

LIMIT PROBABILITIES FOR CRITICAL AGE-DEPENDENT  
BRANCHING PROCESSES WITH IMMIGRATION

BY

HOWARD J. WEINER

TECHNICAL REPORT NO. 27  
JANUARY 15, 1979

PREPARED UNDER GRANT  
DAAG29-77-G-0031  
FOR THE U.S. ARMY RESEARCH OFFICE

Reproduction in Whole or in Part is Permitted  
for any purpose of the United States Government  
Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS  
STANFORD UNIVERSITY  
STANFORD, CALIFORNIA



LIMIT PROBABILITIES FOR CRITICAL AGE-DEPENDENT  
BRANCHING PROCESSES WITH IMMIGRATION

By

Howard J. Weiner

TECHNICAL REPORT NO. 27

January 15, 1979

Prepared under Grant DAAG29-77-G-0031

For the U.S. Army Research Office

Herbert Solomon, Project Director

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS  
STANFORD UNIVERSITY  
STANFORD, CALIFORNIA

Partially supported under Office of Naval Research Contract N00014-76-C-0475  
(NR-042-267) and issued as Technical Report No. 266.

The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

Limit Probabilities for Critical Age-Dependent  
Branching Processes with Immigration

by

Howard J. Weiner

University of California, Davis and Stanford University

1. Introduction.

(1.1) Let  $Z(t)$  denote the number of cells alive at time  $t$  in a standard critical age-dependent branching process ([1], Chapter 4) with absolutely continuous cell lifetime distribution function

$$(1.2) \quad G(t), \quad G(0+) = 0$$

and satisfying

$$(1.3) \quad 0 < \mu \equiv \int_0^\infty t dG(t).$$

Let

$$(1.4) \quad g(t) \equiv G'(t)$$

be the density of  $G$ . Assume

$$(1.5) \quad \int_0^\infty t^{b+1} g(t) dt < \infty$$

with  $b$  given by (1.15).

At the end of each cell life, the original cell disappears, and is replaced by  $k$  new cells with probability  $p_k \geq 0$  and

$$(1.6) \quad \sum_{k=0}^{\infty} p_k = 1 ,$$

satisfying criticality

$$(1.7) \quad \sum_{k=1}^{\infty} k p_k = 1 .$$

Let, for  $0 \leq s \leq 1$

$$(1.8) \quad h(s) \equiv \sum_{k=0}^{\infty} p_k s^k$$

and assume that, for some  $\epsilon > 0$ ,

$$(1.9) \quad h(1+\epsilon) \text{ exists .}$$

This guarantees, in particular, that for  $n \geq 1$ ,

$$(1.10) \quad \sum_{k=1}^{\infty} k^n p_k \text{ exists}$$

and that all derivatives of  $h(s)$  for  $0 \leq s \leq 1$  exist at  $s = 1$  and can be evaluated by interchanging derivatives and summation.

Assume in addition that

$$(1.11) \quad 0 < h''(1) .$$

(1.12) Let  $N(t)$  denote the total progeny born by time  $t$  in a critical age-dependent process satisfying (1.1)-(1.11).

(1.13) Let  $Z_0(t)$  denote the number of cells alive at  $t$  in a cell immigration process in which new-born cells are introduced at renewal epochs. The (random) time between epochs is governed by a continuous distribution function  $G_0(t)$ ,  $G_0(0+) = 0$

with

$$(1.14) \quad 0 < \mu_0 \equiv \int_0^\infty t dG_0(t)$$

and for

$$(1.15) \quad b \equiv \frac{2\mu_0 m_0}{\mu_0 h''(1)} \quad (\text{with } m_0 \text{ defined below})$$

that, as  $t \rightarrow \infty$ ,

$$(1.16) \quad t^{b+2}(1-G_0(t)) \rightarrow 0.$$

At each renewal epoch,  $k$  new cells are introduced with probability  $p_{0k}$  and let, for  $0 \leq s \leq 1+\epsilon$  for some  $\epsilon > 0$

$$(1.17) \quad h_0(s) \equiv \sum_{k=0}^{\infty} p_{0k} s^k < \infty$$

and

$$(1.18) \quad 0 < m_0 = h'_0(1)$$

and

$$h''_0(1) < \infty, \quad h''(1) < \infty.$$

Each new cell introduced at a renewal epoch now is part of the process and initiates, independent of all other cells and the immigration process, a critical age-dependent branching process satisfying (1.1)-(1.11).

