac{1}{4i} H_o^{(2)} \left(k\sqrt{(x-x')^2 + (z-z')^2}\right) . \tag{3.3}$$

Thus the half-space Green's function satisfying the appropriate magnetic boundary condition is given by

$$g(x,x',z,z') = \frac{1}{4i} \left[H_o^{(2)} \left(k\sqrt{(x-x')^2 + (z-z')^2}\right) + H_o^{(2)} \left(k\sqrt{(x-x')^2 + (z+z')^2}\right)\right] . \tag{3.4}$$

Using Green's second identity, the radiation condition at infinity, and continuity of the tangential field components across the aperture, one

obtains

$$2i\hat{H}_y^{inc}(x) = \int_{-L}^{L} H_o^{(2)}\ (k|x-x'|)\ \hat{E}_x(x')\ dx' \ . \tag{3.5a}$$

The incident field is given by

$$2\ \hat{H}_y^{inc}(x) = \iint_{A_I} g(x,x',o,z')\ M_y(x',z')\ dx'dz' \ , \tag{3.5b}$$

where A_I is all of Region I. The above equation is solved numerically by approximating $\hat{E}_x(x)$ by pulses, point matching, and inverting a matrix equation. As illustration, numerical results are given for a normally incident plane wave illuminating an aperture of total width .025λ (Figures 3 and 4). The dashed lines give computed values of E_x when expanded in 20 pulses; the dotted lines give E_x using a 40 pulse expansion.

The Hard Polarization

Techniques similar to those used to solve the easy polarization may be applied to (3.1) to obtain

$$2iH_x^{inc}(x) = \lim_{z\to o} \int_{-L} \frac{\partial^2}{\partial z^2} H_o^{(2)}\ (k\sqrt{(x-x')^2 + z^2}\ )\ \hat{E}_y(x')\ dx' \ , \tag{3.6}$$

where $H_x^{inc}(x)$ is the x component of the incident magnetic field in the aperture. As z approaches zero the kernel becomes highly singular and the limit and integration cannot be interchanged. Noting that

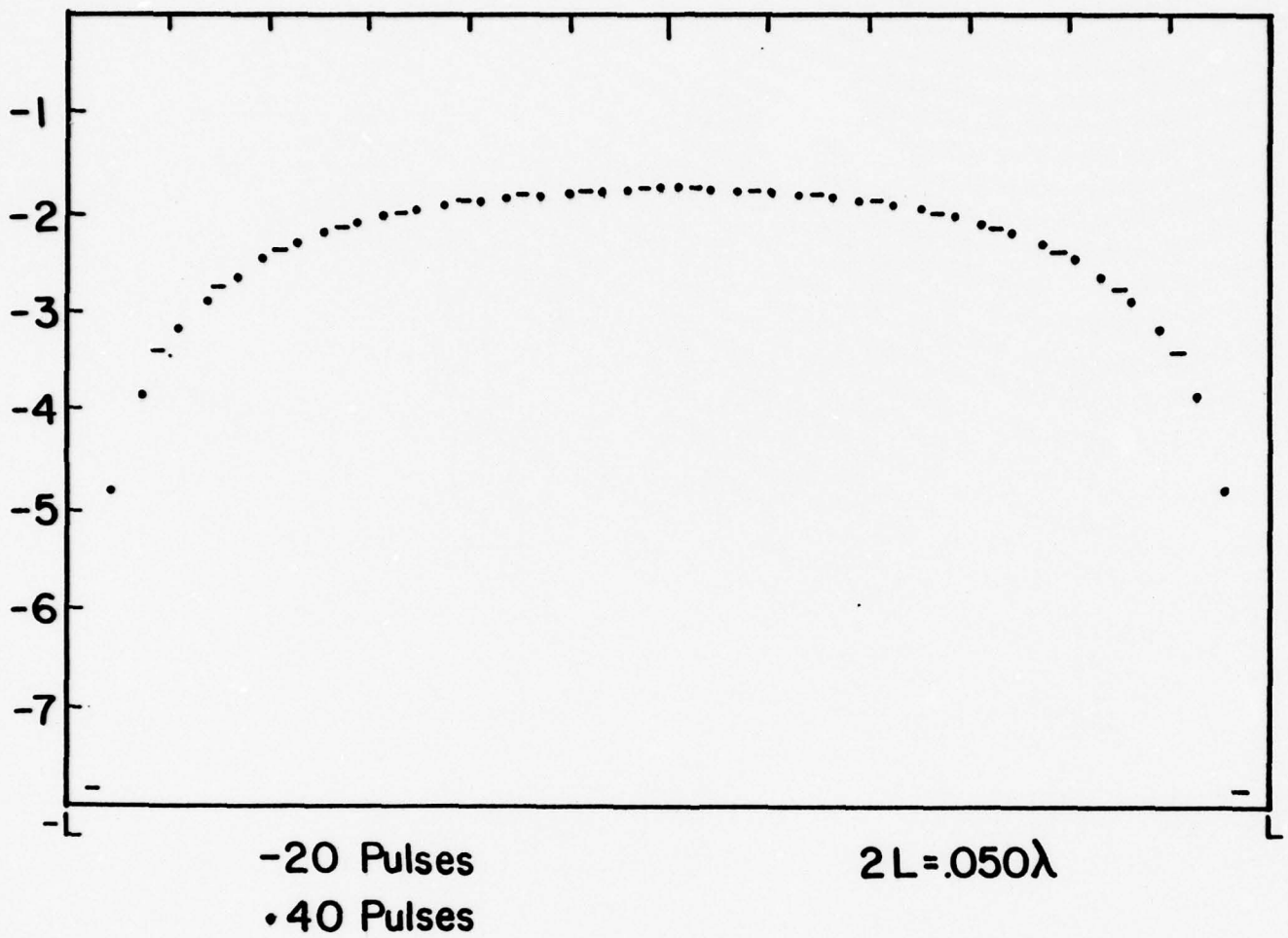


Figure 3. Image E_x^{inc}, Easy Polarization

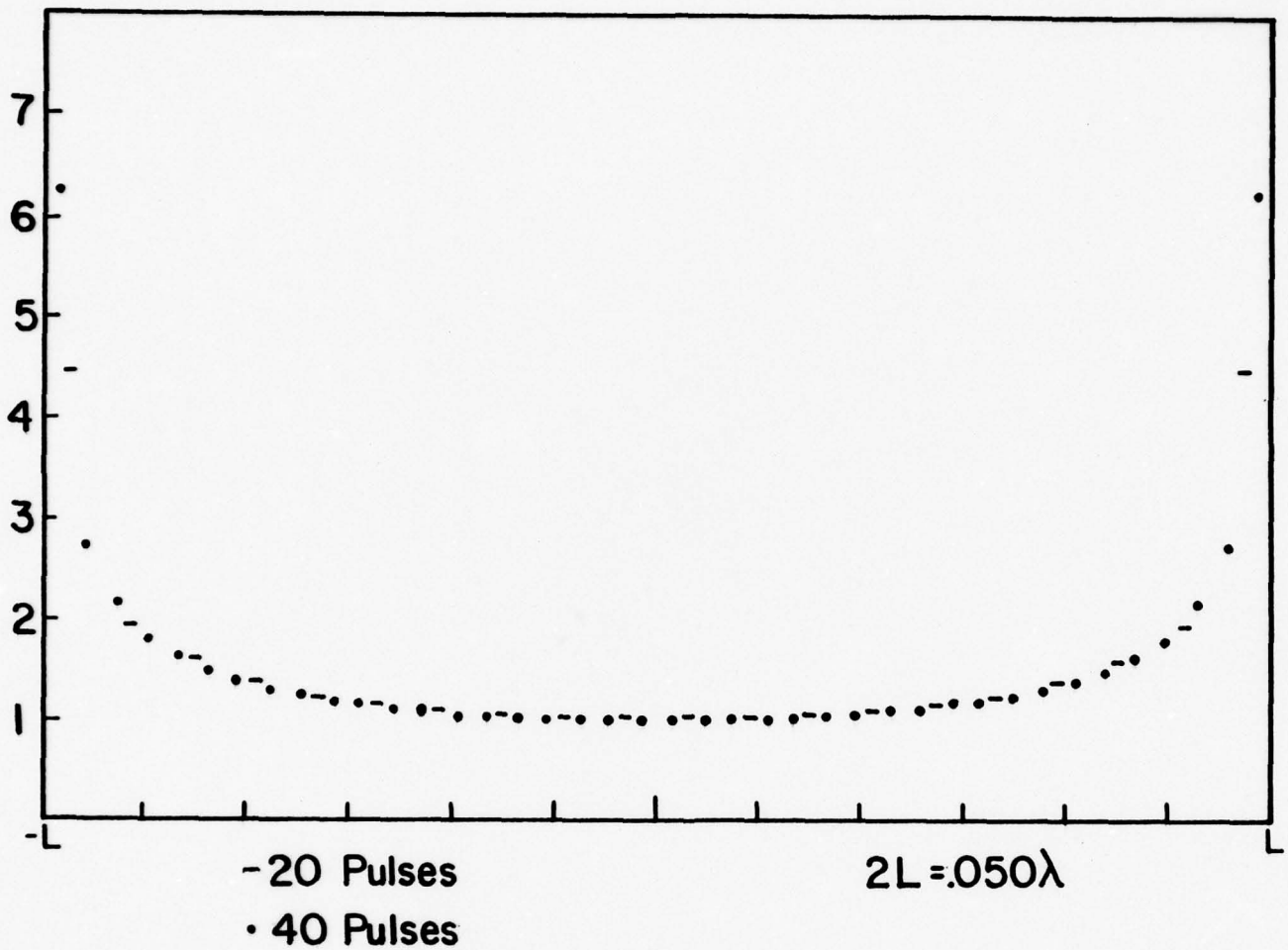


Figure 4. Real E_x^{inc}, Easy Polarization

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_o^{(2)} \left(k\sqrt{(x-x')^2 + z^2}\right) = 0 \ , \ z \neq 0 \qquad (3.7)$$

(3.6) becomes

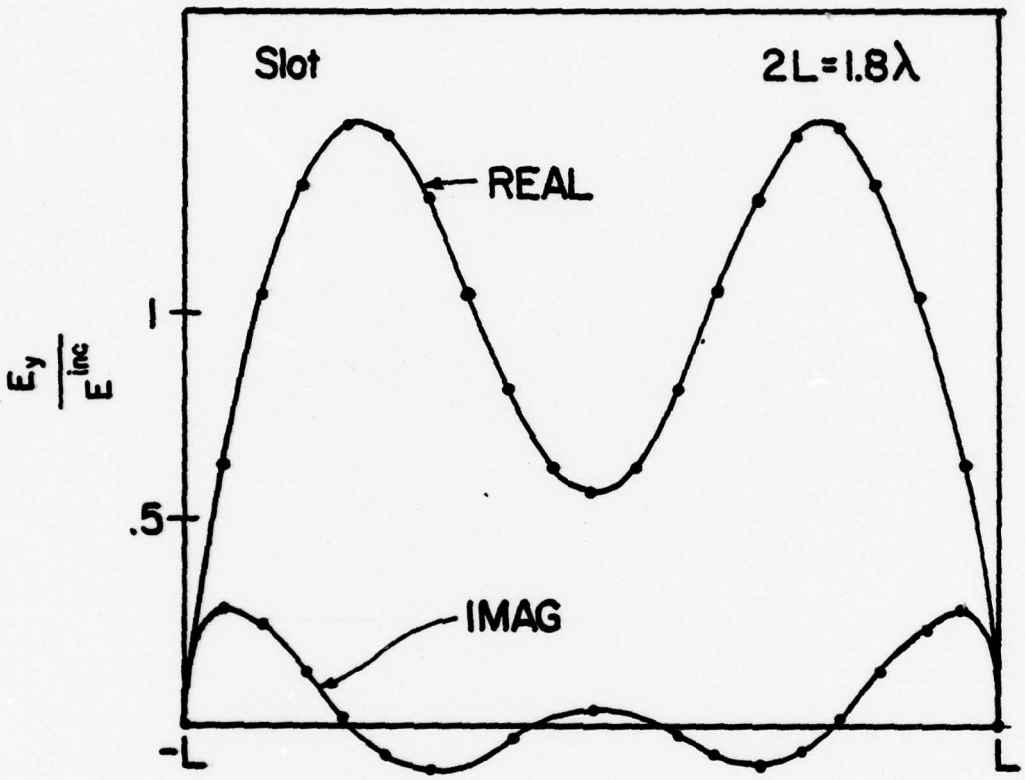
$$- 2iH_x^{inc}(x) = \lim_{z\to o}\left(\frac{\partial^2}{\partial x^2} + k^2\right)\int_{-L}^{L} H_o^{(2)} \left(k\sqrt{(x-x')^2 + z^2}\right)\hat{E}_y(x')dx' \ . \qquad (3.8)$$

Johnson and Dudley [1978] have studied the aperture electric field's singularities and have obtained a method of numerical solution of (3.8) with an interpolary feature that may be useful in the higher dimensional vector problem. Alternately (3.8) may be solved by pulse expansion and piecewise sine testing [Wilton and Butler 1976]. A comparison of Johnson and Dudley's Green function moment method (GFM) and Butler's piecewise sine method (PS) is given for normally incident plane wave excitation. The real and imaginary parts of the aperture electric field versus the aperture coordinate are given for total aperture widths of .01λ (Figure 5) and 1.8λ (Figure 6).

