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NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS OCCURRING IN TW--ETC(U)

JAN 79 R B JACKSON, A R ROBINSON N00014-75-C-0164

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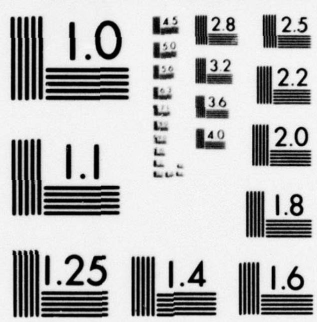
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NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS OCCURRING IN TWO-POINT BOUNDARY VALUE PROBLEMS

By

R. S. JACKSON

A. R. ROBINSON

A Technical Report of
Research Sponsored by
THE OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
Contract No. N00014-75-C-0164
Project No. NR 064-183

UNIVERSITY OF ILLINOIS
at URBANA-CHAMPAIGN
URBANA, ILLINOIS
JANUARY 1979

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(12)

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THE OFFICE OF NAVAL RESEARCH
DEPARTMENT OF THE NAVY
Contract No. N00014-75-C-0614
Project No. NR 064-183

University of Illinois
at Urbana-Champaign
Urbana, Illinois

January 1979

ACKNOWLEDGMENT

This report was prepared as a doctoral dissertation by Mr. Rodger B. Jackson and was submitted to the Graduate College of the University of Illinois at Urbana - Champaign in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering. The work was done under the supervision of Dr. Arthur R. Robinson, Professor of Civil Engineering.

The investigation was conducted as part of a research program supported by the Office of Naval Research, Contract No. 0014-75-C-0164, "Numerical and Approximate Methods of Stress Analysis".

The numerical results were obtained with the use of the IBM 360 computer system of the Department of Computer Science of the University of Illinois at Urbana - Champaign. The cooperation of the staff of the Digital Computer Laboratory is greatefully acknowledged.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

1. INTRODUCTION

1.1 Objective and Scope

The main purpose of this investigation is to develop a numerical procedure for accurately solving two-point boundary value problems using an initial value or "shooting" approach (Roberts, 1962, and Salvadori and Baron, 1961)* when rapidly growing solutions are present. The method is applicable to a wide variety of practical engineering problems, but herein will be demonstrated only in the solution of a fourth-order system of differential equations with two growing solutions in each direction, and in the determination of natural frequencies and mode shapes of sperical shell segments.

It is well known that in the application of a straightforward shooting method, difficulties are encountered when rapidly growing solutions are present. Such rapidly growing solutions always exist when there is an edge effect, such as in long, slender beams on elastic foundations or in the flexural theory of thin shells. Several methods have been proposed to treat cases involving rapidly growing solutions, none of them entirely satisfactory.

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* Names and dates in parenthesis refer to entries in the List of References

What is proposed here is a new method for initial value solution of ordinary differential equations which have edge effects in the solution. In essence, a rapidly growing solution of a differential equation or system of differential equations is first found. Then the differential equation is altered with the aid of the rapidly growing solution, so that the remaining differential equation does not contain this solution. Since the difficulty arising with a rapidly growing solution lies in the fact that the growing solution dominates all shooting solutions, a method which prevents that from occurring is an attractive way to eliminate the difficulty.

1.2 Initial Value Solutions of Two-Point Boundary Value Problems

Two-point boundary value problems occurring in various areas of engineering, physics and applied mathematics generally cannot be solved analytically. Therefore, it becomes necessary to use numerical methods to find the solution. Some of the numerical methods being used include the following: variational methods, the method of collocation, and the finite difference method (Roberts, 1972). The application of these methods, however, normally require large amounts of computer time and storage space.

An alternative way to solve two-point boundary value problems is to replace them with a set of initial value problems. The procedure merely involves numerically integrating the differential equation from the initial boundary to the final boundary a number of times. The integrations start with given and assumed independent initial data, the total number of initial conditions always being equal to the sum of the number of given initial and the number of given final boundary conditions. The solution to the boundary value problem in question will then be obtained either by successive approximations to the unknown initial data or by combining the solutions solving simultaneous equations. This method is referred to as a shooting method (Roberts, 1972). Solutions obtained by this method do not present difficulties as long as no rapidly growing solutions exist. When rapidly growing solutions are present, the application of the shooting method as described above is ineffective. The solutions lose their independence because the rapidly growing part of the solution eventually dominates in every initial value solution. Consequently, since the solutions are not effectively independent, from a numerical point of view, they cannot be accurately combined to form the correct solution.

Variants of the shooting method have been devised to avoid the problem caused by rapidly growing solutions.

Among these are the subinterval technique, the suppression technique, the field method, and the invariant imbedding method.

Cohen (Cohen, 1964) and Kalnins (Kalnins, 1964) developed methods whereby the total interval is divided into a number of subintervals, over each of which initial value problems are solved separately using the same system of differential equations. After the integrations have been performed over each subinterval, continuity conditions are imposed on appropriate quantities at the ends of each segment, thereby forming a simultaneous system of linear equations.

The suppression method developed by Zarghamee and Robinson (Zarghamee, 1967)* re-orthogonalizes the solutions whenever they grow beyond a certain predetermined limit. As in the ordinary shooting method, solutions are obtained corresponding to each set of initial conditions by numerical integration. For example, let

$$W^{IV} - \beta^4 W = 0 \qquad (1.1)$$

be the governing differential equation. Further let

---

* The suppression method was also developed independently by Goldberg (Goldberg, 1965).

$$W(0) = W_o$$

$$W'(0) = 0$$

be the actual initial conditions and

$$W''(0) = 0$$

$$W'''(0) = 1$$

and
$$W''(0) = 1$$

$$W'''(0) = 0$$

be two assumed sets of independent initial conditions. Two solutions are obtained, one for each set of assumed conditions, but both satisfying the actual initial conditions. Integration of equation (1.1) is started at the initial boundary and continued until the solutions grow beyond some predetermined limit. When this happens, a point of suppression is introduced. Two new sets of conditions are assumed. The two independent solutions are then linearly combined to satisfy each set of new conditions, thereby, forming two new independent solutions. These solutions are propagated until another point of suppression is introduced and the solutions are again re-orthogonalized.

After several suppressions, the remaining solutions will be very close to the correct solution which does not contain the growing function. It is obvious that in this method fewer solutions must be carried along than in the Cohen or Kalnins method.

The field method (Miller, 1967, and Berezin and Zhidkov, 1960) uses a technique whereby a differential equation which has a growing solution is replaced by a set of first-order differential equations, involving the field functions, which do not possess growing solutions. A linear differential equation of order 2n with n boundary conditions given at a and n at b is first replaced by a system of n second-order differential equations. This is accomplished by substituting new variables for the dependent variable and its derivatives. The system of n second-order differential equations is in turn, replaced by a system of n(n + 1) first-order differential equations which must be integrated from a to b and a second system of n first-order equations which have to be integrated from b to a. The variables and their first derivatives in the system of second order equations for which initial values are given are expressed as linear functions of the variables for which initial values are not given. These expressions form a system of n first-order differential equations. The coefficients in these expressions are the field functions.

Through a series of differentiations and substitutions of the n first-order equations, the $n(n+1)$ first-order field equations are formed. These equations are integrated from a to b to determine the field functions. Then the n first-order equations are integrated from b to a to determine the solution to the original problem.

Although the computed solutions are forced to satisfy the boundary conditions exactly, it has been shown that with certain boundary conditions the solutions for the internal points could be greatly in error (Zarghamee, 1970). This would be due to a growing solution for the field equations when integrated from a to b. In addition, the integration from b to a would yield erroneous results, because the field functions at b would be in error and the integration from b to a would also give rise to a growing solution.

In the invariant imbedding method (Scott, 1973), which is similar to the field method, an nth-order differential equation, with boundaries at a and b, is first replaced with a set of first-order differential equations. From this set of n first-order equations and with the use of a modified Riccati transformation, two systems of equations involving the invariant imbedding fucntions are derived. One system contains first-order equations in terms of invariant imbedding functions only, and is integrated from a to b to determine the values of these functions. The other system contains only algebraic equations in terms of the invariant imbedding functions and the dependent variables in the original system of n first-order equations. This system is used to determine the values of the original dependent variables between a and b. Here, however, is where a major problem can develop. Expressions for these variables involve taking differences of invariant imbedding functions. In certain cases these differences can be very small numbers. On the other hand, the invariant imbedding functions can be very large numbers; therefore, if not enough significant figures are carried the solutions could be very much in error.

1.3 Organization of the Study

In Chapter 2, the proposed method of analysis is developed. Section 2.1 gives a general description of the process of reduction of order and its use to determine solutions of two-point boundary value problems when growing solutions are present. Sections 2.2, 2.3 and 2.4 deal with the explanation of how numerical reduction of order of differential equations can be effected. First, a single second-order differential equation is used to illustrate the reduction of order process. The method is then extended to a general differential equation of order n and finally applied to a system of n first-order differential equations.

It is shown in section 2.4 how to use initial value solutions and the reduction of order procedure to eliminate difficulties presented by the growing solutions. The treatment of non-homogeneous cases is also discussed.

Chapter 3 presents an example of a two-point boundary value problem which contains four growing solutions, two in each direction. The reduction of order procedure here is used in the determination of four independent growing solutions. Once their values are found, they are then linearly combined to find the correct solution which satisfies the boundary conditions.

In Chapter 4, the natural frequencies and corresponding mode shapes of spherical shells are determined by the Holzer method (Holzer, 1921) and by a modification of the Newton-Raphson method developed by Robinson and Harris (Robinson and Harris, 1971). The application of both methods involve the solution of two-point boundary value problems. These problems are solved using an initial value procedure and the reduction of order method.

The results of the applications of the method of analysis are presented and discussed in Chapter 5 along with results obtained from other methods. In particular, comparisons are made between the proposed method and Zarghamee's method for determining natural frequencies of spherical shells.

In Chapter 6, general conclusions are drawn from the various applications of the proposed method and possible extensions of the procedure are discussed.

1.4 Nomenclature

Each symbol is explained when it is introduced in the text. The following is presented for the convenience of the reader.

a	Shell radius, Chapter 1
a,b	Boundary conditions, Chapter 1
a_i	Weighting coefficient, Chapter 3
A	Linear differential operator, Chapter 4
A_i	Coefficients in general system of n first-order differential equations, Chapter 3
b_i	Boundary condition
B	Linear differential operator, Chapter 4
C	Constant in differential equations of reduced order
C_i	Coefficients in fourth-order system of differential equations, Chapter 3

D	$\frac{Eh^3}{12\ (1-\nu^2)}$, Chapter 4 and Appendices B and C
D	Matrix of the values of the displacements and stress resultants, Appendix D
E	Modulus of elasticity
E'	$\frac{Eh}{(1-\nu^2)}$
f(x)	Right side of general system of n first-order differential equations
F_n	Shear force on two adjacent shell segments, see Fig. 8
F_t	Sin ϕ F_n d_θ
h	Shell thickness, Chapter 4 and Appendices B and C
h	interval size, Chapter 2

k	$1 + \frac{D}{a^2 E'}$
k_1, k_2	Coefficients in fourth-order differential equations, Chapters 3 and 5
L	Interval length, Chapter 5
L	Differential operator, Section 2.2
L_i	Linear operator, Section 2.3
n	Order of system of differential equations, Chapters 2 and 3
n	Modal wave number, Chapter 4 and Appendices B and C
$M_\phi, M_\theta, M_{\phi\theta}$	Moment stress resultants
$M_{\phi n}, M_{\theta n}, M_{\phi\theta n}$	Moment stress resultants corresponding to the value n, Eqs. (4.1)

$N_\phi, N_\theta, N_{\phi\theta}$	Normal stress resultants
$N_{\phi n}, N_{\theta n}, N_{\phi\theta n}$	Normal stress resultants corresponding to the value n, Eqs. (4.1)
p	Differential operator, Chapter 2
p_1, p_2	Particular solutions to fourth-order differential equations
$p(x)$	Right side of fourth-order differential equations, Chapter 5
q, r, s, t	Dependent variables in the fourth-order system of four first-order differential equations, Chapters 3 and 5
q_i, r_i, s_i, t_i	Homogeneous solution to the fourth-order system of differential equations, Chapter 3
$q_1, r_1, s_1, t_1,$ q_2, r_2, s_2, t_2	First and second homogeneous solutions respectively, to the fourth-order system of differential equations, Chapters 3 and 5.

$q_{p1}, r_{p1}, s_{p1}, t_{p1}$ $q_{p2}, r_{p2}, s_{p2}, t_{p2}$	First and second particular solutions repectively, to the fourth-order system of differential equations, Chapter 3
$q_{T1}, r_{T1}, s_{T1}, t_{T1}$ $q_{T2}, r_{T2}, s_{T2}, t_{T2}$	First and second complete solutions to the fourth-order system of differential equations, Chapters 3 and 5
q_i	Dependent variable in the general system of n first-order differential equations, Chapter 2
q_{ij}	Solution to general system of n first-order differential equations, Chapter 2
q_ϕ, q_θ, q_z	Components of the external pressure on the shell
Q_ϕ, Q_θ	Shear stress resultants
r	a + z
$R^{(i)}$	Residual function in the Robinson-Harris method

s	$\frac{dw}{ad\phi}$
t	Time coordinate, Chapter 4
u, u_i	Dependent variable in the differential equations, Chapter 2
u, v, w	Components of the displacement vector of the middle surface in the meridional, circumferential, and normal directions respectively, Chapter 4 and Apendices B and C
$\overline{u}, \overline{v}, \overline{w}$	Components of the displacement vector of any arbitrary point in the shell, not necessarily on the middle surface
u_n, v_n, w_n	Displacement function of the middle surface corresponding to the value n, Eqs. (4.1)
u_ϕ, v_ϕ, w_ϕ	$\frac{\partial u}{\partial \phi}$, $\frac{\partial v}{\partial \phi}$, $\frac{\partial w}{\partial \phi}$ respectively

v, v_i	Dependent variables in the differential equations of reduced order, Chapters 2 and 3
V_ϕ	$Q_\phi + \frac{n}{a \sin \phi M_{\phi\theta}}$
V_θ	$N_{\phi\theta} - \frac{M_{\phi\theta}}{a}$
w, W	Dependent variables in general systems of nth-order differential equations, Chapters 1, 2, 3 and 5
w_p	Particular solution of the fourth-order system of differential equations, Chapter 5
$X, X^{(i)}$	Eigenfunctions in the Robinson-Harris method, Section 4.5.2 and Appendix D
y	Dependent variable in general system of differential equations, Chapter 2
y_i	Dependent variable in shell differential equations of reduced order, Appendix B

z	Thickness coordinate measured positive outward from the middle surface of the shell
z	ln (sin ϕ), Appendix C
α, β	Boundary conditions, Chapters 2 and 3
α_1, α_2	Weighting coefficients, Chapter 3
β	Constant coefficient in fourth-order differential equation, Chapter 1
δ	When preceeding a variable, it indicates an incremental change in that variable
Δ	Determinate of the coefficients of the linear, homogeneous equations of the boundary conditions at the base and apex of the shell

$\gamma_{\phi r}, \gamma_{\theta r}, \gamma_{\phi\theta}$	Shear strains
$\varepsilon_{\phi}, \varepsilon_{\theta}, \varepsilon_{r}$	Strains
λ	ω^2
ν	Poisson's ratio
ρ	Mass density of shell material, Chapter 4 and Appendices A, B and C
ρ, σ, τ	Dependent variables in differential equations of reduced order, Chapters 3 and 5
$\sigma_{\phi}, \sigma_{\theta}, \sigma_{r}$	Stresses
$\tau_{\phi r}, \tau_{\phi r}, \tau_{\phi\theta}$	Shear stresses
ϕ_o	Opening angle of a spherical shell
ϕ, θ	Geometric coordinates, colatitude and azimuth angle, locating points on the middle surface of the shell

ω	natural frequency (rad./sec.)
ω_o	Reference frequency $= \sqrt{\dfrac{E}{\rho a^2(1-\nu^2)}}$

2. DEVELOPMENT OF THE METHOD OF ANALYSIS

2.1 General

Solutions of differential equations occurring in various types of two-point boundary value problems quite often contain rapidly growing solutions which will dominate in any initial value solution. Consequently, solutions obtained by straight forward numerical integreation of the differential equation, even if using independent initial conditions, will not be independent enough for numerical accuracy. As mentioned in Chapter 1, if the solutions are not linearly independent, they cannot be accurately combined to form a solution which satisfies the boundary conditions.

The method of reduction of order of differential equation uses one growing solution to obtain other solutions, either growing or non-growing, which are truly independent. Thus, the solutions can be linearly combined to form a solution which satisfies the boundary conditions.

In Section 2.2, the well-known process of reduction of order of a differential equation will first be illustrated using a homogeneous second-order differential equation. Then, the method will be applied to an nth-order differential equation. It will be seen that difficulties are encountered when using the straightforward method of reduction of order if the first solution obtained passes through zero. A modification to eliminate this difficulty will also be explained in Section 2.2.

The process of reduction of order is extended to the general case of an nth-order system of differential equations in Section 2.3. This new method is applied to an nth-order system consisting of n simultaneous first-order homogeneous differential equations where the first solution may pass through zero. The process of determining a non-growing particular solution of the general nth-order system will be explained in Section 2.4.

Section 2.5 sets forth the general process of obtaining the solution of a boundary value problem where the corresponding initial value problem has growing solutions. It will be seen that for a system of differential equations of order n, n+2 initial value problems must be solved, n/2 homogeneous ones from each end (if n is even) and a particular solution from each end.

2.2 Method of Reducing the Order of a Differential Equation

Consider a linear ordinary differential equation $L(u) = 0$ of order n, with r independent solutions known. It is well known (Ince, 1956) that by using r known independent solutions, the equation can be reduced in order from n to n-r with only n-r solutions remaining. To clarify this procedure and show the nature of the solutions, a second-order system will be studied first.

Let u_1 be the known solution of the second-order differential equation

$$u'' - u = 0 \tag{2.1}$$

corresponding to the following set of initial conditions:

$$u_1(0) = \alpha, \quad u_1'(0) = 0$$

Now let

$$v_1 = u_1$$

and

$$v_2 = \frac{d}{dx}\left(\frac{u_2}{v_1}\right) \tag{2.2}$$

where u_2 is a second, unknown, solution of equation (2.1) and v_2 is also unknown. Solving (2.2) for u_2 yields

$$u_2 = v_1 \int v_2 \, dx \; ^{*} \tag{2.3}$$

---

* The integral is a definite integral, say $\int_0^x v\,dx$. However, in some cases it is convenient to have a different lower limit Therefore, the definite integral will be written as if it were an indefinite integral.

If we differentiate (2.3), we get

$$u_2' = u_1 v + u_1' \int v\,dx \tag{2.4a}$$

A second differentiation of (2.3) yields:

$$u_2'' = 2u_1' v_2 + u_1 v_2' + u_1'' \int v_2\,dx \tag{2.4b}$$

Substitution of (2.3) and (2.4) into Eq. (2.1) gives:

$$2u_1' v_2 + u_1 v_2' + u_1'' \int v_2\,dx - u_1 \int v_2\,dx = 0$$

or

$$\int v_2\,dx\,(u_1'' - u) + 2u_1' v_2 + u v_2' = 0 \tag{2.5}$$

Since u_1 is a solution of Eq. (2.1), Eq. (2.5) reduces to:

$$2u_1' v_2 + u_1 v_2' = 0 \tag{2.6}$$

Thus, Eq. (2.1) of second order has been reduced to Eq. (2.6) of first order with v_2 as the unknown dependent variable, and u_1 and u_1' as known coefficients. Equation (2.6) can now be integrated to determine v_2. The initial condition for the integration is determined by assuming another, and independent, set of initial conditions for Eq. (2.1). Let the second set of initial conditions for Eq. (2.1) be:

$$u_2(0) = 0, \quad u_2(0) = \beta$$

We see from Eq. (2.4a) that at x = 0

$$u_2'(0) = u_1(0)v_2(0) \qquad \left(\int v_2 dx = 0 \text{ at } x = 0 \right)$$

or

$$v_2(0) = \frac{u_2(0)}{u_1(0)} = \frac{\beta}{\alpha}$$

After v_2 has been determined the other solution, u_2, of Eq. (2.1) is obtained by use of Eq. (2.3).

In general, the nth-order equation is not solvable analytically in terms of familiar functions. If a numerical integration of the differential equation is carried out, then u_1 will automatically include a significant component of the most rapidly growing solution. Thus, when the equation is reduced in order, this rapidly growing solution will no longer be a solution of the equation of lower order.

The general case of an nth-order differential equation L(u) = 0 is treated as follows (Ince, 1956):

Let

$$v_1 = u_1$$

$$v_2 = \frac{d}{dx}\left(\frac{u_2}{v_1}\right)$$

$$v_3 = \frac{d}{dx}\left[\frac{1}{v_2} \cdot \frac{d}{dx}\left(\frac{u_3}{v_1}\right)\right]$$

$$\vdots$$

$$v_r = \frac{d}{dx}\left[\frac{1}{v_{r-1}} \cdot \frac{d}{dx}\left(\frac{1}{v_{r-2}} \cdots \frac{d}{v_2 dx} \cdot \frac{u_r}{v_1}\right)\right]$$

$$\vdots$$

$$v_n = \frac{d}{dx}\left[\frac{1}{v_{n-1}} \cdot \frac{d}{dx}\left(\frac{1}{v_{n-2}} \cdots \frac{d}{v_2 dx} \cdot \frac{u_n}{v_1}\right)\right]$$

$$v_{n+1} = \frac{d}{dx}\left[\frac{1}{v_n} \cdot \frac{d}{dx}\left(\frac{1}{v_{n-1}} \cdots \frac{d}{v_2 dx} \cdot \frac{u}{v_1}\right)\right] \qquad (2.7)$$

where

$$L(u) = 0$$

Since the order of the differential equation is n, only n independent solutions exist. Consequently, u must be some linear combination of the n independent solutions u_1, u_2, ..., u_n. It follows then, that v_{n+1} has to be zero.

