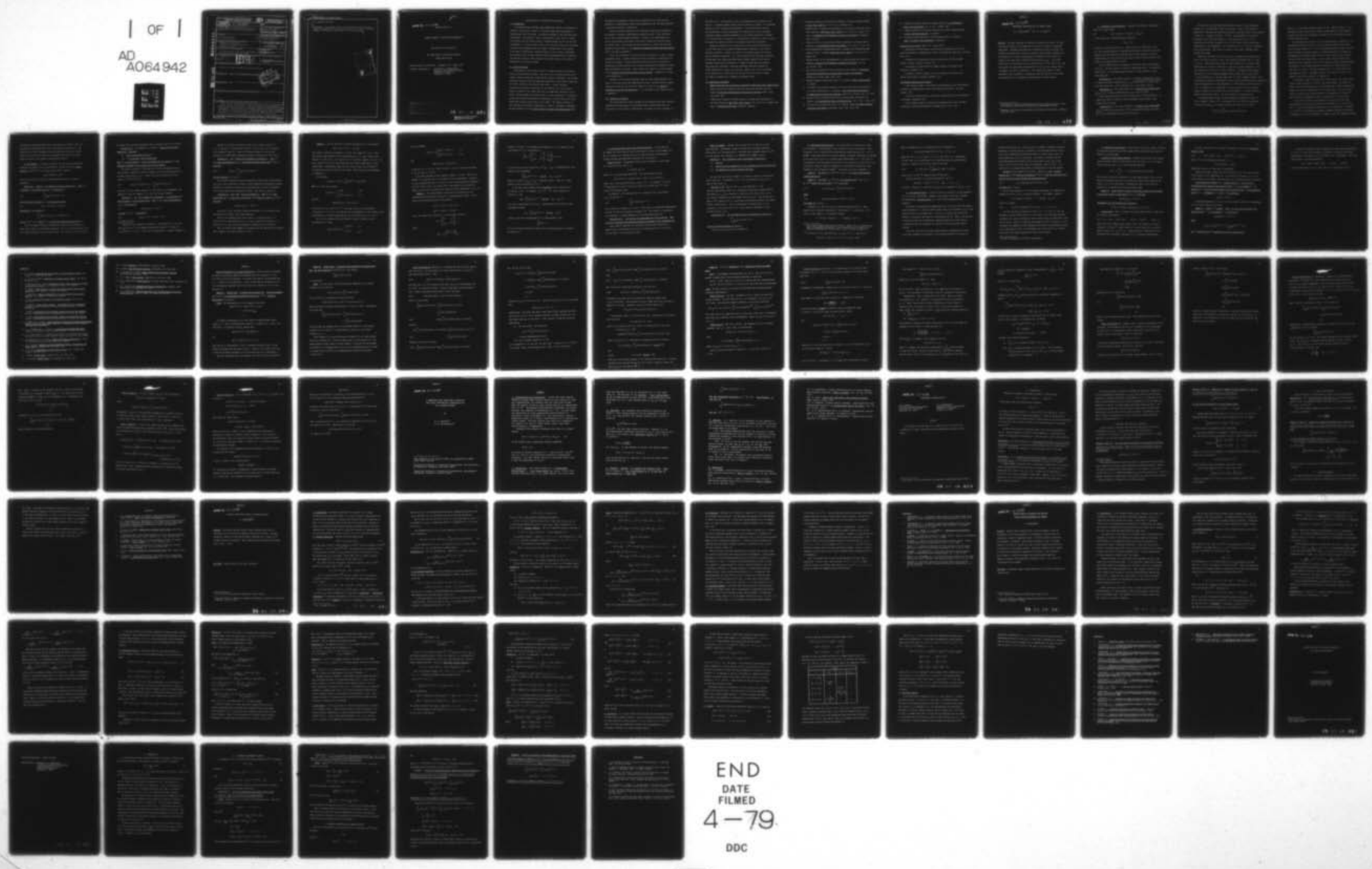
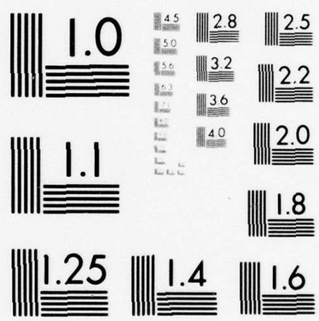
AD-A064 942 NORTHWESTERN UNIV EVANSTON IL DEPT OF MECHANICAL ENG--ETC F/G 17/7
OPTIMAL CONTROL OF SYSTEMS WITH UNCERTAINTY.(U)
1978 W E SCHMITENDORF AFOSR-76-2923
UNCLASSIFIED AFOSR-TR-78-1536 NL

| OF |
AD A064942



END
DATE
FILMED
4-79
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ADA064942

DDC FILE COPY

LEVEL II

DDC FEB 14 1979 RECEIVED

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: AFOSR-TR- 78-1536	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): OPTIMAL CONTROL OF SYSTEMS WITH UNCERTAINTY	5. TYPE OF REPORT & PERIOD COVERED: Interim rept. Sep 77-Aug 78
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): W.E. Schmitendorf	8. CONTRACT OR GRANT NUMBER(s): AFOSR-76-2923
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Northwestern University, Mechanical Engineering Department, Evanston, IL 60201	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F 2304/A1
11. CONTROLLING OFFICE NAME AND ADDRESS: Air Force Office of Scientific Research/NM, Bolling AFB, Washington, DC 20332	12. REPORT DATE: 1978
	13. NUMBER OF PAGES: 87
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 83p.	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The research being performed under AFOSR Grant 76-2923 is concerned with optimally controlling a system to a specified target when disturbances or uncertainties enter the system. Two general problems in this area are being investigated. One problem is that of determining if there exists a control that assures that the system reaches a specified target set for all possible disturbances. If such a control exists, then the next problem is to determine a control which guarantees that the target is reached and it also optimal in the sense of minimizing a specified measure of the system's

DD FORM 1 JAN 73 1473

410 129

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. Abstract continued.

performance. Our research effort has been directed toward obtaining methods for answering the questions raised by these problems and thereby aiding in the design of controllers for uncertain systems.

AFOSR-TR- 78-1536

Progress Report on

OPTIMAL CONTROL OF SYSTEMS WITH UNCERTAINTY

The research was supported by

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

GRANT AFOSR 76-2923

Period Covered By This Report: September 1977 - August 1978

Principal Investigator: Professor W. E. Schmitendorf
Mechanical Engineering Department
Northwestern University
Evanston, IL 60201

79 02 08 076

Optimal Control of Systems With Uncertainty

I. Introduction

The research being performed under AFOSR Grant 76-2923 is concerned with optimally controlling a system to a specified target when disturbances or uncertainties enter the system. Two general problems in this area are being investigated. One problem is that of determining if there exists a control that assures that the system reaches a sepcified target set for all possible disturbances. If such a control exists, then the next problem is to determine a control which guarantees that the target is reached and is also optimal in the sense of minimizing a specified measure of the system's performance. Our research effort has been directed toward obtaining methods for answering the questions raised by these problems and thereby aiding in the design of controllers for uncertain systems.

II. Results Obtained

The problem of controlling a system to a target belongs to the general area of controllability problems. Most results in this area have dealt with linear, constant coefficient systems without control constraints. But in many realistic situations there are magnitude constraints on the control values and the linear system has time-varying coefficients. We have obtained necessary and sufficient conditions for the existence of a control which steers the system to the origin when there are magnitude constraints on the control values and when the linear system is a time varying one. In addition, we have also devised a technique for determining a control which drives from system from a given initial state to the origin. The complete details of this research are included as Appendices A, B and C. The paper in Appendix A has been submitted to the S.I.A.M. Journal on Control and Optimization while the material in Appendices B and C will be presented at the 1978 Allerton Conference on Communication, Control and Computing and the 17th IEEE Conference on Decision and Control, respectively.

We have previously reported on the sufficient conditions we obtained for the problem of optimally controlling an uncertain system. These conditions can be used to design minmax controllers. Recently, our investigation has led to a more general sufficiency result which includes the previous results as special cases. This research has been reported in a paper which has been accepted for publication in the Journal of Optimization Theory and Applications and is included as Appendix D.

Often, the performance of a system cannot be measured by a single, scalar performance index. Instead, multiple criteria are needed to characterize the system's performance. We have obtained results which treat the problem of designing an optimal controller for a system with multiple performance criteria when disturbances are present. These results will appear in an invited paper in the Journal of Optimization Theory and Applications. Appendix E is a copy of this manuscript.

Finally, in a previously published paper on static minmax problems, the proof of the major result was complicated and long. We have devised a simpler and shorter proof and this proof will appear as a note in the Journal of Mathematical Analysis and Application. It is included in this progress report as Appendix F.

III. Research in Progress

The controllability results obtained are more general than those currently available since they apply to time-varying linear systems where there are constraints on the controls. Nevertheless, the results are not as general as they need to be. In particular, we are investigating three extensions of the results: (i) general target (rather than the origin as a target), (ii) nonlinear systems and (iii) systems with disturbances. The controllability problem with disturbances is being investigated in conjunction with Bruce Elenbogen, a graduate student in Applied Mathematics who is being supported by the grant. It is believed that our approach to controllability problems will lead to results that apply to nonlinear systems with a general target and disturbances.

Sufficient conditions for a minmax control are now well developed and it is felt that little can be gained by making minor extensions of the results currently available. Instead, our efforts are concentrating on the development of necessary conditions. It is quite important that necessary conditions be developed since complex problems usually require numerical solution and necessary conditions lend themselves to numerical algorithms more readily than sufficient conditions. Necessary conditions have been obtained for linear systems with a linear cost function. However, this class of problems is too narrow to be of much practical use and we are attempting to extend the ideas to problems with a quadratic performance index.

IV. Additional Information

Papers resulting from the research sponsored by AFOSR under Grant AFOSR 76-2923.

1. Static Multicriteria Problems: Necessary Conditions and Sufficient Conditions, Proceedings IFAC Symposium on Large Scale Systems, Udine, Italy, June 16-20, 1976.

2. A Sufficient Condition for Minmax Control of Systems with Uncertainty in the State Equations, IEEE Trans. Auto. Control, Vol. AC-21, No. 4, August 1976. (Also in Proceedings 1976 JACC, Lafayette, Indiana).

3. Necessary Conditions and Sufficient Conditions for Static Minmax Problems, J. Math. Anal. Applic., Vol. 57, No. 2, February 1977.

4. Minmax Control of Systems with Uncertainty in the Initial State and in the State Equations, IEEE Trans. Auto. Control, Vol. AC-22, No. 2, April 1977 (Also in Proceedings 1976 Conference on Decision and Control, Clearwater Beach, Florida).

5. A Note on the Use of the Direct Sufficient Conditions in Optimal Control Problems, J. of Optimization Theory and Applic., Vol. 23, No. 3, Nov. 1977.

6. Profit Maximization Through Advertising: A Nonzero Sum Differential Game Approach (with G. Leitmann), IEEE Trans. Auto. Control, Vol. AC-23, No. 4, August 1978.

7. Optimal Control of the End-Temperature in a Semi-Infinite Rod (with W. E. Olmstead), Zeitschrift für angewandte Mathematik und Physik, Vol. 28, pp. 697-706, 1977.

8. Multicriteria Optimization With Uncertainty in the Dynamics, Proceedings 1977 Allerton Conference on Communication, Control and Computing, Monticello, Illinois, Sept. 28-30, 1977.

9. Optimal Blowing to Reduce Drag (with W. E. Olmstead), SIAM J. Applied Math, (to appear).

10. A Necessary and Sufficient Condition for Local Constrained Controllability of a Linear System (with B. R. Barmish), Proc. 1978 Allerton Conference on Communication, Control and Computing, Monticello, Illinois, Oct. 4-6, 1978.

11. Optimal Control of Systems with Multiple Criteria When Disturbances are Present, J. of Optimization Theory and Applications, Vol. 27, No. 1, Jan. 1979.

12. Constrained Controllability (with B. R. Barmish), Proc. 17th IEEE Conference on Decision and Control, San Diego, Calif., Jan. 10-12, 1979.

13. A General Sufficiency Theorem for Minmax Control, J. of Optimization Theory and Applications, Vol. 27, No. 3, March, 1979.

14. A Simple Derivation of Necessary Conditions for Static Minmax Problems, J. Math. Analysis and Applic. (to appear).

15. Constrained Controllability of Linear Systems (with B. R. Barmish), SIAM J. On Control and Optimization (submitted).

Conferences and Lectures (Sept. 1977 - August 1978)

I presented an invited paper on multicriteria optimization at the 1977 Allerton Conference on Communication, Control and Computing, Monticello, Illinois, Sept. 1977.

I attended the International Forum On Alternatives For Multivariable Control, Chicago, Illinois, October, 1977.

I presented an invited lecture on minmax problems at Michigan State University, November, 1977.

I chaired a session on games at the 1977 Conference on Decision and Control, New Orleans, Louisiana, December, 1977.

Forthcoming Conferences and Lectures

I will present a paper on constrained controllability at the 1978 Allerton Conference on Communication, Control and Computing, Monticello, Illinois, October, 1978.

I have been invited to give a lecture on controllability at the University of Rochester, November, 1978.

I will present a paper on constrained controllability at the 17th IEEE Conference on Decision and Control, San Diego, Calif., Jan., 1979.

APPENDIX A

AFOSR-TR- 78-1536

CONSTRAINED CONTROLLABILITY OF LINEAR SYSTEMS

W. E. Schmitendorf* and B. R. Barmish**

Abstract. The paper considers the problem of steering the state of a linear time-varying system to the origin when the control is subject to magnitude constraints. Necessary and sufficient conditions are given for global constrained controllability as well as a necessary and sufficient condition for the existence of a control (satisfying the constraints) which steers the system to the origin from a specified initial epoch (x_o,t_o). The global result does not require zero to be an interior point of the control set Ω and the theorem for constrained controllability at (x_o,t_o) only requires that Ω be compact, not that it contain zero. The results are compared to those available in the literature. Furthermore, numerical aspects of the problem are discussed as is a technique for determining a steering control.

* Technological Institute, Northwestern University, Evanston, Illinois 60201
The research of this author was supported by AFOSR Grant 76-2923.

** Department of Electrical Engineering, The University of Rochester, Rochester, New York 14627

1. Introduction and Formulation. Consider the problem of steering the state of a linear system

$$\text{(S)} \qquad \dot{x}(t) = A(t)x(t) + B(t)u(t); \; t \in [t_o, \infty)$$

to the origin from a specified initial condition

$$x(t_o) = x_o$$

by choice of control function $u(\cdot)$. Here $x(t) \in R^n$, $u(t) \in R^m$ and $A(\cdot)$ and $B(\cdot)$ are continuous matrices of appropriate dimension. Unlike the usual controllability problem where the control values at each instant of time are unconstrained, we insist here that the control values at each instant of time belong to a prespecified set Ω in R^m.

Let $\mathfrak{m}(\Omega)$ denote the set of functions from R into Ω that are measurable on $[t_o, \infty)$. Then any control $u(\cdot) \in \mathfrak{m}(\Omega)$ is termed admissible. We now define three notions of constrained controllability or, more precisely, Ω-null controllability.

Definition 1.1. The linear system (S) is Ω-null controllable at (x_o, t_o) if, given the initial condition $x(t_o) = x_o$, there exists a $u(\cdot) \in \mathfrak{m}(\Omega)$ such that the solution $x(\cdot)$ of (S) satisfies $x(t) = 0$ for some $t \in [t_o, \infty)$.

Definition 1.2. The linear system (S) is globally Ω-null controllable at t_o if (S) is Ω-null controllable at (x_o, t_o) for all $x_o \in R^n$.

Our results will pertain to the above two types of controllability. To compare our results to those of other researchers, we also need a local controllability concept.

Definition 1.3. The linear system (S) is locally Ω-null controllable at t_o if there exists an open set $V \subset R^n$, containing the origin, such that (S) is null controllable at (x_o, t_o) for all $x_o \in V$.

The majority of constrained controllability results are for autonomous systems, i.e., systems where A and B are constant. When $\Omega = R^m$, Kalman [1] showed that a necessary and sufficient condition for global R^m-null controllability is rank (Q) = n where $Q \triangleq [B, AB, \cdots, A^{n-1}B]$. Lee and Markus [2] considered constraint sets $\Omega \subset R^m$ which contain u = 0 and showed that r(Q) = n is a necessary and sufficient condition for (S) to be locally Ω-null controllable. Furthermore, if each eigenvalue λ of A satisfies $Re(\lambda) < 0$, then (S) is globally Ω-null controllable. This result is typical of the results available when Ω contains the origin.

Saperstone and Yorke [3] were the first to eliminate the assumption that zero is an interior point of Ω when they considered problems with m = 1 and $\Omega = [0,1]$. Their result states that for these problems (S) is locally Ω-null controllable if and only if rank (Q) = n and A has no real eigenvalues. They also extend this result to m > 1 and $\Omega = \Pi_1^m[0,1]$. Problems with more general constraint sets were studied by Brammer [4] who showed that if there exists a $u \in \Omega$ satisfying Bu = 0 and the convex hull of Ω has a nonempty interior, then necessary and sufficient conditions for local Ω-null controllability are rank (Q) = n and the nonexistence of a real eigenvector v of A^T satisfying $v^T Bu \leq 0$ for all $u \in \Omega$. In addition, if no eigenvalue of A has a positive real part then the theorem becomes one for global Ω-null controllability. A similar result for global controllability when $\Omega = [0,1]$ was obtained by Saperstone [5].

For nonautonomous systems, the most familiar controllability result is that of Kalman [1] when $\Omega = R^m$. He showed that (S) is R^m-null controllable if and only if $W(t_o, t_1)$ is positive definite for some $t_1 \in [t_o, \infty)$ where

$$W(t_o, t_1) \triangleq \int_{t_o}^{t_1} \phi(t_1, \tau) B(\tau) B'(\tau) \phi'(t_1, \tau) d\tau$$

and $\phi(t,\tau)$ is the state transition matrix for (S). When the control is constrained, the major results are by Conti [6] and Pandolfi [7] who obtained necessary and sufficient conditions for global Ω-null controllability when Ω is the closed unit ball.

We also mention the results on Ω-null controllability of Dauer [8], [9] and Chukwu and Gronski [10] for a class of nonlinear systems satisfying a certain "growth condition". In [11], Grantham and Vincent consider the problem of steering a nonlinear system to a target and present a technique for determining the boundary between the set of states which can be steered to the target and those which cannot. More recently, Murthy and Evans [12] obtained results comparable to [3]-[5] for discrete linear systems and Pachter and Jacobson [13] developed sufficient conditions for controllability for case where $A(\cdot)$ and $B(\cdot)$ are time invariant and Ω is a closed convex cone containing the origin. A readable account of the state of the art is contained in the book by Jacobsen [14, Chapter 5].

In contrast to much of the work of previous authors, this paper concentrates on the case where $A(\cdot)$ and $B(\cdot)$ are time-varying. Our results for global Ω-null controllability are for constraint sets Ω that are compact and contain zero (but not necessarily as an interior point). One of our main results on global Ω-null controllability is an extension of a theorem of Conti [6] and it degenerates to Conti's theorem when Ω is a unit ball.