Dyadic Formulation for the Easy Polarization

The dyadic formulation for the easy polarization is now given and shown to reduce to the scalar problem. By image theory the electric vector potential dyad (2.30) for the half-space one or two (Figure 2) is given by

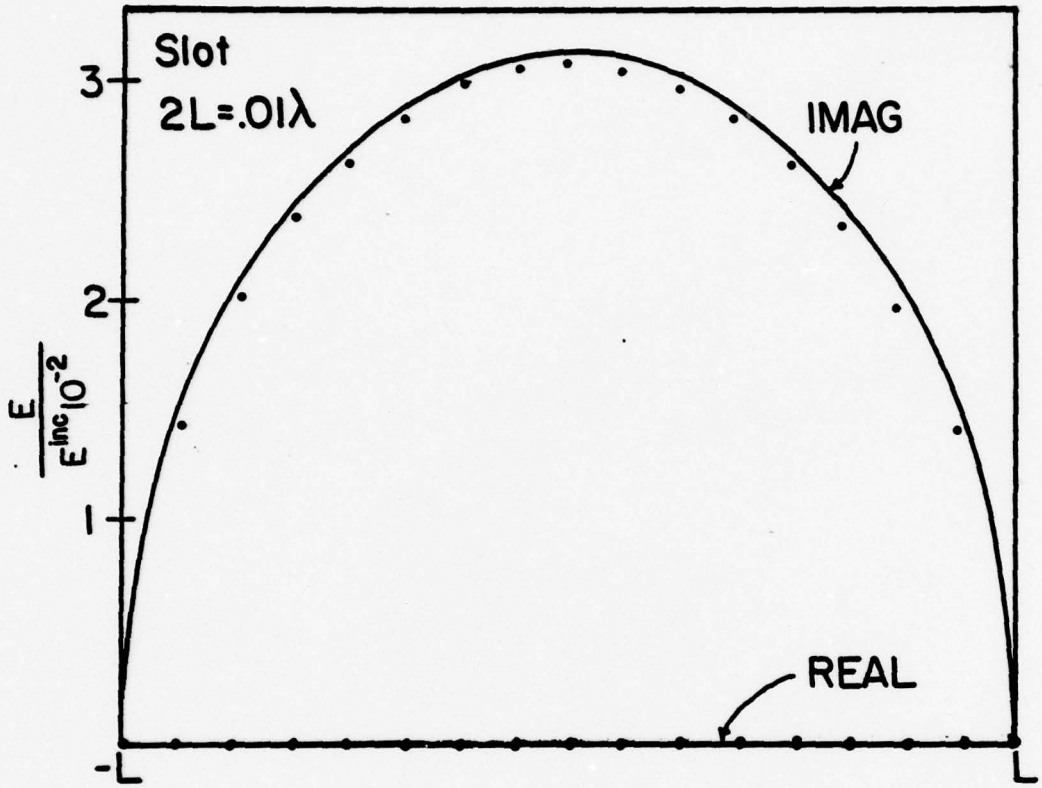
$$\bar{\bar{g}}_F(\bar{r}|\bar{r}') = (\psi + \psi^i)\hat{x}\hat{x} + (\psi + \psi^i)\hat{y}\hat{y} + (\psi - \psi^i)\hat{z}\hat{z} \ , \qquad (3.9a)$$



	# Pulses	Sec	Error
— GFM	10	.8	$\Delta_{10,20} = 1.0\,\%$
• • PS	39	.7	$\Delta_{39,79} = 1.5\,\%$
			$\Delta_{GFM,PS} = 3.0\,\%$

$$\Delta_{n,m} = \max \left| \frac{E_{yn} - E_{ym}}{E_{yn}} \right|$$

Figure 5. Small Slot, Hard Polarization



	# Pulses	Sec	Error
— GFM	2	.19	$\Delta_{2,4}$ =.005%
• • PS	15	.16	$\Delta_{15,31}$ =1.00%
			$\Delta_{GFM,PS}$=4.00%

Figure 6. Medium Slot, Hard Polarization

where

$$\psi = \frac{e^{-ikR}}{4\pi R} \tag{3.9b}$$

$$\psi^i = \frac{e^{-ikR^i}}{4\pi R^i} , \tag{3.9c}$$

$$R = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2} , \tag{3.9d}$$

and

$$R^i = [(x-x')^2 + (y-y')^2 + (z+z')^2]^{1/2} . \tag{3.9e}$$

From (2.27e) one obtains,

$$\bar{\bar{g}}_h(r|r') = (\bar{\bar{I}} + \frac{\nabla\nabla\cdot}{k^2})\ \bar{\bar{g}}_F(\bar{r}|\bar{r}') . \tag{3.10}$$

Equivalently,

$$\bar{\bar{g}}_h(r|r') = (\bar{\bar{I}} - \frac{1}{k^2}\nabla\nabla')(\psi + \psi^i) - 2\ \psi^i\ \hat{z}\hat{z} . \tag{3.11}$$

Equation (2.15c) yields

$$\bar{\hat{H}}_1(\bar{r}) = \iiint_{V_1} \bar{\bar{g}}_h(\bar{r}|\bar{r}') \cdot \hat{y}\ M_y(x',z')dv$$

$$- \iint_A \bar{\bar{g}}_h\ (\bar{r}|\bar{r}') \cdot (\hat{z} \times \bar{\hat{E}}_1(\bar{r}'))ds' \tag{3.12}$$

for $\bar{r}$ in Region 1, where A is the aperture surface. In Region 2,

$$\hat{\bar{H}}_2(\bar{r}) = \iint_A \bar{\bar{g}}_h(\bar{r}|\bar{r}') \cdot (\hat{z} \times \hat{\bar{E}}_2(\bar{r}'))ds' \ . \tag{3.13}$$

Noting the relationship between the two and three dimensional Green's functions [Morse and Feshbach 1953, p. 1323].

$$\frac{1}{4i} H_o^{(2)}(k\sqrt{x^2 + z^2}) = \int_{-\infty}^{\infty} \frac{e^{-ik\sqrt{x^2 + y^2 + z^2}}}{4\pi\sqrt{x^2 + y^2 + z^2}} dy \tag{3.14}$$

the volume integral in (3.12) becomes

$$\hat{y} \iint_{R_1} M_y(x',z') \frac{1}{4i} [H_o^{(2)}(k\sqrt{(x-x')^2 + (z-z')^2})$$

$$+ H_o^{(2)}(k\sqrt{(x-x')^2 + (z+z')^2})]dx'dy' \tag{3.15}$$

Continuity of the tangential field components across the aperture and y independence of the fields yield

$$\hat{y} \iint_{R_1} M_y(x',z') \frac{1}{4i} [H_o^{(2)}(k\sqrt{(x-x')^2 + (z-z')^2})$$

$$+ H_o^{(2)}(k\sqrt{(x-x')^2 + (z+z')^2})]dx'dz'$$

$$= \hat{y} \int_{-L}^{L} E_x(x') \frac{H_o^{(2)}(k|x-x'|)}{i} dx'$$

$$- \hat{x} \lim_{z \to o} \left(\frac{\partial^2}{\partial x^2} + k^2\right) \int_{-L}^{L} E_y(x') \frac{H_o^{(2)}(k\sqrt{(x-x')^2 + z^2})}{ik^2} dx'. \tag{3.16}$$

Since only the x component of the electric field is excited E_y is set to zero and (3.16) reduces to its scalar counterpart (3.5).

Dyadic Formulation for the Hard Polarization

The dyadic formulation for the hard polarization also reduces to its corresponding scalar equation (3.8). By image theory the magnetic vector potential dyad (2.21) for the halfspace is given by

$$\bar{\bar{G}}_A(\bar{r}|\bar{r}') = (\psi - \psi^i)\hat{x}\hat{x} + (\psi - \psi^i)\hat{y}\hat{y} + (\psi + \psi^i)\hat{z}\hat{z} , \tag{3.17}$$

where ψ and ψ^i are defined in (3.9). Equation (2.16e) may be manipulated into the form

$$\hat{\bar{E}}_1(\bar{r}) = \iiint_{V_1} \bar{\bar{G}}_e(\bar{r}|\bar{r}') \cdot \hat{y}\, J_y(x',z')dv' - \iint_A (\hat{z} \times \bar{E}(\bar{r}'))$$

$$\cdot \nabla' \times \bar{\bar{G}}_e(\bar{r}'|\bar{r})ds' \tag{3.18a}$$

in Region 1. Proceeding as before (3.12 - 3.15), (3.18a) may be expressed as

$$\bar{\hat{E}}_1(r) = \hat{y}\iint_{R_1} J_y(x',z') \frac{[H_o^{(2)}(k\sqrt{(x-x')\ +(z-z')^2})-H_o^{(2)}(k\sqrt{(x-x')^2+(z+z')^2})]\ dx'dz'}{4i}$$

$$- \iint (\hat{z} \times \bar{\hat{E}}_1(\bar{r}')) \cdot \nabla' \times \bar{\bar{G}}_e(\bar{r}'|\bar{r})ds' \tag{3.18b}$$

In Region 2 (2.11) yields

$$\bar{\hat{E}}_2(\bar{\bar{r}}) = \iint_A (\hat{z} \times \bar{\hat{E}}_2(\bar{r}')) \cdot \nabla \times \bar{\bar{G}}_e(\bar{r}'|\bar{r})ds' \ . \tag{3.19}$$

Using (3.14) and the y independence of the fields

$$\bar{H}_{1t}(\bar{r}_a) = \hat{x} \iint_{R_1} \bar{J}_y(x',z') \frac{\partial}{\partial z'} \frac{H_o^{(2)}(k\sqrt{(x-x')^2 + z'^2})\ dx'dz'}{2i}$$

$$- \lim_{z\to o} - [\hat{x} \frac{\partial}{\partial z^2} \int_{-L}^{L} \hat{E}_y(x') \frac{H_o^{(2)}(k\sqrt{(x-x')^2 + z^2})\ dx'}{2i}$$

$$+ \hat{y}\ k^2 \int_{-L}^{L} \hat{E}_x(x') \frac{H_o^{(2)}(k\sqrt{(x-x')^2 + z^2})\ dx'}{2i} \quad] \tag{3.20a}$$

Where $\bar{H}_{1t}(\bar{r}_a)$ is the tangential $\bar{H}$ component as $\bar{r}$ approaches the aperture point $\bar{r}_a$ from Region 1. Similarly

$$\bar{H}_{2t}(r_a^-) = \lim_{z\to o} + [\hat{x} \frac{\partial}{\partial z^2} \int_{-L}^{L} \hat{E}_y(x') \frac{H_o^{(2)}(k\sqrt{(x-x^2) + z^2})}{2i} dx'$$

$$+ \hat{y}\, k^2 \int_{-L}^{L} \hat{E}_x(x') \frac{H_o^{(2)}(k\sqrt{(x-x')^2 + z^2})}{2i} dx' \qquad (3.20b)$$

Equating the x components of (3.20a) and (3.20b) yields the scalar equation (3.6). Since the x component of the electric field is not excited it is set to zero. One may check (3.18) or (3.19) to see that indeed only the $\hat{y}$ component of $\hat{z}$ is excited. Thus the dyadic problem has reduced to the scalar problem providing a check on the dyadic formulation.