Thus,

$$0 = \frac{d}{dx}\left[\frac{1}{v_n}\frac{d}{dx}\left(\frac{1}{v_{n-1}} \cdot \cdot \cdot \cdot \cdot \frac{d}{v_2 dx} \cdot \frac{u}{v_1}\right)\right] \qquad (2.8)$$

If the solutions u_1, u_2, .., u_r, $(r<n)$ are known then, Eq. (2.8) may be written as

$$P(v) = 0 \qquad (2.9)$$

where P is the differential operator of order n-r and

$$v = \frac{d}{dx} \cdot \frac{d}{v_r dx} \cdot \cdot \cdot \cdot \cdot \cdot \frac{d}{v_2 dx} \cdot \frac{u}{v_1} \qquad (2.10)$$

solving Eq. (2.10) for u gives

$$u = v_1 \int \left\{ v_2 \int \left[v_3 \cdot \cdot \cdot \cdot \int \left(v_r \int v dx \right) \cdot \cdot \cdot \cdot \right] dx \right\} dx$$

Difficulties arise in the above procedure if at some point any v_i passes through zero. If this happens, it is obvious that the remaining n-i solutions cannot be calculated at that point by numerical integration without introducing signularities in the remaining v_s', v_{i+1}, v_{i+2}, ..., v_n.

For example, with different initial conditions, at some point where

$$u_1 = v_1 = 0 \quad \text{and} \quad u_2 \neq 0$$

$$\int v_2 dx = \frac{u_2}{v_1}$$

is undefined. Any attempt to calculate u_2 at this point using the above expression would involve numerical difficulties. A new method, therefore, had to be devised for handling this particular situation. A constant c (c>0) can be added to u_1 such that $v_1 = u_1 + c$ is always greater than zero. This will, of course, change the procedure for determining v_2, but will not change the final complete solution of Eq. (2.1). Going back to the previous second order example we have:

$$u'' - u = 0 \quad \text{and} \quad v_1 = u_1 \quad \text{as before.}$$

However, now

$$u_2 = \left(u_1 + c\right) \int \bar{v}_2 dx \tag{2.11}$$

$$u_2' = \left(u_1 + c\right) \bar{v}_2 + u_1' \int \bar{v}_2 dx$$

and

$$u_2'' = u_1'' \int \bar{v}_2 dx + 2u_1' \bar{v}_2 + \left(u_1 + c\right) \bar{v}_2' \tag{2.12}$$

where $\bar{v}_2$ is different from v_2.

Substituting Eqs. (2.11) and (2.12) into Eq. (2.1) and simplifying, we get

$$2u_1'\bar{v}_2 + \left(u_1 + c\right)\bar{v}_2' + c\int \bar{v}_2 dx = 0 \qquad (2.13)$$

Thus, again we have reduced Eq. (2.1) to a first order differential equation. Here we see that if $c = 0$, Eq. (2.13) is the same as Eq. (2.6). With the occurrence of the term $c\int v_1 dx$ in Eq. (2.13) one would tend to think this equation is of second order. However, because $\int v_1 dx$ is a definite integral, Eq. (2.13) is of first order; that is, it has a one parameter family of solutions.

The initial conditions for Eq. (2.13) are dictated by the initial conditions for Eq. (2.1) as in the earlier example. The value of $\int v dx$ at the initial boundary is zero as before. The value of $v_2(0)$ is determined by substituting into Eq. (2.13) the independent sets of initial conditions for Eq. (2.1), and in this case is

$$v_2(0) = \frac{\beta}{(\alpha+c)} \qquad (2.14)$$

This is the only initial condition possible for Eq. (2.13) given the two sets of initial conditions for Eq. (2.1).

2.3 Reduction of Order of General Systems of Homogeneous, Ordinary, First-Order Differential Equations

A system of first order differential equations is generally easier to handle than one differential equation of higher order (Goldberg, 1967). The conversion of a differential equation of higher order to a system of first order differential equations merely involves the replacement of the dependent variable and its derivatives by new variables.

Let

$$y^{(n)} + A_1 y^{(n-1)} + \ldots + A_{n-2} y'' + A_{n-1} y' + A_n y = 0 \qquad (2.15)$$

be an nth-order differential equation with the following boundary conditions:*

$$\begin{array}{llll} y(0) = \alpha_1 & & y^{(n-2)}(L) = \alpha_{n/2+1} & \\ y'(0) = \alpha_2 & & y^{(n/2+1)}(L) = \alpha_{n/2+2} & \\ \vdots & & \vdots & \\ y^{((n-2)/2)}(0) = \alpha_{n/2} & & y^{(n-1)}(L) = \alpha_n & (2.16) \end{array}$$

---

* Here, n is assummed to be even, because most engineering problems involve differential equations of even order. However, n could be odd in which case (n-1)/2 conditions would be specified at one boundary and (n+1)/2 would be specified at the other.

Define new variables as follows:

$$u_1 = y$$

$$u_2 = u_1' = y'$$

$$u_3 = u_2' = y''$$

$$\vdots$$

$$u_n = u_{n-1}' = y^{(n-1)}$$

$$u_n' = y^{(n)} \qquad (2.17)$$

If Eqs. (2.17) are substituted into Eq. (2.15), we get one first order differential equation.

$$u_n' + A_1 u_{n-1} + \ldots + A_{n-1} u_2 + A_n u_1 = 0$$

The other n-1 first order equations are taken from Eqs.(2.17):

$$u_1' - u_2 = 0$$

$$u_2' - u_3 = 0$$

$$\vdots$$

$$u_i' - u_{i-1} = 0$$

$$\vdots$$

$$u_{n-1}' - u_n = 0 \qquad (2.18)$$

The boundary conditions for the first-order system of equations would be the same as the boundary conditions for the nth-order single differential equation: that is:

$$u_1(0) = y(0) = \alpha_1 \qquad u_{n/2+1}(L) = y^{(n/2)}(L) = \alpha_{n/2+1}$$

$$u_2(0) = y'(0) = \alpha_2 \qquad \cdots\cdots$$

$$\cdots\cdots \qquad \cdots\cdots$$

$$u_{n/2}(0) = y^{(n/2-1)}(0) = \alpha_{n/2} \qquad u_n(L) = y^{(n-1)}(L) = \alpha_n \tag{2.19}$$

The general system of linear ordinary first order differential equations should be written in the following specified form, solved for the derivatives, in order that numerical integration techniques be readily applicable:

$$q_1' + L_1(q_1, q_2, \ldots, q_n) = 0$$

$$q_2' + L_2(q_1, q_2, \ldots, q_n) = 0$$

$$\cdots\cdots$$

$$q_n' + L_n(q_1, q_2, \ldots, q_n) = 0 \tag{2.20}$$

where $L_1, L_2, \ldots, L_n$ are linear in the q's.

The boundary conditions are:

$$
\begin{aligned}
q_1(0) &= \beta_1 & q_{n/2+1}(L) &= \beta_{n/2+1} \\
q_2(0) &= \beta_2 & q_{n/2+2}(L) &= \beta_{n/2+2} \\
&\vdots & &\vdots \\
q_{n/2}(0) &= \beta_{n/2} & q_n(L) &= \beta_n
\end{aligned}
\qquad (2.21)
$$

Assume that $q_{11}, q_{21}, \ldots, q_{n1}$ is a growing solution determined by straight forward numerical integration of Eq. (2.19) starting with the following set of assumed initial conditions:

$$
\begin{aligned}
q_{11}(0) &= \alpha_1 \\
q_{21}(0) &= \alpha_2 \\
&\vdots \\
q_{n1}(0) &= \alpha_n
\end{aligned}
\qquad (2.22)
$$

Let

$$q_{12} = (q_{11} + c) \int v_1 dx \qquad (q_{11}(x) + c) > 0$$

$$q_{22} = q_{21} \int v_1 dx + v_2 \qquad \text{and } (c > 0)$$

$$q_{32} = q_{31} \int v_1 dx + V_3 \qquad o \leq x \leq = L$$

$$\vdots$$

$$q_{n2} = q_{n1} \int v_1 dx + v_n \qquad (2.23)$$

where $q_{12}, q_{22}, \ldots, q_{n2}$ is a second solution to be determined; that is,

$$q_{12}' + L_1(q_{12}, q_{22}, \ldots, q_{n2}) = 0$$

$$q_{22}' + L_2 (q_{12}, q_{22}, \ldots, q_{n2}) = 0$$

$$\cdots\cdots$$

$$q_{n2}' + L_n (q_{12}, q_{22}, \ldots, q_{n2}) = 0 \qquad (2.24)$$

Substitution of Eqs. (2.23) and their derivatives into Eqs. (2.24) yields

$$(q_{11} + c)v_1 + q_{11}'\int v_1 dx + L_1 \Big[(q_{11} + c)\int v_1 dx,$$
$$q_{21}\int v_1 dx + v_2, \ldots\ldots, q_{n1}\int v_1 dx + v_n\Big] = 0$$

$$q_{21}v_1 + q_{21}'\int v_1 dx + v_2' + L_2 \Big[(q_{11} + c)\int v_1 dx,$$
$$q_{21}\int v_1 dx + v_2, \ldots\ldots, q_{n1}\int v_1 dx + v_n\Big] = 0$$

$$\ldots\ldots\ldots$$

$$q_{n1}v_1 + q_{n1}'\int v_1 dx + v_n' + L_n \Big[(q_{11} + c)\int v_1 dx,$$
$$q_{21}\int v_1 dx + v_2, \ldots\ldots, q_{n1}\int v_1 dx + v_n\Big] = 0$$

Rearranging terms, we get

$$(q_{11} + c)v_1 + \Big[q_{11}' + L_1(q_{11}, q_{21}, \ldots\ldots, q_{n1})\Big]\int v_1 dx$$
$$+ L_1\Big[c \ \ v_1 dx, v_2, v_3, \ldots\ldots, v_n\Big] = 0$$

$$q_{21}v_1 + v_2' + \Big[q_{21}' + L_2(q_{11}, q_{21}, \ldots\ldots, q_{n1})\Big]\int v_1 dx$$
$$+ L_2\Big[c \ \ v_1 dx, v_2, v_3, \ldots\ldots, v_n\Big] = 0$$

$$\ldots\ldots\ldots$$

$$q_{n1}v_1 + v_n' + \Big[q_{n1}' + L_n(q_{11}, q_{21}, \ldots\ldots, q_{n1})\Big]\int v_1 dx$$
$$+ L_n\Big[c \ \ v_1 dx, v_2, v_3, \ldots\ldots, v_n\Big] = 0$$

However,

$$(q_{11} + c)v_1 + L_1\left[c\int v_1 dx,\ v_2,\ v_3,\ \ldots,\ v_n\right] = 0$$

$$v_2' + q_{21}v_1 + L_2\left[c\int v_1 dx,\ v_2,\ v_3,\ \ldots,\ v_n\right] = 0$$

$$\cdots\cdots$$

$$v_n' + q_{n1}v_1 + L_n\left[c\int v_1 dx,\ v_2,\ v_3,\ \ldots,\ v_n\right] = 0$$

Therefore,

$$(q_{11} + c)v_1 + L_1\left[c\int v_1 dx,\ v_2,\ v_3,\ \ldots,\ v_n\right] = 0$$

$$v_2' + q_{21}v_1 + L_2\left[c\int v_1 dx,\ v_2,\ v_3,\ \ldots,\ v_n\right] = 0$$

$$\cdots\cdots$$

$$v_n' + q_{n1}v_1 + L_n\left[c\int v_1 dx,\ v_2,\ v_3,\ \ldots,\ v_n\right] = 0 \qquad (2.25)$$

Hence, the system of differential Eqs. (2.20) of order n has been reduced to the system of differential Eqs. (2.25) or order n-1. The initial conditions for the equations of reduced order, as previously mentioned in Chapter 1, are dictated by the chosen independent sets of initial conditions for Eqs. (2.20). Equations (2.25) can now be numerically integrated to find the variables $v_1, \ldots, v_n$. Once their values are obtained another solution of the system (2.20) can be determined by use of Eqs. (2.23)*.

---

*Further explanation of the numerical procedure used to calculate the second solution will be given in Chapter 3.

If system (2.20) contains only one rapidly growing solution, it will be $q_{11}, q_{21}, \ldots, q_{n1}$. Other solutions of (2.20) will not grow rapidly. On the other hand, if the system (2.20) contains more than one rapidly growing solution, then the second solution $q_{12}, q_{22}, \ldots, q_{n2}$ will be a growing solution also, but independent of the first and more slowly growing than the first. This second solution can be used to reduce the order of Eq. (2.25) and a third solution of Eq. (2.20) can be determined. The process of reducing the order should continue until all growing solutions have been obtained.

2.4 Procedure for Solving nth-Order Systems Using Reduction of Order

In the second-order case, independent initial value solutions for the homogeneous equation can be determined without reduction of order. Numerical integration of the homogeneous second-order differential equation will yield independent solutions, if one solution is obtained by integrating from 0 to L and the other from L to 0. An independent slowly varying particular solution cannot, however, be obtained without reducing the order of the equation. In general, for a system of differential equations of order n containing rapidly growing solutions, a reduction of order is necessary to obtain independent homogeneous solutions as well as an independent particular solution.

A system of differential equations of order n, where n is even, could possibly have n growing solutions. Commonly, n/2 solutions grow from 0 to L and n/2 solutions grow from L to 0. Unless it is known in advance how many growing solutions are present, it is necessary to keep reducing the order of the system of differential equations to find each new solution.

The procedure for determining the initial value solutions for a general system of linear ordinary differential equations of even order n containing n growing solutions is as follows:

1. Integrate the original nth-order system of differential equations from 0 to L to obtain the fastest growing solution from 0 to L, u_1.

2. Use u_1 to reduce the order of the equation to n-1 and integrate the reduced order equations from 0 to L.

3. Use the solution of the n-1 order system of equations, v_1, and the first solution of the nth order differential equation, u_1, to determine another solution, u_2, of the original nth order system as previously described.

4. Test the second solution to the nth order system to see if it grows rapidly. If it does not grow rapidly, then no further reductions of order are necessary to determine the remaining independent solutions. If it does grow rapidly, then another reduction of order is necessary to obtain another independent solution of the nth-order system.

5. Assuming further reductions of order are necessary, use solution v_1, already obtained, to reduce the n-1 order system to an n-2 order system. Solve this system and use its solution, v_2, along with solutions u_1 and v_1 to obtain solution number 3, u_3, of the nth order system.

6. In general, the n-th solution, u_{n-i}, of the original nth-order system is determined using solution u_1 of the nth-order system and solutions $v_1, \ldots, v_{n+1-i}$ of the systems of reduced order.

7. If, at some point, a solution is obtained that does not grow rapidly, further reductions are not necessary to determine the remaining initial value solutions from 0 to L. The remaining solutions are then obtained by

integrating the latest reduced order equation from 0 to L a number of times starting with independent sets of initial conditions.

The initial value solutions which grow from L to 0 are obtained in the same manner as the solutions which grow from 0 to L, however, the direction of integration is reversed.

2.5 Non-growing Particular Solutions

Up until now we have only discussed the numerical determination of independent homogeneous solutions. Many two-point boundary value problems involve differential equations which are non-homogeneous. It then becomes necessary to determine an independent particular solution as well as a full set of independent homogeneous solutions.

A particular solution can be obtained using the same procedure as for determining solutions $q_{i2}, \ldots.,$ $q_{in/2}$. However, in this case the reduced order equations and the original first order system of differential equations will, of course, be non-homogeneous. Equations (2.20) become

$$q_1' + L_1(q_1, q_2, \ldots., q_n) = f_1(x)$$

$$\vdots$$

$$q_1' + L_n(q_1, q_2, \ldots., q_n) = f_n(x) \qquad (2.26)$$

If the right side of Eq. (2.26) is carried along in the reduction process, we find

$$(q_{11} + c)v_1 + L_1(c\int v_1, v_2, \ldots, v_n) = f_1(x)$$

$$v_2' + q_{21}v_1 + L_2(c\int v_1, v_2, \ldots, v_n) = f_2(x)$$

$$\cdots\cdots$$

$$v_n' + q_{n1}v_1 + L_n(c\int v_1, v_2, \ldots, v_n) = f_n(x) \qquad (2.27)$$

where $q_{11}, \ldots, q_{n1}$ is the original homogeneous initial value solution to Eq. (2.20). From this point on, the solution is computed exactly the same way as the other solutions but using, for convenience, homogeneous initial conditions.

The reductions are done i times until the reduced homogeneous equation of order $n-i$ does not contain a growing solution. This n-ith order equation will be used along with the homogeneous solutions to determine a non-growing particular solution, u_{p1}. To assure accuracy of the total solution to the nth-order equation throughout the interval 0 to L, it is convenient to find a second particular solution starting from L and integrating to zero. This particular solution will involve the homogeneous solutions which were found by integrating from L to 0. The two particular solutions are used, along with the homogeneous solutions, to determine two solutions to the original nth-order differential equation. The particular solution which was found by integrating from 0 to L, when combined with the homogeneous solutions, will yield a solution which is most accurate in the first half of the interval. The second solution which contains a particular solution found by integrating from L to 0, will be most accurate in the second half of the interval. The two solutions will match near the middle of the interval. Thus, the total solution to the nth-order system will be a combination of the first half of one solution and the second half of the other solution.

2.6 Numerical Integration Procedure

The specific numerical integration method used was the trapezoidal rule in a predictor-corrector type procedure. A procedure of this form was chosen over other methods such as Runge-Kutta, because it is easy to handle for certain cases and permits the changing of the interval size during the integration. Variation of the interval size is done to correspond to changes in the gradient of the function. The definite integrals in equations of the form of system (2.23) were determined by quadratures using Simpson's Rule.

2.6.1 Method of Integration

There are many techniques which can be used for numerical integration such as Runge-Kutta and various predictor-corrector methods. For the applications given herein, a predictor-corrector method will be used. The algorithm for the method of integration is as follows:

1. The assumed and given initial conditions are substituted into the differential equations at point 1 (x=0) to calculate the value of the first derivative.

2. The value of the first derivative at point 1 becomes the first approximation of the first derivative at point 2. The value of the variable will be determined by numerical integration using the trapezoidal rule; that is,

$q_i = q_{i-1} + 1/2\ h(q_1' + q_{i-1}')$.

3. Values of the variable at point 2 will be substituted back into the differential equations to calculate new values for the first derivatives. These new values will be compared to the old values. If the two values are within a certain tolerance, we proceed to point 3. If they are not, then the new value of the first derivatives will be used in the trapezoidal expression to compute another value of the variable at point 2.

4. The procedure is repeated until a certain tolerance has been obtained between the old and new estimates of the first derivatives.

5. If it takes more than six iterations for the convergence, the size of the interval is decreased and the procedure is repeated at the same point. If the convergence takes less than three interations, the size of the interval is increased for the next interval

2.6.2 Errors

The error in the numerical integration procedure using the trapezoidal rule, in part, depends on how close the actual function is to a straight line in the interval. That is, the error depends on the magnitude of the curvature of the function in the interval. The difference between the approximation and the true solution is called the discretization error. For one interval the error is

$$h/2(x_s + x_{s+1}) - \int_{t_s}^{t_{s+1}} x(t)\,dx = \frac{h^3 d^2 x(t)}{12\ dt^2}$$

However,

$$\Delta t = h = T/s$$

Therefore, over the interval $t_o - t_s$ the error is:

$$h(x_{o/h} + x_1 + x_2 + \ldots\ldots + x_{s-1} + x_{s/2}) - \int_{\tau_o}^{\tau_s} x(t)\,dt = \frac{h^2 T d^2 x(\tau)}{12\ dt^2}$$

where $t_o \leq \tau \leq t_s$. Because the error of one interval is of the order h^3, the error over the total range is of the order h^2 since the number of intervals equals T/h (Thomas 1961).

The error in Simpon's rule can be expressed as

$$h/3(x_{s-1} + 4_s + x_{s+1} - \int_{t_{s-1}}^{t_{s+1}} (x(t))dx = \frac{h^5 d^4 x(t)}{90\ dt^4}$$

Over an extended range T the error would be

$$h/3(x_o + 4x_1 + 2x_2 + 4x_3 + 2x_4 + \ldots . + 2x_{2s-2}$$

$$+ 4x_{2s-1} + x_{2s}) - \int_{\tau_o}^{\tau_{25}} x(t)dx = \frac{Th^4 d^4 x(\tau)}{180\ dt^4}$$

$$(\Delta t = h = T/2s \text{ and } t_o \leq \tau \leq t_{2s})$$

Here the error over one interval is of the order h^5 and the error over the entire range is of the order h^4.

In addition to the discretization error described above, round-off error is also present. Round-off error will increase as the number of intervals increase. The interval size selector is used in the procedure to save execution time and space by keeping the interval size at a maximum length and the number of intervals to a minimum.

3. AN EXAMPLE OF MULTIPLE EDGE EFFECTS

3.1 General

In this chapter the method of reducing the order of a differential equation will be applied to a system of differential equations containing four growing solutions, two in each direction. A fourth-order example is given to test the ability of the procedure to separate growing solutions. The fourth-order equation will first be rewritten as a system of four first-order equations, as in Chapter 2, before the procedure begins.

3.2 Fourth-Order System With Two Growing Solutions in Each Direction

It is easy to write down a single fourth-order system with two growing solutions from each edge. For instance, a general solution of the homogeneous equation having the desired form could be taken as

$$w = C_1 e^{k_1 x} + C_2 e^{k_2 x} + C_3 e^{-k_1 x} + C_4 e^{-k_2 x} \tag{3.1}$$

It is obvious that $e^{k_1 x}$ and $e^{k_2 x}$ will grow from 0 to L and $e^{-k_1 x}$ and $e^{-k_2 x}$ will grow from L to 0. The fourth-order differential equation which has Eq. (3.1) as the homogeneous part of its solution is

$$\left[\frac{d^2}{dx^2} - k_1^{\,2}\right]\left[\frac{d^2}{dx^2} - k_2^{\,2}\right] w = f(x)$$

where

$$k_1 > k_2 \quad \text{and} \quad k_2L >> 1$$

If $k_1 > k_2$ then k_1 represents the more rapdily growing edge effect. Also, $k_2L >> 1$ in order that the edge effects be pronounced. Expanding the above equation yields

$$\frac{d^4}{dx^4} w - (k_1^2 + k_2^2) \frac{d^2}{dx^2} w + k_1^2 k_2^2 w = f(x)$$

or

$$w^{iv} - (k_1^2 + k_2^2) w'' + k_1^2 k_2^2 w = f(x) \qquad (3.2)$$

The fourth-order equation (3.2) can be re-written as a system of four first-order equations by substituting new variables for the dependent variable and its derivatives.