(1.19) Let  $N_0(t)$  denote the total progeny by time  $t$  of the immigration process satisfying (1.1)-(1.18).

It is the purpose of this paper to show that for  $k \geq 1$ , as  $t \rightarrow \infty$ ,

$$(1.20) \quad P_{0k}(t) = P[Z_0(t)=k] \sim \frac{c}{t^b}$$

where

$$b = \frac{2\mu m_0}{\mu_0 h''(1)} \quad \text{and}$$

where  $c > 0$  denotes a constant which may depend on  $k$  and under the additional hypotheses that

$$(1.21) \quad p_{0k} > 0 \quad \text{all } k \geq 0,$$

and that there is a unique  $\alpha > 0$  defined by

$$(1.22) \quad p_{00} \int_0^\infty e^{\alpha y} dG_0(y) = 1$$

that, as  $t \rightarrow \infty$ , for  $k \geq 0$ ,

$$(1.23) \quad Q_{0k}(t) = P[N_0(t)=k] \sim ce^{-\alpha t}$$

for  $c$  (depending on  $k$ ) some positive constant. A multi-dimensional version and extension are indicated in Section 3.

## 2. Integral Equations.

For reference later, some results about  $Z(t)$  are listed. See [1], Chapter 4, for example.

Let, for  $0 \leq s \leq 1$

$$(2.1) \quad E(s^{Z(t)}) \equiv F(s, t) .$$

Then, by notation (1.1)-(1.11)

$$(2.2) \quad F(s, t) = s(1-G(t)) + \int_0^t h(F(s, t-u))dG(u) .$$

Under the hypotheses (1.1)-(1.11), denoting

$$(2.3) \quad P_k(t) \equiv P[Z(t)=k] ,$$

then [3]

$$(2.4) \quad P_1(t) = 1 - G(t) + \int_0^t h'(1-P(t-u))P_1(t-u)dG(u)$$

and in general, for  $k \geq 2$ ,

$$(2.5) \quad P_k(t) = f_k(t) + \int_0^t h'(1-P(t-u))P_k(t-u)dG(u) ,$$

where

$$(2.6) \quad P(t) \equiv P[Z(t) > 0] .$$

By [1], [3] respectively,

$$(2.7) \quad P(t) \sim (2\mu)(h''(1)t)^{-1}$$

and for  $k \geq 1$ ,

$$(2.8) \quad P_k(t) \sim \frac{c_k}{t^2} ,$$

where  $c_k > 0$  is a constant, possibly depending on  $k$ .

Denote, for  $0 \leq s \leq 1$ ,

$$(2.9) \quad F(s, t) \equiv Es^{Z(t)} = \sum_{k=0}^{\infty} P[Z(t)=k] s^k .$$

$$(2.10) \quad F_0(s, t) \equiv Es^{Z_0(t)} = \sum_{k=0}^{\infty} P[Z_0(t)=k] s^k .$$

$$(2.11) \quad H(s, t) \equiv Es^{N(t)} = \sum_{k=1}^{\infty} P[N(t)=k] s^k .$$

$$(2.12) \quad H_0(s, t) \equiv Es^{N_0(t)} = \sum_{k=0}^{\infty} P[N_0(t)=k] s^k .$$

Then the following theorem holds.

Theorem 1. Assume (1.1)-(1.18) hold. Then for  $k \geq 0$ , as  $t \rightarrow \infty$ ,

$$(2.13) \quad P[Z_0(t)=k] \sim \frac{c}{t^b}$$

where  $c > 0$  depends on  $k$ .

