AD-A065 780 NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CONN NEW --ETC F/G 20/1
A GENERAL THEORY OF ACOUSTIC PROPAGATION WITH APPLICATIONS TO S--ETC(U)
JAN 78 J R BRETON
UNCLASSIFIED NUSC-TR-5871 NL

1 OF 5
AD A065780

NUSC Technical Report 5871

DDC FILE COPY AD A065780

LEVEL II

NUSC Technical Report 5871

(10)

A General Theory of Acoustic Propagation With Applications to Strong Acoustic Scattering in the Atmosphere And Ocean

Joseph R. Breton
Design Department

DDC RECEIVED MAR 16 1979 A

16 January 1978

NUSC
NAVAL UNDERWATER SYSTEMS CENTER
Newport, Rhode Island • New London, Connecticut

Approved for public release; distribution unlimited.

79 03 15 026

This work was performed as part of the NUSC Program of Independent Research. The author is grateful to the Naval Underwater Systems Center and particularly to Mr. D. L. Nichols (Code 40) and Dr. W. A. Von Winkle (Code 10) for supporting this study.

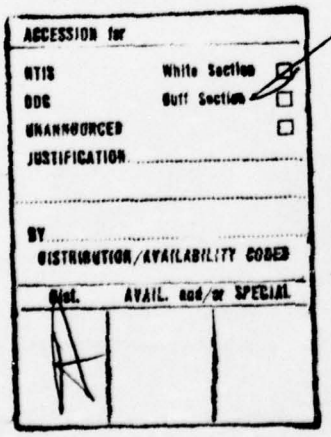
REVIEWED AND APPROVED: 16 January 1978

D. L. Nichols
Associate Technical Director
for Engineering and
Technical Support

W. A. Von Winkle
Associate Technical Director
for Technology

NOTE

This study was submitted by Joseph R. Breton and accepted in 1977 by the University of Rhode Island as a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Electrical Engineering. As a thesis, the material is protected under copyright.



The author of this report is located at the New London Laboratory, Naval Underwater Systems Center, New London, Connecticut 06320.

(14) NUSC

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
TR-5871	(9)	Technical Rept.

(6) 4. TITLE (and Subtitle)

A GENERAL THEORY OF ACOUSTIC PROPAGATION WITH APPLICATIONS TO STRONG ACOUSTIC SCATTERING IN THE ATMOSPHERE AND OCEAN.

5. TYPE OF REPORT & PERIOD COVERED

6. PERFORMING ORG. REPORT NUMBER

(10) 7. AUTHOR(s)

Joseph R. Breton

8. CONTRACT OR GRANT NUMBER(s)

(11) 16 Jan 78

9. PERFORMING ORGANIZATION NAME AND ADDRESS

Naval Underwater Systems Center
New London Laboratory
New London, CT 06320

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

~~710V11~~

11. CONTROLLING OFFICE NAME AND ADDRESS

Naval Underwater Systems Center
(Code 4142)
New London, CT 06320

12. REPORT DATE

(12) 405 p.

13. NUMBER OF PAGES

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)

UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Acoustics, Algorithm, Approximations, Boundaries, Broadband, Communications, Control Theory, Distributed Channel, Dyson's Equation, Field Feed-Back, Filters Models, Narrowband, Operational, Propagation, Scattering, Sea-Surface, Solution. Theory, Wave Equation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report contains the exact (general) solution to adiabatic acoustic propagation in arbitrary media. The exact (general) solution, which depends on the medium, may be obtained by a new algorithm derived from the (new) Field Feed-Back Theorem (FF-BT). Existing theoretical solutions are all subject to at least one of four generic types of approximations. The theory leading to the general solution permits cataloging the models of the literature based on their assumptions and determines quantitatively the circumstances

Go next page

DD FORM 1 JAN 73 1473

405918

20. Abstract (Cont'd)

for which they are valid.

From the illumination shed by the general solution, new insights and (extended) solutions are obtained for the important broadband form of the stochastic wave equation,

$$\left(\nabla^2 - \frac{1}{c^2(\underline{r},t)} \frac{\partial^2}{\partial t^2}\right) p = s,$$

of which the usual narrowband form is a special case. Operational (feedback) solutions not only yield exact solutions for arbitrary or stochastic media with boundaries (moving and stochastic), but also provide new insight into the convergence of the series solutions. As applied to moments, moreover, these solutions allow definition of "equivalent moment media." Dyson's equation is one example.

As an ~~further~~ illustration of the power of these methods, solutions for scattering from interfaces (and particularly the ocean surface) are obtained which agree with and extend the classical solutions. As one specific extension, it is found that for many circumstances the interactive scattering between surface and volume amounts to less than 40 dB of the non-interacting scattering.

The report also establishes specific and applied connections between the study of acoustic propagation and such other disciplines as filter and control theory and statistical communication theory from the viewpoint of modeling the medium as a statistical, physical (distributed) channel.

This report is taken from and reproduces almost exactly the author's doctoral dissertation of the same title, published by the University of Rhode Island in 1977. Professor David Middleton acted as advisor.

TABLE OF CONTENTS

Section		Page
	List of Tables	vii
	List of Figures	ix
	Summary	xiii
	List of Abbreviations and Symbols	xxiii
	Acknowledgement	xxvii
	CHAPTER I - HYDRODYNAMIC AND WAVE EQUATIONS	
1.0	Introduction	1
1.1	Preliminaries	2
1.1.1	The Feedback Theorem	4
1.2	General Theory	5
1.2.1	Hydrodynamic Concepts	6
1.2.1.1	Conservation Laws	10
1.2.1.2	The Divergence Theorem	17
1.3	The First Order Hydrodynamic Equation	19
1.4	The Second Order Hydrodynamic Equation	23
1.5	Hydrodynamic Equations Compared to Wave Equations	29
1.5.1	The Ideal Wave Equation	30
1.5.2	The Stochastic Helmholtz Wave Equation	32
1.5.3	Chernov's Wave Equation	34
1.5.4	Tatarskii's Wave Equations	35

TABLE OF CONTENTS (Cont'd)

Section		Page
1.5.5	Blokhintsev and Mintzer	40
1.5.6	Neubert and Lumley	42
1.5.7	Non-Linear Acoustics	45
1.6	Summary and Remarks	46
	CHAPTER II - APPROXIMATIONS	
2.0	Introduction	51
2.1	Order of Importance: First Order Equation	55
2.2	Order of Importance: Second Order Equation	62
2.3	Comparison with Neubert and Lumley	75
2.4	Circumstances for Validity	77
2.5	Losses and Relaxation Effects	83
2.6	Some Suggested Experiments and Procedures.	86
2.7	Summary	88
	CHAPTER III - THE GENERAL SOLUTION	
3.0	Overview - The Meaning of Solutions	91
3.1	Operational Solution	94
3.2	The General Solution	98
3.2.1	The General Space-Time Solution	98
3.2.2	The General Transform Solution	115
3.2.2.1	The General Transform Solution: First Order	117
3.2.2.2	The General Transform Solution: Second Order	118
3.3	Summary	121

TABLE OF CONTENTS (Cont'd)

Section		Page
	CHAPTER IV - HISTORICAL SOLUTIONS	
4.0	Introduction	123
4.1	Velocity Fields and Linearization: Blokhintsev	125
4.2	Stochastic-Helmholtz Solutions	126
4.2.1	Mintzer	127
4.2.2	Rytov	128
4.2.3	Prokhorov	130
4.3	Quantum Mechanical Solutions	132
4.4	FOM Solution	138
4.5	Summary	141
	CHAPTER V - PROBABALISTIC METHODS	
5.0	Introduction	143
5.1	General Methods	148
5.1.1	Moment Equations	149
5.1.1.1	Means (First Moment)	149
5.1.1.2	Second Moment	150
5.1.1.3	Third Moment	152
5.1.1.4	Fourth Moment and Beyond	153
5.1.2	Operational Solutions	154
5.1.3	Moment Differential Equations	160

TABLE OF CONTENTS (Cont'd)

Section		Page
	CHAPTER V - PROBABALISTIC METHODS (Cont'd)	
5.1.4	Feynman Diagrams	162
5.2	Specialization	164
5.2.1	The Inverse Operator $\hat{G}$	166
5.2.2	Statistical Solutions	174
5.3	Approximations	181
5.3.1	Truncated Sums, B(1)	182
5.3.2	Selected Subsums, B(2)	189
5.3.2.1	First Order Independent Solutions, B(21)	190
5.3.2.2	Second Order Independent Solutions, B(22)	197
5.3.3	Comparison of Solutions	202
5.4	Poisson Statistics	205
5.5	Summary	207
	CHAPTER VI - SURFACE-VOLUME INTERACTION	
6.0	Introduction	213
6.1	Surface Scattering	214
6.1.1	The Average Surface	219
6.1.2	Locally and Globally Coupled Scattering	225
6.1.3	Solutions	226

TABLE OF CONTENTS (Cont'd)

Section		Page
	CHAPTER VI - SURFACE-VOLUME INTERACTION (Cont'd)	
6.1.4	Spectral Response	233
6.1.5	An Alternate Method	234
6.2	Volume Reverberation	237
6.3	Surface x Volume	246
6.3.1	Solutions	248
6.3.2	Criteria	251
6.3.3	The Global Filter	254
6.4	An Example	255
6.5	Summary	259
	CHAPTER VII - CONNECTIONS	
7.0	Introduction	263
7.1	Filters	263
7.2	Control Theory and Estimation	268
7.3	Noise	273
7.4	Communication Theory	276
7.5	Summary	282
	CHAPTER VIII - SUMMARY	285

TABLE OF CONTENTS (Cont'd)

Section	Page
APPENDIX A - THE HYDRODYNAMIC PLANE WAVE	A-1
APPENDIX B - COMPUTABILITY	B-1
APPENDIX C - DIMENSIONS	C-1
APPENDIX D - INVERTING $(\nabla^2 + k_0^2)p$	D-1
APPENDIX E - CORRELATION DECOMPOSITION	E-1
APPENDIX F - THE UNSCATTERED WAVE: $p_u(\underline{r},t)$	F-1
APPENDIX G - EIGENVALUES	G-1
APPENDIX H - CONVERGENCE, INVERTIBILITY, THE RADIATION CONDITION, CAUSALITY AND TAYLOR'S APPROXIMATION	H-1
REFERENCES	R-1
JOURNAL ABBREVIATIONS	R-22
GLOSSARY	GL-1

LIST OF TABLES

Table		Page
1-1	Catalog of Assumptions	47
2-1	Analysis of Local Approximations: General First Order Equation	60
2-2	Analysis of Local Approximations: General Second Order Equation	73
2-3	Circumstances for the Validity of Approximations . . .	80
2-4	Criteria for the Validity of the Stochastic-Helmholtz Equation	84
5-1	Comparison of the General Solution to the Broadband Wave Equation Solution	147
5-2	Approximations for Q Based on the Born Approximations, B(1)	176
5-3	Approximations for $\langle d^n p \rangle$ Based on Selected Subsums B(2)	177
A-1	Plane Wave Parameters (Fixed-Space)	A-18
A-2	Plane Wave Parameters (Material)	A-19
A-3	Plane Wave Parameters (Variational)	A-20

79 03 15 026

LIST OF TABLES (Cont'd)

Table		Page
C-1	Dimensional Relationships	C-2
E-1	The n Structure of k	E-10
F-1	The Unscattered Wave: $p_u = \hat{G}(s)$	F-4

LIST OF FIGURES

Figure		Page
P-1	Approximations in Acoustics	iv
2-1	Relative Importance of $\frac{d(p\|\underline{r})}{dt}$ $\frac{d}{dt}(\ln\rho c^2\|\underline{r})$	64
2-2	Local Effect of Gravity (Equation (2.22a))	68
3-1	Propagation (Deterministic).	92
3-2	Propagation (Stochastic)	92
3-3	Operational Solution: $p_2(\underline{x},t)$	93
3-4	Operational Solution: $p(\underline{x},t)$	93
3-5	Operational Solution: $p(0,t) \rightarrow p(x,t)$	94
3-6	General Operational Solution: $p_2(x,t)$	95
3-7	General Operational Solution: $p(\underline{r},t)$	96
3-8	General Operational Solution: $p_i(\underline{r},t) + p_0$	96
3-9	General Operational Solution with Attenuation	97
3-10	General Propation	114
3-11	The General System Function	114
3-12	The General Operational Solution	115
5-1	Operational Solution to Eq. (5.1)	155
5-2	Operational Solution for <p>	157
5-3	Dyson's Equation, Operational Form	158
5-4	Second Moment Operational Solution	159
5-5	Feynman Diagram of Surface, Volume, and Bottom Interactions	165
5-6	Operational Solution for B(21) Mean	191
5-7	Operational Solution for B(22) Mean	198

LIST OF FIGURES (Cont'd)

Figure		Page
6-1	Geometry (Eqs. (6.4), (6.8))	216
6-2	The Average Surface $\underline{r}_s$	219
6-3	Surface Phase Delay Due to Source	221
6-4	Coordinate System for the Mean Surface	228
6-5	Operational Solution with Absorption	241
6-6	Operational Solution of Moroz Formula	242
6-7	Typical c,d,ε for Ocean and Atmosphere (c_o=1000) . . .	247
6-8	First Moment Solution of S x V	248
6-9	Example Geometry	256
6-10	Relative Effect of S x V Scattering to the Mean Received Pressure (Example)	259
7-1	Transmission as a Filter (I), Eq. (5.1)	263
7-2	Transmission as a Filter (II), Eq. (5.55)	263
7-3	First-Moment Filter, Eq. (5.62)	264
7-4	Feed-Forward Approximation B(11), Eq. (5.81)	265
7-5	The B(21) Approximation, Eq. (5.98)	265
7-6	The B(22) Approximation (I), Eq. (5.125)	266
7-7	The B(22) Approximation (II), Eq. (5.125)	266
7-8	Taylor's Approximation B(21), Eq. (5.105)	267
7-9	Taylor's Approximation B(22), Eq. (5.132)	267
7-10	A Simple Temporal Feedback Loop, Eq. (7.1)	269
7-11	Operational Method for Obtaining Q	272
7-12	Homogeneous Transmission	280

LIST OF FIGURES (Cont'd)

Figure		Page
7-13	Decoding Scheme for Transmission Through Deterministic Media	280
7-14	Decoding Scheme for Transmission Through Stochastic Media	281
A-1	Position Graph	A-13
A-2	Particle Position, $r = x + jae^{xpj(\omega_0 t - k_0 x)}$	A-14
A-3	Particle Velocity, $v = -a\omega_0 e^{xpj(\omega_0 t - k_0 x)}$	A-15
A-4	Dilatation Rate, $\nabla \cdot \underline{v}$	A-16
A-5	Acoustic Pressure, p	A-17
B-1	Operational Solution of Eq. (5.55)	B-1
G-1	Eigenvalue Representation of Harmonic Propagation .	G-6
G-2	Operational Solution - State Variable Form	G-7
H-1	Plot of $\frac{k^2-k_0^2}{(k-k_0)^2}$ versus k	H-6

SUMMARY

Sound, so immediate and general in our experience, lends itself as immediately and generally to direct experimentation. By sound, we communicate, detect dangers, are annoyed, are delighted, recognize and are recognized. Indeed we learn, through experimentation, an entire dimension of life. It is hardly surprising then, that so basic an experience should influence the science of sound, acoustics.

But the experiments of acoustics have not been sure guides. Now, one result is obtained where before, another, perhaps very different, was obtained. This fits our experience well, but it has hampered the science of acoustics. We are accustomed to raising our voice against a stiff breeze; but where is the "breeze" in acoustical theory?

Classical acoustics, finding no solutions to the equations of continuum mechanics, approximated these to obtain the wave equation, $\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} p = 0$, solutions of which are available. The approximation has proven remarkably successful in many situations. But not, of course, for many others. More recently, the wave equation has been expanded and so the expansion has covered many more experimental situations. But, of course, many experiments still deviate substantially from predicted results.

A. General Aims of the Report

This background has set the general purpose of this report: to produce a general theory which explains the validity of previous formulations and which can set circumstances for which these models can be used reasonably. Chapters I and II are devoted to this purpose.

Let us be precise in our terms. A theory we take to be a logically consistent elaboration of a set of axioms. The theory of numbers, of measure, provide illustrations. A model in contrast is a mathematical construction (not necessarily logically consistent) to suit a utilitarian purpose. Because any approximation implies logical inconsistencies, formulations using approximations must necessarily be classed as models rather than theory. So it is that the boundary of a theory are its axioms, whereas the boundary of a model are its approximations. The formulations of classical acoustics fall, without exception, into the class of models, rather than of theory.

The model approximations may be classified into four generic types: hydrodynamic (I); differential (II); truncation/selection (III); and analytical (IV). By the hydrodynamic approximation is meant a (deliberate) misinterpretation of the hydrodynamic derivatives; by differential, the exclusion of terms in the governing differential equation; by truncation/selection, the approximation of exact solutions by various canonical forms; by analytical, the use of approximations usually required to obtain closed form solutions. These approximations are generic in the sense that they may be made independently. Available solutions in the literature of random scattering usually contain elements of I - IV. Solutions by ray tracing certainly contain I, II. Diagramatically, we have Figure P-1.

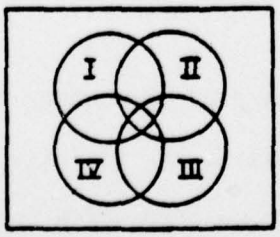


Fig. P-1 - Approximations in Acoustics

Within each class, a subclass of solutions may be exact. For instance, within the class I II, the plane wave solution $p = p_0 e^{j(\omega t - \underline{k}\cdot\underline{r})}$ is an exact solution of the wave equation. The general theory should exhibit the class I and II approximations which keep the plane wave solution from being exact physically.

But of a proper acoustical theory more can be expected than merely ordering the rationalizations of a set of models. We can expect the theory to produce insights and methods of manipulating the models. So this thesis is a thesis of foundations -- it opens lines of investigation into standard problems, suggests extensions, and especially connections with other disciplines. The latter part of the Thesis, Chapters V - VII, are devoted to this purpose. For example in Chapter VI, we set ourselves to determine the effect of coupled surface and volume scattering.

We state the acoustic problem as follows: a physical medium, perhaps in motion, is driven by a source, s, located in a region $\{\underline{r}_0\}$. The acoustic pressure, p, at some point, $\underline{r}$, is required. In the general theory, the answer is given in terms of the source, the initial state of the medium, its subsequent state, and interfering sources.

But a complete description of these latter is scarcely ever available. So the answer generates a second question: What information, H, is required to control the error in p; or stated otherwise, with fixed information, H, of the medium (et al.), what error in p can be expected? Such questions point to relevant experiments and techniques which, though not a primary aim of this report, are occasionally sketched concisely where appropriate.

These considerations suggest a stochastic treatment of the relevant equations. The governing differential equations may be so treated (i.e., as Langevin equations) in the general case. Particular models are the subject of much recent study. Individual solutions of these equations can be considered realizations from an event space whose importance depends on their measure (often zero). Solution in terms of moments provides a meaningful approach.

Our method of proceeding is as follows. To classify and order the approximations of the literature we develop the adiabatic hydrodynamic equations. This procedure reveals type I and II approximations. Moreover, this procedure makes apparent the utility of the stochastic viewpoint, which is again underscored by the treatment of solutions in Chapters III and IV. Type III and IV approximations are more clearly revealed in Chapters V - VII, which deal with solutions to the stochastic wave equations. In Chapter III we give the general solution to acoustic propagation. From the solution, especially in the "feedback" form, we gain new methods and insights for manipulating the usual models and understand more readily connections with other disciplines (Chapter VII).

In terms of the extended literature[1] on this subject (which includes electromagnetic (EM) propagation), our methods are oriented more

---

[1]A recent paper [95] summarizes the EM effort mostly and dwells on transport theory. According to its author, "analytical theory ... is mathematically rigorous in the sense that in principle all the multiple scattering, diffraction and interference effects can be included. In practice, however, ... various theories which yield useful solutions are all approximate ... . Transport theory, on the other hand, ... is heuristic and it may lack the mathematical rigor of the analytical theory." p. 1033. P. Ishimaru will soon have published Wave Propagation and Scattering in Random Media, New York: Academic Press.

to the "multiple scattering" or "analytical" approach than to "radiative-transfer" or "transport theory."

B. New Results

The originality of the Thesis comes first from its statement of the general adiabatic acoustic case and its solution. We note the following important new results:

1. The general solution of adiabatic acoustic propagation (Chp. III)
   a) The general solution in transforms (Sec. 3.2.2)
   b) The exact, closed solutions of plane acoustic isothermal transmission in a physically homogeneous medium. This shows real propagation to be non-linear even in the simplest cases. The physical error of the classical plane wave solution. (Appendix A)
   c) The propagation of non-acoustic fields (Sec. 1.4)
2. The operational form of the general solution (Sec. 3.1)
   a) The general feed-forward operator (Eq. 3.44)
   b) The specific structure of the space/time filter characterizing acoustic transmission (Eq. 3.42)
3. The place of generalized functions and the extended divergence theorem in acoustic theory The results are used to solve the stochastic boundary problem, using a method previously advocated by Middleton [136]. (Sec. 1.2.1.2)
4. The distinction between hydrodynamic and partial derivatives (Sec. 1.2.1)

Secondly, new results have been obtained by comparing the general situation with specific models. We indicate here:

5. A complete listing of conditions and circumstances for the validity of the usual models of the literature (Chp. II)
   a) "Global" effects (Chp. II)
   b) The acoustical effect of gravity, with possible meteorological implications (Fig. 2-2)
   c) A new relaxation effect in air at high acoustic intensity (Fig. 2-1)
   d) Some suggested new experiments (Sec. 2.6)
   e) The effective versus true propagation velocity (Sec. 2.4)
6. The limitations of beam patterns, wave structure and similar notions (Sec. 1.6, Table 2-2)

Finally, new results have been obtained for a specific model, the stochastic wave equation. We signal the following:

7. The exact operational solution; its advantages over closed forms (Sec. 5.1.2)
   a) Separation of the operational solution into purely geometrical and medium operators (Sec. 5.1.2)
      (1) Definition of moment-equivalent media (Sec. 5.1.2)
      (2) An operational solution for Q, the mass operator (Eq. 7.23)
      (3) Application of system identification techniques to propagation (Sec. 7.2)
   b) The feed-forward operator $\hat{G}$ (Sec. 5.2.1)
      (1) The distinction between $(\nabla^2 + k_o^2)e^{-jk_o\rho}/4\pi\rho$ and $\delta(\rho)$ (Appendix H)
      (2) Definition of $\hat{G}'$ which allows operator commutation. (Eq. 5.54)

c) Feed-forward and feed-backward analogies in <p> (Sec. 7.1)

(1) Taylor's approximation in operational form (Figs. 7-8, 7-9)

8. The broadband solution; explicit solutions for the first, second, third and fourth moments (Eq. 55, Sec. 5.5.2)

a) The general solution to the simple refraction problem (deterministic) with boundary (Eq. 6.97)

b) A canonical decomposition of approximations (B) (Sec. 5.2.2.2)

(1) The Born solution with interfaces (Sec. 5.3.1)

(2) A criterion for the validity of the first Born approximation B(11) (Sec. 5.3.2.1)

(3) A convergent series for the mass operator, Q (Sec. 5.3.2.2)

(4) Feynman diagrams which include interfaces (Fig. 5-5)

(5) A new category of approximations: first order independent B(21) (Sec. 5.3.2.1)

(6) Generalization of the Bourret solution (Sec. 5.3.2.2)

(7) A new canonical form for second moments which generalizes to higher moments (Sec. 5.1.1.2)

(8) The recognition that higher moments accumulate the errors made in approximating lower orders (Sec. 5.2.2)

(9) An approximation for higher order moments (Eq. 5.123)

(10) Comparison of approximations: relative errors (Sec. 5.3.3)

(11) The differential equation for the moments (Sec. 5.1.3)

(12) The recognition that the stochastic wave equation is a "noise-free" case and considerations in generalization to the noisy case (Sec. 5.1 and Sec. 7.3)

9. A criterion for surface-volume interactions (Eqs. 6.106, 6.107, 6.110-6.112)

a) The condition for non-interaction of interfaces and volume for the first Born approximation B(11) (Sec. 5.3.1, 6.3.1)

b) A volume reverberation model with frequency dependent loss (Sec. 6.2)

c) The global filter decomposition into surface and volume subfilters (Sec. 6.3.3)

10. The classical acoustic channel characterization as linear, having memory, noiseless, which affects phase variation, and which does not preserve measure. These come from considering the distributed nature of the channel. (Sec. 7.4)

a) The linear filter of the stochastic channel and its impulse function (Eq. 7.26)

b) Formulation of the "decoding" problem of acoustic transmission (Sec. 7.4)

The Thesis turns on two hinges: 1) the development of the general solution and 2) the operational solutions (feedback[1] forms). From the first, the conditions for the validity of the usual models are obtained. From the second we develop new insights and methods of

[1]The feedback forms are explicit, new algorithms which have been recently called field feed-back theorems (FF-BT). An initial exposure has been given in the 1977 International Symposium on Information Theory, David Middleton and J. R. Breton, "Stochastic Channels as Generalized Communication Networks," IEEE Catalog No. 77CH 1277-3 IT Session DI, October 10-14, 1977, Ithaca, NY.

(12) The recognition that the stochastic wave equation is a "noise-free" case and considerations in generalization to the noisy case (Sec. 5.1 and Sec. 7.3)

9. A criterion for surface-volume interactions (Eqs. 6.106, 6.107, 6.110-6.112)
   a) The condition for non-interaction of interfaces and volume for the first Born approximation B(11) (Sec. 5.3.1, 6.3.1)
   b) A volume reverberation model with frequency dependent loss (Sec. 6.2)
   c) The global filter decomposition into surface and volume subfilters (Sec. 6.3.3)
10. The classical acoustic channel characterization as linear, having memory, noiseless, which affects phase variation, and which does not preserve measure. These come from considering the distributed nature of the channel. (Sec. 7.4)
   a) The linear filter of the stochastic channel and its impulse function (Eq. 7.26)
   b) Formulation of the "decoding" problem of acoustic transmission (Sec. 7.4)

The Thesis turns on two hinges: 1) the development of the general solution and 2) the operational solutions (feedback[1] forms). From the first, the conditions for the validity of the usual models are obtained. From the second we develop new insights and methods of

[1]The feedback forms are explicit, new algorithms which have been recently called field feed-back theorems (FF-BT). An initial exposure has been given in the 1977 International Symposium on Information Theory, David Middleton and J. R. Breton, "Stochastic Channels as Generalized Communication Networks," IEEE Catalog No. 77CH 1277-3 IT Session DI, October 10-14, 1977, Ithaca, NY.

extending the classical solutions of the stochastic wave equation to the broadband case and to bounded media (including moving and random surfaces).

Remembering that the medium (here the ocean or atmosphere) is necessarily regarded as a communication channel, we note at once that our propagation results apply to a wide variety of traditional subjects in the general discipline of electrical engineering, among others, telecommunication, detection and associated classification problems, localization, remote sensing, etc. In addition, it is well known that physical channels through which information is passed can be regarded as space-time filters, or generalized networks, analogous to the temporal filters or networks of conventional circuit and control theory. This Thesis presents quantitatively the beginning of an analogous theory for such distributed filters or networks, specifically taking into account the physical (and distributed) nature of the acoustic channels.

Finally, to the extent that electromagnetic propagation may be modeled in scalar form, it also can be described by the broadband propagation and scattering models developed here, particularly in Chapters V - VII.

LIST OF ABBREVIATIONS AND SYMBOLS

a,b	scalar parameters or functions defined in context
$\underline{a},\underline{b}$	vector or vectorial functions defined in context
B	second moment equivalent medium; canonical approximation
c^2	ratio of dynamic material pressure to dynamic material density
C	channel capacity
d	scatterer; differential element
D	differential operator
E	deformation tensor
$\underline{f}$	body force/unit mass
f	any function; a probability density function
$\hat{F}$	Fourier transform
$\hat{G},\hat{G}'$	Operators
H	the system function; the medium; information
i	an index
j	an index; imaginary number (0,1)
J	Jacobian; dilatation
k	wave number
$\hat{L}$	a linear operator
m	mass; an index; moment; criterion
$\hat{n}$	normal unit vector
n	an index; $\int_S \underline{v}\cdot\hat{n}\, dS$
p	dynamic pressure (acoustic)

LIST OF ABBREVIATIONS AND SYMBOLS (Cont'd)

p_0	background pressure
p_1	turbulent pressure
p_2	total pressure
p_u	unscattered pressure (wave)
p_s	scattered pressure (wave)
Q	first moment equivalent medium
$\underline{r}$	particle position, (position vector)
$\underline{r}_s$	surface vector
$\underline{\bar{r}}_s$	mean surface vector
R	correlation
s	acoustic source
S	surface
t	time
$\underline{T}$	stress tensor; third order equivalent medium; a communications channel
u	$-\{\rho c^2(\nabla\cdot\underline{v}) + \underline{v}\nabla\cdot(p_0 + p_1) + \frac{d}{dt}((p_0 + p_1)\|r)\}$; waveform
$\underline{v}$	particle velocity
V	volume
W	fourth order equivalent medium
$\underline{x}$	rest position of particle; particle
$\underline{z}$	surface deviation from mean
Δ	increment
ξ	random variable (operator)
δ	impulse function, delay
θ	angle
λ	wave length

LIST OF ABBREVIATIONS AND SYMBOLS (Cont'd)

$\emptyset$	angle; phase
ρ	density
τ	time delay; temperature
ω	radian
$(\hat{\cdot})$	operator, unit vector
$(\cdot)\cdot(\cdot)$	dot product
$(\cdot)x(\cdot)$	cross product, $(\cdot) \times (\cdot)$
$\frac{d^n}{dt^n}[(\cdot)\|x]$	material derivative (Lagrangian)
$\frac{d^n}{dt^n}[(\cdot)\|r]$	fixed space derivative (Eulerian)

ACKNOWLEDGMENT

J. H. Newman, the great educator of the 19th century, replied, when accosted by a colleague who complained about the antics of his students, that they might be sent to him because he would know how to employ them. The author of this Thesis is deeply grateful to his major advisor, Professor David Middleton, who with great tolerance, tact, and energy knew how to "employ" him. His suggestions, directions, discussions, and experience shaped this effort, especially in its large design, while not neglecting the finer structure.

The particular interest of Mr. Donald L. Nichols and Dr. William Von Winkle has been a constant and solid support. I am likewise indebted to our common employer, the Naval Underwater Systems Center (NUSC), which has provided the major financial support for my graduate program. My special praise and gratitude goes to Nancy O'Neill and her helpers who have coped valiantly with a long and difficult manuscript.

It is also a pleasure to acknowledge the help given by the faculty of the University of Rhode Island, and in a special way Professor Charles Polk, Head of the Department of Electrical Engineering.

In a totally different, but equally important, way I am indebted to my family for their continued patience and support.

And finally I thank J. H. S. who made it all possible.

I. HYDRODYNAMIC AND WAVE EQUATIONS

1.0 Introduction

The study of sound propagation in real media may properly be said to have begun with Christian Huygens in 1678 [94]. Huygens' work, resting on an Aristolelian concept of motion, created considerable controversy, an attribute which has persisted with this subject to the present.[1] [24, 185, 170, 48, 168]

The reasons for the controversy can be partly explained by the numerous and diverse assumptions made. We shall see that much of the literature presumes a perfect fluid; practically all of it presumes a Navier-Stokes fluid.

It is not a little disconcerting then to find an authority like Truesdell [48, p. iv] writing, "...the inability of the Navier-Stokes fluid to support wave motions merely illustrates its exceptional character, untypical of real fluid behavior." Yet much of the theory is solidly useful; certainly many real media over a fairly broad range of circumstances permit more or less accurate representation by the models in current use. It is in judging the limits and accuracy of applicability that we enter into difficulties and controversy.

It is one cardinal objective of this report to examine basic assumptions and thereby reveal conditions and circumstances for the validity of the various formulations found in the literature.

---

[1] [24] contains an interesting account of the controversy applied to diffraction, pp. 370-375.

An adequate theory of wave propagation in a turbulent medium should account for observed effects: beam spreading, beam wander, loss of coherence, intensity fluctuations, angle of arrival variations and short-pulse effects. Furthermore, for small amplitude waves, the propagation should be mainly linear.

Recent articles [62, 168] classify the various models according to their success in predicting measurements of long range laser (i.e., EM) propagation in air. They find the geometric models to be valid only for the shortest ranges; the Rytov and early Tatarskii work for ranges to 1 km; and the more recent work for ranges to 30 km.

In this chapter we present an alternate classification, more suitable for the acoustic case. We develop, with considerable care, a general theory from basic hydrodynamic considerations. From these we initially develop three constitutive equations: state, mass (continuity) and momentum; we continue to canonical formulations of the first and second order hydrodynamic equations. The latter serves as a standard for comparing and classifying the historical models, especially those of the last thirty years. In the later sections of this chapter we find, not surprisingly, that all the historical models may be obtained by suitable approximations from the general theory developed here. Circumstances in which the assumptions are valid (known only from the experimental situation) correspond to the range of the model's validity. For this reason we expect the results of this chapter (and the next) to be useful to experimenters as well as theorists.

1.1 Preliminaries

Certain words, notations or ideas require some initial clarification:

a) The word, "homogeneous," often used to refer either to a physical or a statistical property, is here reserved for the physical property. "Isotropic" is used to refer to the statistical property.

b) Complex notation is used whenever appropriate (i.e., for linear propagation). Real notation is used for non-linear propagation.

c) Operations (such as differentiation) are assumed to hold over a generalized function space.

d) Given $\hat{G}$, an operator, $\hat{G}_1\hat{G}_2$ signifies $\hat{G}_1$ operating on $\hat{G}_2$ while $\hat{G}_1(\hat{G}_2)$ indicates that each operator operates separately on its subject. For instance, if $\hat{G}_1 = \int_0^t (\cdot)dt$,

$$\hat{G}_1\hat{G}_2 f(t_1,t_2) = \int_0^t \int_0^{t_1} f(t_1,t_2)dt_2\, dt_1 ,$$

whereas

$$\hat{G}_1(\hat{G}_2 f(t_1,t_2)) = \int_0^t \int_0^t f(t_1,t_2)dt_1 dt_2 .$$

We abbreviate $\hat{G}_1\hat{G}_2\ldots\hat{G}_n$ as $\hat{G}^{(n)}$, and $\hat{G}_1\,\hat{G}_2(\ldots(\hat{G}_n))$ as $\hat{G}^n$.

Other notation follows mostly conventions of the literature. The reader should consult the glossary for the definition of notation with which he is unfamiliar.

1.1.1 The Feedback Theorem

Suppose a class of functions, {f}, and an associated norm such that $||f|| < \infty$. Then an operator, $\hat{G}$, is called bounded[1] if $\hat{G}f = g \varepsilon \{f\}$ for all f.

The feedback theorem states that for a bounded operator, $\hat{G}$,

$$(1 - \hat{G})^{-1} = \sum_{n=0}^{\infty} (\hat{G})^n = [1 + \hat{G} + \hat{G}\hat{G} + \ldots + \hat{G} \overset{n}{\ldots} \hat{G} + \ldots] \qquad (1.1)$$

This is easily shown by operating on both sides with $1 - \hat{G}$. On the left hand side (LHS) of Eq. 1.1, for any relevant function f we get

$$(1 - \hat{G})(1 - \hat{G})^{-1} f = f$$

while

$$[1 - \hat{G}] \sum_{n=0}^{\infty} (\hat{G})^n [f] = f + \hat{G}f - \hat{G}f + \hat{G}\hat{G}f - \hat{G}\hat{G}f + \ldots = f \ .$$

This last statement is true sufficiently for all absolutely convergent sums and for others as well (e.g., $1 - 1/2 + 1/3 - 1/4 + \ldots$).

We shall need an enlargement of the usual definitions to allow {f} to include generalized (impulsive) functions in space and time. This enlargement also requires an extended notion of convergence (convergence almost everywhere) as may be seen from

$$[1 - G]^{-1} \delta(\underline{r} - \underline{r}_o) = \delta(\underline{r} - \underline{r}_o) + \hat{G}\delta(\underline{r} - \underline{r}_o) + \hat{G}\hat{G}\delta(\underline{r} - \underline{r}_o) + \ldots$$

---

[1] The reader may consult [76] p. 69.

which may be said to converge everywhere except at $\underline{r}_o$. We are thus led to separate from the concept of convergence the idea of form, which may be correct even where the functional value is unbounded.

In Chapter III the feedback theorem is used to obtain the general solution to adiabatic acoustic propagation. It illuminates the problem of convergence in the approximate series solutions of Chapter IV. In Chapters V and VI it is used to obtain approximations for the broadband case. Most importantly, the theorem leads to the core of this thesis, operational (feedback) solutions. Because of this latter relationship, when $\hat{G}$ is restricted to a space-time operator, the feedback theorem is called the Field Feed-Back Theorem (FF-BT).

1.2 General Theory

Noll [156] has shown that the representation of matter as a continuum can be set into entire agreement with the statistical view of matter as composed of classical molecules [194]. We therefore accept the point of view of continuum mechanics, and in particular, the principles of hydrodynamics. Here and throughout we adopt a fundamental set of assumptions:

(1) The medium is a fluid continuum (perhaps moving) (A1)

(2) Propagation is adiabatic. (A2)

The first assumption allows discontinuities in any fluid property except position. Various authors [188, 40, 22] include in their fundamental development the thermodynamic considerations of entropy and temperature. Others [40] have sought to relate fundamental equations to concentrations of salinity. These developments aim laudably at relating inhomogeneities in the medium to oceanographic observables.

Here, however, we limit our concern to only one observable: the dynamic pressure, $p(\underline{r},t)$. The variations in this parameter reflect the many variations of the media, and indeed, the variation in temperature, salinity, currents, tides, gravity, etc., are of interest here only to the extent that they influence the dynamic pressure.

The second assumption does not hold for large amplitudes or whenever heat transport is significant over time intervals of interest. In our development we assume the total pressure to be adiabatic and allow the statistical properties of the acoustic pressure to be non-stationary. Assumption (A2) relies on the observation that small amplitude acoustic waves propagate essentially adiabatically while thermal processes in the ocean and atmosphere are generally very slow (greater than minutes typically).[1]

A medium-related theory has somehow to characterize the medium and incorporate its dynamics. We turn, accordingly, to a brief development of hydrodynamical results necessary to what follows.

1.2.1 Hydrodynamic Concepts

Consider initially the relaxed medium, that is, the medium in equilibrium undergoing no relative motion. Call this the medium "at rest." Because the medium is assumed continuous (A1), the particles are in one-to-one correspondence with their position. This position we call the "rest" position of the particle. The correspondence between position and particle enables one to label or designate each particle

---

[1]The conditions for these assumptions are nicely explained in [92], p. 411.

by its rest position $\underline{x}$.[1] Thus, by the phrase, "the particle $\underline{x}$," we mean "the particle which in the relaxed medium was at rest position $\underline{x}$."

Suppose now the particle $\underline{x}$ is found at some time, t, to be at position $\underline{r} = \underline{r}(\underline{x},t)$. We define the vector

$$\underline{v}(\underline{r}(\underline{x},t)) \equiv \frac{d}{dt}\,(\underline{r}(\underline{x},t)|\underline{x}) = \lim_{\Delta t \to o} \frac{\underline{r}(\underline{x},t+\Delta t) - \underline{r}(\underline{x},t)}{\Delta t} \qquad (1.2)$$

as the velocity of particle $\underline{x}$ at time t (the particle velocity).

The derivative in Eq. (1.1) associated with the particle is termed the material (Lagrangian, convective) derivative. It is commonly associated with another derivative, $\frac{d}{dt}((\cdot)|\underline{r})$, which we have called the fixed-space (Eulerian) derivative. It is usual[2] to denote

$$\frac{d}{d\alpha}\,((\cdot)|\underline{x}) = \frac{d}{d\alpha}\,(\cdot) \qquad (1.3a)$$

and

$$\frac{d}{d\alpha}\,((\cdot)|\underline{r}) = \frac{\partial}{\partial\alpha}\,(\cdot) \qquad (1.3b)$$

and to treat these derivatives by the usual rules of the calculus.

Unfortunately, this leads not only to great confusion, but even to contradiction. For instance, by the calculus, we must have $\frac{\partial r}{\partial r} = 1$, whereas $\frac{d(\underline{r}|\underline{r})}{d\underline{r}} = 0$.

---

[1]The concept of rest position is not strictly necessary since one may start with initial position $\underline{x}(t_0)$. It is important, however, not to confuse "rest" position with "initial" position for, generally, they will not coincide. The differences may be readily appreciated from the development of ideal wave propagation (Appendix A).

[2]See [147], pp. 234-239.