Let

$$q = w$$

$$r = q' = w'$$

$$s = r' = w''$$

$$t = s' = w'''$$

$$t' = w^{iv} \qquad (3.3)$$

Substitute Eq. (3.3) into Eq. (3.2) to find one of the four first-order equations

$$t' - (k_1^2 + k_2^2)s + k_1^2 k_2^2 q = f(x) \qquad (3.4a)$$

The other three first-order equations are

$$q' - r = 0 \qquad (3.4b)$$

$$r' - s = 0 \qquad (3.4c)$$

$$s' - t = 0 \qquad (3.4d)$$

3.3 Reduction of Order

When the system of four first-order differential equations, Eqs. (3.4), is numerically integrated from 0 to L the fastest growing solution in this direction will dominate when x is at all large. Let this solution be q_1, r_1, s_1, t_1. Another solution can be defined by

$$q^2 = (q_1 + c)\int v dx \qquad c > 0^*$$

$$r^2 = r_1 \int v dx + \rho$$

---

* See Eq. (2.9)

$$s_2 = s_1 \int v dx + \sigma$$

$$t_2 = t_1 \int v dx + \tau \qquad (3.5)$$

(ρ, σ and τ are functions of x)

where

$$(q_1 + c) > 0$$

Substituting Eq. (3.5) into Eq. (3.4) and simplifying we find the following thrid-order system of differential equations:

$$p' - (k_1^2 + k_2^2) u + k_1^2 k_2^2 c \int v dx + tv = f(x)$$

$$\sigma' - \tau + sv = 0$$

$$\rho' - \sigma + rv = 0$$

$$v(q + c) - \rho = 0 \qquad (3.6)$$

These equations in turn can be solved for ρ, σ, τ, and v. The second solution q_2, r_2, s_2, t_2 described by Eq. (3.5) will be forced to be independent of the first solution and more slowly growing.

In the numerical integration of the original nth-order differential equation, it is found that the solution q_1, r_1, s_1, t_1 contains significant portions of all the growing and non-growing homogeneous solutions when x is small. However, when x is large the most rapidly growing solution that might be contained in q_2, r_2, s_2, t_2, is forced to equal zero at 1.5L, that is:

$$q_2(1.5L) = 0 = (q_1(1.5L) + c) \int_0^{1.5L} vdx + \alpha_1 q_1(1.5L)$$

Thus, a second, independent, solution is computed as follows:

$$q_2 = (q_1 + c) \int vdx + \alpha_1 q_1$$

$$r_2 = r_1\left(\int vdx + \alpha_1\right) + \rho$$

$$s_2 = s_1\left(\int vdx + \alpha_1\right) + \sigma$$

$$t_2 = t_1\left(\int vdx + \alpha_1\right) + \tau \qquad (3.7)$$

By using the above procedure, but integrating from L to 0, the fastest growing solution from L to 0, q_3, r_3, s_3, t_3, and the slower growing solution from L to 0, q_4, r_4, s_4, t_4, can be computed.

The reason that this procedure works can be understood readily if it is remembered that the desired second solution grows much more slowly than the first. By eliminating the first solution well beyond $x = L$, the proportion of the first solution present in the second is reduced almost to zero over the entire range 0 to L. Experience indicates that the combination at 1.5L works effectively without increasing computational effort greatly.

After all four independent homogeneous solutions are found, the particular solution is calculated using the non-homogeneous third-order system (3.6). To guarantee accuracy of the total solution at both edges, two particular solutions are determined and used to calculate two solutions to the nth-order system.

One particular solution is obtained by first integrating Eqs. (3.6) from 0 to L and replacing q, r, s, t with q_1, r_1, s_1, t_1 to determine ρ_2, σ_2, τ_2, v_2. Then, by using the following system, q_{p_1}, r_{p_1}, s_{p_1}, t_{p_1} is determined:

$$q_{p_1} = (q_1 + c)\int v_2 dx + \alpha_2 q_1 + \alpha_3 q_2$$

$$r_{p_1} = r_1\left(\int v_2 dx + \alpha_2\right) + \rho_2 + \alpha_3 r_2$$

$$s_{p_1} = s_1\left(\int v_2 dx + \alpha_2\right) + \sigma_2 + \alpha_3 s_2$$

$$t_{p_1} = t_1\left(\int v_2 dx + \alpha_2\right) + \tau_2 + \alpha_3 t_2 \qquad (3.8)$$

where α_2 and α_3 are determined from the following two equations:

$$(q_1(L) + c)\int_0^L v_2 dx + \alpha_2\, q_1(L) + \alpha_3 q_2(L) = 0$$

$$r_1(L)\left(\int_0^L v_2 dx + \alpha_2\right) + \rho_2(L) + \alpha_3 r_2(L) = 0 \qquad (3.9)$$

The use of Eqs. (3.9) guarantees that all rapidly growing solutions are eliminated from q_{p_1}, r_{p_1}, s_{p_1}, t_{p_1}.

A second particular solution is obtained using the above procedure, but reversing the direction of integration and replacing q, r, s, t in Eqs. (3.6) with q_3, r_3, s_3, t_3. The second particular solution then is

$$q_{p_2} = (q_3 + c)\int v_3 dx + \alpha_4 q_3 + \alpha_5 q_4$$

$$r_{p_2} = r_3\left(\int v_3 dx + \alpha_4\right) + \rho_3 + \alpha_5 r_4$$

$$s_{p_2} = s_3\left(\int v_3 dx + \alpha_4\right) + \sigma_3 + \alpha_5 s_4$$

$$t_{p_2} = t_3\left(\int v_3 dx + \alpha_4\right) + \tau_3 + \alpha_5 t_4 \qquad (3.10)$$

where α_4 and α_5 are determined as follows:

$$(q_3(0) + c)\int_L^0 v_3 dx + \alpha_4 q_3(0) + \alpha_5 q_4(0) = 0$$

$$r_3(0)\left(\int_L^0 v_3 dx + \alpha_4\right) + \rho_3(0) + \alpha_5 r_4(0) = 0$$

The particular solution q_{p_2}, r_{p_2}, s_{p_2}, t_{p_2} will be most accurate when x is large.

After all six solutions have been found, the four homogeneous solutions are linearly combined and added to each particular solution to satisfy the four boundary conditions. Assume that the boundary conditions are

$$q(0) = b_1 \qquad s(0) = b_3$$

$$r(L) = b_2 \qquad t(L) = b_4$$

and the six solutions are

$$q_i, r_i, s_i, t_i$$

where $i = 1, 2, 3, 4$ are homogeneous solutions and $i = 5, 6$ are the particular solutions.

At the boundaries the linear combination is the solution and must, therefore, satisfy the boundary conditions. Thus, For Solution 1 For Solution 2

$$\sum_{i=1}^{4} q_i(0)a_i + q_5(0) = b_1 \qquad \sum_{i=1}^{4} q_i(0)\bar{a}_i + q_6(0) = b$$

$$\sum_{i=1}^{4} r_i(L)a_i + r_5(L) = b_2 \qquad \sum_{i=1}^{4} r_i(L)\bar{a}_i + r_6(L) = b_2$$

$$\sum_{i=1}^{4} s_i(0)a_i + s_5(0) = b_3 \qquad \sum_{i=1}^{4} s_i(0)\bar{a}_i + s_6(0) = b_3$$

$$\sum_{i=1}^{4} t_i(L)a_i + t_5(L) = b_4 \qquad \sum_{i=1}^{4} t_i(L)\bar{a}_i + t_6(L) = b_4 \qquad (3.11)$$

where a and $\bar{a}$ are the weighting coefficients. Equations (3.11) are solved for the unknown a_i and $\bar{a}_i$. The total solution is then found by linearly combining all the solutions with their respective weighting co-efficients as follows:

Solution 1 Solution 2

$$q_{T_1} = \sum_{i=1}^{4} q_i a_i + q_5 \qquad q_{T_2} = \sum_{i=1}^{4} q_i \bar{a}_i + q_6$$

$$r_{T_1} = \sum_{i=1}^{4} r_i a_i + r_5 \qquad r_{T_2} = \sum_{i=1}^{4} r_i \bar{a}_i + r_6$$

$$s_{T_1} = \sum_{i=1}^{4} s_i a_i + s_5 \qquad s_{T_2} = \sum_{i=1}^{4} s_i \bar{a}_i + s_6$$

$$t_{T_1} = \sum_{i=1}^{4} t_i a_i + t_5 \qquad t_{T_2} = \sum_{i=1}^{4} t_i \bar{a}_i + t_6$$

where q_{T_1}, r_{T_1}, s_{T_1}, t_{T_1} and q_{T_2}, r_{T_2}, s_{T_2}, t_{T_2} are two total solutions. The first solution will be most accurate when x is small and the other solution will be most accurate when x is large. The two solutions will be identical near the middle of the interval. Thus, the total solution to the nth-order system will consist of q_{T_1}, r_{T_1}, s_{T_1}, t_{T_1} for small x and q_{T_2}, r_{T_2}, s_{T_2}, t_{T_2} for large x.

Numerical results for fourth-order example and other applications will be given in Chapter 5.

4. FREE VIBRATION OF THIN SPHERICAL SHELLS

4.1 Introduction

The shells in this study will be limited to linearly elastic spherical shell segments. Love's first approximation (Love, 1881) will be used. This approximation is based on the following assumptions:

(1) The shell is thin ($h/a \leq 1/20$).

(2) The deflections are small.

(3) A condition of approximately plane stress exist.

(4) The normal to the middle surface remains normal, straight and short enough that its extension can be taken as zero during deformation.

A system of eight first-order differential equations of motion for a spherical shell segement will be put into a usable first-order form. The equations will be determined from the following fundamental equations of a shell found in the appendix:

(1) The equations of motion obtained from the balancing of forces acting on some fundamental element.

(2) Strain-displacement relations obtained from purely geometric considerations.

(3) The relationship between stress and strain (Hooke's Law).

4.2 Fundamental Equations of a Shell

The equilibrium equations (A.1) and (A.2) in the appendix can be converted to a system of eight first-order differential equations. For ease of calculation, we have assumed that the motion of the shell is symmetric motion, that is, it does not include torsion. The displacements and stress resultants in the hemispherical shell are either symmetric or anti-symmetric about a meridian from which the azimuth angle θ is measured.

Fourier series will be used to change the partial differential equations to ordinary differential equations. The general expression of a function, say f(x), in a Fourier series is

$$f(x) = \frac{a_o}{2} + a_1 \cos nx + a_2 \cos 2nx + \ldots.$$
$$+ b_1 \sin nx + b_2 \sin 2nx + \ldots.$$

The cosine terms of the Fourier series are symmetrical and the sine terms of the Fourier series are anti-symmetrical about zero. Hence, the displacements and stress resultants can be expressed in terms of either sine or cosine functions of the Fourier series; that is,

$$u = \sum_{n=0}^{\infty} u_n (\phi, t) \cos n\theta \qquad Q_\phi = \sum_{n=0}^{\infty} Q_{\phi n} (\phi, t) \cos n\theta$$

$$v = \sum_{n=1}^{\infty} v_n (\phi, t) \sin n\theta \qquad Q_\theta = \sum_{n=1}^{\infty} Q_{\theta n} (\phi, t) \sin n\theta$$

$$w = \sum_{n=0}^{\infty} w_n (\phi, t) \cos n\theta \qquad M_\phi = \sum_{n=0}^{\infty} M_{\phi n} (\phi, t) \cos n\theta$$

$$N_\phi = \sum_{n=0}^{\infty} N_{\phi n} (\phi, t) \cos n\theta \qquad M_\theta = \sum_{n=0}^{\infty} M_{\theta n} (\phi, t) \cos n\theta$$

$$N_\theta = \sum_{n=1}^{\infty} N_{\theta n} (\phi, t) \cos n\theta \qquad M_{\phi\theta} = M_{\theta\phi} = \sum_{n=1}^{\infty} M_{\phi\theta n} \sin n\theta$$

$$N_{\phi\theta} = N_{\theta\phi} = \sum_{n=1}^{\infty} N_{\phi\theta n} (\phi, t) \sin n\theta \tag{4.1}$$

The displacements u and w are even functions; therefore, the Fourier series for them only involves the even functions $\cos n\theta$. Conversely, the displacement v is an odd function and, therefore, its Fourier series only involves $\sin n\theta$. Likewise, the stress resultants are either even or odd functions; thus, their Fourier series only involve either $\sin n\theta$ or $\cos n\theta$.

Substitution of the Fourier series expressions for the displacements into Equations A.6 yields the following:

$$N_\phi = \frac{Eh}{a(1-\nu^2)} [w_n + u_{n,\phi} + \nu (w_n + \frac{n v_n}{\sin\phi} + u_n \cot\phi)]$$

$$N_\theta = \frac{Eh}{a(1-\nu^2)} [w_n + \frac{n v_n}{\sin\phi} + \cot\phi \, u_n + (w_n + u_{n,\phi})]$$

$$N_{\phi\theta} = \frac{Eh}{2a(1+\nu)} \left[\frac{-nv_n}{\sin\phi} + v_{n,\phi} - v_n \cot\phi \right]$$

$$M_{\phi} = \frac{Eh^3}{12a^2(1-\nu^2)} \left[w_{n,\phi\phi} - u_{n,\phi} + \nu \left(w_{n,\phi} \cot\phi - \frac{n^2 w_n}{\sin^2\phi} - \frac{nv_n}{\sin\phi} - u_n \cot\phi \right) \right]$$

$$M_{\theta} = \frac{Eh^3}{12a^2(1-\nu^2)} \left[-u_n \cot\phi - \frac{nv_n}{\sin\phi} + w_{n,\phi} \cot\phi - \frac{n^2 w_{n,\phi}}{\sin^2\phi} + \nu \left(-u_{n,\phi} + w_{n,\phi\phi} \right) \right]$$

$$M_{\phi\theta} = \frac{Eh^3}{24a^2(1+\nu)} \left[\frac{nu_n}{\sin\phi} - v_{n,\phi} + v_n \cot\phi - \frac{2nw_{n,\phi}}{\sin\phi} + \frac{2nw_n \cos\phi}{\sin^2\phi} \right] \tag{4.2}$$

Using the above equations and the equilibrium equations, we can derive a system of eight first-order differential equations of motion of the shell.

Let

$$s = \frac{dw}{a\, d\phi}^{*}$$

or

$$\frac{dw}{d\phi} = sa \tag{4.3a}$$

---

* For ease in writing, the subscript n will be dropped.

This becomes the first of the eight first-order differential equations. Through a series of substitutions involving the stress resultant equations, the equilibrium equations, derived in Appendix A, and the Fourier series expressions, the other seven first-order equations are determined. Substitution of Equation (4.3a) into (4.2d) yields:

$$M_\phi = \frac{D}{a^2}\left[\frac{ds}{d\phi}\,a - u_\phi + \nu\left(s\;a\cot\phi - \frac{n^2 w}{\sin^2\phi} - \frac{nv}{\sin\phi} - u\cot\phi\right)\right]$$

Solving the above equation for u_ϕ gives

$$u_\phi = a\,\frac{ds}{d\phi} - \frac{a^2 M_\phi}{D} + \nu\left(s\;a\cot\phi - \frac{n^2 w}{\sin^2\phi} - \frac{nv}{\sin\phi} - u\cot\phi\right)$$

Substitution of this expression into 4.2a gives

$$N_\phi = \frac{E'}{a}\left[w + a\,\frac{ds}{d\phi} - a^2\,\frac{M\phi}{D} + \nu\left(s\;a\cot\phi - \frac{n^2 w}{\sin^2\phi}\right)\right]$$

Solving for $a\,\frac{ds}{d\phi}$ yields

$$\frac{ds}{a\,d\phi} = \left(\frac{\nu n^2}{a^2\sin^2\phi} - \frac{(1+\nu)}{a^2}\right)w - \left(\frac{\nu\cot\phi}{a}\right)s + \frac{N_\phi}{a\,E'} + \frac{1}{D}\,M_\phi$$

which is the second equation in the eighth-order system. Similarly, the remaining six first-order differential equations can be found. They are as follows:

$$a \frac{du}{d\phi} = - \frac{\nu \cot \phi}{a} u - \frac{\nu n}{a \sin \phi} v - \frac{(1+\nu)}{a} w + \frac{1}{E'} N_\phi$$

$$a \frac{dv}{d} = \frac{n}{a \sin \phi} u + \frac{\cot \phi}{a} v + \frac{2nD}{a^2 E' \sin \phi} \left(\frac{w \cot \phi}{a} - s\right)$$

$$+ \frac{2}{E'k(1-\nu)} V_\theta$$

$$a \frac{dN_\phi}{d\phi} = \frac{E'}{a^2}\left[(1-\nu^2) \cot \phi \right] u + \frac{nE' \cot \phi}{a^2 \sin \phi} (1-\nu^2) v$$

$$+ \left[\frac{(1-\nu^2) E' \cot \phi}{a^2} - \frac{2n^2 D \cot \phi}{a^4 k \sin^2 \phi} (1-\nu)\right] w$$

$$+ \frac{2n^2 D(1-\nu)}{a^3 k \sin^2 \phi} s - \frac{(1-\nu) \cot \phi}{a} N_\phi + \frac{V_\phi}{a} + \frac{n}{a \sin \phi} (k-2) V_\theta$$

$$- a^2 \sin \phi\, q_\phi + X/a^2$$

$$a \frac{dM_\phi}{d\phi} = - \frac{(1-\nu^2) D \cot \phi}{a^3} \left(u \cot \phi + \frac{n}{\sin \phi} v \right) - \frac{n^2 D(1-\nu) \cot \phi}{a^3 \sin^2 \phi} \left(\frac{2}{k} \right.$$

$$\left. + 1 + \nu \right) w + \frac{(1-\nu) D}{a^2} \left[\frac{2n}{k \sin^2 \phi} + (1+\nu) \cot^2 \phi \right] s$$

$$- (1-\nu) \frac{\cot \phi}{a} M_\phi + V_\phi + \frac{2nD}{a^2 E'k \sin \phi} V_\theta + U/a^2$$

$$\frac{d}{a\, d\phi} (V_\phi \sin \phi) = - \frac{(1-\nu)E'}{a^2} \left[(1+\nu) + \frac{n^2 D(1+\nu)}{a^2 E' \sin^2 \phi} \right] (u \cos \phi + nv)$$

$$- \frac{(1-\nu^2)E'}{a^2 \sin \phi} \left[\sin^2 \phi + \frac{n^4 D}{a^2 E' \sin^2 \phi} + \frac{2n^2 D \cot^2 \phi}{a^2 E'k(1+\nu)} \right] w$$

$$- \frac{(1-\nu)E' \cot \phi}{a} \left[\frac{n^2 (1+\nu) D}{a^2 E' \sin \phi} - \frac{2n^2 D}{a^2 E'k \sin \phi} \right] s$$

$$- \frac{(1+\nu)\sin\phi}{a} N_\phi + \frac{\nu n^2}{a^2 \sin\phi} M_\phi + \frac{2nD\cot\phi}{a^3 E' k} V_\theta$$

$$+ \sin\phi\, q_z + \frac{n}{a^3} V - \frac{\sin\phi}{a^2} Z$$

$$\frac{d}{a\,d\phi}(V_\theta \sin\phi) = \frac{nE'}{a^2}\left[\cot\phi\,(1-\nu^2)k\,u + \frac{n}{\sin\phi}(1-\nu^2)k\,v + \right.$$

$$\frac{1}{E'}\left[E'(1-\nu^2)\left(1 + \frac{n^2 D}{a^2 \sin^2\phi}\right)\right] w + \frac{D(1+\nu)\cot\phi}{naE'} s$$

$$\left. + \frac{na\,\nu}{E'} N_\phi - \frac{\nu}{E'} M_\phi \right] - \frac{\cos\phi}{a} V_\theta - V\frac{\sin\phi}{a^3} - \sin\phi\, q\,\theta$$

$$+ Y/a^2 \sin\phi \qquad (4.3)$$

where

$$E' = \frac{Eh}{1-\nu^2}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

$$k = 1 + \frac{D}{a^2 E'}$$

The terms V_ϕ and V_θ are similar to the Kirchhoff shear terms and are derived in Appendix E. The terms X, Y, Z, U and V are inertia terms from the force and moment equilibrium equations. The first term, X , is expressed as follows:

$$X \sin\phi = \rho \sin\phi \int_{-h/2}^{h/2} r^2 \frac{\partial^2 u}{\partial t^2}\, dz$$

$$= \rho \sin\phi \int_{-h/2}^{h/2} (a+Z)^2 \frac{\partial^2 u}{\partial t^2}\, dz$$

$$= \rho \sin\phi \left[a^2 h + o + O(h^3) \right] \frac{\partial^2 u}{t^2}$$

Neglecting higher order terms we get

$$X \sin\phi = \rho\, a^2 h \sin\phi \frac{\partial^2 u}{\partial t^2}$$

Similarly,

$$Y \sin\phi = \rho\, a^2 \sin\phi \frac{\partial^2 v}{\partial t^2}$$

$$Z \sin\phi = \rho\, a^2 h \sin\phi \frac{\partial^2 w}{\partial t^2}$$

$U \sin\phi = O(h^3)$ (which will be taken as zero)

$V \sin\phi = O(h^3)$ (also taken as zero)

In order to determine the natural frequencies and mode shapes of a shell the actual external loads have to be set to zero, and a harmonic motion assumed. That is,

$$\begin{Bmatrix} u(\phi,t) \\ v(\phi,t) \\ w(\phi,t) \end{Bmatrix} = \begin{Bmatrix} u(\phi) \\ v(\phi) \\ w(\phi) \end{Bmatrix} e^{iwt}$$

and thus,

$$\begin{Bmatrix} N(\phi,t) \\ s(\phi,t) \\ Q_{\phi}(\phi,t) \\ N_{\phi\theta}(\phi,t) \\ M_{\phi\theta}(\phi,t) \end{Bmatrix} = \begin{Bmatrix} N(\phi) \\ s(\phi) \\ Q_{\phi}(\phi) \\ N_{\phi\theta}(\phi) \\ M_{\phi\theta}(\phi) \end{Bmatrix} e^{iwt} \qquad (4.4)$$