CHAPTER 4

INTEGRAL EQUATIONS FOR APERTURE FIELDS AND OBSTACLE CURRENTS

The three dimensional exterior-interior coupling problem of Chapter 1 is now considered. The dyadic formalism of Chapter 2 is used to develop a set of coupled integral equations for unknown aperture fields and obstacle currents. The techniques of Wilton and Dunaway [1975] for aperture penetration through a perfectly conducting infinite screen are followed.

Application of (2.10) to Region 1 of Figure 1 yields

$$\bar{\hat{E}}(\bar{r}) = \bar{\hat{E}}^{sc}(r) + \iint_A [\hat{z} \times \bar{\hat{E}}(\bar{r}')] \cdot \bar{\bar{G}}_h^{hs}(\bar{r}'|\bar{r})\, ds' \tag{4.1a}$$

where

$$\bar{\hat{E}}^{sc}(\bar{r}) = \iiint_{\text{Region 1}} \bar{\bar{G}}_e^{hs}(\bar{r}\ \bar{r}') \cdot \bar{J}(\bar{r}')\, dv'$$

is interpreted distributionally and the superscript "hs" denotes a half space dyad. The short circuited electric field $\bar{\hat{E}}^{sc}$ is the field that would be present if the aperture were not present. Similarly in Region 2 (Figure 1) one obtains

$$\bar{\hat{E}}(\bar{r}) = -\iint_A [\hat{z} \times \bar{\hat{E}}(\bar{r}')] \cdot \bar{\bar{G}}_h^{c}(\bar{r}'|\bar{r})\, ds' - \iint_{\text{Obstacle}} \bar{\bar{G}}_e^{c}(\bar{r}|\bar{r}')$$

$$\cdot\ (\hat{n} \times \bar{H}(\bar{r}')\, ds' \tag{4.1b}$$

where $\hat{n}$ is the unit normal pointing outward from Region 2 and inward to the obstacle. The superscript "c" denotes a cavity dyad. On a perfectly conducting surface the surface current $\bar{J}_s$ equals $\bar{\bar{H}} \times \hat{n}$ [Harrington 1961, p. 106]. Thus

$$\bar{\hat{E}}(\bar{r}) = - \iint_A [\hat{z} \times \bar{\hat{E}}(\bar{r}')] \cdot \bar{\bar{G}}_h(\bar{r}'|\bar{r}) ds' + \iint_{\text{Obstacle}} \bar{\bar{G}}_e(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}')\, ds' \tag{4.1c}$$

in Region 2.

Application of the result

$$[\nabla' \times \bar{\bar{G}}_e(\bar{r}'|\bar{r})]^{\sim} = \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}') \qquad \text{(Appendix A)} \tag{4.2}$$

to (4.1a) plus (2.1a) yields

$$\bar{H}(\bar{r}) = \bar{H}^{sc}(\bar{r}) + \nabla \times \iint_A \nabla \times \bar{\bar{g}}_h^{hs}(\bar{r}|\bar{r}') \cdot (\hat{z} \times \bar{\hat{E}}(\bar{r}'))\, ds' \tag{4.3a}$$

interior to Region 1, where $\bar{H}^{sc}(r) = \nabla \times \bar{E}^{sc}(r)$. Similarly (4.1c) and (4.2) yield

$$\bar{H}(\bar{r}) = - \nabla \times \iint_A \nabla \times \bar{\bar{g}}_h^{c}(\bar{r}|\bar{r}') \cdot [\hat{z} \times \bar{\hat{E}}(\bar{r}')] ds' + \nabla \times \iint_{\text{Obstacle}} \bar{\bar{G}}_e(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}') ds' \tag{4.3b}$$

interior to Region 2.

Now continuity of the tangential components of the aperture fields will be invoked to obtain a coupled set of integral equations for the tangential electric field in the aperture. An integral equation will also be enforced for the obstacle surface currents which are coupled to the tangential components of the aperture electric field. Equation set (4.3) and (2.12c), plus continuity of tangential field components yield

$$\bar{H}_t(\bar{r}) = [\lim_{z\to c^+} - k^2 \iint_A \bar{\bar{g}}_h^{hs}(\bar{r}|\bar{r}') \cdot (\hat{E}_x(\bar{r}')\hat{y} - \hat{E}_y(\bar{r}')\hat{x})ds'$$

$$+ \lim_{z\to c^-} -k^2 \iint_A \bar{\bar{g}}_h^{c}(\bar{r}|\bar{r}') \cdot (\hat{E}_x(\bar{r}')\hat{y} - \hat{E}_y(\bar{r}')\hat{x})ds'$$

$$+ \iint_{\text{Obstacle}} \bar{\bar{G}}_h^{c}(\bar{r}|\bar{r}') \cdot \bar{J}(\bar{r}')ds']_t \,, \qquad (4.4.)$$

in the aperture where the subscript "t" denotes the tangential component. Application of (2.16d) and (2.27e) to (4.4) yields

$$H_x^{sc}(\bar{r}) = \lim_{z\to c} - \frac{\partial}{\partial x}\frac{\partial}{\partial y} \iint_A [\bar{\bar{g}}_F^{hs,yy}(\bar{r}|\bar{r}') + \bar{\bar{g}}_F^{c,yy}(\bar{r}|\bar{r}')]\ \hat{E}_x(\bar{r}')ds'$$

$$+ \lim_{z\to c} (\frac{\partial^2}{\partial x^2} + k^2) \iint_A [\bar{\bar{g}}_F^{hs,xx}(\bar{r}|\bar{r}') + \bar{\bar{g}}_F^{c,xx}(\bar{r}|\bar{r}')]\ \hat{E}_y(\bar{r}')ds'$$

$$+ \iint_{\text{Obstacle}} [\frac{\partial}{\partial y}\ \bar{\bar{G}}_A^{c,zz}(\bar{r}|\bar{r}')\ J_z(\bar{r}')$$

$$- \frac{\partial}{\partial z}\ \bar{\bar{G}}_A^{c,yy}(\bar{r}|\bar{r}')\ J_y(\bar{r}')]ds' \qquad (4.5a)$$

in the aperture interior. Similarly the result

$$\bar{H}_y^{sc}(\bar{r}) = \lim_{z\to c} - \left(\frac{\partial^2}{\partial y^2} + k^2\right) \iint_A [g_F^{hs,yy}(\bar{r}|\bar{r}')$$

$$+ g_F^{c,yy}(\bar{r}|\bar{r}')] \; \hat{E}_x(\bar{r}')ds' + \lim_{z\to c} \frac{\partial}{\partial y}\frac{\partial}{\partial x} \iint_A [g_F^{hs,xx}(\bar{r}|\bar{r}')$$

$$+ g_F^{c,xx}(\bar{r}|\bar{r}')] \; \hat{E}_y(\bar{r}')ds' + \iint_{\text{Obstacle}} [\frac{\partial}{\partial z} G_A^{c,xx}(\bar{r}|\bar{r}') \; J_x(\bar{r}')$$

$$- \frac{\partial}{\partial x} G_A^{c,zz}(\bar{r}|\bar{r}') \; J_z(\bar{r}')]ds' \tag{4.5b}$$

may be obtained in the aperture interior. Equation set (4.5) will be decoupled by use of techniques put forth by Dunaway [1974].

Since $\bar{E}$ and $\bar{H}$ satisfy Maxwell's equations they are not independent. In fact continuity of tangential aperture fields implies that the normal component of $\bar{E}$ is continuous across the aperture [Jones 1974]. From equation set (4.1),

$$\hat{E}_z^{sc}(\bar{r}) = \hat{z} \cdot [- \lim_{z\to c^+} \iint_A \nabla \times g_h^{hs}(\bar{r}|\bar{r}') \cdot [\hat{E}_x(\bar{r}')\hat{y} - \hat{E}_y(\bar{r}')\hat{x}]ds'$$

$$- \lim_{z\to c^-} \iint_A \nabla \times \bar{\bar{g}}_h^c(\bar{r}|\bar{r}') \cdot [\hat{E}_x(\bar{r}')\hat{y} - \hat{E}_y(\bar{r}')\hat{x}]ds'$$

$$+ \iint_{\text{Obstacle}} \bar{\bar{G}}_e^c(r|r') \cdot \bar{J}(r')ds'] \tag{4.6a}$$

Equation (4.6a) may be simplified through the use of (2.16e) and (2.27e), viz:

$$\hat{E}_z^{sc}(r) = - \lim_{z\to c} \frac{\partial}{\partial x} \iint_A [g_F^{hs,yy}(\bar{r}|\bar{r}') + g_F^{c,yy}(\bar{r}|\bar{r}')] \hat{E}_x(\bar{r}')ds'$$

$$- \lim_{z\to c} \frac{\partial}{\partial y} \iint_A [g_F^{hs,xx}(\bar{r}|\bar{r}') + g_F^{c,xx}(\bar{r}|\bar{r}')] \hat{E}_y(\bar{r}')ds'$$

$$+ \frac{1}{k^2} \iint_{\text{Obstacle}} \frac{\partial^2}{\partial z \partial x} G_A^{c,xx}(\bar{r}|\bar{r}') J_x(\bar{r}')$$

$$+ \frac{\partial^2}{\partial z \partial y} G_A^{c,yy}(\bar{r}|\bar{r}') J_y(\bar{r}') + (\frac{\partial^2}{\partial z^2} + k^2)$$

$$\cdot G_A^{c,zz}(\bar{r}|\bar{r}') J_z(\bar{r}')] ds' \qquad (4.6b)$$

Now (4.5) is brought into a more convenient form by use of (4.6b). The $\hat{E}_x$ term in (4.5a) is replaced by the appropriate counterpart from (4.6b). This yields

$$(\nabla_t^2 + k^2) \iint_A [g_F^{hs,xx}(\bar{r}|\bar{r}') + g_F^{c,xx}(\bar{r}|\bar{r}')] \hat{E}_y(\bar{r}')ds'$$

$$= H_x^{sc}(\bar{r}) - \frac{\partial}{\partial y} \hat{E}_z^{sc}(\bar{r})$$

$$+ \frac{1}{k^2} \iint_{\text{Obstacle}} \frac{\partial^3}{\partial x \partial y \partial z} G_A^{c,xx}(\bar{r}|\bar{r}') J_x(\bar{r}') +$$

$$+ \frac{\partial^3}{\partial y^2 \partial z} G_A^{c,yy}(\bar{r}|\bar{r}') \; J_y(\bar{r}') + (\frac{\partial^2}{\partial z^2} + k^2) \frac{\partial}{\partial y}$$

$$\cdot \; G_A^{c,zz}(\bar{r}|\bar{r}') \; J_z(\bar{r}')] \; ds'$$

$$- \iint_{\text{Obstacle}} [\frac{\partial}{\partial y} G_A^{c,zz}(\bar{r}|\bar{r}') \; J_z(\bar{r}') - \frac{\partial}{\partial z} G_Z^{c,yy} \; J_y(\bar{r}')] ds' \; . \tag{4.7a}$$

In (4.7a) the "∇_t^2" operator equals $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Similarly (4.5b) becomes

$$(\nabla_t^2 + k^2) \iint_A [g_F^{hs,yy}(\bar{r}|\bar{r}') + g_F^{hs,yy}(\bar{r}|\bar{r}')] \; \hat{E}_x(r') ds'$$

$$= -H_y^{sc}(\bar{r}) - \frac{\partial}{\partial x} \hat{E}_z^{sc}(\bar{r})$$

$$+ \frac{1}{k^2} \iint_{\text{Obstacle}} \frac{\partial^3}{\partial x^2 \partial z} G_A^{c,xx}(\bar{r}|\bar{r}') \; J_x(\bar{r}')$$

$$+ \frac{\partial^3}{\partial x \partial y \partial z} G_A^{c,yy}(\bar{r}|\bar{r}') \; J_y(\bar{r}') + (\frac{\partial^2}{\partial z^2} + k^2) \frac{\partial}{\partial x}$$

$$\cdot \; G_A^{c,zz}(\bar{r}|\bar{r}') \; J_z(\bar{r}')] ds' + \iint_{\text{Obstacle}} [\frac{\partial}{\partial z} G_A^{c,xx}(\bar{r}|\bar{r}') \; J_x(r')$$

$$- \frac{\partial}{\partial x} G_A^{c,zz}(\bar{r}|\bar{r}') \; J_z(\bar{r}')] ds' \; . \tag{4.7b}$$

Equation set (4.7) is desirable not only for the partial decoupling effect, but also for the presence of the harmonic operator

$(\nabla_t^2 + k^2)$. Although (4.5) plus the correct obstacle surface currents is sufficient to uniquely determine the electromagnetic fields, (4.7) is not. The aperture fields must be subject to the constraint equation (4.6b).