Proof. By [2]

$$(2.14) \quad F_0(s, t) = 1 - G_0(t) + \int_0^t h_0(F(s, t-u)) F_0(s, t-u) dG_0(u) .$$

For  $\ell \geq 0$  an integer, denote by

$$(2.15) \quad P_{0\ell}(t) \equiv P[Z_0(t) = \ell]$$

$$(2.16) \quad P_\ell(t) \equiv P[z(t) = \ell]$$

and

$$(2.17) \quad P(t) = P[Z(t) > 0] .$$

From the assumptions we note that

$$(2.18) \quad \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F_0(s, t) \Big|_{s=0} = P_{0\ell}(t)$$

and

$$(2.19) \quad \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} F(s, t) \Big|_{s=0} = P_\ell(t) .$$

By (2.18) applied to (2.14) for  $\ell = 0$

$$(2.20) \quad P_{00}(t) = 1 - G_0(t) + \int_0^t h_0(1 - P(t-u)) P_{00}(t-u) dG_0(u) .$$

Define

$$(2.21) \quad R(t) \equiv 1 - G_0(t) + \frac{1}{\mu_0} \int_0^t h_0(1-P(t-u))R(t-u)e^{-\frac{(t-u)}{\mu_0}} du$$

or equivalently,

$$R(t) = 1 - G_0(t) + \frac{e^{-\frac{t}{\mu_0}}}{\mu_0} \int_0^t h_0(1-P(u))R(u)e^{-\frac{u}{\mu_0}} du$$

Taking the derivative w.r.t.  $t$  in (2.21) and simplifying leads to the differential equation

$$(2.22) \quad R'(t) + \frac{(1-h_0(1-P(t)))}{\mu_0} R(t) = f(t)$$

where

$$(2.23) \quad f(t) = o(t^{-b-2}).$$

Expanding  $1-h_0(1-P(t))$  in a Taylor series, using (2.7) and the idea of the proof of Claim IV of ([3] pp 480-481), one may solve for  $R(t)$  asymptotically to get

$$(2.24) \quad R(t) \sim ct^{-b}, \quad \text{where } c > 0$$

is a constant whose value may change from equation to equation. From (2.20), (2.21),

$$(2.25) \quad \begin{aligned} P_{00}(t) - R(t) &= \int_0^t h_0(1-P(t-u))(P_{00}(t-u) - R(t-u))dG_0(u) \\ &\quad + \int_0^t h_0(1-P(t-u))R(t-u)(dG_0(u) - dE(u)) \end{aligned}$$

where

$$(2.26) \quad E(t) = 1 - e^{-\frac{t}{\mu_0}}.$$

Define

$$(2.27) \quad \Delta(t) = |P_{00}(t) - R(t)|.$$

Then, iterating (2.25) repeatedly, one obtains

$$(2.28) \quad \Delta(t) \leq \Delta \cdot G_{0n}(t) + R \cdot |G-E| \cdot U_0(t)$$

for all  $n, t$ , and the dots denote convolution integral, where  $G_{0n}(t)$  is the  $n^{\text{th}}$  convolution of  $G_0$  with itself, and

$$U_0(t) = \sum_{\ell=0}^{\infty} G_{\ell}(t) \sim \frac{t}{\mu_0}.$$

Let  $n \rightarrow \infty$ , then  $t \rightarrow \infty$ , and the law of large numbers and the properties of  $R, G-E, U_0$  yield that

$$(2.29) \quad t^b \Delta(t) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

This yields the result of Theorem 1 for  $P_{00}(t)$ .

The argument for  $P_{01}(t)$  is similar and uses the result for  $P_{00}(t)$ . The general result for  $P_{0n}(t)$  follows by induction using Leibniz' rule for successive differentiation, and is omitted.

Remark: The proof of Theorem 1 of [3] on pp. 482-483 is incompletely justified and would go through by an argument as above.

Theorem 2. Assume (1.1)-(1.22) to hold. Then, for  $k \geq 0$  an integer

$$(2.30) \quad Q_{0k}(t) \equiv P[N_0(t)=k] \sim ce^{-\alpha t}$$

for some  $c > 0$  depending on  $k$ , where  $\alpha$  is as given in (1.22).