Substitution of Equations (4.4) into Equations (4.3) yields the following system of eight first-order homogeneous equations:

$$\frac{dw}{ad\phi} = s$$

$$\frac{ds}{ad\phi} = \left(\frac{\nu n^2}{a^2 \sin^2 \phi} - \frac{(1+\nu)}{a^2}\right) w - \left(\frac{\nu \cot \phi}{a}\right) s + \frac{N_{\phi}}{aE'} + \frac{1}{D} M_{\phi}$$

$$\frac{du}{ad\phi} = -\frac{\nu \cot \phi}{a} u - \frac{\nu n}{a \sin \phi} v - \frac{(1+\nu)}{a} w + \frac{1}{E'} N_{\phi}$$

$$\frac{dv}{ad\phi} = \frac{n}{a \sin\phi} u + \frac{\cot\phi}{a} v + \frac{2nD}{a^2 E' \sin\phi}\left(\frac{w \cot\phi}{a} - s\right)$$

$$+ \frac{2}{E'k(1-\nu)} V_\theta$$

$$\frac{dN_\phi}{ad\phi} = \frac{E'}{a^2}\left[(1-\nu^2) \cot^2\phi \right] u + \frac{nE' \cot\phi}{a^2 \sin\phi} (1-\nu^2) v$$

$$+ \left[\frac{(1-\nu^2)E' \cot\phi}{a^2} - \frac{2n^2 D \cot\phi}{a^4 k \sin\phi} (1-\nu)\right] w$$

$$+ \frac{2n^2 D(1-\nu)}{a^3 k \sin^2\phi} s - \frac{(1-\nu) \cot\phi}{a} N_\phi + \frac{V_\phi}{a}$$

$$+ \frac{n}{a \sin\phi} (k-2) V_\theta - (\rho h \omega^2) u$$

$$\frac{dM_\phi}{ad\phi} = - \frac{(1-\nu^2) D \cot\phi}{a^3} \left(u \cot\phi + \frac{n}{\sin\phi} v\right) - \frac{n^2 D(1-\nu) \cot\phi}{a^3 \sin^2\phi}\left(\frac{2}{k}\right.$$

$$\left. + 1 + \nu\right) w + \frac{(1-\nu) D}{a^2}\left[\frac{2n}{k \sin^2\phi} + (1+\nu) \cot^2\phi\right] s$$

$$- (1-\nu) \frac{\cot\phi}{a} M_\phi + V_\phi + \frac{2nD}{a^2 E'k \sin\phi} V_\theta$$

$$\frac{d}{ad\phi} (V_\phi \sin\phi) = - \frac{(1-\nu) E'}{a^2}\left[(1+\nu) + \frac{n^2 D(1+\nu)}{a^2 E' \sin^2\phi}\right] (u \cos\phi + nv)$$

$$- \frac{(1-\nu^2)E'}{a^2 \sin\phi}\left[\sin^2\phi + \frac{n^4 D}{a^2 E' \sin^2\phi} + \frac{2n^2 D \cot^2\phi}{a^2 E'k(1+\nu)}\right] w$$

$$- \frac{(1-\nu)E' \cot\phi}{a}\left[\frac{n^2(1-\nu)D}{a^2 E' \sin\phi} - \frac{2n^2 D}{a^2 E'k \sin\phi}\right] s$$

$$- \frac{(1+\nu) \sin\phi}{a} N_\phi + \frac{\nu n^2}{a^2 \sin\phi} M_\phi + \frac{2nD \cot\phi}{a^3 E'k} V_\theta$$

$$+ \rho h \omega^2 \sin\phi \, w$$

$$\frac{d}{ad\phi}\,(V_\theta \sin\phi) = \frac{nE'}{a^2}\Bigg[\cot\phi\,(1-\nu^2)ku + \frac{n}{\sin\phi}\,(1-\nu^2)kv$$

$$+ \frac{1}{E'}\left[E'(1-\nu^2)\left(1 + \frac{n^2 D}{a^2\sin^2\phi}\right)\right]w + \frac{D(1+\nu)\cot\phi}{naE'}\,s$$

$$+ \frac{na\nu}{E'}N_\phi - \frac{\nu}{E'}M_\phi\Bigg] - \frac{\cos\phi}{a}V_\theta - (\rho h\omega^2 \sin\phi)v \tag{4.5}$$

These equations will be used to find the natural frequencies and mode shapes of the shell.

4.3 Boundary Conditions

The system of eight first-order shell equations contain eight unknowns. Therefore, it is necessary to have eight boundary conditions to determine a unique solution, four at the base and four at the apex. The boundary conditions at the base are generally quite easy to determine. They depend on the type of support of the shell. Here, all the shells will be taken to have a fixed base. This means that at the base there can be no displacements in the meridional, circumferential, or radial directions. Also, the rate of change of the meridional displacement at the base with respect to the meridional angle ϕ must be zero. These conditions hold true from t-initial to t-final. That is,

$$u(\phi,t) = 0$$

$$v(\phi_o,t) = 0$$

$$w(\phi_o,t) = 0$$

$$\frac{\partial w}{\partial \phi}(\phi_o,t) = 0$$

The boundary conditions at the apex of a hemispherical shell are considerably more complex than those at the base. Although the actual physical quantities are not singular at the apex, a singular point exists in the coordinate system chosen. At the apex, a one-to-one correspondence between a point as described by the coordinate system and an actual point on the shell breaks down. For example, the points (ϕ_1, θ_i, r) and $(\phi_1, \theta_i+\Delta\theta, r)$ are the same point if $\phi_1=0$ regardless of the value of $\Delta\theta$.

We know that the apex does not behave any differently than any other part of the shell. Therefore, the stress resultants, the displacements and their derivatives have to be finite at the apex. Examination of the eight first-order equations reveals that when $n=0$

$$u=s=Q_\phi=0$$

for the first derivatives to be finite. From the finiteness requirement of the stress resultants we see that

$$u_1 + v_1 = w_1 = 0 \qquad \text{for } n=1$$

$$u_n = v_n = w_n = \frac{\partial wn}{\partial \phi} = 0 \qquad \text{for } n\ 1$$

Thus, we have three conditions for n=0, two for n=1 and four conditions for n>1. For the first case, n=0, only three conditions at the apex are necessary, because the eighth-order system of equations reduces to sixth-order when n=0. When n>1 we have four conditions at the apex needed to determine a solution. However, for n=1 we are lacking two necessary boundary conditions. Greenbaum (Greenbaum, 1963) added two conditions to the case where n=1 by considering the force-strain relationship and moment-curvature relationship of the shell segment. The results were as follows for n=1:

$$u' = 0$$

$$M_{\phi} = 0$$

These two conditions with the previously determined conditions constitute the four necessary boundary conditions at the apex for n=1.

4.4 Method of Numerical Integration

The system of shell equations derived in Section 4.2 is of eighth-order when $n>1$ and sixth-order when $n=0$. The equations to be solved in this study will be of sixth-order; that is, $n=0$. Therefore, six solutions will be computed numerically and linearly combined to satisfy the six boundary conditions. All of the derivations, however, will be done for the eighth-order system from which the sixth-order system can easily be obtained by setting $n=0$. There will be two different numerical procedures used: one will be used for the neighborhood of the appex where difficulties are encountered due to small angles, and another for the rest of the shell.

4.4.1 Method of Integration Away From the Apex

Included in the total solution to the sixth-order system of shell equations are four non-growing solutions and one growing solution in each direction. One solution grows from the base to the apex and the other grows from the apex to the base. These two growing solutions are obtained by numerically integrating the sixth-order shell equations from a point near the apex to the base and from the base to the same point near the apex. The four non-growing solutions are determined by first integrating the reduced order equations (B.3) two times from the base to near the apex and two times from the base to near the apex and two times from near the apex to the base. The results are then substituted into Eqs. (B.1) to determine the displacements and stress resultants.

4.4.2 Method of Integration Near the Apex

As stated earlier the singularity at the apex, due to the coordinate system chosen, warrants special attention. The problem occurs when numerical integration is carried out in the neighborhood of the apex. Most of the coefficients of the dependent variables are divided by $\sin\phi$ or $\sin^2\phi$, such as

$$\frac{n}{a\sin\phi}u \quad \text{and} \quad \frac{2n^2D(1-\nu)}{a^3k\sin^2\phi}$$

Therefore, these terms are very large near the apex and undefined at $\phi=0$. To eliminate this problem the equations of the shell in the neighborhood of the apex are re-written in terms of a new independent variable Z, defined as follows:

$$z = \mathrm{Ln}(\sin\phi)$$

It is seen, that as $\sin\phi$, and thus ϕ, approaches zero, z approaches minus infinity. This means that for the start of integration at the apex, some finite, negative number for z can be chosen such that ϕ is approximately zero. For example, if $z=-10$ then $\phi=4.5\times10^{-5}$ rad. ≈ 0 (see Table 1). The derivative of z with respect to ϕ is

$$\frac{dz}{d\phi} = \frac{\cos\phi}{\sin\phi}$$

A typical derivative term in the shell equations, say $\frac{du}{d\phi}$ is transformed as follows when the independent variable is changed:

$$\frac{du}{d\phi} = \frac{du}{dz} \cdot \frac{dz}{d\phi} = \frac{du}{dz} \cdot \frac{\cos\phi}{\sin\phi}$$

All of the derivative terms in the shell equations will be of the form

$$f'(z) \cdot \frac{\cos\phi}{\sin\phi}$$

To solve for these terms, each first-order shell equation is divided by $\cos\phi/\sin\phi$. This eliminates all of the $1/\sin\phi$ coefficients in the shell equations which means a term such as $nu/\sin\phi$ becomes $nu/\cos\phi$. It is obvious, that $1/\cos\phi$ will cause no difficulty at or near $\phi=0$. The $1/\sin^2\phi$ coefficients do not cause any difficulty at $\phi=0$ either because the dependent variables by which they are multiplied are zero at $\phi=0$ and do not grow rapidly in the neighborhood of the apex. The interval size needed starts out small, but can be rapidly increased as the integration moves away from the apex. Integration continues from $z=-10$ to $z\simeq-1.75$ which is $\phi\simeq10^{\circ}$. Here, the derivatives with respect to z of the displacements and stress resultants will be used to calculate the derivatives with respect to ϕ. At this point, the integration will continue with the set of differential Eqs. (4.3) where the independent variable is ϕ.

When integrating the reduced order equations from the base to the apex, again at $\phi \approx 10^{\circ}$ derivatives will be calculated in terms of z. From this point to the apex the differential equations written in terms of z will be integrated to find the displacements and stress resultants.

In this interval the solutions do not grow to the point where they lose their independence. Therefore, from $\phi \approx 0$ to $\phi \approx 10^{\circ}$ it is not necessary to use the reduced order equations to find the independent solutions.

4.5 Determination of the Natural Frequency

In determining the natural frequencies of the shell it is necessary to solve the homogeneous portion of the shell equations. A combination of the Holzer method (Holzer, 1921) and a method developed by Robinson and Harris (Robinson and Harris, 1971) will be used to determine the natural frequency. The Holzer method will be used to obtain a first approximation for a frequency, then the Harris method will be used to improve upon that approximation.

4.5.1 Holzer Method

The application of the Holzer method to the shell is carried out as follows:

(1) An initial frequency ω is chosen along with six independent sets of initial values for

the system of six first-order equations. Six independent initial value solutions to the homogeneous equations are determined using the reduction of order procedure.

(2) The six solutions are combined to satisfy the six homogeneous boundary conditions. This is only possible if the determinate of the coefficients, Δ ,is zero. (The coefficients in this case are the values of the displacement and the stress resultants at the boundaries.) Therefore, the object is to get a zero value of the determinant.

(3) Values of ω are chosen to obtain different values of Δ. If Δ changes sign we know that a solution to the homogeneous system of equations lies between the two corresponding values of ω.

The value of ω can be determined only within certain limits with the Holzer method. When ω gets within a certain range of the true ω, the determinent of the coefficients Δ begins to oscillate between positive and negative values because of the small differences encountered when finding the value of the determinant. Therefore, it becomes necessary to go to a more accurate method such as the Robinson-Harris method as described in the next section.

4.5.2 Robinson-Harris Method

The method developed by Robinson and Harris is an application of the Newton-Raphson technique of improving an eigenvalue and the corresponding eigenvector from approximations of the eigenvalue and eigenvector. In this section, the Robinson-Harris method will be used to improve the approximation of the natural frequency of a spherical shell.

The problem to be solved can be described by

$$AX - \lambda BX = 0 \qquad (4.6)$$

where A and B are linear differential operators, λ is the square of the natural frequency and X is the eigenfunction. If an approximate eigenvalue and eigenfunction are substituted into Equation (4.6) we get

$$AX^{(i)} - \lambda^{(i)} BX^{(i)} = R^{(i)} \qquad (4.7)$$

Since $X^{(i)}$ and $\lambda^{(i)}$ are only approximations of the real eigenvalue and eigenfunction, there will be a residual function $R^{(i)}$. The object is to remove the residual.

Equation (4.7) is interperted as a non-linear equation in $\lambda^{(i)}$ and $X^{(i)}$, the non-linear term being $\lambda^{(i)}BX^{(i)}$. Equation (4.7) is linearized about the reference configuration corresponding to the superscript (i) which yields in linearized for

$$A\delta X^{(i)} - \lambda^{(i)}B\delta X^{(i)} - \delta\lambda^{(i)}BX^{(i)} = R^{(i)} \qquad (4.8)$$

where $\delta X^{(i)}$ and $\delta\lambda^{(i)}$ are small unknown incremental changes of $X^{(i)}$ and $\lambda^{(i)}$. The residual $R^{(i)}$ is determined from equation (4.7). Equation (4.8) has an extra unknown, $\delta\lambda^{(i)}$, therefore an additional equation is needed to get a solution. The additional equation is taken as

$$\int_0^L \delta X^{(i)}\, X^{(i)} \quad dx = 0 \qquad (4.9)$$

This guarantees that the allowable change in the vector of eigenfunctions is orthogonal to the latest vector of eigenfunctions. To further explain this procedure the actual shell equations will be used.

The shell equations are first put into the form of Eqs. (4.6), that is,

$$A\left\{u,v,w,s,N_\phi,M_\phi,V_\phi,V_\theta\right\} - \lambda B\left\{u,v,w,s,N_\phi,M_\phi,V_\phi,V_\theta\right\} = 0 \qquad (4.10)$$

where A and B are 8 x 8 matrices composed of differential and other linear operators. An approximate eigenvalue and eigenfunction are then substituted into Equation (4.10) to give:

$$A \left\{u_T^{(i)}, v_T^{(i)}, w_T^{(i)}, s_T^{(i)}, N_{\phi T}^{(i)}, M_{\phi T}^{(i)}, V_{\phi T}^{(i)}, V_{\theta T}^{(i)}\right\} - \lambda^{(i)} B \left\{u_T^{(i)}, v_T^{(i)}, w_T^{(i)}, N_{\phi T}^{(i)}, M_{\phi T}^{(i)}, V_{\phi T}^{(i)}, V_{\theta T}^{(i)}\right\} = \left\{R_1^{(i)}, R_2^{(i)}, R_3^{(i)}, R_4^{(i)}, R_5^{(i)}, R_6^{(i)}, R_7^{(i)}, R_8^{(i)}\right\} \tag{4.11}$$

The approximate eigenvalue is the square of the latest frequency determined in the Holzer method. The approximate eignefunction is composed of stress resultants and displacements corresponding to the latest frequency.* Next, Equations (4.11) are linerized about the reference configuration yielding the following non-homogeneous, first-order differential equations:

$$\delta\left(\frac{dw}{ad\phi}\right)^{(i)} + f_1\left(\delta s^{(i)}\right) = -R_1^{(i)}$$

$$\delta\left(\frac{ds}{ad\phi}\right)^{(i)} + f_2\left(\delta w^{(i)}, \delta s^{(i)}, \delta N_\phi^{(i)}, \delta M_\phi^{(i)}\right) = -R_2^{(i)}$$

$$\delta\left(\frac{du}{ad\phi}\right)^{(i)} + f_3\left(\delta u^{(i)}, \delta v^{(i)}, \delta w^{(i)}, \delta N_\phi^{(i)}\right) = -R_3^{(i)}$$

$$\delta\left(\frac{dv}{ad\phi}\right)^{(i)} + f_4\left(\delta u^{(i)}, \delta v^{(i)}, \delta w^{(i)}, \delta s^{(i)}, \delta V_\theta^{(i)}\right) = -R_4^{(i)}$$

$$\delta\left(\frac{dN_\phi}{ad\phi}\right)^{(i)} + f_5\Big(\delta u^{(i)}, \delta v^{(i)}, \delta w^{(i)}, \delta s^{(i)}, \delta N_\phi^{(i)}, \delta V_\phi^{(i)}, \delta V_\phi^{(i)}, \delta\lambda^{(i)}\Big) = -R_5^{(i)}$$

---

*Appendix E

$$\delta\left(\frac{dM_\phi}{ad\phi}\right)^{(i)} + f_6\left(\delta u^{(i)}, \delta v^{(i)}, \delta w^{(i)}, \delta s^{(i)}, \delta M_\phi^{(i)}, \delta V_\phi^{(i)}, \delta V_\theta^{(i)}\right) = -R_6^{(i)}$$

$$\delta \frac{d}{ad\phi}(V_\phi \sin\phi)^{(i)} + f_7\Big(\delta u^{(i)}, \delta v^{(i)}, \delta w^{(i)}, \delta s^{(i)}, \delta N_\phi^{(i)}, \delta M_\phi^{(i)}, \delta V_\theta^{(i)}, \delta\lambda^{(i)}\Big) = -R_7^{(i)}$$

$$\delta \frac{d}{ad\phi}(V_\theta \sin\phi)^{(i)} + f_8\Big(\delta u^{(i)}, \delta v^{(i)}, \delta w^{(i)}, \delta s^{(i)}, \delta N_\phi^{(i)}, \delta M_\phi^{(i)}, V_\theta^{(i)})\Big) = -R_8^{(i)} \tag{4.12}$$

We now have eight equations in nine unknowns, the unknowns being the incremental changes in the displacements and stress resultants, and the incremental change in the square of the natural frequency. Again, as in Equation (4.8), we have one more unknown than equations; thus, an additional side condition is needed to solve the above system of differential equations. In this case, the side condition is taken as

$$\int_o^{\phi_o}\int_o^{2\pi}\left[\rho h a^2 \sin\phi \left(u^{(i)}\delta u^{(i)} + w^{(i)}\delta w^{(i)} + v^{(i)}\delta v^{(i)}\right)\right] d\theta d\phi = 0$$

which simplifies to

$$\int_o^{\phi_o} \sin\left[\phi \quad u^{(i)}\delta u^{(i)} + w^{(i)}\delta w^{(i)} + v^{(i)}\delta v^{(i)}\right] d\phi = 0 \tag{4.13}$$

This is the equivalent to saying that the incremental change in the displacement vector is orthogonal to the latest displacement vector with respect to the mass. We now have nine equations in nine unknowns.

The boundary value problem described by Equations (4.12) and (4.13) is solved by the method of reduction of order of a system of differential equations as described in Chapter 2, with the exception that for Equation (4.13) there is no reduction of order just quadrature by Simpson's Rule. The initial values used in the solution of the boundary value problem are listed in Table 2. Ten independent solutions, nine homogeneous and one particular are determined using the initial values in the reduction of order method.

The solutions are then combined to satisfy the following eight homogeneous boundary conditions*:

$$0 = \delta u_{10}(0) + \sum_{j=1}^{9} x_j \delta u_j(0) \qquad (u=0 \text{ at } \phi=0)$$

$$0 = \delta w_{10}(0) + \sum_{j=1}^{9} x_j \delta w_j(0) \qquad (w=0 \text{ at } \phi=0)$$

$$0 = \delta v_{10}(0) + \sum_{j=1}^{9} x_j \delta v_j(0) \qquad (v=0 \text{ at } \phi=0)$$

$$0 = \delta s_{10}(0) + \sum_{j=1}^{9} x_j \delta s_j(0) \qquad (s=0 \text{ at } \phi=0)$$

---

*For n=0 or n=1 the boundary conditions would be changed appropriately.

$$0 = \delta u_{10}(\phi_o) + \sum_{j=1}^{9} x_j \delta u_j(\phi_o) \qquad (u=0 \text{ at } \phi=\phi_o)$$

$$0 = \delta w_{10}(\phi_o) + \sum_{j=1}^{9} x_j \delta w_j(\phi_o) \qquad (w=0 \text{ at } \phi=\phi_o)$$

$$0 = \delta v_{10}(\phi_o) + \sum_{j=1}^{9} x_j \delta v_j(\phi_o) \qquad (v=0 \text{ at } \phi=\phi_o)$$

$$0 = \delta s_{10}(\phi_o) + \sum_{j=1}^{9} x_j \delta s_j(\phi_o) \qquad (s=0 \text{ at } \phi=\phi_o) \qquad (4.14)$$

where j is the initial value solution number and x_j is the weighting coefficient. The additional equation needed to determine the weighting coefficients is one involving the side condition and expressed as

$$0 = \int_0^{\phi_o} \sin\phi \left[u^{(i)} \delta u_{10} + w^{(i)} \delta w_{10} + v^{(i)} \delta v_{10} \right] d\phi$$

$$+ \sum_{j=1}^{9} x_j \int_0^{\phi_o} \sin\phi \left[u^{(i)} \delta u_j + w^{(i)} \delta w_j + v^{(i)} \delta v_j \right] d\phi \qquad (4.15)$$

From Equations (4.14) and (4.15) the x_j's are calculated. Once their values are known the total incremental change in eigenvalue and displacements and stress resultants can be computed, that is,

$$\delta u^{(k)} = \delta u_{10}^{(k)} + \sum_{j=1}^{9} x_j \delta u_j^{(k)}$$

The other variables and their derivatives can be similarly expressed. The incremental changes are then added to the total solution to get

$$u^{(k+1)} = u^{(k)} + \delta u^{(k)}$$

The new reference position is, therefore, the $(k+1)^{th}$ approximation. This new approximation is then substituted back into Equation (4.11) to compute new residuals. The new residuals are checked to see if they are within a certain tolerance. If they are, the procedure is terminated. If not, the complete process is repeated and new residuals are calculated and checked for accuracy.