Our results for Ω-null controllability at (x_o,t_o) have even wider applicability since they do not require the existence of a $u \in \Omega$ such that $Bu = 0$. Thus we can analyze controllability of a system with, for example, $m = 1$ and $\Omega = [1,2]$ whereas the presently available theorems do not apply. Furthermore, as will be illustrated by examples, there are autonomous systems

(S) which are neither globally Ω-null controllable nor locally Ω-null controllable but nevertheless are Ω-null controllable at some (x_o, t_o). Our theorem can be used to decompose the state space into two sets. Initial states in one set can be steered to the origin while those in the other cannot be driven to the origin by an admissible control.

2. Main Results. In order to describe our necessary and sufficient conditions for global Ω-null controllability, we make use of the support function $H_\Omega : R^m \to R \cup \{\infty\}$ on Ω which for any $\alpha \in R^n$ is given by

$$H_\Omega(\alpha) \triangleq \sup\{\omega'\alpha : \omega \in \Omega\} \quad .$$

Using this notation, we have the following theorem, which is proved in Apppendix A.

Theorem 2.1. *Suppose* Ω *is a compact set which contains zero. Then,* (S) *is globally* Ω-*null controllable at* t_o *if and only if*

$$\int_{t_o}^{\infty} H_\Omega(B'(\tau)z(\tau))d\tau = +\infty \tag{2.1}$$

for all non-zero solutions $z(\cdot)$ *of the adjoint system*

$$\dot{z}(t) = -A'(t)z(t); \; t \in [t_o, \infty) \quad . \tag{S'}$$

Equivalently, if and only if

$$\int_{t_o}^{\infty} \sup\{\omega'B'(\tau)\phi'(t_o,\tau)\lambda : \omega \in \Omega\}d\tau = +\infty$$

for all $\lambda \in R^n$, $\lambda \neq 0$, *where* $\phi(t,\tau)$ *is the state transition matrix for* (S).

In the following corollary, we examine the special case of Theorem 2.1 which arises under the strengthened hypothesis "zero is an interior point of Ω." As we might anticipate, for this special case, the structure of the set Ω will not matter other than the requirement that it contains zero in its interior.

Corollary 2.2. (See Appendix A for proof): *Suppose there exists a compact set Ω such that*

(i) *zero is an interior point of* Ω;

(ii) (S) *is globally* Ω-*null controllable*.

Then (S) *is also globally* Ω'-*null controllable for any other set* Ω' (*not necessarily compact*) *which contains zero in its interior*.

Our proof of Theorem 2.1 will make use of a more fundamental result (also proven in Appendix A) giving conditions for Ω-null controllability at a fixed initial epoch (x_o, t_o). To meet this end, we define the scalar function $J : R^n \times R \times R^n \to R$ by

$$J(x_o, T, \lambda) \triangleq x_o' \Phi'(T, t_o)\lambda + \int_{t_o}^{T} H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau \tag{2.2}$$

Since we will only consider compact Ω in the sequel, we can guarantee (see Appendix A) that the integrals (2.1) and (2.2) are well defined.

Theorem 2.3. *Let* Ω *be a compact set and suppose* Λ *is any subset of* R^n *which contains* 0 *as an interior point*. *Then* (S) *is* Ω-*null controllable at* (x_o, t_o) *if and only if*

$$\min\{J(x_o, T, \lambda) : \lambda \in \Lambda\} = 0 \tag{2.3}$$

for some $T \in [t_o, \infty)$. *Equivalently*,

$$J(x_o, T, \lambda) \geq 0 \quad \text{for all} \quad \lambda \in \Lambda \tag{2.4}$$

for some $T \in [t_o, \infty)$.

Implicit in the statements of Theorem 2.1 and Theorem 2.3 is the fact that $H_\Omega(B'(\tau)z(\tau))$ is an integrable measurable function of τ along all trajectories $z(\cdot)$ of (S'). This fact is established as a lemma in Appendix A.

Theorem 2.3 can also be stated in terms of the adjoint system (S'). i.e., if we take $\Lambda = R^n$ and notice that $z(t) = \Phi'(t_o,t)z(t_o)$ is the response of the adjoint system (S'), then the following theorem is easily proven. (The proof is established by making the change of variables $z(t) \triangleq \phi'(T,t)\lambda$).

Theorem 2.3'. Let Ω satisfy the hypothesis of Theorem 2.3. Then (S) is Ω-null controllable at (x_o,t_o) if and only if there exists some $T \in [t_o,\infty)$ such that

$$x_o'z(t_o) + \int_{t_o}^{T} H_\Omega(B'(\tau)z(\tau))d\tau \geq 0 \tag{2.5}$$

for all solutions $z(\cdot)$ of (S').

This theorem demonstrates that the question of constrained controllability at (x_o,t_o) can be answered by solving a finite dimensional optimization problem. Moreover, the question of global Ω-null controllability can also be answered via a finite dimensional optimization problem. This result is given as

Corollary 2.4. Let Ω and Λ be as in Theorem 2.3. Then (S) is Ω-null controllable at t_o if and only if for every $x_o \in R^n$ there is a time $T_{x_o} \in [t_o,\infty)$ such that

$$\min\{J(x_o,T_{x_o},\lambda) : \lambda \in \Lambda\} = 0 \quad .$$

The proof of this corollary follows from Theorem 2.3 in conjunction with the definition of global Ω-null controllability.

There is one technical point worth noting. In using Theorem 2.1 to check for Ω-null controllability at t_o, Ω must be compact and contain 0. If Corollary 2.4 is used, only the compactness assumption must be satisfied.

Next, we present some examples to illustrate how our theorems can be applied and to compare our results to those of [3-5].

Example 1. Let x(t) and u(t) be scalars and suppose (S) is described by

$$\dot{x}(t) = x(t) + u(t) \quad , \quad t \in [0,\infty) \quad .$$

This system is R^m-null controllable if $\Omega = R^m$. But suppose $\Omega = [0,1]$. Then the system is not globally Ω-null controllable at $t_o = 0$. This follows from Theorem 2.1 since, for $z_o < 0$, $H_\Omega(B'(\tau)z(\tau)) = 0$ and thus $\int_o^\infty H_\Omega(B'(\tau)z(\tau)d\tau < +\infty$. Also, using [3] or [4] it can be shown that the system is not locally Ω-null controllable. Nevertheless, there do exist initial states x_o from which it is possible to steer the system to the origin. Such states can be determined via Theorem 2.3.

For the above

$$J(x_o,T,\lambda) = x_o e^T\lambda + \int_o^T \sup\{\omega e^{T-\tau}\lambda : \omega \in [0,1]\}d\tau$$

When $\Lambda = [-1,1]$, this becomes

$$J(x_o,T,\lambda) = \begin{cases} x_o e^T\lambda & \lambda \leq 0 \\ x_o e^T\lambda + \lambda(e^T - 1) & \lambda > 0 \end{cases}$$

and thus

$$\min\{J(x_o,T,\lambda) : \lambda \in [-1,1]\} = 0$$

if and only if $x_o \leq 0$ and $x_o > e^{-T} - 1$ for some $T \in [0,\infty)$, or equivalently, if and only if $-1 < x_o \leq 0$. We conclude that even though (S) is not locally Ω-null controllable,it is Ω-null controllable at $(x_o,0)$ if and only if $-1 < x_o \leq 0$.

If $\Omega = [1,2]$, neither [3-6] nor Theorem 2.1 apply. However, we can use Theorem 2.3. Since

$$H_\Omega(B'(\tau)\phi'(T,\tau)\lambda = \begin{cases} 2\lambda e^{(T-\tau)} & \lambda > 0 \\ \lambda e^{(T-\tau)} & \lambda \leq 0 \end{cases}$$

$J(x_o,T,\lambda)$ becomes

$$J(x_o,T,\lambda) = \begin{cases} x_o e^T\lambda + 2\lambda(e^T - 1) & \lambda > 0 \\ x_o e^T\lambda + \lambda(e^T - 1) & \lambda \le 0 \end{cases}$$

and

$$\min\{J(x_o,T,\lambda) : \lambda \in [-1,1]\} = 0$$

if and only if $-2 < x_o \le 0$. Thus (S) with $\Omega = [1,2]$ is Ω-null controllable at (x_o,t_o) when $-2 < x_o \le 0$.

As a final variation of this problem, suppose $\Omega = [-a,a]$. Then [4] or Theorem 2.1, shows that (S) is not globally Ω-null controllable. Using [4], it can be demonstrated that S is locally Ω-null controllable while Theorem 2.3 not only tells us that S is locally Ω-null controllable but also that the states x_o which can be steered to the origin are those satisfying $-a < x_o < a$.

Example 2. Our second example illustrates the application of our constrained controllability criteria for a two-dimensional system. We consider the time-varying system (S) described by

$$\dot{x}_1(t) = -x_1(t) + x_2(t) + u(t) \quad ;$$

$$\dot{x}_2(t) = -tx_2(t) \ ; \ t \in [0,\infty) \quad .$$

First, we compute the solution $z(\cdot)$ of the adjoint system (S')

$$z(t) = \begin{bmatrix} e^t & 0 \\ \theta(t) & e^{t^2/2} \end{bmatrix} z_o$$

where

$$\theta(t) = -\int_0^t e^{\frac{t^2}{2} + \xi - \frac{\xi^2}{2}} d\xi \quad .$$

Consider $\Omega \triangleq [-M,M]$. In accordance with Theorem 2.1, (S) is globally Ω-null controllable at $t_o = 0$ if and only if

$$\int_o^\infty M[0 \quad 1]\begin{bmatrix} e^\tau & 0 \\ \theta(\tau) & e^{\tau^2/2} \end{bmatrix}\begin{bmatrix} z_{01} \\ z_{02} \end{bmatrix} d\tau = +\infty \quad ;$$

for all non-zero initial conditions $z_o \triangleq [z_{01} \quad z_{02}]'$. Expanding above, this reduces to the requirement

$$\int_o^\infty M|z_{01}|e^\tau d\tau = +\infty \quad \underline{\text{for all}} \quad (z_{01} \quad z_{02}) \neq 0$$

which is violated by $z_{01} = 0$; z_{02} = anything non-zero. Hence (S) is <u>not</u> globally Ω-null controllable at $t_o = 0$.

On the other hand, perhaps certain <u>specified</u> initial conditions can be steered to zero. For example, if $x_o = [\eta \quad 0]^T$, then we require by Theorem 2.3' that

$$x_o' z_o + \int_o^T M|z_{01}|e^\tau d\tau \geq 0 \quad \underline{\text{for all}} \quad (z_{01} \quad z_{02}) \neq 0$$

for Ω-null controllability at $(0,x_o)$. Substituting for the given x_o, this requirement becomes

$$\eta z_{01} + \int_o^T M|z_{01}|e^\tau d\tau \geq 0 \quad \underline{\text{for all}} \quad z_{01} \in R \quad .$$

Clearly, this can be accomplished if T is large enough so that

$$\int_o^T Me^\tau d\tau \geq \eta \quad .$$

We also could have derived this same result by invoking Theorem 2.3 instead of Theorem 2.3'.

3. Relationship with Other Controllability Results. In this section, we compare our controllability results with those of Conti [6] and Brammer [4]. We also consider as a limiting case of our theory the usual controllability problem obtained when magnitude constraints are not present.

Result of Conti. An important special case of Theorem 2.1 occurs when Ω is a closed unit ball in R^m i.e.,

$$\Omega = \{\omega \in R^m : \|\omega\| \leq 1\}$$

where $\|\cdot\|$ is a prespecified norm on R^m. For this situation we have

$$H_\Omega(B'(\tau)z(\tau)) = \sup\{\omega' B'(\tau)z(\tau) : \|\omega\| \leq 1\} = \|B'(\tau)z(\tau)\|_*$$

where $\|\cdot\|_*$ is the norm on R^m which is dual to $\|\cdot\|$. (For example $\|\cdot\|_*$ is the ℓ^1 norm when $\|\cdot\|$ is the ℓ^∞ norm and $\|\cdot\|$ and $\|\cdot\|_*$ coincide when $\|\cdot\|$ is the usual ℓ^2 (Euclidean) norm.)

By Theorem 2.1, we conclude that (S) is globally Ω-null controllable at t_o if and only if

$$\int_{t_o}^{\infty} \|B'(\tau)z(\tau)\|_* d\tau = +\infty \tag{3.1}$$

for all non-zero solutions $z(\cdot)$ of (S'). This result is established independently in Conti [6] and also discussed in Pandolfi [7]. This result, in conjunction with Corollary 2.2 leads immediately to the following Proposition.

Proposition 3.1. Let Ω be any set containing zero in its interior. Then (3.1) is a necessary and sufficient condition for global Ω-null controllability.

Thus, Conti's condition is a necessary and sufficient condition for global Ω-null controllability for any set Ω containing zero in its interior, not just when Ω is the closed unit ball.

Result of Brammer. Consider the case when $A(t) \equiv A$ and $B(t) \equiv B$ are time-invariant. Furthermore, assume Ω satisfies the following conditions: there exists a $u \in \Omega$ satisfying $Bu = 0$ and $CH(\Omega)$ has a nonempty interior in R^m. For this special case, Theorem 2.1 is comparable to Brammer's result.

Theorem 3.2. (S) is globally Ω-null controllable if and only if

i) $r(Q) = n$

ii) there is no real eigenvector v of A' satisfying $v'B\omega \leq 0$ for all $\omega \in \Omega$

iii) no eigenvalue of A has a positive real part.

We note that the system of Example 1 of Section 2 does not satisfy these three conditions. Nevertheless, it is Ω-null controllable at (x_o, t_o) for some initial states x_o.

The Case $\Omega = R^m$. When $\Omega = R^m$, it is well known [15, p. 171] that the time-varying system (S) is completely controllable (globally R^m-null controllable at t_o in our notation) if and only if the rows of $\phi(t_o,\cdot)B(\cdot)$ are linearly independent on some bounded interval $[t_o,T]$. Here we show that when $\Omega = R^m$, equation (2.1) is a necessary and sufficient condition for global R^m-null controllability. This is accomplished by showing that (2.1) is equivalent to the rows of $\phi(t_o,\cdot)B(\cdot)$ being linearly independent on some bounded interval $[t_o,T]$.

Proposition 3.3. (S) is globally R^m-null controllable if and only if

$$\int_{t_o}^{\infty} H_{R^m}(B'(\tau)z(\tau))d\tau = +\infty$$

for all non-zero solutions $z(\cdot)$ of (S').

The proof of this result is in Appendix B.

4. Some Computational Aspects. In some problems, one may have to resort to the computer to check whether or not a system is Ω-null controllable. When using Equ. (2.3), a solution of the minimization problem $\min\{J(x_o,T,\lambda) : \lambda \in \Lambda\}$ is needed. Direct application of so-called gradient or descent algorithms is precluded by the fact that $J(x_o,T,\lambda)$ is in general not differentiable in λ. This fact is a consequence of the sup operation involved in the definition of $H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)$. Fortunately, however, numerical computation is nevertheless feasible as a consequence of the following two lemmas.† The proofs are given in Appendix C.

Lemma 4.1. For fixed $(x_o,T) \in R^n \times R$, $J(x_o,T,\lambda)$ is a lower semicontinuous convex function of λ.

Lemma 4.2. For fixed $(x_o,T) \in R^n \times R$, the subdifferential†† of $J(x_o,T,\cdot)$ at $\lambda \in R^n$ consists of all vectors $\lambda_* \in R^n$ of the form

$$\lambda_* = \Phi(T,t_o)x_o + \int_{t_o}^{T} \Phi(T,\tau)B(\tau)\omega_*(\tau)d\tau \tag{4.1}$$

where

$$\omega_*(\tau) \in \arg\max\{\omega'B'(\tau)\Phi'(T,\tau)\lambda : \omega \in \Omega\} \tag{4.2}$$

for almost all $\tau \in [0,T]$.

Formulae (4.1) and (4.2) hold for arbitrary compact-convex Ω. Often, however, more structural information is known about Ω. In such cases, (4.1) and (4.2) may simplify. To illustrate, suppose

$$\Omega = [-M_1,M_1] \times [-M_2,M_2] \times \dots \times [-M_m,M_m]; \quad (M_i > 0)$$

† The so-called "generalized steepest descent" schemes rely on subdifferential rather than gradient information. Hence computations can be carried out using the subdifferential description of $J(x_o,T,\lambda)$ given in Lemma 4.2.

†† $\lambda_* \in \partial J(x_o,T,\lambda)$, the subdifferential of $J(x_o,T,\cdot)$ at λ, if and only if

$$J(x_o,T,z) \geq J(x_o,T,\lambda) + (z-\lambda)'\lambda_* \quad \text{for all } z \in R^n .$$

Then, the maximum in (4.2) is achieved in the i^{th} component by

$$[\omega_*(\tau)]_i \in M_i \ \mathrm{sgn}[B'(\tau)\Phi'(T,\tau)\lambda]_i \ ; \ i = 1,2,\ldots,m$$

where $\mathrm{sgn}\ x \triangleq 1$ if $x > 0$; $\mathrm{sgn} \triangleq -1$ if $x < 0$; $\mathrm{sgn}\ 0 \triangleq [-1,1]$. Consequently, for this case, we can substitute in (4.1) and show that the subdifferential $\partial J(x_o,T,\lambda)$ consists of all vectors $\lambda_* \in R^n$ of the form

$$(4.3) \qquad \lambda_* = \Phi(T,0)x_o + \int_o^T \sum_{i=1}^m M_i h_i(T,\tau)\,\mathrm{sgn}\ \lambda' h_i(T,\tau)\,d\tau$$

where $h_i(T,\tau)$ is the i^{th} column of $H(T,\tau) \triangleq \Phi(T,\tau)B(\tau)$.

We also note that λ_* is uniquely specified by (4.3) if

$$\mathrm{measure}\{\tau : \lambda' h_i(T,\tau) = 0\} = 0 \quad \underline{\text{for}} \quad i = 1,2,\ldots,m \quad .$$

For such λ, $\partial J(x_o,T,\lambda)$ is precisely $\nabla_\lambda J(x_o,T,\lambda)$, the gradient of $J(x_o,T,\cdot)$ at λ.

5. The Steering Control. Using the results of Section 2, we can determine if (S) is Ω-null controllable. However, those results do not give a method for determining a steering control $u_*(\cdot) \in \mathfrak{M}(\Omega)$ which accomplishes this objective.

One method of determining an appropriate $u_*(\cdot)$ is to solve the time optimal control problem, i.e., find $u_*(\cdot) \in \mathfrak{M}(\Omega)$ which steers (S) from given (x_o,t_o) to the origin and does so in minimum time. If there is a control which steers the system to the origin, then there is a time optimal one [2]. Hence, in principle, a steering control can be numerically computed using any of a wide variety of algorithms which are available for solution of the time optimal control problem.