Again using Dunaway's technique [Dunaway 1974] the harmonic operators of (4.7) will be inverted. To invert the harmonic operators consider the two dimensional Helmholtz problem described in Figure 7. The problem is given as

$$- (\nabla_t^2 + k^2)\ \Phi(\bar{\rho}) = \begin{cases} v(\bar{\rho}), & \bar{\rho}\varepsilon\ A_1 \\ 0\ , & \bar{\rho}\varepsilon\ A_2 \end{cases} \tag{4.8}$$

where $\Phi(\bar{\rho})$ is assumed continuous across the boundary and satisfies the radiation condition at infinity. Use will be made of Green's second identity in two dimensions given by

$$\iint (\psi\ \nabla_t^2\ \Phi - \Phi\ \nabla_t^2\ \psi)ds = \oint(\psi\ \frac{\partial\Phi}{\partial\eta} - \Phi\ \frac{\partial\psi}{\partial\eta})d\ell \tag{4.9}$$

[Harrington 1961]. Also use is made of the two dimensional Green's function satisfying the radiation condition at infinity. This is given by

$$g_o(\bar{\rho}|\bar{\rho}') = \frac{H_o^{(2)}(k|\bar{\rho}-\bar{\rho}'|)}{4i} \tag{4.10a}$$

and satisfies

$$- (\nabla_t^2 + k^2)\ g_o(\bar{\rho}|\bar{\rho}') = \delta(\bar{\rho}-\bar{\rho}') \tag{4.10b}$$

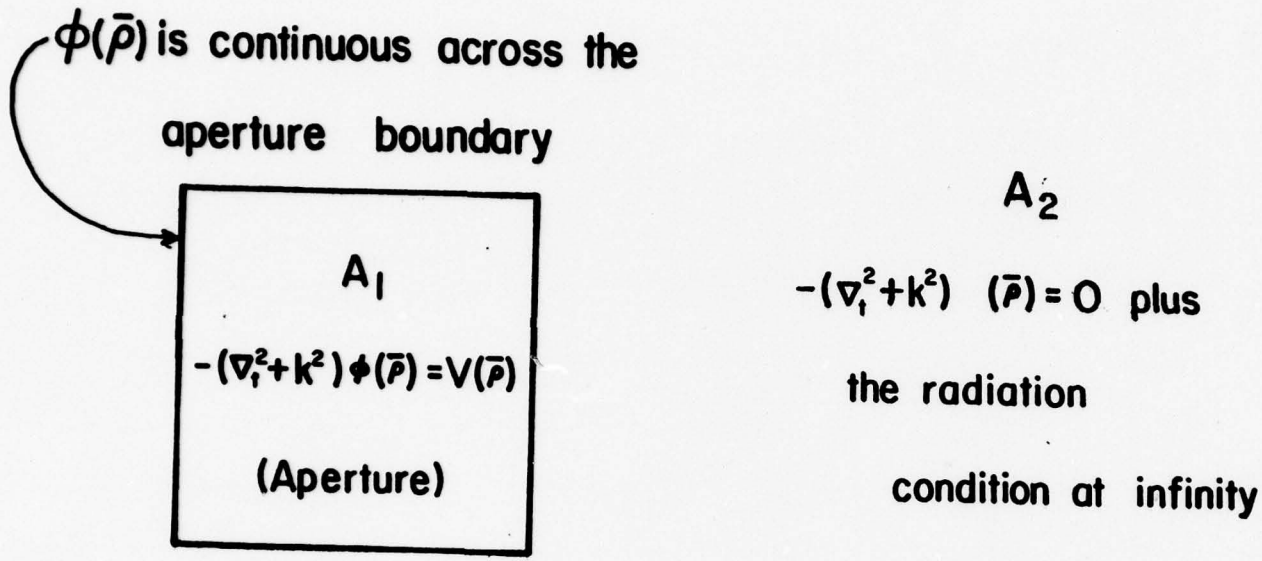


Figure 7. Inversion of Helmholtz Operator

Application of Green's second identity, (4.9), to A_2 of Figure 8 yields

$$\int_{\partial A_1} [g_o(\bar{\rho}|\bar{\rho}') \frac{\partial \Phi}{\partial n'}(\bar{\rho}') - \Phi(\bar{\rho}') \frac{\partial}{\partial n'} g_o(\bar{\rho}|\bar{\rho}')] d\ell' = 0 \tag{4.11a}$$

provided $\bar{\rho}$ is interior to A_1 where $\hat{n}$ is the outward pointing unit normal from A_1. Similarly application of (4.9) to A_1 yields

$$\Phi(\bar{\rho}) = \iint_{\partial A_1} g_o(\bar{\rho}|\bar{\rho}') \, v(\bar{\rho}') ds' + \int_{\partial A_1} [g_o(\bar{\rho}|\bar{\rho}') \frac{\partial \Phi(\bar{\rho}')}{\partial n'}$$

$$- \Phi(\bar{\rho}') \frac{\partial g_o(\bar{\rho}|\bar{\rho}')}{\partial n'}] d\ell' \, . \tag{4.11b}$$

Under the assumption that $\Phi(\bar{\rho})$ is continuous across the boundary, subtraction of (4.11a) from (4.11b) yields

$$\Phi(\bar{\rho}) = \iint_{A_1} g_o(\bar{\rho}|\bar{\rho}') \, v(\bar{\rho}') ds' + \int_{\partial A_1} g_o(\bar{\rho}|\bar{\rho}') \, \varepsilon(\bar{\rho}') d\ell' \tag{4.12}$$

where

$$\varepsilon(\bar{\rho}') = \frac{\partial \Phi}{\partial n} A_1(\bar{\rho}') - \frac{\partial \Phi}{\partial n} A_2(\bar{\rho}')$$

the subscripts "A_1" and "A_2" denote limiting values as $\bar{\rho}'$ approaches the boundary from the respective region, and $\bar{\rho}$ belongs to the interior of A_1. Since the aperture fields satisfy Meixner's edge condition [Van Bladel 1964], one may show that the functions operated on by the Helmholtz operators (4.7) satisfy the appropriate continuity conditions

across the aperture boundary. Thus the Helmholtz operators in (4.7) may be inverted by means of equation (4.12). Thus (4.7a) becomes

$$\iint_A [g_F^{hs,xx}(\bar{r}|\bar{r}') + g_F^{c,xx}(\bar{r}|\bar{r}')]\ \hat{E}_y(\bar{r}')ds' = \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \psi_y(r')d\ell'$$

$$+ \iint_A g_o(\bar{r}|\bar{r}')\ [H_x^{sc}(r') - \frac{\partial}{\partial y}\hat{E}_z^{sc}(r') + T_1(r') + T_2(r')]ds' \ , \tag{4.13a}$$

where

$$T_1(r) = \frac{1}{k^2} \iint_{\text{Obstacle}} [\frac{\partial^3}{\partial x \partial y \partial z} G_A^{c,xx}(\bar{r}|\bar{r}')\ J_x(r')$$

$$+ \frac{\partial^3}{\partial y^2 \partial z} G_A^{c,yy}(\bar{r}|\bar{r}')\ J_y(\bar{r}')$$

$$+ \frac{\partial}{\partial y}(\frac{\partial^2}{\partial z^2} + k^2)\ G_A^{c,zz}(\bar{r}|\bar{r}')\ J_z(\bar{r}')]ds' \ ,$$

$$T_2(r) = - \iint_{\text{Obstacle}} [\frac{\partial}{\partial y} G_A^{c,zz}(\bar{r}|\bar{r}')\ J_z(r')$$

$$- \frac{\partial}{\partial z} G_A^{c,yy}(\bar{r}|\bar{r}')\ J_y(\bar{r}')]ds'$$

and $\bar{r}$ belongs to the aperture surface A. Equation (4.7b) becomes

$$\iint_A [g_F^{hs,yy}(\bar{r}|\bar{r}') + g_F^{c,yy}(\bar{r}|\bar{r}')]\ \hat{E}_x(\bar{r}')ds'$$

$$= \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \psi_x(\bar{r}')d\ell' + \iint_A g_o(\bar{r}|\bar{r}')[-H_y^{sc}(\bar{r}')$$

$$- \frac{\partial}{\partial x} E_z^{sc}(\bar{r}') + T_3(\bar{r}') + T_4(\bar{r}')]ds' , \qquad (4.13b)$$

where

$$T_3(\bar{r}) = \frac{1}{k^2} \iint_{\text{Obstacle}} [\frac{\partial^3}{\partial x^2 \partial z} G_A^{c,xx}(\bar{r}|\bar{r}')\ J_x(\bar{r}')$$

$$+ \frac{\partial^3}{\partial x \partial y \partial z} G_A^{c,yy}(\bar{r}|\bar{r}')\ J_y(\bar{r}') + \frac{\partial}{\partial x} (\frac{\partial^2}{\partial z^2} + k^2)$$

$$\cdot\ G_A^{c,zz}(\bar{r}|\bar{r}')\ J_z(\bar{r}')]ds' ,$$

$$T_4(\bar{r}) = - \iint_{\text{Obstacle}} [\frac{\partial}{\partial z} G_A^{c,xx}(\bar{r}|\bar{r}')\ J_x(\bar{r}')$$

$$- \frac{\partial}{\partial x} G_A^{c,zz}(\bar{r}|\bar{r}')\ J_z(\bar{r}')]\ ds' ,$$

and $\bar{r}$ belongs to the aperture surface.

Equations (3.13a) and (3.13b) are coupled by means of the obstacle currents and the auxiliary unknowns ψ_x and ψ_y. Before proceeding to express the constraint equation in terms of obstacle currents and

the auxiliary unknowns, it is convenient to express the short circuited aperture fields in terms of the incident fields.