5. NUMERICAL RESULTS OF THE APPLICATION OF THE METHOD OF REDUCTION OF ORDER

5.1 General Remarks

In this Chapter, the results of the application of the proposed method to a system with multiple edge effects is presented. In addition, the results of the calculations of free vibration of spherical shell segments is also presented. The multiple edge effect example is solved using the methods presented in Chapters 2 and 3, and the shell example is solved using the methods presented in Chapters 2 and 4. The main objective of presenting these examples is to demonstrate some of the possible uses of the proposed method.

The multiple edge effect example,

$$w^{iv} - (k_1^2+k_2^2)w'' + k_1^2k_2^2w = p(x) \qquad (3.2)$$

given in Section 5.2 contains two rapidly growing solutions which are effectively separated numerically by the proposed method. The results of the calculations are compared with the exact solution of Equation (3.2).

The examples given in Section 5.3 are spherical shell segments where the natural frequencies and mode shapes have been computed for $\phi_o = 5^o, 10^o, 18^o$. The natural frequencies and the mode shapes are found for these shell segments using the Holzer method and the method developed by Robinson and Harris as presented in Chapter 4. In the application of the Holzer, and Robinson and Harris methods, it is necessary to repeatly solve two-point boundary value problems. The reduction of order method is used to solve these boundary value problems.

5.2 Example With Multiple Edge Effects

In solving a problem with multiple edge effects, the difficulty lies in first trying to seperate the two growing solutions and then trying to obtain a non-growing particular solution. The example described by Equation (3.2) was first re-written as four first order equations, as described in Chapter 3, for conveninece. The equations are

$$t' - (k_1^2 + k_2^2)s + k_1^2k_2^2q = p(x)$$

$$q' - r = 0$$

$$r' - s = 0$$

$$s' - t = 0$$

The values of k were

$$k_1 = 10$$

$$k_2 = 7$$

The interval length was 1 and p(x) was 10,000. Since the solutions were of the form e^{kx}, it is obvious, that both solutions, $w_1=e^{10x}$ and $w_2=e^{7x}$, grow rapidly. The boundary conditions used are as follows:

$$w(0) = 1 \qquad w'(L) = -4$$

$$w''(L) = 0 \qquad w'''(0) = 0 \qquad (5.1)$$

The exact solution was determined using Equation (3.1). If this equation is differentiated three times we have

$$w = C_1 e^{k_1 x} + C_2 e^{k_2 x} + C_3 e^{-k_1 x} + C_4 e^{-k_2 x}$$

$$w' = k_1 C_1 e^{k_1 x} + k_2 C_2 e^{k_2 x} - k_1 C_3 e^{-k_1 x} - k_2 C_4 e^{-k_2 x}$$

$$w'' = k_1^2 C_1 e^{k_1 x} + k_2^2 C_2 e^{k_2 x} + k_1^2 C_3 e^{-k_1 x} + k_2^2 C_4 e^{-k_2 x}$$

$$w''' = k_1^3 C_1 e^{k_1 x} + k_2^3 C_2 e^{k_2 x} - k_1^3 C_3 e^{k_1 x} - k_2^3 C_4 e^{-k_2 x} \qquad (5.2)$$

This is the homogeneous solution of Eq. (3.1). The particular solution is

$$w_p = p(x)/(k_1^2+k_2^2)$$

The coefficients in Eq. (5.2) are determined by setting the homogeneous and particular solutions and their derivatives equal to boundary conditions (5.1). Thus, once the coefficients are determined, the exact solution can be calculated.

5.3 Numerical Results for Multiple Edge Effects Example

Data given in Table 3 shows the rate of growth of the fastest growing solution. This solution was obtained by straightforward numerical integration of the homogeneous part of Eq. (3.2). It is seen that when x is at all large

$$10^3 w_1 = 10^2 w_1' = 10 w_1'' = w_1'''$$

This is exactly what we have in the exact homogeneous solution, that is,

$$w = e^{k_1 x}$$

$$w' = k_1 e^{k_1 x}$$

$$w'' = k_1^2 e^{k_1 x}$$

$$w''' = k_1^3 e^{k_1 x}$$

The above is also true for the solution, w_3, which grows from L to 0.

Data given in Table 4 is the slower growing solutions. This solution was obtained using Eqs. (3.6) and Eqs. (3.7). It is seen that

$$w_2 = e^{k_2 x}$$

$$w_2' = k_2 e^{k_2 x}$$

$$w_2'' = k_2^2 e^{k_2 x}$$

$$w_3''' = k_2^3 e^{k_2 x}$$

where $k_2 = 7$

Here we have

$$7^3 w_2 = 7^2 w_2' = 7w'' = w'''$$

The above is also true for the slower growing solution, w_4, which grows from L to 0.

From experience it has been found that a value of c of about three times the maximum value of w_1 yields the best results. The value of c should be chosen such that the quantity q_1+c does not vary greatly. A value of c much larger than just what is needed to make (q_1+c) constant increases round-off error and decreases the accuracy of the results.

The data given in Tables 3 and 4 clearly indicate that the two growing solutions from each end of the interval are accurately separated. Thus, if they are accurately separated they then can be linearly combined when determining the total solution to Eq. (3.1).

The non-growing particular solutions listed in Tables 5 and 6 were determined using Eqs. (3.8) and (3.10), respectively. Both tables list particular solutions to Eq. (3.6).

In Table 7 total solution number 1, containing particular solution 1 and total solution number 2, containing particular solution 2 are listed. As can be seen, when compared to the exact solution, total solution number 1 is more accurate when x is small and total solution 2 is more accurate when x is large. Also, it is seen that solutions 1 and 2 are almost identical in the middle of the interval and both compare very well with the exact solution at this point.

Table 8 shows comparisons between the exact solution and the solution obtained by the reduction of order method. The solutions obtained by reduction of order contain portions of two different solutions as described above.

The calculations for the reduction of order method were done on an IBM 360 computer, using single precision. The exact solution was calculated on the same computer, but using double precision.

5.4 Natural Frequencies and Mode Shapes of Spherical Shell Segments

The numerical results of the method of analysis used in the determination of natural frequencies and mode shapes of spherical shells is presented in this section. The Holzer method, and the Robinson-Harris method are used to determine the natural frequencies and mode shapes of the spherical shell segments. The reduction of order method is used to solve the two-point boundary value problems which occur in both methods.

The shell segments used in this study had opening angles of 5^{o}, 10^{o}, and 18^{o}. The h/a ratio was 1/100 and the base of the shell was clamped, that is,

$$u = w = s = 0 \qquad (s = dw/d\phi)$$

The Holzer method is used to determine an approximate value of the natural frequency parameter

$$\bar{\omega} = \frac{\omega}{\omega_o} \qquad \omega_o = \sqrt{\frac{E}{\rho a^2 (1-\nu^2)}}$$

This approximation is improved upon by the Robinson-Harris method. Table 9 gives a comparison between the natural frequency parameter $\bar{\omega}$ determined by the two methods.

When two successive values of $\bar{\omega}$ are within a certain tolerance, the last value of $\bar{\omega}$ determined in the Holzer method becomes the first approximation of $\lambda (\lambda = \bar{\omega}^2)$ in the Robinson-Harris method. Table 10 lists the values of $\bar{\omega}$ determined by the Robinson-Harris method from the first approximation to the final value and the values of the residuals R_i. As can be seen from the table, there is a very rapid convergence when starting with a fairly good approximation of λ.

The problem at the apex, due to the coordinate system chosen, is treated by changing the indpendent variable from ϕ to z as described in Section 4.4.2. That is, $z = \ln(\sin \phi)$. By changing the independent variable to z the integration can begin very close to $\phi=0$ ($\phi=4.5 \times 10^{-5}$ rad.) without difficulty. The size of the intervals is very small at $\phi \approx 0$, but is increased rapidly as ϕ increases. Table 11 shows how the intervals are increased as ϕ increases and also shows the values of the displacements.

A comparison is made between the method applied in this study and work done by Zarghamee in Table 12. It can be observed that there is a fairly good comparison between the values of the natural frequency parameter $\bar{\omega}$.

6. GENERAL CONCLUSIONS

6.1 Multiple Edge Effects

The reduction of order method developed in this study provides a procedure for determining accurate solutions of two-point boundary value problems when edge effects are present. Extremely good agreement was obtained when the exact solution was compared to the reduction of order solution of the multiple edge effect example. Although this method was only applied to examples of problems with one and two edge effects it can be extended to problems where there are more than two edge effects. A general procedure was described for the solution of an nth-order system with possibly n/2 growing solutions. The procedure works well for homogeneous as well as non-homogeneous problems.

6.2 Free Vibrations of Spherical Shells

The method of analysis used in this study for determining the natural frequencies and mode shapes of spherical shells was a combination of the Holzer and Robinson-Harris method. The reduction of order method was used to solve the two-point boundary value problems which occurred in these methods. The difficulties created at the apex by the coordinate system were effectively dealt with by changing the independent variable in the shell differential equations. This allowed numerical integration to begin at a point close enough to $\phi = 0$ such that for all practical purposes ϕ could be

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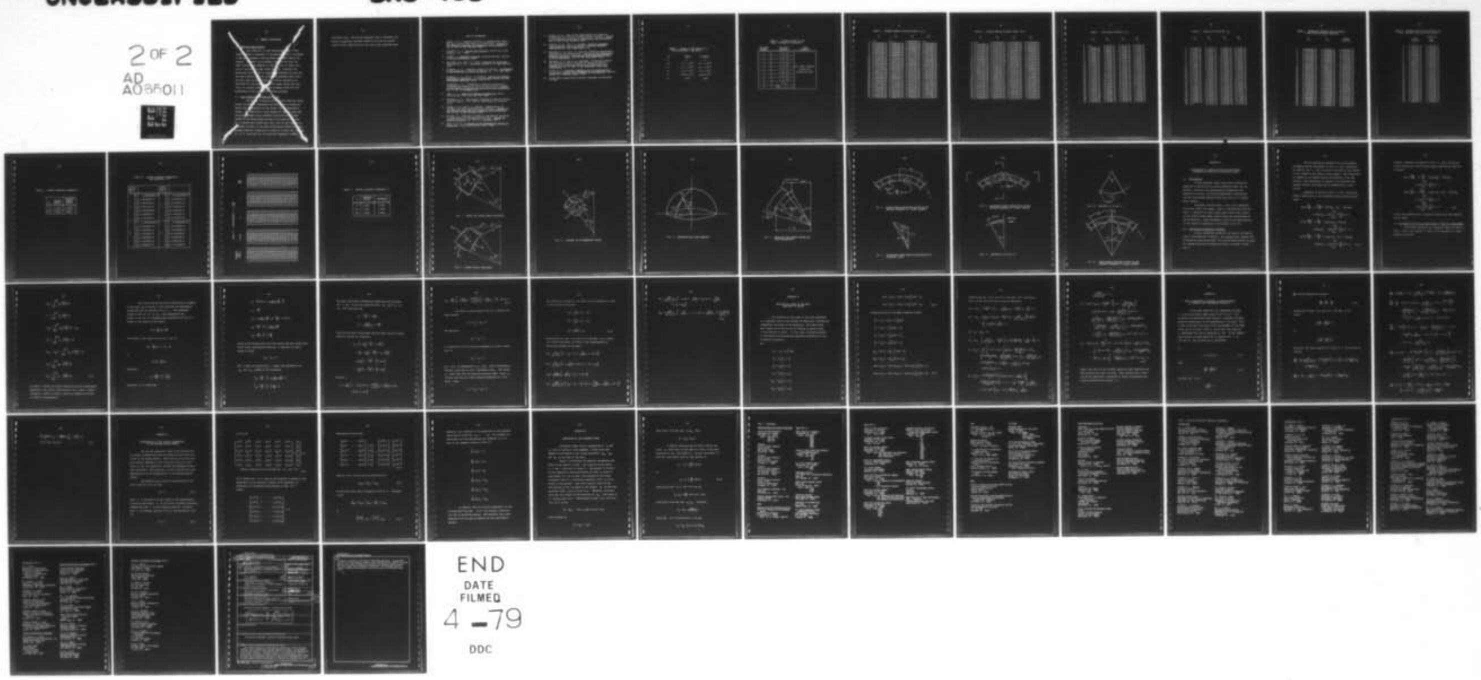

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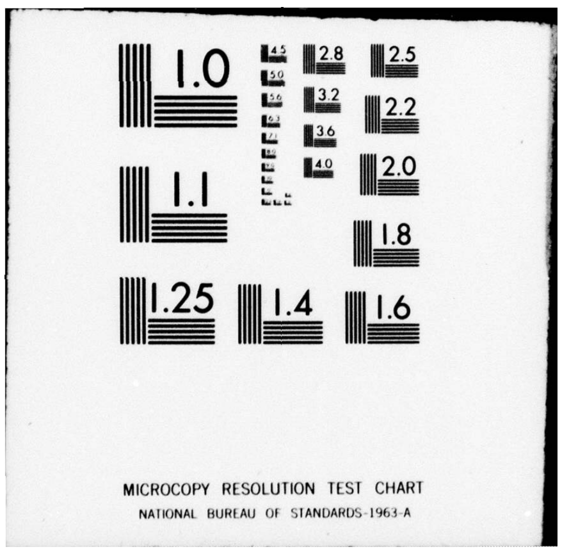
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TABLE 1. VALUES OF THE QUANTITY z, THE COLATITUDE ϕ AND SIN ϕ

Z	sin ϕ	ϕ (rad.)
$-\infty$	0	0
-10	4.5×10^{-5}	4.5×10^{-5}
-5	6.7×10^{-3}	6.7×10^{-3}
-4	1.8×10^{-2}	1.8×10^{-2}
-3	4.98×10^{-2}	4.98×10^{-2}
-2	.1353	.1353

TABLE 2. INITIAL VALUES OF THE SHELL DIFFERENTIAL EQUATIONS

SOLUTION NUMBER	NON-ZERO QUANTITY	ZERO QUANTITY
1	δu_1 (0)	
2	δw_2 (0)	
3	δs_3 (0)	
4	δv_4 (0)	
5	δu_5 (ϕo)	All other initial-values not listed are zero
6	δw_6 (ϕo)	
7	δs_7 (ϕo)	
8	$\delta \;_8$ (ϕo)	
9	$\delta \lambda_9$	
10	$-R_{10}^{(i)}$, i=1, ---, 8	

TABLE 3. FASTEST GROWING SOLUTION FROM 0 TO L

	w_1	w_1'	w_1''	w_1'''
1	0.0	0.0	0.9999999E-09	0.0
2	0.4582567E-12	0.3078551E-10	0.1068107E-08	0.4564470E-08
3	0.2122414E-11	0.7007898E-10	0.1312822E-08	0.1022545E-07
4	0.5274820E-11	0.1209362E-09	0.1766964E-08	0.1721998E-07
5	0.1042904E-10	0.1912656E-09	0.2489540E-08	0.2644401E-07
6	0.1840207E-10	0.2914942E-09	0.3573659E-08	0.3906096E-07
7	0.3041961E-10	0.4358829E-09	0.5157670E-08	0.5663972E-07
8	0.4827243E-10	0.6442988E-09	0.7441709E-08	0.8133958E-07
9	0.7454230E-10	0.9446541E-09	0.1071142E-07	0.1161645E-06
10	0.1129249E-09	0.1376300E-08	0.1537175E-07	0.1653132E-06
11	0.1686872E-09	0.1994767E-08	0.2199467E-07	0.2346686E-06
12	0.2493132E-09	0.2878414E-08	0.3138624E-07	0.3324797E-06
13	0.3654148E-09	0.4137735E-08	0.4468085E-07	0.4703170E-06
14	0.5320133E-09	0.5928442E-08	0.6347335E-07	0.6644074E-06
15	0.7703413E-09	0.8469794E-08	0.9000433E-07	0.9375022E-06
16	0.1110374E-08	0.1207026E-07	0.1274198E-06	0.1321489E-05
17	0.1594377E-08	0.1716359E-97	0.1801344E-06	0.1861063E-05
18	0.2281902E-08	0.2435923E-07	0.2543407E-06	0.2618828E-05
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20	0.4636888E-08	0.4882660E-07	0.5054470E-06	0.5174872E-05
21	0.6587982E-08	0.6898506E-07	0.7115705E-06	0.7267845E-05
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23	0.1322844E-07	0.1372426E-06	0.1407127E-05	0.1431426E-04
24	0.1870341E-07	0.1933000E-06	0.1976853E-05	0.2007559E-04
25	0.2641126E-07	0.2720309E-06	0.2775732E-05	0.2814534E-04
26	0.3725426E-07	0.3825495E-06	0.3895539E-05	0.3944570E-04
27	0.5249710E-07	0.5376176E-06	0.5464695E-05	0.5526657E-04
28	0.7391185E-07	0.7551022E-06	0.7662891E-05	0.7741200E-04
29	0.1039811E-06	0.1060011E-05	0.1074151E-04	0.1084048E-03
30	0.1461816E-06	0.1487343E-05	0.1505218E-04	0.1517726E-03
31	0.2053816E-06	0.2086076E-05	0.2108664E-04	0.2124478E-03

TABLE 4. SLOWER GROWING SOLUTION FROM 0 TO L

	w_2	w_2'	w_2''	w_2'''
0	0.0	0.9999999E-09	0.4893369E-08	-0.9999999E-09
1	0.3233699E-10	0.1150013E-08	0.5173288E-08	0.1902207E-07
2	0.7334304E-10	0.1337381E-08	0.6137736E-08	0.3832426E-07
3	0.1222734E-09	0.1567526E-08	0.7725976E-08	0.5665011E-07
4	0.1794046E-09	0.1861117E-08	0.9936436E-08	0.7580849E-07
5	0.2477141E-09	0.2239733E-08	0.1282402E-07	0.9739614E-07
6	0.3304412E-09	0.2727664E-08	0.1649552E-07	0.1229456E-06
7	0.4317087E-09	0.3353642E-08	0.2110981E-07	0.1540483E-06
8	0.5566982E-09	0.4152621E-08	0.2688151E-07	0.1924676E-06
9	0.7118923E-09	0.5167777E-08	0.3408834E-07	0.2402354E-06
10	0.9053860E-09	0.6452762E-08	0.4308220E-07	0.2997586E-06
11	0.1147280E-08	0.8074451E-08	0.5430320E-07	0.3739297E-06
12	0.1450176E-08	0.1011610E-07	0.6829748E-07	0.4662297E-06
13	0.1829791E-08	0.1268134E-07	0.8573915E-07	0.5808624E-06
14	0.2305712E-08	0.1589892E-07	0.1074567E-06	0.7228800E-06
15	0.2902335E-08	0.1992818E-07	0.1334640E-06	0.8983016E-06
16	0.3649991E-08	0.2496602E-07	0.1679959E-06	0.1114311E-05
17	0.4586326E-08	0.3125524E-07	0.2095467E-06	0.1379308E-05
18	0.5757990E-08	0.3909298E-07	0.2609177E-06	0.1702996E-05
19	0.7222596E-08	0.4884231E-07	0.3242503E-06	0.2096480E-05
20	0.9051252E-08	0.6094376E-07	0.4020740E-06	0.2571745E-05
21	0.1133129E-07	0.7593042E-07	0.4973326E-06	0.3141613E-05
22	0.1416966E-07	0.9444091E-07	0.6134014E-06	0.3818728E-05
23	0.1769661E-07	0.1172341E-06	0.7540557E-06	0.4614005E-05
24	0.2206974E-07	0.1452017E-06	0.9234027E-06	0.5534937E-05
25	0.2747902E-07	0.1793760E-06	0.1125707E-05	0.6581395E-05
26	0.3415118E-07	0.2209254E-06	0.1364999E-05	0.7740222E-05
27	0.4235375E-07	0.2711431E-06	0.1644526E-05	0.8977018E-05
28	0.5239974E-07	0.3314017E-06	0.1965818E-05	0.1022033E-04
29	0.6464904E-07	0.4030799E-06	0.2327171E-05	0.1134491E-04
30	0.7950808E-07	0.4874310E-06	0.2721223E-05	0.1214235E-04

TABLE 5. PARTICULAR SOLUTION, w_{p1}

	w_{p_2}	w'_{p_2}	w''_{p_3}	w'''_{p_4}
0	0.0	0.1097E 02	-0.4511E 02	-0.1097E 02
1	0.3099E 00	0.9624E 01	-0.4419E 02	0.6559E 02
2	0.6076E 00	0.8193E 01	-0.4114E 02	0.1110E 03
3	0.8593E 00	0.6884E 01	-0.3706E 02	0.1295E 03
4	0.1070E 01	0.5718E 01	-0.3266E 02	0.1317E 03
5	0.1243E 01	0.4698E 01	-0.2836E 02	0.1242E 03
6	0.1385E 01	0.3817E 01	-0.2441E 02	0.1115E 03
7	0.1500E 01	0.3060E 01	-0.2093E 02	0.9645E 02
8	0.1591E 01	0.2411E 01	-0.1796E 02	0.8059E 02
9	0.1662E 01	0.1853E 01	-0.1553E 02	0.6492E 02
10	0.1716E 01	0.1366E 01	-0.1362E 02	0.4988E 02
11	0.1754E 01	0.9365E 00	-0.1218E 02	0.3571E 02
12	0.1779E 01	0.5464E 00	-0.1121E 02	0.2240E 02
13	0.1791E 01	0.1810E 00	-0.1068E 02	0.9793E 01
14	0.1791E 01	0.1738E 00	-0.1056E 02	-9.2391E 01
15	0.1780E 01	-0.5305E 00	-0.1083E 02	-0.1379E 02
16	0.1756E 01	-0.9031E 00	-0.1147E 02	-0.2472E 02
17	0.1719E 01	-0.1298E 01	-0.1248E 02	-0.3482E 02
18	0.1668E 01	-0.1733E 01	-0.1375E 02	-0.4348E 02
19	0.1602E 01	-0.2218E 01	-0.1522E 02	-0.4823E 02
20	0.1519E 01	-0.2759E 01	-0.1685E 02	-0.4879E 02
21	0.1417E 01	-0.3357E 01	-0.1856E 02	-0.4321E 02
22	0.1295E 01	-0.4004E 01	-0.1997E 02	-0.3107E 02
23	0.1153E 01	-0.4667E 01	-0.2046E 02	0.1367E 00
24	0.9852E 00	-0.5336E 01	-0.1977E 02	0.5426E 02
25	0.7903E 00	-0.5889E 01	-0.1610E 02	0.1528E 03
26	0.5767E 00	-0.6255E 01	-0.7973E 01	0.3146E 03
27	0.3672E 00	-0.6219E 01	0.8320E 01	0.5879E 03
28	0.1724E 00	-0.5641E 01	0.3617E 02	0.9978E 03
29	0.2710E-01	-0.3688E 01	0.7590E 02	0.1611E 04
30	0.9033E-02	0.2148E 00	0.1461E 03	0.2464E 04