Since the solution of the time optimal problem is determined by solving a two point boundary value problem, it can be quite difficult to obtain the steering control this way. In this section, a "simpler" alternative method for generating a steering control is presented. This technique does not involve a two point boundary value problem and leads to a control which steers the system arbitrarily close to the origin. Our result is obtained from the following minimum norm problem:† Given initial point (x_o,t_o) and a final time T, find $u(\cdot) \in \mathfrak{m}(\Omega)$ which leads to the smallest value of $\|x(T)\|$. The solution of this minimum norm problem is characterized in the next theorem.

Theorem 5.1. (See Appendix D for proof). Let (x_o,t_o) and T be given. Suppose that $\lambda_* \in R^n$ achieves the minimum of $J(x_o,T,\lambda)$ over the closed unit ball. Then any solution of the minimum norm problem satisfies

$$(5.1) \qquad u_*(\tau) \in \arg\max\{\omega' B'(\tau)\phi'(T,\tau)\lambda_* : \omega \in \Omega\}$$

for almost all $\tau \in [t_o,T]$.

We note that condition (5.1) will uniquely determine $u_*(\cdot)$ whenever the minimum of $\omega' B'(\tau)\phi'(T,\tau)\lambda_*$ is uniquely achieved. For example, suppose

$$\Omega = [-M_1,M_1] \times [-M_2,M_2] \times \cdots \times [-M_m,M_m] \qquad (M_i > 0) \quad .$$

Then (5.1) requires

$$(5.2) \qquad [u_*(\tau)]_i \in M_i \operatorname{sgn}[B'(\tau)\phi'(T,\tau)\lambda_*]_i \quad , \quad i = 1,2,\cdots,m \quad .$$

For the case when the minimum of $\|x(T)\| = 0$, $\lambda_* = 0$ and (5.1) will not determine a control which steers (S) to the origin. The following heuristic procedure can be used to determine a control which steers (S) arbitrarily close to the origin: Choose a T such that the minimum of $\|x(T)\|$ is nonzero. As T is increased, the minimum of $\|x(T)\|$ approaches zero and the corresponding solution $u_*(\cdot)$, generated via (5.2), of the minimum norm problem results in a control which steers the system progressively closer to the origin.

† (S) here is required to be R^m-null controllable.

6. Additional Applications. In this section, we use our results to obtain an existence theorem for the time optimal control problem and also apply our results to a pursuit game.

Existence of Time Optimal Controls. Consider the following time optimal control problem: Find $u(\cdot) \in \mathfrak{M}(\Omega)$ which drives the state $x(\cdot)$ of (S) from an initial position $x(t_o) = x_o$ to the origin and minimizes

$$C(u(\cdot)) = \int_{t_o}^{t_f} dt \quad ; \quad t_f = \text{arrival time at the origin.}$$

The classical theorem for existence of a time optimal control (e.g., Lee and Markus [2])requires that there is at least one control which transfers the state $x(\cdot)$ of (S) to the origin. Combining the result of [2] with our Theorem 2.3, we obtain the following existence lemma.

Lemma 6.1. *There exists a solution to the time optimal control problem if and only if there is some finite* $t_f \in [t_o, \infty)$ *such that*

$$\min\{J(x_o, t_f, \lambda) : \lambda \in \Lambda\} = 0 \quad .$$

Furthermore, the time optimal cost is given by

$$C^*(u_*(\cdot)) = \inf\{t_f : \min[J(x_o, t_f, \lambda) : \lambda \in \Lambda] = 0\} \quad .$$

Pursuit Games. Next, we consider the pursuit game studies by Hajek [14]. The system is described by

$$(6.1) \quad \dot{x}(t) = Ax(t) - p(t) + q(t) \quad ; \quad p(t) \in P \quad , \quad q(t) \in Q \qquad x(t_o) = x_o$$

where P and Q are compact convex subsets of R^n. The pursuer $p(\cdot)$ seeks a strategy $\sigma : Q \times [t_o, \infty) \to P$ which steers $x(\cdot)$ to the origin for all possible quarry controls $q(\cdot) : [t_o, \infty) \to Q$. A quarry control is admissible if it is measurable and a strategy is admissible if $\sigma(\cdot)$ preserves measurability.

In [16], a solution to this problem is obtained in terms of the associated control system

(6.2) $$\dot{y}(t) = Ay(t) - u(t) \quad ; \quad u(t) \in P \overset{*}{-} Q \quad ; \quad y(t_o) = 0$$

where $P \overset{*}{-} Q$ is the Pontryagin difference. i.e.,

$$P \overset{*}{-} Q \overset{\Delta}{=} \{x \in R^n : x + Q \subseteq P\} \quad .$$

Admissible controls $u(\cdot)$ above must be measurable.

Simply put, Hajek's result says that the state $x(\cdot)$ of (6.1) can be forced to the origin, for all admissible $q(\cdot)$, if and only if the state $y(\cdot)$ of (6.2) can be steered to the origin. More precisely, the following theorem is available.

First Reciprocity Theorem [16]. *Initial position x_o in (6.1) can be (stroboscopically) forced to the origin at time $T \geq t_o$ by a strategy $\sigma(\cdot)$ if and only if, x_o in (6.2) can be steered to the origin at time T by an admissible control $u(\cdot)$. Furthermore, $\sigma(\cdot)$ and $u(\cdot)$ are related by*

(6 .) $$\sigma(q,t) = u(t) + q \quad .$$

By applying Theorem 2.3 to (6.2), we obtain another condition for determining if (6.1) can be forced to the origin.

Lemma 6.2. *Assume $P \overset{*}{-} Q$ compact. Then x_o in (6.1) can be forced to the origin at time $T \geq t_o$ by a strategy $\sigma(\cdot)$ if and only if*

$$\min\{K(x_o,T,\lambda) : \lambda \in \Lambda\} = 0$$

where

$$K(x_o,T,\lambda) \overset{\Delta}{=} x_o' e^{A'(T-t_o)} + \int_{t_o}^{T} H_{P \overset{*}{-} Q}(e^{A'(T-\tau)}\lambda)\,d\tau$$

and Λ is any subset of R^n containing zero as an interior point.

It should be pointed out that in addition to pursuit game interpretation of (6.1), (6.1) can also be viewed as a problem of steering a system with disturbances to the origin if $q(\cdot)$ is thought of as a disturbance. Also, the results apply to systems described by

$$\dot{x}(t) = Ax(t) + Bp(t) + Cq(t) \quad ; \quad p(t) \in P \quad , \quad q(t) \in Q$$

if one replaces $Bp(t)$ by $p'(t)$, $Cq(t)$ by $-q'(t)$, P by BP and Q by CQ.

References

1. R. E. Kalman, Mathematical description of linear dynamical system, this Journal, 1 (1963), pp. 152-192.

2. E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley, New York, 1967.

3. S. Saperstone and J. Yorke, Controllability of linear oscillatory systems using positive controls, this Journal, 9 (1971), pp. 253-262.

4. R. Brammer, Controllability in linear autonomous systems with positive controllers, this Journal, 10 (1972), pp. 339-353.

5. S. Saperstone, Global controllability of linear systems with positive controls, this Journal, 11 (1973), pp. 417-423.

6. R. Conti, Teoria del Controllo e del Controllo Ottimo, UTET, Torino, Italy 1974.

7. L. Pandolfi, Linear control systems: controllability with constrained controls, Journal of Optimization Theory and Applications, 19 (1976), pp. 577-585.

8. J. Dauer, Controllability of nonlinear systems using a growth condition, Journal of Optimization Theory and Applications, 9 (1972), pp. 90-98.

9. J. Dauer, Controllability on nonlinear systems with restrained controls, Journal of Optimization Theory and Applications, 14 (1974), pp. 251-262.

10. E. Chukwu and J. Gronski, Controllability of Nonlinear Systems with Restrained Controls to Closed Convex Sets, Department of Mathematics Report CSUMD 45, Cleveland State University.

11. W. J. Grantham and T. L. Vincent, A controllability minimum principle, Journal of Optimization Theory and Applications, 17 (1975), pp. 93-114.

12. M. E. Evans and D. N. P. Murthy, Controllability of discrete-time systems with positive controls, IEEE Automatic Control, AC-22 (1977), pp. 942-945.

13. M. Pachter and D. H. Jacobson, Control with conic constraint set, Journal of Optimization Theory and Applications (to appear).

14. D. H. Jacobson, Extension of Linear-Quadratic Control, Optimization and Matrix Theory, Academic Press, New York, 1977.

15. C. T. Chen, Introduction to Linear System Theory, Holt, Rinehart and Winston, New York, 1970.

16. O. Hájek, Pursuit Games, Academic Press, New York, 1975.

17. P. R. Halmos, Measure Theory, Van Nostrand, New York, 1950.

18. S. Lang, Analysis I, Addison-Wesley, New York, 1969.

19. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

20. I. Ekeland and R. Teman, Convex Analysis and Variational Problems, North-Holland, Amsterdam, 1976.

21. H. L. Roydon, Real Analysis, MacMillan Co., New York, 1968.

22. R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1972.

23. R. T. Rockafellar, Conjugate Duality and Optimization, monograph, SIAM Publications, Philadelphia, Pa., 1974.

24. R. T. Rockafellar, Duality and stability in extremum problems involving convex function, Pacific J. Math., 21 (1967), pp. 167-187.

APPENDIX A

Proof of Theorems 2.1, 2.3 and Corollary 2.2. Before proving our theorems, we need two preliminary lemmas which guarantee that the integrals in (2.1), (2.2) and (2.5) are well-defined. To simplify our notation, we henceforth take $t_o = 0$, without loss of generality. First, we show that the integrand in (2.1) is a non-negative measurable function and hence the integral in (2.1) is well-defined [17].

Lemma A.1. Suppose that Ω is compact and contains zero. Then the integrand $H_\Omega(B'(\tau)z(\tau))$ is a non-negative measurable function of $\tau \in [0,\infty)$ along all trajectories $z(\cdot)$ of (S').

Proof: The non-negativity of the integrand follows from

$$\begin{aligned} H_\Omega(B'(\tau)z(\tau)) &= \sup\{\omega'B'(\tau)z(\tau) : \omega \in \Omega\} \\ &\geq \omega'B'(\tau)z(\tau)\big|_{\omega=0} \\ &= 0 \quad . \end{aligned}$$

To establish measurability, we first extract a countable dense subset $\langle\omega_n\rangle_{n=1}^{\infty}$ of Ω. This is always possible because Ω is compact [18, p. 146]. For each fixed τ, the density of the ω_n implies that

$$H_\Omega(B'(\tau)z(\tau)) = \sup_n \omega_n' B_n'(\tau)z(\tau) \quad .$$

Let

$$H_n(\tau) = \omega_n' B'(\tau)z(\tau); \; n = 1,2,\ldots$$

Then each $H_n(\cdot)$ is measurable (in fact, continuous) and $H_\Omega(B'(\cdot)z(\cdot)$ is the pointwise supremum over the family of $H_n(\cdot)$. Theorem 1.14 of Rudin [19] tells us that the pointwise supremum of this collection must also be measurable. □

The next lemma shows that the integrals (2.2) and (2.5) are well-defined.

Lemma A.2. Suppose that Ω is compact (not necessarily containing zero). Then, for each fixed pair $(\lambda,T) \in R^n \times [0,\infty)$, the integral

$$\int_0^T H_\Omega(B'(\tau)\phi'(T,\tau)\lambda)d\tau$$

is well-defined.

Proof. For any fixed $u(\cdot) \in \mathfrak{M}(\Omega)$ and initial condition x_o, the unique solution of (S) is given by

$$x(t) = \phi(t,0)x_o + \int_0^t \phi(t,\tau)B(\tau)u(\tau)d\tau \quad .$$

Let $u_\lambda(\cdot) \in \mathfrak{M}(\Omega)$ be a measurable selection such that

$$\sup\{\omega'B'(\tau)\phi'(T,\tau)\lambda : \omega \in \Omega\} \equiv u_\lambda'(\tau)B'(\tau)\phi'(T,\tau)\lambda \quad .$$

Such a selection is possible using Theorem 1.2 of [20, p. 236]. Consequently, it follows that

$$\int_0^T H_\Omega(B'(\tau)\phi'(T,\tau)\lambda)d\tau = \int_0^T u_\lambda'(\tau)B'(\tau)\phi'(T,\tau)\lambda\, d\tau$$

$$= \lambda' \int_0^T \phi(T,\tau)B(\tau)u_\lambda'(\tau)d\tau \quad .$$

We notice that the integral above is well-defined because it is the unique solution of (S) at time $t = T$, corresponding to input $u_\lambda(\cdot)$ and initial condition $x_o = 0$. □

Since Theorem 2.3 is used in the proof of Theorem 2.1, we first present the proof of Theorem 2.3. There are various ways to prove Theorem 2.3. Our chosen method of proof seems to be most natural -- our proof not only decides on existence of a steering control $u_*(\cdot)$, but also characterizes $u_*(\cdot)$ as half of a saddle point $(\lambda_*;u^*(\cdot))$ of an appropriately constructed functional $V : R^n \times \mathfrak{M}(\Omega) \rightarrow R$.

Proof of Theorem 2.3 (Necessity): We suppose that (S) is Ω-null controllable at (x_o,t_o). Let $u_*(\cdot)\in\mathfrak{M}(\Omega)$ be a control which drives x_o to zero at some future time $T\in[0,\infty)$. Then

$$x_*(T) = 0 = \Phi(T,0)x_o + \int_o^T \Phi(T,\tau)Bu_*(\tau)d\tau \quad . \tag{A.1}$$

For this value of T, we are going to show that $J(x_o,T,\lambda)$ is non-negative for all $\lambda\in R^n$. We proceed as follows: Using the measurable selection theory of [20], we make a measurable selection $\tilde{u}_\lambda(\cdot)\in\mathfrak{M}(\Omega)$ such that

$$\sup\{\lambda'\Phi(T,\tau)B(\tau)\omega : \omega\in\Omega\} \equiv \lambda'\Phi(T,\tau)B(\tau)\tilde{u}_\lambda(\tau) \quad . \tag{A.2}$$

Hence, it follows that

$$\begin{aligned}\int_o^T H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau &= \int_o^T \sup\{\lambda'\Phi(T,\tau)B(\tau)\omega : \omega\in\Omega\}d\tau \\ &= \int_o^T \lambda'\Phi(T,\tau)B(\tau)\tilde{u}_\lambda(\tau)d\tau \\ &\le \sup\left\{\int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau : u(\cdot)\in\mathfrak{M}(\Omega)\right\} \quad . \qquad \text{(A.3)}\end{aligned}$$

But also

$$\begin{aligned}\sup\left\{\int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau : u(\cdot)\in\mathfrak{M}(\Omega)\right\} &\le \int_o^T \sup\left\{\lambda'\Phi(T,\tau)B(\tau)\omega : \omega\in\Omega\right\} d\tau \\ &= \int_o^T H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau \quad . \qquad \text{(A.4)}\end{aligned}$$

Equations (A.3) and (A.4) imply

$$\int_o^T H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau = \sup\left\{\int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau : u(\cdot)\in\mathfrak{M}(\Omega)\right\} \quad . \tag{A.5}$$

Now, for any $\lambda \in R^n$, we have

$$J(x_o,T,\lambda) = \lambda'\Phi(T,0)x_o + \int_o^T H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau$$

$$= \lambda'\Phi(T,0)x_o + \sup\left\{\int_o^T \lambda'\Phi(t,\tau)B(\tau)u(\tau) : u(\cdot) \in \mathfrak{m}(\Omega)\right\}$$

$$\geq \lambda'\Phi(T,0)x_o + \int_o^T \lambda'\Phi(T,\tau)B(\tau)u_*(\tau)d\tau$$

$$= \lambda' x_*(T)$$

$$= 0 \quad .$$

Therefore $J(x_o,T,\lambda) \geq 0$ for all $\lambda \in \Lambda$. Since $0 \in \Lambda$ and $J(x_o,t,0) = 0$, we conclude that

$$\min\{J(x_o,T,\lambda) : \lambda \in \Lambda\} = 0 \quad .$$

(Sufficiency): We assume that there exists some $T \in [0,\infty)$ such that the minimum in (2.3) is zero. We are going to construct a control function $u_*(\cdot)$ in $\mathfrak{m}(\Omega)$ which steers x_o to 0 at time T. First, however, we make the following observations:

(i) For each fixed λ, the functional

$$h_\lambda(u(\cdot)) \triangleq \int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau$$

is lower semicontinuous on $\mathfrak{m}(\Omega)$.

(ii) $\mathfrak{m}(\Omega)$ is weakly compact [2, p. 157].

As a consequence of (i) and (ii), for each fixed λ, $\sup\{h_\lambda(u(\cdot)) : u(\cdot) \in \mathfrak{m}(\Omega)\}$. is attained. Hence, we can change "sup" to "max" in (A.5) and we have

(A.6) $$\int_o^T H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau = \max\left\{\int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau : u(\cdot)\in\mathfrak{m}(\Omega)\right\} .$$

Thus

(A.7) $$J(x_o,T,\lambda) = \lambda'\Phi(T,0)x_o + \max\left\{\int_o^T \lambda'(T,\tau)B(\tau)u(\tau)d\tau : u(\cdot)\in\mathfrak{m}(\Omega)\right\} .$$

Now, let $\mathcal{B}$ be any closed ball and define $V : \mathcal{B}\times\mathfrak{m}(\Omega)\to R$ by

(A.8) $$V(\lambda,u(\cdot)) = \lambda'\Phi(T,0)x_o + \int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau .$$

We assume for now that $V(\lambda,u(\cdot))$ possesses at least one saddle point $(\lambda_*,u_*(\cdot))\in\mathcal{B}\times\mathfrak{m}(\Omega)$. (The existence of such a saddle point will be proven in Lemma A.3 to follow). Therefore, for all $u(\cdot)\in\mathfrak{m}(\Omega)$ and all $\lambda\in\mathcal{B}$

(A.9) $$V(\lambda_*,u(\cdot)) \leq V(\lambda_*,u_*(\cdot)) \leq V(\lambda,u_*(\cdot)) .$$

By hypothesis, there is a $T\in[0,\infty)$ and a set Λ containing 0 as an interior point such that

$$0 = \min\{J(x_o,T,\lambda) : \lambda\in\Lambda\}$$

Since 0 is an interior point of Λ, there is a closed ball $\mathcal{B}\subset\Lambda$ such that

$$\begin{aligned} 0 &= \min\{J(x_o,\tau,\lambda) : \lambda\in\mathcal{B}\} \\ &= \min\left\{\max\left[\lambda'\Phi(T,0)x_o + \int_o^T \lambda'\Phi(T,\tau)B(\tau)u(\tau)d\tau : u\in\mathfrak{m}(\Omega)\right] : \lambda\in\mathcal{B}\right\} . \end{aligned}$$

Hence, $V(\lambda_*,u_*(\cdot)) = 0$. Using this in conjunction with (A.9) leads to

$$0 \leq \lambda'\Phi(T,0)x_o + \int_o^T \lambda'\Phi(T,\tau)B(\tau)u_*(\tau)d\tau \quad \underline{\text{for all}}\ \lambda\in\mathcal{B}$$

and

(A.10) $$0 \leq \lambda' x_*(T) \quad \underline{\text{for all}}\ \lambda\in\mathcal{B}$$

where $x_*(\cdot)$ is the state response of (S) resulting from input $u_*(\cdot)$. The only way that (A.10) can hold for <u>all</u> $\lambda\in\mathcal{B}$ is if $x_*(T) = 0$ and $u_*(\cdot)$ steers (S) from x_o at $t_o = 0$ to zero at time T. □

Lemma A.3. $V(\lambda,u(\cdot))$, defined in (A.8), possesses at least one saddle point.