Clarity consequently requires a distinction be made between the hydrodynamic derivatives and those of the calculus (variational). To avoid confusion, we have introduced a specific notation for the hydrodynamic derivatives and now make the following definitions:[1]

$$\frac{d}{d\alpha}((\cdot)|\underline{r}) \equiv \text{the derivative of } (\cdot) \text{ with respect to } \alpha, \text{ holding the spatial coordinate, } \underline{r}, \text{ fixed. (fixed-space, Eulerian).}$$

$$\frac{\partial}{\partial\alpha}(\cdot) \equiv \text{the derivative of } (\cdot) \text{ with respect to } \alpha, \text{ holding all other variables fixed (partial).}$$

$$\frac{d}{d\alpha}((\cdot)|\underline{x}) \equiv \text{the derivative of } (\cdot) \text{ with respect to } \alpha, \text{ holding the material coordinate, } \underline{x}, \text{ fixed. (material, Lagrangian, corrective).}$$

$$\frac{d}{d\alpha}(\cdot) \equiv \text{the derivative of } (\cdot) \text{ with respect to } \alpha, \text{ with other variables changing with respect to } \alpha, \text{ (whole).} \qquad (1.3c)$$

The material and fixed-space temporal derivatives are related through a celebrated relationship attributed to Euler. The relationship involves three different types of derivatives which may be seen from the following derivation. For any function $f = f(\underline{r}(\underline{x}),t)$ we have

$$\frac{d(f|\underline{x})}{dt} = \lim_{\Delta t\to 0} \frac{f[((\underline{r}+\Delta\underline{r})|\underline{x}),t+\Delta t] - f(\underline{r}|\underline{x},t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{f[((\underline{r}+\Delta\underline{r})|\underline{x}),t+\Delta t] - f(\underline{r}|\underline{y},t+\Delta t)+f(\underline{r}|\underline{y},t+\Delta t) - f(\underline{r}(\underline{x}),t)}{\Delta t}$$

[1]These definitions may be better appreciated in the context of Appendix A.

$$= \lim_{\Delta t \to o} \frac{f[((\underline{r}+\Delta\underline{r})|\underline{x}),t+\Delta t] - f[(\underline{r}|\underline{y}),t+\Delta t]}{|\Delta(\underline{r}|\underline{x})|} \frac{|\Delta(\underline{r}|\underline{x})|}{\Delta t}$$

$$+ \lim_{\Delta t \to o} \frac{f(\underline{r}|\underline{y},t+\Delta t) - f(\underline{r}|\underline{x},t)}{\Delta t}$$

$$= |\nabla\underline{f}|(\cos\theta)\ |\underline{v}(\underline{r}(\underline{x}))| + \frac{d(f|\underline{r})}{dt}$$

where y is the particle at $\underline{r}$ at time $t+\Delta t$, and θ is the angle between $\nabla\underline{f}$ and $\underline{v}$. The last step can be appreciated from

$$\cos\theta = (\nabla\underline{f})\cdot\hat{\underline{v}} = \lim_{\Delta\underline{r}\to 0}\left(\frac{\dfrac{f(\underline{r}+\Delta\underline{r}|\underline{x},t+\Delta t) - f(\underline{r}|\underline{y},t+\Delta t)}{\Delta\underline{r}}}{|\nabla\underline{f}|}\right)$$

and remembering that $\Delta\underline{r}$ has the same direction as $\underline{v}$ in the limit. Here $(\hat{\cdot})$ denotes a unit vector. We may thus write

$$\frac{d(f|\underline{x})}{dt} = \frac{d(f|\underline{r})}{dt} + \underline{v}\cdot\nabla\underline{f} \qquad (1.4)$$

as the Eulerian relation.

We make three observations. First, the derivative $\frac{d(f|\underline{r})}{dt}$ generally involves more than one particle. Second, if $f(\underline{r},t)$ is a surface of material particles, then

$$\frac{d(f|\underline{r})}{dt} = -\underline{v}\cdot\nabla\underline{f} \qquad (1.5)$$

(i.e., $\frac{d(f|\underline{x})}{dt} \equiv 0$ is a necessary condition for a material surface). Lastly, we note that assumption (A1) does not guarantee the existence of derivates. Still, we may restrict f to a generalized function (impulsive) in both space and time, so that Euler's relationship will always hold in the completed function space.

1.2.1.1 Conservation Laws

In a realistic fluid, the conservation of mass and momentum yield a continuity equation and an equation of motion (Cauchy's), respectively. Because approximations from the general theory usually involve these equations, they will be developed here from first principles.

Let V_o be a material neighborhood at rest of the particle $\underline{x}$, and V_t be same material neighborhood at time t. Define $J(\underline{x}) \equiv \lim_{V \to \underline{x}} \frac{V_t}{V_o}$ called the dilation of $\underline{x}$. We first demonstrate the Euler expansion formula, $\frac{1}{J}\frac{d(J|\underline{x})}{dt} = \nabla \cdot \underline{v}$.

Consider

$$\lim_{V \to \underline{x}} \int_S \underline{v} \cdot \hat{\underline{n}} ds = \lim_{V \to \underline{x}} \int_S \left(\frac{d(\underline{r}|\underline{x},t)}{dt} \right) \cdot \hat{\underline{n}} \, dS$$

$$= \lim_{V \to \underline{x}} \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_S [\underline{r}(\underline{x},t+\Delta t) - \underline{r}(\underline{x},t)] \cdot \hat{\underline{n}} \, dS$$

where S is the surface of V_t. For differential movements the integral approximates the volume $V(\underline{x},t+\Delta t) - V(\underline{x},t)$ and thus

$$\lim_{V\to x} \int_S \underline{v}\cdot\hat{n}dS = \lim_{V\to \underline{x}} \lim_{\Delta t\to 0} \left[\frac{V(\underline{x},t+\Delta t) - V(\underline{x},t)}{\Delta t}\right]$$

$$= \lim_{V\to \underline{x}} \frac{d}{dt}\left[V(\underline{x},t)|\underline{x}\right] . \qquad (1.6)$$

The expansion formula follows from Eq. (1.6)

$$\frac{1}{J}\frac{d}{dt}(J|\underline{x}) = \lim_{V\to\underline{x}} \frac{V_o}{V_t}\left(\frac{d}{dt}\frac{V(\underline{x}|t)}{V_o}\right) = \lim_{V\to\underline{x}} \frac{1}{V_t}\frac{d}{dt}(V(\underline{x}|t))$$

$$= \lim_{V\to x} \frac{\int_S \underline{v}\cdot\hat{n}\, dS}{V_t} \equiv \nabla\cdot\underline{v} \quad .$$

The expansion formula is often written

$$\boxed{\frac{d}{dt}(J|\underline{x}) = J\nabla\cdot\underline{v} \qquad (1.7)}$$

Now let $f(\underline{r},t)$ be any function (not necessarily material) and V_t be a moving material volume. Form the integral

$$\int_{V_t} f(\underline{r},t)dV = f_o(t) .$$

Then

$$\frac{d}{dt}[f_o(t)|V_o] = \frac{d}{dt}\left(\int_{V_t} f(\underline{r},t)dV\ |V_o\right) , \quad \underline{x}\in V_o$$

$$= \frac{d}{dt}\left(\int_{V_o} f(\underline{r},t)JdV_o|V_o\right) = \int_{V_o}\left(\frac{d}{dt}[f(\underline{r},t)JdV_o\ |\underline{x}]\right)$$

$$= \int_{V_o} \left(J \frac{d}{dt}(f|\underline{x}) + f(\underline{r},t) \frac{d}{dt}(J|\underline{x}) \right) dV_o$$

$$= \int_{V_o} \left(\frac{d}{dt}(f|\underline{x}) + f(\underline{r},t)\, \nabla\cdot(\underline{v}) \right) J dV_o$$

or finally,

$$\frac{d}{dt} \left(\int_{V_t} f(\underline{r},t) dV \,|V_o \right) = \int_{V_t} \left(\frac{d}{dt}(f|\underline{x}) + f\, \nabla\cdot\underline{v} \right) dV \qquad (1.8a)$$

where $(d/dt)(f_o(t)|V_o)$ is the obvious extension of $(d/dt)(f_o(t)|\underline{x})$ to a material volume.

Equation (1.8a) is often called Reynold's Transport Theorem. It may be written in two other forms:

$$\frac{d}{dt} \left(\int_{V_t} f(\underline{r},t) dV | V_o \right) = \int_{V_t} \left(\frac{d}{dt}\,(f|\underline{r}) + \nabla\cdot f\underline{v} \right) dV \,, \qquad (1.8b)$$

using the relationship Eq. (1.4) between material and fixed-space derivatives, and

$$\frac{d}{dt} \left(\int_{V_t} f(\underline{r},t) dV | V_o \right) = \int_{V_t} \left(\frac{d}{dt}(f(\underline{r},t)|r) \right) dV + \int_{S_t} f\underline{v}\cdot\hat{n} dS \,, \qquad (1.8c)$$

provided the divergence theorem holds. In Eq. (1.8c), S_t is the surface of V_t and $\hat{n}$ is its outward unit normal. For $f(\underline{r},t) = p(\underline{r},t)$, the pressure at a point, we have over a moving surface

$$\int_{S_t} p\underline{v}\cdot\hat{n} dS = \frac{d}{dt} \left(\int_{V_t} p(\underline{r},t) dV | V_o \right) - \int_{V_t} \frac{d}{dt}(p(\underline{r},t)|\underline{r}) dV, \qquad (1.9)$$

provided the volume V_t contains no sources (or discontinuities).

Of course, in the ordinary sense none of these entities need exist. The velocity may not be integrable and the limit may not exist. Thus, if the particle becomes two or more particles, or is annihilated, or any neighborhood fragments, etc., which conditions may pertain to certain physical problems, Eq. (1.7) cannot be used. In such cases, it is clear that J^{-1} is likewise meaningless. This being so, we agree to limit our consideration to those media sufficiently well-behaved to allow the use of Eq. (1.6) (which is to say that the limits exist and integration is definable in some sense). In colloquial terms, we allow discontinuities, but not nuclear explosions.

We now generalize the transport theorem to show the effects of a discontinuity. Suppose a material volume, V, partitioned by an internal surface, S_o, which moves with velocity $\underline{u}$. Suppose that $f(\underline{r},t)$ suffers a discontinuity across S_o. Then

$$\frac{d}{dt}\left(\int_{V_t} f dV \,|\, V_o\right) = \int_{V_t}\left(\frac{df}{dt}\,|\,\underline{x}\right)dV + \int_{S_t} f\underline{v}\cdot\hat{\underline{n}}dS + \int_{S_o}\Delta f\underline{u}\cdot\hat{\nu}dS \ , \qquad (1.10)$$

where $\hat{\nu}$ is unit normal to S_o and Δf is the discontinuity across S_o. To show this, divide V_t into V_1 and V_2 and f into f_1 and f_2 such that $S_1 = S_o \cup (S \tilde{\cap} S_2)$ encloses V_1. Define S_2 similarly for V_2. Then over V_1 and V_2 the transport theorem for continuous functions applies, namely:

$$\frac{d}{dt}\left(\int_{V_1} f_1(\underline{r},t)\,|\,V_o\right) = \int_{V_1}\left(\frac{df_1}{dt}\,|\,\underline{x}\ dV_1\right) + \int_{S_1-S_o} f_1\underline{v}\cdot\hat{\underline{n}}dS + \int_{S_o} f_1\underline{u}\cdot\hat{\nu}dS \ ,$$

and

$$\frac{d}{dt}\left(\int_{V_2} f_2(\underline{r},t)|V_o\right) = \int_{V_2}\left(\frac{df_2}{dt}|\underline{x}\right)dV_2 + \int_{S_2-S_0} f_2\ \underline{v}\cdot\hat{\underline{n}}dS + \int_{S_0} f_2\ \underline{u}\cdot(-\hat{\underline{v}})dS\ .$$

We obtain Eq. (1.10) by addition. Thus, comparing Eq. (1.10) and Eq. (1.8c), we see that the presence of singularities profoundly alters the hydrodynamic results.

These preliminaries completed, we now develop equations which conserve mass and momentum. Let $\rho(\underline{x},t)$ be the mass per unit volume at $\underline{r}(\underline{x})$. Then the total mass m in a material volume V_t is

$$m = \int_{V_t} \rho(\underline{x},t)dV\ . \tag{1.11}$$

By the transport theorem

$$\frac{d(m|\underline{x})}{dt} = \int_{V_t}\left(\frac{d(\rho|\underline{x})}{dt} + \rho\nabla\cdot\underline{v}\right)dV\ .$$

But m is constant for mass conservation. Hence,

$$\int_{V_t}\left(\frac{d(\rho|\underline{x})}{dt} + \rho\nabla\cdot\underline{v}\right)dV = 0\ .$$

Since V_t is arbitrary, we must have

$$\boxed{\frac{d(\rho|\underline{x})}{dt} + \rho\nabla\cdot\underline{v} = 0\ .} \tag{1.12}$$

Two immediate corrolaries follow:

$$\frac{d(\rho|\underline{x})}{dt} + \rho\nabla\cdot\underline{v} = 0 = \frac{d(\rho|\underline{r})}{dt} + \nabla\cdot\rho\underline{v}\ , \tag{1.13}$$

and for any function $f(\underline{r},t)$,

$$\frac{d}{dt}\left(\int_{V_t} \rho f dV \;|\underline{x}\right) = \int_{V_t} \rho \frac{d(f|\underline{x})}{dt} dV \;, \tag{1.14}$$

which can be seen by substituting $f = \rho f$ in the Transport Theorem Eq. (1.8a).

Applying the conservation of linear momentum, namely that the rate of change of linear momentum of a material volume equals the sum of the body forces, the external forces, and the internal forces exerted through its bounding surface, we have

$$\frac{d}{dt}\left(\int_{V_t} \rho \underline{v} dV \;|\underline{x}\right) = \int_{V_t} \rho \underline{f} dV + \int_{S_t} \hat{\underline{n}}\cdot\underline{T} dS, \tag{1.15}$$

where $\underline{T}$ is the stress tensor, and $\underline{f}$ is the external force vector per unit mass. Then

$$\frac{d}{dt}\left(\int_{V_t} \rho \underline{v} dV|\underline{x}\right) = \int_{V_t} \rho \frac{d(\underline{v}|\underline{x})}{dt} = \int_{V_t} (\rho \underline{f} + \nabla\cdot\underline{T}) dV.$$

Since V_t is arbitrary,

$$\boxed{\rho \frac{d(\underline{v}|x)}{dt} = \rho \underline{f} + \nabla\cdot\underline{T} \;.} \tag{1.16}$$

Equations (1.12) and (1.16) represent the conservation laws of mass and momentum in the fluid. They are often termed the "continuity" equation and Cauchy's law of motion.

In general, the stress tensor, $\underline{T}$, incorporates viscous and thermodynamic effects. Polar fluids are defined[1] by a non-symmetric stress tensor. The assumption of a non-polar fluid (often made) is, therefore, equivalent to requiring $\underline{T}$ to be symmetric.

For compressible fluids at rest, it is well known that $\underline{T}$ is diagonal with diagonal elements, $d_i = -p$, the local static pressure. Thus,

$$\nabla \cdot \underline{T} = -\underline{\nabla p}$$

for compressible fluids at rest. For non-static cases, the assumed symmetry of $\underline{T}$ defines a preferred coordinate system where the mean local pressure is the mean of the diagonal elements (often called principal stresses). A perfect fluid is defined by

$$\underline{T}_{ij} = -p\delta_{ij} ,$$

so that,

$$\nabla \cdot \underline{T} = -\underline{\nabla p}$$

is assumed to describe dynamic as well as static cases.

A general Stokesian fluid is characterized by

$$\underline{T} = (-p + \alpha)\underline{I} + \beta\underline{E} + \gamma\underline{E}^T\underline{E}$$

where $\underline{E}$ is the deformation tensor, $\underline{I}$ is the unit dyadic and α, β, γ are scalar weights. A perfect fluid is then a type of Stokesian fluid.

---

[1]For this and the definitions below, see [6], pp. 105-109.

The remark by Truesdell[1] means that no Stokesian fluid can physically support a wave (or conversely that since real fluids support waves, they are non-Stokesian).

1.2.1.2 The Divergence Theorem

For a vector field $\underline{a}(\underline{r})$ the divergence theorem,

$$\int_V \nabla\cdot\underline{a}dV = \int_S \underline{a}\cdot\hat{\underline{n}}dS$$

is usually demonstrated for volumes with (piece-wise) smooth surfaces where the vector field is continuously space-differentiable. It figures prominently in Eq. (1.16) and certain auxiliary forms of earlier equations. It, likewise, forms the basis of Green's functions to be used subsequently.

It is recognized [38] that the divergence theorem may be extended to a larger class of functions. This extension, however, is not presently available in easily recognizable form. For the purposes of this thesis, we simply indicate that the extension is not empty and leave others to discover the actual topography. Our results apply to the extension.

We give heuristic argument for the extension to a class of non-continuous fields $\underline{a}$. The divergence theorem, in one dimension, is the fundamental theorem of calculus

$$\int_X \frac{d}{dx} a(x)dx = a(x_1) - a(x_0)$$

---

[1][48], p. iv.

$$= \int_{\text{Boundary}(X)} a(x)\cdot\hat{n}\,, \qquad x = [x_1, x_0]$$

Suppose, now, that $a(x) = U(x)$, the unit step function, and $X = [-1,1]$. Then, $U(1) - U(-1) = 1$, but

$$\int_{[-1,1]} \frac{d}{dx}[U(x)]dx = \int_{[-1,1]} \delta(x)dx$$

has no meaning. If we choose to define

$$\int_X U(x)dx \triangleq \int_{-\infty}^{\infty} \delta(x)dx = 1\,,$$

then we achieve both meaning and consistency, that is, we have extended the fundamental theorem to a class of non-continuous $\underline{a}(\underline{x})$. This argument is easily carried to three dimensions.

Suppose, for instance, a unit sphere centered on the origin and $a(\underline{r}) = \hat{r}U(\underline{r})$. Then,

$$\oint_S U(\underline{r})\hat{r}\cdot\hat{n} = 4\pi\,.$$

Also

$$\int_V \nabla\cdot[\underline{r}\,U(\underline{r})] = 4\pi\int_0^1 \left(|r|^2 \frac{\partial U|r|}{\partial|r|} + \frac{2}{|r|}U|\underline{r}|\right)d|\underline{r}|$$

$$= 4\pi\int_0^1 \frac{\partial}{\partial|\underline{r}|}\,[|r|^2U(|r|)]d|r|$$

$$= 4\pi \left[|\underline{r}|^2 U(\underline{r}) \right] \Big|_0^1$$

$$= 4\pi \ .$$

(In later chapters we will sometimes put $\rho = |\underline{r}|$, where no confusion with density arises.) Thus generalized functions, like $U(\underline{r})$ and their derivatives, may, under circumstances not completely known, obey the divergence theorem.

Although our treatment here is not profound, we have no intention of slighting the extension problem which is crucially important to understanding the hydrodynamics of wave propagation in realistic media. The extension, in itself, however, is a consideration for mathematicians in the way of extended theories of integration (so useful in physics).

1.3 The First Order Hydrodynamic Equation

We return to developing the general hydrodynamic equation. Given assumption (A2), material changes in the pressure are related only to material changes in density. We thus write

$$\frac{d(p_2|\underline{x})}{d(\rho|x)} \equiv c^2(\underline{r},t) \ ,$$

or

$$\boxed{\frac{d(p_2|\underline{x})}{dt} = c^2 \frac{d(\rho|x)}{dt}} \qquad (1.17)$$

for the equation of state, where c^2 so defined has dimension of velocity squared.

Here and throughout we distinguish the categories of pressure as

$p_2(\underline{r},t)$ ≡ the total pressure

$p(\underline{r},t)$ ≡ the acoustic pressure

$p_0(\underline{r})$ ≡ the background pressure (deterministic)

$p_1(\underline{r},t)$ ≡ the turbulent pressure (stochastic).

Evidently[1]

$$p_2 = p + p_0 + p_1 \ .$$

Equation (1.17) is immediately solvable as

$$p_2(\underline{x},t) - p_0 = \int_t c^2(\underline{x},t) \frac{d(\rho|\underline{x})}{dt} dt \ . \qquad (1.18)$$

In Eq. (1.18) the integral is taken with reference to a fixed particle (fixed-particle integration). In a homogeneous real fluid where $c^2 = c_0^2$, constant, we have immediately

$$p_2(\underline{x},t) - p_0 = c_0^2[\rho(\underline{x},t) - \rho_0] \ . \qquad (1.19a)$$

The quantities, p_0 and ρ_0 are the rest pressure and density, respectively. In Appendix A, Eq. (1.19a) is used to obtain the ideal propagation in a real homogeneous fluid for a plane wave. For the plane

---

[1]As stated, this statement restates the fact that any real (complex) number (or function) can be decomposed into three other numbers (or functions). The total pressure, p_2, is directly observable. The acoustic pressure, p, is due to the (assigned) acoustic source(s), and thus is absent when the (assigned) source no longer contributes to the total pressure. The decomposition into p_0 and p_1 is somewhat arbitrary. One may put p_0, for instance, as either $p_0(\underline{r},t) = p_0$ constant, or $p_0(\underline{r},t) = p_0(\underline{r})$ or $p_0(\underline{r},t)$. We have suggested $p_0(\underline{r},t) = p_0(\underline{r})$ as capturing the entire deterministic structure of the remainder, $p_2(\underline{r},t) - p(\underline{r},t)$. Then $p_1(\underline{r},t)$ is the stochastic remainder. The sequel is valid, however, for any consistent decomposition.

case we show that

$$p_2 = \frac{\rho_o c_o^2}{1 - k_o a \cos(\omega_o t - k_o x)}$$

$$\rho = \frac{\rho_o}{1 - k_o a \cos(\omega_o t - k_o x)}$$

$$\nabla \cdot \underline{v} = \frac{\omega_o k_o a \sin(\omega_o t - k_o x)}{1 - k_o a \cos(\omega_o t - k_o x)}$$

$$\underline{v} = a\omega_o \cos(\omega_o t - k_o x)$$

$$\nabla p_2 = \frac{\rho_o c_o^2 k_o^2 a \sin(\omega_o t - k_o x)}{[1 - k_o a \cos(\omega_o t - k_o x)]^3} \qquad (1.19b)$$

where a is the maximum particle displacement, and $k_o = \omega_o / c_o$.

The simplicity of Eq. (1.17) is deceiving. In addition to accepting it in a generalized sense, the particle positions may not be simply related to source motion as in ideal cases. Even here the solution is non-linear. Further still, solutions are commonly expressed in fixed-space rather than material terms.

To the first point above, it must be understood that Eq. (1.17), for all its seeming simplicity, must be interpreted in an extended class of functions for which an extended notion of derivative always exists. Thus, Eq. (1.17) rests on an extension not yet fully understood and, to that extent Eq. (1.17), though simply stated, is not

fully comprehended.

To the second point, we note that Eq. (1.17) deals with the state of things at a point (or particle). Exceptionally, a numerically identical relationship will allow continuation to a source (or boundary) as in the classical solutions. The form of Eq. (1.17) makes no such requirement, however. In this latter case, we must answer precisely what we mean by a solution to Eq. (1.17). This discussion we leave to Chapter III on solutions.

To the third point, we write Eq. (1.17) in fixed-space form. Using Eq. (1.4), we have

$$\frac{d(p_2|\underline{r})}{dt} + \underline{v}\cdot(\nabla p_2 - c^2\nabla\rho) - c^2\frac{d(\rho|\underline{r})}{dt} = 0 \qquad (1.20)$$

From Eq. (1.20), it is obvious that the result does not depend on a unique particle velocity $\underline{v}$. A simplified form may be obtained from the mass Eq. (1.12).

$$\frac{d(p_2(\underline{r},t)|\underline{r})}{dt} + \underline{v}(\underline{r},t)\cdot\nabla p_2(\underline{r},t) + \rho(\underline{r},t)c^2(\underline{r},t)\nabla\cdot\underline{v}(\underline{r},t) = 0 \qquad (1.21)$$

It should be noted that the substitution has not eliminated any of the variables.

The variables in Eq. (1.21) generally represent the effects complicated processes. The particle motion $\underline{v}$ and dynamic density $\rho(\underline{r},t)$, for instance, arise not only from acoustic influences but also from tidal movements, salinity and temperature gradients, etc.

The ideal solution mentioned above shows, furthermore, that Eq. (1.21) is non-linear. For instance,

$$\rho c_o^2 \nabla\cdot v = \frac{\rho_o c_o^2}{1 - k_o a \cos(\omega_o t - k_o x)} \quad \frac{k_o a \sin(\omega_o t - k_o x)}{1 - k_o a \cos(\omega_o t - k_o x)}$$

$$= \frac{k_o a \sin(\omega_o t - k_o x)}{\rho_o c_o^2} \left(p_2\right)^2 \quad .$$

Similarly, the term $\underline{v}\cdot\nabla p_2$ yields non-linear forms.

Equation (1.21) describes the hydrodynamic situation at the point $\underline{r}$ at time t, (the space-time point $(\underline{r},t)$). Letting $p_2 = p + p_0 + p_1$, where p is the acoustic pressure, p_0 the unperturbed (background) pressure, and p_1 the turbulent pressure, we rewrite Eq. (1.21) as

$$\frac{d(p|\underline{r})}{dt} + \underline{v}\cdot(\nabla p + \nabla(p_0 + p_1)) + \rho c^2 \nabla\cdot\underline{v} = -\frac{d}{dt}((p_0 + p_1)|\underline{r}) \quad . \qquad (1.22)$$

Here, $\underline{v}$ and ρ remain the total particle velocity and total dynamic density, respectively.

One appreciates immediately that the quantity $\underline{v}\cdot\nabla(p_0 + p_1)$ is associated with beam spreading and beam wander; $\nabla\cdot\underline{v}$ will influence spatial coherence; $\{(d/dt)[(p_0 + p_1)|\underline{r}]\}$ will influence temporal coherence; short pulse effects will depend on initial conditions, etc. In fact, in Eq. (1.22), it is possible to discern qualitatively all observed effects, just as we should expect from an exact equation.

1.4 The Second Order Hydrodynamic Equation

To this point, we have employed a first order differential equation. Because most acousticians employ a second order equation, we

develop the exact equations to the second order to facilitate comparisons. Differentiating Eq. (1.22) with respect to time (fixed-space), we obtain

$$\frac{d^2(p_2|\underline{r})}{dt^2} + \underline{v}\cdot\left(\frac{d(p_2|\underline{r})}{dt}\right) + (\nabla p_2)\cdot\frac{d(\underline{v}|\underline{r})}{dt} + \rho c^2\nabla\cdot\left(\frac{d\underline{v}|\underline{r}}{dt}\right)$$

$$+ (\nabla\cdot\underline{v})\,\frac{d}{dt}\,[(\rho c^2)|\underline{r}] \qquad = 0\ . \tag{1.23}$$

Substituting now the momentum Eq. (1.16), we have

$$\frac{d^2(p_2|\underline{r})}{dt^2} + \underline{v}\cdot\left(\nabla\frac{d(p_2|\underline{r})}{dt}\right) + \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot\underline{T}}{\rho}\right)\cdot\nabla p_2$$

$$+ \left(\frac{d}{dt}\,(\rho c^2|\underline{r})\right)\left(\nabla\cdot\underline{v}\right) + \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right) = 0\ . \tag{1.24}$$

Separating the acoustic quantities ($\underline{T}_2 = \underline{T} + \underline{T}_0 + \underline{T}_1$) to the LHS gives

$$\frac{d^2(p|\underline{r})}{dt^2} + \underline{v}\cdot\nabla\frac{d(p|\underline{r})}{dt} + \left(\underline{f} + \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot\nabla p$$

$$+ \nabla(p_0 + p_1)\cdot\frac{(\nabla\cdot\underline{T})}{\rho} + \rho c^2\nabla\cdot\frac{(\nabla\cdot\underline{T})}{\rho}$$

$$= -\left\{\frac{d^2}{dt^2}\,((p_0 + p_1)|\underline{r}) + \underline{v}\cdot\nabla\frac{d}{dt}\,(p_0 + p_1|\underline{r})\right.$$

$$+ \left(f - \underline{v}\nabla\cdot\underline{v} + \nabla\cdot\frac{(\underline{T}_0 + \underline{T}_1)}{\rho}\right)\cdot\nabla(p_0 + p_1)$$

$$\left. + \left[\frac{d}{dt}\,(\rho c^2|\underline{r})\right][\nabla\cdot\underline{v}] + \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \nabla\cdot\frac{(\underline{T}_0 + \underline{T}_1)}{\rho}\right)\right\} \tag{1.25}$$

A symmetric form of Eq. (1.25) may be obtained from the term

$$\left(\frac{d}{dt}\,\rho c^2|\underline{r}\right)\left(\nabla\cdot\underline{v}\right)\ ,$$

and the relation

$$\nabla\cdot\underline{v} = -\frac{1}{\rho c^2}\left(\frac{d(p_2|\underline{r})}{dt} + \underline{v}\cdot\nabla p_2\right)$$

$$= -\frac{1}{\rho c^2}\left(\frac{dp|\underline{r}}{dt} + \underline{v}\cdot\nabla p\right) - \frac{1}{\rho c^2}\left(\frac{d((p_0 + p_1)|\underline{r})}{dt} + \underline{v}\cdot\nabla(p_0 + p_1)\right)$$

from Eq. (1.21).

Then, we obtain

$$\frac{d^2(p|\underline{r})}{dt^2} - \left(\frac{d}{dt}\,\ell n\rho c^2|\underline{r}\right)\frac{d(p|\underline{r})}{dt} + \underline{v}\cdot\left(\nabla\frac{d(p|\underline{r})}{dt}\right)$$

$$+ \left(f - \underline{v}\cdot\nabla\underline{v} - \left(\frac{d}{dt}\,(\ell n\rho c^2|\underline{r})\right)\underline{v} + \frac{\nabla\cdot T_2}{\rho}\right)\cdot\nabla p$$

$$+ \nabla(p_0 + p_1)\cdot\left(\frac{\nabla\cdot T}{\rho}\right) + \rho c^2\nabla\cdot\left(\frac{\nabla\cdot T}{\rho}\right)$$

$$= -\left\{\frac{d^2}{dt^2}\,((p_0 + p_1)|\underline{r}) + \underline{v}\,\frac{d}{dt}\left(\nabla(p_0 + p_1)|\underline{r}\right)\right.$$

$$+ \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \left(\frac{d}{dt}\,\ell n\rho c^2|\underline{r}\right)\,\underline{v} + \frac{\nabla\cdot(T_0 + T_1)}{\rho}\right)\cdot\nabla(p_0 + p_1)$$

$$- \left(\frac{d}{dt}\,\ell n\rho c^2|\underline{r}\right)\left(\frac{d}{dt}\,(p_0 + p_1)|\underline{r}\right)$$

$$\left. + \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\nabla\cdot\underline{v} + \frac{\nabla\cdot(T_0 + T_1)}{\rho}\right)\right| \qquad (1.26)$$

Another form may be obtained by associating acoustic terms,

$$\frac{d^2(p|\underline{r})}{dt^2} - \frac{d}{dt}\left(\ell n\rho c^2|\underline{r}\right)\left(\frac{d(p|\underline{r})}{dt}\right) + \underline{v}\cdot\left(\frac{d}{dt}\nabla p + \nabla(p_0 + p_1)\right)$$

$$+ \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \left(\frac{d}{dt}\ell n\rho c^2|\underline{r}\right)\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot\left(\nabla p + \nabla(p_0 + p_1)\right)$$

$$+ \rho c^2\nabla\cdot\left(\frac{\nabla\cdot\underline{T}}{\rho}\right)$$

$$= -\left\{\frac{d^2}{dt^2}(p_0 + p_1|\underline{r}) - \left(\frac{d}{dt}\ell n\rho c^2|\underline{r}\right)\left(\frac{d}{dt}(p_0 + p_1)|\underline{r}\right)\right.$$

$$\left. + \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot\underline{T}_0 + \underline{T}_1}{\rho}\right)\right\}. \qquad (1.27)$$

An exact solution of Eq. (1.27) requires knowledge of ρc^2 and its fixed-space derivative; the turbulent and unperturbed pressure and associated stress tensor; the particle velocity, its dilatations and rotations; body forces; knowledge of ideal conditions; initial conditions, and, finally, how all these elements change in space and time. Such detailed information about the field is rarely obtainable. We are, thus, led to interpret Eq. (1.27) as defining a stochastic (space-time) process for which meaningful solutions exist only in a statistical sense. This implies the mathematical conceit of a universe of similarly generated media and turbulent sources, together with associated probability measures. Our ability to obtain knowledge of this universe will thus be conditioned on the stationarity (or better, rapidity of non-stationarity) of the medium regarded as a space-time process. With non-stationary processes, we can accommodate non-adiabatic conditions

excluded in fundamental assumption (A2). Diurnal and seasonal variations in the medium, for example, can be so treated.

We thus consider Eq. (1.27) to be a stochastic (Langevin) equation of the form

$$\frac{d^2(p|\underline{r})}{dt^2} + a_0\frac{d(p|\underline{r})}{dt} + \underline{a}_1 \cdot \frac{d}{dt}\left[(\nabla p + \underline{b}_1)|\underline{r}\right] + \underline{a}_2\cdot(\nabla p + \underline{b}_2) + \hat{\underline{a}}_3\cdot(\nabla p + \underline{b}_3) = q_0, \tag{1.28}$$

where the random variables a_i, b_i are defined as

$$a_0 \equiv -\frac{d}{dt}\left[(\ell n \rho c^2)|\underline{r}\right]$$

$$\underline{a}_1 \equiv \underline{v}$$

$$\underline{b}_1 \equiv \nabla(p_0 + p_1)$$

$$\underline{a}_2 \equiv \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \left(\frac{d}{dt}\,\ell n \rho c^2|\underline{r}\right)\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)$$

$$\underline{b}_2 \equiv \underline{b}_1 = \nabla(p_0 + p_1)$$

$$\hat{\underline{a}}_3 \equiv -c^2(\nabla - \nabla\ell n\rho) \qquad \text{(an operator)}$$

$$\underline{b}_3 \equiv -(\nabla\cdot\underline{T} + \nabla p)$$

$$q_0 \equiv -\left\{\frac{d^2}{dt^2}\left((p_0 + p_1)|\underline{r}\right) - \left(\frac{d}{dt}\,\ell n\rho c^2|\underline{r}\right)\frac{d}{dt}\left((p_0 + p_1)|\underline{r}\right) + \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho}\right)\right\}.$$

In this form, it is easy to recognize that the $\{\underline{b}_i\}$ are related to scattering out of the beam and, thus, to spatial incoherence. Similarly, the $\{\underline{a}_i\}$ relate to temporal incoherence.

In particular, a_0 produces a temporal relaxation effect which we may expect to depend on frequency; a_1 is influenced by tides, currents, as well as small scale motion caused by propagation; $\underline{a}_2$, a non-linear term, is influenced by gravity, medium motion, and material properties; $\hat{\underline{a}}_3$ is a random vector operator which depends on the medium material properties and its distribution; $\underline{b}_1$ accounts for background and turbulent spatial variation; $\underline{b}_3$ is associated with the acoustic stress; and g_0 acts as a rather complicated distributed noise source. Further physical description of the a_i, b_i and their effect on propagation in both the atmosphere and ocean are given in Chapter II.

It is interesting to note that in the absence of an acoustic pressure ($p = 0$), the turbulence propagates analogously to the acoustic case. Here, we have

$$\frac{d^2 p_1}{dt^2} - \left(\frac{d}{dt}\, \ell n \rho c^2 | \underline{r}\right)\left(\frac{d}{dt}\, p_1 | \underline{r}\right) + \underline{v}\cdot\frac{d}{dt}\,\nabla(p_0 + p_1)|\underline{r}$$

$$+ \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \frac{d}{dt}\, \ell n \rho c^2 | \underline{r}\ \ \underline{v} + \frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho}\right)\cdot \nabla(p_0 + p_1)$$

$$+ c^2\nabla\cdot\left(\frac{\nabla\cdot\underline{T}_1}{\rho}\right)$$

$$= -\left\{\frac{d^2}{dt^2}\,(p_0|\underline{r}) - \left(\frac{d}{dt}\, \ell n \rho c^2 | \underline{r}\right)\left(\frac{dp_0|\underline{r}}{dt}\right)\right.$$

$$\left. + \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot T_0}{\rho}\right)\right\}. \qquad (1.29)$$

Similarly, even when no turbulence exists, propagation is still possible because of the motion produced by the body forces, i.e.,

$$\frac{d^2(p_o|\underline{r})}{dt^2} - \left(\frac{d}{dt}\,\ell n\rho c^2|\underline{r}\right)\left(\frac{d(p_o|\underline{r})}{dt}\right) + \underline{v}\cdot\frac{d}{dt}\,\nabla p_o$$

$$+ \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \left(\frac{d}{dt}\,\ell n\rho c^2|\underline{r}\right)\underline{v} + \frac{\nabla\cdot\underline{T}_o}{\rho}\right)\cdot\nabla p_o$$

$$+ \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot\underline{T}_o}{\rho}\right) = 0 \quad . \tag{1.30}$$

These non-acoustic propagating pressure fields, acting very much as distributed secondary sources of the acoustic field, underscore the reason for considering the latter as a stochastic process.

1.5 Hydrodynamic Equations Compared to Wave Equations

The formulations of acoustics (i.e., the models established to study a given range of experiments) are very many. In Chapter IV we will attempt to pick representative models from different schools. In contrast, the number of wave equations which have emerged in the study of sound (compared to proposed solutions) are few. Here we study eight formulations which have enjoyed a following in the last thirty years.

Each deals with partial rather than hydrodynamic derivatives. They are second order equations. Each emphasizes the nature of the propagation as a wave (i.e., repeated--in some sense--in space and time).

It will be convenient to use Eq. (1.27) as a basis for cataloging the assumptions made (explicitly or implicitly) in each case.

1.5.1 The Ideal Wave Equation

We term the equation

$$\frac{\partial^2 p(\underline{r},t)}{\partial t^2} - c_0^2 \nabla^2 p(\underline{r},t) = 0 \qquad (1.31)$$

the ideal wave equation. It is clearly a linear relation in p.

In the ideal case c_0^2 is constant. Deterministic extensions to $c^2(\underline{r})$ have been successful in modeling the refractive effects of the ocean.

The relationship between the ideal Eq. (1.31) and the general, non-linear Eq. (1.27) is easily shown by noting that a perfect fluid is one for which $\underline{I}$ is always diagonal. Then, for $\nabla\cdot\underline{I} = -\nabla p$,

$$\rho c^2 \nabla\cdot\left(\frac{\nabla\cdot\underline{I}}{\rho}\right) = c^2\nabla\cdot(-\nabla p) = -c^2\nabla^2 p \quad , \qquad \rho \text{ constant.}$$

If we further let the background and turbulent pressures equal zero, we have

$$\frac{d^2}{dt^2}(p|\underline{r}) - \frac{d}{dt}((\ell n \rho c^2)|\underline{r})\frac{d(p|\underline{r})}{dt} + \underline{v}\cdot\nabla\frac{d(p|\underline{r})}{dt} + \left(\underline{f} - \underline{v}\cdot\nabla\underline{v}\right.$$

$$\left. - \left(\frac{d}{dt}(\ell n \rho c^2|\underline{r})\right)\underline{v}\right)\cdot\nabla p - c^2\nabla^2 p = \rho c^2 \nabla\cdot(\underline{f} - \underline{v}\cdot\nabla\underline{v}).$$

If we further assume ρ is constant, $\underline{v}$ is negligible and no body forces (like gravity) act, we obtain for $c = c_0$

$$\frac{d^2}{dt^2}(p|\underline{r}) - c_o^2\nabla^2 p = 0 \ .$$

Upon equating the hydrodynamic fixed-space derivative with the partial derivative, we obtain Eq. (1.31).

The ideal case thus implies:

p_0	= constant	a_0	= 0
p_1	= constant	a_1	= 0
$\underline{T}_0$	= 0	a_2	= 0
$\underline{T}_1$	= 0	$\hat{a}_3$	= $(-c_o^2\nabla)$
$\nabla\cdot\underline{T}$	= $-\nabla p$	b_1	= 0
ρ	= spatially constant	b_2	= 0
$\frac{d}{dt}(\ell n\rho c^2\|\underline{r})$	= 0	b_3	= 0
$\underline{f}$	= 0 (no body forces)	g_0	= 0
$\underline{v}$	= negligible		(1.32)

Thus, the ideal case assumes a perfect fluid at rest (static) with no body forces. It takes no account of turbulence losses, relaxation, body forces, or fluid motion. One cannot expect this model to be successful in accounting for beam wander, loss of coherence, and other observed effects. The medium has, with the exception of the propagation velocity, c_o, been totally ignored.