Proof. By arguments similar to those used to establish (2.14) by the law of total probability,

$$(2.31) \quad H_0(s, t) = 1 - G_0(t) + \int_0^t h_0(H(s, t-u))H_0(s, t-u)dG_0(u) .$$

The assumptions of the theorem allow derivatives with respect to  $s$  to be taken under the summation sign in (2.11)-(2.12) and that for  $\ell \geq 0$ ,

$$(2.32) \quad \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} H(s, t) \Big|_{s=0} = P[N(t)=\ell] \equiv Q_\ell(t)$$

and

$$(2.33) \quad \frac{1}{\ell!} \frac{\partial^\ell}{\partial s^\ell} H_0(s, t) \Big|_{s=0} = P[N_0(t)=\ell] = Q_{0\ell}(t) ,$$

and note that

$$(2.34) \quad Q_0(t) = P[N(t)=0] = 0 .$$

Applying (2.32)-(2.34) to (2.31) for  $\ell = 0$  yields

$$(2.35) \quad Q_{00}(t) = 1 - G_0(t) + p_{00} \int_0^t Q_{00}(t-u)dG_0(u) .$$

But (2.35) is in the standard form of the integral equation for the mean number of cells at time  $t$  in a Bellman-Harris age-dependent branching process with cell lifetime distribution function  $G_0(t)$  and mean number of progeny per parent of  $0 < p_{00} < 1$ , the subcritical case. (See [1] pp 162-168). Hence [1] as  $t \rightarrow \infty$ ,

$$(2.36) \quad Q_{00}(t) \sim ce^{-\alpha t},$$

where  $c > 0$  may be explicitly evaluated [1], but since no general tractable expression for corresponding constants in the asymptotic form for  $Q_{0\ell}(t)$  seems obtainable, such constants will not be evaluated explicitly, although this proof indicates how they may be obtained recursively.

Applying (2.32)-(2.34) to (2.31) for  $\ell = 1$  yields

$$(2.37) \quad Q_{01}(t) = p_{01} \int_0^t Q_1(t-u)Q_{00}(t-u)dG_0(u) + p_{00} \int_0^t Q_{01}(t-u)dG_0(u),$$

which can be expressed in the form

$$(2.38) \quad Q_{01}(t) = f(t) + p_{00} \int_0^t Q_{01}(t-u)dG_0(u),$$

where, from [1] and (2.37), it follows that, as  $t \rightarrow \infty$ ,

$$(2.39) \quad f(t) \sim ce^{-\alpha t}.$$

By Theorem 1 (i) of ([1] p. 145) and the argument of equation (9)-(11) on page 146 of [1], one then obtains

$$(2.40) \quad Q_{01}(t) \sim ce^{-\alpha t},$$

for a  $c > 0$  which may be evaluated, as indicated in the remark following (2.37).

The rest of the argument proceeds by induction analogous to that used in Theorem 1.

### 3. Multidimensional Case.

Let

(3.1)  $Z_{ij}(t)$  = the number of cells of type  $j$  at time  $t$  starting with one new-born cell of type  $i$  at  $t = 0$  with  $1 \leq i \leq m$  in an  $m$ -type critical age-dependent branching process described as follows. At time  $t = 0$ , one newly born cell of type  $i$  starts the process, for some  $1 \leq i \leq m$ . The cell lives a random time described by a continuous distribution function

$$(3.2) \quad G_i(t), \quad G_i(0+) = 0.$$

At the end of its life, cell  $i$  is replaced by  $j_1$  new daughter cells of type 1,  $j_2$  new cells of type 2, ...,  $j_m$  cells of type  $m$  with probability  $\mathcal{P}_{ij_1 j_2 j_3 \dots j_m}$ .

Define the generating functions, for  $\underline{s} = (s_1, \dots, s_m)$ ,  $\underline{j} = (j_1, \dots, j_m)$ ,  $\underline{s}^{\underline{j}} \equiv (s_1^{j_1}, \dots, s_m^{j_m})$ .

$$(3.3) \quad h_i(s_1, \dots, s_m) \equiv h_i(\underline{s}) = \sum_{(j_1 \dots j_m)} \mathcal{P}_{ij_1 \dots j_m} s_1^{j_1} \dots s_m^{j_m} \equiv \sum_{\underline{j}} p_{i\underline{j}} \underline{s}^{\underline{j}}.$$

Each daughter cell proceeds independently of the state of the system, with each cell type  $j$  governed by  $G_j(t)$  and  $h_j(\underline{s})$ .