Figure 8 illustrates the incident plane wave electric field. The incident electric field is given by

$$\bar{E}^{inc}(x,y,z) = [(E_\theta^{inc} \cos\theta \cos\phi - E_\phi^{inc} \sin\phi)\hat{x}$$

$$+ (E_\theta^{inc} \cos\theta \sin\phi + E_\phi^{in} \cos\phi)\hat{y}$$

$$- E^{inc} \sin\theta \, \hat{z}] \, e^{ik \sin\theta \cos\phi \, x} \, e^{ik \sin\theta \sin\phi \, y} \, e^{ik \cos\theta \, z} \tag{4.14}$$

Since the short circuited fields satisfy

$$\nabla \times \hat{\bar{E}}^{sc} = \bar{H}^{sc} , \tag{4.15}$$

one obtains

$$H_x^{sc} - \frac{\partial}{\partial y} \hat{E}_z^{sc} = - \frac{\partial}{\partial z} \hat{E}_y^{sc} = - 2 \frac{\partial}{\partial z} \hat{E}_y^{inc} \Big|_{z=c} \tag{4.16a}$$

and

$$H_y^{sc} + \frac{\partial}{\partial x} \hat{E}_z^{sc} = \frac{\partial}{\partial z} \hat{E}_x^{sc} = 2 \frac{\partial}{\partial z} E_x^{inc} \Big|_{z=c} \tag{4.16b}$$

in the aperture. These equations may be used to replace expressions for short circuited aperture fields in equation set (4.13) by their incident field counterparts.

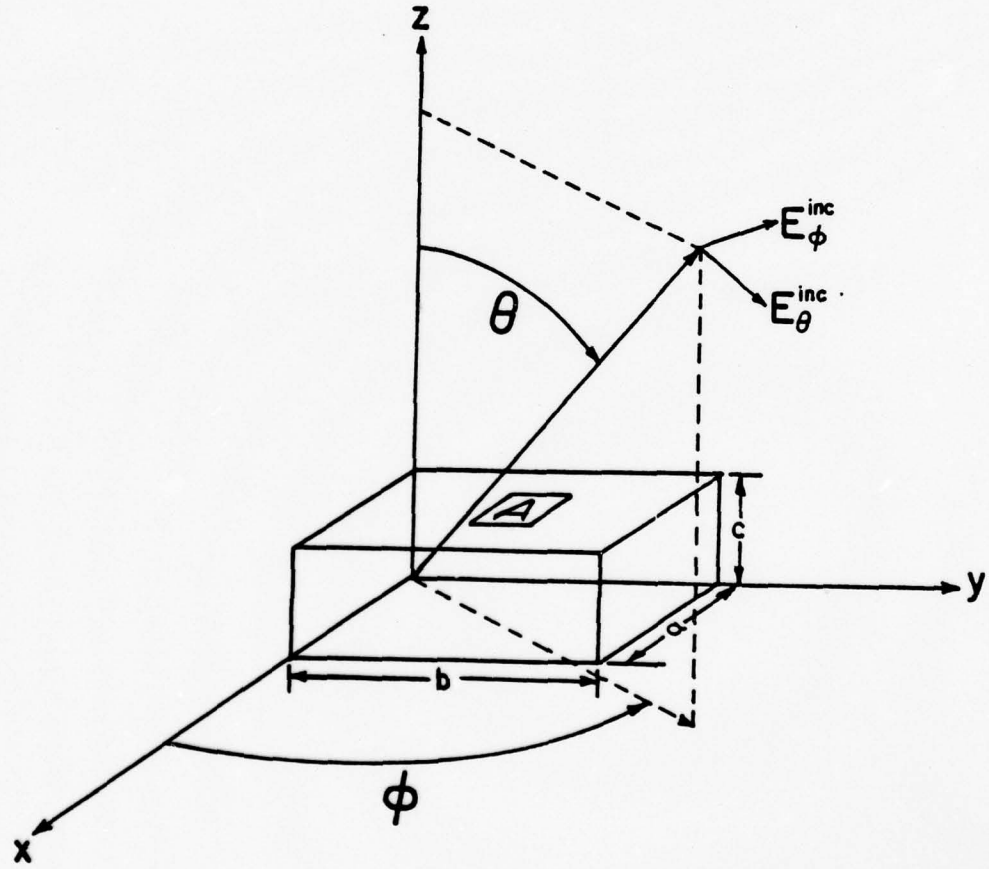


Figure 8. Incident Field

Although (4.5) plus the boundary condition that $\bar{n} \times \hat{E} = 0$ on the perfectly conducting obstacle (4.1c) are sufficient to determine the electromagnetic fields, the constraint equation (4.6b) must be enforced if the more tractable set (4.13) is to replace (4.5) [Mittra et al. 1973]. The constraint equation (4.6b) will now be given in terms of the auxiliary unknowns plus the obstacle currents.

Substitution of (4.13) into (4.6b) yields

$$
\begin{aligned}
\hat{E}_z^{sc}(\bar{r}) = & - \frac{\partial}{\partial x} \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \psi_x(\bar{r}')d\ell' \\
& - \frac{\partial}{\partial y} \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \psi_y(\bar{r}')d\ell' \\
& + 2\frac{\partial}{\partial x} \iint_A g_o(\bar{r}|\bar{r}')\ \frac{\partial}{\partial z}\hat{E}_x^{inc}(\bar{r}')ds' \\
& + 2\frac{\partial}{\partial y} \iint_A g_o(\bar{r}|\bar{r}')\ \frac{\partial}{\partial z}\hat{E}_y^{inc}(\bar{r}')ds' \qquad (4.17) \\
& - \frac{\partial}{\partial x} \iint_A g_o(\bar{r}|\bar{r}')[T_3(\bar{r}') + T_4(\bar{r}')]ds' \\
& - \frac{\partial}{\partial y} \iint_A g_o(\bar{r}|\bar{r}')[T_1(\bar{r}') + T_2(\bar{r}')]ds' \\
& + \frac{1}{k^2} \iint_{\text{Obstacle}} [\frac{\partial^2}{\partial z \partial x} G_A^{c,xx}(\bar{r}|\bar{r}')\ J_x(\bar{r}') +
\end{aligned}
$$

$$+ \frac{\partial^2}{\partial z \partial y} G_A^{c,yy}(\bar{r}|\bar{r}') \; J_y(\bar{r}')$$

$$+ \left(\frac{\partial^2}{\partial z^2} + k^2\right) G_A^{c,zz}(\bar{r}|\bar{r}') \; J_z(\bar{r}')]ds'$$

Next the integration by parts formula,

$$\int_D u \frac{\partial v}{\partial x_i} dx = - \int_D v \frac{\partial u}{\partial x_i} dx + \int_D u \, v \, n_{x_i} d\ell' \tag{4.18}$$

where n_{x_i} is the component of the outward unit normal in the $\hat{x}_i$ direction, is applied to the $\hat{E}^{inc}$ terms. This yields

$$\frac{\partial}{\partial x} \iint_A g_o(\bar{r}|\bar{r}') \frac{\partial}{\partial z} E_x^{inc}(\bar{r}')ds' = \iint_A g_o(\bar{r}|\bar{r}') \frac{\partial^2}{\partial x \partial z} E^{inc}(\bar{r}')ds'$$

$$- \int_{\partial A} g_o(\bar{r}|\bar{r}') \frac{\partial}{\partial z} E_x^{inc}(\bar{r}') \; n_x(r')ds' \tag{4.19a}$$

and

$$\frac{\partial}{\partial y} \iint_A g_o(\bar{r}|\bar{r}') \frac{\partial}{\partial z} E_y^{inc}(\bar{r}')ds' = \iint_A g_o(\bar{r}|\bar{r}') \frac{\partial^2}{\partial y \partial z} E_y^{inc}(\bar{r}')ds'$$

$$- \int_{\partial A} g_o(\bar{r}|\bar{r}') \frac{\partial}{\partial z} E_y^{inc}(\bar{r}') \; n_y(\bar{r}')ds' \; . \tag{4.19b}$$

Addition of the first two terms on the right of (4.19a) and (4.19b) yields

$$T(\bar{r}) = \iint_A g_o(\bar{r}|\bar{r}') \left[\frac{\partial^2}{\partial x \partial z} \hat{E}_x^{inc}(\bar{r}') + \frac{\partial^2}{\partial y \partial z} \hat{E}_y^{inc}(\bar{r}')\right] ds' \tag{4.20}$$

Since $\hat{E}^{inc}$ is divergence-free away from the impressed source, Green's second identity (4.9) gives

$$T(\bar{r}) = - \hat{E}_z^{inc}(\bar{r}) + \int_{\partial A} [g_o(\bar{r}|\bar{r}') \frac{\partial}{\partial n'} \hat{E}_z^{inc}(\bar{r}')$$

$$- \hat{E}_z^{inc}(\bar{r}') \frac{\partial}{\partial n'} g_o(\bar{r}|\bar{r}')] d\ell' \tag{4.21}$$

Thus (4.17) becomes

$$\frac{\partial}{\partial x} \int_{\partial A} g_o(\bar{r}|\bar{r}') \psi_x(\bar{r}') d\ell' + \frac{\partial}{\partial y} \int_{\partial A} g_o(\bar{r}|\bar{r}') \psi_y(\bar{r}') d\ell'$$

$$= - 4 \hat{E}_z^{inc}(\bar{r}) + 2 \int_{\partial A} [g_o(\bar{r}|\bar{r}') \frac{\partial}{\partial n'} \hat{E}_z^{inc}(r')$$

$$- \hat{E}_z^{inc}(\bar{r}') \frac{\partial}{\partial n'} g_o(\bar{r}|\bar{r}')] d\ell'$$

$$- 2 \int_{\partial A} g_o(\bar{r}|\bar{r}') \left[\frac{\partial}{\partial z} \hat{E}_y^{inc}(\bar{r}') n_y(\bar{r}') + \frac{\partial}{\partial z} \hat{E}_x^{inc}(\bar{r}') n_x(\bar{r}')\right] ds'$$

$$- \frac{\partial}{\partial y} \iint_A g_o(\bar{r}|\bar{r}') [T_1(\bar{r}') + T_2(\bar{r}')] d\ell'$$

$$- \frac{\partial}{\partial x} \iint_A g_o(\bar{r}|\bar{r}') [T_3(\bar{r}') + T_4(\bar{r}')] ds' +$$

$$+ \frac{1}{k^2} \iint_{\text{Obstacle}} [\frac{\partial^2}{\partial z \partial x} G_A^{c,xx}(\bar{r}|\bar{r}') \; J_x(\bar{r}')$$

$$+ \frac{\partial^2}{\partial z \partial y} G_A^{c,yy}(\bar{r}|\bar{r}') \; J_y(\bar{r}') + (\frac{\partial^2}{\partial z^2} + k^2) G_A^{c,zz}(\bar{r}|\bar{r}') J_z(\bar{r}')] ds'. \tag{4.22}$$

Similarly (4.18) may be used to eliminate the derivatives on the integrals involving the T_1, T_2, T_3, and T_4 terms in (4.22).

In conclusion (4.13) and the constraint equation (4.22) enforced in the aperture, plus the boundary condition that the tangential component of $\bar{E}$ vanish on the perfectly conducting obstacle (4.1c) as well as on the aperture boundary is sufficient to determine the electromagnetic fields.