Here and subsequently, it is not very difficult to point out inconsistencies. For example, if ρ is constant, Eq. (1.21) becomes

$$\frac{d(p|\underline{r})}{dt} = -\underline{v}\cdot\nabla p \tag{1.33}$$

which governs the propagation in an incompressible fluid. But from Eq. (1.33), we cannot obtain Eq. (1.31), although in this case $\nabla\cdot\underline{T} = -\nabla p$ as well. Whenever c_0^2 is large, the equation of state, Eq. (1.17) says that small changes in density effect large changes in pressure. Considering ρ constant because its variation is small ($\approx p/c_0^2$) may satisfy a utilitarian point of view where inconsistencies do not matter. This view will not do theoretically, however, since by considering the effect and ignoring the cause, we eliminate the very basis of understanding.

The ideal propagation is, thus, seen to be fictitious, as Truesdell remarked. Any acoustic disturbance must physically cause a disturbance in the medium.

1.5.2 The Stochastic Helmholtz Wave Equation

The equation,

$$\nabla^2 p(\underline{r}) + k^2 p(\underline{r}) = 0 \tag{1.34}$$

where k is a random variable, is called the stochastic Helmholtz equation for its similarity to the non-stochastic form. With k random, the solution p is a random variable; thus Eq. (1.34) governs a random process in space. The stochastic Helmholtz equation, more than any other, constitutes the basis of much of the theory of wave propagation

in random media, [168, 62]. A more general form is used as the basis of the Rytov method [174], i.e.,

$$\frac{(1 + \mu^2(\underline{r}))}{c_0^2} \frac{\partial^2}{\partial t^2} p(\underline{r},t) - \nabla^2 p(\underline{r},t) = 0 \ . \qquad (1.35)$$

The parameter μ is a random variation of the index of refraction, and c_0 is the unperturbed (wavefront) propagation velocity. The relationship between Eqs. (1.34) and (1.35) and the general Eq. (1.27) thus implies

$$\begin{aligned}
p_0 &= \text{constant} & a_0 &= 0 \\
p_1 &= \text{constant} & \underline{a}_1 &= 0 \\
\underline{T}_0 &= 0 & \underline{a}_2 &= 0 \\
\underline{T}_1 &= 0 & \hat{a}_3 &= \frac{-c_0^2}{1 + \mu^2} \nabla \\
\nabla \cdot \underline{T} &= -\nabla p \text{ (a perfect fluid)} & \underline{b}_1 &= 0 \\
\rho &= \text{constant} & \underline{b}_2 &= 0 \\
\frac{d}{dt} (\ell n \rho c^2 | \underline{r}) &= 0 & \underline{b}_3 &= 0 \\
\underline{f} &= 0 & g_0 &= 0 \ . \qquad (1.36)
\end{aligned}$$

The stochastic Helmholtz equation differs from the harmonic ideal wave equation only in interpretation. It likewise takes no account of losses, relaxation effects, body forces or fluid motion. Equation (1.34) will clearly not account for temporal or mixed (space-time) effects. Some turbulent effects, however, may be modeled in the statistics of k.

1.5.3 Chernov's Wave Equation

L.A. Chernov [41] has investigated solutions to the wave-like equation

$$\frac{1}{c^2(\underline{r})}\frac{\partial^2}{\partial t^2}p(\underline{r},t) - \nabla^2 p(\underline{r},t) + \nabla \ell n \rho(\underline{r})\cdot\nabla(p(\underline{r},t)) = 0 \qquad (1.37)$$

From Eq. (1.27), we easily find that this equation implies

$$\begin{aligned}
p_0 &= \text{constant} & a_0 &= 0\\
p_1 &= \text{constant} & \underline{a}_1 &= 0\\
\underline{T}_0 &= 0 & \underline{a}_2 &= 0\\
T_1 &= 0 & \hat{a}_3 &= c^2(\nabla - \nabla \ell n \rho_0)\\
\nabla\cdot\underline{T} &= -\nabla p & \underline{b}_1 &= 0\\
\rho &= \rho(\underline{r}) & \underline{b}_2 &= 0\\
\frac{d}{dt}(\ell n \rho c^2 | \underline{r}) &= 0 & \underline{b}_3 &= 0\\
\underline{f} &= 0 & g_0 &= 0
\end{aligned} \qquad (1.38)$$

Here we recognize a clear extension from the ideal wave equation. The density is allowed a spatial variation. Still, turbulent effects, background pressure, losses, body forces and fluid motion are not accounted for. The term $\nabla \ell n \rho_0(\underline{r})\cdot\nabla p$ will contribute a spatial relaxation effect.

We note that Chernov's equation does not allow the acoustic disturbance to influence the medium. It, thus, describes a non-material surface propagating through a static but spatially inhomogeneous medium.

1.5.4 Tatarskii's Wave Equations

V.I. Tatarskii has published two formulations of fundamental equations which we investigate here. Both assume narrowband radiation.

The first[1] states

$$\nabla^2 p + k_o^2 p = \frac{jk_o}{c_0}\,\underline{v}\cdot\nabla p + k_o^2\left(\frac{\tau_1}{\tau}\right)p, \qquad k_o = \omega_o/c_o \tag{1.39}$$

where τ is the mean temperature and τ_1 its deviation (from the mean).

A straightforward comparison is easily made for the harmonic realization of the general Eq. (1.28) which is,

$$-\omega_o{}^2 p + ja_0\omega_o p + j\omega_o \underline{a}_1\cdot(\nabla p) + \frac{d}{dt}(\underline{b}_1|r) + \underline{a}_2\cdot(\nabla p + b_2)$$

$$+ \hat{\underline{a}}_3\cdot(\nabla p + \underline{b}_3) = g_0 .$$

Then, for $a_0 = 0$; $\underline{b}_1 = \underline{b}_2 = \underline{a}_2 = [0]$, $g = 0$ and $a_3 = -c^2\nabla$ we have

$$k^2 p + \nabla^2 p - \frac{j\omega_o}{c^2}\,\underline{v}\cdot\nabla p = 0, \qquad k = \omega_o/c$$

The result follows from $k^2 \equiv k_o{}^2\left(1 - \frac{\tau_1}{\tau}\right)$

Thus, in Eq. (1.34) Tatarskii assumes the following

p_0	= constant	a_0	= 0
p_1	= constant	$\underline{a}_1$	= $\underline{v}$
$\underline{T}_0$	= 0	$\underline{a}_2$	= 0

[1] See [188], Eq. 5.5.

$$\underline{T}_1 = 0 \qquad \hat{\underline{a}}_3 = -c^2\nabla$$

$$\nabla\cdot\underline{T} = -\nabla p \qquad \underline{b}_1 = 0$$

$$\rho = \text{constant} \qquad \underline{b}_2 = 0$$

$$\frac{d}{dt}\left(\ell n \rho c^2 | \underline{r}\right) = 0 \qquad \underline{b}_3 = 0$$

$$\underline{f} = 0 \qquad g_0 = 0$$

$$k^2 = k_0^2\left(1 - \frac{\tau_1}{\tau}\right) \tag{1.40}$$

The term,

$$k_0\left(\frac{j\underline{v}\cdot\nabla p}{c_0} + k_0\,\frac{\tau_1}{\tau}\,p\right),$$

acts as a distributed random source, dependent, however, on the acoustic pressure. Thus the medium, appearing in the particle velocity $\underline{v}$, the temperature τ, and the wave number k_0, acts to influence the result. Body forces, independent noise sources, and other medium variables are not considered. We shall see (in Chapter II) that the factor $(jk_0/c_0)\underline{v}\cdot\nabla p$ is considerably more important in air than in water.

It is clear that from Eq. (1.39) temporal and mixed effects cannot be obtained.

A second equation published[1] subsequent to Eq. (1.39) attempts to link the propagation of sound directly to the turbulence factors of temperature and wind velocity. It is given as

$$\nabla^2 p + k_0^2 p = -\nabla\cdot\left(\frac{\tau_1}{\tau}\nabla p\right) + \frac{2}{j\omega_0}\nabla\cdot\nabla\cdot(\underline{u}\nabla p) \tag{1.41}$$

[1]See [187], p. 156, Eq. (15).

where $\underline{u}$ is the turbulent velocity and $\underline{u}\nabla p$ is a second order tensor. This equation stems from the deriving equations:

$$\frac{\partial \underline{v}}{\partial t} = -\frac{\nabla p}{\rho} - \underline{v}\cdot\nabla\underline{v} \tag{T1}$$

$$\frac{\partial p}{\partial t} = -\nabla\cdot\rho\underline{v} \tag{T2}$$

$$\frac{dS}{dt} = 0 \tag{T3}$$

$$S = C_v \ell n\rho - C_p \ell n\rho \quad \text{(ideal gas law)} \tag{T4}$$

where $c^2 = \frac{p}{\rho}\frac{C_p}{C_v}$.

Equation (1.41) can be established from the harmonic form of the general equation with $a_0 = 0$, $\underline{a}_1 = \underline{a}_2 = \underline{b}_1 = \underline{b}_2 = \underline{b}_3 = [0]$, ρ constant and $g = -\rho c^2 \nabla\cdot(\underline{v}\cdot\nabla\underline{v})$.

In this event, we have

$$\omega_0{}^2 p + c^2(\nabla\cdot\nabla p) = -\rho c^2 \nabla\cdot(\underline{v}\cdot\nabla\underline{v})$$

Now let $\underline{v} = \underline{u} + v_a$, the sum of turbulent and acoustic material velocities with $\nabla\cdot\underline{u} = 0$ (an incompressible turbulence).

With no body forces ($\underline{f} = 0$) in a perfect fluid, the momentum equation, Eq. (1.16) (in partial derivatives), becomes ($\nabla\cdot T = -\nabla p$)

$$(\underline{v}\cdot\nabla\underline{v}) = -\frac{\partial \underline{v}}{\partial t} - \frac{\nabla p}{\rho}$$

or

$$(\underline{v}_a + \underline{u})\cdot\nabla(\underline{v}_a + \underline{u}) = -\frac{\partial(\underline{v}_a + \underline{u})}{\partial t} - \frac{\nabla p}{\rho} .$$

For the incompressible turbulent fluid we have

$$\underline{u}\cdot\nabla\underline{u} = \frac{\partial \underline{u}}{\partial t} = 0$$

so that, if we linearize by neglecting $\underline{v}_a\cdot\nabla\underline{v}_a$, we get

$$\underline{u}\cdot\nabla\underline{v}_a + \underline{v}_a\cdot\nabla\underline{u} = -\frac{d\underline{v}_a}{dt} - \frac{\nabla p}{\rho} .$$

In the harmonic case, then, it follows that,

$$\underline{v}_a = \frac{1}{j\omega_o}\left(-(\underline{u}\nabla\cdot\underline{v}_a + \underline{v}_a\cdot\nabla u) - \frac{\nabla p}{\rho}\right) - \frac{\nabla p}{j\omega_o\rho} .$$

These conclusions may be used to approximate (again by linearizing)

$$\rho c^2\nabla\cdot(\underline{v}\cdot\nabla\underline{v}) \doteq \rho c^2\nabla\cdot(\underline{u}\cdot\nabla\underline{v}_a + \underline{v}_a\cdot\nabla\underline{u}) \doteq \rho c^2\nabla\cdot\left[\underline{u}\cdot\left(\frac{\nabla}{j\omega_o}\frac{\nabla p}{\rho}\right) - \frac{1}{j\omega_o}\left(\frac{\nabla p}{\rho}\right)\cdot\nabla u\right]$$

$$\frac{-\rho c^2}{j\omega_o}\nabla\cdot\left[\underline{u}\cdot\nabla\left(\frac{\nabla p}{\rho}\right) + \frac{\nabla p}{\rho}\cdot\nabla\underline{u}\right].$$

Using Tartarskii's reasoning that, since $\nabla\cdot u = 0$, $\underline{u}$ may be brought into the derivatives, (i.e., $\underline{u}\nabla\nabla p = \nabla\cdot(\underline{u}\nabla p)$, we have

$$\rho c^2\nabla\cdot(\underline{v}\cdot\nabla\underline{v}) \quad \frac{-\rho c^2}{j\omega_o}\nabla\cdot\left[\nabla\cdot\left(\underline{u}\frac{\nabla p}{\rho}\right) + \nabla\cdot\underline{u}\frac{\nabla p}{\rho} - \underline{u}\nabla^2 p\right] .$$

Neglecting the last term and considering ρ constant, we arrive at

$$\rho c^2\nabla\cdot(\underline{v}\cdot\nabla\underline{v}) \quad \frac{-2c^2}{j\omega_o}\nabla\cdot\nabla\cdot(\underline{u}\nabla p) .$$

This approximation in hand, the general equation becomes, with $c^2 = c_o^2\,(1 + \tau_1/\tau)$,

$$\omega_o{}^2 p + c_o^2\left(1 + \frac{\tau_1}{\tau}\right)\nabla^2 p = \frac{2c^2}{j\omega_o}\,\nabla\cdot\nabla\cdot(\underline{u}\nabla p) \quad .$$

Now, letting $\nabla\,\dfrac{\tau_1}{\tau} = 0$, and dividing by c_o^2, we have

$$k_o^2 p + \nabla^2 p = -\nabla\cdot\left(\frac{\tau_1}{\tau}\,\nabla p\right) + \frac{2c^2}{j\omega_o c_o^2}\,\nabla\cdot\nabla\cdot(\underline{u}\nabla p), \qquad k_o = \omega_o/c_o$$

$$-\nabla\cdot\left(\frac{\tau_1}{\tau}\,\nabla p\right) + \frac{2}{j\omega_o}\,\nabla\cdot\nabla\cdot\,(u\nabla p)$$

which is Eq. (1.41).

The derivation illustrates the tortuous reasoning underpinning Eq. (1.41). The value of the neglected terms in relation to those retained depends strongly on the situation. Obviously, $|\underline{v}_a\cdot\nabla\underline{v}_a| \geq \underline{u}\cdot\nabla\underline{v}_a$ whenever the turbulence is small (i.e., $\underline{u} \sim \underline{0}$). Chapter II will give some insight into those situations where Eq. (1.41) may be useful.

We have found that in Eq. (1.41), it is assumed that,

p_0	= constant	a_0	= 0
p_1	= constant	$\underline{a}_1$	= $\underline{0}$
$\underline{T}_0$	= [0]	$\underline{a}_2$	= $\underline{0}$
$\underline{T}_1$	= [0]	$\hat{\underline{a}}_3$	= $c^2\nabla$ (random)
$\nabla\cdot\underline{T}$	= $-\nabla p$	$\underline{b}_1$	= 0
ρ	= constant	$\underline{b}_2$	= 0
$\dfrac{d}{dt}\,(\ell n\rho c^2\,\vert\,\underline{r})$	= 0	$\underline{b}_3$	= 0

$$\underline{f} = 0 \qquad g_0 = -\rho c^2(\underline{v}\cdot\nabla\underline{v})$$

$$k^2 = k_0^2\left(1 + \frac{\tau_1}{\tau}\right). \tag{1.42}$$

Equation (1.41) also assumes the turbulence to be incompressible and the randomness to be spatially homogeneous. It is apparent that this equation is limited to spatial effects.

1.5.5 Blokhintsev and Mintzer

In the mid-1940's, D. Blokhintsev published [22] wave equations for propagation of sound in an inhomogeneous and moving medium. His deriving equations are:

$$\frac{\partial \underline{v}}{\partial t} + \frac{\nabla \underline{v}^2}{2} + (\nabla \times \underline{v}) \times \underline{v} = \frac{-\nabla p_2}{\rho} \tag{B1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\underline{v} = 0 \tag{B2}$$

$$\frac{\partial S}{\partial t} + \underline{v}\cdot\nabla S = 0 \tag{B3}$$

$$p_2 = Z(\rho, S) \tag{B4}$$

where S is the entropy and Z is the equation of state. He further uses $c^2 = (\partial p_2/\partial\rho)_S$, the square of the adiabatic sound velocity, and ignores effects of viscosity and heat conduction.

The momentum Eq. (1.16), can be reduced to (B1) under the assumption $\nabla\cdot\underline{T}_2 = -\nabla p_2$ by using the relation

$$\nabla(\underline{v}\cdot\underline{v}) = 2\underline{v}\cdot\nabla\underline{v} - 2(\nabla \times \underline{v}) \times \underline{v} = \nabla\underline{v}^2 \ .$$

Then

$$\frac{\partial \underline{v}}{\partial t} = \underline{f} + \frac{\nabla\cdot\underline{T}_2}{\rho} - \underline{v}\cdot\nabla\underline{v} = f - \frac{\nabla p_2}{\rho} - \frac{1}{2}\nabla v^2 - (\nabla \times v) \times v$$

which for $\underline{f} = 0$ reduces to (B1).

Solutions are obtained from small perturbations of the deriving equations.

Mintzer, somewhat later [141-143] used these relationships to obtain solutions for homogeneous and isotropic media.

While direct comparison with Eq. (1.27) is difficult for a perturbation theory, it is nevertheless clear that $\nabla\cdot\underline{T}_2 = -\nabla p_2$ and $\underline{f} = 0$ are used. We put down as typical for this approach

$$\begin{aligned}
p_0 &= p_0 & a_0 &= 0 \\
p_1 &= p_1 & \underline{a}_1 &= 0 \\
\nabla\cdot\underline{T}_0 &= -\nabla p_0 & \underline{a}_2 &= \frac{\nabla\cdot\underline{T}_2}{\rho} - \underline{v}(\nabla\cdot\underline{v}) \\
\nabla\cdot\underline{T}_1 &= -\nabla p_1 & \hat{\underline{a}}_3 &= -c^2\nabla \\
\nabla\cdot\underline{T} &= -\nabla p & \underline{b}_1 &= 0 \\
\rho &= \text{small variations} & \underline{b}_2 &= 0 \\
\frac{d}{dt}(\ell n \rho c^2 | \underline{r}) &= 0 & \underline{b}_3 &= 0 \\
\underline{f} &= 0 & g_0 &= 0 \ . \qquad (1.43)
\end{aligned}$$

Physically, these constitutive equations imply an ideal Navier-Stokes fluid with consideration given to its motion and thermodynamic state.

1.5.6 Neubert and Lumley

In an interesting paper J. A. Neubert and J. L. Lumley [151] seek the conditions for the validity of the stochastic Helmholtz equation

$$(\nabla^2 + k^2(\underline{r}))p(\underline{r}) = 0 \ . \tag{1.34}$$

The deriving equations are:

$$\frac{\partial \rho}{\partial t} = \nabla\cdot\rho\underline{v} = 0 \tag{N1}$$

$$\frac{\partial}{\partial t}(\rho\underline{v}) + \nabla\cdot(\rho\underline{v}\underline{v}^T) = -\nabla p_2 + \frac{1}{3}\mu\nabla(\nabla\cdot\underline{v}) + \mu\nabla\cdot(\nabla\underline{v}) \tag{N2}$$

$$\frac{\partial p}{\partial \rho_a} = c^2(\underline{r}) \ , \ \text{(random)} \ . \tag{N3}$$

(N2) is a linear form of the Navier-Stokes equation.[1] Equation (N3) is a simplified equation of state. From these equations, a basic stochastic equation is obtained:

---

[1]The non-linear form of the Navier-Stokes equation is

$$\underline{T}_2 = (-p_2 + \alpha)I + \beta E + \gamma E^T E$$

N2 is, as given, spatially non-linear in $\underline{v}$, but linear in p_2. Not surprisingly, these same characteristics are found in equation (1.44) following. Materially, N2 is linear in $\underline{v}$ as well, because $\partial/\partial t(\rho\underline{v}) + \nabla\cdot(\rho\underline{v}\underline{v}^T) = d/dt(\rho\underline{v})$. The non-linearity of N2 is thus a hydrodynamic perspective.

$$\nabla^2 p - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \underline{u}\nabla\right)^2 p$$

$$= 2(\nabla\cdot\rho_0\underline{v}_a)\ \underline{u}\cdot\frac{\nabla c}{c} - \nabla\underline{u}:\nabla(2\rho_0\underline{v}_a - \rho_a\underline{u}) \qquad (1.44a)$$

where $\underline{v} = \underline{u} + \underline{v}_a$ and $\rho = \rho_0 + \rho_a$ are the turbulent and acoustic decomposition; the symbol (:) denotes a double contraction.[1]

Neubert's basic equation is obtained by approximations based on physcial considerations. A cardinal approximation ignores terms that scale like $\lambda u^2/L_u c^2$ relative to unity, where λ is the wavelength and L_u is the turbulent integral scale for velocity fluctuations,

$$L_u = \int_0^\infty \langle u(\underline{r})u(r_0)\rangle d(|\underline{r} - \underline{r}_0|) \ . \qquad (1.45)$$

As initial approximations, $\nabla\cdot\underline{u}$ and $\underline{b}_3$ are set to zero, that is, the turbulent fluid is considered incompressible and the acoustic stress tensor is assumed diagonal.

The left hand side (LHS) of Eq. (1.36) and the fundamental Eq. (1.27) are the same under the conditions $(d/dt)\ell n\rho c^2 = 0$, $\nabla\ell n\rho = 0$, $v \doteq u$. The right hand side (RHS) of Eq. (1.36) comes from the term

$$\rho c^2\nabla\cdot\left(\underline{v}\cdot\nabla\underline{v} - \frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho}\right)$$

of g_0. Since

---

[1]Double contraction takes a second order tensor into a scalar, often written $A_{ij}B_{ji}$.

$$c^2\nabla\cdot\ \rho\underline{v}\cdot\nabla\underline{v} - \nabla\cdot(\underline{T}_0 + \underline{T}_1) \ = c^2\nabla\cdot(\rho\underline{v}\cdot\nabla\underline{v}) - c^2\ \nabla\cdot\nabla\cdot(\underline{T}_0 + \underline{T}_1),$$

and the term

$$c^2\nabla\cdot\nabla\cdot(\underline{T}_0 + \underline{T}_1) \approx c^2\nabla\cdot\rho\ \frac{\partial\underline{u}}{\partial t} \approx \nabla\cdot\ \rho c^2\frac{\partial\underline{u}}{\partial t} - \rho\ \frac{\partial\underline{u}}{\partial t}\ \cdot\nabla c^2$$

$$\approx \rho\ \frac{\partial\underline{u}}{\partial t}\cdot\nabla c^2$$

from scaling, the first term of Eq. (1.44) can be obtained from

$$\rho\ \frac{d\underline{u}}{dt} = \rho\ \frac{\partial\underline{u}}{\partial t} + \underline{v}\cdot\nabla\underline{u} \approx \rho_0\underline{v}_a\cdot\nabla\underline{u}$$

$$\approx \nabla\cdot(\rho_0\underline{v}_a\cdot\underline{u}) - \underline{u}\cdot\nabla(\rho_0\underline{v}_a) \approx u\cdot\nabla\rho_0\underline{v}_a\ .$$

These manipulations clearly demonstrate the importance of the scaling in this development.

The second LHS term of Eq. (1.44) can be manipulated from $c^2\nabla\cdot(\rho\underline{v}\cdot\nabla\underline{v})$. We can thus set the conditions and assumptions for Eq. (1.44).

$p_0 = p_0$ $\qquad$ $a_0 = 0$

$p_1 = p_1$ $\qquad$ $\underline{a}_1 = \underline{u}$

$\underline{T}_0 = \underline{T}_0$ $\qquad$ $\underline{a}_2 = -\underline{v}\cdot\nabla\underline{u} + \dfrac{\nabla\cdot\underline{T}_2}{\rho}$

$\underline{T}_1 = \underline{T}_1$ $\qquad$ $\hat{\underline{a}}_3 = -c^2(\underline{r})\nabla$

$\nabla\cdot\underline{T} = -\nabla p_2 + \frac{\mu}{3}\nabla(\nabla\cdot\underline{v}) + \mu\nabla^2\underline{v}$, $\underline{b}_1 = 0$

$$\rho = \text{constant} \qquad \underline{b}_2 = 0$$

$$\frac{d}{dt}\,(\ell n \rho c^2 | \underline{r}) = 0 \qquad \underline{b}_3 = 0 \qquad (1.46)$$

$$\underline{f} = 0 \qquad g_0 = \rho c^2 \nabla\cdot\left(\frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho} - \underline{v}\cdot\nabla\underline{v}\right).$$

The Navier-Stokes assumption plays no role in deducing Eq. (1.44). Physically, we see that Eq. (1.44) incorporates certain non-linear characteristics of the moving medium, considered deterministic save for random parameter $\underline{c}^2(\underline{r})$. Even given Eq. (1.44), the justification of the stochastic Helmholtz equation requires further approximation. shall return to [151] in Chapter II where criteria for the validity of assumptions are given.

1.5.7 Non-Linear Acoustics

Finally following Blackstock [21] we give the basic equations used to model non-linear acoustics. These are the following:

$$\frac{\partial \lambda}{\partial t} + v\,\frac{\partial \lambda}{\partial r} + c\,\frac{\partial v}{\partial r} = 0\ ,\ c = c(\underline{r},t), \qquad (B1\text{-}1)$$

$$\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial r} + c\,\frac{\partial \lambda}{\partial r} = 0\ , \qquad (B1\text{-}2)$$

where

$$\lambda = \int_{\rho_0}^{\rho} (c/\rho)\,d\rho\ .$$

These basic equations stem from an assumed plane (one dimensional) wave propagation in an ideal fluid. Following Earnshaw, λ is set equal to v, so that equation (B1-1) reduces to equation (B1-2). Solutions are

obtained by assuming a perfect gas law (e.g., $p/p_o = (\rho/\rho_o)^\gamma$, and evaluating the parameter, λ.

Because this method does not express a differential equation in p, it is somewhat difficult to categorize. We put down

$$
\begin{aligned}
p &= p_o & a_o &= 0 \\
p_1 &= 0 & \underline{a}_1 &= \underline{v} \\
\underline{T}_o &= [0] & \underline{a}_2 &= [0] \\
\underline{T}_1 &= [0] & \hat{\underline{a}}_3 &= -c^2(\underline{r},t)\nabla \\
\nabla\cdot T &= -\nabla P & \underline{b}_1 &= 0 \\
\rho &= \rho(\underline{r},t) & \underline{b}_2 &= 0 \\
\frac{d}{dt}\,(\ell n \rho c^2 | \underline{r}) &= 0 & \underline{b}_3 &= 0 \\
\underline{f} &= 0 & g_o &= 0.
\end{aligned}
\qquad (1.47)
$$

The focus of attention here is the dynamic character of the propagation velocity $c(\underline{r},t)$ and the density $\rho(\underline{r},t)$. It can be seen that turbulence (gusts of wind, for example) is not accounted for.

1.6 Summary and Remarks

In Table 1-1, we summarize the observations made in the preceding section. One notes that most acoustic theory has been based on ideal media. The measurable effects of turbulence and body forces, however, are attributes of real media, so it is hardly surprising that the subject of wave propagation in turbulent fluids has been controversial.

Many of our concepts stem from the assumptions of ideal media. The notion of "beam patterns," for instance, implies a linear, homogeneous medium. As turbulence comes to the fore, it becomes increasingly difficult to define a "beam pattern." The general Eq. (1.27)

TABLE 1-1. CATALOG OF ASSUMPTIONS[1,2]

	Ideal	Stochastic Helmholtz	Chernov	Tatarskii I	Tatarskii II	Blokhintsev	Neubert	Blackstock	General
p_0	constant	constant	constant	constant		p_0	p_0	constant	p_0
p_1	constant	constant	constant	constant		p_1	p_1	0	p_1
$\underline{T}_0$	constant	constant	constant	constant		$-\nabla p_0$	$\underline{T}_0$	[0]	$\underline{T}_0$
$\underline{T}_1$	constant	constant	constant	constant		$-\nabla p_1$	$\underline{T}_1$	[0]	$\underline{T}_1$
$\nabla\cdot T$	$-\nabla p$	$-\nabla p$	$-\nabla p$	$-\nabla p$		$-\nabla p$	$-\nabla p + \frac{1}{3}\nabla(\nabla\cdot\underline{v}) + \mu\nabla\cdot(\nabla\underline{v})$	$-\nabla p$	$\nabla\cdot\underline{T}_2$
ρ	constant	constant	$\rho(\underline{r})$	constant		perturbation	constant	$\rho(\underline{r},t)$	ρ
$\frac{d}{dt}(\ln\rho c^2\|\underline{r})$	0	0	0	0		0	0	0	$\frac{d}{dt}(\ln\rho c^2\|\underline{r})$
$\underline{f}$	0	0	0	0		0	0	0	$\underline{f}$
a_0	0	0	0	0	0	0	0	0	a_0
$\underline{a}_1$	0	0	0	$\underline{v}$	0	0	0	$\underline{v}$	$\underline{a}_1$
$\underline{a}_2$	0	0	0	0	0	$\frac{\nabla\cdot\underline{T}_2}{\rho} - \underline{v}\cdot\nabla\underline{v}$	$\frac{\nabla\cdot\underline{T}_2}{\rho} - \underline{v}\cdot\nabla\underline{v}$	0	$\underline{a}_2$
a_3	$-c^2\nabla$	$-(c_0^2/(1+\mu)\nabla$	$-c^2(\nabla-\nabla\ln\rho)$	$-c^2\nabla$	$-c^2\nabla$	$-c^2\nabla$	$-c^2\nabla$	$c^2(\underline{r},t)\nabla$	a_3
$\underline{b}_1$	0	0	0	0	0	0	0	0	$\underline{b}_1$
$\underline{b}_2$	0	0	0	0	0	0	0	0	$\underline{b}_2$
$\underline{b}_3$	0	0	0	0	0	0	0	0	$\underline{b}_3$
g_0	0	0	0	0	$\rho c^2\nabla\cdot(\underline{v}\cdot\nabla\underline{v})$	0	$\rho c^2\nabla\cdot\frac{\nabla\cdot(\underline{T}_0+\underline{T}_1)}{\rho} - \underline{v}\cdot\nabla\underline{v}$	0	g_0
Basic Equation	(1.31)	(1.34)	(1.37)	(1.39)	(1.41)	(B)	(1.44)	(BL)	(1.27)/(1.28)

[1]All except the general equation use partial derivatives in place of the hydrodynamic derivatives.

[2]All except the general equation and Neubert assume a perfect fluid.

has, for another instance, no well defined wave structure. Moreover, it is non-linear. Evidently, it is important to know in what circumstances our usual concepts apply and to what extent they no longer hold in the physical medium.

To answer such questions, a naive utilitarianism breaks down. Nothing will do but to search for a theory, logically consistent, and broad enough to embrace our interests. With such a view in hand, we possess the means of judging quantitatively, and thus the knowledge to apply approximations effectively.

Tatarskii remarks [188, pg. 153] that "...many of the early studies of the scattering of sound in a turbulent atmosphere proceeded from exceedingly crude starting equations." This chapter has reviewed some prominent recent efforts and has found that many of these depart widely from a strict derivation based on hydrodynamic principles. Fortunately, the fluids of most interest to us usually are weakly turbulent. It is when the turbulence is strong, or when turbulent effects have accumulated because of propagation over long distances, that the various theories lead to conflicting or incomplete results.

The method selected has led to several original results. Foremost is the comparison of the usual formulations with the general adiabatic case (Table 1-1). This yields, when added to the axioms (A1) and (A2), a complete set of assumptions for the usual formulations. Secondly, we have distinguished between the hydrodynamic and partial derivatives (see Eq. (1.4)). In Appendix A this distinction becomes crucial in obtaining the first known exact solutions to the adiabatic hydrodynamic equations. Our method has likewise derived the propagation of non-acoustic fields (see Section 1.4) and the general stochastic (Langevin)

Eq. (1.28). Important, likewise, but mathematically oriented, is our framing the extension of the divergence theorem (Sec. 1.2.1.2).

In concluding this section, we comment on some historical trends. The difficulty of solving the hydrodynamic equations has long been recognized. Consequently, early researchers, in their quest for solutions, simplified the constitutive equations. Emphasis was placed on solutions in the classical sense using idealized boundary values. As the basic stochastic nature of the problem came to be appreciated, researchers attempted to add stochastic variables to the simplified equations. Thus, solutions were produced which accounted for some, but not all, of the observed effects.

Relying on this considerable historical effort, we can now see the problem in a different perspective. Viewing propagation as essentially a random process considerably eases the need for simplifying approximations. Further, it removes the task of medium identification from a practically impossible task to one that is at least tractable. On the other hand, this view forces a new meaning to the solution of the problem, for we can no longer insist on deterministic solutions. Meaningful solutions, rather, are given in terms of process (ensemble) characteristics, namely moments, distributions, and similar ensemble properties.

But, before turning to the solution of the hydrodynamic and wave equations, let us first determine the conditions for the validity of wave equations in the media of most interest.

II. APPROXIMATIONS

2.0 Introduction

Having formally derived the assumptions of the usual models, we propose in this chapter to examine quantitatively the circumstances under which they are valid. Because the medium itself will not be denied, we concentrate our attention on the atmosphere and the ocean.

The most satisfying method of judging an approximation to the general equation, (1.22) or (1.27), would be to condition a medium experimentally and compare results. Unfortunately, except for a few laboratory environments, it is practically impossible to condition the medium dynamically.

Another satisfactory method would compare the solution of the general equation to that of the approximation. Thus, for the ideal, physically homogeneous medium, the hydrodynamic equation for a harmonic plane wave[1] is

$$p(\underline{r},t) = \frac{\rho_0 c_0^2}{1 - k_0 a \cos(\omega_0 t - k_0 x(r))}, \quad k_0 = \omega_0/c_0, \qquad (2.1)$$

which, when contrasted to the classical solution,

$$\hat{p}(\underline{r},t) = \rho_0 c_0^2 (1 + k_0 a \cos(\omega_0 t - k_0 r)), \qquad (2.2)$$

[1]See Appendix A.

shows the latter to be in error by

$$e \equiv p - \hat{p} = \rho_0 c_0^2 \left(\sum_{n=2}^{\infty} (k_0 a)^n \cos^n(\omega_0 t - k_0 x) + k_0 a(\cos(\omega_0 t - k_0 x) - \cos(\omega_0 t - k_0 r)) \right) . \qquad (2.3)$$

Since the general solution is exact for all adiabatic propagation, approximations for many cases of interest may be found. Still, this method is mostly inconvenient. It presupposes solutions to the general equation, which, in turn, require exact knowledge of the medium and, also importantly, a precise description of the geometry (including interfaces). Change the medium or the geometry, the solution also changes, sometimes dramatically, so that what was a good approximation becomes questionable.

A less satisfying method, but one which yields more insight, focuses on the situation at a point. A description of the situation at any point is available from the differential equation. One may readily inquire into the magnitude of the terms comprising the general differential equation and so come to a judgment (sometimes more qualitative than quantitative) of a given approximation. Thus, for example, the element $(d/dt)(\ell n \rho c^2 | \underline{r})$ may be compared with $(d^2/dt^2)(p|\underline{r})$ for a particular medium (which comparison inexorably involves the physics of the medium) and so a judgment may be formed of any approximation which neglects it. Contrariwise, conditions may be found under which a neglected element of the general equation begins to compare in magnitude with retained elements in an approximation. Under these conditions, one cannot hope the approximation to be useful.

Contrasted to approximation by results (either experimental or integrated solutions) approximations at a point require no knowledge of geometry or interfaces. In this sense, the conclusions drawn thus enjoy a somewhat more general application.

On the other hand, a strong caveat is in order. Given an effect negligible at a point, it does not follow that it is negligible in the result. The result integrates the effect at a point over a region so that the integrated result may be appreciable. For example, we may find tidal motion insignificant at a point, but consequential in the result. In the sequel, when there is reason to believe such to be the case, we shall term the effect a "global effect."

In addition to the caveat mentioned above, our analysis suffers from several other defects not intrinsic to the method. One can appreciate the first of these by noticing that most physical descriptions are given in terms of a static medium whereas we consider the dynamic medium. The speed of sound in the ocean, for instance, is given in terms of temperature, salinity and depth [210] (a static representation). Dynamically, it is expressed as the ratio of change in pressure and density. Density is affected, of course, by temperature, salinity and depth; it is, likewise, affected by the motion of the medium. We must perforce accept such physical data as are available.

Further, because the point description is given in terms of the operators $(d/dt)(\cdot|\underline{r})$, $d^2(\cdot|\underline{r})/dt^2$, ∇ and ∇^2, we must have some consistent way of evaluating the effect of these operators. We use the following approximations (suggested by Fourier transforms):

$$\frac{d}{dt}(\cdot|\underline{r}) \rightarrow \omega(\cdot)$$

$$\frac{d^2}{dt^2}(\cdot|\underline{r}) \rightarrow \omega^2(\cdot)$$

$$\nabla(\cdot) \rightarrow \underline{k}(\cdot)$$

$$\nabla^2(\cdot) \rightarrow k^2(\cdot) \qquad (2.4)$$

so that validity is expressed in terms of frequency and wave number (not necessarily restricted to the source).

The approximations given in Eq. (2.4) can be appreciated from the exact solutions for a plane wave (Appendix A). For the acoustic pressure

$$p = \frac{\rho_0 c_0^2 k_0 a \cos(\omega_0 t - k_0 x)}{1 - k_0 a \cos(\omega_0 t - k_0 x)} ,$$

we have

$$\frac{d(p|\underline{r})}{dt} = -\frac{\rho_0 c_0^2 \omega_0 k_0 a \sin(\omega_0 t - k_0 x)}{[1-k_0 a \cos(\omega_0 t - k_0 x)]^3} = -\frac{\omega_0 p}{[1-k_0 a \cos(\omega_0 t - k_0 x)]^2} \quad \omega_0 p$$

$$\nabla p = \frac{\rho_0 c_0^2 k_0^2 a \sin(\omega_0 t - k_0 x)}{[1-k_0 a \cos(\omega_0 t - k_0 x)]^3} = \frac{k_0 p}{[1-k_0 a \cos(\omega_0 t - k_0 x)]^2} \quad k_0 p .$$

Thus, in those areas of the field where the phenomenon acts locally like a plane wave, the approximate transformations represented by Eq. (2.4) may be safely used. For many cases of interest, this is, of course, the situation for much of the field.

In the sequel, we will often check our results with reference to the plane wave case. In so doing, the approximation may be judged against a simple but exact comparison.

The upshot of these decisions is to give the results of this chapter a certain coarseness. We will, consequently, conclude by suggesting some experiments for refining our results.

2.1 Order of Importance: First Order Equation

The first-order general Eq. (1.22) may be rendered as

$$\frac{d(p|\underline{r})}{dt} + \underline{v}\cdot\nabla p + \rho c^2 \nabla\cdot\underline{v} + \frac{d(p_0|\underline{r})}{dt} + \frac{d(p_1|\underline{r})}{dt} + \underline{v}\cdot\nabla p_0 + \underline{v}\cdot\nabla p_1 = 0 \ . \tag{2.5}$$

For clarity we specify (in contrast to $p_2 = p_0 + p_1 + p$), the decomposition

$$\underline{v} = \underline{v}_0 + \underline{v}_1 + \underline{v}_a$$

$$\underline{k} = \underline{k}_0 + \underline{k}_1 + \underline{k}_a$$

$$\rho = \rho_0 + \rho_1 + \rho_a$$

to designate the total parameter and its decomposition into background, turbulent and acoustic components, respectively. Let us assume an unchanging background pressure, i.e., $[d(p_0|\underline{r})]/dt = 0$. Using Eq. (2.4), we obtain for Eq. (2.5)

$$\omega_a p + p\underline{k}_a \cdot \underline{v} + \rho c^2 \, \underline{v} \cdot (\underline{k}_0 + \underline{k}_1 + \underline{k}_a) + \omega_1 p_1 + p_0 \underline{k}_0 \cdot \underline{v}$$

$$+ p_1 \underline{k}_1 \cdot \underline{v} = 0 \, . \qquad (2.6)$$

Each term may now be compared to the first term.

1. The term $p\underline{k}_a \cdot \underline{v}$ compares to $\omega_a p$ as

$$\frac{p\underline{k}_a \cdot \underline{v}}{p\omega_a} = \frac{\underline{k}_a \cdot \underline{v}}{\underline{k}_a \cdot \underline{c}} \quad \frac{|\underline{v}|}{|\underline{c}|} \equiv \frac{v}{c} \, , \text{ with } \omega_a = \underline{k}_a \cdot \underline{c} \, . \qquad (2.7)$$

The particle velocity, $\underline{v}$, is due to background motion, turbulence, and acoustic influences. In the sea, we may find currents of up to 4 kt or 0.5 percent the propagation velocity. Thus, if <u>globally present</u> (i.e., present along the entire wave traverse), we might expect to discern an effect at 200. In the air, winds up to 50 km/hr are not too unusual; this is an effect of 4.2 percent, noticeable at 25 m. Gusts and turbulent eddies may be expected to produce effects of the same order.