Assume, for  $\underline{1} + \epsilon \equiv (1+\epsilon, \dots, 1+\epsilon)$  and  $\underline{1} = (1, \dots, 1)$ ,  $m$ -vectors,

$$(3.4) \quad h_i(\underline{1} + \epsilon) < \infty \quad \text{for } 1 \leq i \leq m.$$

This insures that all moments of  $h_i(\underline{s})$  evaluated at  $\underline{s} = \underline{1}$  may be computed by partial differentiations under the summation sign.

Define, for  $1 \leq i, j \leq m$ ,

$$(3.5) \quad m_{ij} \equiv \left. \frac{\partial h_i(\underline{s})}{\partial s_j} \right|_{\underline{s}=\underline{1}} \equiv h_{ij}(\underline{1})$$

and assume

$$(3.6) \quad m_{ij} > 0 \quad \text{all } 1 \leq i, j \leq m,$$

and let the first moment  $m \times m$  matrix be

$$(3.7) \quad M = (m_{ij}).$$

By standard Frobenius theory ([1], p. 185), there is a largest eigenvalue in absolute value, denoted  $\rho$ , which is positive.

The basic assumption of criticality is that

$$(3.7)(i) \quad \rho = 1.$$

It follows that there are strictly positive eigenvectors  $\underline{u} > 0$ ,  $\underline{v} > 0$  such that (see [4]),

$$(3.7)(ii) \quad \underline{M}\underline{u} = \underline{u} \ , \quad \underline{v}\underline{M} = \underline{v} \ ,$$

$$\sum_{i=1}^m u_i = 1 = \underline{u} \cdot \underline{1} \ ,$$

and

$$\underline{u} \cdot \underline{u} \equiv \sum_{\ell=1}^m u_{\ell} v_{\ell} = 1 \ .$$

Assume

$$(3.7)(iii) \quad \infty > \frac{\partial^2 h_i(\underline{1})}{\partial s_j \partial s_k} > 0 \ , \quad 1 \leq j, k \leq m \ .$$

Denote

$$(3.7)(iv) \quad Q(\underline{u}) \equiv \frac{1}{2} \sum_{i=1}^m \sum_{\ell=1}^m \sum_{r=1}^m \frac{\partial^2 h_i(\underline{1})}{\partial s_{\ell} \partial s_r} u_{\ell} u_r v_i < \infty \ ,$$

where, for  $1 \leq i \leq m$ , for  $a > 0$  (3.9)

$$(3.8)(i) \quad \int_0^{\infty} t^{a+4} dG_i(t) < \infty \ ,$$

and denote,  $0 \leq i \leq m$

$$(3.8)(ii) \quad 0 < u_i \equiv \int_0^{\infty} t dG_i(t) \ ,$$

where  $a > 0$  is given by

$$(3.9) \quad a \equiv \frac{\left( \sum_{\ell=1}^m h_{0\ell}(\underline{1}) u_{\ell} \right) \left( \sum_{r=1}^m \mu_{\ell}^u v_{\ell} \right)}{\mu_0 Q(\underline{u})} \ ,$$

with  $h_{0\ell}(\underline{1}) \equiv \frac{\partial}{\partial s_\ell} h_0(\underline{1})$ , assumed to exist.