Proof. In accordance with Proposition 2.3, [20, p. 175], $V(\lambda,u(\cdot))$ will possess a saddle point if the following conditions are satisfied:

(A.3.1) For all $\lambda \in \mathcal{B}$, $u(\cdot) \to V(\lambda,u(\cdot))$ is concave and upper semicontinuous.

(A.3.2) For all $u(\cdot) \in \mathfrak{M}(\Omega)$, $\lambda \to V(\lambda,u(\cdot))$ is convex and lower semicontinuous.

(Note: *We shall view* $\mathfrak{M}(\Omega)$ *as a subset of* $L^2[0,T,\mathcal{R}^m]$ when verifying the semicontinuity requirements.)

Proof of (A.3.1): Fix $\lambda \in \mathcal{B}$. Concavity of $V(\lambda,\cdot)$ follows trivially from affine linearity. We shall show that $V(\lambda,\cdot)$ is continuous. Let $u_n(\cdot) \to u(\cdot)$ in L^2 norm. Then using the Schwartz inequality, we can easily show that

$$|V(\lambda,u_n(\cdot)) - V(\lambda,u(\cdot))| \le \left\{ \int_0^T \|\lambda'\Phi(T,\tau)B(\tau)\|^2 d\tau \right\}^{1/2} \left\{ \int_0^T \|u_n(\tau) - u(\tau)\|^2 d\tau \right\}^{1/2}$$

The first term on the right hand side is finite since $\lambda'\Phi(T,\cdot)B(\cdot)$ is continuous. The second term converges to zero by hypothesis. Hence, $V(\lambda,u_n(\cdot)) \to V(\lambda,u(\cdot))$ as $n \to \infty$.

Proof of A.3.2: Now fix $u(\cdot) \in \mathfrak{M}(\Omega)$. The linearity of $V(\cdot,u(\cdot))$ implies convexity. Let $\lambda_n \to \lambda$ in $\mathcal{B}$. Then, we can show that

$$|V(\lambda_n,u(\cdot)) - V(\lambda,u(\cdot)| \le K\|\lambda_n - \lambda\|$$

where

$$K = \|\Phi(T,0)x_0\| + \left\|\int_0^T \Phi(T,\tau)B(\tau)u(\tau)d\tau\right\| \;;\; K < \infty \quad .$$

Hence, $V(\lambda_n,u(\cdot)) \to V(\lambda,u(\cdot))$ as $n \to \infty$. □

Next, we present the proof of Theorem 2.1. In the proof, Theorem 2.3 is used.

Proof of Theorem 2.1 (Necessity): We suppose that (S) is globally Ω-null controllable at $t_o = 0$. Let $z(\cdot)$ be any non-zero solution of (S') and we must prove that

$$\text{(A.11)} \qquad \int_o^\infty H_\Omega(B'(\tau)z(\tau))d\tau = +\infty \quad .$$

Proceeding by contradiction, suppose there is a non-zero solution $\hat{z}(\cdot)$ such that

$$\int_o^\infty H_\Omega(B'(\tau)\hat{z}(\tau))d\tau = \beta \ ; \ \beta < \infty \quad .$$

From Lemma A.1, $H_\Omega(B'(\tau)\hat{z}(\tau))d\tau \geq 0$ and $\beta \geq 0$. Suppose $\beta \neq 0$ and define

$$x_o^* = \frac{-2\beta\hat{z}(0)}{\hat{z}'(0)\hat{z}(0)} \quad ; \quad x_o^* \neq 0 \quad .$$

We now claim that x_o^* cannot be steered to zero by an admissible input $u(\cdot) \in \mathfrak{M}(\Omega)$. To prove our claim, for each $t \in [0,\infty)$, define

$$\lambda_t \triangleq \Phi'(0,t)\hat{z}(0) \quad ; \quad \lambda_t \neq 0 \quad .$$

Now

$$\begin{aligned} J(x_o^*,t,\lambda_t) &= x_o^{*\prime}\Phi'(t,0)\lambda_t + \int_o^t H_\Omega(B'(\tau)\Phi'(t,\tau)\lambda_t)d\tau \\ &= x_o^{*\prime}\hat{z}(0) + \int_o^t H_\Omega(B'(\tau)\hat{z}(\tau))d\tau \\ &\leq -2\beta + \beta \\ &< 0 \quad . \end{aligned}$$

Taking Λ to be the closed unit ball in Theorem 2.3, it now follows that $\rho_t\lambda_t \in \Lambda$ for sufficiently small ρ_t. We have

$$\begin{aligned} \min\{J(x_o^*,t,\lambda) : \lambda \in \Lambda\} &\leq J(x_o^*,t,\rho_t\lambda_t) \\ &< 0 \end{aligned}$$

for all $t \in [0,\infty)$. By Theorem 2.3, (S) is not Ω-null controllable at (x_o^*,t_o).

Now suppose $\beta = 0$. Then, for all $t \in [0,\infty)$,

$$\int_0^t H_\Omega(B'(\tau)\phi'(t,\tau)\hat{z}(0))d\tau = 0$$

Choose any x_o such that $x_o'\hat{z}(0) < 0$. Then

$$J(x_o,t,\lambda_t) = x_o'\hat{z}(0) < 0$$

for all $t \in [0,\infty)$. Again by Theorem 2.3, (S) is <u>not</u> Ω-null controllable at (x_o,t_o). This contradicts the hypothesis that (S) is globally Ω-controllable.

(Sufficiency): Now, we assume that (A.11) holds. Again, we proceed by contradiction. i.e., suppose (S) is <u>not</u> globally Ω-null controllable at $t_o = 0$. Hence, there exists an initial condition $x_o^* \neq 0$ which cannot be steered to zero. By Theorem 2.3 (with $\Lambda = R^n$), we can find a sequence of times $\langle t_k \rangle_{k=1}^{\infty}$ and a sequence of vectors $\langle \lambda_k \rangle_{k=1}^{\infty}$ having the following properties:

P1. $\lim_{k\to\infty} t_k = +\infty$;

P2. $J(x_o^*,t_k,\lambda_k) < 0$ for $k = 1,2,3...$

We are going to construct an initial condition $\tilde{z}_o \neq 0$ for (S') which makes the integral in (A.11) finite. To meet this end, let

$$z_k = \frac{\Phi'(t_k,0)\lambda_k}{\|\Phi'(t_k,0)\lambda_k\|} \quad ; \quad k = 1,2,\ldots \quad ; \quad z_k \neq 0 \quad .$$

Then $\langle z_k \rangle_{k=1}^{\infty}$ is a sequence in R^n belonging to the set

$$S \triangleq \{z \in R^n : \|z\| = 1\} \quad .$$

Since S is compact, we can extract a subsequence $\langle z_{k_j} \rangle_{k_j=1}^{\infty}$ which converges to some vector $\tilde{z}_o \in S$. We will now show that $\tilde{z}_o$ is the initial condition which we seek. Let $\tilde{z}(\cdot)$ be the trajectory of (S') generated by $z(0) \triangleq \tilde{z}_o$; let

$\langle t_{k_j} \rangle_{k_j=1}^{\infty}$ denote the subsequence of times corresponding to $\langle z_{k_j} \rangle_{k_j=1}^{\infty}$. By P1, we have

$$\lim_{k_j \to \infty} t_{k_j} = +\infty$$

and by P2, it follows that

$$x_o^{*\prime}\Phi'(t_{k_j},0)\lambda_{k_j} + \int_o^{t_{k_j}} H_\Omega(B'(\tau)\Phi'(t_{k_j},\tau)\lambda_{k_j})d\tau < 0 \quad \underline{\text{for}} \quad k_j = 1,2,3,\ldots$$

Dividing by $\|\Phi'(t_{k_j},0)\lambda_{k_j}\|$ and noting that H_Ω is positively homogeneous, we obtain

$$\int_o^{t_{k_j}} H_\Omega(B'(\tau)\Phi'(0,\tau)z_{k_j})d\tau \le \|x_o^*\| \, \|z_{k_j}\| \quad \underline{\text{for}} \quad k_j = 1,2,3,\ldots$$

$$\le \|x_o^*\| \quad \underline{\text{for}} \quad k_j = 1,2,3,\ldots$$

We would like to obtain an inequality involving $\tilde{z}_o$ with an infinite upper limit on this integral. To accomplish this, we define

$$f_{k_j}(\tau) \triangleq H_\Omega(B'(\tau)\Phi'(0,\tau)z_{k_j}) \quad \text{if} \quad \tau \in [0,t_{k_j}] \quad ;$$

$$\triangleq 0 \quad \text{otherwise} \quad .$$

$$f(\tau) \triangleq H_\Omega(B'(\tau)\Phi'(0,\tau)\tilde{z}_o) \quad ; \quad \tau \in [0,\infty)$$

and make the following observations:

(i) $\int_o^{\infty} f_{k_j}(\tau)d\tau$ is bounded (by $\|x_o^*\|$) for $k_j = 1,2,3,\ldots$

(ii) $f_{k_j}(\tau)$ converges pointwise to $f(\tau)$ on $[0,\infty)$. This observation is proven using the facts that $z_{k_j} \to \tilde{z}_o$, $t_{k_j} \to +\infty$ and H_Ω depends continuously on its argument.

Applying Fatou's Lemma [21, p. 83], we have

$$\int_0^\infty f(\tau)d\tau \le \liminf_{k_j \to \infty} \int_0^\infty f_{k_j}(\tau)d\tau$$

$$\le \limsup_{k_j \to \infty} \int_0^\infty f_{k_j}(\tau)d\tau$$

$$\le \|x_o^*\| \quad .$$

Substitution for $f(\tau)$ above gives

$$\int_0^\infty H_\Omega(B'(\tau)\Phi'(0,\tau)\tilde{z}_o)d\tau \le \|x_o^*\| \quad .$$

i.e.,

$$\int_0^\infty H_\Omega(B'(\tau)\tilde{z}(\tau))d\tau \le \|x_o^*\|$$

$$< \infty$$

which is the contradiction that we seek. This completes the proof of the theorem. □

Proof of Corollary 2.2. Suppose Ω and Ω' satisfy the hypotheses of the corollary. We are going to show that (S) is globally Ω'-null controllable. To prove this, it is sufficient to find a subset $\Omega'_\delta \subseteq \Omega'$ such that (S) is globally Ω'_δ-null controllable: Pick $\delta > 0$ such that

$$\Omega'_\delta \triangleq \{\omega : \|\omega\| \le \delta\} \subseteq \Omega'$$

(This can be accomplished because zero is interior to Ω.) Now, to prove that Ω'_δ has the desired property, we pick $R > 0$ such that

$$\Omega_R \triangleq \{\omega : \|\omega\| \le R\} \supseteq \Omega$$

(This can also be done since Ω is compact, hence bounded.) Let $z(\cdot)$ be any non-zero solution of (S'). Then we have

$$\int_0^\infty H_{\Omega'_\delta}(B'(\tau)z(\tau))d\tau = \int_0^\infty \sup\{\omega' B'(\tau)z(\tau) : \|\omega\| \le \delta\}$$

$$= \delta \int_0^\infty \|B'(\tau)z(\tau)\| d\tau$$

$$= \frac{\delta}{R} \int_0^\infty R\|B'(\tau)z(\tau)\| d\tau$$

$$= \frac{\delta}{R} \int_0^\infty \sup\{\omega' B'(\tau)z(\tau) : \|\omega\| \le R\} d\tau$$

$$= \frac{\delta}{R} \int_0^\infty H_{\Omega_R}(B'(\tau)z(\tau))d\tau$$

$$= +\infty$$

since (S) is globally Ω_R-null controllable. (Ω_R-null controllability follows from Ω-null controllability in conjunction with the fact that $\Omega_R \supseteq \Omega$.) By Theorem 2.1, we conclude that (S) must be globally Ω'_δ-null controllable and hence Ω'-null controllable. □

[redacted]

Proof of Proposition 3.3. (Necessity): Suppose (S) is globally R^m-null controllable. Then there is a finite interval $[0,T]$ on which the rows of $\phi(0,\cdot)B(\cdot)$ are linearly independent. Thus, for every non-zero vector $z_o \in R^n$, it follows that $B'(t)\phi'(0,t)z_o \neq 0$ for some $t \in [0,T]$. Since, $B'(\cdot)\phi'(0,\cdot)z_o$ is continuous, there must be an interval $I = [t-\delta, t+\delta]$ on which $B'(\tau)\phi'(0,\tau)z_o \neq 0$ for all $\tau \in I$. On this interval, we have

$$\sup\{\omega' B'(\tau)\phi'(0,\tau)z_o : \omega \in R^m\} = +\infty \quad .$$

Hence, using the non-negativity of $H_\Omega(\cdot)$, we conclude that

$$\int_o^\infty H_{R^m}(B'(\tau)z(\tau))d\tau \geq \int_I H_{R^m}(B'(\tau)\phi'(0,\tau)z_o)d\tau$$

$$= \int_I \sup\{\omega' B'(\tau)\phi'(0,\tau)z_o)d\tau$$

$$= +\infty$$

(Sufficiency): Proceeding by contradiction, we suppose that for all non-zero solutions $z(\cdot)$ of (S'), we have

$$\int_o^\infty H_{R^m}(B'(\tau)z(\tau))d\tau = +\infty$$

but the columns of $B'(\cdot)\phi'(0,\cdot)$ are linearly dependent on every bounded interval $[0,T]$. Let $\langle T_n \rangle_{n=1}^\infty$ be a monotone increasing sequence of times such that $T_n \to \infty$. Then, for each n, we can find a non-zero vector $\tilde{z}_n$ such that $B'(\tau)\phi'(0,\tau)\tilde{z}_n \equiv 0$ on $[0,T_n]$. Let

$$z_n \triangleq \frac{\tilde{z}_n}{\|\tilde{z}_n\|} \quad \text{for} \quad n = 1,2,\ldots \quad .$$

Then, $\langle z_n \rangle_{n=1}^{\infty}$ is sequence in the (compact) unit ball. Hence, we can extract a subsequence z_{n_j} converging to some $\hat{z}_o$, $\|\hat{z}_o\| = 1$. We notice that the corresponding subsequence of times T_{n_j} still converges to $+\infty$. Furthermore, for each fixed $\tau \in [0,\infty)$, we have

$$B'(\tau)\phi'(0,\tau)\hat{z}_o = \lim_{n_j \to \infty} B'(\tau)\phi'(0,\tau)z_{n_j}$$

$$= 0$$

Consequently, if $\hat{z}(\tau)$ is the trajectory mate of $\hat{z}_o$,

$$\int_o^{\infty} H_{R^m}(B'(\tau)\hat{z}(\tau))d\tau = \int_o^{\infty} \sup\{\omega' B'(\tau)\phi'(0,\tau)\hat{z}_o : \omega \in R^m\}d\tau$$

$$= 0$$

which contradicts the assumed hypothesis. □

[redacted]

Proof of Lemma 4.1. For (x_o,T) fixed, $J(x_o,T,\lambda)$ can be expressed as

$$J(x_o,T,\lambda) = \sup\{H_\omega(\lambda) : \omega(\cdot) \in \mathfrak{M}(\Omega)\}$$

where

$$H_\omega(\lambda) \triangleq \lambda'\Phi(T,0)x_o + \int_o^T \lambda'\Phi(T,\tau)B(\tau)\omega(\tau)d\tau \quad .$$

Consequently, $J(x_o,T,\cdot)$ is the pointwise supremum over an indexed collection of continuous linear (hence convex) functions. Hence $J(x_o,T,\cdot)$ itself must be convex and at least lower semicontinuous (in fact, continuous). □

Proof of Lemma 4.2. We prove this lemma using some of the standard properties of subdifferentials given in Rockafellar [22], [23]. Since both functions in the definition of $J(x_o,T,\lambda)$ are finite and convex $\lambda_* \in \partial J(x_o,T,\lambda)$ if and only if

$$\lambda_* \in \partial(x_o'\Phi'(T,0)\lambda) + \partial \int_o^T H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau \quad \text{(by Theorem 23.8 of [23])}$$

$$= \Phi(T,0)x_o + \int_o^T \partial H_\Omega(B'(\tau)\Phi'(T,\tau)\lambda)d\tau \quad \text{(by Theorem 23 of [23])}$$

$$= \Phi(T,0)x_o + \int_o^T \Phi(T,\tau)B(\tau)\cdot\partial H_\Omega(\hat\omega(\tau))\Big|_{\hat\omega(\tau)=B'(\tau)\Phi'(T,\tau)\lambda} d\tau$$

(by Theorem 23.9 of [22])

Now, by Corollary 23.5.3 of [22], $\omega_*(\tau) \in \partial H_\Omega(\hat\omega(\tau))$ if and only if $\omega_*(\tau) \in \arg\max\{\omega'\hat\omega(\tau) : \omega \in \Omega\}$. Substituting the required form for $\hat\omega$ above, we obtain our desired representation for λ_*. □

[redacted]

Proof of Theorem 5.1. Let $f : L^1(0,T;R^m) \to R^n$, $g : R^n \to R$, $\Lambda_T : L^1(0,T;R^m) \to R^n$ be given by

$$f(u) \triangleq 0 \text{ if } u(\cdot) \in \mathfrak{M}(\Omega) \quad ; \quad f(u) \triangleq +\infty \text{ otherwise} \quad ;$$

$$g(z) \triangleq -\|\Phi(T,0)x_o + z\| \quad ; \quad z \in R^n \quad ;$$

$$\Lambda_T u \triangleq \int_o^T \Phi(T,\tau)B(\tau)u(\tau)d\tau \quad .$$

Then, using the notation above

$$\inf(MN) \triangleq \inf\{\|x(T)\| : u(\cdot) \in \mathfrak{M}(\Omega)\}$$

$$= \inf\{f(u) - g(\Lambda_T u) : u \in L^1(0,T;R^m)\} \quad .$$

Written in this way, inf(MN) is in the standard form for application of Rockafellar's extension of Fenchel's Duality Theorem (cf. [24], Theorem 1). The functionals f and g are respectively proper convex and concave functions; it can be easily shown that inf(MN) is "stably set" -- a technical precondition for Rockafellar's Theorem.