Particle motion due to acoustic effects may be estimated from displacements of transducer heads. For harmonic motion, the time required for one cycle is λ/c, during which transducer head has traveled 4a. The head velocity can thus be approximated as:

$$v_a = \frac{4ca}{\lambda} \, ,$$

or

$$\frac{v}{c} = \frac{|\underline{v}_a|}{|\underline{c}|} = \frac{4a}{\lambda} = k_a a \, .$$

Head displacement is obviously a function of power and frequency. As a reference, one may expect a transducer emitting 500 W into sea water at 3 kHz to produce a displacement of 0.15×10^{-4} m. Thus, $|\underline{v}_a|/|\underline{c}| \simeq 10^{-4} = 0.01$ percent. Moreover, this action may be purely local, the sinusoidal motion cancelling any cumulative effect. The motion of a transducer head, as we see from here and Appendix A, figures as an important medium descriptor.

The motion induced by speakers in air is quite sensible, certainly exceeding 10^{-3} m at high power. The effect in air of this term then exceeds that in sea water by at least an order of magnitude. In either case one expects the displacement to be a function of pressure, amplitude, and frequency.

2. The third term of Eq. (2.6), $\rho c^2 \nabla \cdot \underline{v}$, compares with the first as

$$\frac{\rho c^2 \underline{v} \cdot (\underline{k}_a + \underline{k}_0 + \underline{k}_1)}{\underline{k}_a \cdot \underline{c} p} \simeq \frac{\rho c^2 |v|}{p|c|} + \frac{\rho c^2 \underline{v} \cdot \underline{k}_0}{p\underline{c} \cdot \underline{k}_a} + \frac{\rho c^2 \ \underline{v} \cdot \underline{k}_1}{p\underline{c} \cdot \underline{k}_a} \qquad (2.8)$$

For many cases $(\rho c^2/p)(|v|/|c|) \simeq 1$. Letting, for instance, $p = \rho_a c^2$, where $\rho = \rho_0 + \rho_a \simeq \rho_0 + k_a \rho$, we have

$$\frac{\rho c^2}{p} = \frac{\rho c^2}{\rho_a c^2} \quad \frac{\rho}{\rho(ka)} \quad \frac{|c|}{|v|}$$

from above. Thus, for acoustic pressures, this term is strictly comparable to $d(p|\underline{r})/dt$.

Tatarskii and others treat the turbulence as incompressible ($\nabla \cdot \underline{v}_1 = 0$), so that for them, the term $\rho c^2 \nabla \cdot \underline{v}$ is predominately acoustic.

For propagating internal waves, however, the turbulent terms would compare as the frequencies involved. This means that at the lower "acoustic" frequencies (i.e., less than 1 Hz), the effects of the higher frequency internal waves can be pronounced.

3. The term $\omega_1 p_1$ of Eq. (2.6) compares as

$$\frac{\omega_1 p_1}{\omega_a p} \simeq \frac{|k_1| p_1}{|k_a| p} \quad . \tag{2.9}$$

Except for very low frequency transmissions, the acoustic wave number is usually larger than the turbulent "wave number," (i.e., $k >> k_1$). However p_1, the turbulent pressure may be much greater than the acoustic pressure p. Few acoustic waves, for example, exert sufficient pressure to blow off one's hat. The importance of this term comes from the strength of the turbulence and acoustic frequency employed.

4. Because k_0 (the wave number of the background pressure) is an awkward concept, we consider the term $p_0 \underline{k}_0 \cdot \underline{v}$ in its original form $\underline{v} \cdot \nabla p_0$. This term compares as

$$\frac{\underline{v} \cdot \nabla p_0}{\omega_a p} \simeq \frac{\underline{v} \cdot \nabla p_0}{(\underline{c} \cdot \underline{k}_a) p} \simeq \frac{(\underline{k}_0 + \underline{k}_1 + \underline{k}_a) a}{|k|} \cdot \frac{\nabla p_0}{p}$$

$$\simeq \frac{a(\underline{k}_0 + \underline{k}_1 + \underline{k}_a)}{|k_a|} \cdot \frac{\nabla p_0}{\rho_0 c^2 k_a a} \simeq \frac{\lambda |\nabla p_0|}{2\pi \rho_0 c^2} \quad ; \quad k_a = \frac{2\pi}{\lambda} \quad .$$

In air $|\nabla \underline{p}_0|$ may be taken as 10 Pa/m, while in the sea $|\nabla p_0| \simeq 10^4$ Pa/m. In water $\rho_0 c^2 \simeq 225 \times 10^7$ Pa; in air $\rho_0 c^2 \simeq 15 \times 10^4$ Pa. Thus, this element is insignificant locally, but quite important globally since it almost always accumulates. For the acoustic condition then, we have

$$\frac{\lambda \nabla p_0}{2\pi \rho_0 c^2} \begin{cases} \simeq (7 \times 10^{-4})\lambda & \text{for water} \\ \simeq (1.5 \times 10^{-5})\lambda & \text{for air.} \end{cases} \quad (2.10)$$

5. Finally, the term $p_1 \underline{k}_1 \cdot \underline{v}$ compares to the first term of Eq. (2.6) as

$$\frac{p_1 \underline{v} \cdot \underline{k}_1}{\omega_a p} = \frac{p_1 \underline{v} \cdot \underline{k}_1}{p \underline{c} \cdot \underline{k}_a} \simeq \frac{p_1 (\underline{k}_0 + \underline{k}_1 + \underline{k}_a) a |k_1|}{p\, k_a} = \frac{p_1 |k_1|}{\rho_0 c^2 |k_a|} \cdot \left(1 + \frac{|k_0 + k_1|}{|k_a|} \right). \quad (2.11)$$

The effect is appreciable locally when $p_1 a k_1 \geq p$. The global effects depend strongly on the strength, nature and extent of the turbulence.

These approximations, gross as they are, show the great differences between the sea and the atmosphere in acoustic propagation. The differences between sea and pure water are much more subtle and require better approximations (including losses in propagation). Our results are summarized in Table 2-1.

We now compare these approximations with the known conditions for an ideal hydrodynamic plane wave. In this case

$$\frac{d(p_2 | \underline{r})}{dt} = \frac{\rho_0 c_o^2 \omega_a k_a \sin(\omega_a t - k_a x)}{[1 - k_a a \cos(\omega_a t - k_a x)]^3} = \frac{-\omega_a p \tan(\omega_a t - k_a x)}{[1 - k_a a \cos(\omega_a t - k_a x)]^2} \quad (2.12a)$$

$$v_a = a \omega_a \cos(\omega_a t - k_a x) \quad (2.12b)$$

$$\nabla \cdot \underline{v}_a = \frac{\omega_a k_a a \sin(\omega_a t - k_a x)}{1 - k_a a \cos(\omega_a t - k_a x)} \quad (2.12c)$$

TABLE 2-1. ANALYSIS OF LOCAL APPROXIMATIONS:

GENERAL FIRST ORDER EQUATION

Element	Local Approx.	Comparability to dp/dt	Typical Values Air Ac.	Air Tur.	Sea Water Ac.	Sea Water Tur.	Global Conditions
$\frac{d(p\|\underline{r})}{dt}$	$\omega_a p$	1	-	-	-	-	-
$\underline{v}\cdot\nabla p$	$p\underline{k}\cdot\underline{v}_a$	$\frac{v}{c}$	0.1%	4.2%	0.01%	0.5%	Turbulent effect at near ranges
$\rho c^2 \nabla\cdot\underline{v}$	$\rho c^2 k\cdot\underline{v}$	1	100%	*	100%	*	Turbulent effects at low frequencies ≈ .1 Hz
$\frac{d(p_1\|\underline{r})}{dt}$	$\omega_1 p_1$	$\frac{k_1 p_1}{kp}$	-	*	-	*	Important when $p_1 > p$; when $p_1 < p$, important at moderate ranges.
$\underline{v}\cdot\nabla p_0$	$\underline{v}\cdot\nabla p_0$	$\frac{\|\nabla p_0\|}{k\rho_0 c^2}$	.002λ%	**	.07λ%	**	Acoustic effect appreciable at long ranges.
$\underline{v}\cdot\nabla p_1$	$\underline{v}\cdot\underline{k}_1 p_1$	$\frac{k_1 p_1}{k\rho_0 c^2}$	-	**	-	**	Effect important only in strong turbulences

*depends on frequency
**depends on the strength and extent of the turbulence

$$\nabla p_2 = \frac{\rho_0 c_0^2 k_a^2 a \sin(\omega_a t - k_a x)}{[1-k_a a \cos(\omega_a t - k_a x)]^3} \tag{2.12d}$$

$$p_1 = 0 \ . \tag{2.12e}$$

$$\text{Then } \frac{\underline{v}\cdot\nabla p}{\frac{d(p|\underline{r})}{dt}} = \frac{a\omega_a \sin(\omega_a t-k_a x)\rho_0 c_0^2 k_a^2 a \sin(\omega_a t-k_a x)[1-k_a a \cos(\omega_a t-k_a x)]^3}{[1-k_a a \cos(\omega_a t-k_a x)]^3 \ (-\rho_0 c_0^2 \omega_a k_a a \sin(\omega_a t-k_a x))}$$

$$= -k_a a \sin(\omega_a t - k_a x) \tag{2.13}$$

which, in magnitude, is the result obtained for the acoustic case of Eq. (2.7).

Similarly, we have

$$\frac{\rho c_0^2 \nabla\cdot\underline{v}}{\frac{d(p|\underline{r})}{dt}} = \frac{\rho_0 c_0^2 \ \omega_a k_a a \sin(\omega_a t-k_a x) \ [1-k_a a \cos(\omega_a t-k_a x)]^3}{[1-k_a a \cos(\omega_a t-k_a x)]^2 \ (-\rho_o c_0^2 \omega_a k_a a \sin(\omega_a t-kx))}$$

$$= -[1 \ - k_a a \cos(\omega_a t - k_a x)] \ , \tag{2.14}$$

so that this term, as we found in Eq. (2.8), is strictly comparable to $\frac{d(p|r)}{dt}$. For this homogeneous case, the other terms of Eq. (2.5) equal zero.

2.2 Order of Importance: Second Order Equation

While the previous section gives some insight into the order of importance, it does not set conditions for the approximations of Chapter I. For this, we need the second order general Eq. (1.28). Using Eq. (2.4), we set down the elements of the equation:

$$1)\quad \frac{d^2(p|\underline{r})}{dt^2} \to \omega^2 p$$

$$2)\quad a_0 \frac{d(p|\underline{r})}{dt} \to \omega p \frac{d}{dt}(\ell n \rho c^2|\underline{r})$$

$$3)\quad \underline{a}_1 \cdot \frac{d}{dt}(\nabla p|\underline{r}) \to \omega p \underline{k}\cdot\underline{v}$$

$$3a)\quad \underline{a}_1 \cdot \frac{d}{dt}(\nabla(p_0 + p_1)|\underline{r}) \to \omega_1 p_1 \underline{k}_1\cdot\underline{v}$$

$$4)\quad \underline{a}_2\cdot\nabla p \to \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \frac{d}{dt}(\ell n\rho c^2|r)\,\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot \underline{k}p \qquad (2.15)$$

$$4a)\quad \underline{a}_2\cdot\underline{b}_1 \to \left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \frac{d}{dt}(\ell n\rho c^2|\underline{r})\,\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot (\nabla p_0 + \underline{k}_1 p_1)$$

$$5)\quad \hat{\underline{a}}_3\cdot\nabla p \to c^2k^2p - c^2(\nabla\ell n\rho)\cdot(\underline{k}p)$$

$$5a)\quad \hat{\underline{a}}_3\cdot b_3 = -c^2(\nabla - \nabla\ell n\rho)\cdot\underline{b}_3$$

$$6)\quad g_0 \to \omega_1^2 p_1 - \frac{d}{dt}(\ell n\rho c^2|\underline{r})\,\omega_1 p_1 - \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho}\right)$$

Each term of Eq. (2.15) will be compared to the first term.

1. The first term compares as 1.
2. The second term compares to the first as

$$\frac{\omega p \dfrac{d}{dt}(\ell n \rho c^2 | \underline{r})}{\omega^2 p} = \frac{\dfrac{d}{dt}\ell n(\rho c^2 | \underline{r})}{\omega} = \frac{\dfrac{d}{dt}(\rho c^2 | \underline{r})}{\omega \rho c^2} \quad . \tag{2.16}$$

Let $\rho(\underline{r},t) = \rho_0(1+\sigma)$ and $c^2(\underline{r}) = c_0^2(1+\mu)$, with ρ_0 and c_0^2 constant. Then

$$\frac{d}{dt}(\rho c^2 | \underline{r}) = \rho_0 c_0^2 \frac{d}{dt}[(1+\sigma)(1+\mu)|\underline{r}]$$

$$= \rho_0 c^2 \left[(1+\sigma)\frac{d(\mu|\underline{r})}{dt} + (1+\mu)\frac{d(\sigma|\underline{r})}{dt}\right],$$

so that (2.16) becomes (with $\dfrac{d(\mu|\underline{r})}{dt} \to 0$ and $\dfrac{d(\sigma|\underline{r})}{dt} \to j\omega\sigma$)

$$\frac{\dfrac{d}{dt}(\rho c^2 | \underline{r})}{\omega \rho c^2} \quad \frac{\rho_0 c_0^2 (1+\mu)\omega\sigma}{\omega \rho_0 c^2 (1+\mu)(1+\sigma)} \quad \frac{\sigma(\underline{r},t)}{1+\sigma(\underline{r},t)} \quad . \tag{2.17}$$

In the ocean (if we except interface discontinuities) $\sigma < .04$[1], so that this term may generally be disregarded. This means that the temporal relaxation is a very weak phenomenon in the oceans. The result is generally true for air as well, except for strong sounds (an over-pressure of about 2 atmospheres). In this case, temporal relaxation becomes an important factor. See Fig 2-1. A practical conclusion is that because of this relaxation, high intensity acoustic ranging is likely to be more effective in water than in air.

---

[1]Equivalently ρ = 1000 to 1040 kg/m^3. See [57]

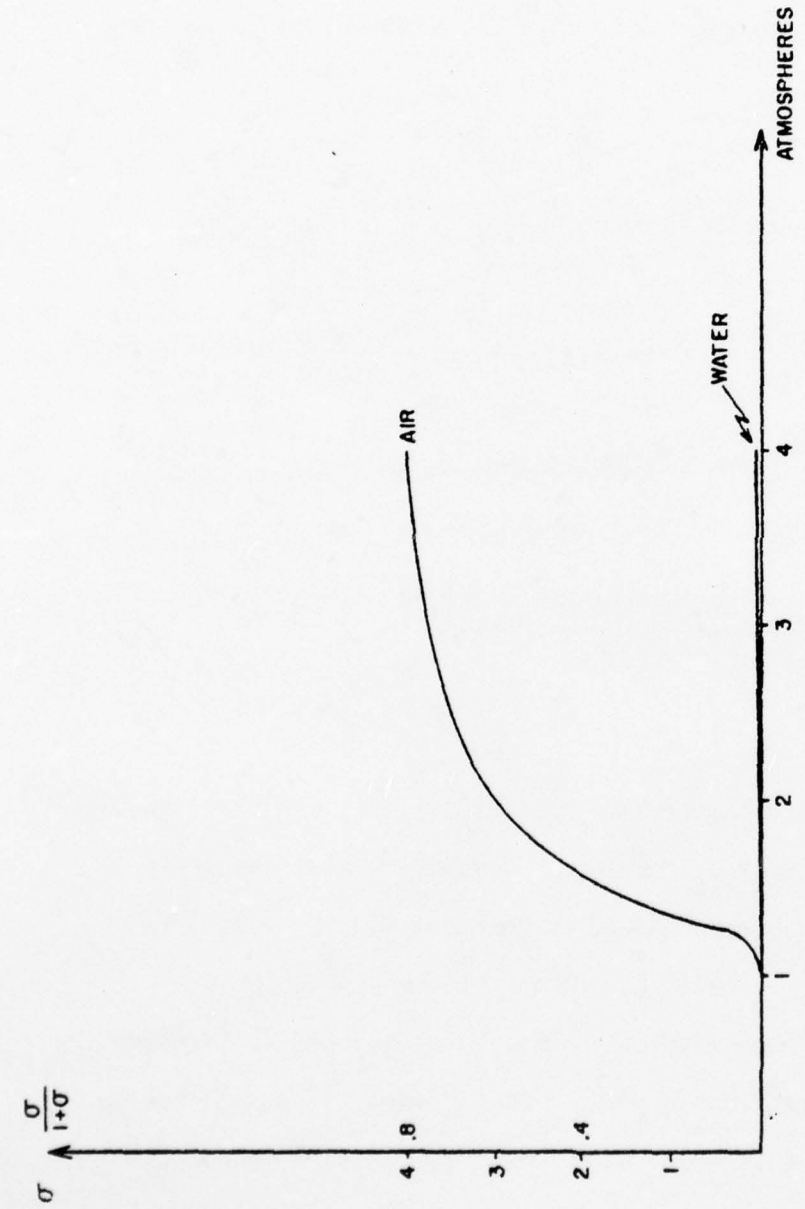


Fig. 2-1. Relative Importance of $\left(\frac{dp|\underline{r}}{dt}\right)\left(\frac{d}{dt}\left(\ell n \rho c^2|\underline{r}\right)\right)$

The plane wave solution (Appendix A) illuminates the above comparison. For this case we have

$$\frac{\frac{d}{dt}\left(\ell n \rho c^2 | \underline{r}\right) \frac{d(p|\underline{r})}{dt}}{\frac{d^2}{dt^2}(p|\underline{r})} = \frac{\frac{1}{p_2}\left(\frac{d}{dt}(p_2|\underline{r})\right)\left(\frac{d(p|\underline{r})}{dt}\right)}{\frac{d^2}{dt^2}(p|\underline{r})} = \frac{\frac{1}{p_2}\left(\frac{d}{dt}(p|\underline{r})\right)^2}{\frac{d^2}{dt^2}(p|\underline{r})}$$

$$= \frac{[\rho_o c_o^2 \omega_a k_a a \sin(\omega_a t - k_a x)]^2}{p_2 \omega_a^2 p[1 - k_a a \cos\omega_a t - k_a x]^4}$$

$$= \frac{\rho_o c_o^2}{p_2} k_a a \quad k_a a \qquad . \tag{2.18}$$

Thus, so long as the acoustic pressure remains only a small fraction of the total pressure, this term may safely be neglected. In this rendition, we write

$$1 + \sigma = \frac{1}{1 - k_a a \cos(\omega_a t - k_a x)}$$

so that

$$\frac{\sigma}{1 + \sigma} = \frac{k_a a \cos(\omega_o t - k_o x)\,[1 - k_a a \cos(\omega_a t - k_a x)]}{[1 - k_a a \cos(\omega_a t - k_a x)]}$$

$$= ka \cos(\omega_a t - k_a x) \tag{2.19}$$

3. The third element, $\underline{v} \cdot \frac{d}{dt}(\nabla p|\underline{r})$ compares with the first as

$$\frac{\underline{v} \cdot \underline{k} \omega p}{\omega_a{}^2 p} = \frac{\underline{v} \cdot \underline{k}_a}{\omega a} = \frac{\underline{v} \cdot \underline{k}}{\underline{c} \cdot \underline{k}_a} a \simeq \frac{|v|}{|c|} \simeq k_a a \tag{2.20}$$

For air and water, the naturally occurring turbulent velocites (due to currents, winds, etc.) rarely reaches .01c and is often not aligned with $\underline{k}_p$. These may thus be neglected, (to 1 percent).

For instance, a 100 kt wind moves at 44.7 m/sec compared with $|c|$ = 331 m/sec. Likewise, a 20 knot current moves at 9 m/sec compared with $|c|$ = 1450 m/sec. We are obviously much safer here with water.

The global effects of this term depend greatly on the strength and extent of the turbulence. The acoustic effects should not accumulate with range.

3a). A similar conclusion holds for the term $\underline{v}\cdot\frac{d}{dt}(p_1|\underline{r})$ which compares as

$$\frac{\underline{v}\cdot p_1\omega_1\underline{k}_1}{\omega^2_a p} = \frac{\underline{v}\cdot\underline{k}_1}{\underline{c}\cdot\underline{k}_a}\left(\frac{\omega_1}{\omega_a}\right)\frac{p_1}{p} \quad . \tag{2.21}$$

Usually $k > k_1$ and $\omega > \omega_1$; often, however, $p_1 > p$. Thus, if we except strong turbulence, the term is safely neglected locally for all but the extremely low frequencies. But note that we are dealing here with a term whose effect may accumulate over the course of a propagation path.

4. The next term $\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \frac{d}{dt}(\ell n\rho c^2|\underline{r})\,\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot\nabla p$ compares as

$$\frac{\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \frac{d}{dt}(\ell n\rho c^2|\underline{r})\,\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot\underline{k}_a p}{\omega^2_a p}$$

$$= \frac{|\underline{f}|}{\omega_a|c|} - \frac{|\underline{v}\cdot\nabla\underline{v}|}{\omega_a|c|} - \frac{\left(\frac{d}{dt}(\ell n\rho c^2|\underline{r})\right)|\underline{v}|}{\omega_a\ \ |\underline{c}|} + \frac{|\nabla\cdot\underline{T}_2|}{\rho\omega_a|\underline{c}|} \tag{2.22}$$

The exterior (body) force $\underline{f}$, the force/unit mass, influences propagation. If $\underline{f}$ is the force of gravity

$$\frac{f}{\omega_a c} = \frac{9.8}{\omega_a c} = \begin{cases} 6.5 \times 10^{-3}/\omega_a & \text{for water} \\ 3.0 \times 10^{-2}/\omega_a & \text{for air} \end{cases} \qquad (2.22a)$$

Thus, at very low frequencies, gravity may become considerably important. The effect, shown in Fig. 2-2, weighs more heavily in air than in water. Tides, which may be considered low frequency acoustic propagation, are well known to be dominated by gravity.

Besides gravity, one might also add the Coriolis force due to the earth's rotation.

All body forces may be expected to accumulate globally.

The element $\underline{v}\cdot\nabla\underline{v} = \frac{\nabla v^2}{2} + \underline{v}\times(\nabla\times\underline{v})$ and so represents a contribution from the local vorticity of the medium. Thus, this term may become prominent in eddies, wakes, etc. If we separate acoustic from turbulent terms and assume that the acoustic propagation induces no vorticity (as in a plane wave), then

$$\underline{v}\cdot\nabla\underline{v} = \frac{\nabla v_a^2}{2} + \nabla(\underline{v}_a\cdot\underline{v}_1) + \underline{v}_a\times(\nabla\times\underline{v}_1) + \underline{v}_1\times(\nabla\times\underline{v}_1) + \frac{\nabla v_1^2}{2}$$

Again, the nature of the turbulence (its force, direction, vorticity) is felt here. For the ideal hydrodynamic plane wave

$$\frac{\nabla \underline{v}_a^2}{2\omega_a c} = \frac{2k_a a\omega_a^2 \cos(\omega_a t - k_a x)\sin(\omega_a t - k_a x)}{2\omega_a c(1 - k_a a \cos(\omega_a t - k_a x))} \approx (k_a a)^2 , \qquad (2.23)$$

so that this term is not important save for high wave numbers.

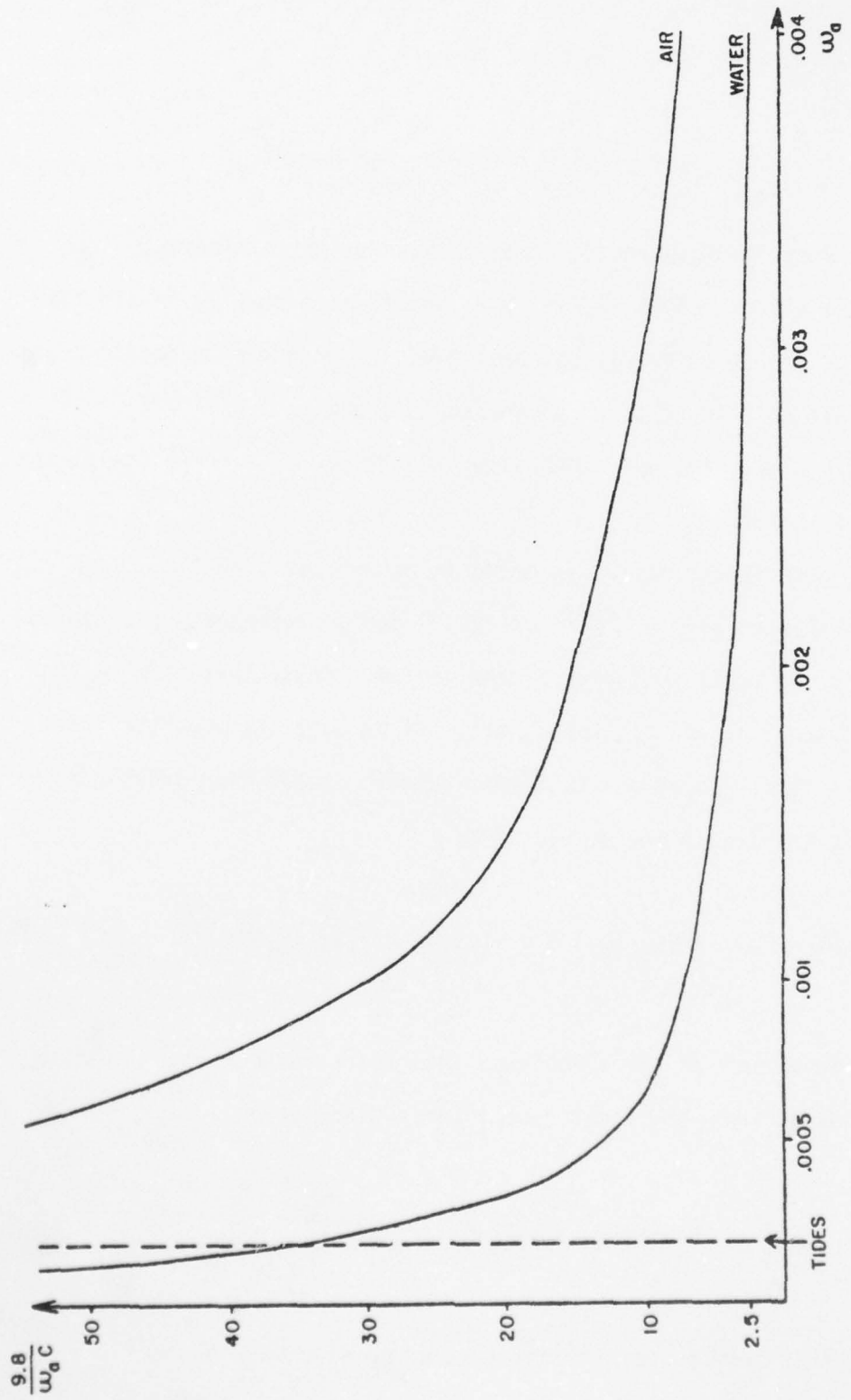


Fig. 2-2. Local Effect of Gravity (Equation 2.22a)

AD-A065 780 NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CONN NEW --ETC F/G 20/1
A GENERAL THEORY OF ACOUSTIC PROPAGATION WITH APPLICATIONS TO S--ETC(U)
JAN 78 J R BRETON

UNCLASSIFIED NUSC-TR-5871 NL

2 OF 5
AD
AD65780

The element $\{d/dt\ [(\ell n \rho c^2 |\underline{r})\underline{v}]\}/\omega_a c$ compares as

$$\frac{\sigma}{1+\sigma}\left|\frac{v}{c}\right| \simeq \frac{\sigma}{1+\sigma}\ k_a a \simeq (k_a a)^2, \tag{2.24}$$

from Eqs. (2.16) and (2.20). Thus, this term is largely ignorable in usual situations.

The element $\nabla\cdot\underline{T}_2/\rho\omega_a c$ in Eq. (2.22) compares as

$$\frac{-\nabla^2(p_1+p)}{\rho\omega_a c} - \frac{\nabla\cdot\underline{b}_3}{\rho\omega_a c} = \frac{k_I^2 p_1 + k_a^2 p}{k_a p_2} - \frac{\nabla\cdot\underline{b}_3}{\omega_a \rho c} \tag{2.25}$$

The first term, $(k_I^2 p_1 + k_a^2 p)/k_a p_2$, may be ignored except for high acoustic pressures. The second part, $\nabla\cdot\underline{b}_3/\rho\omega_a c$ comes from that part of the stress tensor which differs from a perfect fluid. It thus incorporates viscosity and loss mechanisms in realistic fluids. In the ideal, lossless, plane wave case, we have

$$\underline{b}_3 = -\nabla\cdot\underline{T} - \nabla p = -\rho\frac{d(\underline{v}|\underline{x})}{dt} - \nabla p$$

$$= \frac{\rho_0 a\omega_a^2 \sin(\omega_a t - k_a x)}{[1-k_a a\cos(\omega_a t - k_a x)]} - \frac{\rho_o c_o^2 k_a^2 a \sin(\omega_a t - k_a x)}{[1-k_a\cos(\omega_a t - k_a x)]^3}$$

$$= \frac{\rho_0 a\omega_a^2 \sin(\omega_a t - k_a x)}{[1-k_a a\cos(\omega_a t - k_a x)]^3}\ [1-k_a a\cos(\omega_a t - k_a x)]^2 - 1$$

$$= \nabla p\{[1-\kappa_a\cos(\omega_a t - k_a x)]^2 - 1\}, \tag{2.26}$$

which reveals clearly enough that the ideal hydrodynamic fluid is not a perfect fluid.[1] Here, $\underline{b}_3$ has a "reactance" effect, producing leads and lags to ∇p. A loss mechanism would introduce a net reduction in the vector; viscosity, a change in direction, etc.

4a.) The associated term in 4a., Eq. (2.15), compares as

$$\frac{\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} - \frac{d}{dt}\left(\ell n\rho c^2|\underline{r}\right) + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot\left(\nabla p_0 + \underline{k}_1 p_1\right)}{k_a^2 c^2 p}. \qquad (2.27)$$

In addition to the above considerations, the ratios $\nabla p_0/k_a p$ and $\omega_1 p_1/\omega_a p$ govern the effect. In many cases $|\nabla p| > |\nabla(p_0 + p_1)|$, so we should expect this effect to be less than the former. Unfortunately for analysis, however, many such effects tend to become important globally, although they remain insignificant locally.

5. The next term, $\hat{\underline{a}}_3\cdot\nabla p$, compares as

$$\frac{c^2(k_a{}^2 p - \nabla\ell n\rho\cdot\underline{k}_a p)}{\omega_a{}^2 p} = 1 - \frac{\nabla\ell n\rho}{k_a}. \qquad (2.28)$$

The first item, $c^2\nabla^2 p$, is of the same order as $d^2(p|\underline{r})/dt$ and figures in all the classical approximations. The second item does not have much importance locally except at interfaces and discontinuities in the medium. For the plane wave case, we have

$$\frac{|\nabla\ell n\rho|}{k_a} = \frac{k_a a\,\sin(\omega_a t - k_a x)}{[1 - k_a a\,\cos(\omega_a t - k_a x)]^2} \simeq k_a a\,.$$

---

[1] See section 1.2.1.1.

Globally, however, this item contributes principally to refractive effects.

5a. The associated term, $\hat{a}_3 \cdot \underline{b}_3$, goes as the properties of $\underline{b}_3$.

6. The final term, g_0, acts as a local forcing term and compares as

$$\frac{\omega_1^2 p_1}{\omega_a^2 p} - \left(\frac{d}{dt} \ell n \rho c^2 | \underline{r}\right) \frac{\omega_1 p_1}{\omega_a^2 p} + \frac{\rho c^2}{\omega_a^2 p} \nabla \cdot \left(\underline{f} - \underline{v} \cdot \nabla \underline{v} + \frac{\nabla \cdot (\underline{T}_0 + \underline{T}_1)}{\rho} \right). \quad (2.29)$$

In the first item, the usual situation has $\omega_1 < \omega_a$ but $p_1 > p$. Thus, this item depends greatly on the frequency regions and the turbulent strength.

The second item of Eq. (2.29) follows the corresponding acoustic term, Eq. (2.16), but is usually diminished.

The third item of Eq. (2.29) may be more closely compared with $\frac{d^2(p|\underline{r})}{dt^2}$ as

$$\frac{p_2}{\omega_a^2 p} \nabla \cdot \left(\underline{f} - \underline{v} \cdot \nabla \underline{v} + \frac{\nabla \cdot (\underline{T}_0 + \underline{T}_1)}{\rho} \right).$$

The factor, $p_2/\omega_a^2 p \simeq 1/(\omega_a^2 ka)$ is usually small for sea water ($\simeq 2 \times 10^{-6}$ at 3 kHz), but very sensitive to frequency ($\simeq 5 \times 10^{-2}$ at 100 kHz). Depending on the strength of the gradient, it may become an appreciable factor, especially at low frequencies. For instance, the item

$$\frac{\rho c^2}{\omega_a^2 p} \nabla \cdot \left(\frac{\nabla \cdot \underline{T}_1}{\rho} \right) \simeq \frac{\rho c^2}{\omega_a^2 p} \frac{\nabla^2 p_1}{\rho} \simeq \frac{\rho c^2 k_1^2 p_1}{\rho c^2 k_a^2 p} \simeq \frac{k_1^2 p_1}{k_a^2 p}. \quad (2.30)$$

Usually, $k_1 < k$, but often $p_1 > p$. Thus, low frequency acoustic propagation in strong, high frequency turbulence brings this item into effect.

This final term, g_0, almost always accumulates in real fluids, eventually forcing a total degeneration of the acoustic wave.

The results of this section are summarized in Table 2-2, where typical values are given.

From the discussion in this section, a set of critical ratios has emerged. These are

$$|\underline{v}|/|\underline{c}|$$

$$|\underline{k}_1|/|\underline{k}|$$

$$p_1/p \ .$$

Of lesser importance, we have $|f|/\omega|c|$, $\nabla\cdot b_3/\rho\omega c$, $|\nabla p_0|/|k|p$, $\nabla \ell n\rho|/|k|$, etc. These, alone or in combination, yield a large number of "criteria" for abandoning selective terms of Eq. (1.27). In applying these criteria, one needs still to remember possible global effects.

As a rule one can view acoustic propagation in three progressive stages.

Stage I: In the first stage $k_a\ p > k_1 p_1$, $p > p_1$, $k_a > k_1$, so that the acoustic wave dominates the turbulence.

Stage II: As the propagation progresses into the second stage its amplitude diminishes to where $p < p_1$, but $k_a p > k_1 p_1$. This is the stage of most acoustic processing, for the signal is buried in the turbulence, but recoverable. Thus we commonly talk and are understood even in heavy winds.

TABLE 2-2
ANALYSIS OF LOCAL APPROXIMATIONS:

GENERAL SECOND ORDER EQUATION

Element	Local Approximation	Typical Values (%) Air Acoustic	Air Tur	Sea Water Acoustic	Sea Water Tur	Comment and Global Effect
$\frac{d^2(p\|\underline{r})}{dt^2}$	1	-----		-----		-----
$\frac{d}{dt}(\varepsilon n\rho c^2\|\underline{r})\ \frac{d}{dt}(p\|\underline{r})$	$\frac{v}{c}$	1	0.001	0.1	.001	Large intensities in air
$\underline{v}\cdot\frac{d}{dt}(\nabla p\|\underline{r})$	$\frac{vk}{ck_a}$	1	4.2	0.1	0.5	Global; high frequencies
$\underline{v}\cdot\frac{d}{dt}(\nabla(p_c+p_1)\|\underline{r})$	$\frac{v\hat{k}_1 p_1}{ck_a^2 p}$	0.1	0.4	0.01	0.05	Global
$\nabla p\cdot\underline{f}$	$\lvert f\rvert/\omega_a c$	0.001	-	0.0001		Low frequencies - Global
$\nabla p\cdot\underline{v}\cdot\nabla\underline{v}$	v^2/c^2		depends on vorticity			Eddies, wakes; high frequency
$\nabla p\cdot\frac{d}{dt}\varepsilon n\rho c^2\|\underline{r}\ \underline{v}$	v^2/c^2	0.01	0.01	0.01	0.001	Global
$\nabla p\cdot\frac{\nabla\cdot\bar{T}_2}{\rho}$	$\frac{k_1^2 p_1 + k_a^2 p}{k_a p^2}$	5	-	0.5	-	Large intensities Losses and viscosity - Global
$(\nabla(p_0+p_1))\cdot\underline{f}$	$\frac{\lvert f\rvert}{\omega_a c}\ \frac{k_1 p_1}{k_a p}$	-	0.0001	-	0.0001	Very low frequencies - Global
$(\nabla(p_0+p_1))\cdot\underline{v}\cdot\nabla\underline{v}$	$k_1 p_1 \underline{v}/\underline{k}_a pc$		depends on vorticity			Eddies, wakes; high frequency
$(\nabla(p_0+p_1))\cdot\frac{d}{dt}\varepsilon n\rho c^2\|\underline{r}\ \underline{v}$	$k_1 p_1 \underline{v}/k_a pc$	-	0.001	-	0.0001	Global
$(\nabla(p_0+p_1))\cdot\frac{\nabla\cdot T}{\rho}$	$\frac{(\nabla p_0+k_1 p_1)(k_1^2 p+k_a^2 p)}{(k_a p)(k_a p_2)} \doteq \frac{\nabla\cdot b_3}{k_a p_2}$	-	0.5	-	0.05	Global

TABLE 2-2
ANALYSIS OF LOCAL APPROXIMATIONS

GENERAL SECOND ORDER EQUATION (CONT'D)

Element	Local Approximation	Typical Values (%) Air Acoustic	Air Tur	Sea Water Acoustic	Sea Water Tur	Comment and Global Effects
$c^2 \begin{bmatrix} \nabla^2 p \\ \nabla \ell n \rho \cdot \nabla p \end{bmatrix}$	1	100	-	100	-	-----
	$\nabla \ell n c / k_a$	0.001		0.0001		Refractive - Global high frequency
$c^2 \begin{bmatrix} \nabla \cdot \underline{b}_3 \\ \nabla \ell n \rho \cdot \underline{b}_3 \end{bmatrix}$	$\nabla \cdot b_3 / k_a^2 p$		depends on b_3			Viscosity - Global
	$\nabla \ell n \rho \cdot \nabla b_3 / k_a^2 p$		depends on b_3			Refractive - Global
$\frac{d^2}{dt^2} ((p_0 + p_1) \mid r)$	$k_1^2 p_1 / k_a^2 p$		depends on turbulent pressure			
$\frac{d}{dt} (\ell n \rho c^2 \mid \underline{r}) \ \frac{d}{dt} (p_0 + p_1 \mid \underline{r})$	$\frac{v}{c} \frac{k_1}{k_a} \frac{p_1}{p}$	-	0.001	-	0.0001	Strong turbulence
$\rho c^2 \nabla \cdot \begin{bmatrix} \underline{f} \\ \underline{v} \cdot \nabla \underline{v} \\ \frac{\nabla \cdot (T_0 + T_1)}{\rho} \end{bmatrix}$	$\frac{\rho \nabla \cdot \underline{f}}{k_a^2 p}$	-	-	-	-	Low frequencies - Global
	$\frac{\rho}{k_a^2 p} \nabla (\underline{v} \cdot \nabla \underline{v})$		depends on vorticity			Eddies, wakes - Global
	$\frac{k_1^2 p_1}{k_a^2 p}$	-	0.1	-	0.1	Strong turbulence - Global

Stage III: The final stage has $p < p_1$ and $k_a p < k_1 p_1$, so that the propagation is entirely dominated by turbulence.

2.3 Comparison with Neubert and Lumley

Neubert and Lumley, in an important paper [151], have sought to find conditions for the validity of the Stochastic Helmholtz equation, Eq. (1.34). From a starting point based on a linear Navier-Stokes equation (N1-N3, Chap. I), a reduction is made in which the important parameters are frequency, a medium parameter (the Prandtl number σ_K), and the turbulent scale.