Let

$$(3.10) \quad \underline{Z}_i(t) = (Z_{i1}(t), Z_{i2}(t), \dots, Z_{im}(t)) .$$

Let

$$(3.11) \quad \underline{N}_i(t) = (N_{i1}(t), N_{i2}(t), \dots, N_{im}(t))$$

denote the  $m$ -vector with entries

$$(3.12) \quad N_{ij}(t) = \text{total progeny of type } j \text{ born by } t \text{ in} \\ \text{the above critical } m\text{-type process starting with one} \\ \text{new cell of type } i.$$

An  $m$ -type branching process with immigration is defined as follows.  
At renewal epochs with inter-arrival time continuous distribution

$$(3.13) \quad G_0(t) ,$$

$$(3.14) \quad G_0(0+) = 0, G_0(t) < 1 \text{ for all } t > 0 ,$$

satisfying

$$(3.15) \quad t^{4+a}(1-G_0(t)) \rightarrow 0 \text{ as } t \rightarrow \infty$$

$m$ -types of new cells are introduced such that there are  $i_1$  new cells of type 1,  $i_2$  new cells of type 2, ...,  $i_m$  cells of type  $m$  introduced with probability  $p_{0i_1, \dots, i_m}$ . Denote

$$(3.16) \quad h_0(\underline{s}) \equiv \sum_{(i_1, \dots, i_m=0)}^{\infty} P_{0i_1 \dots i_m} s_1^{i_1} \dots s_m^{i_m} \equiv \sum_{\underline{l}} P_{0\underline{l}} \underline{s}^{\underline{l}},$$

and assume

$$(3.17) \quad h_0(\underline{1+\epsilon}) \text{ exists}$$

for some  $\epsilon > 0$ .

Each new cell of type  $i$  initiates an  $m$ -type critical age-dependent branching process [1] independent of all other cells and of the renewal process, satisfying (3.1)-(3.12).

Define, for  $1 \leq i \leq m$ ,

$$(3.18) \quad Z_{0i}(t) \text{ and } N_{0i}(t)$$

to be the number of cells of type  $i$  alive at  $t$  and the total progeny born by  $t$ , respectively, in the  $m$ -type branching process satisfying (3.1)-(3.17), called an  $m$ -type critical age-dependent branching process with immigration.

Denote

$$(3.19) \quad \underline{Z}_0(t) \equiv (Z_{01}(t), Z_{02}(t), \dots, Z_{0m}(t))$$

$$(3.20) \quad \underline{N}_0(t) \equiv (N_{01}(t), N_{02}(t), \dots, N_{0m}(t)) .$$

Theorem 3. Under assumptions (3.1)-(3.12), for  $\underline{k} = (k_1, \dots, k_m)$  a vector of non-negative integers, at least one of which is strictly positive,

$$(3.21) \quad \lim_{t \rightarrow \infty} t^2 P[\underline{Z}_i(t) = \underline{k}] = c > 0$$

$$(3.22) \quad \lim_{t \rightarrow \infty} P[\underline{N}_i(t) = \underline{k}] = d > 0$$

where  $c, d$  are constants which may depend on  $i, k$ .

Proof. The proof follows the one-dimensional case using [4] and is omitted.

Theorem 4. Under assumptions (3.11)-(3.20), for  $\underline{\ell} = (\ell_1, \dots, \ell_m)$  a vector of non-negative integers,

$$(3.23) \quad \lim_{t \rightarrow \infty} t^a p[\underline{Z}_0(t) = \underline{\ell}] = c > 0$$

for some constants  $c$ .

If

$$(3.24) \quad p_{0\underline{\ell}} > 0$$

and there is a unique  $\alpha > 0$  defined by

$$(3.25) \quad h_0(0) \int_0^\infty e^{\alpha u} dG_0(u) = 1,$$

then

$$(3.26) \quad \lim_{t \rightarrow \infty} e^{\alpha t} P[\underline{N}_0(t) = \underline{\ell}] = c > 0.$$

Proof. Theorem 4 follows from Theorem 3 in a proof similar to Theorems 1 and 2, respectively.

Remark: If the quantities  $\underline{Z}_1(t)$ ,  $\underline{N}_1(t)$ ,  $\underline{Z}_0(t)$ ,  $\underline{N}_0(t)$ ,  $\underline{k}$ ,  $\underline{\ell}$  in Theorems 3 and 4 are replaced by corresponding marginal vectors of dimension  $1 \leq d < m$ , the corresponding results of Theorems 3 and 4 hold and are of the same form, since the method of proof is the same, with expressions of the same form.