TABLE 6. PARTICULAR SOLUTION, w_{p2}

	w_{p_2}	w'_{p_2}	w''_{p_3}	w'''_{p_4}
0	0.9033E-02	-0.2148E 00	0.1461E 03	-0.2464E 04
1	0.1709E 01	0.3457E 01	0.8196E 02	-0.1684E 04
2	0.1565E 00	0.5508E 01	0.3883E 02	-0.1047E 04
3	0.3452E 00	0.6297E 01	0.9965E 01	-0.6186E 03
4	0.5544E 00	0.6267E 01	0.6617E 01	-0.3392E 03
5	0.7693E 00	0.5944E 01	-0.1548E 02	-0.1655E 03
6	0.9671E 00	0.5396E 01	-0.1955E 02	-0.6207E 02
7	0.1137E 01	0.4747E 01	-0.2044E 02	-0.4578E 01
8	0.1282E 01	0.4072E 01	-0.1999E 02	0.2909E 02
9	0.1406E 01	0.3419E 01	-0.1869E 02	0.4249E 02
10	0.1510E 01	0.2818E 01	-0.1706E 02	0.4863E 02
11	0.1594E 01	0.2269E 01	-0.1536E 02	0.4848E 02
12	0.1662E 01	0.1789E 01	-0.1390E 02	0.4397E 02
13	0.1715E 01	0.1341E 01	-0.1260E 02	0.3587E 02
14	0.1753E 01	0.9407E 00	-0.1156E 02	0.2575E 02
15	0.1778E 01	0.5675E 00	-0.1087E 02	0.1495E 02
16	0.1791E 01	0.2087E 00	-0.1056E 02	0.3559E 01
17	0.1792E 01	-0.1455E 00	-0.1064E 02	-0.8637E 01
18	0.1781E 01	-0.5089E 00	-0.1114E 02	-0.2111E 02
19	0.17575 01	-0.8961E 00	-0.1207E 02	-0.3435E 02
20	0.1721E 01	-0.1321E 01	-0.1345E 02	-0.4841E 02
21	0.1669E 01	-0.1801E 01	-0.1532E 02	-0.6338E 02
22	0.1599E 01	-0.2352E 01	-9.1770E 02	-0.7901E 02
23	0.1510E 01	-0.2991E 01	-0.2061E 02	-0.9487E 02
24	0.1398E 01	-0.3736E 01	-0.2404E 02	-0.1101E 03
25	0.1259E 01	-0.4604E 01	-0.2795E 02	-0.1231E 03
26	0.1089E 01	-0.5609E 01	-0.3222E 02	-0.1313E 03
27	0.8822E 00	-0.6761E 01	-0.3663E 02	-0.1304E 03
28	0.6348E 00	-0.8056E 01	-0.4076E 02	-0.1138E 03
29	0.3419E 00	-0.9477E 01	-0.4396E 02	-0.7173E 02
30	0.0	-0.1097E 02	-0.4511E 02	0.1097E 02

TABLE 7. COMPARISON BETWEEN TOTAL SOLUTIONS 1 AND 2 AND THE EXACT SOLUTION

	w_{T_1}	w_{T_2}	EXACT SOLUTION
0	0.9999999E 00	0.1000000E 01	0.1000000D 01
1	0.1159199E 00	0.11185422 01	0.1159139D 01
2	0.1311897E 01	0.1267189E 01	0.1311907D 01
3	0.1441156E 01	0.1395608E 01	0.1441141D 01
4	0.1549276E 01	0.1508801E 01	0.1549242D 01
5	0.1638834E 01	0.1613165E 01	0.1638786D 01
6	0.1712343E 01	0.1698983E 01	0.1712292D 01
7	0.1772130E 01	0.1762797E 01	0.1772083D 01
8	0.1820264E 01	0.1811711E 01	0.1820218D 01
9	0.1858508E 01	0.1851007E 01	0.1858473D 01
10	0.1888365E 01	0.1882613E 01	0.1888334D 01
11	0.1910976E 01	0.1906458E 01	0.1911009D 01
12	0.1927374E 01	0.1923646E 01	0.1927449D 01
13	0.1938210E 01	0.1935847E 01	0.1938364D 01
14	0.1943997E 01	0.1942519E 01	0.1944242D 01
15	0.1944908E 01	0.1943964E 01	0.1945359D 01
16	0.1941240E 01	0.1940779E 01	0.1941795D 01
17	0.1932470E 01	0.1932685E 01	0.1933434D 01
18	0.1918213E 01	0.1919416E 01	0.1919973D 01
19	0.1898513E 01	0.1900434E 01	0.1900917D 01
20	0.1872685E 01	0.1875299E 01	0.1875584D 01
21	0.1839049E 01	0.1842897E 01	0.1843108D 01
22	0.1798345E 01	0.1802300E 01	0.1802448D 01
23	0.1748493E 01	0.1752319E 01	0.1752421D 01
24	0.1684845E 01	0.1691689E 01	0.1691750D 01
25	0.1602128E 01	0.1619122E 01	0.1619162D 01
26	0.1503093E 01	0.1533532E 01	0.1533556D 01
27	0.1400103E 01	0.1434243E 01	0.1434261D 01
28	0.1286058E 01	0.1321447E 01	0.1321465D 01
29	0.1167562E 01	0.1196843E 01	0.1196354D 01
30	0.1078046E 01	0.1064647E 01	0.1064606D 01

TABLE 8. SOLUTION OBTAINED BY REDUCTION OF ORDER COMPARED WITH THE EXACT SOLUTION

	EXACT SOLUTION	REDUCTION OF ORDER
0	0.100000D 01	0.100000E 01
1	0.115914D 01	0.115911E 01
2	0.131191D 01	0.131190E 01
3	0.144114D 01	0.144116E 01
4	0.154924D 01	0.154928E 01
5	0.163879D 01	0.163883E 01
6	0.171229D 01	0.171234E 01
7	0.177208D 01	0.177213E 01
8	0.182022D 01	0.182026E 01
9	0.185847D 01	0.185851E 01
10	0.188833D 01	0.188837E 01
11	0.191101D 01	0.191098E 01
12	0.192745D 01	0.192737E 01
13	0.193836D 01	0.193821E 01
14	0.194424D 01	0.194400E 01
15	0.194536D 01	0.194491E 01
16	0.194179D 01	0.194124E 01
17	0.193343D 01	0.193269E 01
18	0.191997D 01	0.191942E 01
19	0.190092D 01	0.190048E 01
20	0.187558D 01	0.187530E 01
21	0.184310D 01	0.184290E 01
22	0.180245D 01	0.180230E 01
23	0.175242D 01	0.175232E 01
24	0.169175D 01	0.169169E 01
25	0.161916D 01	0.161912E 01
26	0.153356D 01	0.153353E 01
27	0.143426D 01	0.143424E 01
28	0.132146D 01	0.132145E 01
29	0.119685D 01	0.119684E 01
30	0.106461D 01	0.106465E 01

TABLE 9. NATURAL FREQUENCY PARAMETER $\bar{\omega}$

ϕ_o	HOLZER METHOD	ROBINSON-HARRIS METHOD
5^o	4.1620	4.09406
10^o	1.6603	1.6463

TABLE 10. NATURAL FREQUENCY PARAMETER $\bar{\omega}$ AND RESIDUALS, $R^{(i)}$

Iteration No.			Iteration No.		
1	$\bar{w}$	0.409459400E 01	3	$\bar{w}$	0.409406853E 01
	R_1	0.7008921E-11		R_1	0.1064426E-13
	R_2	0.4893371E-12		R_2	0.1623701E-14
	R_3	-0.1103119E-07		R_3	-0.1102431E-07
	R_4	-0.3493263E-08		R_4	-0.2739130E-11
	R_5	-0.3018936E-10		R_5	-0.4452652E-14
	R_6	0.1511309E-11		R_6	-0.1260710E-14
2	$\bar{w}$	0.409406662E 01	4	$\bar{w}$	0.40946853E 01
	R_1	-0.6915302E-13		R_1	-0.1276756E-14
	R_2	0.1003364E-13		R_2	0.2636780E-15
	R_3	-0.1102694E-07		R_3	-0.1102374E-07
	R_4	0.1389229E-10		R_4	0.7851226E-12
	R_5	-0.7243799E-13		R_5	0.3940427E-15
	R_6	-0.2829605E-14		R_6	-0.1261198E-14

TABLE 11. DISPLACEMENT u, w, $\frac{\partial w}{a \partial \phi}$

INTERVAL SIZE (RAD.)	ϕ(rad)	u	w	$\frac{\partial w}{a \partial \phi}$
0.1649474E-05	0.4499999E-04	-0.6938894E-16	-0.9967970E-11	0.1465494E-13
0.2364235E-05	0.6449943E-04	0.5230191E-15	-0.9967712E-11	0.1592060E-11
0.3388704E-05	0.9244845E-04	0.8439430E-15	-0.9967241E-11	0.1127098E-11
0.4857255E-05	0.1325089E-03	0.1046038E-14	-0.9967856E-11	0.9945378E-12
0.6962073E-05	0.1899294E-03	0.1332268E-14	-0.9968459E-11	0.8504308E-12
0.9978423E-05	0.2722295E-03	0.2011412E-03	-0.9969263E-11	0.6579182E-12
0.1430255E-04	0.3901913E-03	0.2552429E-14	-0.9969981E-11	0.7722711E-12
0.2050051E-04	0.5592690E-03	0.2878232E-14	-0.9970678E-11	0.9431900E-12
0.2938393E-04	0.8016140E-03	0.3636631E-14	-0.9971223E-11	0.1325065E-11
0.4211697E-04	0.1148977E-02	0.5282179E-14	-0.9970624E-11	0.1981512E-11
0.6036763E-04	0.1646867E-02	0.7610720E-14	-0.9969390E-11	0.2814499E-11
0.8652732E-04	0.2360509E-02	0.1092719E-13	-0.9966940E-11	0.4005921E-11
0.1240228E-03	0.3383400E-02	0.1571562E-13	-0.9961911E-11	0.5715817E-11
0.1777634E-03	0.4849523E-02	0.2251124E-13	-0.9951695E-11	0.8175088E-11
0.2547950E-03	0.6950978E-02	0.3224002E-13	-0.9930837E-11	0.1169437E-10
0.3652237E-03	0.9963132E-02	0.4601475E-13	-0.9887991E-11	0.1671349E-10
0.5235150E-03	0.1428073E-01	0.6555802E-13	-0.9800368E-11	0.2382003E-10
0.7504523E-03	0.2046977E-01	0.9277399E-13	-0.9622032E-11	0.3370630E-10
0.1075894E-02	0.2934216E-01	0.1294915E-12	-0.9262804E-11	0.4705708E-10
0.1542844E-02	0.4206345E-01	0.1755862E-12	-0.8553736E-11	0.6385246E-10
0.2213534E-02	0.6030971E-01	0.2237285E-12	-0.7214613E-11	0.8148983E-10
0.3179014E-02	0.8649874E-01	0.2472527E-12	-0.4919029E-11	0.9040112E-10
0.4575551E-02	0.1241453E 00	0.1853913E-12	-0.1811479E-11	0.6848812E-10
0.9599999E-03	0.1411468E 00	0.1291060E-12	-0.8143600E-12	0.4809105E-10
0.9599999E-03	0.1507468E 00	0.9295310E-13	-0.4156051E-12	0.3483071E-10
0.9599999E-03	0.1603467E 00	0.5532679E-13	-0.1478190E-12	0.2088209E-10
0.9599999E-02	0.1699466E 00	0.1762779E-13	-0.1537434E-13	0.6712456E-11
0.3137183E-01	0.1745325E 00	0.2338162E-18	-0.2234037E-16	-0.1091195E-15

TABLE 12. NATURAL FREQUENCY PARAMETER, $\bar{\omega}$

ϕ_o	ROBINSON-HARRIS	ZARGHAMEE
5^o	4.0941	4.1134
10^o	1.6463	1.6556
18^o	1.2606	1.2649

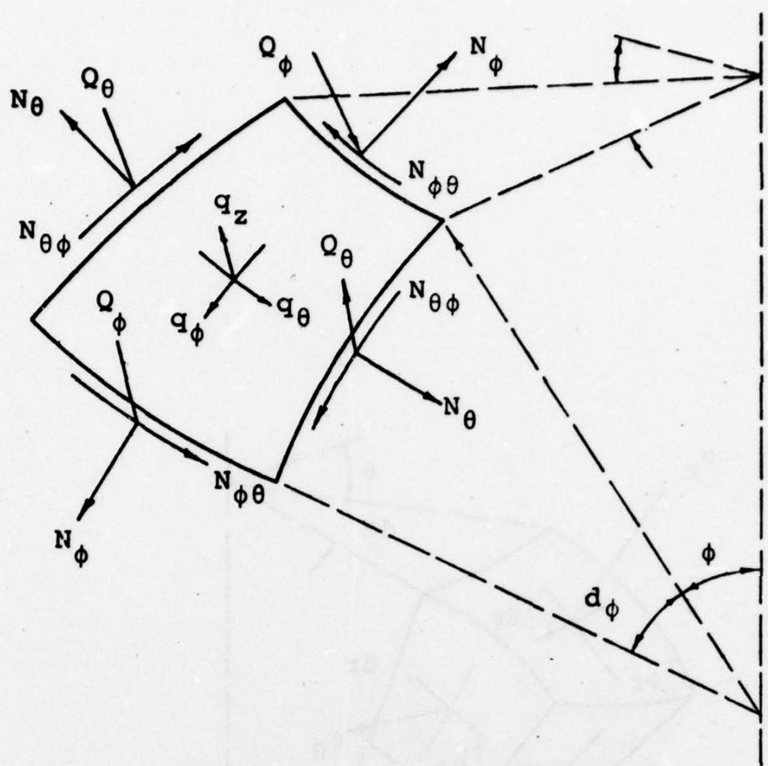

FIG. 1. NORMAL AND SHEAR STRESS RESULTANTS

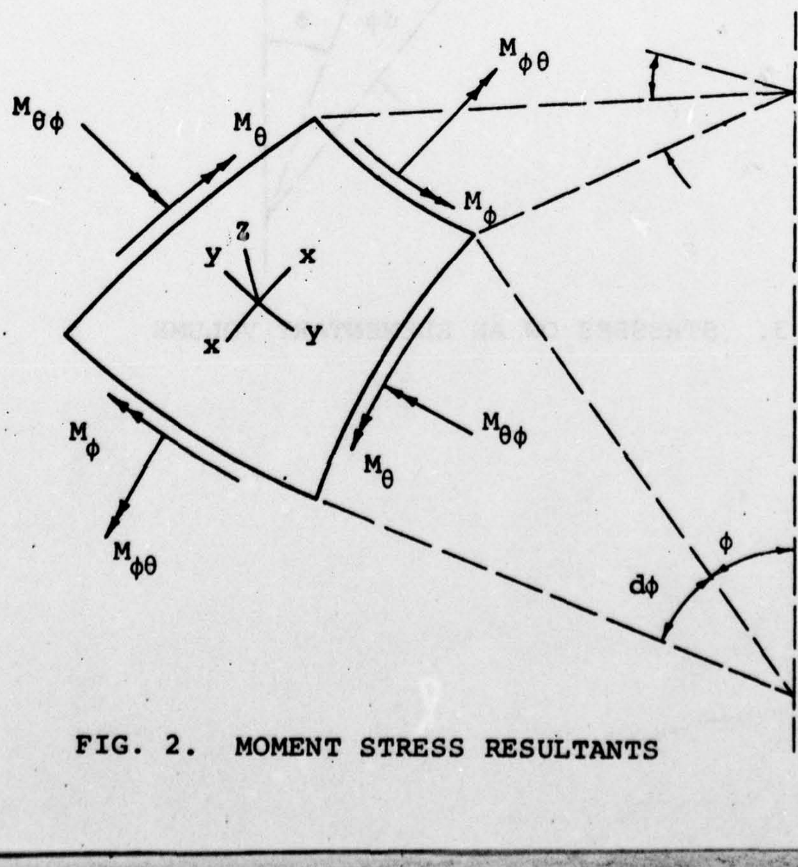

FIG. 2. MOMENT STRESS RESULTANTS

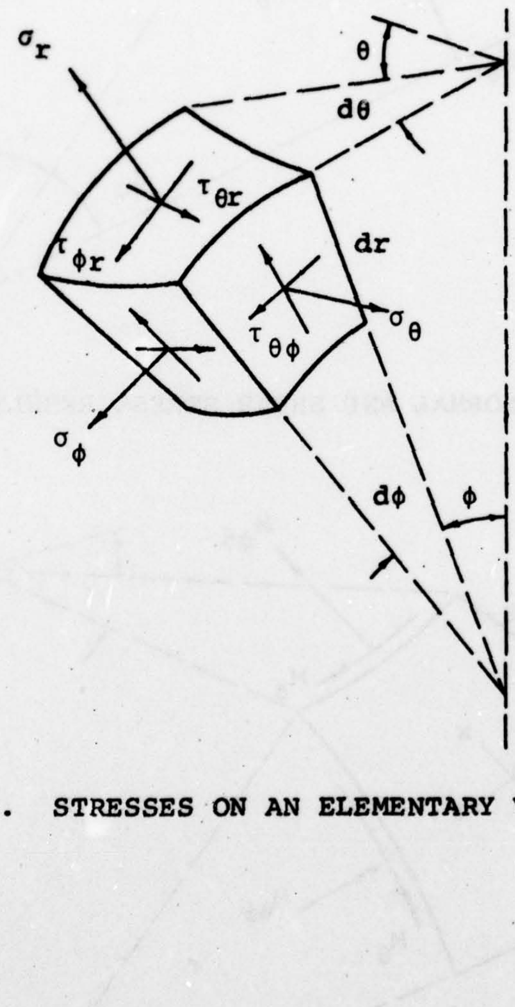

FIG. 3. STRESSES ON AN ELEMENTARY VOLUME

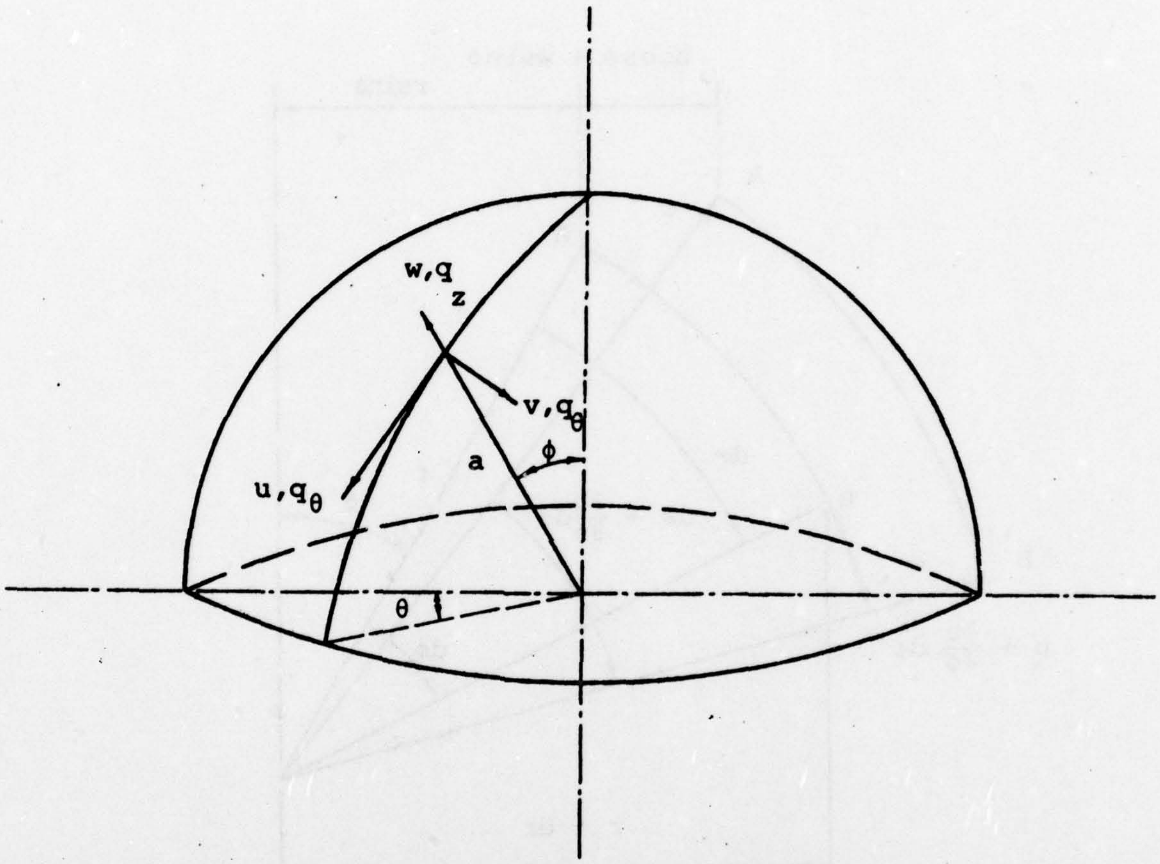

FIG. 4. DEFLECTION AND LOAD NOTATION

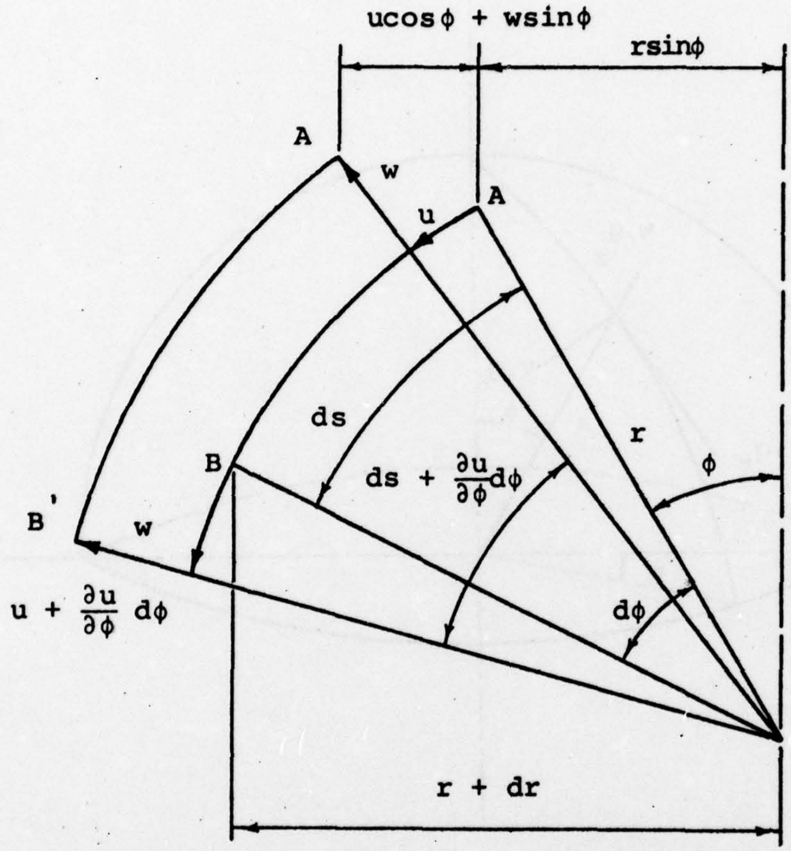

FIG. 5. MERIDIAN LINE ELEMENT BEFORE AND AFTER DEFORMATION.