We outline the assumptions made:

1) $v \propto e^{j(\omega t - kx)}$

2) $\rho = \rho_0 + \rho_a, \quad \rho_0 >> \rho_a$,

3) $\underline{v} = 0\left(\frac{c\rho_a}{\rho_0}\right)$,

4) small perturbation theory

5) $\nabla \cdot \underline{v}_1 = 0$, (incompressible turbulent fluid)

6) $\nabla \cdot (\rho_a \underline{v}_a) = 0$,

7) $\nabla^2(\mu \underline{v}) \simeq 0 \simeq \nabla(\mu \nabla \cdot \underline{v})$ (viscosity terms $\rightarrow 0$)

8) $\frac{\partial \rho_a}{\partial t} + \nabla \cdot (\rho_0 \underline{v}_a) + (\nabla \rho_a) \cdot \underline{v}_1 = 0$

For scaling approximations:

$\underline{v} = 0$ (rms turbulent velocity fluctuations)

$$\frac{\partial}{\partial t} = 0\left(\frac{v_1}{L_{v_1}}\right) \qquad \text{for turbulence}$$

$$\frac{\partial}{\partial t} = 0\,\frac{c}{\lambda} = 0\,\frac{\omega_a}{2\pi} \qquad \text{for acoustic wave}$$

$$\nabla = 0\,\frac{1}{\lambda_g} \qquad \text{for turbulent fluid}$$

$$\nabla = 0\,\frac{1}{\lambda_\theta} \qquad \text{for thermal fluctuation}$$

$$\nabla = 0\,\frac{1}{\lambda} = 0\,\frac{k_a}{2\pi} \qquad \text{for acoustic wave.}$$

Neubert and Lumley use $\sigma_k = \lambda_g^2/\lambda^2$ as a medium differentiator (σ_k = 0.7 for air and 7.0 for water). They next scale by neglecting terms such that $\lambda v_1^2/(L_{v_1} c^2) < 1$ and $\lambda v_1/(L_{v_1} c) < 1$. L_{v_1} is the turbulent integral scale defined as

$$L_{v_1} \equiv \int_0^\infty d|(\underline{r}'-\underline{r})|\ \langle v_1(\underline{r}')\ v_1(\underline{r})\rangle\ .$$

Neubert and Lumley use as starting point (for approximations) Eq. (1.44), namely,

$$\nabla^2 p - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \underline{v}\cdot\nabla\right)^2 p$$

$$= \left(2\nabla\cdot(\rho_0 v_a)\right)\left(\frac{v_1}{c}\cdot\nabla c\right) - (\nabla\underline{v}_1):\nabla(2\rho_0\underline{v}_a - \rho_a\nabla\cdot\underline{v}_1)\ .$$

This reduces to the stochastic Helmholtz equation

$$\nabla^2 p - \frac{1}{c^2(\underline{r})}\,\frac{\partial^2 p}{\partial t^2} = 0\ ,$$

if

$$\sigma_k^{1/2} \left(\frac{v_1}{c}\right) \frac{\lambda}{\lambda_g} << 1 \qquad \text{when} \quad \lambda \leq O(\lambda_g) , \qquad (2.31a)$$

and

$$\sigma_k^{1/2} \left(\frac{v_1}{c}\right) \frac{\lambda^2}{\lambda_g^2} << 1 \qquad \text{when} \quad \lambda > O(\lambda_g) . \qquad (2.31b)$$

Neubert and Lumley contrast this situation with Tatarskii who uses the conditions $\lambda << \lambda_g$ and $v_1/c << 1$ for validity (in air).

These conditions, Eq. (2.31), compare with

$$\frac{v}{c} \frac{k_1}{k_a} \frac{p_1}{p}$$

and

$$\frac{v}{c} \frac{k_1^2}{k_a^2} \frac{p_1}{p}$$

of the previous section. As might be expected, reduction from our general equation, Eq. (1.28), both confirms and illuminates the Neubert results. We shall find, in the next section, those circumstances which Neubert and Lumley ignore and, consequently, those which must be excepted under Eq. (2.31). Finally, it is well to point out that the statistical indicator L_{v_1}, while admirably accentuating the stochastic nature of $\underline{v}$, nevertheless presumes an isotropicity which, in turn, leads to a neglect of "global" considerations (i.e., at what range do the approximations fail?).

2.4 Circumstances for Validity

The results of Section 2.2 indicate that no set of approximations will satisfy every circumstance. In this section we indicate those

circumstances which validate the various approximations of Chapter I.

In the ideal case, Eq. (1.31), the only medium variable is c_o^2 which, given local variation, can be formulated into a refractive model. Beyond this, $c^2(\underline{r})$ may be given a temporal history and temporal averages may be taken. In addition, one may enter the stochastic Helmholtz approximation by giving $c^2(\underline{r})$ a statistical representation.

In the simplest of these interpretations, the ideal case, $(c_0^2\nabla^2 - \partial^2/\partial t^2)p = 0$, can account for no turbulent phenomena, no inhomogeneity, no divergent frequency effects, no signal degradation with range, no viscosity effects, no loss effects; only a simplified refraction (when $c_0 \rightarrow c_0(\underline{r})$).

The attractiveness of the ideal case lies in its ability to be adapted to a wide variety of realistic conditions. If one allows a full panoply of spatial and temporal variation in c^2, it is possible to induce many observed effects: beam spreading; beam wander; loss of coherence intensity fluctuations; short pulse effects; and angle of arrival fluctuations.

Accordingly, let us consider both the simplest ideal case, Eq. (1.31), (where c_0^2 is constant) and the furthest extension. Both instances neglect influences due to some losses, movement of the medium, refraction, gravity, turbulence of various sorts. Since most of these influences increase with range, we should expect both forms of the ideal wave equation to become ultimately unacceptable.

Yet the derived extension can be made to cover, partially, a great range of conditions. By collecting terms containing c^2, we may write Eq. (1.27), approximately, as

$$\frac{\partial^2 p}{\partial t^2} - c^2\left[\nabla^2 p + \nabla\cdot\underline{b}_3 + \nabla \ln\rho\cdot\nabla p + \nabla \ln\rho\cdot\underline{b}_3 + \rho\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho}\right)\right] = 0 .$$

Then, defining a fictitious effective wavefront velocity c_x according to

$$c_x^2 = c^2\left[1 + \frac{\nabla\cdot\underline{b}_3 + \nabla \ell n\rho\cdot\nabla p + \nabla \ell n\rho\cdot\underline{b}_3 + \rho\nabla\cdot\left(\underline{f} - \underline{v}\cdot\nabla\underline{v} + \frac{\nabla\cdot(\underline{T}_0 + \underline{T}_1)}{\rho}\right)}{\nabla^2 p}\right]$$

we may write formally

$$\frac{\partial^2 p}{\partial t^2} - c_x^2\nabla^2 p = 0 . \qquad (2.33)$$

In this interpretation,[1] Eq. (2.33) is seen to include a wide range of situations including low frequencies, viscosity, refraction, wakes, strong local turbulence. In addition, by insisting on Eq. (2.33), the experimenter is likely to measure c_x^2 rather than c_0^2.

In Table 2-3, we include the simplest version of the ideal case and this "extreme" extension. The table shows which terms of the general equation are included and which omitted. Physical situations which embrace an omitted term significantly (see Table 2-2) lie beyond the reach of the ideal equation.

---

[1]The reader should note that $c_x^2 \equiv c_x^2(p)$, so that Eq. (2.33) is nonlinear in p. Moreover, the definition of c_x^2 shows an "anatomization" of the structure of $c_x^2(\underline{r},t)$. Solutions to Eq. (2.33), of course, are difficult to obtain.

TABLE 2-3
CIRCUMSTANCES FOR THE VALIDITY OF APPROXIMATIONS

Equation	(1.31)	(2.33)	(1.39)	(1.44)	
Element	Simple Ideal Case	Extended Ideal Case	Tatarskii I	Neubert	Significant Circumstances
$\frac{d^2(p\|\underline{r})}{dt^2}$	✓[1]	✓	✓	✓	
$\frac{d}{dt}(\rho_0 c^2\|\underline{r})\frac{d}{dt}(p\|\underline{r})$	-[2]	-	-	-	Large intensities in air
$\underline{v}\cdot\frac{d}{dt}(\nabla p\|\underline{r})$	-	-	✓	✓	Currents, high platform speeds; high frequencies
$\underline{v}\cdot\frac{d}{dt}(\nabla(p_0+p_1)\|\underline{r})$	-	-	-	-	Currents, high platform speeds; high frequencies
$\nabla p\cdot\underline{f}$	-	-	-	-	Gravitation: low frequency
$\nabla p\cdot\underline{v}\cdot\nabla\underline{v}$	-	-	-	✓	Vorticity
$\nabla p\cdot\frac{d}{dt}(\rho_0 c^2\|\underline{r})\,\underline{v}$	-	-	-	-	---
$\nabla p\cdot\frac{\nabla\cdot T_2}{\rho}$	-	-	-	✓	Large acoustic pressures

[1] A check (✓) indicates that the approximation incorporates the element.
[2] A dash (-) indicates that the approximation does not incorporate the element.

TABLE 2-3. CIRCUMSTANCES FOR THE VALIDITY OF APPROXIMATIONS (CONT'D)

Element	Equation (1.31) Simple Ideal Case	(2.33) Extended Ideal Case	(1.39) Tatarskii I	(1.44) Neubert	Significant Circumstances
$\nabla(p_0+p_1)\cdot \begin{bmatrix} \underline{f} \\ \underline{v}\cdot\nabla\underline{v} \\ \frac{d}{dt}(\ell n\rho c^2/\underline{r})\,\underline{v} \\ \frac{\nabla\cdot\underline{T}_2}{\rho} \end{bmatrix}$	-	-	-	-	Low frequency
	-	-	-	-	Vorticity
	-	-	-	-	---
	-	-	-	-	Large acoustic, or turbulent pressures flux
$\begin{bmatrix} \nabla^2 p \\ c^2 \\ \nabla\ell n\rho\cdot\nabla p \end{bmatrix}$	✓	✓	✓	✓	
	-	✓	-	-	Refraction; high frequencies
$c^2 \begin{bmatrix} \nabla\cdot\underline{b}_3 \\ \nabla\ell n\rho\cdot b_3 \end{bmatrix}$	-	✓	-	✓	Refraction, losses
	-	✓	-	-	Refraction

TABLE 2-3. CIRCUMSTANCES FOR THE VALIDITY OF APPROXIMATIONS (CONT'd)

Equation	(1.31)	(2.33)	(1.39)	(1.44)	
Element	Simple Ideal Case	Extended Ideal Case	Tatarskii I	Neubert	Significant Circumstances
$\frac{d^2}{dt^2}(p_0 + p_1)\lvert\underline{r}$	-	-	-	-	Turbulence
$\frac{d}{dt}(\rho c^2\lvert\underline{r})\ \frac{d}{dt}(p_0 + p_1)\lvert\underline{r}$	-	-	-	-	Turbulence
$\rho c^2 \nabla\cdot\left[\underline{f}\right]$	-	✓	-	-	Gravitational change
$\left[\underline{v}\cdot\nabla\underline{v}\right]$	-	✓	-	✓	Vorticity change
$\left[\frac{\nabla(\underline{T}_0 + \underline{T}_1)}{\rho}\right]$	-	✓	-	✓	Turbulent change

For the other historical models, it is easy to see that the stochastic Helmholtz equation, Eq. (1.34), Chernov's equation, Eq. (1.37), the second Tatarskii equation, Eq. (1.41), are adaptations of the extended ideal equation. The first Tatarskii equation, Eq. (1.39), adds the term $\underline{a}_1 = \underline{v}_1 \neq 0$. Equation (1.44) further adds $\underline{a}_2$.

In the absence of specified information about the turbulence no sure frequency regimes can be specified. Much depends on the strength of the turbulence as well as its frequency content. This may be seen from Table 2-4. For the validity of the stochastic Helmholtz equation each item must be much less than 1 and should not accumulate significantly over the region of integration. It is obvious that $(v/c)^2 < (v/c)$. However, the relative effect of $(v/c)(k_1/k_a)(p_1/p)$ vis-à-vis $(v/c)(k_1/k_a)^2 (p_1/p)$ depends greatly not only on the acoustic frequency but also on the turbulent frequency. Similarly $(v/c)(k/k_a)$ vis-à-vis $(v/c)(k_1/k_a)(p_1/p)$ depends strongly on the strength of the turbulence (p_1). Analagous statements may be made for the other criteria.

Of the sixteen criteria given in Table 2-4, only numbers 4 and 5 have been previously reported (analogously).[1]

2.5 Losses and Relaxation Effects

The relaxation[2] (loss) terms in second order differential equations

---

[1]See Eq. (2.31).

[2]The term relaxation is not well defined in the literature, and the application of "relaxation" has not been wholly successful in explaining observations. The approach tends to curve-fitting (Cf. Hunt, F.V., in [93], pp 3-55). Nevertheless, these methods have often exposed very real loss mechanisms.

TABLE 2-4

CRITERIA[1] FOR THE VALIDITY OF THE STOCHASTIC-HELMHOLTZ EQUATION

1. $v/c \ll 1$
2. $v^2/c^2 \ll 1$
3. $(v/c)\,(k/k_a) \ll 1$
4. $(v/c)\,(k/k_a)\,(p_1/p) \ll 1$
5. $(v/c)\,(k_1/k_a)^2\,(p_1/p) \ll 1$
6. $(k_1^2/k_a^2)\,(p_1/p) \ll 1$
7. $[(k_1^2 p_1 + k_a^2 p)/(k_a p_2)] \ll 1$
8. $[(\nabla p_o + k_1 p_1)(k_1^2 p_1 + k_a^2 p)/(k_a^2 p_2)] \ll 1$
9. $|\underline{f}|/\omega c \ll 1$
10. $(\underline{f}/\omega c)\,(k_1/k_a)\,(p_1/p) \ll 1$
11. $(\rho \nabla \cdot \underline{f}) \,/\, (k_a^2 p) \ll 1$
12. $[\rho \nabla \cdot (\underline{v} \cdot \nabla \underline{v})]/(k_a^2 p) \ll 1$
13. $\nabla \ell n \rho / k_a \ll 1$
14. $(\nabla \ell n \rho \cdot \underline{b}_3)/(k_a^2 p) \ll 1$
15. $(\nabla \cdot \underline{b}_3)/(k_a^2 p) \ll 1$
16. $\nabla \cdot \underline{b}_3 / k_a p_2 \ll 1$

---

[1]The quantities given in this table are all magnitudes (vectorial or phasor).

are associated with first order terms. In space-time equations, these are various types.

Lindsay [121], for example, mentions three types: molecular, structural, and chemical. Since, in the general equation, Eq. (1.27), the medium is characterized only with respect to its mechanical properties, one may not hope for other than this insight from it. Further explanations must be left to other methods and disciplines.

Still, from the general equation, some interesting points may be made. The term $a_0 = (-d/dt)(\ell n\rho c^2)$ is associated uniquely with the temporal nature of the field and so may be termed a purely temporal relaxation. We may group the temporal terms as

$$\left[\frac{d^2((\cdot)|\underline{r})}{dt^2} - \frac{d}{dt}(\ell n\rho c^2|\underline{r})\,\frac{d((\cdot)|\underline{r})}{dt}\right]p. \tag{2.34}$$

In this formulation, a_0 is seen to "relax" the importance of the second order temporal term. The discussion of Section 2.2 shows that the second term is almost always negligible in sea water, and important in air only for high acoustic intensities (or at high frequencies).[1]

Passing to spatial considerations, we note the term

$$\hat{\underline{a}}_3\cdot(\nabla p + \underline{b}_3) = -c^2(\nabla - \nabla\ell n\rho)\cdot(\nabla p + \underline{b}_3)$$

$$= -c^2[\nabla^2 p + \nabla\cdot\underline{b}_3 - \nabla\ell n\rho\cdot\nabla p - \nabla\ell n\rho\cdot\underline{b}_3] ,$$

where $\underline{b}_3 = -\nabla p - \nabla\cdot\underline{T}$.

[1]See Fig. 2-1.

Supposing, as is likely, $\underline{b}_3 = \underline{b}_3(p)$, we may write the above as

$$\hat{\underline{a}}_3 \cdot (\nabla p + \underline{b}_3) = -c^2[\nabla^2 + \nabla \cdot \hat{\underline{b}}_3(\cdot) - (\nabla \ell n \rho) \cdot \nabla - (\nabla \ell n \rho) \cdot \underline{b}_3(\cdot)]p \ .$$

These terms associated with the propagation velocity, c, may be termed the "radiation" relaxation. If we ignore the $\nabla \ell n \rho$ terms (associated with refraction), we have

$$-c^2[\nabla^2 + \nabla \cdot (\underline{b}(\cdot))]p \tag{2.35}$$

analogous to Eq. (2.34). If, for example, $\underline{b}_3(p) \equiv a\nabla \frac{d}{dt} p$, then

$$c^2 \nabla \cdot (\nabla p + \underline{b}_3) = c^2 \left(1 + a \frac{d}{dt}\right) \nabla^2 p$$

for constant a.

We meet another spatial term in

$$\underline{a}_2 \cdot \nabla p = \left(\underline{f} - \underline{v} \cdot \nabla \underline{v} - \left(\frac{d}{dt} \ell n(\rho c^2)\right) \underline{v} + \frac{\nabla \cdot \underline{T}_2}{\rho}\right) \cdot \nabla p \tag{2.36}$$

which is likewise a relaxation term but unassociated with the radiation velocity c. We may consider this a "local" spatial relaxation.

Finally, the term g_0, (Eq. 1.28), contains only background and turbulent pressures and so acts as a distributed source (or sink) which we distinguish from relaxation. Of course, a sink is a <u>loss</u> in the field and may be interpreted as a relaxation, if so desired.

2.6 Some Suggested Experiments and Procedures

Ultimately, nothing will do but to measure the medium. The foregoing discussion, while oriented theoretically, suggests certain

experiments for determining an approximation's validity. Subsequently, when solutions are considered, other experiments will be suggested.

There can be no question here of any but a rough sketch. Obtaining accurate physical information is, in itself, a formidable undertaking.

1. The solution to the hydrodynamic plane wave (Appendix A) indicates that ka is an important medium descriptor. One may observe a directly in the motion of a rigid piston. In addition to being a medium descriptor, a may be presumed to be a function of pressure, amplitude, and frequency.

2. Currents, winds, tidal movements and other factors which give a gross bias to the turbulent particle velocity are often known. Tables 2-1 to 2-4 allow a quantitative assessment of the effect of these circumstances on the propagation.

3. Specifically, direct measurement of $k_1 p_1$, in at least a portion of the field, may be possible. It gives an indication of turbulent effects beyond the models of the simple wave equation.

4. Section 2.2 suggests modifying or, at least, disclaiming static physical data. Theoretically, this introduces one or more dimensions to the static measuremetts. For given fluids like air and sea water, statistical results may be useful.

5. The temporal relaxation due to high acoustic intensity in air might be investigated as important in blasting, etc.

6. Low frequency experiments may be constructed to include gravitational effects.

7. The presence of substantial mixing should alert the experimenter to possible vorticity effects, especially for high frequency experiments.

8. Much information may be obtainable from spatial considerations. Placing two receivers even approximately locally in a field, will indicate something of the spatial wandering in the field. Similarly, comparing low and high frequency results may prove useful in isolating turbulent terms.

9. Finally, the experimenter has the onus, in direct measurement of c^2, to indicate whether he measures c^2 or c_x^2, the "effective" propagation velocity. The difference between the two is, in itself, an excellent measure of turbulence.

2.7 Summary

This chapter answers its purpose in Table 2-4 which completes, for the first time, a listing begun by Tatarskii and continued by Neubert and Lumley for determining the circumstances under which the Stochastic-Helmholtz equation, Eq. (1.34), is valid. Possessing Tables 2-2 and 2-3, both theoreticians and experimentalists may judge whether or not any given acoustical model can adequately describe the circumstrnces at hand.

The method used to obtain the complete listing starts from the general equations of Chapter I, and continues by comparing each successive term with the first term (a specific acoustical quantity of known importance). Several novel and specific results are uncovered thereby. We mention: (1) an important relaxation effect in air at high acoustical intensities, and (2) the local effect of gravity (predicting a

larger effect in air than in water). The analysis provides new insights into the effect of turbulent pressure versus its frequency content (the principal stages of propagation, Sec. 2.2). Reflection on the results has allowed the definition of an effective wavefront velocity c_x (Eq. (2.33)) and the proposal of several new experiments and procedures (Sec. 2.6). From a purely theoretical viewpoint, we mention that Eq. (2.26) shows the mechanism of a lossless non-ideal stress tensor beyond any possible Navier-Stokes formulation, which in the case stipulated would require $\nabla \cdot \underline{T} = -\nabla p$.

Finally, it should be noted that our results apply to broadband radiation by interpreting k_a, k_1, ω_1, ω_a as bands.

Reverse Blank

III. THE GENERAL SOLUTION

3.0 Overview - The Meaning of Solutions

The problem of finding solutions to the propagation of waves in turbulent media suffers a charming paradox. From the controversy surrounding the fundamental equations and the practical complexity of the hydrodynamic equations, one might have expected few, if any, solutions. However, the sometimes benign and ever changing nature of the usual media have allowed solutions of many different forms to be constructed and somewhat justified.

In this section, we obtain the solution to the exact hydrodynamic equation. The exact solution requires, however, exact knowledge of the medium, for the solution is medium-related. Because such knowledge is practically impossible, we underscore and corroborate the recent trend of viewing the propagating medium as a spatial-temporal random variable whose statistics influence the output statistics of the process.

Our approach will first cast the relevant hydrodynamic equation into operational form (operator unknown). We next elaborate a series solution to obtain the operator explicitly. The use of the feedback theorem (Sec 1.1.1) yields equivalent forms. Finally Fourier transform solutions are given.

Before developing the general solution, it will be instructive to consider the form of a solution from information already at hand. We suppose a system $\hat{H}$ (the medium), a source at $(\underline{r}_0, t_0)$ and a receiver at $(\underline{r},t)$

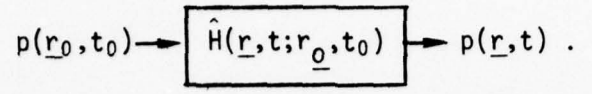


Fig. 3-1. Propagation (Deterministic)

Here and subsequently, the caret (^) denotes an operator. In general, we may expect $\hat{H}$ to be non-linear. Because the output is known to vary statistically, Eq. (3.1) must be augmented. While not disclaiming the usual method of introducing channel noise, we choose to represent the stochastic nature of the medium by an ensemble of systems $\{\hat{H}_i\}$. Hence, at the space time point $(\underline{r},t)$ and for a given realization H_i (from an appropriate event space), we write

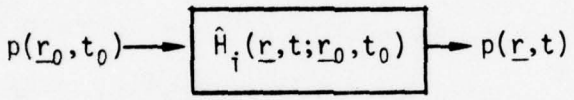


Fig. 3-2. Propagation (Stochastic)

In contrast to Fig. 3-1, where a single system function operates, Eq. (3.2) operates with any one of the ensemble system functions according to some probabilistic measure. Since individual realizations are often of little probabilistic weight, we seek to obtain the statistics of the output process from the presumably known statistics of the medium.

Depending on the realizations, dispersion in both space and time may occur in the output.

The nature of the operator $\hat{H}$ may be appreciated from an illustration based on a known situation. We use the ideal hydrodynamic plane wave solution,[1] which we write

$$p_2(x,t) = p_0 - bk_0a\cos(\omega_0 t - k_0 x) + k_0 a p_2(x,t)\cos(\omega_0 t - k_0 x), \quad (3.1)$$

[1]See Appendix A.

or operationally, as in Fig. 3-3.

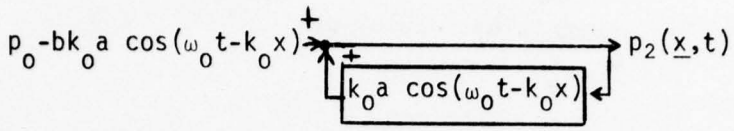


Fig. 3-3. Operational Solution: $p_2(\underline{x},t)$

The acoustic pressure, p, may be given as

$$p(x,t) = \rho_0 c^2 \, k_o a \cos(\omega_o t - k_o x) + k_o a p(x,t) \cos(\omega_o t - k_o x) \, , \quad (3.2)$$

or operationally, as in Fig. 3-4.

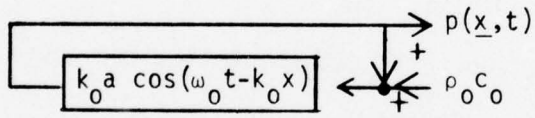


Fig. 3-4. Operational Solution: $p(\underline{x},t)$

For $p(\underline{0},t) = \rho_o c_o^2 \cos(\omega_o t)/(1 - k_o a \cos(\omega_o t))$, we can show a "source" explicitly as

$$p(x,t) = \frac{(1 - k_o \cos(\omega_o t))[(\cos(\omega_o t - k_o x))/(\cos\omega_o t)]p(\underline{0},t)}{1 - k_o a \cos(\omega_o t - k_o x)} \, . \quad (3.3)$$

Operationally, this is given by Fig. 3-5.

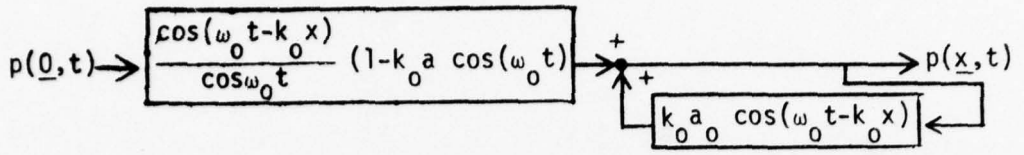
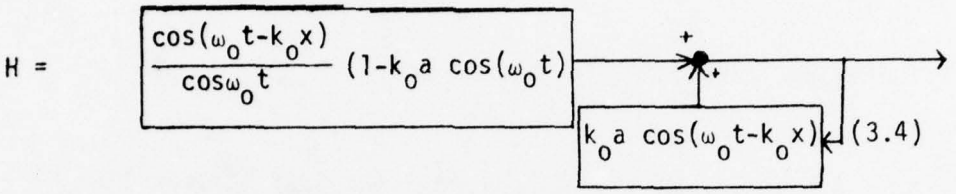


Fig. 3-5. Operational Solution: $p(\underline{0},t) \rightarrow p(\underline{x},t)$

Thus



When $k_0 a$ is negligible, $p(x,t)\cos(\omega_0 t - k_0 x) = p(0,t)\cos(\omega_0 t)$ so that $\hat{H}$ effects a simple phase shift.

In this illustration, $\hat{H}$ is deterministic, the given medium being homogeneous (i.e., deterministic).

3.1 Operational Solution

A general solution, similar to the above, may be set down. Such a solution we call an "operational" solution. We shall meet such solutions subsequently in dealing with approximations.

Write the first order general equation, Eq. (1.22), as

$$\frac{d(p_2|\underline{r})}{dt} + \nabla\cdot(\underline{v}p_2) = p_2\nabla\cdot\underline{v} - \rho c^2\nabla\cdot\underline{v} \ . \tag{3.5}$$

Now define an operator $\hat{L}$,

$$\hat{L} \equiv \frac{d}{dt}(\cdot|\underline{r}) + \nabla\cdot(v(\cdot)) \ . \tag{3.6}$$

Here $\hat{L}$ is a space-time operator. Suppose further that $\hat{L}^{-1} = \hat{G}$ exists. Then, applying $\hat{G}$ to Eq. (3.5) gives

$$(\hat{G}\hat{L})p_2 = p_2 = \hat{G}[(p_2 - \rho c^2)\nabla\cdot\underline{v}] \ . \qquad (3.7)$$

Equation (3.7) may be rendered operationally as in Fig. 3-6.

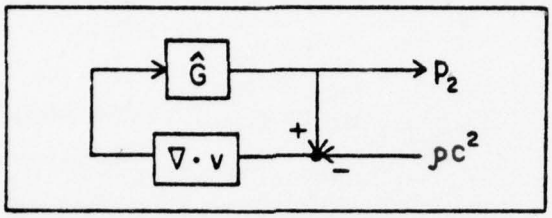


Fig. 3-6. General Operational Solution: $p_2(\underline{r},t)$

In general, the operators $\hat{L}$ and $\hat{G}$ and the quantities p^2, $\nabla\cdot\underline{v}$ and ρc^2 are stochastic.

Figure 3-6, although simple in form, immediately offers some important insights.

1. By analogy with the temporal case, we can characterize the pressure field on the basis of the feedback parameter

$\nabla\cdot\underline{v} < 0$: a decaying pressure field (negative feedback)

$\nabla\cdot\underline{v} = 0$: a non-decaying, non-increasing pressure field - (no feedback)

$\nabla\cdot\underline{v} > 0$: an increasing pressure field (focusing) - (positive feedback)

The complex questions of loop stability (which we later distinguish from convergence problems associated with approximations to $\hat{G}$) can be

appreciated. For instance, in an otherwise decaying pressure field, it is possible to have a local instability ($\nabla\cdot v(\underline{r},t) > 0$ for some $\{\underline{r}\}$) causing the pressure field to concentrate or focus.

2. The feedback representation gives insight into the interaction of the pressure and velocity fields. Such an interaction, often neglected, must exist physically since the dynamic pressure depends on fluid motion.

3. Decomposing Eq. (3.6) into its acoustic and turbulent parts as in Eq. (1.22), ($p_2 = p_0 + p_1 + p$), we obtain Fig. 3-7.

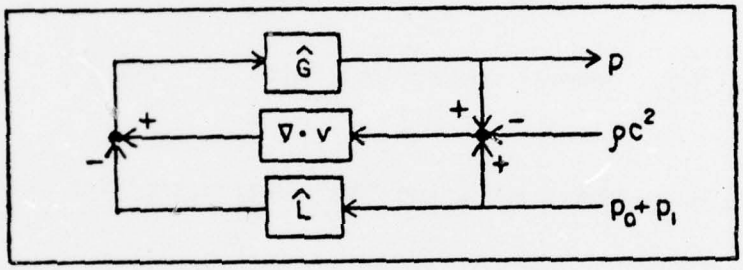


Fig. 3-7. General Operational Solution: $p(\underline{r},t)$

The diagram shows the way turbulent pressures (in addition to their effect on particle velocity) influence the acoustic pressure.

For probing the turbulent medium acoustically, the diagram may be inverted to obtain Fig. 3-8.

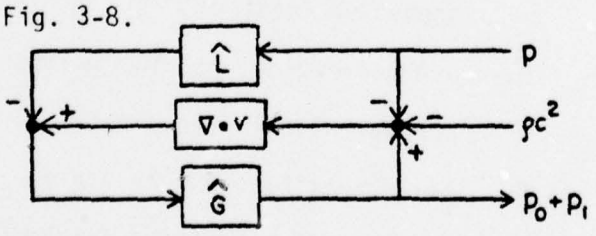


Fig. 3-8. General Operational Solution $p_1(\underline{r},t) + p_0$

From this point of view, it is not surprising to find experimental results[1] that link the turbulent and acoustic pressure fields.

4. The role of attenuation may also be appreciated from Eq. (3.6). Put

$$\nabla\cdot\underline{v} = \nabla\cdot\underline{v}' - a(\underline{r},t) \tag{3.8}$$

where $\nabla\cdot\underline{v}'$ represents the spatial derivative of the unattenuated particle velocity and $\underline{a}$ is the local (space-time) attenuation. The dimension of this $a(\underline{r},t)$ is frequency. Physically, $a(\underline{r},t)$ may arise from medium properties (ionic disassociation, molecular resonance, etc.). Fig. 3-6 may then be written as Fig. 3-9.

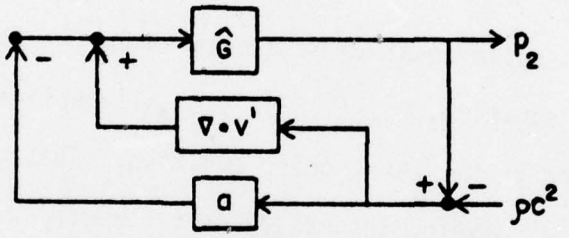


Fig. 3-9. General Operational Solution with Attenuation

Here it is not hard to see that the attenuation acts like a low-pass (space-time) filter.

We note, finally, that in view of Eq. (3.5), observed effects may be categorized as purely spatial, purely temporal, and mixed. The same categories may be applied to solutions. This may be better

---

[1]As an example (among many possible), we mention: [11] p. 595: "The spectra of the amplitude fluctuations of such transmissions (270 Hz)... are remarkably similar to the spectra of the vertical internal-wave motions...."

appreciated in terms of the general solution and the associated historical approximations.

3.2 The General Solution

We give the general solution first in a space-time form and afterwards in Fourier transforms. From this development emerges, in explicit form, the operator $\hat{G}$ of Eq. (3.9).

3.2.1 The General Space-Time Solution

Because the first order general equation

$$\frac{d(p|\underline{r})}{dt} + \underline{v}\cdot(\nabla p + \nabla p_0 + \nabla p_1) + \rho c^2 \nabla\cdot\underline{v} = \frac{-d}{dt}\,(p_0 + p_1|\underline{r}) \qquad (3.9)$$

has a solution space equal to or greater than the corresponding second order general equation, Eq. (1.27), and is symbolically simpler, we base a solution on the first order equation. This departs from the usual historical developments based on the second order equation.

We can write Eq. (3.19) in a number of different forms. All variables here are considered functions of both space and time (i.e., $c=c(\underline{r},t)$; $p = p(\underline{r},t)$, etc). We write the following:

$$1.\quad \frac{d(p|\underline{r})}{dt} + \underline{v}\cdot\nabla p = -\{\rho c^2(\nabla\cdot\underline{v}) + \underline{v}\cdot\nabla(p_0 + p_1) + \frac{d}{dt}\,[(p_0 + p_1)|\underline{r}]\}, \qquad (3.9a)$$

$$2.\quad \frac{d(p|\underline{r})}{dt} + \nabla\cdot(\underline{v}p) = p(\nabla\cdot\underline{v}) - \{\rho c^2\nabla\cdot\underline{v} + v\cdot\nabla(p_0 + p_1) + \frac{d}{dt}\,[(p_0 + p_1)|\underline{r}]\}, \qquad (3.9b)$$

$$3.\ \frac{d(p|\underline{r})}{dt} + \nabla\cdot(vp) = p(\nabla\cdot\underline{v}) + u \quad . \tag{3.9c}$$

Thus, we can define two operators, $\hat{D}_0$ and $\hat{D}_1$ by:

$$4.\ \left[\frac{d}{dt}(\cdot|r) + \nabla\cdot(\underline{v}(\cdot))\right]p \equiv \hat{D}_0 p = p\nabla\cdot\underline{v} + u, \tag{3.9d}$$

$$5.\left[\frac{d}{dt}(\cdot|r) + \underline{v}\cdot\nabla(\cdot)\right]p \equiv \hat{D}_1 p = u, \tag{3.9e}$$

where $u = -\{\rho c^2\nabla\cdot(\underline{v}) + \underline{v}\cdot\nabla(p_0 + p_1) + \frac{d}{dt}\ ((p_0 + p_1)|\underline{r})\}$, and $\hat{D}_0$ and $\hat{D}_1$ finding an inverse operator $\hat{G}_1$ such that

$$\hat{G}_1\hat{D}_1 = \delta(\underline{r}-\underline{r}_0, t-t_0), \tag{3.10}$$

since

$$\hat{G}_1\hat{D}_1 p = p(\underline{r},t) = \hat{G}_1(u) \ . \tag{3.11}$$

Here δ is the space-time impulse function.

If $\underline{v}$ is considered a random variable, then

$$\hat{G}_1 \equiv \left[\frac{d(\cdot|\underline{r})}{dt} + \underline{v}\cdot\nabla(\cdot)\right]^{-1} \tag{3.12}$$

is a stochastic space-time operator. The space-time function, u, is also a random variable. Moreover, since u is a function of ρ, c, $\underline{v}$, p_0, p_1, the solution will be a function of the medium variables, that is,

$$p(\underline{r},t) = f(\rho,c,\underline{v},p_0,p_1) \ . \qquad (3.13)$$

The operational solutions suggest that we may conveniently start with form (3.9d)

$$\left[\frac{d}{dt}\,((\cdot)|\underline{r}) + \nabla\cdot(\underline{v}(\cdot))\ p\right] = p(\nabla\cdot\underline{v}) + u \ .$$

As stated, this relationship applies at a "space-time" point $(\underline{r},t)$, i.e.,

$$\left[\frac{d}{dt}\,((\cdot)|\underline{r}) + \nabla\cdot(v(\cdot))\right]p(\underline{r},t) = p(\underline{r},t)(\nabla\cdot\underline{v}(r,t)) + u(\underline{r},t) \ .$$

However, $(\underline{r},t)$ is any "space-time" point, so that the form (3.14d) holds everywhere in the field with $\underline{v}$ and u valued differently, perhaps, for each $(\underline{r},t)$. Our problem is to relate the source pressure $p(\underline{r}_0,t_0)$ to the ambient field pressure $p(\underline{r},t)$ in terms of the field variables $\underline{v}$ and u.

Power has dimensions of force x $\frac{\text{length}}{\text{time}}$; therefore it follows that the elements of the general equation (3.14) deal with a power density.[1] The temporal integral of this density is pressure. The spatial integral of this density is power, not pressure. This "power" may be reduced to "pressure" by suitable normalization.

We observe first that

$$\left[\frac{d}{dt}\,((\cdot)|\underline{r}) + \nabla\cdot(\underline{v}(\cdot))\right] \equiv D_0 \qquad (3.14)$$

[1] See Appendix C on physical dimensions.

is the sum of a temporal and spatial operator. Inverses for each of these operators may be found separately. Denoting these latter by $\hat{G}_t$ and $\hat{G}_{\underline{r}}$, we note that their operation on a space-time function $a(\underline{r},t)$ is given by

$$\hat{G}_t: a(\underline{r},t_1,) \rightarrow b_0(\underline{r},t), \tag{3.15}$$

$$\hat{G}_{\underline{r}}: a(\underline{r}_1,t) \rightarrow b_1(\underline{r},t), \tag{3.16}$$

so that

$$\hat{G}_t\hat{G}_r: a(\underline{r}_1,t_1) \rightarrow b_2(\underline{r},t), \tag{3.17}$$

and

$$\hat{G}_r\hat{G}_t: a(\underline{r}_1,t_1) \rightarrow b_3(\underline{r},t) \tag{3.18}$$

for some b_i.

This is, however, not the desired solution, because

$$\hat{G}_t\hat{G}_r\left[\hat{D}_t + \hat{D}_r\right] = \hat{G}_r + \hat{G}_t \neq \delta(\underline{r}-\underline{r}_0,t-t_0) . \tag{3.19}$$

From Eq. (3.19), we see that the pseudo-solutions Eqs. (3.17) and (3.18) contain the action of both spatial and temporal terms, but not their mutual interaction. The solutions to the stochastic Helmholtz equation, $(\nabla^2 + k^2)p = 0$, to be considered in Chapter IV, can be seen as partial or pseudo-solutions giving spatial, but neither temporal nor mixed terms.

For simplicity's sake, we introduce the following definitions:

$$a(\underline{r}_i, t_j) \equiv a(ij) \quad \text{(for any function a)}, \tag{3.20}$$

$$a(\underline{r}, t) \equiv a(00), \tag{3.21}$$

$$\hat{G}_t \equiv \int_{t_0}^{t} (\cdot)dt \qquad \text{(fixed-space, generalized, temporal integration)}, \tag{3.22}$$

$$\hat{G}_r \equiv \left(\int_V \underbrace{(\cdot)dV}_{\underline{r}} \Big/ \int_S \underline{v}\cdot\hat{n}dS \right) \equiv \frac{\int_V \underbrace{(\cdot)dV}_{\underline{r}}}{n(ij)} \qquad \text{(fixed time, generalized spatial integration)}, \tag{3.23}$$

$$\hat{G}_{rt} = \hat{G}_r + \hat{G}_t, \tag{3.24}$$

$$\hat{G}_{rtD} = \hat{G}_r \frac{d((\cdot)|r)}{dt} + \hat{G}_t \nabla\cdot\underline{v}(\cdot) \quad . \tag{3.25}$$

In Eq. (3.23), the volume V and surface S are defined by a <u>connected equipressure surface containing r</u>. The integrals are intimately connected with the extension of the divergence theorem (Section 1.2.1.2).

Thus, we have

$\hat{G}_t$: power density → pressure,

$\hat{G}_r$: power density → pressure,

and

$\int_V (\cdot)dV$: power density → power.

The integral $\int_S \underline{v}\cdot n\, dS \equiv n(ij)$ in Eq. (3.23) is the required spatial normalizing factor.

In Eq. (3.24), we follow the convention established in Eqs. (3.15) and (3.16), namely

$$\hat{G}_{rt}[a] = \left[\hat{G}_r + \hat{G}_t\right]a = \hat{G}_r[a(10)] + \hat{G}_t[a(01)] \tag{3.26}$$

where

$$\hat{G}_r\colon a(10) \to b(00) \tag{3.26a}$$

and

$$\hat{G}_t\colon a(01) \to b(00)\ . \tag{3.26b}$$

In the sequel we will find it convenient to use the substantial compression in notation allowed by Eqs. (3.20) to (3.25).