By carrying out the computations involved in Theorem 1 of [24], it can be shown that the problem

$$\min(MN)^* \triangleq \min\{J(x_o,T,\lambda) : \lambda \in \Lambda\}$$

is dual to inf(MN) in the following sense:

$$\inf(MN) + \min(MN)^* = 0 \quad .$$

The "extremality condition" in Rockafellar's theorem provides a necessary condition which must be satisfied by all solution pairs λ_* solving $(MN)^*$ and $u_*(\cdot)$ solving (MN). The extremality condition requires

$$\Lambda_T^* \lambda_* \in \partial f(u_*)$$

where Λ_T^* is the adjoint of Λ_T and $\partial f(u_*)$ is the subdifferential of f at u_*. For our choice of f, this necessary condition particularizes to

$$\lambda_*' \Phi(T,\tau) B(\tau) \in (\text{Normal cone of } \mathfrak{M}(\Omega) \text{ at } u_*(\cdot)) \quad .$$

We denote this normal cone at u_* by $N_c(u_*)$. By definition of the normal cone, we have $v(\cdot) \in N_c(u^*)$ if and only if

$$\int_0^T u_*'(\tau) B'(\tau) \Phi'(T,\tau) \lambda_* d\tau = \int_0^T \sup\{\omega' B'(\tau) \Phi'(T,\tau) \lambda_* : \omega \in \Omega\} d\tau \quad .$$

This is possible only if $\omega = u_*(\tau)$ achieves the supremum of $\omega' B'(\tau) \Phi'(T,\tau) \lambda_*$ for almost all $\tau \in [0,T]$. Equivalently, we must have

$$u_*(\tau) \in \arg\max\{\omega' B'(\tau) \Phi'(T,\tau) \lambda_* : \omega \in \Omega\}$$

for almost all $\tau \in [0,T]$. □

APPENDIX B

AFOSR-TR- 78-1536

A NECESSARY AND SUFFICIENT CONDITION FOR LOCAL CONSTRAINED CONTROLLABILITY OF A LINEAR SYSTEM[1]

by

B. R. Barmish[2]
W. E. Schmitendorf[3]

[1]The research of the second author was supported by AFOSR under Grant 76-2923.

[2]Associate Professor of Electrical Engineering, The University of Rochester, Rochester, New York 14627

[3]Associate Professor of Mechanical Engineering, Northwestern University, Evanston, Illinois 60201

Summary

1. Introduction and Formulation. Unlike the usual controllability problem where control values at each instant of time are unconstrained, we consider here the case which arises when the control values at each instant belong to a prescribed set Ω in R^m. Constrained controllability problems, for linear systems, are examined in References [1] - [5]. In this paper, our definition of local constrained controllability is identical to that of Brammer [1]. Our new results, however, generalize those of [1] from a time-invariant to a time-varying linear system. The main results here serve as a companion for those of [6] and [7] where global rather than local controllability is emphasized.

Consider the problem of steering the state of a linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t); \; t\varepsilon[t_0,\infty) \qquad (S)$$

to the origin from a specified initial condition

$$x(t_0) = x_0$$

by choice of control function $u(\cdot)$. Here $x(t)\varepsilon R^n$, $u(t)\varepsilon R^m$ and $A(\cdot)$ and $B(\cdot)$ are continuous matrices of appropriate dimension. Let $M(\Omega)$ denote the set of functions from R into Ω that are measurable on $[t_0,\infty)$.

2. Definition. The linear system (S) is constrained controllable (Ω - null controllable) at (x_0,t_0) if for given initial condition $x(t_0) = x_0$, there exists a $u(\cdot)\varepsilon M(\Omega)$ such

that the solution $x(\cdot)$ of (S) satisfies $x(T) = 0$ for some $T \varepsilon [t_0, \infty)$. We say that (S) is locally Ω - null controllable at t_0 if there exists an open set $V \subset R^n$, containing the origin, such that (S) is Ω - null controllable at (x_0, t_0) for all $x_0 \varepsilon V$.

3. Notation. Our necessary and sufficient condition for local Ω - null controllability will be expressed in terms of $A(\cdot)$, $B(\cdot)$ and $H_\Omega : R^m \to R$, the support function on Ω, which is given by

$$H_\Omega(\alpha) \triangleq \sup\{\omega'\alpha : \omega \varepsilon \Omega\}$$

for $\alpha \varepsilon R^m$. We need some further notation: Suppose $f(\cdot)$ is an n-dimensional continuous function on $[t_0, \infty)$ such that $f(0) \neq 0$. Then $f_*(\cdot)$ will denote the normalized version of $f(\cdot)$ which is given by

$$f_*(t) \triangleq \left\|\frac{f(t)}{f(0)}\right\|$$

for $t \varepsilon [t_0, \infty)$. In the theorem to follow, the adjoint system

$$\dot{z}(t) = -A'(t)z(t); \ t \varepsilon [t_0, \infty)$$

will be denoted by (S') and $\phi(t,\tau)$ will be the state transition matrix for (S).

4. Theorem. Suppose Ω is compact and contains zero. Then (S) is locally Ω - null controllable at t_0 if and only if there exists an $\varepsilon > 0$ such that

$$\int_{t_0}^{\infty} H_{\Omega}(B'(\tau)z_*(\tau)d\tau \geq \varepsilon$$

for all normalized solutions $z_*(\cdot)$ of (S'). *Equivalently, if and only if*

$$\int_{t_0}^{\infty} \sup\{\omega' B'(\tau)\phi'(t_0,\tau)\lambda : \omega\varepsilon\Omega\} d\tau \geq \varepsilon$$

for all $\lambda \varepsilon R^n$, $||\lambda|| = 1$.

5. *Remarks*. (i) Implicit in the statement of the theorem is the fact that $H_{\Omega}(B'(\tau)z_*(\tau))$ is an integrable measurable function of τ along all normalized trajectories $z_*(\cdot)$ of (S'). This fact is established in the paper.

(ii) The theorem demonstrates that the question of local constrained controllability can be answered by solving a finite-dimensional problem in the variable $\lambda\varepsilon R^n$ -- or equivalently, in the variable $z_*(0)\varepsilon R^n$.

(iii) If Ω and Ω' are two compact sets having the same convex closure, we know that $H_{\Omega}(\alpha) = H_{\Omega'}(\alpha)$ for *all* $\alpha\varepsilon R^n$. Consequently, we can immediately conclude (from the theorem) that (S) is locally Ω-null controllable if and only if (S) is locally Ω'-null controllable.

(iv) In the paper, we examine some interesting special cases which arise when Ω is endowed with additional structure (over and above "$0\varepsilon\Omega$" and "Ω compact").

6. *References*

[1] R. Brammer, "Controllability of Linear Autonomous Systems with Positive Controllers," *SIAM J. Control*, vol. 10, pp. 339-353, 1972.

[2] S. Saperstone and J. Yorke, "Controllability of Linear Oscillatory Systems Using Positive Controls," *SIAM J. Control*, vol. 9, pp. 253-262, 1971.

[3] S. Saperstone, "Global Controllability of Linear Systems with Positive Controls," SIAM J. Control, vol. 11, pp. 417-423, 1973.

[4] R. Conti, Teoria del Controllo e del Controllo Ottimo, UTET, Torino, Italy, 1974.

[5] L. Pandolfi, "Linear Control Systems: Controllability with Constrained Controls," J. Optimization Theory and Appl., Vol. 19, pp. 577-585, 1976.

[6] W. E. Schmitendorf and B. R. Barmish, "Constrained Controllability of Linear Systems," to appear in print.

[7] B. R. Barmish and W. E. Schmitendorf, "Constrained Controllability," to appear in print.

APPENDIX C

AFOSR-TR- 78-1536

CONSTRAINED CONTROLLABILITY[1]

B. R. Barmish
Electrical Engineering Department
The University of Rochester
Rochester, NY 14627

W. E. Schmitendorf
Mechanical Engineering Department
Northwestern University
Evanston, IL 60201

Abstract

In this paper we survey some of our recent results [1,2] on the controllability of linear systems when there are magnitude constraints on the control.

[1] The research of the second author was supported by AFOSR under Grant 76-2923.

1. Introduction

Consider the problem of steering the state of a linear system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad ; \quad t \in [t_o, \infty)$$

to the origin from a specified initial condition

$$x(t_o) = x_o$$

by choice of control function $u(\cdot)$. Here $x(t) \in R^n$, $u(t) \in R^m$ and $A(\cdot)$ and $B(\cdot)$ are continuous matrices of appropriate dimension. Unlike the usual controllability problem where the control values at each instant of time are unconstrained, we insist here that the control values at each instant belong to a prescribed set Ω in R^m.

Let $\mathfrak{M}(\Omega)$ denote the set of functions from R into Ω that are measurable on $[t_o, \infty)$. Then any control $u(\cdot) \in \mathfrak{M}(\Omega)$ is termed admissible. We now define the concept of constrained controllability or, more precisely, Ω-null controllability.

<u>Definition 1.</u> The linear system (S) is <u>constrained controllable at (x_o, t_o) (Ω-null controllable at (x_o, t_o)</u> if for given initial condition $x(t_o) = x_o$ there exists a $u(\cdot) \in \mathfrak{M}(\Omega)$ such that the solution $x(\cdot)$ of (S) satisfies $x(T) = 0$ for some $T \in [t_o, \infty)$.

<u>Definition 2.</u> (S) is <u>completely constrained controllable at t_o (globally Ω-null controllable at t_o</u>) if (S) is Ω-null controllable at (x_o, t_o) for all $x_o \in R^n$.

A simple example illustrates that a system may be controllable if the control values are unconstrained but not if they are constrained. Let $x(t)$ and $u(t)$ be scalars and $\dot{x}(t) = x(t) + u(t)$, $x(0) = x_o$. If there are no constraints on $u(\cdot)(\Omega = R)$, the system is completely controllable. However, if $\Omega = \{\omega : |\omega| \le a\}$, then for $x_o \ge a$ it is not possible to drive the system to the origin with a control satisfying $u(t) \in \Omega$.

In the next section, we present a necessary and sufficient condition for (S) to be constrained controllable at (x_o,t_o) and a necessary and sufficient condition for complete constrained controllability at t_o. Unlike much of the work of previous authors [3-8], our results concentrate on the case where the matrices $A(\cdot)$ and $B(\cdot)$ are time-varying rather than constant matrices. Our conditions for global Ω-null controllability require that Ω be compact and contain zero, but not that zero be an interior point. Furthermore, in the case of Ω-null controllability at (x_o,t_o), we need not assume that there exists a $u \in \Omega$ such that $Bu = 0$.

2. Necessary and Sufficient Conditions

First we present a necessary and sufficient condition for Ω-null controllability at (x_o,t_o). To this end, define the scalar function $J : R^n \times R \times R^n \to R$ by $J(x_o,T,\lambda) = x_o'\phi'(T,t_o)\lambda + \int_{t_o}^{T} \sup\{\omega'B'(\tau)\phi'(T,\tau)\lambda : \omega \in \Omega\}d\tau$ where $\phi(t,\tau)$ is the state transition matrix for (S).

<u>Theorem 1. [Ref. 1].</u> <u>Let Ω be compact and suppose Λ is any subset of R^n which contains 0 as an interior point. Then (S) is Ω-null controllable at (x_o,t_o) if and only if</u>

$$\min\{J(x_o,T,\lambda) : \lambda \in \Lambda\} = 0$$

<u>for some $T \in [t_o,\infty)$.</u>

This theorem demonstrates that the question of constrained controllability can be answered by solving a finite dimensional optimization problem. Moreover, the system is globally Ω-null controllable at t_o if, and only if, for every $x_o \in R^n$ there is a $T < \infty$ (which may depend on x_o) such that $\min\{J(x_o,T,\lambda) : \lambda \in \Lambda\} = 0$. However, checking for global Ω-null controllability this way can be quite tedious. A simpler technique is embodied in the following theorem.

Theorem 2. [Ref. 1]. Suppose Ω is a compact set which contains 0. Then (S) is globally Ω-null controllable at t_o is and only if

$$\int_{t_o}^{\infty} \sup\{\omega' B'(\tau) z(t) : \omega \in \Omega\} d\tau = +\infty$$

for all non-zero solutions $z(\cdot)$ of the adjoint system

$$\dot{z}(t) = -A'(t) z(t) \quad ; \quad t \in [t_o, \infty) \tag{S'}$$

Consider again the example with $\dot{x}(t) = x(t) + u(t)$, $\Omega = \{\omega : |\omega| \leq a\}$. Applying Theorem 2 with $z_o \neq 0$ as initial condition for (S'), we compute

$$\int_{o}^{\infty} \sup\{\omega' B'(\tau) z(\tau) : \omega \in \Omega\} d\tau = \int_{o}^{a} a|e^{-\tau} a_o| d\tau < \infty$$

and this system is not globally Ω-null controllable.

Next, we use Theorem 1 to determine if there are any values of x_o such that the system is Ω-null controllable at $(x_o, 0)$. First, we determine that

$$J(x_o, T, \lambda) = \begin{cases} [x_o e^t - a(1 - e^t)]\lambda & \text{if} \quad \lambda \geq 0 \\ [x_o e^t + a(1 - e^t)]\lambda & \text{if} \quad \lambda < 0 \end{cases}$$

Taking $\Lambda = \{\lambda : |\lambda| \leq 1\}$ in Theorem 1, it follows that the minimum of $J(x_o, T, \lambda)$ will be zero if and only if

$$-a(1 - e)^T \leq x_o \leq a(1 - e^{-T})$$

and the system is Ω-null controllable if and only if $|x_o| < a$.

If the system equation is $\dot{x}(t) = -x(t) + u(t)$, then

$$\int_{o}^{\infty} \sup\{\omega' B'(\tau) z(t) : \omega \in \Omega\} = \int_{o}^{\infty} a|e^{\tau} z_o| d\tau$$

Since this integral is $+\infty$ for all $z_o \neq 0$, the system is globally Ω-null controllable.

3. Local Controllability

One other concept of controllability is that of local controllability.

Definition 3. (S) is locally Ω-null controllable at t_o if there exists an open set $V \subset R^n$, containing the origin, such that (S) is Ω-null controllable at (x_o, t_o) for all $x_o \in V$.

In the next theorem, normalized solutions $z_*(\cdot)$ of $z(\cdot)$ are needed. Such a solution is defined by

$$z_*(t) \overset{\Delta}{=} \frac{z(t)}{\|z(0)\|}$$

Theorem 3. [Ref. 2]. Suppose Ω is compact and contains zero. Then (S) is locally Ω-null controllable at t_o if and only if there exists an $\epsilon > 0$ such that

$$\int_{t_o}^{\infty} \sup\{\omega' B'(\tau) z_*(\tau) : \omega \in \Omega\} d\tau \geq \epsilon$$

for all normalized non-trivial solutions $z_*(\cdot)$ of (S').

As an example, consider the scalar system $\dot{x}(t) = tx(t) + u(t)$, $\Omega = [-1,1]$. For this system,

$$\int_0^{\infty} \sup\{\omega' B(\tau) z_*(\tau) : \omega \in \Omega\} d\tau = \int_0^{\infty} \sup_{|\omega| \leq 1} \frac{\omega e^{-\tau^2/2} z(0)}{\|z(0)\|} d\tau$$

$$= \int_0^{\infty} e^{-\tau^2/2} d\tau = \sqrt{\frac{\pi}{2}}$$

Thus the system is locally Ω-null controllable but not globally Ω-null controllable.

4. Concluding Remarks

The above theorems can be used to determine if (S) is Ω-null controllable. However, they do not give a method for determining a control which steers (S) to the origin. One method of determining a steering control is to solve the time optimal problem of finding $u_*(\cdot) \in \mathfrak{M}(\Omega)$ which steers (S) from given (x_o, t_o) to the origin in minimum time. If there is a control which steers the system to the origin then there is a time optimal one [3]. Hence, the algorithms available for solution of the time optimal problems can be used to compute a steering control.

Another technique which involves simpler computations and leads to a control which steers the system arbitrarily close to the origin is based on a minimum norm optimal control problem. Full details are given in [1].

References

1. W. E. Schmitendorf and B. R. Barmish, "Constrained Controllability of Linear Systems," SIAM J. on Control and Optimization (submitted).

2. B. R. Barmish and W. E. Schmitendorf, "A Necessary and Sufficient Condition for a Local Constrained Controllability of A Linear System," Proceedings 1978 Allerton Conference on Communication, Control and Computing, Monticello, Illinois, 1978.

3. E. Lee and L. Markus, Foundations of Optimal Control Theory, John Wiley, New York, 1967.

4. S. Saperstone and J. Yorke, "Controllability of Linear Oscillatory Systems Using Positive Controls," SIAM J. Control, Vol. 9, pp. 253-262, 1971.

5. R. Brammer, "Controllability of Linear Autonomous Systems with Positive Controllers," SIAM J. Control, Vol. 10, pp. 339-353, 1972.

6. S. Saperstone, "Global Controllability of Linear Systems with Positive Controls," SIAM J. Control, Vol. 11, pp. 417-423, 1973.

7. R. Conti, Teoria del Controllo e del Controllo Ottimo, UTET, Torino, Italy, 1974.

8. L. Pandolfi, "Linear Control Systems: Controllability with Constrained Controls," J. Optimization Theory and Appl., Vol. 19, pp. 577-585, 1976.

APPENDIX D

AFOSR-TR- 78-1536

A General Sufficiency Theorem for Minmax Control[1]

W. SCHMITENDORF[2]

Abstract. This paper considers optimal control problems where there is uncertainty in the differential equations describing the system. A minmax optimality criterion is used and sufficient conditions for a control to be a minmax control are presented. These conditions are more general than those given in Refs. 1 and 2.

Key Words: Minmax problems, sufficient conditions.

1 This research was supported by AFOSR under Grant 76-2923.

2 Associate Professor, Mechanical Engineering Department, Northwestern University, Evanston, Illinois 60201

1. Introduction. Sufficient conditions for a control to be a minmax control have been presented in Refs. 1-2. Here we present a more general sufficiency result, Theorem 3.1, and also show that the sufficient conditions of Refs. 1-2 are just special cases of this new set of conditions. Theorem 3.1 applies to problems with more general terminal conditions and cost functions than does Ref. 2 and can be used to verify that a control is a minmax control for some problem where the results of Refs. 1-2 fail to yield any information.

2. Problem Formulation. Consider the differential system

$$\dot{x}(t) = f(x(t),u(t),v(t)) \tag{1}$$

where the state $x(t) \in R^n$, control variables $u(t) \in R^{m_1}$ and the disturbance $v(t) \in R^{m_2}$. We assume $f(\cdot,\cdot,\cdot)$ is C^1 on $R^n \times R^{m_1} \times R^{m_2}$. The playing space is denoted by $X \times T$. X is an open set in R^n and $T = [t_o,t_f]$ where t_o and t_f are specified. At t_o the initial state x_o is specified. At the final time, we require $x(t_f) \in \theta$, where the target set θ is a given set in the closure of X.