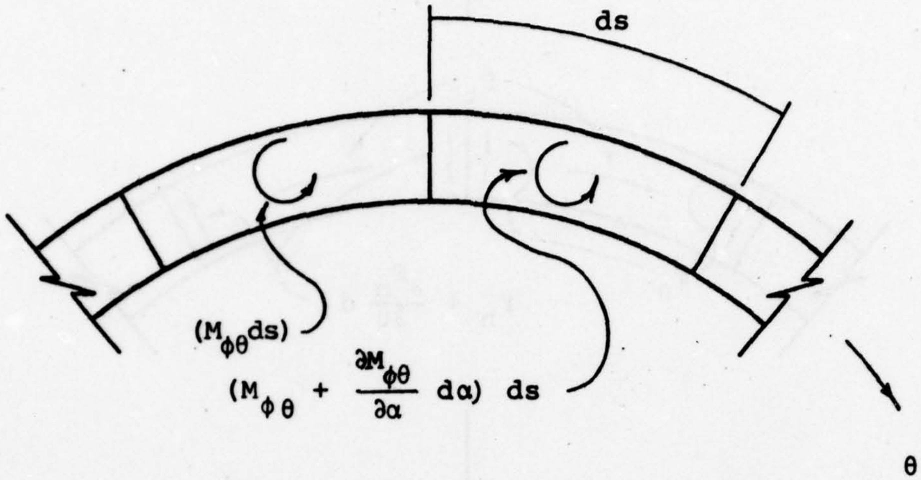

FIG. 6. MOMENT STRESS RESULTANTS ACTING ON TWO ADJACENT ELEMENTS OF A SHELL SEGMENT.

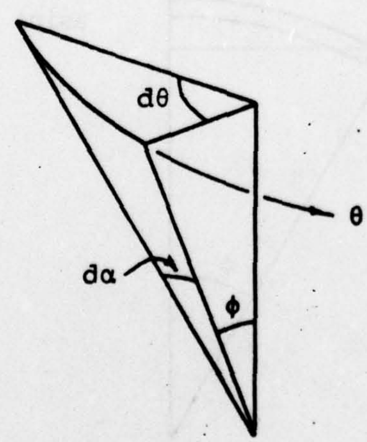

FIG. 7. COORDINATE SYSTEM USED FOR DERIVATION OF KIRCHHOFF FORCE.

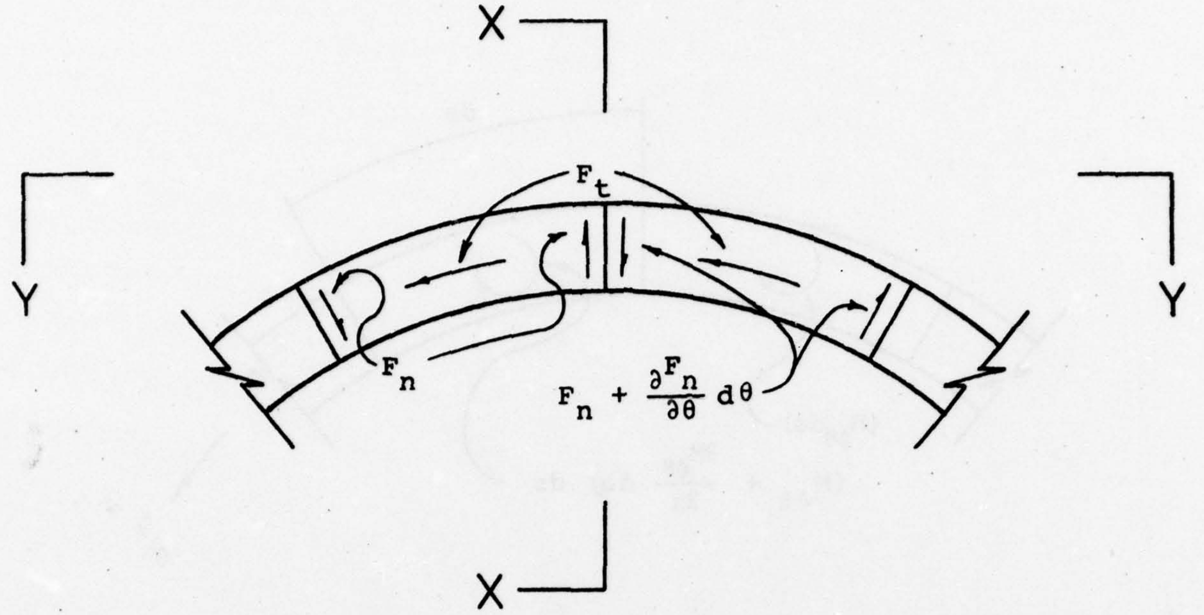

FIG. 8. EQUIVALENT FORCE SYSTEM ACTING ON TWO ADJACENT ELEMENTS OF A SHELL SEGMENT

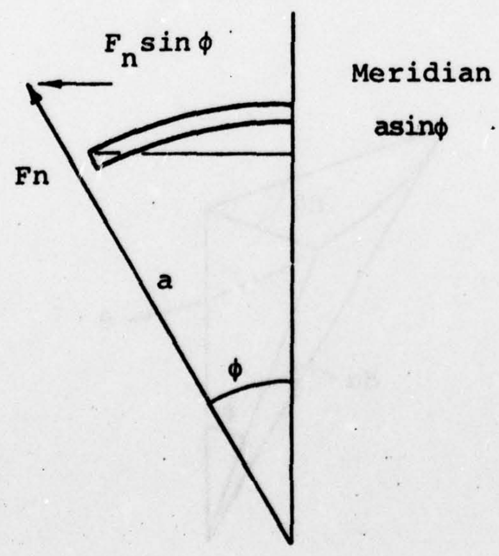

FIG. 9. SECTION X-X OF FIG. 9.

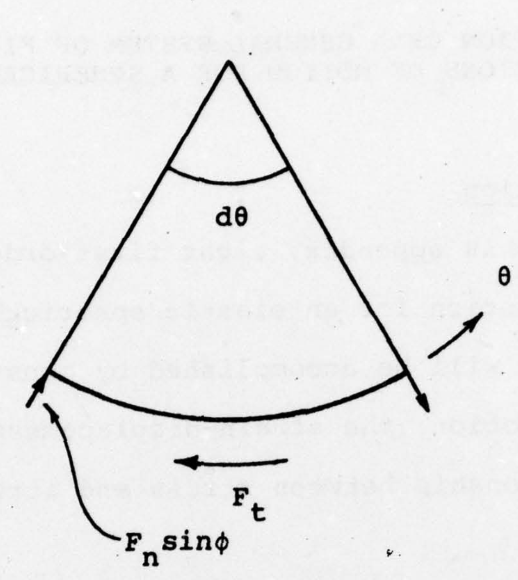

FIG. 10. SECTION Y-Y OF FIG. 9.

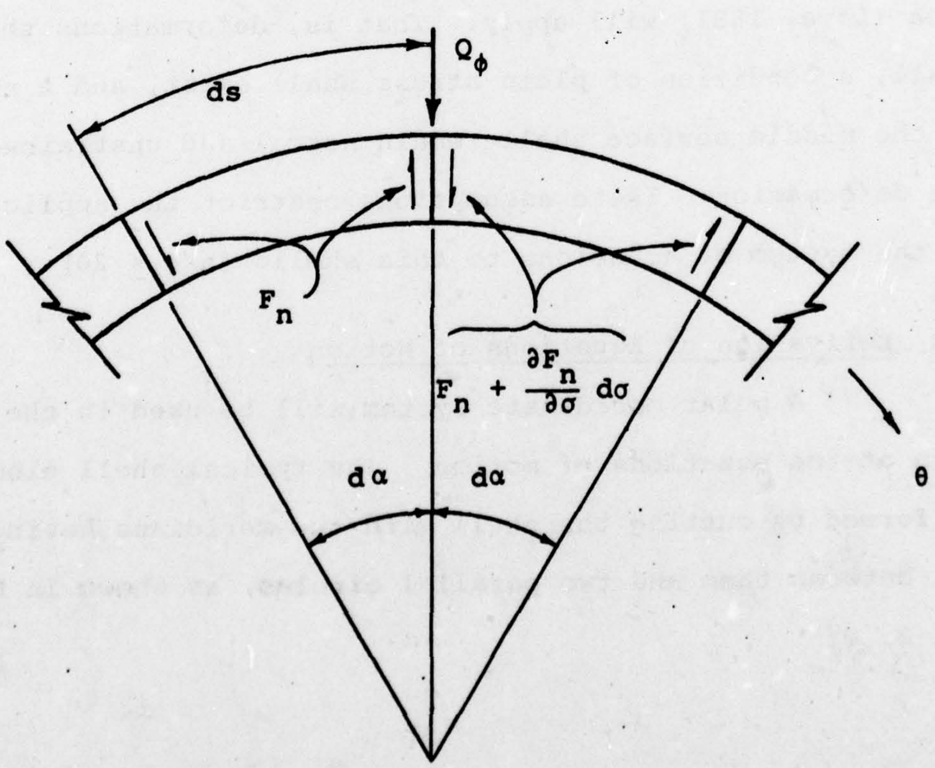

FIG. 11. SHEAR STRESS RESULTANT ACTING ON TWO ADJACENT ELEMENTS OF A SHELL SEGMENT.

APPENDIX A

DERIVATION OF A GENERAL SYSTEM OF FIRST-ORDER EQUATIONS OF MOTION FOR A SPHERICAL SHELL

A.1 Introduction

In this appendix, eight first-order differential equations of motion for an elastic spherical shell will be derived. This will be accomplished by considering the equations of motion, the strain-displacement relationships and the relationship between stress and strain of a typical shell element.

Throughout the derivation, Love's first approximation (Love, 1881) will apply. That is, deformations shall be small, a condition of plain stress shall exist, and a normal to the middle surface shall remain normal and unstrained during deformation. These assumptions restrict the applicability of the system of equations to thin shells ($h/a \leq 20$).

A.2 Derivation of Equations of Motion

A polar coordinate system will be used in the derivation of the equations of motion. The typical shell element will be formed by cutting the shell with two meridians having an angle $d\theta$ between them and two parallel circles, as shown in Figs. 1 and 2.

The six equilibrium equations will be determined by summing forces and moments in the x, y, and z directions as shown in Fig. 2. The z direction is normal to the surface of the element at the center of the element. The y-direction is tangent to the middle surface and parallel to the two circles. The x-direction is tangent to the meridian that equally divides the element and is perpendicular to the circle.

Summation of forces in the x, y, and z directions yields respectively in the following force equilibrium equations:

$$a\sin\phi \frac{\partial N_\phi}{\partial \phi} + a \frac{\partial N_{\theta\phi}}{\partial \theta} + a\cos\phi(N_\phi - N_\theta) - a\sin\phi\, Q_\phi + a^2\sin\phi\, q_\theta - \rho\sin\phi \int_{-h/2}^{h/2} r^2 \frac{\partial^2 u}{\partial t^2}\, dz = 0$$

$$a\sin\phi \frac{\partial N_{\phi\theta}}{\partial \phi} + a \frac{\partial N_\theta}{\partial \theta} + 2a\cos\phi\, N_{\phi\theta} - a\sin\phi\, Q_\theta + a^2\sin\phi\, q_\theta - \rho\sin\phi \int_{-h/2}^{h/2} r^2 \frac{\partial^2 v}{\partial t^2}\, dz = 0$$

$$a\sin\phi \frac{\partial Q_\phi}{\partial \phi} + a \frac{\partial Q_\theta}{\partial \theta} + a\sin\phi\,(N_\theta + N_\phi) + a\cos\phi\, Q_\phi - a^2\sin\phi\, q\, Z - \rho\sin\phi \int_{-h/2}^{h/2} r^2 \frac{\partial^2 w}{\partial t^2}\, dz = 0 \qquad \text{(A.1)}$$

Likewise, summation of moments in the x, y, and z directions yields respectively the following moment equilibrium equations of motion:

$$a\sin\phi \frac{\partial M_{\phi\theta}}{\partial \theta} + a \frac{\partial M_{\theta}}{\partial \theta} + 2a\cos\phi M_{\phi\theta} - a^2\sin\phi Q_{\theta} - \rho\sin\phi \int_{-h/2}^{h/2} r^2 z \frac{\partial^2 v}{\partial t^2} dz = 0$$

$$a\sin\phi \frac{\partial M_{\phi}}{\partial \phi} + a \frac{\partial M_{\phi\theta}}{\partial \theta} + a\cos\phi (M_{\phi} - M_{\theta}) - a^2\sin\phi Q_{\phi} - \rho\sin\phi \int_{-h/2}^{h/2} r^2 z \frac{\partial^2 u}{\partial t^2} dz = 0$$

$$- \rho\sin\phi \int_{-h/2}^{h/2} r^2 z \frac{\partial^2 \omega}{\partial t^2} dz = 0 \qquad (A.2)$$

It has been assumed that no external moments have been applied to the shell.

A.3 Determination of Stress Resultants in Terms of Displacement

The stress resultants of a spherical shell as shown in Figs. 1 and 2, are defined in terms of the stresses of a shell segment as follows:

$$N_\phi = \int_{-h/2}^{h/2} \sigma_\phi \left(\frac{a+z}{a}\right) dz$$

$$N = \int_{-h/2}^{h/2} \sigma_\theta \left(\frac{a+z}{a}\right) dz$$

$$N_{\phi\theta} = \int_{-h/2}^{h/2} \tau_{\phi\theta} \left(\frac{a+z}{a}\right) dz = N_{\theta\phi}$$

$$M_\phi = -\int_{-h/2}^{h/2} \sigma_\phi \left(\frac{a+z}{a}\right) z\, dz$$

$$M_{\phi\theta} = M_{\theta\phi} = -\int_{-h/2}^{h/2} \tau_{\phi\theta} \left(\frac{a+z}{a}\right) z\, dz$$

$$Q_\phi = \int_{-h/2}^{h/2} \tau_{\phi r} \left(\frac{a+z}{z}\right) dz$$

$$Q_\theta = \int_{-h/2}^{h/2} \tau_\theta \left(\frac{a+z}{a}\right) dz \qquad \text{(A.3)}$$

In order to relate the stress resultants and the displacement quantities, the elastic stress-strain law is used to obtain stresses in terms of strains, which are themselves written in terms of displacements.

The strains can be derived by considering an element of the shell as in Figure 5 which describes the deformation associated with the meridian strain ε_ϕ. The undeformed length of the arc AC is ds. Upon deformation the length of the arc is increased proportionately with the increase in the length of the radius

$$(ds + \frac{\partial u}{\partial \phi} d\phi) \frac{r+w}{r}$$

The change in the length of the arc Δ ds is

$$(ds + \frac{\partial u}{\partial \phi} d\phi) \; 1 + \frac{w}{r} \quad -ds$$

or

$$ds \frac{w}{r} + \frac{\partial u}{\partial \phi} d\phi$$

Therefore,

$$\varepsilon_\phi = \frac{\Delta ds}{ds} = \frac{\overline{w}}{r} + \frac{\partial \overline{u}}{\partial \phi} \frac{1}{r}$$

Similarly, it is found that

$$\varepsilon_\theta = \frac{\bar{u}}{r} \cot\phi + \frac{1}{r \sin\phi} \frac{\partial \bar{v}}{\partial \theta} + \frac{\bar{w}}{r}$$

$$\varepsilon_r = \frac{\partial \bar{w}}{\partial r}$$

$$\gamma_{\phi\theta} = \frac{1}{r \sin\phi} \frac{\partial \bar{u}}{\partial \theta} + \frac{1}{r} \left(\frac{\partial \bar{v}}{\partial \phi} - \bar{v} \cot\phi\right)$$

$$\gamma_{\theta r} = \frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r} + \frac{1}{r \sin\phi} \frac{\partial \bar{w}}{\partial \theta}$$

$$\gamma_{\phi r} = \frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r} + \frac{1}{r} \frac{\partial \bar{w}}{\partial \phi}$$

Since we are dealing only with thin shells and have stated that Love's first approximation applies, a condition of strain exists in which

$$\gamma_{\theta r} = \gamma_{\phi r} = 0$$

and w does not change with r. Hence, the expression for $\gamma_{\theta r}$ and $\gamma_{\phi r}$ reduce to the following:

$$\gamma_{\theta r} = \frac{\partial \bar{v}}{\partial r} - \frac{\bar{v}}{r} + \frac{1}{r \sin\phi} \frac{\partial w}{\partial \phi} = 0$$

$$\gamma_{\phi r} = \frac{\partial \bar{u}}{\partial r} - \frac{\bar{u}}{r} + \frac{1}{r} \frac{\partial w}{\partial \phi} = 0$$

The above two partial differential equations can be solved for $\bar{u}$ and $\bar{v}$ with the condition that $\bar{u}=u$ and $\bar{v}=v$ at $r=a$. The solutions are

$$\bar{u} = -\frac{w_\phi z}{a} + u\,\frac{a+z}{a}$$

$$\bar{v} = -\frac{w_\phi z}{a \sin\phi} + v\,\frac{a+z}{a}$$

Substituting these values back into the three non-zero strain equations yields the following:

$$\varepsilon_\phi = \frac{w}{r} + \frac{\partial}{\partial\phi}\left[-\frac{w_\phi z}{a} + u\,\frac{a+z}{a}\right]$$

$$= \frac{w}{a+z} + \frac{1}{a+z}\left[-\frac{w_{\phi\phi} z}{a} + u_\phi\left(\frac{a+z}{a}\right)\right]$$

$$= \frac{1}{a+z}\left[w - \frac{w_{\phi\phi} z}{a} + \frac{u_\phi}{a}(a+z)\right]$$

$$= \frac{1}{a+z}\left[w - \frac{w_{\phi\phi} z}{a} + \frac{u_\phi}{a}(a+z)\right]$$

Similarly,

$$\varepsilon_\theta = \frac{1}{a+z}\left[w - w_\phi \frac{z}{a}\cot\phi - \frac{w_{\theta\theta} z}{a \sin^2\phi} + \frac{v_\theta}{a \sin\phi}(a+z) + \frac{u}{a}\cot\phi\,(z+a)\right]$$

$$\gamma_{\phi\theta} = \frac{2z}{a+z}\left[-\frac{w_{\theta\phi}}{a \sin\phi} + \frac{w_\theta \cos\phi}{a \sin^2\phi}\right] + \frac{u_\theta}{a \sin\phi} + \frac{v_\phi}{a} - \frac{v}{a}\cot\phi \tag{A.4}$$

The shell is now assumed to be in a condition of plane stress,

$$\sigma_r = \tau_{\theta r} = \tau_{\phi r} = 0$$

The condition

$$\tau_{\phi r} = \tau_{\theta r} = 0$$

is consistent with out previous assumption of plain strain, that is,

$$\gamma_{\phi r} = \gamma_{\theta r} = 0$$

but $\sigma_r=0$ is inconsistent with $\varepsilon_r=0$. This inconsistency, however, gives way to only a negligible error. The stress σ_r would only come from applied external loads. Also, σ_r is much less than the other internal pressures of a thin shell. Thus,

$$\sigma_r = \tau_{\phi r} = \tau_{\theta r} = 0$$

The remaining stresses of the shell can be expressed in terms of the strains as follows:

$$\sigma_\phi = \frac{E}{1-\nu^2}(\varepsilon_\phi + \nu\varepsilon_\theta)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_\phi)$$

$$\tau_{\phi\theta} = \frac{E}{2(1+\nu)}\gamma_{\phi\theta} \qquad \text{(A.5)}$$

Substitution of Eqs. (A.4) and (A.5) into Eqs. (A.3) yields the stress resultants in terms of the displacements of the middle surface of the shell.

$$N_\phi = \frac{Eh}{a(1-\nu^2)}\left[w + u_\phi + \nu(w + \frac{v_\theta}{\sin\phi} + u \cot\phi)\right]$$

$$N_\theta = \frac{Eh}{a(1-\nu^2)}\left[w + \frac{v_\theta}{\sin\phi} + \cot\phi\, u + \nu(w + u_\phi)\right]$$

$$N_{\phi\theta} = \frac{Eh}{2a(1+\nu)}\left[\frac{u_\theta}{\sin\phi} + v_\phi - v\cot\phi\right]$$

$$M_\phi = \frac{Eh^3}{12a^2(1-\nu^2)}\left[w_{\phi\phi} - u_\phi + \nu(w_\phi \cot\phi + \frac{w_{\theta\theta}}{\sin^2\phi} - \frac{v_\theta}{\sin\phi} u \cot\phi)\right]$$

$$M_{\theta} = \frac{Eh^3}{12a^2(1-\nu^2)} \left[- u \cot \phi - \frac{v_{\theta}}{\sin \phi} + w_{\phi} \cot \phi + \frac{w_{\theta\theta}}{\sin^2 \phi} + \nu(-u_{\phi} + w_{\phi\phi}) \right]$$

$$M_{\phi\theta} = \frac{Eh^3}{24a^2(1+\nu)} \left[- \frac{u_{\theta}}{\sin \phi} - v_{\phi} + v \cot \phi + \frac{2\, w_{\phi\theta}}{\sin \phi} - \frac{2\, w_{\theta} \cos \phi}{\sin^2 \phi} \right]$$

(A.6)

APPENDIX B

REDUCTION OF ORDER OF THE SHELL EQUATIONS OF MOTION

The reduction of the order of the shell equations is a necessary step in the process of numerically determining independent solutions to the equations. The eighth order shell equations of motion can be reduced to seventh order if one solution is known. In this case, the known solution is obtained by straightforward numerical integration of the differential equation.