These conventions and definitions can be made clearer by illustrating an initial construction. Apply first temporal integration $\int_{t_0}^{t}(\cdot)dt$ to

$$\frac{d(p|\underline{r})}{dt} + \nabla\cdot(\underline{v}p) = p(\nabla\cdot\underline{v}) + u \tag{3.14d}$$

which yields

$$p(\underline{r},t) = p(\underline{r},t_0) + \int_{t_0}^{t} p(\underline{r},t_1)\nabla\cdot(\underline{v}(\underline{r},t_1))dt_1 + \int_{t_0}^{t} u(\underline{r},t_1)dt$$

$$- \int_{t_0}^{t} \nabla\cdot(p(\underline{r},t_1))\underline{v}(\underline{r},t_1)dt_1 \tag{3.27a}$$

or

$$p(0,0) = p(0,t_0) + \hat{G}_t p(01)\, \nabla\cdot\underset{\sim}{v}(01) + \hat{G}_t u(01)$$

$$-\hat{G}_t\, \nabla\cdot(p(01)\underline{v}(01)). \tag{3.27b}$$

Next, integrate Eq. (3.14d) spatially to obtain

$$\int_V \nabla\cdot(\underline{v}p)dV = \int_V p\nabla\cdot\underline{v}dV + \int_V udV - \int_V \frac{d}{dt}\, pdV. \tag{3.28}$$

For $\underline{v}$ and p continuous in V the divergence theorem holds. We are not guaranteed this continuity in our problem, however. Our theory here is limited to the type of discontinuities which may be treated by an extended divergence theorem.[1] We have assumed this extension implicitly in the temporal integration, Eq. (3.27).

Under this limitation, we may then write

$$\int_V \nabla\cdot[\underset{\sim}{v}(\underline{r},t)p(\underline{r},t)]dV = \int_S p\underline{v}\cdot\hat{\underline{n}}ds + s(\underline{r}_i,t)$$

$$= pn(\underline{r},t) + s(\underline{r}_i,t) \tag{3.29}$$

since S in an equipressure surface and $s(\underline{r}_i,t) = \int_{\underline{r}\ \text{discontinuous}} \nabla\cdot(vp)dV$ is a residual due to discontinuities. From the definition, one sees that these discontinuities are both "active" and passive (inasmuch as these latter have an effect only when illuminated by a pressure field). The passive discontinuities have the effect of additional (distributed) sources and must not be neglected. The acoustic source, in particular, is represented in s as an active discontinuity.

---

[1] See Chapter I, Sections 1.2.1.1 and 1.2.1.2.

Rewriting Eq. (3.28) now and isolating $p(\underline{r},t)$, we have

$$p(\underline{r},t) = \frac{1}{n(\underline{r},t)} \left\{ \int_V [(p(\underline{r},t))\nabla\cdot v(\underline{r},t) + u(\underline{r},t)]dV \right.$$

$$\left. - \frac{d}{dt} \left(\left(\int_V p(\underline{r},t)dV|\underline{r} \right) \right) - s(\underline{r},t) \right\}. \qquad (3.30a)$$

Here, by using standard notation, we confuse the fixed space variable $\underline{r}$ (LHS) with the variable of integration (RHS). It is thus convenient to introduce another notation, Eq. (3.16), and rewrite Eq. (3.30a) as

$$p(\underline{r},t) = \frac{1}{n(\underline{r},t)} \left\{ \int_{\underline{r}_1} p(\underline{r}_1,t)\nabla\cdot\underline{v}(\underline{r}_1,t) + u(\underline{r}_1,t) \right.$$

$$\left. - \frac{d}{dt} p(\underline{r}_1,t|\underline{r})\, d\underline{r}_1 - s(\underline{r},t) \right\}, \qquad (3.30b)$$

or

$$p(00) = \frac{1}{n(00)} \left\{ \int_{r_1} [p(10)\nabla\cdot v(10) + u(10) \right.$$

$$\left. - \frac{d}{dt} p(10|r)] dr_1 - s(00) \right\} \qquad (3.30c)$$

$$= \hat{G}_r [p\nabla\cdot v + u - \frac{d}{dt}(p|\underline{r})] - \frac{s(00)}{n(00)} . \qquad (3.30d)$$

Adding Eq. (3.30d) with Eq. (3.27b), we have

$$2p(00) = p(0,t_0) + \hat{G}_t[p(01)\nabla\cdot\underline{v}(01)] + \hat{G}_t u(01)$$

$$- \hat{G}_t \nabla\cdot(v(01)p(01)) + \hat{G}_r[p(10)\nabla\cdot v(10)] + G_r u(10)$$

$$- G_{r\frac{d}{dt}} p(10) - s(00)/n(00), \qquad (3.31a)$$

or

$$2p(00) = 0(0,t_0) + \hat{G}_{rt}[(p(00)\nabla\cdot\underline{v}(00))] + \hat{G}_{rt} u(00)$$

$$- \hat{G}_{rt_0} p(00) - \frac{s(00)}{n(00)} \quad . \qquad (3.31b)$$

This first application of the operators is called the m = 0 iteration. We point out, parenthetically, that Eq. (3.31) may contain additional terms due to the temporal integration $\hat{G}_t$ over discontinuities in time ($\hat{G}_t$ is defined as a <u>generalized</u> integral) in a manner analogous to the spatial case.

Many ways of iterating are possible. We next produce the four possible iterations of the generator, $p\nabla\cdot v$. Iterating Eq. (3.27) into Eq. (3.27), we have

$$p(0,0) = p(0,t_0) + \hat{G}_t \nabla\cdot\underline{v}(01)p(0,t_0)$$

$$- \hat{G}_t\hat{G}_t(\nabla\cdot\underline{v}(01))(\nabla\cdot\underline{v}(02))p(02) - \hat{G}_t(\nabla\cdot\underline{v}(01))p(01)$$

$$+ \hat{G}_t\hat{G}_t p(02)\nabla\cdot\underline{v}(01) + \hat{G}_t u(01) + \hat{G}_t\hat{G}_t(\nabla\cdot\underline{v}(01))u(02); \qquad (3.32a)$$

for Eq. (3.27) into Eq. (3.30),

$$p(0,0) = \hat{G}_r(\nabla\cdot\underline{v}(10))p(1,t_0) - \hat{G}_r\hat{G}_t(\nabla\cdot\underline{v}(10))\ \nabla\cdot\ (p(11)\underline{v}(11))$$

$$- \hat{G}_r \frac{d}{dt} p(10) + \hat{G}_r\hat{G}_t(\nabla\cdot\underline{v}(10)) \; p(11)\nabla\cdot\underline{v}(11)$$

$$+ \hat{G}_r u(10) + \hat{G}_r\hat{G}_t(\nabla\cdot\underline{v}(10))u(11) - s(00/n(00)); \qquad (3.32b)$$

for Eq. (3.30) into Eq. (3.27)

$$p(00) = p(0,t_0) - \hat{G}_t\hat{G}_r \; (\nabla\cdot\underline{v}(01)) \frac{d}{dt} p(11)$$

$$- \hat{G}_t\nabla\cdot \underline{v}(01) \; (p(01)) - \hat{G}_t \; (\nabla\cdot\underline{v}(01)) \frac{s(01)}{n(01)}$$

$$+ \hat{G}_t\hat{G}_r(\nabla\cdot\underline{v}(01)) \; (p(11)\nabla\cdot\underline{v}(11))$$

$$+ \hat{G}_t u(01) + \hat{G}_t\hat{G}_r(\nabla\cdot\underline{v}(01)u(11)) \qquad (3.32c)$$

for Eq. (3.30) into Eq. (3.30)

$$p(00) = - \hat{G}_r\hat{G}_r\nabla\cdot\underline{v}(10) \frac{d}{dt} p(20) - \hat{G}_r \frac{d}{dt} p(10)$$

$$- \hat{G}_r(\nabla\cdot\underline{v}(10)) \; \frac{s(10)}{n(10)} - \frac{s(00)}{n(00)}$$

$$+ \hat{G}_r\hat{G}_r(\nabla\cdot\underline{v}(10))\nabla\cdot(p(20)\underline{v}(20))$$

$$+ \hat{G}_r u(10) + \hat{G}_r\hat{G}_r\nabla\cdot v(10)u(20) \; . \qquad (3.32d)$$

Adding, we have

$$4p(00) = 2(p_0,t_0) + \hat{G}_t(\nabla\cdot\underline{v}(01))p(0,t_0) + \hat{G}_r(\nabla\cdot\underline{v}(10))p(1,t_0)$$

$$+ \hat{G}_t\hat{G}_t(\nabla\cdot\underline{v}(01))p(02)\nabla\cdot\underline{v}(02) + \hat{G}_r\hat{G}_t(\nabla\cdot\underline{v}(01))p(11)\cdot\nabla\cdot\underline{v}(11)$$

$$+ \hat{G}_r\hat{G}_r(\nabla\cdot\underline{v}(10))p(20)\nabla\cdot\underline{v}(20) + \hat{G}_t\hat{G}_t(\nabla\cdot\underline{v}(01))\ \ p(11)\nabla\cdot\underline{v}(11)$$

$$+ \hat{G}_t u(01) + \hat{G}_t\hat{G}_t(\nabla\cdot\underline{v}(01))u(02)$$

$$+ \hat{G}_r u(10) + \hat{G}_r\hat{G}_t(\nabla\cdot\underline{v}(10))u(11)$$

$$+ \hat{G}_t u(01) + \hat{G}_t\hat{G}_r(\nabla\cdot\underline{v}(01))u(11)$$

$$+ \hat{G}_r u(10) + \hat{G}_r\hat{G}_r(\nabla\cdot\underline{v}(10))u(20)$$

$$- \hat{G}_t\nabla\cdot(\underline{v}(01)p(01)) - \hat{G}_t(\nabla\cdot\underline{v}(01))p(01) - \hat{G}_r\frac{d}{dt}p(10)$$

$$- \hat{G}_r\frac{d}{dt}p(10) - 2\frac{s(00)}{n(00)}$$

$$- \hat{G}_r(\nabla\cdot\underline{v}(10))\frac{s(10)}{n(10)} - \hat{G}_t(\nabla\cdot\underline{v}(01))\frac{s(01)}{n(01)}$$

$$- \hat{G}_t\hat{G}_t\ (\nabla\cdot\underline{v}(01))(\nabla\cdot p(02)\underline{v}(02)) - \hat{G}_r\hat{G}_t\nabla\cdot\underline{v}(10)\nabla\cdot(p(11)\underline{v}(11))$$

$$- \hat{G}_t\hat{G}_r(\nabla\cdot\underline{v}(01))\cdot\frac{d}{dt}p(11) - \hat{G}_r\hat{G}_r(\nabla\cdot\underline{v}(10))\frac{d}{dt}p(20). \qquad (3.33a)$$

Now, using the convention in Eq. (3.26), we get

$$4p(00) = 2p(0,t_0) + \hat{G}_{rt}(\nabla\cdot\underline{v})p(0,t_0) + \hat{G}_{rt}(\nabla\cdot\underline{v})\hat{G}_{rt}(\nabla\cdot\underline{v})p$$

$$+ 2\hat{G}_{rt}u + \hat{G}_{rt}(\nabla\cdot\underline{v})G_{rt}u$$

$$- 2\hat{G}_{rtD}P - \hat{G}_{rt}(\nabla\cdot\underline{v}) \cdot \hat{G}_{rtD}P$$

$$-2s/n - \hat{G}_{rt}(\nabla\cdot\underline{v})\,\frac{s}{n}\,. \qquad (3.33b)$$

This result involves a double factorization, for example,

$$\hat{G}_{rt}[\nabla\cdot\underline{v}]\hat{G}_{rt}u = [\hat{G}_r + \hat{G}_t]\,(\nabla\cdot\underline{v})\,[\hat{G}_r + \hat{G}_t]\,u(00)$$

$$= [\hat{G}_r + \hat{G}_t]\,(\nabla\cdot v)\,[\hat{G}_r u(10) + \hat{G}_t u(01)]$$

$$= \hat{G}_r\hat{G}_r(\nabla\cdot\underline{v}(10))u(20) + \hat{G}_r\hat{G}_t\nabla\cdot\underline{v}(10)u(11)$$

$$+ \hat{G}_t\hat{G}_r(\nabla\cdot\underline{v}(01))u(11) + \hat{G}_t\hat{G}_t(\nabla\cdot\underline{v}(01)u(02)) \qquad (3.34)$$

which may be found in Eq. (3.33a).

At this point, the reader may easily comprehend why so many solutions to the acoustic problem abound. Each term of Eq. (3.33a) contains one or more potential approximations.

A second reiteration on the same generator will produce eight separate and distinct expressions for $p(\underline{r},t)$. Each expression will, of course, be considerably more complicated than the above.

Thus, for m = 2, the second iteration, we obtain in compact notation,

$8p(00) = 4p(0,t_0)$

$$+ 2\hat{G}_{rt}(\nabla\cdot\underline{v})p(0,t_0) + \hat{G}_{rt}(\nabla\cdot\underline{v})\hat{G}_{rt}(\nabla\cdot\underline{v})\ (p(0,t_0))$$

$$+ 4\hat{G}_{rt}u + 2\hat{G}_{rt}(\nabla\cdot\underline{v})G_{rt}u + \hat{G}_{rt}(\nabla\cdot\underline{v})\quad \hat{G}_{rt}(\nabla\cdot\underline{v})\hat{G}_{rt}u$$

$$+ \hat{G}_{rt}(\nabla\cdot\underline{v})\hat{G}_{rt}(\nabla\cdot\underline{v})\hat{G}_{rt}[p\nabla\cdot\underline{v}]$$

$$- \hat{G}_{rt}(\nabla\cdot v)\hat{G}_{rt}(\nabla\cdot\underline{v})\ \hat{G}_r\left[\frac{d}{dt}\ (\cdot) + \hat{G}_t\nabla\cdot(\underline{v}(\cdot))\right]p$$

$$- 2\hat{G}_{rt}(\nabla\cdot\underline{v})\ \hat{G}_r\left[\frac{d}{dt}\ (\cdot) + \hat{G}_t\nabla\cdot(\underline{v}(\cdot))\right]p$$

$$- 4\hat{G}_r\left[\frac{d}{dt}\cdot(\cdot) + \hat{G}_t\nabla(\underline{v}(\cdot))\right]p$$

$$- 4\ \frac{s}{n} - 2\hat{G}_{rt}(\nabla\cdot\underline{v})\ \frac{s}{n} - \hat{G}_{rt}(\nabla\cdot\underline{v})\hat{G}_{rt}(\nabla\cdot\underline{v})\ \frac{s}{n}\ . \qquad (3.35)$$

The m^{th} iteration can be written (using a right to left order of operators) as

$$p(00) = \sum_{i=0}^{m} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}}\ \hat{G}_{rt}u$$

$$+ \sum_{i=0}^{m} \frac{\left(\hat{G}_{rt}(\nabla\cdot v)\right)^{(i)}}{2^{i+1}}\ p(0,t_0)$$

$$- \sum_{i=0}^{m} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}}\ \hat{G}_{rtD}\ p(0,0)$$

$$- \sum_{i=0}^{m} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}} s/n$$

$$+ \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(m)}}{2^{m+1}} p\nabla\cdot\underline{v} \quad . \tag{3.36}$$

For convergence, the term $\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(m)} p\nabla\cdot\underline{v}/2^{m+1}$ collapses to zero with increasing m, (except, possibly, in the case of reacting fluids).

Continuing the iteration indefinitely, we thus obtain

$$p(0,0) = \sum_{i=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot v)\right)^{(i)}}{2^{i+1}} \left[p(0,t_0) + G_{rt}u - G_{rtD}(p) - \frac{s}{n}\right] . \tag{3.37}$$

Note that the initial condition, $p(\underline{r}_0,t_0)$, indicates the initial condition of the acoustic pressure field throughout the volume at time t_0. In many practical cases, this initial condition is zero; in others, it may not be neglected.

The final pressure term in the RHS of Eq. (3.37) may be eliminated formally as follows:

$$\left[1 + \sum_{i=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}} \hat{G}_{rtD}\right] p$$

$$= \sum_{i=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}} \left[p(0,t_0) + \hat{G}_{rt}u - \frac{s}{n}\right]$$

so that

$$p = \left[1 + \sum_{k=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(k)}}{2^{i+1}} \hat{G}_{rtD}\right]^{-1}$$

$$\cdot \sum_{i=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}} \left[p(0,t_0) + \hat{G}_{rt}u - \frac{s}{n}\right] \qquad (3.38)$$

or

$$p(\underline{r},t) = \sum_{j=0}^{\infty} (-1)^j \left(\sum_{k=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(k)}}{2^{i+1}} \hat{G}_{rtD}\right)^{(j)}$$

$$\cdot \sum_{i=0}^{\infty} \frac{\left(\hat{G}_{rt}(\nabla\cdot\underline{v})\right)^{(i)}}{2^{i+1}} \left[p(0,t_0) + \hat{G}_{rt}u - \frac{s}{n}\right]. \qquad (3.39)$$

This is the desired general solution to the hydrodynamic equations.

These results have been derived painstakingly (begging the reader's tolerance) to underscore the meaning of the symbols. They may be derived much more simply by using the feedback theorem.[1] From Eq. (3.31), we have

$$\left[1 - \frac{\hat{G}_{rt}\nabla\cdot\underline{v}(\cdot)}{2}\right] p = \frac{p(0,t_0) + G_{rt}u - G_{rtD}p - \frac{s}{n}}{2}$$

or

$$p = \sum_{i=0}^{\infty} \left(\frac{G_{rt}\nabla\cdot\underline{v}}{2}\right)^{(i)} \left[\frac{p(0,t_0) + G_{rt}u - G_{rtD}p - \frac{s}{n}}{2}\right].$$

[1]See Preliminaries Chapter 1, Section 1.1.

Repeating now as before

$$\left[1 + \sum_{i=j}^{\infty}\left(\frac{\hat{G}_{rt}\nabla\cdot\underline{v}}{2}\right)^{(i)} \frac{\hat{G}_{rtD}}{2}\right] p = \sum_{i=0}^{\infty}\left(\frac{\hat{G}_{rt}\nabla\cdot\underline{v}}{2}\right)^{(i)}\left[\frac{p(0,t_0) + \hat{G}_{rt}u - \frac{s}{n}}{2}\right],$$

or

$$p = \sum_{j=0}^{\infty}(-1)^j\left(\sum_{k=0}^{\infty}\left(\frac{\hat{G}_{rt}(\nabla\cdot\underline{v})}{2}\right)^{(k)}\frac{\hat{G}_{rtD}}{2}\right)^{(j)}\sum_{i=0}^{\infty}\left(\frac{\hat{G}_{rt}\nabla\cdot\underline{v}}{2}\right)^{(i)} \cdot\left[\frac{p(0,t_0) + \hat{G}_{rt}u - \frac{s}{n}}{2}\right]$$

as in Eq. (3.39).

Similar arguments provide two alternate forms

$$1.\quad p(\underline{r},t) = \sum_{i=0}^{\infty}\left(\frac{\hat{G}_{rt}(\nabla\cdot\underline{v}) - \hat{G}_{rtD}}{2}\right)^{(i)}\left[\frac{p(0,t_0) + \hat{G}_{rt}u - \frac{s}{n}}{2}\right] \quad (3.40)$$

and

$$2.\quad p(r,t) = \sum_{j}\left(\sum(-1)^k\left(\frac{\hat{G}_{rtD}}{2}\right)^{(k)}\frac{\hat{G}_{rt(\nabla\cdot\underline{v})}}{2}\right)^{(j)}\sum_{i}(-1)^i\left(\frac{\hat{G}_{rtD}}{2}\right)^{(i)} \cdot\left[\frac{p(0,t_0) + G_{rt}u - \frac{s}{n}}{2}\right]. \quad (3.41)$$

If now we segregate s_0, the acoustic source, ($s = s_1 + s_0$), we may find the transfer function H_i of Fig. 3-2 from Eq. (3.40), rewritten as

$$p(r,t) = \frac{\hat{G}(s_0)}{2n} + \hat{G}\left[\frac{p(0,t_0) + \hat{G}_{rt}u - \frac{s_1}{n}}{2}\right] \qquad (3.42)$$

with

$$\hat{G} \equiv \sum_{i=0}^{\infty} \left(\frac{\hat{G}_{rt}(\nabla\cdot\underline{v}) - G_{rtD}}{2}\right)^{(i)} . \qquad (3.43)$$

Diagramatically, this is given in Fig. 3-10.

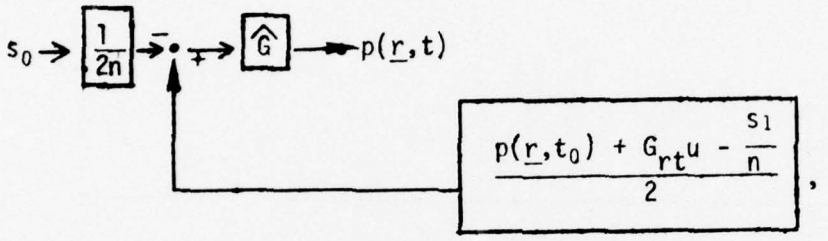


Fig. 3-10 General Propation

The general system function, H_i, is given in Fig. 3-11.

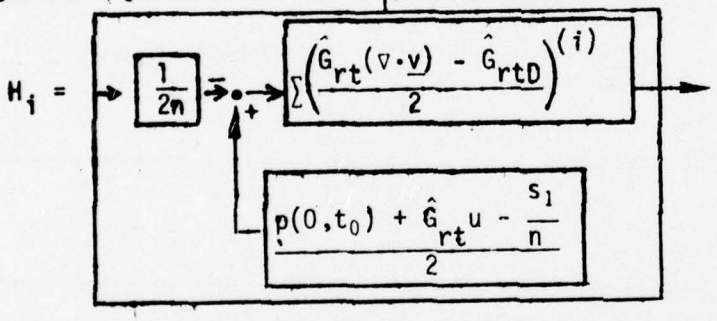


Fig. 3-11 The General System Function

Note that the random field contained in u and s_1 acts as a secondary source. These are physically the interfering noise sources of all types and their propagated effects,

The stochastic nature of H_i (given $\underline{v}$, u and s_1, stochastic) is plainly evident.

Likewise, since $\hat{G}_{rt}u = \hat{G}_{rt}[-\rho c^2 \nabla\cdot v - \underline{v}\cdot\nabla(p_0+p_1) - (d/dt)((p_0+p_1)|r)]$, we may diagram Eq. (3.40) as

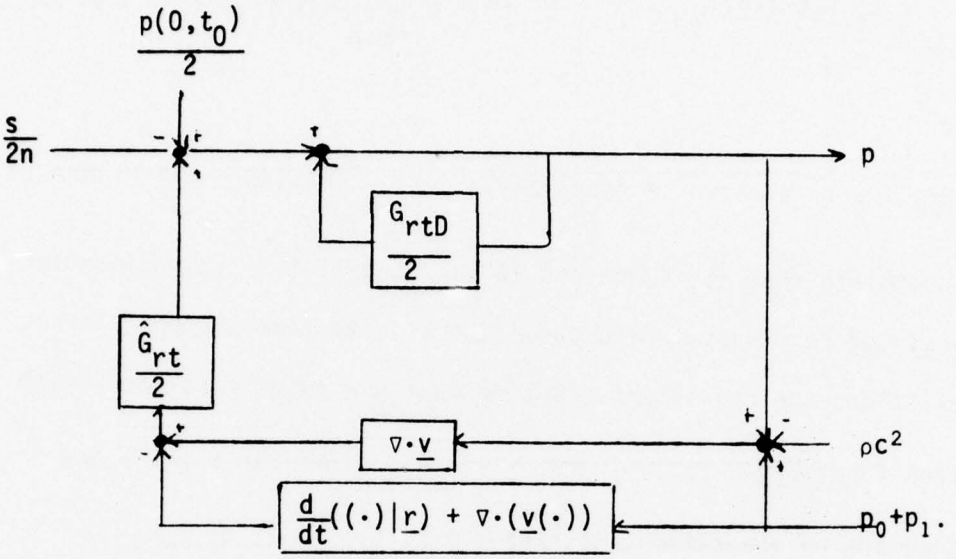
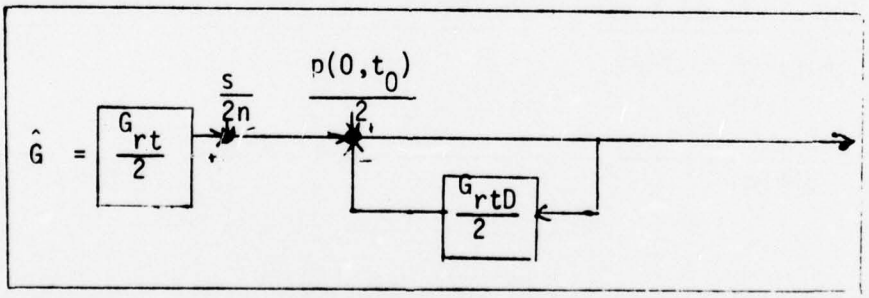


Fig. 3-12. The General Operational Solution

It follows from Eq. (3.10), that the $\hat{G}$ of Eq. (3.7) is



(3.44)

3.2.2 The General Transform Solution

It is a property of generalized functions always to possess Fourier transforms. It is not surprising, therefore, that transform methods are widely used in the literature. We give a brief summary of transform theory needed for this analysis. First definitions:

1. $\hat{F}_t \equiv \int_{-\infty}^{\infty} (\cdot)e^{j\omega t}dt$ 1'. $\hat{F}_\omega \equiv \hat{F}_t^{-1} \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} (\cdot)e^{-j\omega t}d\omega$

2. $\hat{F}_{\underline{r}} \equiv \int_{-\infty}^{\infty} (\cdot) e^{-j\underline{k}\cdot\underline{r}}\, d\underline{r}$ 2'. $\hat{F}_{\underline{k}} \equiv \hat{F}^{-1}_{\underline{r}} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} (\cdot) e^{j\underline{k}\cdot\underline{r}}\, d\underline{r}$

3. $\hat{F} \equiv \int_{-\infty}^{\infty} (\cdot) e^{j(\omega t - \underline{k}\cdot\underline{r})}\, d\underline{r}dt$ 3'. $\hat{F}^{-1} \equiv \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} (\cdot) e^{-j(\omega t - \underline{k}\cdot\underline{r})}\, d\underline{k}d\omega$.

The integrals are repeated as necessary, i.e., $\int_{-\infty}^{\infty} (\cdot) d\underline{r} \equiv \iiint_{-\infty}^{\infty} (\cdot) dr^3$.

We postulate that the functions to be integrated in each dimension are <u>generalized</u> functions, which condition assures that the operators always exist and are consistent. But we must use rules consistent with generalized functions, <u>e.g.</u>, $\int_{-\infty}^{\infty} d\underline{r}dt = \delta(\underline{k},\omega)$. Then, if a and b are scalar function of space-time, we have

$$\hat{F}(ab) = \hat{F}(a) * \hat{F}(b) = \int_{-\infty}^{\infty} a(\underline{k}-\underline{k}_0, \omega-\omega_0) b(k_0, \omega_0) d\underline{k}_0 d\omega_0 \tag{3.45}$$

$$\hat{F}\ \frac{d}{dt}\ (a|\underline{r}) = -j\omega \hat{F}(a) \tag{3.46}$$

$$\hat{F}(\nabla a) = j\underline{k}\hat{F}(a)\ . \tag{3.47}$$

If $\underline{a}$ and $\underline{b}$ are vector functions of space-time, we have

$$\hat{F}(\underline{a}\ \underline{b}) = \hat{F}(\underline{a}) * \hat{F}(\underline{b}) \equiv \hat{F}(a_x b_x) + \hat{F}(a_y b_y) + \hat{F}(a_z b_z) \tag{3.48}$$

$$\hat{F}(\nabla \cdot \underline{a}) = j\underline{k} \cdot \hat{F}(\underline{a})\ . \tag{3.49}$$

These in hand, we give solutions based on the first and second order general Eqs. (1.22) and (1.27).

3.2.2.1 The General Transform Solution: First Order

Write the first order Eq. (1.22) as

$$\frac{d}{dt}(p|\underline{r}) + \underline{v}\cdot\nabla p = -\rho c^2\nabla\cdot\underline{v} - \frac{d}{dt}((p_0+p_1)|\underline{r}) - \underline{v}\cdot\nabla(p_0+p_1) \equiv u. \quad (3.50)$$

Apply $\hat{F}$ to obtain

$$-j\omega\hat{F}(p) + j\underline{k}\hat{F}(p) * F(\underline{v}) = F(u) . \quad (3.51)$$

Note that p and $\underline{v}$ are generally inextricable.

Now

$$p(\underline{k},\omega) - \frac{1}{\omega}\int_{\omega_0,\underline{k}_0} \underline{k}_0 p(\underline{k}_0,\omega_0)\cdot\underline{v}(\underline{k}-\underline{k}_0,\omega-\omega_0)d\underline{k}_0 d\omega_0 = \frac{u(\underline{k},\omega)}{j\omega} . \quad (3.52)$$

Let

$$\hat{G} \equiv \frac{1}{\omega}\int k_0(\cdot)\cdot\underline{v}(\underline{k}-\underline{k}_0,\omega-\omega_0)d\underline{k}_0 d\omega_0 , \quad (3.53)$$

to obtain by the feedback theorem,

$$\boxed{p(\underline{k},\omega) = \hat{F}(p) = \sum_{n=0}^{\infty} \hat{G}(n)\left[\frac{\underline{u}(k,\omega)}{j\omega}\right] .} \quad (3.54)$$

Inverting, we have

$$p(\underline{r},t) = \hat{F}^{-1} \sum_{n=0}^{\infty} \hat{G}(n)\left[\frac{u(\underline{k},_{\omega})}{j_{\omega}}\right] . \quad (3.55)$$

Note that the space-time solution

$$\sum_{n=0}^{\infty} \left(\frac{\hat{G}_{rt}\nabla\cdot v - \hat{G}_{rtD}}{2} \right)^{(n)} [u] = \sum_{n=0}^{\infty} \hat{F}^{-1}\, \hat{G}^{(n)}\, \frac{[u(k,\omega)]}{j\omega} \qquad (3.56)$$

cannot be compared to Eq. (3.55) in individual addends.

It may be observed that both forms have the same general characteristics: an infinite sum of nested operators, each of which is a random space-time integral.

3.2.2.2 The General Transform Solution: Second Order

If we apply $\hat{F}$ to the second order general equation, Eq. (1.28), we obtain

$$(-j\omega)^2\hat{F}(p) - \hat{F}(a_0)*j\omega\hat{F}(p) + \hat{F}(\underline{a}_1)\dot{*}(-j\omega)\,(j\underline{k})\hat{F}(p) + \hat{F}(\underline{b}_1)$$

$$+ \hat{F}(a_2)\dot{*}\left(j\underline{k}\,\hat{F}(p) + \hat{F}(b_1)\right) - \hat{F}(\hat{a}_3\cdot\nabla\cdot T) = \hat{F}(g_0),$$

or

$$\omega^2\hat{F}(p) + j\hat{F}(a_0)*\omega\hat{F}(p) - \hat{F}(\underline{a}_1)\dot{*}\omega\left(\underline{k}\,\hat{F}(p) + \hat{F}(\underline{b}_1)\right)$$

$$- j\hat{F}(a_2)\,\dot{*}\,\left(k\,\hat{F}(p) + \hat{F}(\underline{b}_1)\right) - \hat{F}\left(\hat{a}_3\cdot(\nabla\cdot T)\right)$$

$$= -\hat{F}(g_0)\ . \qquad (3.57)$$

The constituent elements are given by:

1. $$\hat{F}(a_0) = j\omega\hat{F}(\ln\rho c^2) = j\hat{F}\left(\frac{1}{\rho c^2}\right) *\omega\hat{F}(\rho c^2) \qquad (3.58)$$

$$2.\quad \hat{F}(\underline{a}_1) = \hat{F}(\underline{v}) \tag{3.59}$$

$$3.\quad \hat{F}(\underline{a}_2) = \hat{F}(\underline{f}) - j\hat{F}(\underline{v})*\underline{k}\hat{F}(\underline{v}) - \hat{F}(a_0)*\hat{F}(\underline{v}) + j\hat{F}\left(\frac{1}{\rho}\right)*\underline{k}\cdot\hat{F}(T_2) \tag{3.60}$$

$$4.\quad \hat{F}(\underline{a}_3) = j\hat{F}(c^2)*\underline{k}(1-\hat{F}(\ln\rho)) \tag{3.61}$$

$$5.\quad \hat{F}(\underline{b}_1) = j\underline{k}\cdot\hat{F}(p_0 + p_1) \tag{3.62}$$

$$6.\quad \hat{F}(g_0) = \omega^2\hat{F}(p_0+p_1) - j\hat{F}(a_0)*\omega\hat{F}(p_0+p_1) - j\hat{F}(\rho c^2)*\underline{k}\cdot\hat{F}(\underline{f}) - \hat{F}(\underline{v})*\underline{k}\hat{F}(\underline{v}) + \hat{F}\left(\frac{1}{\rho}\right)*k\cdot\hat{F}(T_0+T_1)\ . \tag{3.63}$$

In Eqs. (3.60) and (3.63), the item $\underline{k}\hat{F}(\underline{v})$ is an outer product (second order tensor).

To find $F(\hat{a}_3\cdot(\nabla\cdot\underline{T}))$, put

$$\nabla\cdot\underline{T} = -\nabla p - \underline{b}_3, \tag{3.64}$$

so that

$$\hat{\underline{a}}_3\cdot(\nabla\cdot\underline{T}) = -c^2(\nabla - \nabla\ln\rho)\cdot(-\nabla p - \underline{b}_3) = c^2(\nabla^2 p - \nabla\ln\rho\cdot\nabla p + \nabla\cdot\underline{b}_3 - \nabla\ln\rho\cdot\underline{b}_3)\ . \tag{3.65}$$

Hence,

$$\hat{F}\,\hat{a}_3\cdot(\nabla\cdot\underline{T}) = -\hat{F}(c^2)*[\underline{k}^2\hat{F}(p) - \underline{k}\hat{F}(\ln\rho)*\underline{k}F(p)$$

$$- jk\hat{F}(b_3) + j\underline{k}\hat{F}(\ln\rho)*F(b_3)] \ . \qquad (3.66)$$

The feedback theorem gives the general solution. Collecting the acoustic terms of Eq. (3.57), we have

$$\omega^2\hat{F}(p) + j\hat{F}(a_0)*\omega\hat{F}(p) - \hat{F}(\underline{a}_1)*\omega k\hat{F}(p) - j\hat{F}(\underline{a}_2)*\underline{k}\hat{F}(p)$$

$$- \hat{F}(c^2)*\ k^2\hat{F}(p) - \underline{k}\hat{F}(\ln\rho)*k\ \hat{F}(p) = \hat{F}(u') \ , \qquad (3.67)$$

where

$$\hat{F}(u') \equiv - \hat{F}(g_0) + \hat{F}(\underline{a}_1)*\omega\underline{k}\hat{F}(\underline{b}_1) + j\hat{F}(\underline{a}_1)*\underline{k}F(\underline{b}_2)$$

$$- \hat{F}(c^2)*[j\underline{k}\hat{F}(\underline{b}_3) - j\underline{k}\hat{F}\ell n\rho*F(\underline{b}_3)] \ . \qquad (3.68)$$

Now define

$$\hat{G} \equiv - \frac{1}{\omega^2}\Big\{j\hat{F}(a_0)*\omega\hat{F}(\cdot) - \hat{F}(\underline{a}_1)*\ \underline{k}\hat{F}(\cdot) - j\hat{F}(\underline{a}_2)*\underline{k}\hat{F}(\cdot)$$

$$- \hat{F}(c^2)*[k^2\hat{F}(\cdot) - \underline{k}\hat{F}\ell n\rho*\underline{k}\hat{F}(\cdot)]\Big\}, \qquad (3.69)$$

so that,

$$\hat{F}(p) = \sum_{n=0}^{\infty} \hat{G}^{(n)}\,\frac{\hat{F}(u)}{\omega^2} \ . \qquad (3.70)$$

Again, the characteristic of the general solution is evident: an infinite sum of nested operators. Equation (3.70) is convenient for comparing solutions obtained by transforms of approximations to the general solution.

3.3 Summary

For the most part the contents of this chapter are original. We underscore:

1. The operational solutions - Fig. 3-3 to Fig. 3-9.
2. The general system function - Eq. (3.42).
   a.) The homogeneous system function - Eq. (3.4).
3. The general feed-forward operator, $\hat{G}$ - Eq. (3.44).
4. The general space-time solutions - Eqs. (3.39), (3.40), and (3.41).
5. The general transform solutions - Eqs. (3.55) and (3.70).

It can be seen that the solutions to adiabatic acoustic propagation are complicated, as they must be to embrace the many sound fields (including interfering sources, tides, wakes, internal waves, etc.) which they describe.

Before attempting probabilistic approaches (Chapter V), however, let us complete our theory by reviewing the historical solutions of the recent past.

IV. HISTORICAL SOLUTIONS

4.0 Introduction

Excepting propagation in solids, physiological and psychological acoustics, and similar topics, the results of Chapters I-III, obtained as they were from very general principles, may be said to be pertinent to a broad spectrum of acoustical modelling and experiments.[1] One might include theoretical work on random systems, [23, 38, 114, 117, 175], continuum mechanics, [18, 43, 44, 89, 103] and the extensive work in EM propagation [10, 27, 39, 62, 74, 225], this latter when based on a scalar wave equation.

We will attempt to indicate succinctly major areas of recent acoustic and related research which propose to some degree solutions to acoustic propagation. Much work is summarized in the texts of Morse and Ingard [147], Skudryzyk [181] and Tolstoy and Clay [194].

Notable in the study of general random inhomogeneities are Tatarskii [188-191], Keller [97, 99-101], Middleton [136-138], Bourret [26], Chernov [40-42], Dashen [52], Foldy [69], Frisch [72], Neubert [150, 151], Wenzel [203, 204], and earlier, Mintzer [141,143] and Kraichnan [108-110], who have largely discovered approximate solutions to the stochastic Helmholtz equation. Since much of this work assumes monochromatic propagation, it is especially relevant to the

---

[1]By itself the contemporary effort in acoustics is voluminous. We instance F.E. White and D.C. Teas, References to Contemporary Papers on Acoustics, JASA Suppl. 2. 1977, which covers 283 quarto pages of bibliography.

reverberation studies of Fauré [63], Jobst [96], Ol'shevskii [158], Middleton [138] and Vollerner [197]. This general work also leads to studies of coherence and fluctuation spectra, studies which have attracted a large following. We mention in particular Nichols [152-155], Porter [166, 167], Fitelsen [65-67], Adams [1], Jobst [2], Stanford [185], Tuteur [196], DeFarrari [53], Dyer [58], Glotov [77], Neklyudov and Chuprov [149], Baxter [11], Nefedôv [148], Beran [15], and Aikin [3]. Particular note should be made of Garrett and Munk [75], who propose a model for the effects of internal waves.

One may also distinguish research on the basis of media. Whereas much of the work mentioned above has the oceans in view, Brown and Clifford [29], Monin [144], Piercy [161], and earlier Obukhov [157], have directed their attention to the atmosphere.

Of course, the statistics of the given medium must be obtained experimentally. A large body of literature has grown up around this subject. We mention Dyer [59], Burnett [37], Mellen and Browning [132], Williams [209, 210], Shvachko [179, 180], Sanford [176], Pochapsky [162], Chuprov [45], Watson, Siegmann and Jacobson [201].

In more specialized areas, the acoustic effects of loss mechanisms have been investigated by Schulkin and Marsh [178], Fisher [64], Eby and Einstein [60], Mellen, Browning, and Ross [133] and Thorp [193]. Non-linear effects have concerned Blackstock [19-21], Westerveldt [205-207], Lighthill [118, 119] and Beyer [16], who has also written a text on ultrasonics [17].

Conspicuous among the scatterers of sound is the surface of the sea. A large community of research has developed around solutions to this scattering problem. We mention Clay [46-47], Medwin [130-131],

Middleton [135], Horton [90, 91], Marsh [127], Lysanov [122-125], McDonald, Tuteur, Zornig [129, 215], Brown [31, 33], Gulin [79-84], Harper and Labianca [86, 87], Parkins [159], Fortuin [70-71], Beckmann [14], Barrick [8], Konrady [107], Boyd and Deavenport [25], Smirnov and Tonakonov [182], where much of this work stems from earlier work by Eckart [61]. This area of research will engage our attention in Chapter VI.

Because of the great variety of proposed solutions, it will be impossible to give in this chapter a comprehensive account of this development. Rather, we will select certain developments which, in our opinion, represent many others of similar nature in the hope that, while individual selections themselves may show bias, the overall picture will be credited a fair and accurate account of our subject.