CHAPTER 5

NUMERICAL FORMULATION FOR CAVITY WITHOUT OBSTACLE

Numerical solution to the exterior-interior coupling problem of Chapter 4 could be accomplished by discretizing the system of integral equations. However for arbitrary obstacle geometrics the problem requires inordinate matrix sizes. However if the obstacle were taken to be a thin-wire parallel to a coordinate axis only one component of the current would have to be taken into account. This would reduce the matrix size, as well as the number of computations needed to solve the problem discussed in Chapter 4. Since the numerical difficulties associated with the integral equation obtained by setting the tangential $\hat{E}$ (4.1c) to zero on the thin-wire surface have been overcome by Seidel [1977], we consider the exterior-interior coupling problem (Figure 1) with no scattering obstacle. Equation set (4.13) for the aperture fields becomes

$$\iint_A [g_F^{hs,xx}(\bar{r}|\bar{r}') + g_F^{c,xx}(\bar{r}|\bar{r}')]\, E_y(\bar{r}')ds'$$

$$= \int_{\partial A} g_o(\bar{r}|\bar{r}')\, \tilde{\psi}_y(\bar{r}')d\ell'$$

$$- 2 \iint_A g_o(\bar{r}|\bar{r}')\, \frac{\partial}{\partial z} E_y^{inc}(r')ds' \qquad (5.1a)$$

$$\iint_A [g_F^{hs,yy}(\bar{r}|\bar{r}') + g_F^{c,yy}(\bar{r}|\bar{r}')]\ E_x(\bar{r}')ds'$$

$$= \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \tilde{\psi}_x(\bar{r}')d\ell'$$

$$- 2 \iint_A g_o(\bar{r}|\bar{r}')\ \frac{\partial}{\partial z} E_x^{inc}(\bar{r}')ds' \ . \tag{5.1b}$$

The constraint equation (4.22) reduces to

$$\frac{\partial}{\partial x} \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \tilde{\psi}_x(\bar{r}')d\ell' + \frac{\partial}{\partial y} \int_{\partial A} g_o(\bar{r}|\bar{r}')\ \tilde{\psi}_y(\bar{r}')d\ell' = -4\ E_z^{inc}(\bar{r})$$

$$+ 2 \int_{\partial A} [g_o(\bar{r}|\bar{r}')\ \frac{\partial}{\partial n'} E_z^{inc}(\bar{r}') - E_z^{inc}\ \frac{\partial}{\partial n'} g_o(\bar{r}|\bar{r}')]d\ell'$$

$$- 2 \int_{\partial A} g_o(\bar{r}|\bar{r}')\ [\frac{\partial}{\partial z} E_y^{inc}(\bar{r}')\ n_y(\bar{r}')$$

$$+ \frac{\partial}{\partial z} E_x^{inc}(\bar{r}')\ n_x(\bar{r}')]d\ell' \ . \tag{5.2}$$

In the above equations the "^" have been removed from the $\hat{E}$'s by multiplication by $-i\omega\mu$ (2.1c). Similarly the auxiliary unknowns ψ_x and ψ_y have been replaced by $\tilde{\psi}_x$ and $\tilde{\psi}_y$ which equal $-i\omega\mu$ times ψ_x and ψ_y respectively. The electric field must also satisfy the boundary condition that its tangential component vanish along the perfectly conducting aperture edges.

The method of moments will now be used to solve (5.1) subject to (5.2) plus the appropriate boundary conditions. First the aperture fields are expanded in pulses.

$$E_x(\bar{r}) = \sum_{i=1}^{NX(NY-2)} e_{xi}\, P_{xi}(\bar{r}) \qquad P_{xi}(\bar{r}) = \begin{cases} 1, & \bar{r} \in \text{ith x pulse} \\ 0, & \text{otherwise} \end{cases} \tag{5.3a}$$

$$E_y(\bar{r}) = \sum_{i=1}^{NY(NX-2)} e_{yi}\, P_{yi}(\bar{r}) \qquad P_{yi}(\bar{r}) = \begin{cases} 1, & \bar{r} \in \text{ith y pulse} \\ 0, & \text{otherwise} \end{cases} \tag{5.3b}$$

where NX and NY denote the number of divisions along aperture sides parallel to the $\hat{x}$ and $\hat{y}$ axes respectively (Figure 9). In general the pulses are assumed rectangular in shape. Similarly the auxiliary unknowns are expanded in boundary pulses.

$$\tilde{\psi}_x(\bar{r}) = \sum_{i=1}^{NT} c_{xi}\, P_{si}(\bar{r}) \qquad \tilde{\psi}_y(\bar{r}) = \sum_{i=1}^{NT} c_{yi}\, P_{si}(\bar{r}) \tag{5.4}$$

where

$$P_{si}(\bar{r}) = \begin{cases} 1, & \bar{r} \in \text{ith boundary pulse} \\ 0, & \text{otherwise} \end{cases} \text{ and } NT = 2(NX + NY).$$

Next the system of integral equations is converted to a matrix equation. The total number of unknowns in (5.1) and (5.2) is $2NX \cdot NY + 2NX + 2NY$. First (5.1b) is enforced at nonzero aperture pulse midpoints. Then (5.1a) is enforced at nonzero aperture pulse midpoints. This yields a total of $2NX \cdot NY - 2NX - 2NY$ equations. Next

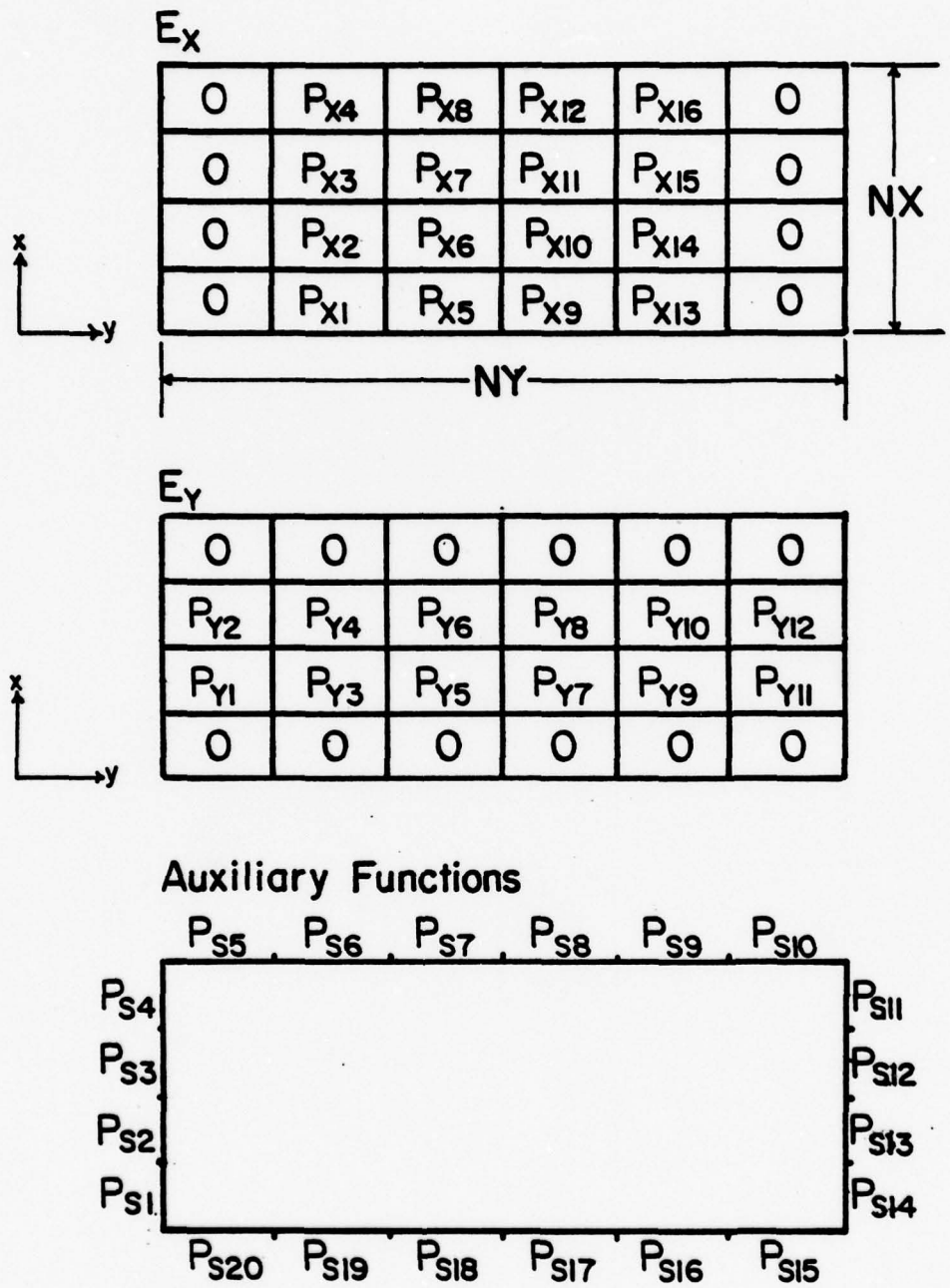


Figure 9. Expansion Functions

(5.1b) is enforced at the zero side pulse midpoints and likewise for (5.1a). The total number of equations is now 2 NX•NY for 2NX•NY + 2NX + 2NY unknowns. It is sufficient to enforce (5.2) only along the aperture boundary [Mittra et al., 1973]. Enforcing (5.2) at the boundary pulse midpoints yields the remaining 2NX + 2NY equations. The resulting matrix equation is given by

$$\begin{bmatrix} GYY & 0 & G\psi X & 0 \\ 0 & GXX & 0 & G\psi Y \\ GYS & 0 & G\psi XS & 0 \\ 0 & GXS & 0 & G\psi YS \\ 0 & 0 & CX & CY \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ c_x \\ c_y \end{bmatrix} = \begin{bmatrix} FX \\ FY \\ FXS \\ FYS \\ FC \end{bmatrix} . \tag{5.5}$$

The partitioned matrices are given below.

$$GYY(i,j) = \iint_{P_{xj}} [g_F^{hs,yy}(\bar{r}_{cxi}|\bar{r}') + g_F^{c,yy}(\bar{r}_{cxi}|\bar{r}')]ds' \tag{5.6a}$$

$$GXX(i,j) = \iint_{P_{yj}} [g_F^{hs,xx}(\bar{r}_{cyi}|\bar{r}') + g_F^{c,xx}(\bar{r}_{cyi}|\bar{r}')]ds' \tag{5.6b}$$

$$G\psi X(i,j) = -\iint_{P_{xj}} g_o(\bar{r}_{bi}|\bar{r}')ds' \tag{5.6c}$$

$$G\psi Y(i,j) = -\iint_{P_{yj}} g_o(\bar{r}_{bi}|\bar{r}')ds' \tag{5.6d}$$

$$GYS(i,j) = \iint_{P_{xj}} [g_F^{hs,yy}(\bar{r}_{xsi}|\bar{r}') + g_F^{c,yy}(\bar{r}_{xsi}|\bar{r}')]ds' \tag{5.6e}$$

$$GXS(i,j) = \iint_{P_{yj}} [g_F^{hs,xx}(\bar{r}_{ysi}|\bar{r}') + g_F^{c,xx}(\bar{r}_{ysi}|\bar{r}')]ds' \tag{5.6f}$$

$$G\psi XS(i,j) = - \iint_{P_{xj}} g_o(\bar{r}_{xsi}|\bar{r}')ds' \tag{5.6g}$$

$$G\psi YS(i,j) = - \iint_{P_{yj}} g_o(\bar{r}_{ysi}|\bar{r}')ds' \tag{5.6h}$$

$$CX(i,j) = \frac{\partial}{\partial x} \iint_{P_{sj}} g_o(\bar{r}_{bi}|\bar{r}')ds' \tag{5.6i}$$

$$CY(i,j) = \frac{\partial}{\partial y} \iint_{P_{sj}} g_o(\bar{r}_{bi}|\bar{r}')ds' \tag{5.6j}$$

In the above equations $\bar{r}_{cxi}$, $\bar{r}_{cyi}$, $\bar{r}_{bi}$, $\bar{r}_{xsi}$, and $\bar{r}_{ysi}$ denote the centerpoints of the ith x pulses, ith y pulses, ith boundary pulses, ith zero x side pulses, and ith zero y side pulses, respectively. The forcing vectors are given by

$$FX(i) = - 2 \iint_A g_o(\bar{r}_{cxi}|\bar{r}') \frac{\partial E_x^{inc}}{\partial z} (\bar{r}')ds' \tag{5.7a}$$

$$FY(i) = - 2 \iint_A g_o(\bar{r}_{cyi}|\bar{r}') \frac{\partial E_y^{inc}}{\partial z} (\bar{r}')ds' \tag{5.7b}$$

$$FXS(i) = 2 \iint_A g_o(\bar{r}_{xsi}|\bar{r}') \frac{\partial E_x^{inc}}{\partial z} (\bar{r}')ds' \tag{5.7c}$$

$$FYS(i) = -2 \iint_A (g_o(\bar{r}_{ysi}|\bar{r}') \frac{\partial E_y^{inc}}{\partial z} (\bar{r}')ds' \tag{5.7d}$$

$$FC(i) = -4 E_z^{inc}(\bar{r}_{bi}) + 2 \int_{2A} [g_o(\bar{r}_{bi}|\bar{r}') \frac{\partial E_z^{inc}}{\partial n'} (\bar{r}')$$

$$- E_z^{inc}(\bar{r}') \frac{\partial g_o}{\partial n'} (\bar{r}_{bi}|\bar{r}')]d\ell' \tag{5.8e}$$

$$- 2 \int_{\partial A} g_o(\bar{r}_{bi}|\bar{r}') [\frac{\partial E_y^{inc}}{\partial z} (\bar{r}')n_y(\bar{r}') \frac{\partial E_x^{inc}}{\partial z} (\bar{r}')n_x(\bar{r}')]d\ell' .$$

The major obstacle in filling the needed matrices is that the cavity dyads in (5.6a), (5.6b), (5.6e) and (5.6f) given by Table 3 are slowly converging as source and observation point become close. The summation of these series and the techniques used are considered important enough to merit a seperate chapter (Chapter 6). The remaining matrix elements are filled first by analytical integration of singular contributions, then by Gaussian quadrature integration of the remaining parts [Abramowitz and Stegun 1970]. Evaluation of the Hankel function of the second kind, needed for $g_o(\bar{r}|\bar{r}')$, is done by means of a "polynomial approximation" [Abramowitz and Stegun 1970]. To evaluate the forcing functions (5.7) the incident fields (4.14) are approximated by the centerpoint values on each aperture pulse (Figure 7). For normally

incident excitation this approximation becomes exact. The only other obstacle to matrix fill is the bookkeeping necessary to insure that all elements are computed with the least amount of computations as possible.