#### References

- [1] ATHREYA, K.B. and NEY, P.E. Branching Processes. Springer-Verlag, New York (1970).
- [2] JAGERS, P. Age-dependent branching processes allowing immigration, Theory Prob. and Appl. 13 (1968), 225-236.
- [3] WEINER, H. Asymptotic probabilities in a critical age-dependent branching process. Jour. Appl. Prob. 13, (1976), 476-485.
- [4] WEINER, H. On a multi-type critical age-dependent branching process. J. Appl. Prob. 7 (1970), 523-543.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE                                                                                                                                                                                                                                                                                                                 |                       | READ INSTRUCTIONS<br>BEFORE COMPLETING FORM                                     |
|-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------|---------------------------------------------------------------------------------|
| 1. REPORT NUMBER<br>27                                                                                                                                                                                                                                                                                                                    | 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER                                                   |
| 4. TITLE (and Subtitle)<br>Limit Probabilities for Critical Age Dependent<br>Branching Processes with Immigration                                                                                                                                                                                                                         |                       | 5. TYPE OF REPORT & PERIOD COVERED<br>TECHNICAL REPORT                          |
|                                                                                                                                                                                                                                                                                                                                           |                       | 6. PERFORMING ORG. REPORT NUMBER                                                |
| 7. AUTHOR(s)<br><br>Howard J. Weiner                                                                                                                                                                                                                                                                                                      |                       | 8. CONTRACT OR GRANT NUMBER(s)<br><br>DAAG29-77-G-0031                          |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Department of Statistics<br>Stanford University<br>Stanford, CA 94305                                                                                                                                                                                                                      |                       | 10. PROGRAM ELEMENT, PROJECT, TASK<br>AREA & WORK UNIT NUMBERS<br><br>P-14435-M |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>U.S. Army Research Office<br>Post Office Box 12211<br>Research Triangle Park, NC 27709                                                                                                                                                                                                         |                       | 12. REPORT DATE<br>January 15, 1979                                             |
|                                                                                                                                                                                                                                                                                                                                           |                       | 13. NUMBER OF PAGES<br>17                                                       |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)                                                                                                                                                                                                                                                               |                       | 15. SECURITY CLASS. (of this report)<br><br>UNCLASSIFIED                        |
|                                                                                                                                                                                                                                                                                                                                           |                       | 15a. DECLASSIFICATION/DOWNGRADING<br>SCHEDULE                                   |
| 16. DISTRIBUTION STATEMENT (of this Report)<br><br>Approved for Public Release; Distribution Unlimited.                                                                                                                                                                                                                                   |                       |                                                                                 |
| 17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)                                                                                                                                                                                                                                                |                       |                                                                                 |
| 18. SUPPLEMENTARY NOTES<br>The findings in this report are not to be construed as an official Department<br>of the Army position, unless so designated by other authorized documents.<br>This report partially supported under Office of Naval Research Contract<br>N00014-76-C-0475 (NR-042-267) and issued as Technical Report No. 266. |                       |                                                                                 |
| 19. KEY WORDS (Continue on reverse side if necessary and identify by block number)<br><br>Limit                      Process<br>Probability              Immigration<br>Branching                                                                                                                                                         |                       |                                                                                 |
| 20. ABSTRACT (Continue on reverse side if necessary and identify by block number)<br><br>PLEASE SEE REVERSE SIDE                                                                                                                                                                                                                          |                       |                                                                                 |

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 65 IS OBSOLETE  
S/N 0102-014-66011

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

Let  $Z_0(t)$ ,  $N_0(t)$  denote, respectively, the number of cells alive at  $t$  and the total progeny born by  $t$  in a process with a random number of new cells introduced at renewal epochs, each new cell initiating a critical age-dependent branching process. As  $t \rightarrow \infty$ , the forms of  $P[Z_0(t) = k]$  and  $P[N_0(t) = k]$  are obtained for  $k = 1, 2, \dots$  and  $k = 0, 1, 2, \dots$ , respectively. A multi-dimensional version and extension are indicated.