Let $\mathcal{U}$ denote the set of piecewise continuous functions from T into R^{m_1} and let U be a given subset of R^{m_1}. Define

$$\mathcal{M}_1 = \{u(\cdot) : u(\cdot) \in \mathcal{U} \quad \text{and} \quad u(t) \in U,\ t \in T\}$$

Let V be a given subset of R^{m_2} and let $\mathcal{V}$ denote the set of piecewise continuous functions $q(\cdot,\cdot) : X \times T \to V$ with respect to some decomposition of $X \times T$.[1] Define

$$\mathcal{M}_2 = \{q(\cdot,\cdot) : q(\cdot,\cdot) \in \mathcal{V} \quad \text{and} \quad q(x,t) \in V \quad \text{for all} \quad (x,t) \in X \times T\}$$

For a given pair $[u(\cdot),q(\cdot,\cdot)]$, $u(\cdot) \in \mathcal{M}_1$ and $q(\cdot,\cdot) \in \mathcal{M}_2$, a solution of (1) from (x_o,t_o) will be denoted by $x(\cdot)$ and called a trajectory. A terminating trajectory is a trajectory satisfying $x(t) \in X$ for all $t \in [t_o,t_f]$ and $x(t_f) \in \theta$. A pair $[u(\cdot),q(\cdot,\cdot)]$ is playable at (x_o,t_o) if it generates a terminating trajectory.

[1] e.g., see page 42 of [3].

The pair $[u(\cdot),q(\cdot,\cdot)]$ may generate more than one terminating trajectory from (x_o,t_o) and $\Phi(u(\cdot),q(\cdot,\cdot))$ will denote the set of all such trajectories.

For $u(\cdot)\in\mathfrak{M}_1$, $\mathfrak{h}(u(\cdot))$ is the set of all $q(\cdot,\cdot)\in\mathfrak{M}_2$ such that $[u(\cdot),q(\cdot,\cdot)]$ is playable at (x_o,t_o). We shall say that $u(\cdot)$ is admissible if $u(\cdot)\in\mathfrak{M}_1$ and $\mathfrak{h}(u(\cdot))\neq\emptyset$.

For playable pair $[u(\cdot),q(\cdot,\cdot)]$ and trajectory $x(\cdot)\in\Phi(u(\cdot),q(\cdot,\cdot))$ the cost is defined by

$$J(u(\cdot),q(\cdot,\cdot),x(\cdot)) = g(x(t_f)) + \int_{t_o}^{t_f} L(x(t),u(t),q(x(t),t))dt \qquad (2)$$

For an admissible $u(\cdot)$, $(q(\cdot,\cdot),x(\cdot))\in\mathcal{K}(u(\cdot))$ iff $q(\cdot,\cdot)\in\mathfrak{h}(u(\cdot))$ and $x(\cdot)\in\Phi(u(\cdot),q(\cdot,\cdot))$. We now define a minmax control.

Definition 2.1. Let $u^*(\cdot)$ be admissible. Then $u^*(\cdot)$ is a minmax control iff

$$\sup_{(q(\cdot,\cdot),x(\cdot))\in\mathcal{K}(u^*(\cdot))} J(u^*(\cdot),q(\cdot,\cdot),x(\cdot))$$

$$\leq \sup_{(q(\cdot,\cdot),x(\cdot))\in\mathcal{K}(u(\cdot))} J(u(\cdot),q(\cdot,\cdot),x(\cdot))$$

for all admissible $u(\cdot)$.

3. A Sufficient Condition.

Before presenting a sufficient condition for a minmax strategy, some additional nomenclature is needed. The set $\mathcal{L}(u(\cdot))$ is defined by

$$\mathcal{L}(u(\cdot)) = \{[q(\cdot,\cdot),x(\cdot)]\in\mathcal{K}(u(\cdot)) : J(u(\cdot),q(\cdot,\cdot),x(\cdot))$$

$$\geq J(u(\cdot),r(\cdot,\cdot),y(\cdot)) \quad \text{for all} \quad [r(\cdot,\cdot),y(\cdot)]\in\mathcal{K}(u(\cdot))\}$$

The set $\mathcal{L}(u(\cdot))$ consists of the disturbances and corresponding trajectories which maximize the cost when Player 1 uses $u(\cdot)$.

In the case where $v(\cdot)$ is not present in (1) or (2) we have the usual optimal control problem. Then we say that $u^*(\cdot)\in\mathfrak{M}_1$ is an optimal control iff it generates a terminating trajectory $x^*(\cdot)$ and

$$J(u^*(\cdot),x^*(\cdot)) \leq J(u(\cdot),x(\cdot))$$

for all $u(\cdot) \in \mathfrak{M}_1$ that generate terminating trajectories.

Let $z(\cdot,\cdot,\cdot)$ be a function from $X \times T \times U \to V$ and, for $u(\cdot) \in \mathfrak{M}_1$, $q_u(\cdot,\cdot)$ is the function defined by $q_u(x,t) = z(x,t,u(t))$. Such a function $z(\cdot,\cdot,\cdot)$ will be called a <u>response function</u>. The set of admissible response functions is

$$Z = \{z(\cdot,\cdot,\cdot) \mid q_u(\cdot,\cdot) \in \mathfrak{n}(u(\cdot)) \quad \text{for all admissible } u(\cdot)\}$$

For positive integer γ, scalars $\alpha_i > 0$, $i = 1,\cdots,\gamma$ and functions $z^i(\cdot,\cdot,\cdot) \in Z$, $i = 1,\cdots,\gamma$ we define an optimal control problem $P(\gamma,\alpha_1,\cdots,\alpha_\gamma, z^1(\cdot,\cdot,\cdot),\cdots, z^\gamma(\cdot,\cdot,\cdot))$. This problem has state equations

$$\dot{x}^i(t) = f(x^i(t),u(t),z^i(x(t),t,u(t))) \ ; \ x^i(t_o) = x_o, \ i = 1,\cdots,\gamma$$

and cost

$$\hat{J}(u(\cdot),x^1(\cdot),\cdots,x^\gamma(\cdot)) = \sum_{i=1}^{\gamma} \alpha_i J(u(\cdot),q_u^i(\cdot,\cdot),x^i(\cdot))$$

Here $q_u^i(x,t) \triangleq z^i(x,t,u(t))$. The terminal conditions are $x^i(t_f) \in \theta$, $i = 1,\cdots,\gamma$.

We now present a sufficient condition for a control to be a minmax control.

<u>Theorem 3.1</u>. Let $u^*(\cdot)$ be admissible. Then $u^*(\cdot)$ is a minmax control if there exists

i) a positive integer γ

ii) scalars $\alpha_i > 0$, $i = 1,\cdots,\gamma$

iii) admissible responses $z^i(\cdot,\cdot,\cdot) \in Z$, $i = 1,\cdots,\gamma$

such that

a) $u^*(\cdot)$ is an optimal control for the problem $P(\gamma,\alpha_1,\cdots,\alpha_\gamma, z^1(\cdot,\cdot,\cdot), \cdots,z^\gamma(\cdot,\cdot,\cdot))$

b) for $i = 1,\cdots,\gamma$, $[q_{u^*}^i(\cdot,\cdot),x^{i*}(\cdot)] \in \mathcal{L}(u^*(\cdot))$ where $q_{u^*}^i(x,t) = z^i(x,t,u^*(t))$ and $x^{i*}(\cdot)$ is the solution of

$$\dot{x}^i(t) = f(x^i(t),u^*(t),q_{u^*}^i(x^i(t),t)), \quad x^i(t_o) = x_o$$

Proof. Consider any admissible $\hat{u}(\cdot)$. Since $u^*(\cdot)$ is a solution of $P(\gamma,\alpha_1,\cdots,\alpha_\gamma, z^1(\cdot,\cdot,\cdot),\cdots,z^\gamma(\cdot,\cdot,\cdot))$

$$\hat{J}(u^*(\cdot),x^{1*}(\cdot),\cdots,x^{\gamma*}(\cdot)) \le \hat{J}(\hat{u}(\cdot),\hat{x}^1(\cdot),\cdots,\hat{x}^\gamma(\cdot))$$

or

$$\sum_{i=1}^{\gamma} \alpha_i J(u^*(\cdot),q^i_{u*}(\cdot,\cdot),x^i(\cdot)) \le \sum_{i=1}^{\gamma} \alpha_i J(\hat{u}(\cdot),q^i_{\hat{u}}(\cdot,\cdot),\hat{x}^i(\cdot))$$

where

$$\hat{q}^i_{\hat{u}}(x,t) = z^i(x,t,\hat{u}(t))$$

Since $\alpha_i > 0$, $i = 1,\cdots,\gamma$, either

$$J(u^*(\cdot),q^i_{u*}(\cdot\ \cdot),x^{i*}(\cdot)) = J(\hat{u}(\cdot),q^i_{\hat{u}}(\cdot,\cdot),\hat{x}^i(\cdot)), \quad i = 1,\cdots,\gamma \tag{3a}$$

or, for at least one $i \in \{1,2,\cdots,\gamma\}$,

$$J(u^*(\cdot),q^i_{u*}(\cdot,\cdot),x^{i*}(\cdot)) < J(\hat{u}(\cdot),q^i_{\hat{u}}(\cdot,\cdot),\hat{x}^i(\cdot)) \tag{3b}$$

Since

$$[q^i_{u*}(\cdot,\cdot),x^{i*}(\cdot)] \in \mathcal{L}(u^*(\cdot)) \quad ,$$

$$\sup_{[q(\cdot,\cdot),x(\cdot)] \in \mathcal{K}(u^*(\cdot))} J(u^*(\cdot),q(\cdot,\cdot),x(\cdot)) = J(u^*(\cdot),q^i_{u*}(\cdot,\cdot),x^{i*}(\cdot)) \tag{4}$$

for $i = 1,\cdots,\gamma$. Also

$$\sup_{[q(\cdot,\cdot),x(\cdot)] \in \mathcal{K}(\hat{u}(\cdot))} J(\hat{u}(\cdot),q(\cdot,\cdot),x(\cdot)) \ge J(\hat{u}(\cdot),q^i_{\hat{u}}(\cdot,\cdot),\hat{x}^i(\cdot)) \tag{5}$$

for $i = 1,\cdots,\gamma$.

From (3)-(5) it follows that

$$\sup_{[q(\cdot,\cdot),x(\cdot)] \in \mathcal{K}(u^*(\cdot))} J(u^*(\cdot),q(\cdot,\cdot),x(\cdot))$$

$$\le \sup_{[q(\cdot,\cdot),x(\cdot)] \in \mathcal{K}(\hat{u}(\cdot))} J(\hat{u}(\cdot),q(\cdot,\cdot),x(\cdot))$$

Since this inequality holds for all admissible $\hat{u}(\cdot)$, $u^*(\cdot)$ is a minmax control. □

4. Discussion. Condition (a) of Theorem 3.1 requires $u^*(\cdot)$ to be the solution of an optimal control problem. Two approaches are available for verifying that a control is an optimal one: (i) the field theorem approach (see, for example, Refs. 4-5) and (ii) the direct sufficient condition approach (see, for example, Refs. 6-7).

If the final time is specified, the final state is free and $g(\cdot)$ is convex, the direct sufficiency approach can be used to establish (a) of Theorem 3.1. This leads directly to the sufficient condition of Ref. 2 and the result presented there is just a corollary of Theorem 3.1. A direct sufficient condition for problems with more general terminal conditions is available in Ref. 8 and it can also be used to verify (a).

While the direct sufficient conditions are relatively easy to apply, they are limited in application since they may yield no information. A more powerful technique is the field theorem approach and it can be used to establish (a) of Theorem 3.1. When $\gamma = 1$ and the field theorem approach is used to verify (a), Theorem 3.1 becomes Theorem 4.1 of Ref. 1 and that theorem is also just a corollary of Theorem 3.1. However, Theorem 3.1 is more general than the theorem in Ref. 1 since it is not restricted to $\gamma = 1$. In using the field theorem approach, $h(x,u,t) \triangleq f(x,u,z(x,t,u),t)$ may not be C^1 and the usual assumption of the field theorem is not met. In this case the more general field theorem of Ref. 9 can be used. There $h(\cdot,\cdot,\cdot)$ is not required to be C^1.

5. Concluding Remarks It has been shown that the problem of verifying that a control is a minmax control can be accomplished by establishing that certain functions satisfy related ordinary optimal control problems. In particular, $u^*(\cdot)$ is a minmax control if it is an optimal solution of $P(\gamma,\alpha_1,\cdots,\alpha_\gamma,z^1(\cdot,\cdot,\cdot),\cdots,z^\gamma(\cdot,\cdot,\cdot))$ and if $q^i_{u^*}(\cdot,\cdot)$ is a solution of the optimal control problem

(1)-(2) with $u(\cdot) = u^*(\cdot)$. All the sufficient conditions available from optimal control theory can be used on these optimal control problems. Previous sufficient conditions for minmax control are just corollaries of this general result given here.

A comment on feedback minmax strategies is appropriate. In Ref. 1, a sufficient condition for a minmax feedback strategy is given as well as the sufficient condition for an open minmax control. Attempts to extend Theorem 3.1 so that it applies to feedback minmax problems have been unsuccessful. The reason for this is that the feedback solution to the problem $P(\gamma,\alpha_1,\cdots,\alpha_\gamma, z^1(\cdot,\cdot,\cdot),\cdots,z^\gamma(\cdot,\cdot,\cdot))$ will depend on $x^i(\cdot)$, $i = 1,\cdots,\gamma$. But the $x^i(\cdot)$ are artificial variables and do not have any physical meaning. There is no way to implement a control based on the x^i and a feedback control dependent on the x^i is meaningless.

Also, the theorem does not apply to problems where the final time is unspecified. In such a case, each trajectory $x^i(\cdot)$, $i = 1,2,\cdots,\gamma$, might reach the target θ at different times and $P(\gamma,\alpha_1,\cdots,\alpha_\gamma,z^1(\cdot,\cdot,\cdot),\cdots,z^\gamma(\cdot,\cdot,\cdot))$ can not be treated by standard optimal control theory.

References

1. SCHMITENDORF, W. E., Differential Games Without Pure Strategy Saddle Point Solutions, Journal of Optimization Theory and Applications, Vol. 18, No. 1, 1976.

2. SCHMITENDORF, W. E., A Sufficient Conditions for Minmax Control of Systems with Uncertainty in the State Equations, IEEE Transactions on Automatic Control, Vol. AC-21, No. 4, 1976.

3. BLAQUIERE, A., GERARD, F. and LEITMANN, G., Quantitative and Qualitative Games, Academic Press, N.Y., 1969.

4. LEITMANN, G., Sufficiency Theorems for Optimal Control, Journal of Optimization Theory and Applications, Vol. 2, No. 5, 1968.

5. LEITMANN, G., A Note on a Sufficiency Theorem for Optimal Control, Journal of Optimization Theory and Applications, Vol. 3, No. 1, 1969.

6. EWING, G. M., Sufficient Conditions for Global Extrema in the Calculus of Variations, Journal of the Astronautical Sciences, Vol. 12, No. 3, 1965.

7. LEITMANN, G. and STALFORD, H., A Sufficiency Theorem for Optimal Control, Journal of Optimization Theory and Applications, Vol. 8, No. 3, 1971.

8. MEREAU, P. M. and POWERS, W. T., A Direct Sufficient Condition for Free Final Time Optimal Control Problems, SIAM Journal on Control, Vol. 14, No. 4, 1976.

9. STALFORD, H., Sufficient Conditions for Optimal Control with State and Control Constraints, Journal of Optimization Theory and Applications, Vol. 7, pp. 118-135, 1971.

AFOSR-TR- 78-1536

Optimal Control of Systems with Multiple Criteria When Disturbances are Present

W. SCHMITENDORF[2]

Abstract. Optimal control problems with a vector performance index and uncertainty in the state equations are investigated. Nature chooses the uncertainty, subject to magnitude bounds. For these problems a definition of optimality is presented. This definition reduces to that of a minmax control in the case of a scalar cost and to Pareto optimality when there is no uncertainty or disturbance present. Sufficient conditions for a control to satisfy this definition of optimality are derived. These conditions are in terms of a related two-player zero-sum differential game and suggest a technique for determining the optimal control. The results are illustrated with an example.

Key Words: Differential games, Pareto-optimality, multicriteria optimization, control theory.

1 This research was supported by AFOSR under Grant 76-2923.

2 Associate Professor, Mechanical Engineering Department, Northwestern University, Evanston, Illinois.

79 02 08 081

1. Introduction. In the standard optimal control problem, the system to be controlled is modelled by ordinary differential equations, the cost is a scalar and one player selects the control. Isaacs (Ref. 1) introduced conflict into the situation with the addition of a second player who chooses his controls to maximize the criterion that the first player want to minimize. This led to two player zero-sum differential game theory.

One application of this theory is to optimal control problems where disturbances are present in the differential equation model of the system. In such problems, it is assumed that nature can choose the disturbance or uncertainty, subject to magnitude constraints. Then, given any control, there is a guaranteed upper value to the cost and the objective is to choose a control which gives the lowest guaranteed cost. A control design based on this philosophy is termed a worst case design.

Looking at nature as a second player this formulation leads to a two player zero-sum differential game. If this game has a saddle point solution (and certain playability assumptions are satisfied), then the corresponding control gives the lowest guaranteed cost. Often, however, there will not exist a saddle point solution for the resulting differential game and the theory of Ref. 1 cannot be used. Some results, in the form of sufficient conditions, are available for determining controls which result in the lowest guaranteed cost (Ref. 2-3).

Another extension of the work of Ref. 1 has been to multicriteria optimization problem. In these problems, there are many players, each having his own performance index. The players may not be in direct conflict and may, in fact, be willing to cooperate.

Here we treat multicriteria optimal control problems where there is uncertainty in the state equations. An optimality criterion is defined and then sufficient conditions for an optimal control are derived. In addition, it is shown how these results may be used to determine an optimal solution.

2. Problem Formulation. Consider a control system described by the differential equations

$$\dot{x}(t) = f(x(t),u(t),v(t)) \tag{1}$$

where $x(t) \in R^n$ is the state, $u(t) \in R^{m_1}$ is the control and $v(t) \in R^{m_2}$ is the disturbance. We assume that $f(\cdot,\cdot,\cdot)$ is a C^1 function. The time interval $[t_o,t_f]$ is specified as is the initial state

$$x(t_o) = x_o \tag{2}$$

The disturbance $v(\cdot)$ is not known exactly; the only information available about $v(\cdot)$ is that $v(t)$ belongs to a known set $V \subset R^{m_2}$. Rather than modelling the system in a stochastic manner, we assume nature can choose any disturbance, subject only to $v(\cdot)$ being piecewise continuous and $v(t) \in V$.

Let

$$M_1 = \{u(\cdot) : u(\cdot) \in \mathcal{U} \text{ and } u(t) \in U \ , \ t \in [t_o,t_f]\}$$

where $\mathcal{U}$ is the set of piecewise continuous functions from $[t_o,t_f] \to R^{m_1}$ and U is a given subset of R^{m_1}. Similarly, let

$$M_2 = \{v(\cdot) : v(\cdot) \in \mathcal{V} \text{ and } v(t) \in V \ , \ t \in [t_o,t_f]\}$$

where $\mathcal{V}$ is the set of piecewise continuous functions from $[t_o,t_f] \to R^{m_2}$. For $u(\cdot) \in M_1$ and $v(\cdot) \in M_2$, the solution of (1), (2) will be denoted by $x(\cdot)$. The pair $[u(\cdot),v(\cdot)]$ is playable if it generates a solution of (1), (2) such that $x(t_f) \in \theta$ where the target set θ is a given set in R^n.