The matrix of the linear system (5.5) can be numerically inverted in a manner that makes advantage of the large blocks of zeros. Once this matrix is inverted it may be used to solve the aperture field responses due to different polarizations and incidence angles of the incident plane wave field. The method of partitioning is used to invert the matrix.

It is straightforward to show [Faddeev and Faddeeva 1963] that if

$$S = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{5.9}$$

where A and D square matrices of order p and q respectively, then

$$S^{-1} = \left[\begin{array}{c|c} K & L \\ \hline M & N \end{array}\right] \tag{5.10}$$

where K and N are square matrices of order p and q respectively. The partitioned matrices of S^{-1} are given by

$$\begin{aligned} N &= (D - CA^{-1}B)^{-1} & L &= -A^{-1}BN \\ M &= -NCA^{-1} & K &= A^{-1} - A^{-1}BM \end{aligned} \tag{5.11}$$

provided that each inverse exists.

This scheme can be applied to the matrix of (5.5) where

$$A = \left[\begin{array}{c|c} GYY & 0 \\ \hline 0 & GXX \end{array}\right] \qquad B = \left[\begin{array}{c|c} G\psi X & 0 \\ \hline 0 & G\psi Y \end{array}\right]$$

$$C = \left[\begin{array}{c|c} GYS & 0 \\ \hline 0 & GXS \\ \hline 0 & 0 \end{array}\right] \qquad D = \left[\begin{array}{c|c} G\psi XS & 0 \\ \hline 0 & G\psi YS \\ \hline CX & CY \end{array}\right] \tag{5.12}$$

GYY and GXX are square matrices of order NX(NY-2) and NY(NX-2). For large NX and NY these matrices will dominate considerations of computer storage. Matrices B, C, and D are [NX(NY-2) + NY(NX-2)] x 4(NY + NY), 4(NX + NY) x [NX(NY-2) + NY(NX-2)], and square 4(NX + NY) matrices respectively. Once K, L, M and N are found the fields can be obtained by matrix multiplications.

CHAPTER 6

EVALUATION OF CAVITY POTENTIAL TERMS

The final obstacle to solution of the exterior-interior coupling problem of Chapter 5 is the matrix fill of elements involving the cavity vector potential g_F^c in (5.6). Table 3 gives this dyad represented as a threefold infinite eigenfunction expansion. For source and observation points distant, it will be seen that this threefold infinite eigenfunction expansion reduces to a rapidly exponentially convergent double series. However as source and observation points become close, the series becomes slowly convergent. This chapter develops an efficient means of evaluation of the cavity potential dyad $\bar{\bar{g}}_F^c$ for source and observation point close under the restriction that ka (Figure 1) is less than π.

Consideration is now given to the $\hat{x}\hat{x}$ component of the cavity dyad $g_F^{c,xx}$. Similiar techniques hold for $g^{c,yy}$. From Table 3 with $z = z' = c$ (Figure 1)

$$g_F^{c,xx}(\bar{r}|\bar{r}') = \frac{1}{abc} \sum_{\substack{\ell=0 \\ m=1 \\ n=0}}^{\infty} \frac{\varepsilon_\ell \varepsilon_m \varepsilon_n}{K_{\ell mn}^2 - k^2} \sin(k_x x)\sin(k_x x')\cos(k_y y)\cos(k_y y') \tag{6.1a}$$

$$= S(|x-x'|,|y-y'|) + S\,(|x-x'|,\ y+y') - S\,(x+x',\ y+y')$$

$$- S\,(x+x',\ |y-y'|) \tag{6.1b}$$

where

$$S(X,Y) = \frac{1}{4abc} \sum_{\substack{\ell=0 \\ m=1 \\ n=0}}^{\infty} \frac{\varepsilon_\ell \varepsilon_m \varepsilon_n}{K_{\ell mn}^2 - k^2} \cos(k_x X) \cos(k_y Y) \ . \tag{6.1c}$$

Since the terms of (6.1b) depend only on sums and differences of the respective source and observation points' components, the number of calculations in evaluating (5.6b) and (5.6f) is less than if (6.1a) were applied directly.

For source and observation points distant the threefold infinite series of (6.1c) may be reduced to a double sum that yields itself to rapid evaluation by digital computer. Equation (6.1c) is closed in Y [Gradshteyn and Ryzhik 1965] which yields

$$S(X,Y) = \frac{1}{2ac} \sum_{\substack{\ell=0 \\ m=1}}^{\infty} \frac{\varepsilon_\ell \cosh[\sqrt{k_x^2 + k_y^2 - k^2}\,(b-Y)]}{\sqrt{k_x^2 + k_y^2 - k^2}\,\sinh[\sqrt{k_x^2 + k_y^2 - k^2}\,b]} \cos(k_x X) \tag{6.2a}$$

with ka (Figure 1) less than π. The series in (6.2a) is exponentially converging and readily evaluated provided that Y is not too small with respect to the aperture dimensions and not too close to 2b. Likewise if (6.1c) is closed in X one obtains

$$S(X,Y) = \frac{1}{4bc} \sum_{\substack{\ell=0 \\ n=0}}^{\infty} \varepsilon_\ell \varepsilon_n \, T(X) \cos(k_y Y) - \frac{1}{4abc} \sum_{\substack{\ell=0 \\ n=0}}^{\infty} \frac{\varepsilon_\ell \varepsilon_n \cos(k_y Y)}{k_y^2 + k_z^2 - k^2} \tag{6.2b}$$

where

$$T(X) = \begin{cases} \dfrac{\cosh[\sqrt{k_y^2 + k_z^2 - k^2}\,(a-X)]}{\sqrt{k_y^2 + k^2 - k^2}\,\sinh[\sqrt{k_y^2 + k_z^2 - k^2}\,a]}, & \text{if } k_y^2 + k_z^2 - k^2 > 0 \\[2ex] -\dfrac{\cos[\sqrt{k^2 - k_y^2 - k_z^2}\,(a-X)]}{\sqrt{k^2 - k_y^2 - k_z^2}\,\sin(\sqrt{k^2 - k_y^2 - k_z^2}\,a)}, & \text{if } k_y^2 + k_z^2 - k^2 < 0 \end{cases}$$

Similar techniques may be used to close the second term of (6.2b). The result is a single infinite series with known asymptotic form [Abramowitz and Stegun 1970, p. 1005]. Thus this series may readily be evaluated. First termwise subtraction of the corresponding asymptotic term from the original series term yields a rapidly converging new series. Next the otherwise slowly converging asymptotic series is added to the modified series by use of its known analytical form. The first term of (6.2b) is exponentially convergent provided X is not too small or too close to 2a. In the aperture geometries considered (Figure 1), the only case where neither (6.2a) or (6.2b) yields itself readily to numerical evaluation is for both X and Y small compared to the cavity dimensions. Otherwise, Seidel [1977] has obtained an efficient method for numerical evaluation of these sums.

The remainder of this chapter is concerned with evaluation of (6.1c) near the source, that is for $X = |x-x'|$ and $Y = |y-y'|$ small. Equation (6.1c) is closed in z, making use of Spiegel [1964, p. 189] and Gradshteyn and Ryzhik [1965, p. 40]. This yields

$$S(X,Y) = \frac{1}{2ab} \sum_{\substack{m=1 \\ n=0}}^{\infty} \varepsilon_n \, T(k_x^2 + k_y^2 - k^2) \cos(k_x X) \cos(k_y Y) \qquad (6.3)$$

where

$$T(k_x^2 + k_y^2 - k^2) = \begin{cases} \dfrac{\coth(c\sqrt{k_x^2 + k_y^2 - k^2}}{\sqrt{k_x^2 + k_y^2 - k^2}} & , \text{ if } k_x^2 + k_y^2 - k^2 > 0 \\[2ex] \dfrac{-\cot(c\sqrt{k^2 - k_x^2 - k_y^2}}{\sqrt{k^2 - k_x^2 - k_y^2}} & , \text{ if } k_x^2 + k_y^2 - k^2 < 0 \, . \end{cases}$$

This series in (6.3) has an asymptotic form

$$S_{ASY}(X,Y) = \frac{1}{2ab} \sum_{\substack{m=1 \\ n=0}}^{\infty} \frac{\varepsilon_n}{\sqrt{k_x^2 + k_y^2 - k^2}} \cos(k_x X) \cos(k_y Y) \, . \qquad (6.4)$$

The difference series whose terms consists of the original series terms (6.3) minus the corresponding asymptotic series terms is readily evaluated by digital computer. Thus (6.3) may be evaluated if a method of evaluating the otherwise slowly converging asymptotic series may be found.

To evaluate $S_{ASY}(X,Y)$ (6.4) consider the single series

$$S_o(Y) = \sum_{n=0}^{\infty} \frac{\varepsilon_n}{\sqrt{k_y^2 + u^2}} \cos(k_y Y) \qquad (6.5)$$

where Y is greater than zero and u^2 equals $k_x^2 - k^2$ which is greater than zero under the restriction that ka is less than π. An alternate

expression for $S_o(Y)$ will be obtained from Poisson's formula [Papoulis 1962]

$$\sum_{n=-\infty}^{\infty} f(nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F\left(\frac{2n\pi}{T}\right) \tag{6.6a}$$

where f(t) is continuous and its Fourier transform $F(\omega)$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \ . \tag{6.6b}$$