4.1 Velocity Fields and Linearization: Blokhintsev

The solutions found by Blokhintsev [22] are based on classical assumptions concerning the velocity vector field of the particle. After exhibiting his deriving equations (B1-4, Chapter I), Blokhintsev proceeds by linearizing (small perturbations) to obtain the expression

$$p = c^2 \nabla^2 \rho + h \nabla S \tag{4.1}$$

where $h = \left(\frac{dp}{dS}\right)_\rho = \rho c^2 \frac{\beta \tau}{C_p}$; β the coefficient of volume expansion; τ, the temperature, and C_p the heat capacity for constant p_2. Pointing out that "the propagation of sound may be said to be adiabatic but not isentropic i.e., $\frac{d\nabla^2 S}{dt} \neq 0$, he nevertheless proposes solutions based on velocity potential, $\emptyset$, (which applies when the entropy is zero). For this case, he obtains a solution,

$$p = \rho \frac{\partial \phi}{\partial t} \quad , \tag{4.2}$$

where

$$\frac{\partial \phi^2}{\partial t^2} = c^2 \nabla^2 \phi + \nabla \left(\int \frac{dp_2}{\rho} \right) \cdot \nabla \phi + \frac{d\phi}{dt} \underline{v} \cdot \nabla \ln c^2 \quad , \tag{4.3}$$

which, in the case of salt water, reduces to

$$p = \rho \int \left(c^2 \nabla^2 \phi + \frac{\nabla (p_2 - c^2 \nabla \rho) \cdot \nabla \phi}{\rho} \right) dt \tag{4.4}$$

a reduction which he justified from measurements of physical constants.

The Blokhintsev solutions are seen to relate the pressure at a spatial point ($\underline{r}$) to the previous temporal history at that point.[1] The method has been outlined here as an example of the approach using velocity potentials and small perturbations.

4.2 Stochastic-Helmholtz Solutions

The solutions to the Stochastic-Helmholtz equation, Eq. (1.34), enjoy the preference of much theoretical research on propagation through a turbulent medium [62, 168]. The popularity of Eq. (1.34) has been so great, in fact, that methods originally devoted to it have been developed which apply to a larger class of solutions. These methods are discussed in section 4.3 rather than here.

---

[1] Blokhintsev also solves $\left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = 0$ for a system of moving coordinates related to $\beta = v/c$. Not surprisingly, solutions for an initially spherical emanation are found to be $f(t + r'/c)$, where $r' = (\beta r \pm (1 - \beta^2) r)/(1 - \beta^2)$. It is from this point of view that he generalizes Kirchoff's theorem.

4.2.1 Mintzer

Mintzer [141-143], motivated by a discrepancy between ray theory predictions and some oceanographic experimental data, solved the stochastic Helmholtz equation using the first Born approximation. He found that the resulting expressions evaluated as a time average (over pulse length) matched, in terms of statistical variation, the experimental data.

Given a random index of refraction $\mu(\underline{r}) = 1 + \alpha\eta(\underline{r})$, $\alpha \ll 1$, the stochastic Helmholtz equation can be linearized to

$$(\nabla^2 + k_0^2 + 2k_0^2\alpha\eta)p = 0 \ . \tag{4.5}$$

For a spherical wave of unit amplitude, Eq. (4.5) has a solution

$$p(\underline{r}) = \frac{e^{jk_0 r}}{r} + \frac{2k_0^2\alpha}{4\pi} \int_{\underline{V}} \frac{\eta(r')e^{jk_0|r-r'|}}{|r'-r|} p(r')dr'^3 \ . \tag{4.6}$$

The (first) Born approximation uses $p(\underline{r}') = e^{jk_0 r'}/r'$, so that we have

$$p(\underline{r}) = \frac{e^{jk_0 r}}{r} + \frac{2k_0^2\alpha}{4\pi} \int_{\underline{V}} \eta(r') \frac{e^{jk_0(|r-r'|)}}{|r'-r|} \frac{e^{jk_0 r'}}{r'} dr'^3 \ . \tag{4.7}$$

Since $\langle\eta\rangle = 0$, $\langle p_s\rangle = 0$. Mintzer calculates the coefficient of variation

$$V = \frac{\langle|\tilde{p}|^2\rangle - |\langle p\rangle|^2}{\langle\bar{p}\rangle^2} = \frac{\alpha^2 k_0^2}{8\pi(2\pi)^\pi} \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \cdot$$

$$\int_0^{\infty} k^2 dk N(k,\theta,\phi) \int_{-r}^{r} dx' \int_{-r}^{r} dx'' \left[\exp \frac{1}{2} jk\cos\theta \ .\right.$$

$$\cdot (x'-x")] \frac{\exp jk^2(x'^2-x"^2)}{8k_0 r} \qquad (4.8)$$

where N is the Fourier transform of the autocorrelation function of n expressed in spherical coordinates and $\bar{p}$ is the time average over pulse length.

One notices that the solution to the turbulent propagation involves the medium variable $n(\underline{r})$.

Mintzer's model, while verified by a well controlled experiment, is not altogether satisfactory since, even as Mintzer pointed out [143, p. 189], the theory predicts that the signal correlation function will have no range dependence - a result contradicted by other data then available.

4.2.2 Rytov

We follow Fante [62] and Chernov [40] in explaining the Rytov Method [174]. This method, originally developed in connection with the theory of diffraction of light by an ultrasonic lattice, has been shown to give good agreement with optical experiments over propagation paths less than 1 km in the atmosphere. However, in the late 1960's, it was found deficient for horizontal propagation paths much greater than 1 km [78].

This method postulates an incident plane wave

$$p_0(\underline{r},t) = a_0 e^{-j(\omega t - \underline{k}\cdot\underline{r})} \quad , \; |\underline{r}| > 0, \quad a_0 \text{ constant} \qquad (4.9)$$

and a solution

$$p(\underline{r},t) = a(\underline{r}) e^{-j(\omega t - s(\underline{r}))} \quad , \qquad (4.10)$$

also written

$$p(\underline{r},t) = a_0 e^{-j(\omega t - \psi(\underline{r}))} , \tag{4.11}$$

so that,

$$\psi(\underline{r}) = s(\underline{r}) - j\ln\left(\frac{a(\underline{r})}{a_0}\right) . \tag{4.12}$$

Using $p(\psi)$ in the stochastic Helmholtz equation, one obtains for ψ

$$\nabla\psi\cdot\nabla\psi - j\nabla^2\psi = k^2 = k_0^2(1+\mu^2) \equiv k_0^2 n^2 . \tag{4.13}$$

Then, setting $\psi = \psi_1 + \psi_0$, where ψ_0 satisfies the equation for the homogeneous medium, $[(\nabla)^2 - j\nabla^2]\psi_0 = k_0^2$, and assuming that

$$\frac{|\nabla\psi_1|}{|\nabla\psi_0|} << 1, \quad \frac{|\nabla\psi_1|}{k} << 1, \text{ and } \mu << 1, \tag{4.14}$$

one obtains

$$2\nabla\psi_0\cdot\nabla\psi_1 - j\nabla^2\psi_1 = 2\mu k^2 \tag{4.15}$$

for the perturbed field ψ_1. The solution for ψ_1 is thus found to be

$$\psi_1(\underline{r}) = \frac{-jk_0^2}{2\pi \exp j\psi_0(\underline{r})} \int_V \mu(\underline{r}')\exp j\psi_0(\underline{r}') \frac{\exp j\underline{k}\cdot(\underline{r}-\underline{r}')}{|\underline{r}-\underline{r}'|} d\underline{r}' . \tag{4.16}$$

The amplitude and phase fluctuations can thus be obtained by separating the real and imaginary parts of ψ_1.

The assumed form of the solutions shows that temporal interactions are neglected, while the restrictions limit these solutions to

situations where the cumulative effect of turbulence is relatively insignificant.

4.2.3 Prokhorov

A.M. Prokhorov and his associates [168] have commented recently on a stochastic approach suitable for optical beams. Here, the stochastic Helmholtz equation to be solved is given by

$$\nabla^2 p + k_0^2(1 + 2\mu(\underline{r}))p = 0. \tag{4.17}$$

The solution is assumed to be a plane wave propagation along the z-axis, of the form,

$$p(\underline{r}) = a(\underline{r})\exp jkz \quad . \tag{4.18}$$

For mostly forward scattering over small angles

$$\nabla^2 p = \nabla_{xy}^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} ,$$

so that $a(\underline{r})$ satisfies

$$2jk_0 \frac{\partial a(\underline{r})}{\partial z} + \nabla_{xy} a(\underline{r}) + 2k_0^2 \mu a = 0 \tag{4.19}$$

Rather than seek approximate solutions which directly link $a(\underline{r})$ and μ (as in the Rytov method), this method proceeds directly to statistical moments across z planes. For the second moment

$$\langle m_{12} \rangle = \langle a(x_1, y_1) a(x_2, y_2) \rangle ,$$

for example, one obtains

$$2jk_o \frac{\partial}{\partial z} \langle m_{12} \rangle + \nabla^2_{x_1 y_1} - \nabla^2_{x_2 y_2} \langle m_{12} \rangle + 2k_o^2 \langle \mu_1 m_{12} \rangle$$

$$- 2k_o^2 \langle \mu_2 m_{12} \rangle = 0 \ . \qquad (4.20)$$

In order to deal with the cross terms $\langle \mu_i m_{12} \rangle$, the local method of small perturbations is used. For the field between z_0 and $z_0 + \Delta z$, let

$$a(z) = a^0(z) + a^1(z) \ , \qquad (4.21)$$

where a^0 is the homogeneous component and a^1 is the first order scattered amplitude. Using this and other assumptions (no backscatter, small inhomogeneities, etc.) one arrives at the equation:

$$\frac{\partial}{\partial t} \langle m_{12} \rangle - \frac{j}{2k_0} \nabla^2_{x_1 y_1} \langle m_{12} \rangle + \frac{j}{2k_0} \nabla^2_{x_2 y_2} \langle m_{12} \rangle + k^2 [R(0)$$

$$- R(1,2)] \langle m_{12} \rangle = 0 \ , \qquad (4.22)$$

where

$$R(1,2) = \int_{z_0}^{z} \langle \mu(r_{\sim x_1 y_1}; z) \mu(r_{\sim x_2 y_2}; z') \rangle dz'$$

is related to the correlation function of μ. It may be seen that $\langle m_{12} \rangle$ characterizes the angular spread of scattered radiation.

Similarly, one obtains for the first moment $\langle m_1 \rangle$, (the average field)

$$\frac{\partial}{\partial z} \langle m_1 \rangle - \frac{j}{2k_0} \nabla^2_{xy} \langle m \rangle + k_0^2 R(0) \langle m_1 \rangle = 0 \ , \qquad (4.23)$$

and for the fourth moment $\langle m_{24} \rangle$ (which characterizes fluctuations of intensity)

$$\frac{\partial}{\partial t}\langle m_{24}\rangle - \frac{j}{2k_0}\left(\nabla^2_{x_1y_1} + \nabla^2_{x_2y_2} - \nabla^2_{x_3y_3} - \nabla^2_{x_4y_4}\right)\langle m_{24}\rangle$$

$$+ k_0^2\langle m_{24}\rangle[2R(0) - R(1,2) + R(1,3) + R(1,4)$$

$$+ R(2,3) + R(2,4) - R(3,4)] = 0 \ . \qquad (4.24)$$

Solutions to the non-stochastic equations (Eqs. 4.22 to 4.24) may be obtained routinely and applied to the effects under study (for instance, the limitations on focal ability due to turbulent propagation).

4.3 Quantum Mechanical Solutions

Analytical solutions to Chernov's equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p + \nabla \ln \rho_0 \cdot \nabla p = 0 \qquad (4.25)$$

are comparatively rare. This is to be expected since, in addition to the information needed for the stochastic Helmholtz equation, a knowledge of the density, ρ_0, and its variation in space are required. In this section, we will present: (1) an abortive attempt at solution by Chernov, and (2) a general method of solution usually applied to the stochastic Helmholtz equation, but applicable here as well.

We follow Chernov [41]. Assuming $p = p_0 + p_1$ where $p_0 = a_0 \exp[-j(\omega t - k_0 r)]$, a plane wave, and that ρ and c deviate only slightly

from their mean values, one obtains for p,

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = -\left(2k^2\frac{\nabla c}{c_0} + \frac{jk}{\rho_0}\frac{\partial \nabla \rho}{\partial x}\right) a_0 \exp -j(\omega t - kr) = 4\pi Q \quad (4.26)$$

where $k = \omega/c_0$. Under the influence of the primary wave, each element of the inhomogeneous medium acts as a distributed, secondary source of radiation. Q may be interpreted as the density of these elementary secondary sources. The solution for the scattered wave is then given by

$$p_1 = \frac{-a_0}{4\pi}\int_V \left(2k^2\frac{\nabla c}{c_0} + \frac{jk}{\rho_0}\frac{\partial \nabla \rho}{\partial r'}\right)\frac{1}{r}\exp jk(r+r')dr' \; . \quad (4.27)$$

for the harmonic case.

Chernov continues by relating propagation velocity and density changes to temperature and, finding velocity change an order of magnitude larger, drops the density terms. He notes, honestly enough, "...that neglecting density fluctuations is equivalent to neglecting the additional term $\nabla \ln \rho \cdot \nabla p$ in the original wave equation."[1]

A wave propagating through a turbulent medium has something about it redolent of quantum field theory. Pursuit of this analogy, in addition to the application of new methods, brings other benefits. As Tatarskii remarks,[2] the "...application of the mathematical tools of quantum field theory enables us to leave the limitations of the theory of small perturbations and to obtain a solution valid in the case of strong fluctuations."

---

[1] [41] p. 50.

[2] [187] p. 335.

Many researchers have taken this lead. We follow an especially fine and lucid exposition by D. Dence and J. Spence [55]. One begins by separating the deterministic from the random parts of the problem. Let $\hat{L}$ be the deterministic operator and ξ a random point function, so that the problem may be stated as

$$\hat{L}p(\underset{\sim}{r},\underset{\sim}{r}') = \xi(\underset{\sim}{r})p(\underset{\sim}{r},\underset{\sim}{r}') + s_0(\underset{\sim}{r},\underset{\sim}{r}') \tag{4.28}$$

where s_0 is a deterministic source at r'. If $\hat{L}$ has an inverse, $\int g(\cdot)dv$, and if $\langle\xi\rangle = 0$, a solution is easily found to be

$$p(\underset{\sim}{r},\underset{\sim}{r}') = \int_V \hat{g}(\underset{\sim}{r},\underset{\sim}{r}')s_0(\underset{\sim}{r}_1,\underset{\sim}{r}')dr_1^3 + \int_V \hat{g}(\underset{\sim}{r},\underset{\sim}{r}')\xi(r_1)p(\underset{\sim}{r}_1,\underset{\sim}{r}')dr_1^3 \ . \tag{4.29}$$

One may now reiterate this solution in a manner analogous to continued fractions, and so produce an infinite series, only the last term of which contains p. All other terms contain interactions of the random field quantities. Assuming the necessary existence properties, one may now calculate the mean $\langle p(\underset{\sim}{r},\underset{\sim}{r}')\rangle$ and other moments of the scattered field in terms of the n^{th} dimensional correlation functions of the random component, i.e., $\langle\xi_1\ \xi_2\ \ldots\ \xi_n\rangle$. If ξ is normally distributed,[1] then these n^{th} dimensional correlations can all be decomposed into products of two point correlation B(i,j). In this latter case, one may introduce the following notation:

---

[1]Middleton has shown (July 1977) in a private correspondence with the author that this will be the case whenever a comparatively large number of scatters interact per unit domain. In general, as Middleton also shows, the statistics will be multi-point poisson whose order depends on the complexity of interaction.

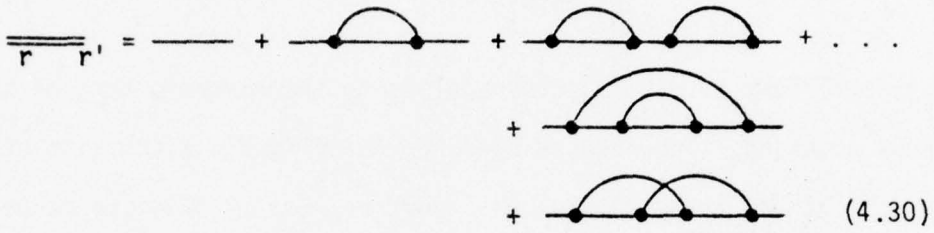
● ≡ a dummy variable over which integration is taken

——— ≡ $\hat{G}$

═══ ≡ $\langle p \rangle$

⌒ ≡ The correlation function, B

Then, the infinite series can be represented in Feynman diagrams as



(4.30)

One may "factor" the diagram to obtain Dyson's equation

$$\langle p(\underline{r},\underline{r}') \rangle = \int_V G(\underline{r},\underline{r}_1) s_0(\underline{r}_1,\underline{r}') dr_1^3 +$$

$$\int_V G(\underline{r},\underline{r}_1)\, Q(\underline{r}_1,\underline{r}_2) \langle p(\underline{r}_2,\underline{r}') \rangle dr_1^3 dr_2^3 \qquad (4.31)$$

where Q represents an infinite subsequence called the "mass operator." Extension of these techniques may be used to obtain higher order moments.

We first show that this approach is suitable for Chernov's equation, Eq. (4.25).

Noticing that the general form Eq. (4.28) contains no time variable, we reduce the Chernov equation to its harmonic form:

$$k^2 p + (\nabla - \nabla \ln \rho_0) \cdot \nabla p = -s(\underline{r}'), \quad k = \omega_0/c \qquad (4.32)$$

with harmonic source, $s(\underline{r}')$.

For $k^2 = k_0^2(1 + \mu)^2$, this reduces to

$$k_0^2 + (\nabla - \nabla \ln \rho_0)\cdot\nabla\, p = -k_0^2\mu(\mu + 2)p + s \qquad (4.33)$$

so that

$$\hat{L} = k_0^2 - (\nabla - \nabla \ln \rho_0)\cdot\nabla \qquad (4.33a)$$

$$\xi = -k_0^2\mu(\mu + 2)\ . \qquad (4.33b)$$

The general form Eq. (4.28) thus applies to the harmonic form of the Chernov equation. The inverse operator is obviously a function of ρ_0 here. If it can be found, Dyson's equation, Eq. (4.31), can be used immediately.

Solutions have been found for $\hat{L} \equiv \nabla^2 + k_0^2$, $\xi = -k_0^2\nu$, which is the stochastic Helmholtz equation for small variation in the index of refraction.

Here

$$\hat{g} = \frac{e^{-jk_0|\underset{\sim}{r}-\underset{\sim}{r}'|}}{-4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|}\ .$$

Putting

$Q \equiv$

representing, in expanded form, the "bilocal" approximation and an assumption of local independence, i.e.,

$$\langle \xi(\underline{r})\xi(\underline{r}_2)p(\underline{r}_2)\rangle = \langle \xi(\underline{r})\xi(\underline{r}_2)\rangle\langle p(\underline{r}_2)\rangle$$

one obtains

$$(\nabla^2 + k_0^2)\langle p(\underline{r})\rangle + k_0^4 \int_r \frac{e^{-jk_0|\underline{r}-\underline{r}|}}{4\pi|\underline{r}-\underline{r}_1|} B(\underline{r},\underline{r}_1)\langle p(\underline{r}_1)\rangle dr_1^3 = \delta(\underline{r}) \qquad (4.34)$$

where B is the second order correlation function of $\underset{\sim}{\nu}$.

If $B(\underset{\sim}{r},\underset{\sim}{r}')$ is stationary, Fourier transform methods may be used to obtain

$$(-k^2 + k_0^2)P(\underset{\sim}{k}) + k_0^4 K(\underset{\sim}{k})P(\underset{\sim}{k}) = 1$$

where P and K are transformations, i.e., $K = \int_V (e^{-jk_0\rho}/4\pi\rho)B(\rho)e^{-j\underset{\sim}{k}\cdot\underset{\sim}{r}}d\underset{\sim}{r}^3$. One must now know $B(\rho)$, the correlation statistic of ν in order to invert. Assuming

$$B = B_0^2 \exp|\underset{\sim}{r}-\underset{\sim}{r}_0|a,$$

then we have as solution

$$p(\underset{\sim}{r}) = a_1 e^{-j\underset{\sim}{k}_1\cdot\underset{\sim}{r}} + a_2 e^{-j\underset{\sim}{k}_2\cdot\underset{\sim}{r}} \tag{4.35}$$

where

$$a_1 = -\frac{1}{8\pi}\left[1 + (1 + 4B_0^2k_0^2a^2/(1 + 2jk_0a)^2)\right]^{-\frac{1}{2}}, \tag{4.35a}$$

$$a_2 = \frac{-1}{8\pi}\left[1 - (1 + 4B_0^2k_0^2a^2/(1 + 2jk_0a)^2)\right]^{-\frac{1}{2}}, \tag{4.35b}$$

$$k_1 = \frac{1}{\sqrt{2}a}\left[k_0^2a^2 - (1 + jk_0^2a) - [(1 + 2jk_0a)^2 + 4B_0^2k_0^4a^4]^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{4.35c}$$

$$k_2 = \frac{1}{\sqrt{2}a}\left[k_0^2a^2 - (1 + jk_0a)^2 - [(1 + 2jk_0a)^2 + 4B_0^2k_0^4a^4]^{\frac{1}{2}}\right]^{\frac{1}{2}}. \tag{4.35d}$$

Thus, depending on conditions in the media (especially the relative measure of k_0 and a), the randomness excites a sound wave, perhaps evanescent. Moreover, each wave, and in particular the coherent wave,

has a complex wave number, which indicates that it is subject to a turbulence loss, independent of other attenuation. Also the real part of the wave number increases, "...causing the coherent wave to travel slower than if these were no randomness."[1]

The existence of limits in these representations (the convergence problem) is a source of major theoretical difficulty.[2] We note here, for later comment in Chapter V on this matter, that (1) only selections of the correlation structure are used in the bilocal approximation and that (2) the assumption of local independence amounts to the use of the first term of a Taylor's series. Both these matters must be duly considered in assessing the convergence of sums (or subsums) expressed in Eq. (4.30). Furthermore, it has been recognized[3] that the formulations based on the bilocal approximation are not energy conserving.

4.4 FOM[4] Solution

The "classical" approach to our subject generally taken, and indeed taken up to this point, attempts to describe the entire pressure field from knowledge of the physical laws locally applicable (as differential equations). One must add to the local phenomenon the boundaries, geometry, waveforms, transmitting and receiving apertures, etc.

---

[1] [55] p. 140.

[2] [128] p. 6, 45.

[3] Rosenbaum, 1968. "On energy-conserving formulations in a randomly fluctuating medium," in Proc. Symp. Turbulence Fluids Plasmas, Polytechnic Press of the Polytechnic Inst. of Brooklyn, p. 163.

[4] FOM ≡ "Fauré - Ol'shevskii - Middleton" after Fauré [63], Ol'shevskii [158], and Middleton [138].

for practical engineering results. This "bottom-up" approach may be contrasted with a more recent "top-down" method, in which the general engineering problem is initially modelled to include all major relevant factors, each of which is then developed in detail as warranted. This "top down" approach is analogous to methods used in communication theory, or system design where the methods have met with clear success.

We follow a vigorous exponent of the latter method [38] in commenting on the FOM model, applicable not without justification [95] to a domain of discrete scatters.

Our interest in these models centers on a waveform U measured through an aperture (perhaps moving). For an active scattering region, Λ, in which a Poisson (spatial) scattering process operates, one obtains as the characteristic function of the received process, X,

$$F_n(j\xi_1,t_1,\ldots\ j\xi_n,t_n|t',\Lambda) = \exp \int_\Lambda \rho <\exp j \sum_{\ell=1}^{n} \xi_\ell U(t_\ell;\lambda,\theta) - 1>_\theta d\underline{\lambda} \quad (4.36)$$

where $\rho(\underline{\lambda})$ is the process "density," U is the scattered waveform associated with each individual scattering event, and $\underline{\lambda}$ is a positional variable. The moments of the received process may then be taken directly from the characteristic function in the usual way. For example, the mean and covariance are given by

$$<X> = \int_\Lambda \rho(\underline{\lambda},t')<u(t_1,\underset{\sim}{\lambda})>d\underset{\sim}{\lambda} \quad (4.37)$$

$$K_X = \int_\Lambda \rho(\underset{\sim}{\lambda},t')<U(t_1,\underset{\sim}{\lambda})U(t_2,\underset{\sim}{\lambda})>d\underset{\sim}{\lambda} \ . \quad (4.38)$$

Under the assumption that the scatterers are in the far field to and from which ideal propagation in a homogeneous medium (constant density) holds, one has here

$$U_j = \frac{1}{4\pi R_{R_j}} \int_{V_R} d\underset{\sim}{\eta} \int_{-\infty}^{\infty} \alpha_R(\underset{\sim}{\eta}, t-\tau + \underset{\sim}{i}_{R_j} \cdot \underset{\sim}{\eta}/c)d\tau$$

$$\cdot \int_{-\infty}^{\infty} h'_M(\tau-\tau_1-t_{dj}[\tau_1,\tau],\tau-r_{R_j}\frac{(\tau_1,\tau)}{c}|R_{T_j})d\tau_1$$

$$\cdot \frac{1}{4\pi R_{Tj}} \int_{V_T} d\underset{\sim}{\xi} \int_{-\infty}^{\infty} \alpha_T(\underset{\sim}{\xi},\tau_1-\tau_2 + \underset{\sim}{i}_{T_j} \cdot \frac{\underset{\sim}{\xi}}{c})S_{in}(\tau_2,\underset{\sim}{\xi})d\tau_2 \qquad (4.39)$$

where α_R and α_T denote the receiving and transmitting apertures, (which act like linear space-time filters); S_{in}, the input signal, h'_M, the filtering effect of a typical scatterer at R_j. This relation, Eq. (4.39), includes the effects of both doppler due to receiver and transmitter motions and doppler due to scatterer motion.

Thus, the major segments of a practical acoustical problem have been sketched in considerable generality. While this approach is interesting in its results, it is still more interesting in its methodology. We expect to find, under differing conditions, results quite different from this early model, yet the methodology can be expected to hold.

It should be noted that, in addition to the limitations of a first order (single scattering) model this early work depends heavily on the independence of the individual scatterers. It thus may be expected to model weak scattering regimes, where fish, for example, provide a physical scattering mechanism, or again a high frequency model of

surface scatter. Where, however, the scattering mechanisms are correlated this early model encounters difficulties. We shall see in Chapter VI Middleton's further development of this model with new methods that allow far greater applicability.

4.5 Summary

In this Chapter we have attempted to give an overview of recent historical solutions to acoustic propagation by first indicating the extensiveness of research effort devoted to these solutions and then by selecting certain typical solutions for exposition. In the main, the reader will note that the effort is directed to solving the (narrow-band) stochastic Helmholtz equation in two broad categories: initial plane wave and initial spherical wave propagation. The more recent work formulates the solutions in a statistical sense, that is, the solution is expressed in terms of the moments of the received pressure viewed as a random process.

V. PROBABILISTIC METHODS

5.0 Introduction

In the middle life history of a pressure wave, that age subsequent to when its acoustic properties dominated the medium, and yet prior to its decadence into background noise[1], the wave may be thought of as composed of two parts. The first relates to the source (correlated); the second does not (uncorrelated). It is this intermediate epoch of the wave's history which we concentrate upon below.

In this chapter we propose to develop the statistics of the received wave (first, second, third, and fourth moments) in terms of the statistics of the medium. The first moment, $\langle p(\underline{r},t)\rangle$, gives the ensemble (not the temporal) mean, and generally does not equal zero. The second moment, $\langle p(\underline{r}_1,t_1)p_2(\underline{r}_2,t_2)\rangle$, refers to various physical properties (spatial or temporal correlation, intensity) depending on the choice of $(\underline{r}_2,t_2)$, or conjugates, etc. The fourth moment is important for determining fluctuations in intensity.

Our development is made in light of the methods used successfully to obtain the general solution of Chapter III. By insisting on approximations which model temporal and mixed effects, we may hope to advance the solutions given in the literature. Both as an illustration and because of its practical importance, we specialize in this

---

[1]See Ch. II, Sec. 2.2.

and the remaining chapters to the following form[1] of the wave equation

$$\left(\nabla^2 - \frac{1}{c_0^2}\,(1+d(\underline{r},t))\,\frac{\partial^2}{\partial t^2}\right) p(\underline{r}_2 t) = s(\underline{r}_0, t_0) \tag{5.1}$$

The spatial-temporal function $d(\underline{r},t)$ is defined by

$$c^2(\underline{r},t)\,[1 + d(\underline{r},t)] \equiv c_0^2 \quad . \tag{5.1a}$$

The conditions for the validity of Eq. (5.1) may be appreciated from Chapter I, Sec 1.5.2. Equation (5.1) can be seen to provide spatial, temporal and mixed space-time effects. The reader will likewise notice that (5.1) is a linear relationship which contains no independent noise sources.[2]

In the sequel, we shall occasionally use Eq. (5.1) in a different form, the so-called M-form,[3] given as

$$\left(\nabla^2 - \frac{1}{c_0^2}\,(1 + d_s + d_v + d_B)\,\frac{\partial^2}{\partial t^2}\right) p = s_0 \ , \tag{5.2}$$

so as to accent the acoustic source, s_0, and to distinguish surface (d_s), volume (d_v) and bottom (d_B) scattering. The M-form frees solutions from the practically impossible problem of boundary conditions (classical method) in accordance with the understanding of solutions (Chap. III). In this conception, interfaces behave as localized scatterers and are so treated formally.

---

[1] In this form harmonic solutions are not generally possible because $d(\underline{r},t)$ is allowed a temporal variation.

[2] See Sec. 7.3 for a treatment of "noisy" models.

[3] From a development made by Middleton [136].

An instructive comparison between Eq. (5.1) and the general solution can be made with transforms.

Applying the transform $\hat{F}$ to Eq. (5.1), we obtain

$$-k^2\hat{F}(p) + \frac{1}{c_0^2}(\delta(\underline{k},\omega) + \hat{F}(d)) * \omega^2\hat{F}(p) = \hat{F}(s_0), \tag{5.3a}$$

or

$$k^2\hat{F}(p) - \frac{1}{c_0^2}\omega^2\hat{F}(p) - \frac{1}{c_0^2}\hat{F}(d)* \omega^2\hat{F}(p) = -F(s_0) . \tag{5.3b}$$

Equation (5.3b) may be put in the form

$$\hat{F}(p) - \frac{\hat{F}d * \omega^2\hat{F}(p)}{c_0^2k^2 - \omega^2} = \frac{c_0^2\hat{F}(-s_0)}{c_0^2k^2 - \omega^2} . \tag{5.4}$$

Applying the feedback theorem with $\hat{G} = \frac{(\hat{F}d * \omega^2F(\cdot))}{c_0^2k^2 - \omega^2}$, we have

$$\hat{F}(p) = \sum_{n=0}^{\infty} (\hat{G})^{(n)} \left[\frac{c_0^2\hat{F}(-s_0)}{c_0^2k^2 - \omega^2}\right] \tag{5.5}$$

for solution.

The formal differences between solutions to Eq. (5.1) and the general equation, Eq. (3.70), may thus be set down[1] as in Table 5-1.

---

[1]This comparison is awkward from Eq. (5.1) being a pressure flux relationship while the general equation in Sec. 3.2.2.2 is given in terms of temporal variation of power density. Further, $c^2(1+d) = c_0^2$ implies

$$\hat{F}(c^2)*\hat{F}(1 + d) = c_0^2\delta(k,\omega) = \hat{F}(c^2) + \hat{F}(c^2)*\hat{F}(d),$$

so that

$$\hat{F}(c^2)*\hat{F}(d) = c_0^2\delta(\underline{k},\omega) - \hat{F}(c^2) = c_0^2\hat{F}(c^2)*\hat{F}\left(\frac{1}{c^2}\right) - \hat{F}(c^2) .$$

It follows that

One can see that even in the perfect medium ($\nabla \cdot T = -\nabla p$) postulated for Eq. (5.1), the introduction of a medium related variable brings on a feedback solution.

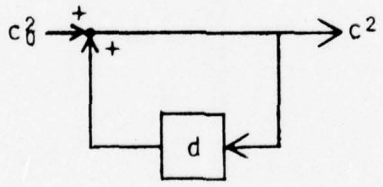
Space-time considerations provide insight as well. Let

$$\hat{G} = \left[\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t}\right]^{-1} , \tag{5.6}$$

so that Eq. (5.1) has for solution

$$\hat{F}(c^2) = c_0^2\left(1 - \hat{F}(d) + \hat{F}(d)*\hat{F}(d) + \ldots + (-1)^{(n)}(\hat{F}d*)^{(n)} + \ldots\right)\delta(k,\omega) .$$

One may argue to the result directly from



from which

$$c^2 = c_0^2 \sum_{n=0}^{\infty} (-1)^n d^n , \text{ if } |d| < 1.$$

From power density relations one would have

$$\hat{F}(p) = \sum_{n=0}^{\infty} \hat{G}_1^{(n)}\left[\frac{\hat{F}(s_0)}{\omega^2}\right]$$

and

$$\hat{G}_1 = \frac{1}{c_0^2}\left[\hat{F}(c^2)*k^2\hat{F}(\cdot)\right] .$$

TABLE 5-1. COMPARISON OF THE GENERAL SOLUTION TO THE BROADBAND WAVE EQUATION SOLUTION

	General Form	Equation (5.1)
$F(p)$	$\sum_{n=0}^{\infty} \hat{G}(n) \quad \frac{\hat{F}(u)}{\omega^2}$	$\sum_{n=0}^{\infty} \hat{G}(n) \quad \frac{c_o^2 F(-s_o)}{c_o^2 k^2 - \omega^2}$
$\hat{G}$	$-\frac{1}{\omega^2}\Big(j\hat{F}(a_o) * \omega\hat{F}(\cdot) - \hat{F}(\underline{a}_1) * \omega(\underline{k}\hat{F}(\cdot)) - j\hat{F}(\underline{a}_2) * \underline{k}\hat{F}(\cdot) + \hat{F}(c^2) * (k^2\hat{F}(\cdot) - \hat{k}\ell n\rho * \hat{F}(\cdot))\Big)$	$\frac{\hat{F}(d) * \omega^2\hat{F}(\cdot)}{c_o^2 k^2 - \omega^2}$
$\frac{\hat{F}(u)}{\omega^2}$	$\frac{1}{\omega^2}\Big(-\hat{F}(g_o) + \hat{F}(a_1) * \omega\underline{k}\hat{F}(\underline{b}_1) + j\hat{F}(\underline{a}_1) * \underline{k}\hat{F}(b_1) - \hat{F}(c^2) * (j\underline{k}\hat{F}(\underline{b}_2) - j\underline{k}\hat{F}(\ell n\rho) * \hat{F}(\underline{b}_3))\Big)$	$\frac{c_o^2\hat{F}(-s_o)}{c_o^2 k^2 - \omega^2}$
$\hat{F}(c^2)$	$c_o^2 \sum(-1)^n(\hat{F}d*)^{(n)}\delta(\underline{k},\omega)$	$c_o^2 \sum(-1)^n(\hat{F}d*)^{(n)}\delta(\underline{k},\omega)$

$$p = \hat{G}(s_0) + \hat{G}\left(\frac{d}{c_0^2}\frac{\partial^2}{\partial t^2}\, p\right) \tag{5.7}$$

$$= \sum_{n=0}^{\infty} \left(\hat{G}\,\frac{d}{c_0^2}\frac{\partial^2}{\partial t^2}\right)^{(n)} \hat{G}(s_0)\ . \tag{5.8}$$

In Eq. (5.7), we recognize $\hat{G}(s_0)$ as the unscattered wave, p_u, (solution for d=0), while

$$P_s \equiv \sum_{n=1}^{\infty} \left(\hat{G}\,\frac{d}{c_0^2}\,\frac{\partial^2}{\partial t^2}\right)^{(n)} \hat{G}\,(s_0) \tag{5.9}$$

may be termed the scattered wave, p_s. Comparing Eq. (5.7) with Eq. (3.40) highlights three assumptions made in the model described by Eq. (5.7):

1. a medium assumption: $-s/n \to G(s_0)$, deterministic;
2. a subject assumption: $[p(o,t_0) + \hat{G}_{rt}u - s/n] \to \hat{G}(s_0)$;
3. an operator assumption: $\frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{\hat{G}_{rt}\nabla\cdot v - \hat{G}_{rtD}}{2}\right)^{(n)}$

$$\to \sum_{n=0}^{\infty} \left(\hat{G}\,\frac{d}{c_0^2}\,\frac{\partial^2}{\partial t^2}\right)^{(n)}\ .$$

5.1 General Methods

In Eq. (5.1) let us now consider $d(\underline{r},t)$ a random variable whose statistics are known. We turn now to calculating the statistics of the output random variable $p(\underline{\theta},t)$.

5.1.1 Moment Equations

This path will be travelled more than once. Here, we pass quickly for direction. Subsequently, we will look for detail.

We start with the linear formulation

$$\hat{L}\, p(\underline{r}_1,\underline{r}_0;t_1,t_0) = \hat{\xi}\,(\underline{r}_1,t_1)\, p(\underline{r}_1,\underline{r}_0;t_1,t_0) + s(\underline{r}_0,t_0) \qquad (5.10)$$

where $\hat{L}$ is a deterministic linear operator; $\hat{\xi}$ is a random operator; p is the random process and s is a source at $\underline{r}_0$.

5.1.1.1 Means (First Moment)

Taking ensemble averages of Eq. (5.10), we have

$$L\langle p\rangle = \langle\hat{\xi}p\rangle + s \ . \qquad (5.11)$$

Putting $\langle\hat{\xi}p\rangle \equiv Q\langle p\rangle$ and applying $\hat{L}^{-1} = \hat{G}$, gives

$$\boxed{\langle p\rangle = \hat{G}(s) + \hat{G}Q\langle p\rangle} \qquad (5.12)$$

which is Dyson's Equation.[1] Q is the "mass operator." Thus

$$\boxed{\langle p\rangle = \sum_{n=0}^{\infty} (\hat{G}Q)^{(n)}\hat{G}(s) = \sum_{n=0}^{\infty} \langle\hat{G}\hat{\xi}\rangle^{(n)}\hat{G}(s) \ .} \qquad (5.12a)$$

Note that $\langle\hat{\xi}\rangle = 0$ does not imply that $\langle\hat{\xi}p\rangle = 0$, or that $\langle p\rangle = 0$. Indeed if $\hat{\xi}\equiv 0$, $\langle p\rangle = \hat{G}(s) = p_u$. In some cases writing $\hat{\xi} = \hat{\xi} - \langle\hat{\xi}\rangle + \langle\hat{\xi}\rangle$, to produce a zero mean process $L\langle p\rangle = \langle(\hat{\xi}-\langle\hat{\xi}\rangle)p\rangle + \langle\hat{\xi}\rangle\langle p\rangle + s$ may be advisable depending on the structure of $\langle\hat{G}(\hat{\xi}-\langle\hat{\xi}\rangle + \langle\hat{\xi}\rangle)\rangle^{(n)}$. Also, the definition $Q\langle p\rangle \equiv \langle\hat{\xi}p\rangle$ may always be made uniquely whenever $\langle p\rangle \neq 0$ because $\langle p\rangle$ and $\langle\hat{\xi}p\rangle$ are simply real (complex) numbers for any space-time point $(\underline{r},t)$. We shall return to the interpretation of Q in Sec. 5.1.2.

---

[1] See [55], p. 136.

5.1.1.2 Second Moment

Several choices are available. We mention:

$$\langle p_1 p_2 \rangle \equiv \langle p(\underline{r},t)p^*(\underline{r},t)\rangle$$

$$\langle p_1 p_2 \rangle \equiv \langle p(\underline{r}_1,t_1)p(\underline{r}_2,t_2)\rangle$$

$$\langle p_1 p_2 \rangle \equiv \langle p(\underline{r}_1,t_1)p(\underline{r}_1,t_2)\rangle$$

$$\langle p_1 p_2 \rangle \equiv \langle p(\underline{r}_1,t_1)p(\underline{r}_2,t_1)\rangle$$

For complex conjugation, we have from Eq. (5.10)

$$[\hat{L}p]^* = \hat{\varepsilon}^* p^* + s^* ,$$

which has a solution

$$p^* = \hat{G}^*(s^*) + \hat{G}^*\hat{\varepsilon}^* p^* . \qquad (5.13)$$

The various choices pertain to different physical realities: peak pressure, spatial correlation, temporal (ensemble) correlation, etc., to be chosen as needed.[1] In the sequel we develop $\langle p_1 p_2 \rangle$ as a type for any of the choices.