For $u(\cdot) \in M_1$, $N(u(\cdot))$ is the set of all $v(\cdot) \in M_2$ such that $[u(\cdot),v(\cdot)]$ is playable. A control $u(\cdot)$ is <u>admissible</u> if $u(\cdot) \in M_1$ and $N(u(\cdot)) \neq \emptyset$.

For a playable pair $[u(\cdot),v(\cdot)]$, the cost is a vector with two components,

$$J_i(u(\cdot),v(\cdot)) = g_i(x(t_f)) + \int_{t_o}^{t_f} L_i(x(t),u(t),v(t))dt \quad , \quad i = 1,2 \qquad (3)$$

Here $x(\cdot)$ denotes the trajectory corresponding to $u(\cdot),v(\cdot)$ starting at (x_o,t_o). (The results presented below can easily be extended to the case where the cost is a K vector. For simplicity of presentation, we concentrate on the case K = 2).

Before defining optimality, consider first the case of a single criterion. Given any $u(\cdot) \in M_1$, $\sup_{v(\cdot) \in N(u(\cdot))} J(u(\cdot),v(\cdot))$ is the guaranteed upper value of the cost when control $u(\cdot)$ is used. A control $u^*(\cdot)$ which gives the lowest guaranteed cost is sought, i.e., find $u^*(\cdot) \in M_1$ satisfying for all $u(\cdot) \in M_1$

$$\sup_{v(\cdot) \in N(u^*(\cdot))} J(u^*(\cdot),v(\cdot)) \leq \sup_{v(\cdot) \in N(u(\cdot))} J(u(\cdot),v(\cdot))$$

A control satisfying this condition is called a minmax control.

Now consider the multicriteria case when there is no disturbance. A useful optimality criterion is Pareto optimality. A control $u^*(\cdot)$ is Pareto optimal if, and only if, for every $u(\cdot) \in M_1$ either $J_i(u(\cdot)) = J_i(u^*(\cdot))$, $i = 1,2$, or, for at least one $i \in \{1,2\}$, $J_i(u^*(\cdot)) < J_i(u(\cdot))$.

In the following definition, we combine these two concepts into a definition of optimality which applies to multicriteria problems with disturbances.

<u>Definition 2.1</u>. A control $u^*(\cdot)$ is Pareto optimal if, and only if, it is admissible and for all admissible $u(\cdot)$ either

$$\sup_{v(\cdot)\in N(u^*(\cdot))} J_i(u^*(\cdot),v(\cdot)) = \sup_{v(\cdot)\in N(u(\cdot))} J_i(u(\cdot),v(\cdot)) \quad , \quad i = 1,2$$

or, for at least one $i \in \{1,2\}$,

$$\sup_{v(\cdot)\in N(u^*(\cdot))} J_i(u^*(\cdot),v(\cdot)) < \sup_{v(\cdot)\in N(u(\cdot))} J_i(u(\cdot),v(\cdot))$$

When the cost has only one component the problem is a two player zero-sum differential game and the definition becomes that of a minmax solution while, if there is no disturbance, the definition is the usual definition of Pareto optimality. If an admissible control, $u_1(\cdot)$, is not Pareto optimal, then there is another admissible control, $u_2(\cdot)$, such that

$$\sup_{v(\cdot)\in N(u_2(\cdot))} J_i(u_2(\cdot),v(\cdot)) \le \sup_{v(\cdot)\in N(u_1(\cdot))} J_i(u_1(\cdot),v(\cdot)) \quad , \quad i = 1,2$$

with the strict inequality holding for at least one i. Thus, if an admissible control is not Pareto optimal there is a control that reduces the guaranteed cost of at least one of the components of the cost vector without increasing the rest.

Static versions of this problem are treated in Ref. 4-6. In Ref. 7, a many player differential game problem with coalitions was studied and that problem can be interpreted as an optimal control problem with disturbances. There, however, in deriving sufficient conditions, it was assumed that a related two-player zero-sum game has a saddle point solution. We do not make that assumption here.

In the next section two sufficient conditions for Pareto optimal control are presented. These conditions are not directly working conditions, but only a step in that direction. In Sec. 4, we use the second of these conditions to obtain other conditions which may be used for determining Pareto optimal controls.

3. Preliminary Results. Sufficient conditions for Pareto optimality will be obtained in terms of the following two-player zero-sum differential game.

$$\hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot)) = \alpha_1 J_1(u(\cdot),v_1(\cdot)) + \alpha_2 J_2(u(\cdot),v_2(\cdot)) \tag{4}$$

$$\dot{x}_1(t) = f(x_1(t),u(t),v_1(t)) \ , \quad x_1(t_o) = x_o \tag{5}$$

$$\dot{x}_2(t) = f(x_2(t),u(t),v_2(t)) \ , \quad x_2(t_o) = x_o \tag{6}$$

Here $x_1(t),x_2(t) \in R^n$, $v_1(\cdot),v_2(\cdot) \in M_2$ and admissible $u(\cdot)$ are defined as in Sec. 2. One player controls $u(\cdot)$ while the other controls $v_1(\cdot)$ and $v_2(\cdot)$. Note that $x_i(\cdot)$ and $v_i(\cdot)$ only enter $\hat{J}$ through J_i while $u(\cdot)$ enters through J_1 and J_2. The time interval $[t_o,t_f]$ is fixed and we require $x_1(t_f),x_2(t_f) \in \theta$.

The controls $[u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)]$ are a saddle point solution for this game is they satisfy

$$\hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot)) \le \hat{J}(u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)) \le \hat{J}(u(\cdot),v_1^*(\cdot),v_2^*(\cdot)) \tag{7}$$

for $v_1(\cdot),v_2(\cdot) \in N(u^*(\cdot))$ and for all $u(\cdot)$ that are playable against $v_1^*(\cdot)$ and $v_2^*(\cdot)$.

In Theorem 3.1 which follows, we assume all $u(\cdot) \in M_1$ are playable against $(v_1^*(\cdot),v_2^*(\cdot))$.

Theorem 3.1. If $[u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)]$ is a saddle point solution to the differential game (4)-(6) for some $\alpha_1,\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, then $u^*(\cdot)$ is a Pareto optimal control for the problem (1)-(3).

Proof. The saddle point inequalities (7) imply

$$\hat{J}(u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)) = \sup_{v_1(\cdot),v_2(\cdot)\in N(u^*(\cdot))} \hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot))$$

and that for any admissible $u(\cdot)$

$$\hat{J}(u^*(\cdot),v_1^*(\cdot),v_2^*(\cdot)) \le \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot)) \quad v_1(\cdot),v_2(\cdot)\in N(u(\cdot))$$

Thus

$$\sup_{v_1(\cdot),v_2(\cdot)\in N(u^*(\cdot))} \hat{J}(u^*(\cdot).v_1(\cdot),v_2(\cdot)$$

$$\le \sup_{v_1(\cdot),v_2(\cdot)\in N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot) \tag{8}$$

for all admissible $u(\cdot)$. Since $v_i(\cdot)$ and $x_i(\cdot)$ only affect J_i

$$\sup_{v_1(\cdot),v_2(\cdot)\in N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot)) = \sum_{i=1}^{2} \alpha_i \sup_{v_i(\cdot)\in N(u(\cdot))} J_i(u(\cdot),v_i(\cdot)) \tag{9}$$

From (8), (9) it follows that

$$\sum_{i=1}^{2} \alpha_i \sup_{v_i(\cdot)\in N(u^*(\cdot))} J_i(u^*(\cdot),v_i(\cdot)) \le \sum_{i=1}^{2} \sup_{v_i(\cdot)\in N(u(\cdot))} J_i(u(\cdot),v_i(\cdot)) \tag{10}$$

Since $\alpha_1,\alpha_2 > 0$, (10) implies that $u^*(\cdot)$ is Pareto optimal. □

From this theorem, it is seen that Pareto optimal solutions can be found by determining open loop saddle point solutions for the two-player zero-sum game (4)-(6). As α_1 and α_2 vary, different saddle point solutions may be obtained. Each of these solutions has the Pareto optimal property. Techniques for finding open loop saddle point solutions can be found in

Ref. 2,7-10. Unfortunately, many of the differential games (4)-(6) which arise from problems with a disturbance (1)-(3) do not have saddle point solutions. They do, however, have minmax solutions.

Definition 3.1. An admissible control $u^*(\cdot)$ is a minmax solution to the game (4)-(6) if it satisfies for all admissible $u(\cdot)$

$$\sup_{v_1(\cdot),v_2(\cdot)\in N(u(\cdot))} \hat{J}(u^*(\cdot),v_1(\cdot),v_2(\cdot)) \le \sup_{v_1(\cdot),v_2(\cdot)\in N(u(\cdot))} \hat{J}(u(\cdot),v_1(\cdot),v_2))$$

Theorem 3.2. If $u^*(\cdot)$ is a minmax solution to the game (4)-(6) for some $\alpha_1,\alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$, then $u^*(\cdot)$ is a Pareto optimal control for the problem (1)-(3).

Proof. Since $u^*(\cdot)$ is a minmax solution, it satisfies (8). The proof then follows in exactly the same manner as the proof of Theorem 3.1. □

Theorems similar to Theorem 3.1 and Theorem 3.2 were obtained for problems involving coalitions in a differential game (Ref.7). There, use was made of Theorem 3.1 to find coalitive Pareto optimal controls. Here, we shall concentrate on Theorem 3.2 and use it, in conjunction with a sufficient condition for minmax control (Ref. 11) to obtain sufficient conditions for a Pareto optimal solution to the problem with disturbances (1)-(3). Also a possible technique for determining Pareto optimal solutions based on these sufficient conditions will be discussed.

4. Main Result. In Ref.(2,11)there are sufficient conditions for a control to be a minmax control. Based on these results, as well as Theorem 3.2, we obtain sufficient conditions for a Pareto optimal control for the original problem (1)-(3). First, we need some definitions. For i = 1,2, the set $\mathcal{L}_i(u(\cdot))$ consists of the admissible disturbances that are playable against

$u(\cdot)$ amd maximize J_i.

$\mathcal{L}_i(u(\cdot)) = \{\hat{v}(\cdot) : \hat{v}(\cdot) \in N(u(\cdot))$ and

$$J_i(u(\cdot),\hat{v}(\cdot)) = \sup_{v(\cdot)\in N(u(\cdot))} J_i(u(\cdot),v(\cdot))\}$$

$$i = 1,2$$

Let $w(\cdot,\cdot)$ be a function from $R^n \times R^{m_1} \to R^{m_2}$ and, for $u(\cdot) \in M_1$, $q_w(\cdot)$ is the function defined by $q_w(t) = w(x(t),u(t))$ where $x(\cdot)$ is the solution of $\dot{x}(t) = f(x(t),u(t),w(x(t),u(t))), x(t_o) = x_o$. The functions $w(\cdot,\cdot)$ will be called response functions. The set of admissible response function is

$$\mathcal{W} = \{w(\cdot,\cdot) : q_w(\cdot) \in N(u(\cdot)) \text{ for all admissible } u(\cdot)\}$$

Let γ be a positive integer, $y^i(\cdot,\cdot) \in \mathcal{W}$, $i = 1,\cdots,\gamma$, $z^i(\cdot,\cdot) \in \mathcal{W}$, $i = 1,\cdots,\gamma$. Consider an optimal control problem with cost

$$K(u(\cdot),y^i(\cdot,\cdot),z^i(\cdot,\cdot))$$

$$= \sum_{i=1}^{\gamma} \mu_i\{\alpha_1 J_1(u(\cdot),y^i(\cdot,\cdot)) + \alpha_2 J_2(u(\cdot),z^i(\cdot,\cdot))\} \tag{11}$$

and state equations

$$\dot{x}_1^i(t) = f[x_1^i(t),u(t),y^i(x_1^i(t),u(t))] , \quad x_1^i(t_o) = x_o \quad i = 1,\cdots,\gamma \tag{12a}$$

$$\dot{x}_2^i(t) = f[x_2^i(t),u(t),z^i(x_2^i(t),u(t))] , \quad x_2^i(t_o) = x_o \quad i = 1,\cdots,\gamma \tag{12b}$$

The terminal conditions are $x_1^i(t_f)$, $x_2^i(t_f) \in \theta$, $i = 1,\cdots,\gamma$.

In the following, we shall require $\alpha_1 > 0$, $\alpha_2 > 0$. Letting $\rho = \alpha_2/\alpha_1$, the cost can be rewritten as

$$K(u(\cdot),y^i(\cdot,\cdot),z^i(\cdot,\cdot))$$

$$= \sum_{i=1}^{\gamma} \mu_i J_1(u(\cdot),y^i(\cdot,\cdot)) + \rho \sum_{i=1}^{\gamma} \mu_i J_2(u(\cdot),z^i(\cdot,\cdot)) \quad (11')$$

We now present a sufficient condition for a control $u^*(\cdot)$ to be a min-max control for the problem (4)-(6) and thus, from Theorem 3.2, a Pareto optimal control for the problem (1)-(3).

Theorem 4.1. Let $u^*(\cdot)$ be admissible. If there exists

i) a positive integer γ

ii) scalars $\mu_i > 0$, $i = 1,\cdots,\gamma$, $\sum_{i=1}^{\gamma} \mu_i = 1$ and a scalar $\rho > 0$

iii) admissible response $y^i(\cdot,\cdot),z^i(\cdot,\cdot) \in \mathfrak{W}$, $i = 1,\cdots,\gamma$

such that

a) $v_1^{i*}(\cdot) \in \mathcal{L}_1(u^*(\cdot))$, $v_2^{i*}(\cdot) \in \mathcal{L}_2(u^*(\cdot))$, $i = 1,\cdots,\gamma$

where $v_1^{i*}(t) = y^i(x_1^{i*}(t),u^*(t))$, $v_2^{i*}(t) = z^i(x_2^{i*}(t),u^*(t))$ and $x_1^{i*}(\cdot)$, $x_2^{i*}(\cdot)$ are solutions of

$$\dot{x}_1^i(t) = f[x_1^i(t),u^*(t),y^i(x_1^i(t),u^*(t))] \ , \ x_1^i(t_o) = x_o \quad i = 1,\cdots,\gamma$$

$$\dot{x}_2^i(t) = f[x_2^i(t),u^*(t),z^i(x_2^i(t),u^*(t))] \ , \ x_2^i(t_o) = x_o \quad i = 1,\cdots,\gamma$$

b) $u^*(\cdot)$ is an optimal control for the problem $P(\gamma,\rho,\mu_i,y^i(\cdot,\cdot), z^i(\cdot,\cdot))$

then $u^*(\cdot)$ is a Pareto optimal control for the problem (1)-(3).

Proof. Consider any admissible $\hat{u}(\cdot)$. Since $u^*(\cdot)$ is a solution of the problem $P(\gamma,\rho,\mu_i,y^i(\cdot,\cdot)), z^i(\cdot,\cdot))$, for any $\hat{u}(\cdot) \in M_1$

$$\sum_{i=1}^{\gamma} \mu_i J_1(u^*(\cdot),v_1^{i*}(\cdot)) + \rho \sum_{i=1}^{\gamma} \mu_i J_2(u^*(\cdot),v_2^i(\cdot))$$

$$\leq \sum_{i=1}^{\gamma} \mu_i J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot)) + \rho \sum_{i=1}^{\gamma} \mu_i J_2(u^*(\cdot),p_{\hat{u}}^i(\cdot))$$

where

$$q_{\hat{u}}^i(t) = y^i(\hat{x}_1^i(t),\hat{u}(t)) \quad i = 1,\cdots,\gamma$$

$$p_{\hat{u}}^i(t) = z^i(\hat{x}_2^i(t),\hat{u}(t)) \quad i = 1,\cdots,\gamma$$

Since $\rho > 0$, $\mu_i > 0$ $i = 1,\cdots,\gamma$, either

$$J_1(u^*(\cdot),v_1^{i*}(\cdot)) = J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot)) \qquad i = 1,\cdots,\gamma$$

and (13)

$$J_2(u^*(\cdot),v_2^{i*}(\cdot)) = J_2(\hat{u}(\cdot),p_{\hat{u}}^i(\cdot)) \qquad i = 1,\cdots,\gamma$$

or

$$J_1(u^*(\cdot),v_1^{i*}(\cdot)) < J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot)) \qquad \text{for some } i \in \{1,\cdots,\gamma\} \tag{14}$$

or

$$J_2(u^*(\cdot),v_2^{i*}(\cdot)) < J_2(\hat{u}(\cdot),p_{\hat{u}}^i(\cdot)) \qquad \text{for some } i \in \{1,\cdots,\gamma\} \tag{15}$$

From (a)

$$\sup_{v \in N(u^*(\cdot))} J_1(u^*(\cdot),v(\cdot)) = J_1(u^*(\cdot),v_1^{i*}(\cdot)) \qquad i = 1,\cdots,\gamma \tag{16}$$

$$\sup_{v \in N(u^*(\cdot))} J_2(u^*(\cdot),v(\cdot)) = J_2(u^*(\cdot),v_2^{i*}(\cdot)) \qquad i = 1,\cdots,\gamma \tag{17}$$

Also,

$$J_1(\hat{u}(\cdot),q_{\hat{u}}^i(\cdot)) \le \sup_{v(\cdot) \in N(\hat{u}(\cdot))} J_1(\hat{u}(\cdot),v(\cdot)) \tag{18}$$

$$J_2(\hat{u}(\cdot),p_{\hat{u}}^i(\cdot)) \le {}_{v(\cdot) \in N(u(\cdot))} J_2(\hat{u}(\cdot),v(\cdot)) \tag{19}$$

Using (16)-(19), each of the three cases (13), (14), and (15) imply $u^*(\cdot)$ is Pareto optimal.

5. Discussion. For a given initial state and time, one can proceed as follows to obtain Pareto optimal solutions. Form the two-player zero-sum game (4)-(6). For those values of α_1 and α_2 with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$ for which there is an open loop saddle point solution, the corresponding $u(\cdot)$ is, according to Theorem 3.1, a Pareto optimal solution.

If there does not exist a saddle point solution, we must resort to Theorem 4.1. Choose a real integer $\gamma \geq 2$ and functions $y^i(\cdot,\cdot)$, $z^i(\cdot,\cdot)$, $i = 1,\cdots,\gamma$. Then determine the optimal control for the problem $P(\gamma,\rho,\mu_i,y^i(\cdot,\cdot),z^i(\cdot,\cdot))$ in terms of ρ and μ_i, i.e., $u(\cdot,\rho,\mu_1,\cdots,\mu_\gamma)$. If possible, choose the μ_i so that

$$J_1(u(\cdot),y^i(\cdot,\cdot)) + \rho\, J_2(u(\cdot),z^i(\cdot,\cdot))$$

$$= J_1(u(\cdot),y^j(\cdot,\cdot)) + \rho\, J_2(u(\cdot),z^j(\cdot,\cdot))$$

for all i, $j \in \{1,\cdots,\gamma\}$. For every $\rho > 0$ for which this can be done, we then have a candidate for a Pareto optimal solution of the problem (1)-(3). It may then be possible to verify that these candidates are solutions by using Theorem 4.1. In checking (a) and (b), of Theorem 4.1, standard sufficiency results for optimal control, such as Ref. 12-16, can be used. Of course, choosing γ, $y^i(\cdot,\cdot)$ and $z^i(\cdot,\cdot)$ may be difficult and some ingenuity as well as trial and error must be used.