Equation (6.5) is rewritten as

$$S_o(Y) = Y \sum_{n=-\infty}^{\infty} \frac{\cos\left(n \frac{\pi Y}{b}\right)}{\sqrt{\left(\frac{n\pi Y}{b}\right)^2 + v^2}} \ , \tag{6.7}$$

where v equals Yu. In (6.6a) f(t) is identified as $\frac{\cos(t)}{\sqrt{t^2 + v^2}}$. Its Fourier transform is given by

$$F(\omega) = \int_{-\infty}^{\infty} \frac{e^{-i\omega\tau} \cos(t)}{\sqrt{\tau^2 + v^2}} dt \tag{6.8a}$$

$$= \int_{0}^{\infty} \frac{\cos[(1 + \omega)t]}{\sqrt{t^2 + v^2}} dt + \int_{0}^{\infty} \frac{\cos[(\omega-1)t]}{\sqrt{\tau^2 + v^2}} dt \tag{6.8b}$$

$$= K_o(|1 + \omega| v) + K_o(|\omega - 1| v) \tag{6.8c}$$

Equation (6.8b) follows from (6.8a) plus symmetry considerations. The integrals of (6.8b) result in the modified Bessel functions of (6.8c) [Morse and Feshbach 1953, p. 1323]. Application of (6.6a) to (6.7) yields

$$S_o(Y) = \frac{b}{\pi} \sum_{n=-\infty}^{\infty} [K_o(|Y + 2nb| \sqrt{k_x^2 - k^2}) + K_o(|2nb - Y| \sqrt{k_x^2 - k^2})] \tag{6.9a}$$

$$= \frac{2b}{\pi} K_o(\sqrt{k_x^2 - k^2}\, Y) + \frac{2b}{\pi} \sum_{n=1} [K_o([2nb + Y]\sqrt{k_x^2 - k^2})$$

$$+ K_o([2nb - Y]\sqrt{k_x^2 - k^2})] \ . \tag{6.9b}$$

Application of (6.9b) to (6.4) yields

$$S_{ASY}(X,Y) = \frac{1}{a\pi} \sum_{m=1}^{\infty} K_o(\sqrt{k_x^2 - k^2}\, Y) \cos(k_x X)$$

$$+ \frac{1}{a\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [K_o([2nb + Y]\sqrt{k_x^2 - k^2})$$

$$+ K_o([2nb - Y]\sqrt{k_x^2 - k^2})] \cos(k_x X) \ . \tag{6.10}$$

The second term on the right of (6.10) is very rapidly convergent due to the exponential behavior of K_o's asymptotic expansion [Arfken 1971, p. 517]. The first term on the right of (6.10) is exponentially convergent for $Y > 0$. However, as Y becomes smaller the series converges

more and more slowly. A summation formula for the first term is given Lewin [1951, p. 83]. Thus

$$\frac{1}{a\pi}\sum_{m=1}^{\infty} K_o(\sqrt{k_x^2 - k^2}\, Y)\cos(k_x X) = \frac{Y_o(kY)}{4a}$$

$$+ \frac{1}{2\pi}\frac{\cos[k\sqrt{X^2 + Y^2}]}{\sqrt{X^2 + Y^2}} \tag{6.11}$$

$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{\cos[k\sqrt{Y^2 + (2na + X)^2}]}{\sqrt{Y^2 + (2na + X)^2}} + \frac{\cos[k\sqrt{Y^2 + (2na - X)^2}]}{\sqrt{Y^2 + (2na - X)^2}} .$$

Note that the dominant singularity in (6.11) is the same as that of the half space dyadic $\bar{\bar{g}}_F$ of (3.9a). Unfortunately the infinite series (6.11a) is slowly convergent. However if one approximates each term of this series by a finite Taylor series in X and Y about the origin, the summation over n may be obtained by analytical methods. For example, the first term obtained in the Taylor series is given by

$$\sum_{n=1}^{\infty} \frac{\cos(2kan)}{na} = -\ \ell n[2\sin(ka)] \tag{6.12}$$

[Abramowitz and Stegun 1970, p. 1005]. Similar formulas may be used to evaluate the higher order terms in the Taylor series approximation.

Although the above discussion has been restricted to the evaluation of $g_F^{c,xx}$ only minor changes are needed to evaluate $g_F^{c,yy}$. Also the terms that must be evaluated (5.6) are integrals of the $\hat{x}\hat{x}$ and $\hat{y}\hat{y}$ components of $\bar{\bar{g}}_F^c$. The sums of (6.2) may be evaluated by term by term

integration [Titchmarsh 1939, pp. 36-45]. The exponentially convergent terms (6.10) are also evaluated termwise. The only remaining question is the accuracy of the previously discussed Taylor series approximation (6.11). A sixth order Taylor series was used for numerical testing and yielded no relative error worse than $1 \cdot 10^{-6}$ in these terms when $\max(X,Y)/a \leq .05$. This procedure allows rapid evaluation of the series on the right of (6.11), provided that both X and Y are small with respect to the cavity dimensions in the aperture plane. Otherwise this series would not readily yield itself to numerical evaluation on a digital computer.

CHAPTER 7

CONCLUSION

In this paper, the task was undertaken to formulate the problem of an aperture in a perfectly conducting infinite plane backed by a rectangular cavity with a wire obstacle. The work of Seidel [1977, 1978] was extended to include apertures large enough so that the dipole moment approximation is no longer necessary. Although the theory presented is valid for more general shapes of obstacles, the particular problem of interest is a thin wire parallel to a coordinate axis. This will reduce matrix sizes and computation time.

One essential aspect of this formulation has been the dyadic Green's functions. Their completeness properties and singularities received special attention. Another important result was the summation technique of Chapter 6 under the restriction that ka (Figure 1) is less than π. This technique allowed summation of magnetic source vector potential dyads with source and observation points close. Without this technique the dyadic series would become very slowly convergent as source and observation points become close. This result made consideration of large apertures and the removal of dipole approximations possible.

Future Work

At this time numerical results should be possible for the problem without the wire obstacle (Chapter 5). The computer program

should be checked by removal of cavity and comparison with the literature, especially the case of small apertures. The problem of large matrix sizes and large computer run times could be reduced by symmetry considerations.

Addition of an obstacle of general shape would greatly increase matrix sizes. However if a thin wire parallel to two cavity sides were used, the problem should be feasible. The major numerical problems involved in addition to this thin-wire have been overcome by Seidel [1977, 1978]. The major remaining work is computation of coupling terms between wire and aperture fields. In addition to symmetry considerations, the use of a slot with the appropriate edge behavior of the fields built into the formulation (Chapter 3) would enable one to consider a one dimensional array of unknown aperture field elements. This procedure may somewhat complicate numerical considerations, but it would greatly reduce matrix sizes.

Finally the transient problem needs to be considered. The time domain solution can be obtained by inverse Fourier transformation. It is noted that many frequency values are needed. Some type of symmetry considerations or the previously discussed slot considerations should be considered to reduce computation time.

APPENDIX A

SOME SYMMETRY PROPERTIES OF DYADIC GREEN'S FUNCTIONS

The symmetry properties for $\bar{\bar{G}}_e$ and $\bar{\bar{g}}_h$ given by (2.9) and (2.14) are now verified. Consider $\bar{\bar{G}}_e$. Equation (2.6) with

$$\bar{P} = \bar{a} \cdot \tilde{\bar{\bar{G}}}_e(\bar{r}|\bar{r}') ,$$

where $\bar{a}$ is an arbitrary constant vector and

$$\bar{\bar{Q}} = \bar{\bar{G}}_e(\bar{r}|\bar{r}'')$$

yields

$$\iiint_V [\bar{a} \cdot \tilde{\bar{\bar{G}}}_e(\bar{r}|\bar{r}') \cdot \nabla \times \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}'') - (\nabla \times \nabla \times \bar{a} \cdot \bar{\bar{G}}_e(\bar{r}|\bar{r}'))$$

$$\cdot \bar{\bar{G}}_e(\bar{r}|\bar{r}'')]dv = 0 ,$$

since the surface integral vanishes by (2.11b) and the radiation condition. Noting

$$\bar{a} \cdot \tilde{\bar{\bar{G}}}_e(\bar{r}|\bar{r}'') = \bar{\bar{G}}_e(\bar{r}|\bar{r}'') \cdot \bar{a} ,$$

and using (2.7) yields

$$\bar{a} \cdot \tilde{\bar{\bar{G}}}_e(\bar{r}''|\bar{r}') = \bar{a} \cdot \bar{\bar{G}}_e(\bar{r}'|\bar{r}'') .$$

Sine $\bar{a}$ is arbitary we have the desired result,

$$\tilde{\bar{\bar{G}}}_e(\bar{r}''|\bar{r}') = \bar{\bar{G}}_e(\bar{r}'|\bar{r}'') \ .$$

The same method applied to $\bar{\bar{g}}_h$ with boundary condition (2.15a) and the radiation condition show

$$\tilde{\bar{\bar{g}}}_h(\bar{r}''|\bar{r}) = \bar{\bar{g}}_h(\bar{r}'|\bar{r}'') \ .$$

Another important symmetry property is

$$[\nabla' \times \bar{\bar{G}}_e(\bar{r}'|\bar{r})]^{\sim} = \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}') \ .$$

Use is made of the identity

$$\nabla \cdot (\bar{A} \times \bar{B}) = \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B}) \ .$$

It follows that

$$\nabla \cdot (\bar{Q} \times \nabla \times \bar{P}) = (\nabla \times \bar{P}) \cdot (\nabla \times \bar{Q}) - \bar{Q} \cdot \nabla \times \nabla \times \bar{P} \ .$$

Thus

$$\iiint_V [(\nabla \times \bar{P}) \cdot (\nabla \times \bar{Q}) - \bar{Q} \cdot (\nabla \times \nabla \times \bar{P})]dv$$

$$= \iint_S (\bar{Q} \times \nabla \times \bar{P}) \cdot \bar{n} \, ds \ .$$

If

$$\bar{P} = \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b} \quad \text{and} \quad \bar{Q} = \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a}$$

one obtains

$$\iiint_V [(\nabla \times \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) - k^2\bar{\bar{G}}_e(\bar{r}|\bar{r}_1)) \cdot \bar{a} \cdot \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b}$$

$$- \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a} \cdot (\nabla \times \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) - k^2\bar{\bar{g}}_h(\bar{r}|\bar{r}_2)) \cdot \bar{b}]dv$$

$$= - k^2 \iiint_V [\bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a} \cdot \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b}$$

$$- \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a} \cdot \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b}]dv$$

$$+ \iint_S [(\nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a}) \times (\nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b})] \cdot \bar{n}\, ds \ .$$

The surface term vanishes for $\bar{r}_1$ and $\bar{r}_2$ interior to v by the radiation condition plus the boundary condition that $\bar{n} \times \bar{\bar{g}}_e(\bar{r}|\bar{r}_2) = 0$ on a perfectly conducting surface. By (2.7c) and (2.12c) one obtains

$$\bar{a} \cdot \nabla_1 \times \bar{\bar{g}}_h(\bar{r}_1|\bar{r}_2) \cdot \bar{b} - \bar{b} \cdot \nabla_2 \times \bar{\bar{G}}_e(\bar{r}_2|\bar{r}_1) \cdot \bar{a}$$

$$= - k^2 \iiint_V [\bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a} \cdot \nabla \times \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b}$$

$$- \nabla \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a} \cdot \bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b}]dv \ .$$

Once the volume integral is shown to be zero we will have our final result

$$[\nabla_2 \times \bar{\bar{G}}_e(\bar{r}_2|\bar{r}_1')] = \nabla_1 \times \bar{\bar{g}}_h(\bar{r}_1|\bar{r}_2) .$$

By the divergence theorem the volume integral becomes

$$- k^2 \iint_S [(\bar{\bar{g}}_h(\bar{r}|\bar{r}_2) \cdot \bar{b}) \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) \cdot \bar{a}] \cdot \bar{n} \, ds$$

which is zero for $\bar{r}_2$ and $\bar{r}_1$ interior to v since

$$\bar{n} \times \bar{\bar{G}}_e(\bar{r}|\bar{r}_1) = 0$$

on a perfectly conducting surface and the radiation condition at infinity holds.

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In this work, the exterior-interior coupling problem of a cavity backed aperture in a perfectly conducting infinite sheet is considered. The cavity is assumed to be rectangular and contains a perfectly conducting obstacle. For numerical considerations the obstacle is taken to be a straight, thin wire, oriented perpendicular to one of the cavity walls. The problem is formulated in the frequency domain with an $e^{i\omega t}$ time dependence.

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The dyadic formulation of this vector electromagnetic boundary value problem is given. The controversy over the longitudinal wave functions and their contribution to field dyads is resolved. Specific attention is given to the singularities and completeness of the Green's dyads' eigenfunction expansions.

Numerically tractable integral equations for the aperture electric fields are obtained. A summation method is developed for evaluation of otherwise slowly converging eigenfunction expansions of potential dyads when source and observation points become close. Suggestions for future extensions of this work are discussed.