Let

$$u_2 = (u_1 + c)\int y_1 dx$$

$$v_2 = v_1\int y_1 + y_2$$

$$w_2 = w_1\int y_1 + y_3$$

$$s_2 = s_1\int y_1 + y_4$$

$$N_{\phi 2} = N_{\phi 1}\int y_1 + y_5$$

$$M_{\phi 2} = M_{\phi 1}\int y_1 + y_6$$

$$(\sin\phi\, V_\phi)_2 = (\sin\phi\, V_\phi)_1 \int y_1 dx + y_7$$

$$(\sin\phi\, V_\theta)_2 = (\sin\phi\, V_\theta)_1 \int y_1 dx + y_8 \qquad \text{(B.1)}$$

Differentiation of the above equations yields:

$$u_2' = (u_1 + c)y_1 + u_1' \int y_1 dx$$

$$v_2' = v_1 y_1 + v_1' \int y_1 dx + y_2'$$

$$w_2' = w_1 y_1 + w_1' \int y_1 dx + y_3'$$

$$s_2' = s_1 y_1 + s_1' \int y_1 dx + y_4'$$

$$N_{\phi_2}' = N_\phi\, y_1 + N_{\phi_1}' \int y_1 dx + y_5'$$

$$M_{\phi 2}' = M_{\phi_1} y_1 + M_{\phi_1}' \int y_1 dx + y_6'$$

$$(\sin\phi\, V_\phi)_2' = (\sin\phi\, V_\phi)_1 y_1 + (\sin\phi\, V_\phi)_1' \int y_1 dx + y_7'$$

$$(\sin\phi\, V_\theta)_2' = (\sin\phi\, V_\theta)_1 y_1 + (\sin\phi\, V_\theta)_1' \int y_1 dx + y_8' \qquad \text{(B.2)}$$

Substituting Eqs. (B.1) and (B.2) into Eqs. (4.5) and simplifying, we get the following system of equations:

$$(u_1 + c)y_1 + \frac{\nu \cot\phi}{a} c\int y_1 - \frac{\nu n}{a \sin\phi} y_2 - \frac{(1+\nu)}{a} y_3 + \frac{y_5}{E'} = 0$$

$$y_2' + \nu y_2 - \frac{n}{a \sin\phi} c\int y_1 dx - \frac{\cot\phi}{a} y_2 - \frac{2nD}{a^2 E' \sin\phi}\left[\frac{y_3 \cot\phi}{a} - y_4\right] - \frac{2}{E' k(1-\nu)} \frac{y_8}{\sin\phi} = 0$$

$$y_3' - y_4 + w_1 y_1 = 0$$

$$y_4' + s_1 y_1 - \left[\frac{\nu n^2}{a^2 \sin^2\phi} - \frac{(1+\nu)}{a^2}\right] y_3 + \left(\frac{\nu \cot\phi}{a}\right) y_4 + \frac{y_5}{aE'} + \frac{y_6}{D} = 0$$

$$y_5' + N_{\phi_1} y_1 - \frac{E'}{a^2}\left[(1-\nu^2)\cot^2\phi\right] c\int y_1 dx - \frac{nE' \cot\phi}{a^2 \sin\phi}(1-\nu^2) y_2$$
$$- \left[\frac{(1-\nu^2)E' \cot\phi}{a^2} - \frac{2n^2 D \cot\phi}{a^4 k \sin^2\phi}(1-\nu)\right] y_3 - \frac{2n^2 D(1-\nu)}{a^3 k \sin^2\phi} y_4$$
$$+ \frac{(1-\nu)\cot\phi}{a} y_5 + \frac{y_7}{a \sin\phi} - \frac{n}{a \sin^2\phi}(k-2) y_8$$
$$- (\rho h \omega^2) c\int y_1 dx = 0$$

$$y_6' + M_{\phi_1} y_1 + \frac{(1-\nu^2) D \cot\phi}{a^3}\left[c\int y_1 dx \cot\phi + \frac{n}{\sin\phi} y_2\right]$$
$$+ \frac{n^2 D(1-\nu)\cot\phi}{a^2 \sin^2\phi}\left(\frac{2}{k} + 1 + \nu \quad y_3\right) - \frac{(1-\nu)D}{a^2}\left[\frac{2n}{k \sin^2\phi}\right.$$
$$\left. + (1+\nu)\cot^2\phi\right] y_4 + (1-\nu)\frac{\cot\phi}{a} y_6 - \frac{y_7}{\sin\phi}$$

$$- \frac{2nDy_8}{a^2E'k\sin^2} = 0$$

$$y_7' + (\sin\phi\, V_\phi)y_1 + \frac{(1-\nu)E'}{a^2}\left[(1-\nu) + \frac{n^2D(1+\nu)}{a^2E'\sin^2\phi}\right]$$

$$(c\int y_1 dx \cos\phi + ny_2) + \frac{(1-\nu^2)E'}{a^2\sin\phi}\left[\sin^2\phi + \right.$$

$$\left.\frac{n^4D}{a^2E'\sin^2\phi} + \frac{2n^2D\cot^2\phi}{a^2E'k(1+\nu)}\right]y_3 + \frac{(1-\nu)E'\cot\phi}{a}$$

$$\left[\frac{n^2(1-\nu)D}{a^2E'\sin\phi} - \frac{2nD}{a^2E'k\sin\phi}\right]y_4 + \frac{(1+\nu)\sin\phi}{a}y_5$$

$$+ \frac{\nu n^2}{a^2\sin\phi}y_6 + \frac{2nD\cot\phi}{a^3E'k\sin\phi}y_8 + \rho h\omega^2\sin\phi\, y_3 = 0$$

$$y_8' + (\sin\phi\, V_\theta)y_1 - \frac{nE'}{a^2}\left[\cot\phi\,(1-\nu^2)k\,c\int y_1 dx + \frac{n}{\sin\phi}(1-\nu^2)ky_2\right.$$

$$- \frac{1}{E'}\left[E'(1-\nu^2)\left(1 + \frac{n^2D}{a^2\sin^2\phi}\right)\right]y_3 - \frac{D(1+\nu)\cot\phi}{naE'}y_4$$

$$\left. - \frac{na\nu}{E'}y_5 + \frac{\nu}{E'}y_6\right] - \frac{\cos\phi}{a\sin\phi}y_8 + (\rho hw^2\sin\phi)y_2 = 0 \qquad \text{(B.3)}$$

Hence, the order of the original system of shell equations has been reduced from eight to seven. This seventh-order system can now be numerically integrated to obtain non-growing solutions as described in section 2.3.

APPENDIX C

SHELL DIFFERENTIAL EQUATIONS OF MOTION WRITTEN IN TERMS OF THE INDEPENDENT VARIABLE Z

In the shell equations the independent variable ϕ is the colatitude, which ranges from 0^o to 90^o. Difficulties are encountered when trying to integrate the shell equations numerically in the neighborhood of the apex. This is due to the fact that many of the coefficients in the equations are of the form $c/\sin\phi$, which gets very large as ϕ approaches zero and is undefined at $\phi=0$. If the independent variable in these equations is changed from ϕ to $z=\ln(\sin\phi)$ this problem can be eliminated.

If

$$z = \ln(\sin\phi) \tag{C.1}$$

then

$$\frac{dz}{d\phi} = \frac{\cos\phi}{\sin\phi} \tag{C.2}$$

Consider Eqs. (4.5a)

$$\frac{1}{a}\frac{dw}{d\phi} = s$$

$\frac{dw}{d\phi}$ can be expressed as follows:

$$\frac{dw}{d\phi} = \frac{dw}{dz} \cdot \frac{dz}{d\phi} \qquad \text{(C.3)}$$

Substitution of Eqs. (C.2) and (C.3) into Eqs. (4.5a) yields

$$\frac{1}{a}\frac{dw}{dz} \cdot \frac{\cos\phi}{\sin\phi} = s$$

or

$$\frac{1}{a}\frac{dw}{dz} = \frac{\sin\phi}{\cos\phi}\, s$$

Similarly, the other equations in terms of z are written as follows:

$$\frac{ds}{adz} = \left(\frac{\nu n^2}{a\cos\phi\sin\phi} - \frac{(1+\nu)\tan\phi}{a^2}\right) w - \left(\frac{\nu}{a}\right) s + \left(\frac{\tan\phi}{aE'}\right) N_\phi$$

$$+ \left(\frac{\tan\phi}{D}\right) M_\phi$$

$$\frac{du}{ad\phi} = -\frac{\nu}{a}u - \frac{\nu n}{a\cos\phi}v - \frac{(1+\nu)\tan\phi}{a}w + \left(\frac{\tan\phi}{E'}\right)N_\phi$$

$$\frac{dv}{ad\phi} = \frac{n}{a \cos \phi} u + \frac{v}{a} + \frac{2nD}{a^2 E' \cos \phi} \left(\frac{w \cot \phi}{a} - s \right) + \frac{2 \tan \phi}{E'k(1-\nu)} V_\theta$$

$$\frac{dN_\phi}{ad\phi} = \frac{E'}{a^2} \left[(1-\nu^2) \cot \phi \right] u + \frac{nE'}{a^2 \sin \phi} (1-\nu^2) v + \left[\frac{(1-\nu^2)E'}{a^2} - \frac{2n^2 D(1-\nu)}{a^4 k \sin^2 \phi} \right] w + \frac{2n^2 D(1-\nu)}{a^3 k \cos \phi \sin \phi} s - \frac{(1-\nu)}{a} \tan \phi$$

$$+ \frac{\tan \phi \, V_\phi}{a} + \frac{n}{a \cos \phi} (k-2) V_\theta - (\rho h \omega^2 \tan \phi) u$$

$$\frac{dM_\phi}{ad\phi} = - \frac{(1-\nu^2)D}{a^3} \left(u \cot \phi + \frac{n}{\sin \phi} \right) v - \frac{n^2 D(1-\nu)}{a^3 \sin^2 \phi} \left(\frac{2}{k} + 1 + \nu \right) w$$

$$+ \frac{(1-\nu)D}{a^2} \left[\frac{2n}{k \cos \phi} + (1+\nu) \cot \phi \right] s - (1-\nu) \frac{M_\phi}{a}$$

$$+ \tan \phi \, V_\phi + \frac{2nD}{a^2 k E' \cos \phi} V_\theta$$

$$\frac{d}{adz} (V_\phi \sin) \phi = - \frac{(1-\nu)E' \tan \phi}{a^2} \left[(1+\nu) + \frac{n^2 D(1+\nu)}{aE' \sin^2 \phi} \right] (u \cos \phi + nv)$$

$$- \frac{(1-\nu^2)E'}{a^2 \cos \phi} \left[\sin^2 \phi + \frac{n^4 D}{a^2 E' \sin^2 \phi} + \frac{2n^2 D \cot^2 \phi}{a^2 E' k(1+\nu)} \right] w$$

$$- \frac{(1-\nu)E'}{a} \left[\frac{n^2(1-\nu)D}{a^2 E' \sin \phi} - \frac{2n^2 D}{a^2 E' k \sin \phi} \right] s - \frac{(1+\nu) \sin^2 \phi}{a \cos \phi} N_\phi$$

$$+ \frac{n^2}{a^2 \cos \phi} M_\phi + \frac{2nD}{a^3 E' k} V_\theta + \rho h \omega^2 \tan \phi \sin \phi \; w$$

$$\frac{d}{adz} \; V_\theta \sin \phi = \frac{nE'}{a^2} \left[(1-\nu^2) k u + \frac{n}{\cos \phi} (1-\nu^2) k v + \frac{\tan \phi}{E'} \right.$$

$$\left. \left[E'(1-D^2) \left(1 + \frac{n^2 D}{a^2 \sin^2 \phi} \right) \right] w + \frac{D(1+\nu)}{naE'} s \right.$$

$$+ \frac{na\,\nu \tan\phi}{E'} N_{\phi} - \frac{\nu \tan\phi}{E'} M_{\phi} \Bigg] - \frac{\sin\phi}{a} V_{\theta}$$

$$- (\rho h \omega^{2} \tan\phi \sin\phi) v \qquad \text{(C.4)}$$

APPENDIX D

DETERMINATION OF THE INITIAL EIGENVECTOR IN THE ROBINSON-HARRIS METHOD

The initial eigenvector used in the Robinson-Harris method is determined from the latest initial-value solutions in the Holzer method. These initial value solutions are linearly combined to form a solution which will come close to, but not completely, satisfy the homogeneous boundary conditions. This solution, in turn, becomes the first approximation of the eigenfunction in the Robinson-Harris method.

The problem to be solved in determining the complete solution is of the form

$$DX = T \tag{D.1}$$

where D is the matrix of the values of the displacements and stress resultants, X is the vector of unknown weighting coefficients and T is the complete solution. To determine X we evaluate Equation (D.1) at the boundaries, that is,

$$DX = 0 \tag{D.2}$$

or for n=0

$$\begin{vmatrix} u_1(0) & u_2(0) & u_3(0) & u_4(0) & u_5(0) & u_6(0) \\ s_1(0) & s_2(0) & s_3(0) & s_4(0) & s_5(0) & s_6(0) \\ Q_{\phi_1}(0) & Q_{\phi_2}(0) & Q_{\phi_3}(0) & Q_{\phi_4}(0) & Q_{\phi_5}(0) & Q_{\phi_6}(0) \\ u_1(\phi_o) & u_2(\phi_o) & u_3(\phi_o) & u_4(\phi_o) & u_5(\phi_o) & u_6(\phi_o) \\ w_1(\phi_o) & w_2(\phi_o) & w_3(\phi_o) & w_4(\phi_o) & w_5(\phi_o) & w_6(\phi_o) \\ s_1(\phi_o) & s_2(\phi_o) & s_3(\phi_o) & s_4(\phi_o) & s_5(\phi_o) & s_6(\phi_o) \end{vmatrix} \times \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{vmatrix} = 0$$

(similar equations can be written for n=1 and n>1)

(D.3)

If we assume that $x_1=-1$ then we can formulate a system of five equations in five unknowns in terms of the remaining x's. Performing the indicated multiplication in Eqs. (D.3) yields

$$\begin{vmatrix} u_1(0)x_1 + \ldots + u_6(0)x_6 \\ s_1(0)x_1 + \ldots + s_6(0)x_6 \\ Q_{\phi_1}(0)x_1 + \ldots + Q_{\phi_1}(0)x_6 \\ u_1(\phi_o)x_1 + \ldots + u_1(\phi_o)x_6 \\ w_1(\phi_o)x_1 + \ldots + w_1(\phi_o)x_6 \\ s_1(\phi_o)x_1 + \ldots + s_1(\phi_o)x_6 \end{vmatrix} = 0$$

Rearranging terms we get

$$\begin{vmatrix} u_2(0) & \cdots & u_6(0) \\ s_2(0) & \cdots & s_6(0) \\ Q_{\phi_2}(0) & \cdots & Q_{\phi_6}(0) \\ u_2(\phi_o) & \cdots & u_6(\phi_o) \\ w_2(\phi_o) & \cdots & w_6(\phi_o) \\ s_2(\phi_o) & \cdots & s_6(\phi_o) \end{vmatrix} \begin{vmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{vmatrix} = - \begin{vmatrix} u_1(0)x_1 \\ s_1(0)x_1 \\ Q_{\phi_1}(0)x_1 \\ u_1(\phi_o)x_1 \\ w_1(\phi_o)x_1 \\ s_1(\phi_o)x_1 \end{vmatrix} = \begin{vmatrix} u_1(0) \\ s_1(0) \\ Q_{\phi_1}(0) \\ u_1(0) \\ w_1(0) \\ s_1(0) \end{vmatrix} \tag{D.4}$$

Equation (D.4) can be written symbolically as

$$D_{6x5} \times E_{5x1} = C_{6x1} \tag{D.5}$$

Pre-multiply both sides of Equation (D.5) by D transpose to get

$$D^T_{5x6} \times D_{6x5} \times E_{5x1} = D^T_{5x1} \times C_{6x1}$$

or

$$\left[D^T DE\right]_{5x1} = \left[D^T C\right]_{5x1} \tag{D.6}$$

Equation (D.6) consists of five equations in five unknowns which can be solved for $x_2, \ldots, x_6$. The weighing coefficients are then substituted into Equation (D.1) to give us the complete solution, which is,

$$\sum_{i=1}^{6} u_i x_i = u_T$$

$$\sum_{i=1}^{6} s_i x_i = s_T$$

$$\sum_{i=1}^{6} w_i x_i = w_T$$

$$\sum_{i=1}^{6} N_{\phi_i} x_i = N_{\phi_T}$$

$$\sum_{i=1}^{6} M_{\phi_i} x_i = M_{\phi_T}$$

$$\sum_{i=1}^{6} Q_{\phi_i} x_i = Q_{\phi_T}$$

In general, when the initial eigenvector is substituted back into Eqs. (D.2), the boundary conditions will not be satisfied exactly. The residuals from these equations will be used to compute the new distribution factors.

APPENDIX E

DERIVATION OF THE KIRCHHOFF FORCE

Kirchhoff's shear forces, represented by V_ϕ and V_θ, will be derived in this appendix. These forces are caused by the effects of the stress resultants $N_{\phi\theta}$, $M_{\phi\theta}$, and Q_ϕ at the edge of the shell.

Figure 6 illustrates two adjacent incremental segments of the edge of a shell. The location of the angles α, ϕ and θ are given in Figure 7. The moments in Figure 6 can be replaced by force equivalents as shown in Figure 8. From Figure 8 it can be seen, that because of the slight divergence there is a horizontal resultant force F_n sin d acting on the section. Also from Figures 8 and 11 we see that because of the curvature of the element ds we have the resultant force $F_t = \sin\phi\ F_n d\theta$ (E.2). Therefore, the total force per unit length in the direction of $N_{\phi\theta}$ (see Figure 1) is $V_\theta = N_{\phi\theta} - F_t/ds$ (E.3). Substitution of Eqs. (E.1) and (E.2) into (E.3) yields

$$V_\theta = N_{\phi\theta} - (\sin\phi\ F_n d\theta)/(a \sin\phi\ d\theta)$$

which reduces to

$$V_\theta = N_{\phi\theta} - F_n/a$$

From Figure C we see that $F_n = M_{\phi\theta}$ Thus,

$$V_\theta = N_{\phi\theta} - M_{\phi\theta}/a$$

A similar reasoning may be used to derive the force V_ϕ which acts on the edge of a shell in the same direction as Q_ϕ (see Figure 1). We see from Figure 11 that the total shear force on that section is

$$V_\phi = Q + \left(\frac{\partial Fn}{\partial \alpha} d\alpha\right)/ds$$

or

$$V_\phi = Q + \left(\frac{\partial Fn}{\partial \theta} d\theta\right)/ds \tag{E.4}$$

Substituting Eqs. (E.1) into (E.4) we get

$$V_\phi = Q_\phi + \frac{\partial Fn}{\partial \theta} (d\theta/a \sin \phi \, d\phi)$$

From Figure 8 we see that $F_n = M_{\phi\theta}$. Therefore,

$$V_\phi = Q_\phi + \frac{\partial M\phi\theta}{a \sin \phi \partial \theta}$$

Using Eqs. (4.1) from Section 4.2 we get

$$V_\phi = Q_\phi + \left[n/(a \sin \phi)\right] M_{\phi\theta}$$

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REPORT DOCUMENTATION PAGE

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1. REPORT NUMBER: UILU-ENG-78-2027

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle): (6) NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS OCCURRING IN TWO-POINT BOUNDARY VALUE PROBLEMS.

5. TYPE OF REPORT & PERIOD COVERED: (9) Technical Report.

6. PERFORMING ORG. REPORT NUMBER: SRS No. 458

7. AUTHOR(s): R. B. Jackson, A. R. Robinson

8. CONTRACT OR GRANT NUMBER(s): (15) N00014-75-C-0164

9. PERFORMING ORGANIZATION NAME AND ADDRESS:
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University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Project No. NR 064-183

11. CONTROLLING OFFICE NAME AND ADDRESS:
Office of Naval Research, CODE N00014
Department of the Navy
Arlington, Virginia 22217

12. REPORT DATE: (11) January 1979

13. NUMBER OF PAGES: 135 (12)

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office):
Material Science Division
Structural Mechanics Program (CODE 474)
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15. SECURITY CLASS. (of this report): UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release: Distribution Unlimited

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17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Differential equations; numerical integration; edge effects

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An accurate procedure is described for numerically solving two-point boundary value problems which contain growing solutions. The procedure involves the process of reducing the order of a differential equation when one solution is known. Two applications of the procedure are given, a fourth order differential equation with two growing solutions and a system of eighth order differential equations of motion for a hemispherical shell. In both examples before the procedure is started, the equations are re- → next page

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written as a system of first order differential equations. It was found that when solving two-point boundary value problems by the reduction of order method, first order differential equations were generally easier to work with than higher order differential equations. For both applications a computer program was developed to solve the system of differential equations.