The general idea of the procedure is to let α_1 take on all possible values in the interval (0,1). For every value of α_1 in this interval we hope to get a Pareto optimal solution. If there is an open loop saddle point solution corresponding to a particular α_1, we use Theorem 3.1; if not, we find a minmax solution and use Theorem 4.1.

<u>6. Example</u>. Consider the following problem with scalar $x(\cdot)$, $u(\cdot)$ and $v(\cdot)$.

$$J_1(u(\cdot),v(\cdot)) = x^2(1), \quad J_2(u(\cdot),v(\cdot)) = -x(1) \tag{20}$$

$$\dot{x}(t) = u(t)v(t) \quad , \quad x(0) = \frac{1}{2} \tag{21}$$

$$U = \{u : u^2 \leq 1\} \quad , \quad V = \{v : 1 \leq v \leq 2\} \tag{22}$$

First we form the two player differential game (4)-(6)

$$\hat{J}(u(\cdot),v_1(\cdot),v_2(\cdot)) = \alpha_1 x_1^2(1) - \alpha_2 x_2(1)$$

$$\dot{x}_1(t) = u(t)v_1(t) \quad ; \quad x_1(0) = \frac{1}{2}$$

$$\dot{x}_2(t) = u(t)v_2(t) \quad ; \quad x_2(0) = \frac{1}{2}$$

Checking for open loop saddle solutions as α_1 ranges between 0 and 1, we find that there are such solutions for $\alpha_1 \in (0, 6/7)$ and that the controls $u(\cdot)$, $v_1(\cdot)$ and $v_2(\cdot)$ are constant. These results are summarized in Table 1.

TABLE 1

α	$\hat{u}(\cdot)$	$\hat{v}_1(\cdot)$	$\hat{v}_2(\cdot)$
$0 < \alpha_1 \leq \frac{1}{11}$	1	2	1
$\frac{1}{11} < \alpha_1 \leq \frac{1}{3}$	$\frac{1 - 2\alpha_1}{8\alpha_1}$	2	1
$\frac{1}{3} < \alpha_1 \leq \frac{1}{2}$	0	$\frac{1 - \alpha_1}{\alpha_1}$	1
$\frac{1}{2} < \alpha_1 \leq \frac{2}{3}$	0	$\frac{2(1 - \alpha_1)}{\alpha_1}$	2
$\frac{2}{3} < \alpha_1 \leq \frac{6}{7}$	$\frac{1 - \frac{3}{2}\alpha_1}{\alpha_1}$	1	2

The inequality $\hat{J}(\hat{u}(\cdot),\hat{v}_1(\cdot),\hat{v}_2(\cdot)) \leq \hat{J}(u(\cdot),\hat{v}_1(\cdot),\hat{v}_2(\cdot))$ was verified via the direct sufficient conditions (Ref. 13,14) while the inequality $\hat{J}(\hat{u}(\cdot),\hat{v}_1(\cdot),\hat{v}_2(\cdot)) \geq \hat{J}(\hat{u}(\cdot),\hat{v}_1(\cdot),v_2(\cdot))$ was established using field type sufficiency theorems (Ref. 10,11). That the $\hat{u}(\cdot)$ part of the saddle point solution is a Pareto optimal solution follows from Theorem 3.1.

For $6/7 < \alpha_1 < 1$, there is no open loop saddle point solution and we must resot to Theorem 4.2. To this end, we consider the control problem given by (11)-(12) with $\gamma = 2$, $y^1(\cdot) = 1$, $y^2(\cdot) = 1$, $z^1(\cdot) = 2$, $z^2(\cdot) = 2$, $0 < \rho < \frac{1}{6}$, $\mu_1 = 2\rho \div \frac{2}{3}$ and $\mu_2 = 1 - \mu_1$.

$$K(u(\cdot),y^i(\cdot),z^i(\cdot)) = \mu_1[x_1^1(1)]^2 + \mu_2[x_1^2(1)]^2 - \rho\mu_1 x_2^1 - \rho\mu_2 x_2^2(1)$$

$$\dot{x}_1^1(t) = u(t) \quad , \quad \dot{x}_2^1(t) = 2u(t)$$

$$\dot{x}_1^2(t) = 2u(t) \quad , \quad \dot{x}_2^2(t) = 2u(t)$$

$$x_1^1(0) = x_1^2(0) = x_2^1(0) = x_2^2(0) = \frac{1}{2}$$

Since $u^*(t) = -1/3$ is a solution to this optimal control problem and since $y^i(\cdot) \in \mathcal{L}_1(u^*(\cdot))$, $z^i(\cdot) \in \mathcal{L}_2(u^*(\cdot))$, $i = 1,2$, $u^*(\cdot)$ is a Pareto optimal solution for this problem. Also, since $\rho = \alpha_2/\alpha_1$ and $0 < \rho < 1/6$, this solution corresponds to $6/7 < \alpha_1 < 1$. Note that Theorem 3.1 does not apply when $6/7 < \alpha_1 < 1$ since there is no saddle point solution and Theorem 4.1 is needed.

6. Concluding Remarks

In the definition of optimality in Sec. 2, for a given $u(\cdot)$, the maximization operation is over all $v(\cdot)$ that lead to termination against $u(\cdot)$. This may be considered unrealistic since there is no assurance that nature will choose $v(\cdot)$ to terminate on θ. For problems where the differential equations (1) satisfy a growth condition and the terminal set $\theta = R^n$, every $u(\cdot) \in M_1$ is playable with every $v(\cdot) \in M_2$ and this difficulty does not arise. In the general case, restricting M_1 and M_2 so that all pairs $(u(\cdot),v(\cdot))$ with $u(\cdot) \in M_1$ and $v(\cdot) \in M_2$ are playable, would eliminate the necessity of the playability assumption on $v(\cdot)$. If it is felt that the condition that the maximization operation is over all $v(\cdot)$ that lead to termination is unrealistic, then the theory should be applied only to problems where M_1 and M_2 are such that all pairs $(u(\cdot),v(\cdot))$ with $u(\cdot) \in M_1$ and $v(\cdot) \in M_2$ are playable.

References

1. ISAACS, R., Differential Games, John Wiley and Sons, New York, 1965.

2. SCHMITENDORF, W. E., A Sufficient Condition for Minmax Control of Systems with Uncertainty in the State Equations, IEEE Transactions on Automatic Control, Vol. AC-15, No. 3, 1970.

3. SCHMITENDORF, W. E., Minmax Control of Systems with Uncertainty in the Initial State and in the State Equations, IEEE Transactions on Automatic Control, Vol. AC-21, No. 4, 1976.

4. GOFFIN, J. AND HAURIE, A., Necessary Conditions and Sufficient Conditions for Pareto Optimality in a Multicriterion Perturbed System, Proceedings of the 5th IFIP Conference on Optimization, 1973.

5. GOFFIN, J., Conditions for Pareto Optimality in a Multicriterion System with Nondifferentiable Functions, Proceedings of the 11th Annual Allerton Conference on Circuits and System Theory, 1973.

6. SCHMITENDORF, W. E., Static Multicriteria Problems: Necessary Conditions and Sufficient Conditions, Proceedings IFAC Symposium on Large Scale Systems, Udine, Italy, June 16-20, 1976.

7. SCHMITENDORF, W. E. AND MORIARTY, G., A Sufficient Condition for Coalitive Pareto-Optimal Solutions, Journal of Optimization Theory and Applications, Vol. 18, No. 1, 1976.

8. BRYSON, A. E. AND HO, Y. C., Applied Optimal Control, Blaisdell, Waltham, Mass., 1969.

9. SCHMITENDORF, W. E., Existence of Optimal Open-Loop Strategies for a Class of Differential Games, Journal of Optimization Theory and Applications, Vol. 5, No. 5, 1970.

10. SCHMITENDORF, W. E., Differential Games with Open-Loop Saddle Point Conditions, IEEE Transactions on Automatic Control, Vol. AC-15, No. 3, 1970.

11. SCHMITENDORF, W. E., A General Sufficiency Condition for Minmax Control, (to be submitted for publication).

12. LEITMANN, G., Sufficiency Theorems for Optimal Control, Journal of Optimization Theory and Applications, Vol. 2, pp. 285-292, 1968.

13. LEITMANN, G., A Note on a Sufficiency Theorem for Optimal Control, Journal of Optimization Theory and Applications, Vol. 3, pp. 76-78, 1969.

14. STALFORD, H., Sufficient Conditions for Optimal Control with State and Control Constraints, Journal of Optimization Theory and Applications, Vol. 7, pp. 118-135, 1971.

15. MANGASARIAN, O. L., Sufficient Conditions for the Optimal Control of Nonlinear Systems, SIAM Journal on Control, Vol. 4, No. 1, 1966.

16. LEITMANN, G., AND STALFORD, H., A Sufficiency Theory for Optimal Control, Journal of Optimization Theory and Applications, Vol. 8, No. 3, 1971.

APPENDIX F

A Simple Derivation of Necessary Conditions for Static Minmax Problems*

W. E. Schmitendorf

The Technological Institute
Northwestern University
Evanston, Illinois 60201

* This research was supported by the Air Force Office of Scientific Research under Grant 76-2923.

Proposed Running Head: Minmax Problems

Mailing Address: Professor W. E. Schmitendorf
Department of Mechanical Engineering
and Astronautical Sciences
Northwestern University
Evanston, IL 60201

1. INTRODUCTION

In this paper we present a simple derivation of necessary conditions for static minmax problems. The problem is to choose $x \in X \subset R^n$ to minimize

$$f(x) = \sup_{y \in Y} \phi(x,y) \tag{1}$$

where $\phi(\cdot,\cdot) : R^n \times R^m \to R$ is C^1. Y is a specified subset of R^m and $X = \{x | C(x) \le 0\}$. The function $C(\cdot) : R^n \to R^p$ is C^1 on R^n.

Necessary conditions which the minimizing point must satisfy are given in [1-3] in terms of the directional derivative of $f(\cdot)$. This results in a Lagrange multiplier rule which is an inequality. In [4], the necessary conditions are in the form of a Lagrange multiplier rule that is an equality. Here, we derive the results of [4] in an alternate and simpler fashion.

Our approach is to replace the above minmax problem, as suggested in [5], by a related nonlinear programming problem where the subsidiary conditions consist of an infinite number of inequalities. The resulting programming problem is called the Fritz John problem [6,7]. A relationship between the solution of the Fritz John problem and the minmax problem is presented. This relationship, in conjunction with the necessary conditions for the Fritz John problem, lead directly to necessary conditions for the minmax problem which are identical to those in [4].

We shall assume that Y is compact. Then for every $\hat{x} \in X$ there exists a $\hat{y} \in Y$ with the property that $\phi(\hat{x},\hat{y}) = \sup_{y \in Y} \phi(\hat{x},y)$ and we can replace "sup" by "max". The vector of partial derivatives $\partial\phi(x,y)/\partial x$ will be denoted by $\phi_x(x,y)$. Similarly, $C_{ix}(x) = \partial C_i(x)/\partial x$.

2. A NONLINEAR PROGRAMMING PROBLEM

As suggested in [5], we consider the problem choosing $z \in R^{n+1}$ to minimize

$$L(z) = z_{n+1} \tag{2}$$

subject to

$$C_i(z_1, z_2, \cdots, z_n) \leq 0 \quad , \quad i = 1,2,\cdots,p \tag{3}$$

and

$$\phi(z_1, z_2, \cdots, z_n, y) - z_{n+1} \leq 0 \quad \text{for all} \quad y \in Y \tag{4}$$

The relationship between the solution of this problem and the minmax problem is given by the following propositions.

PROPOSITION 1. If z^o is a solution to the problem (2)-(4), then $x^o = [z_1^o, z_2^o, \cdots, z_n^o]^T$ is a solution to the minmax problem.

Proof. Suppose x^o is not a solution to the minmax problem. Then there exists an $\hat{x}$ satisfying

$$C_i(\hat{x}) \leq 0 \quad , \quad i = 1,2,\cdots,p$$

such that

$$\sup_{y \in Y} \phi(\hat{x}, y) < \sup_{y \in Y} \phi(x^o, y) \leq z_{n+1}^o$$

Let $\hat{z}_{n+1} = \sup_{y \in Y} \phi(\hat{x}, y)$ and $\hat{z}^T = [\hat{x}^T \; z_{n+1}]$. Then

$$\hat{z}_{n+1} < z_{n+1}^o$$

$$C_i(\hat{z}_1, \cdots, \hat{z}_n) \leq 0 \qquad i = 1,2,\cdots,p$$

$$\phi(\hat{z}_1, \cdots, \hat{z}_n, y) - \hat{z}_{n+1} \leq 0 \quad \text{for all} \quad y \in Y$$

which contradicts the hypothesis that z^o is a solution to the problem (2)-(4). □

PROPOSITION 2. If x^* is a solution to the minmax problem and $z^*_{n+1} = \sup_{y \in Y} \phi(x^*,y)$, then $z^{*T} = [x^{*T} \; z^*_{n+1}]$ is a solution to the problem (2)-(4).

Proof. Suppose z^* is not a solution to the problem (2)-(4). Then there exists a $\hat{z}$ such that

$$\hat{z}_{n+1} < z^*_{n+1} = \sup_{y \in Y} \phi(x^*,y) \tag{5}$$

$$C(\hat{z}_1,\cdots,\hat{z}_n) \le 0$$

$$\phi(\hat{z}_1,\cdots,\hat{z}_n,y) - \hat{z}_{n+1} \le 0 \quad \text{for all} \quad y \in Y$$

The last inequality is equivalent to

$$\sup_{y \in Y} \phi(\hat{z}_1,\cdots,\hat{z}_n,y) \le \hat{z}_{n+1} \tag{6}$$

Then (5) and (6) imply

$$\sup_{y \in Y} \phi(\hat{z}_1,\cdots,\hat{z}_n,y) < \sup_{y \in Y} \phi(x^*,y)$$

which contradicts the hypothesis that x^* is a solution to the minmax problem. □

Thus, the minmax problem is equivalent to a Fritz John problem. In the next section, we state the necessary conditions for the Fritz John problem. Then we apply the conditions to the problem (4)-(6) and thereby obtain necessary conditions which the minmax solution must satisfy.

3. NECESSARY CONDITIONS FOR A MINMAX SOLUTION

The Fritz John problem we are interested in is to determine $z \in R^{n+1}$ which minimizes

$$L(z)$$

subject to

$$C_i(z) \le 0 \qquad i = 1,2,\cdots,p$$

and

$$\theta(z,y) \le 0 \quad \text{for all} \quad y \in Y$$

where $\theta(\cdot,\cdot) : R^{n+1} \times R^m \rightarrow R$ is a C^1 function. Necessary conditions which a minimizing point must satisfy are given in [6] or [7].

THEOREM 1. Let z^* be a minimizing point. Then there exists an integer α, scalars $\lambda_o \ge 0$, $\lambda_1 > 0, \cdots, \lambda_\alpha > 0$, scalars $\mu_i \ge 0$ $i = 1,\cdots,p$ and vectors $y^k \in \{y | \theta(z^*,y) = 0\}$, $k = 1,\cdots,\alpha$ such that

$$\lambda_o L_z(z^*) + \sum_{i=1}^{\alpha} \lambda_i \theta_z(z^*,y^i) + \sum_{i=1}^{p} \mu_i C_{iz}(z^*) = 0$$

$$\mu_i C_i(z^*) = 0 \qquad i = 1,2,\cdots,p$$

$$\theta(z^*,y) \le 0 \qquad \text{for} \quad y \in Y$$

Furthermore, if β is the number of nonzero μ_i, $1 \le \alpha + \beta \le n + 1$.

Applying this theorem to the problem (2)-(4) leads to the conditions

$$\sum_{i=1}^{\alpha} \lambda_i \phi_{z_j}(z_1^*,\cdots,z_n^*,y^i) + \sum_{i=1}^{p} \mu_i C_{iz_j}(z_1^*,\cdots,z_n^*) = 0 \qquad i = 1,\cdots,n$$

$$\lambda_o - \sum_{i=1}^{\alpha} \lambda_i = 0$$

$$\mu_i C_i(z_1^*,\cdots,z_n^*) = 0 \qquad i = 1,\cdots,p$$

$$\phi(z_1^*,\cdots,z_n^*,y) - z_{n+1}^* \le 0 \quad \text{for all} \quad y \in Y$$

where each y^i satisfies

$$\phi(z_1^*,\cdots,z_n^*,y^i) - z_{n+1}^* = 0 \quad \text{for all} \quad y \in Y$$

Restating these results in terms of x rather than z results in necessary conditions for the minmax problem which are identical to those in [4]. Specifically we have

<u>THEOREM 2</u>. <u>Let x^* be a solution to the minmax problem. Then there exists a positive integer α, scalars $\lambda_i > 0$, $i = 1,\cdots,\alpha$, scalar $\mu_i \geq 0$, $i = 1,\cdots,p$, vectors $y^i \in \hat{Y}(x^*) \triangleq \{y \in Y | \phi(x^*,y) = \sup_{z \in Y} \phi(x^*,z)\}$, $i = 1,\cdots,\alpha$ such that</u>

$$\sum_{i=1}^{\alpha} \lambda_i \phi_x(x^*,y^i) + \sum_{i=1}^{p} \mu_i C_{ix}(x^*) = 0 \quad ,$$

$$\mu_i C_i(x^*) = 0 \quad , \quad i = 1,2,\cdots,p \quad ,$$

<u>Furthermore, if β is the number of nonzero μ_i, $1 \leq \alpha + \beta \leq n + 1$.</u>

References

1. J. M. Danskin, The theory of max-min, with applications, J. SIAM Appl. Math., 14 (1966), 641-644.

2. J. Bram, The Lagrange multiplier theorem for max-min with several constraints, J. SIAM Appl. Math., 14 (1966), 655-667.

3. J. M. Danskin, "The Theory of Max-Min and Its Applications to Weapons Allocation Problems," Springer-Verlag, Berlin, 1967.

4. W. E. Schmitendorf, Necessary conditions and sufficient conditions for static minmax problems, J. Math. Analysis and Applications, 57 (1977), 683-693.

5. B. M. Kwak and E. J. Haug, Jr., Optimum design in the presence of parametric uncertainty, J. Optimization Theory and Applic., 19 (1976), 527-546.

6. F. John, Extremum problems with inequalities as subsidiary conditions, in "Studies and Essays, Courant Anniversary Volume," pp. 187-204, Interscience, New York, 1948.

7. K. R. Gehner, Necessary and sufficient conditions for the Fritz John problem with linear equality constraints, SIAM J. Control, 12 (1974), 140-149.