---

[1]The reader should not confuse correlation with coherence. With respect to the source, s, define R_{sr} as

$$R_{sr} \equiv \langle sp \rangle = s\langle p \rangle .$$

We see that the correlation, R_{sr}, generally does not equal zero, because $\langle p \rangle \neq 0$ generally. The coherence (mutual coherence function), however, is defined

$$C_{sr} \equiv \langle (s-\langle s \rangle)(p-\langle p \rangle)\rangle \equiv 0$$

for a deterministic source. Both notions differ, of course, from the average over time.

Multiplying p_1 and p_2 gives

$$p_1p_2 = (\hat{G}_1(s_1) + \hat{G}_1\hat{\xi}_1p_1)(\hat{G}_2(s_2) + \hat{G}_2\hat{\xi}_2p_2)$$

$$= (\hat{G}_1(s_1))(\hat{G}_2(s_2)) + (\hat{G}_1(s_1))(\hat{G}_2\hat{\xi}_2p_2)$$

$$+ (\hat{G}_2(s_2))\ (\hat{G}_1\hat{\xi}_1p_1) + \hat{G}_1(\hat{G}_2(\hat{\xi}_1p_1\hat{\xi}_2p_2))\ , \qquad (5.14)$$

so that

$$\boxed{\begin{aligned}\langle p_1p_2\rangle &= p_{u_1}p_{u_2} + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle \qquad (5.15)\\ &\quad + \hat{G}_1\ (\hat{G}_2\langle\hat{\xi}_1\hat{\xi}_2p_1p_2\rangle)\ .\end{aligned}}$$

Defining B by

$$B_{12}\langle p_1p_2\rangle \equiv \langle\hat{\xi}_1\hat{\xi}_2p_1p_2\rangle \quad , \qquad (5.16)$$

allows us to write Eq. (5.15) as

$$\boxed{\begin{aligned}\langle p_1p_2\rangle &= p_{u_1}p_{u_2} + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle\\ &\quad + \hat{G}_1(\hat{G}_2B_{12}\langle p_1p_2\rangle)\\ &= \sum_{n=0}^{\infty}(\hat{G}_1(\hat{G}_2B_{12}[p_{u_1}p_{u_2} + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle]))^{(n)}\ . \qquad (5.17)\end{aligned}}$$

Equation (5.17) is not the Bethe-Salpeter expression

$$\langle p_1p_2\rangle = \langle p_1\rangle\langle p_2\rangle + \langle p_2\rangle\langle p_2\rangle B\langle p_1p_2\rangle \qquad (5.18)$$

which may be obtained by another choice[1] of B. This latter, Eq. (5.18), is of less use to us, since it does not generalize to higher order moments as easily as Eq. (5.17).

5.1.1.3 Third Moment

We develop the third order moment, $\langle p_1 p_2 p_3 \rangle$, principally for its influences on the fourth moment. Form

$$p_1 p_2 p_3 = (\hat{G}_1 \hat{\xi}_1 p_1 + \hat{G}_1(s_1))(\hat{G}_2 \hat{\xi}_2 p_2 + \hat{G}(s_2))(\hat{G}_3 \hat{\xi}_3 p_3 + \hat{G}(s_3))$$

$$= p_{u_1} p_{u_2} p_{u_3} + p_{u_1} \hat{G}_2(\hat{G}_3(\hat{\xi}_2 \hat{\xi}_3 p_2 p_3)) + p_{u_2} \hat{G}_1(\hat{G}_3(\hat{\xi}_1 \hat{\xi}_3 p_1 p_3))$$

$$+ p_{u_3} \hat{G}_1(\hat{G}_2(\hat{\xi}_1 \hat{\xi}_2 p_1 p_2)) + p_{u_1} p_{u_2} \hat{G}_3 \hat{\xi}_3 p_3 + p_{u_1} p_{u_2} \hat{G}_2 \hat{\xi}_2 p_2$$

---

[1]Recognizing that

$$\langle p_1 \rangle \langle p_2 \rangle = p_{u_1} p_{u_2} + p_{u_1} \hat{G}_2 Q_2 p_{u_2} + p_{u_2} \hat{G}_1 Q_1 p_{u_1} + \hat{G}_1(\hat{G}_2 Q_1 Q_2 p_{u_1} p_{u_2}),$$

the second moment, upon taking ensemble averages, becomes

$$\langle p_1 p_2 \rangle = \langle p_1 \rangle \langle p_2 \rangle - (\hat{G}_1 Q_1 \langle p_1 \rangle)(\hat{G}_2 Q_2 \langle p_2 \rangle) + \hat{G}_1(\hat{G}_2 \langle \hat{\xi}_1 \hat{\xi}_2 p_1 p_2 \rangle) .$$

The Bethe-Salpeter equation follows immediately upon defining

$$-(\hat{G}_1 Q_1 \langle p \rangle)(\hat{G}_2 Q_2 \langle p_2 \rangle) + \hat{G}_1(\hat{G}_2 \langle \hat{\xi}_1 \hat{\xi}_1 p_1 p_2 \rangle) \equiv \langle p_1 \rangle \langle p_2 \rangle B \langle p_1 p_2 \rangle .$$

Unfortunately, the Bethe-Salpeter expression generalizes awkwardly; as a consequence, we have preferred to give a simpler definition of B which results in Eq. (5.17).

$$+ p_{u_2}p_{u_3}\hat{G}_1\hat{\xi}_1p_1 + \hat{G}_1(\hat{G}_2(\hat{G}_3(\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3p_1p_2p_3))) \quad . \tag{5.19}$$

Taking ensemble averages and defining $T_{123} \equiv \langle\hat{\xi}_1\hat{\xi}_2\hat{\xi}_3p_1p_2p_3\rangle/\langle p_1p_2p_3\rangle$, in the usual way, we have

$$\langle p_1p_2p_3\rangle = p_{u_1}p_{u_2}p_{u_3} + p_{u_1}p_{u_2}\hat{G}_3Q_3\langle p_3\rangle + p_{u_1}p_{u_2}\hat{G}_2Q_2\langle p_2\rangle$$

$$+ p_{u_2}p_{u_3}\hat{G}_1Q_1\langle p_1\rangle + p_{u_1}\hat{G}_2(\hat{G}_3B_{23}\langle p_2p_3\rangle) + p_{u_2}\hat{G}_1(\hat{G}_3B_{13}\langle p_1p_3\rangle)$$

$$+ p_{u_3}\hat{G}_1(\hat{G}_2B_{12}\langle p_1p_2\rangle) + \hat{G}_1(\hat{G}_2(\hat{G}_3T_{123}\langle p_1p_2p_3\rangle)) \quad . \tag{5.20}$$

5.1.1.4 Fourth Moment and Beyond

Whenever $p_1p_2 = p_1p_1^*$, that is, "intensity," the quantity $(p_1p_1^*p_2p_2^* - \langle p_1p_1^*\rangle\langle p_2p_2^*\rangle)$ describes the mean square fluctuation in intensity. Fluctuations in intensity represent a limitation imposed by the medium on image forming, beamforming, etc. In general, the fourth moment $\langle p_1p_2p_3p_4\rangle$ will contain $2^4 = 16$ terms. Defining $W\langle p_1p_2p_3p_4\rangle \equiv \langle\xi_1\xi_2\xi_3\xi_4p_ip_2p_3p_4\rangle$, we put down immediately

$$\langle p_1p_2p_3p_4\rangle = p_{u_1}p_{u_2}p_{u_3}p_{u_4}$$

$$+ p_{u_1}p_{u_2}p_{u_3}\hat{G}_4Q_4\langle p_4\rangle + p_{u_1}p_{u_2}p_{u_4}\hat{G}_3Q_3\langle p_3\rangle$$

$$+ p_{u_1}p_{u_3}p_{u_4}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}p_{u_3}p_{u_4}\hat{G}_1Q_1\langle p_1\rangle$$

$$+ p_{u_1}p_{u_2}\hat{G}_3(\hat{G}_4B_{34}\langle p_3p_4\rangle) + p_{u_1}p_{u_3}\hat{G}_2(\hat{G}_4B_{24}\langle p_2p_4\rangle)$$

$$+ p_{u_1}p_{u_4}\hat{G}_2(\hat{G}_3B_{23}\langle p_2p_3\rangle) + p_{u_2}p_{u_3}\hat{G}_1(\hat{G}_4B_{14}\langle p_1p_4\rangle)$$

$$+ p_{u_2}p_{u_4}\hat{G}_1(\hat{G}_3B_{13}\langle p_1p_3\rangle) + p_{u_3}p_{u_4}\hat{G}_1(\hat{G}_2B_{12}\langle p_1p_2\rangle)$$

$$+ p_{u_1}\hat{G}_2(\hat{G}_3(\hat{G}_4T_{234}\langle p_2p_3p_4\rangle)) + p_{u_2}\hat{G}_1(\hat{G}_3(\hat{G}_4T_{134}\langle p_1p_3p_4\rangle))$$

$$+ p_{u_3}\hat{G}_1(\hat{G}_2(\hat{G}_4T_{124}\langle p_1p_2p_4\rangle)) + p_{u_4}\hat{G}_1(\hat{G}_2(\hat{G}_3T_{123}\langle p_1p_2p_3\rangle))$$

$$+ \hat{G}_1(\hat{G}_2(\hat{G}_3(\hat{G}_4W\langle p_1p_2p_3p_4\rangle))) \quad . \tag{5.21}$$

In this scheme, higher order moments may be defined immediately.

$$\langle \prod_{i=1}^{n} p_i \rangle = \prod_{i=1}^{n} p_{u_i} + \sum_{j}^{n} \prod_{j}^{n} p_{u_i}\hat{G}_jQ_j\langle p_j\rangle + \sum_{j,k} \prod_{j,k} p_{u_i}\hat{G}_j(\hat{G}_kB_{jk}\langle p_jp_k\rangle) + \ldots$$

$$+ \ldots \hat{G}_1(\hat{G}_2(\ldots(\hat{G}_nW\langle p_1\ldots p_n\rangle))) \tag{5.22}$$

where

$$\prod_{j}^{n} p_{u_i} := \prod_{\substack{i=1 \\ i\neq j}}^{n} p_{u_i} \; ; \quad \prod_{j,k} p_{u_i} = \prod_{\substack{i=1 \\ i\neq j,k}}^{n} p_{u_i} \quad . \tag{5.22a}$$

5.1.2 Operational Solutions

Reasoning by analogy to the general operational solution (Chapter III, Sec. 3.2), we may expect some valuable insights from the operational solution to Eq. (5.1). For this reason we write Eq. (5.7) as

$$p(\underline{r},t) = \hat{G}\,\hat{\xi}(\underline{r}_1,t_1)\,p(\underline{r}_1,t_1) + \hat{G}(s_0) \quad ,$$

$$\hat{\xi}(\underline{r}_1,t_1) \equiv \frac{d(\underline{r}_1,t_1)}{c_0^2}\frac{\partial^2}{\partial t_1^2} \quad , \text{ random} \tag{5.23}$$

for some operator $\hat{G}$. The operational solution is given in Fig. 5-1.

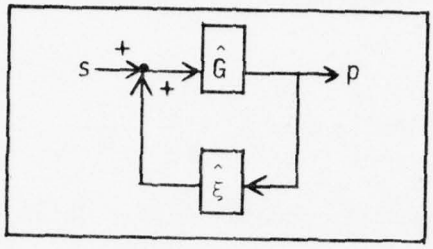


Fig. 5-1. Operational Solution to Eq. (5.1)

Since the operator $\hat{\xi}$ is random, the output p of this system is also random.

Correctly understood, Fig. 5-1 elucidates the controversy concerning the convergence of the series solutions, Eqs. (5.8) and (5.12a). If we concentrate on the loop's sequential operation, we see that on the

$$\text{First pass:} \qquad \hat{p}_1 = \hat{G}(s)$$

$$\text{Second pass:} \qquad \hat{p}_2 = \hat{G}(s) + \hat{G}\hat{\xi}\hat{G}(s) = \hat{G}\,[s+\hat{\xi}\hat{G}(s)]$$

$$\text{Third pass:} \qquad \hat{p}_3 = \hat{G}(s) + \hat{G}\hat{\xi}\hat{G}\hat{\xi}\hat{G}(s) = \hat{G}\,[s+\hat{\xi}\hat{G}(s)+\hat{\xi}\hat{G}\hat{\xi}\hat{G}(s)]$$

. . . . .

$$\text{Nth pass:} \qquad \hat{p}_n = \sum_{n=0}^{N} (G\hat{\xi})^{(n)}\,\hat{G}(s)\ . \qquad (5.24)$$

Thus Eq. (5.23) says that $p = \sum_{n=0}^{\infty} (G\xi)^{(n)}\,\hat{G}(s)$ is a correct representation always, provided only $\hat{G}$ is closed and the functions s and $\hat{\xi}G(s)$ come from a suitable set of functions. These are exactly the conditions for the validity of the feedback theorem.[1] The question of convergence,

[1]Chapter I, Preliminaries, Sec. 1.1.1.

however, embraces not only form but also a numerical limit. For instance, the form

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \quad \ldots$$

is valid for x=-1, but does not converge (i.e., $\ln(0) = -\infty = -\sum_{n=1}^{\infty} \frac{1}{n}$).

This second consideration, when judged in the light of the operational solution, is simply the loop stability. One can expect many types of stability problems. A local finite instability results whenever and wherever the feedback is positive. This results in a concentration or focusing of the field. One may also generate an infinite local instability. This condition characterizes a sharp focus $p(\underline{r}_1,t_1) = \infty$ for some $(\underline{r}_1,t_1)$ at which the energy density--but not the total energy--is infinite). By manipulating the loop, it should be possible to produce all the usual types of instability (neutral, local, global) in both space and time. These varying types of instabilities correspond to the physical situations represented by the proper $\hat{\xi}$, s and $\hat{G}$.

The operational solution, therefore, teaches us to resolve the question of convergence into questions of form and stability. Approximations,[1] however, bring other problems. Finite truncations of the series solution correspond to an incomplete settling of the loop. They will thus converge whenever the true solution converges (and sometimes even when it does not!). Selected subsums, however, create artificial problems. The sum, $\sum_{n=1}^{\infty} (-1)^n(1/n)=\ln(2)$, whereas the subsum

[1]See Section 5.3.

$$1 + 1/3 + 1/5 + \ldots$$

is unbounded. The question of convergence in these cases depends not only on the original series but also on the selected subsum. This kind of approximation may produce spurious instabilities. It may also converge when the original series does not.

Usually, we may expect convergence because $\xi \equiv \frac{d}{c_0^2}\frac{\partial^2}{dt^2} \simeq -\frac{d\omega^2}{c_0^2}$ produces a negative feedback.

The mean value of $p(\underline{r},t)$ in Eq. (5.23) may be found by taking the output mean of Fig. 5-1.

This may be set down as in Fig. 5-2.

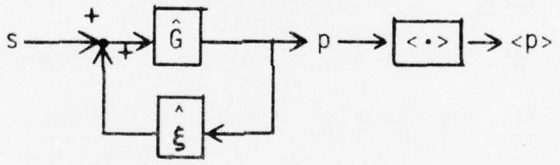


Fig. 5-2. Operational Solution for <p>

The reader will notice that in terms of input and output Fig. 5-2 is deterministic. The Dyson equation, Eq. (5.12), replaces the system of Fig. 5-2 with a deterministic, first-order equivalent medium. Taking the mean of Eq. (5.23), we have

$$\langle p\rangle = \hat{G}\langle\hat{\xi}p\rangle + \hat{G}(s) \,, \tag{5.25}$$

a deterministic equation. Let Q be the equivalent medium statistic i.e., $Q\langle p\rangle \equiv \langle\hat{\xi}p\rangle$, so that

$$\langle p \rangle = \hat{G}Q\langle p \rangle + \hat{G}(s), \qquad (5.26)$$

or operationally as in Fig. 5-3.

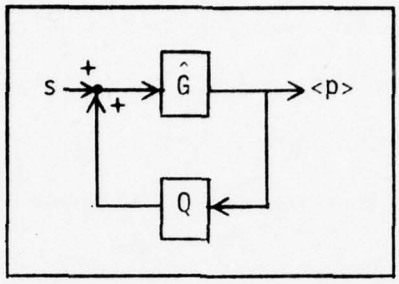


Fig. 5-3. Dyson's Equation, Operational Form

We see at once that the mass operator Q plays the role of a deterministic, equivalent medium, replacing the real medium $\hat{\xi}$. We thus call Q the first moment equivalent medium.

Higher order moments may likewise be interpreted in terms of equivalent media. We may, for the second moment, generate an equivalent medium by letting $B\langle p_1 p_2 \rangle = \langle \hat{\xi}_1 \hat{\xi}_2 p_1 p_2 \rangle$. The second-order equivalent then is described by two deterministic parameters: (Q, B) as in Fig. 5-4. Similarly, the third and fourth order equivalent media are described by (Q, B, T) and (Q, B, T, W).

The opposite problem[1] of finding suitable approximations for moments turns on sequences of approximations[2] $\hat{Q}_1, \ldots, \hat{Q}_n$; $\hat{B}_1, \ldots, \hat{B}_n$; $T_1, \ldots, \hat{T}_n$; and $\hat{W}_1, \ldots, \hat{W}_n$. In other words, the validity of the fourth order approximations depends on the validity of

---

[1]This is the usual problem of the literature, e.g. the bilocal approximation of Chap. IV, Sec. 4.3.

[2]See Sec. 5.3.2.2 where we exhibit methods of constructing sequences of approximations which converge to the correct equivalent medium.

of the lower order approximations. Thus, we find that these approximations tend to build significant error in the higher order moments.

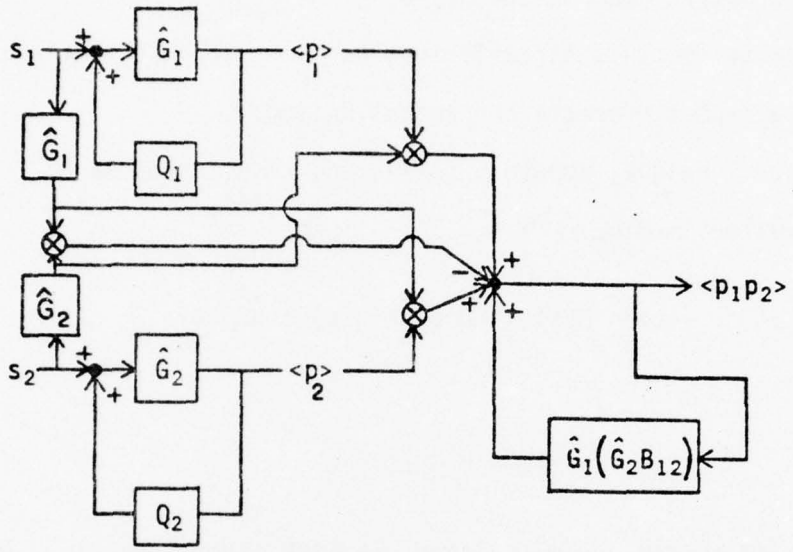


Fig. 5-4. Second Moment Operational Solution

The operational solutions, however, suffer little from this inconvenience. When the input/output structure given in Fig. 5-3 is available, the estimates may be made to any desired precision and errors can be readily computed.[1] We can, thus, approach the moment-equivalent media in a controlled way.

The difference between Fig. 5-1 and Fig. 5-3 is a vast quantity of information. Figure 5-1, as a process generating p, requires the knowledge of the randomness throughout the region -- in other words, a knowledge of the entire ensemble of the space-time random variable. If we possessed such knowledge, we could generate p and from p, <p>.

---

[1]This is the system identification problem. See Chap. VII, Sec. 7.2.

Figure 5-3, however, requires only knowledge of an equivalent medium (reducing the information requirement). We need know only the input and the output mean to specify Q.[1]

The penalty for this streamlining, however, is that from a knowledge of Q, we cannot recreate the actual medium.

Again, Q is unique, modulo a null space of $\hat{G}$. Suppose Q' another equivalent medium. Then,

$$\langle p \rangle = \hat{G}(s) + \hat{G}Q\langle p \rangle = \hat{G}(s) + \hat{G}Q'\langle p \rangle$$

or

$$\hat{G}[Q - Q']\langle p \rangle = 0 \ . \tag{5.27}$$

The reader should recognize that, as with other equivalent media, the moment equivalent media, Q, B, T, W, etc., have only a tenuous relationship to the physical medium embodied in $\hat{\xi}$.

5.1.3 Moment Differential Equations

In this section and the following, we briefly develop two techniques employed in the literature for obtaining moment solutions.

---

[1]An analogy: Let $\{a_i\}$ be an infinite sum of real numbers converging to a_∞. Between a_0 and a_∞, we may construct a geometric series using $a_\infty = a_0 + \rho a_\infty = a_0/(1 - \rho)$. Then the equivalent sum is

$$a_0 + \rho a_0 + \rho^2 a_0 + \ldots = \sum_{n=0}^{\infty} \rho^n a_0 = a_\infty \ .$$

Given a_0, the parameter ρ allows calculation of a_∞, but gives little information about the original sum $\{a_i\}$.

Recognizing that moment solutions are deterministic solutions suggests a direct approach to moment differential equations. Unfortunately, except for restricted cases,[1] such a development suffers the inconveniences mentioned below.

Define

$$m_n \equiv \prod_{i=1}^{n} p(\underline{r}_i) \ . \tag{5.28}$$

Then multiply each

$$[\nabla^2 + k_0^2]p(r_i) = \xi(r_i)p(r_i) + s(r') \tag{5.29}$$

by $m_{i_0} = \prod_{i \neq i_0} p(r_i)$. Summing the results over i_0 and taking ensemble averages gives

$$\sum_i \nabla_i^2 \langle m_n \rangle = \sum_i \langle \xi_i m_n \rangle - s(r') \sum_i \langle m_{i_0} \rangle - nk_0^2 \langle m \rangle \ . \tag{5.30}$$

Equation (5.30) is no longer a stochastic equation and may be solved recursively using any available techniques (e.g., numerical methods) once the coupled term $\langle \hat{\xi} m \rangle$ has been approximated. For reference, we put down the first, second, and fourth moments equations. These are

$$n=1: \quad \nabla^2 \langle p \rangle + k_0^2 \langle p \rangle = \langle \xi p \rangle - s \ ; \tag{5.31}$$

$$n=2: \quad \nabla_i^2 \langle p_i p_j \rangle + \nabla^2{}_j \langle p_i p_j \rangle + 2k_0^2 \langle p_i p_j \rangle = \langle \hat{\xi}_i p_i p_j \rangle$$

$$+ \langle \hat{\xi}_j p_i p_j \rangle - s_i \langle p_j \rangle - s_j \langle p_j \rangle \ ; \tag{5.32}$$

---

[1]See the development in ref. [168].

$$n=4:\ \nabla^2_i\,(m_{(4)}) + \nabla^2_j(m_{(4)}) + \nabla^2_k(m_{(4)}) + \nabla^2_l(m_{(4)}) + 4k_0^2\langle m_{(4)}\rangle$$

$$= \langle\hat{\xi}_i m_{(4)}\rangle + \langle\hat{\xi}_j m_{(4)}\rangle + \langle\hat{\xi}_k m_{(4)}\rangle + \langle\hat{\xi}_l m_{(4)}\rangle - s\sum 4\langle m_{(4)}\rangle\ . \tag{5.33}$$

It is obvious that the coupled terms $\langle\hat{\xi}_i m(\cdot)\rangle$ are troublesome and usually involve approximations for explicit solutions. Also, the results apply to a harmonic case only, and so models only spatial variation.

The stochastic terms associated with s do not contribute outside the source region (considered deterministic) and may thus be omitted.

5.1.4 Feynman Diagrams

One of the many advantages of the operational solutions is the pictorial representation given the solutions. This advantage is even better exhibited in Feynman Diagrams [55, 72, 187].

These diagrams are shorthand expressions for writing equations like[1]

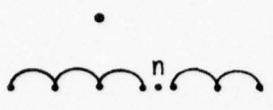
$$\langle p\rangle = \hat{G}(s) + \hat{G}\langle d\sum_{n=0}^{\infty}(\hat{G}'d)^{(n)}\rangle\hat{G}'(s)\ . \tag{5.34}$$

If we let

$\equiv\!\equiv\!\equiv$ $\equiv \langle p\rangle$

$\equiv\!\equiv$ $\equiv \hat{G},\hat{G}'$

[1]This form will be developed in Sec. 5.2 below.



• ≡ an integration node

≡ an n-point correlation $\langle d_1 \ldots \overset{n}{\cdot} \ldots d_n \rangle$,

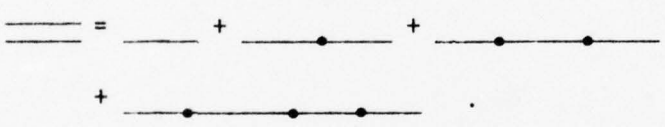
equation (5.34) is thus diagrammed:

(5.34a)

The approximate solution[1]

$$\langle p \rangle = \hat{G}(s) + \hat{G}\langle d \rangle \sum_{n=0}^{\infty} (\hat{G}'\langle d \rangle)^{(n)} \hat{G}'(s) \qquad (5.35)$$

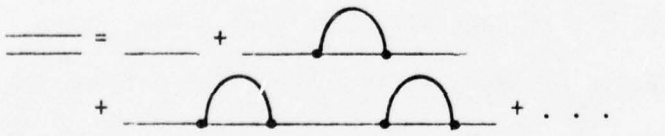
is diagrammed:



(5.35a)

The relationship of Eq. (5.35) to Eq. (5.34) is readily appreciated.

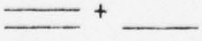
Again, the bilocal (Bourret) approximation[2] is diagrammed by



(5.36)

where the odd-order correlations are set to zero.

The Born approximations[3] are given

n=0:  (5.37a)

n=1: (First Born) (5.37b)

---

[1]See Sec. 5.3.2.1

[2]See Sec. 5.3.2.2.

[3]See Sec. 5.2.2.2.

n=2: (5.37c)

so that the truncation can be readily appreciated.

In the M-form,[1] the diagrams give some appreciation of the surface volume interactions. For $d = d_S + d_V + d_B$, we have

$$\langle d \rangle = \langle d_S \rangle + \langle d_V \rangle + \langle d_B \rangle$$

$$\langle d_1 d_2 \rangle = \langle d_{S_1} d_{S_2} \rangle + \langle d_{S_1} d_{V_2} \rangle + \langle d_{S_1} d_{B_2} \rangle$$

$$+ \langle d_{V_1} d_{S_2} \rangle + \langle d_{V_1} d_{V_2} \rangle + \langle d_{V_1} d_{B_2} \rangle$$

$$+ \langle d_{B_1} d_{S_2} \rangle + \langle d_{B_1} d_{V_2} \rangle \langle d_{B_1} d_{B_2} \rangle .$$

$\langle d_1 d_2 d_3 \rangle \rightarrow 3^3$ terms.

We see here, of course, the "cross terms" signifying volume - surface (V x S) scattering interactions, etc. This breakdown may likewise be diagrammed as in Fig. 5-5.

One readily appreciates that generally solutions dependent on second and higher order correlations involve interaction between scattering classes.

5.2 Specialization

We turn now to detailed structure of the solution. Equation (5.1) requires for solution a specific inverse operator $\hat{G}$. In this section we discover $\hat{G}$ explicitly; once obtained, we will find new formulations of the solution.

---

[1] See Introduction Sec. 5.0, Eq. (5.2).

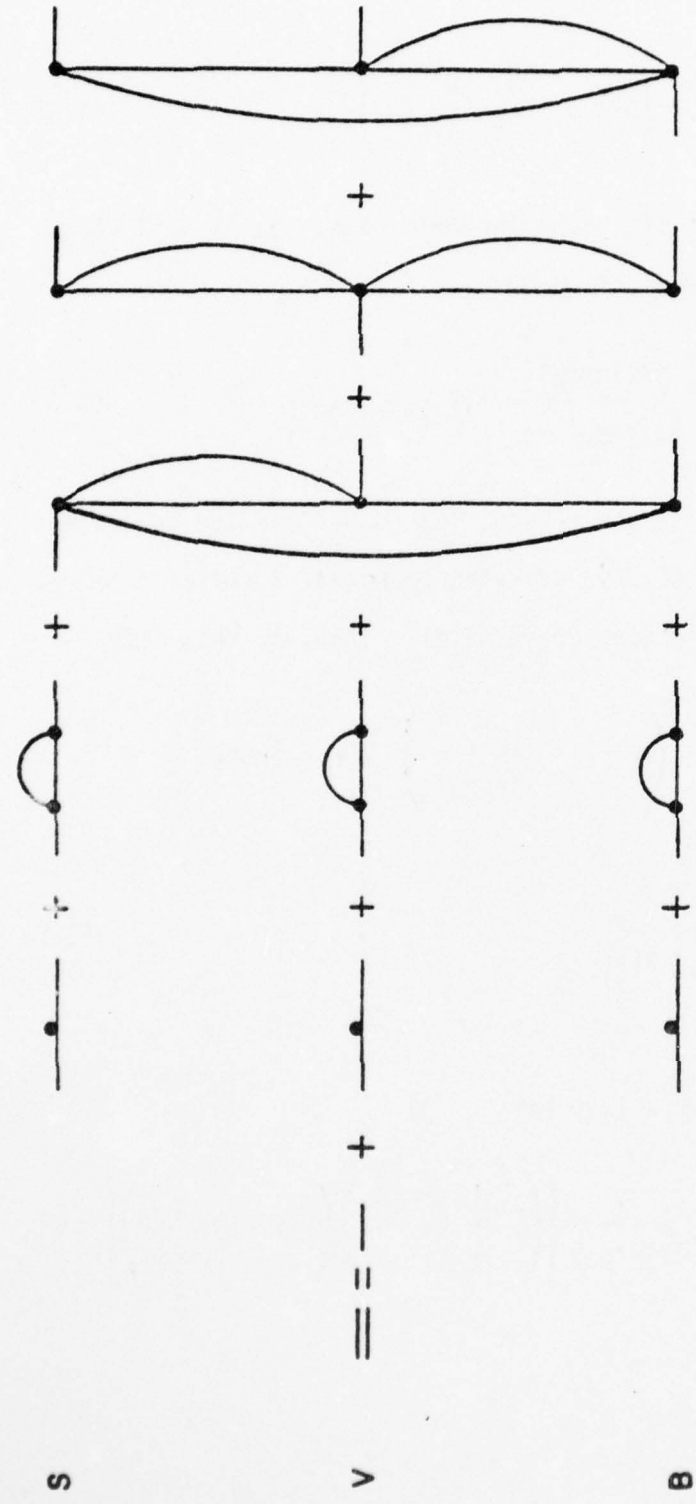


Fig. 5-5. Feynman Diagram of Surface, Volume, and Bottom Interactions

5.2.1 The Inverse Operator $\hat{G}$

We seek the inverse operator for[1]

$$\hat{L} = \left[\nabla^2(\cdot) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(\cdot)\right] \quad . \tag{5.38}$$

Our approach extends the usual harmonic case. It is well known that if $\partial^2/\partial t^2 = -\omega_0^2$, constant, then

$$(\nabla^2 + k_0^2)\frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|} = \delta(\underline{r}-\underline{r}_0) \ , \ k_0 = \frac{\omega_0}{c_0} \quad . \tag{5.39}$$

Since this assumption is so often made, let us perform the indicated operations. Because Eq. (5.36) indicates propagation radial from $\underline{r}_0$, we rewrite the LHS in spherical coordinates. Then, in this case

$$\nabla_\rho^2 \equiv \frac{1}{|\underline{r}-\underline{r}_0|^2}\frac{\partial}{\partial|\underline{r}-\underline{r}_0|}\left(|\underline{r}-\underline{r}_0|^2\frac{\partial}{\partial|\underline{r}-\underline{r}_0|}\right) \ , \ \rho = |\underline{r}-\underline{r}_0|$$

$$\equiv \frac{\partial^2(\cdot)}{\partial|\underline{r}-\underline{r}_0|^2} + \frac{2}{|\underline{r}-\underline{r}_0|}\frac{\partial(\cdot)}{\partial|\underline{r}-\underline{r}_0|} \quad . \tag{5.40}$$

For $g \equiv e^{-jk_0|\underline{r}-\underline{r}_0|}/4\pi|\underline{r}-\underline{r}_0|$, we have

$$\nabla_\rho^2\left[\frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|}\right] = \frac{1}{4\pi}\frac{\partial}{\partial|\underline{r}-\underline{r}_0|}\left[\left(\frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|}\right)\left(-jk - \frac{1}{|\underline{r}-\underline{r}_0|}\right)\right]$$

---

[1]See Sec. 5.1.

$$+ \frac{2}{4\pi} \frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|^2} \left(-jk_0 - \frac{1}{|\underline{r}-\underline{r}_0|}\right) ,$$

or

$$\nabla_\rho^2 \left[\frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|}\right] = \frac{1}{4\pi} \left\{-jk_0 e^{-jk_0|\underline{r}-\underline{r}_0|} \left(\frac{-jk_0}{|\underline{r}-\underline{r}_0|} - \frac{1}{|\underline{r}-\underline{r}_0|^2}\right)\right.$$

$$+ e^{-jk_0|\underline{r}-\underline{r}_0|} \left(\frac{jk_0}{|\underline{r}-\underline{r}_0|^2} + \frac{2}{|\underline{r}-\underline{r}_0|^3}\right)$$

$$\left. + 2 \frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|^2} \left(-jk_0 - \frac{1}{|\underline{r}-\underline{r}_0|}\right)\right\}$$

$$= \frac{1}{4\pi} \frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|} \left(-k_0^2 + \frac{jk_0}{|\underline{r}-\underline{r}_0|} + \frac{jk_0}{|\underline{r}-\underline{r}_0|^2} + \frac{2}{|\underline{r}-\underline{r}_0|^2}\right.$$

$$\left. - \frac{2jk_0}{|\underline{r}-\underline{r}_0|} - \frac{2}{|\underline{r}-\underline{r}_0|^2}\right) .$$

Then $\nabla_\rho^2 g$ is given by

$$\nabla_\rho^2 g = \frac{-k_0^2}{4\pi} \frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|} . \qquad (5.41)$$

We must have, then, for every $\underline{r} \neq \underline{r}_0$

$$(\nabla_q^2 + k_0^2) \left[\frac{e^{-jk|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|}\right] = (-k_0^2 + k_0^2) \frac{e^{-jk|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|} = 0 .$$

AD-A065 780 NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CONN NEW --ETC F/G 20/1
A GENERAL THEORY OF ACOUSTIC PROPAGATION WITH APPLICATIONS TO S--ETC(U)
JAN 78 J R BRETON

UNCLASSIFIED NUSC-TR-5871 NL

3 OF 5
AD
A065780

On the other hand, when $\underline{r} = \underline{r}_0$, the function is unbounded.[1] The troublesome constant $(4\pi)^{-1}$ comes from the normalization of

$$a\int_{\underline{r}} \delta|\underline{r}-\underline{r}_0|\ \underline{dr} = 1 \ .$$

Because $x = |r-r_0|$ represents a spherical shell, if we interpret $\delta|r-r_0| = \delta(x)$, we must have

$$a\int_{r} \delta|\underline{r}-\underline{r}_0|d\underline{r} = \int_{x=-\infty}^{\infty} 4\pi\delta(x)dx = 4\pi \ . \qquad (5.42)$$

In other words, the assumption of radial propagation allows the three-dimensional problem to be considered in one-dimension, with suitable normalization.

The considerations of Eq. (5.39) do not suffice to invert $\hat{L}$. This happens because in general $q\hat{L}p \neq p\hat{L}q$. However, by requiring regularity and a radiation condition (see Appendix D), we may put

$$\frac{e^{-jk|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|}[\nabla_\rho^2 + k_0^2]p(\underline{r}_0,\underline{r}')$$

$$= p(\underline{r}_0,\underline{r}')[\nabla_\rho^2 + k_0^2]\ \frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|}$$

$$= p(\underline{r}_0,\underline{r}')\delta(\underline{r}-\underline{r}_0) \quad . \qquad (5.43)$$

---

[1]We put here, as in the literature $(\nabla^2 + k_0^2)e^{-jk_0\rho}/4\pi\rho = \delta(\underline{\rho})$, but not without misgiving. See Appendix H.

Physically, Eq. (5.43) says that the dynamic condition at a point, radiated spherically, is the contribution to the received pressure from that point.[1]

We conclude the harmonic case by noting that given Eq. (5.43)

$$\hat{G} = \hat{L}^{-1} = \int_{\underline{r}_0} \frac{e^{-jk|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|} (\cdot) d\underline{r}_0 \ . \tag{5.44}$$

For the harmonic case, the solution to $\hat{L}p = \hat{\xi}p+s$ is

$$p = \sum_{n=0}^{\infty} (\hat{G}\hat{\xi})^{(n)}\hat{G}(s) \tag{5.45}$$

with $\hat{G}$ given in Eq. (5.44) and $\hat{\xi} = -k_0^2 d(\underline{r})$.

We return to our search for an inverse to

$$\hat{L} = \left[\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right] .$$

This may be obtained readily from the harmonic case. First note[2] that

---

[1]Huygen's principle states: "...each element of a wavefront may be treated as the center of a secondary disturbance which gives rise to spherical wavelets; and moreover - that the position of the wavefront at any later time is the envelope of all such wavelets." [24], p. 132.

[2]The reader should notice that for the harmonic case $k^2 = \omega_0^2/c^2$, random; for the non-harmonic case $k^2 = \omega^2/c^2$, also random. The non-harmonic case assumes a given reference velocity c_0, so that we may put $k_i^2 = \omega_i^2/c_0^2$. In particular for the non-harmonic case, we put $k_0^2 = \omega_0^2/c_0^2$, variable (but not stochastic). The table shows our conventions.

Wavefront Velocity, c	Harmonic, (ω_0 constant)	Non-Harmonic (ω_0 variable)
c_0, constant	$k_0^2 = \omega_0^2/c_0^2$, constant	$k_i^2 = \omega_i^2/c_0^2$, variable ($k_0^2 = \omega_0^2/c_0^2$)
c, variable or stochastic	$k^2 = \omega_0^2/c^2$ variable or stochastic	$k^2 = \omega_i^2/c^2$, variable or stochastic

$$\left[\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right]\frac{e^{j[\omega_i(t-t_0)-k_i|\underline{r}-\underline{r}_0|]}}{4\pi|\underline{r}-\underline{r}_0|}$$

$$= e^{j\omega_i(t-t_0)}(\nabla^2 + k_i^2)\left[\frac{e^{-jk_i|\underline{r}-\underline{r}_0|}}{4\pi|\underline{r}-\underline{r}_0|}\right] ,$$

$$= e^{j\omega_i(t-t_0)}\delta|\underline{r}-\underline{r}_0|; \qquad k_i^2 = \frac{\omega_i^2}{c_0^2} . \tag{5.46}$$

Now it is well-known that

$$\int_{\omega_i=-\infty}^{\infty} e^{-j\omega_i(t-t_0)}d\omega_i = 2\pi\delta(t-t_0) , \tag{5.47}$$

so that

$$\int_{\omega_i=-\infty}^{\infty} d\omega_i\left[\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right]\frac{e^{j[\omega_i(t-t_0)-k_i|\underline{r}-\underline{r}_0|]}}{8\pi^2|\underline{r}-\underline{r}_0|} = \delta(t-t_0)\delta(\underline{r}-\underline{r}_0). \tag{5.48}$$

Thus,

$$\int_{\underline{r}_0}\int_{t_0}\int_{\omega_i} dt_0 d\underline{r}_0 d\omega_i \; p(\underline{r}_0,t_0)\left[\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right]\frac{e^{-j[\omega_i(t-t_0)-k_i|\underline{r}-\underline{r}_0|]}}{8\pi^2|\underline{r}-\underline{r}_0|}$$

$$= \int_{\underline{r}_0}\int_{t_0} p(\underline{r}_0,t_0)\delta(t-t_0)\delta(\underline{r}-\underline{r}_0)dt\, d\underline{r}_0 = p(\underline{r},t) . \tag{5.49}$$

The object of our search is therefore:

$$\hat{G} = \hat{L}^{-1} = \int_{\underline{r}_0}\int_{t_0}\int_{\omega_0} d\omega_0 d\underline{r}_0 dt_0(\cdot)\frac{e^{j[\omega_0(t-t_0)-k_0|\underline{r}-\underline{r}_0|]}}{8\pi^2|\underline{r}-\underline{r}_0|} . \tag{5.50}$$

The formal solution to Eq. (5.1) can be written immediately as

$$p = \sum_{n=0}^{\infty} (\hat{G}\hat{\xi})^{(n)}\hat{G}(s) = \hat{G}(s) + \hat{G}\hat{\xi}\hat{G}(s) + \hat{G}\hat{\xi}\hat{G}\hat{\xi}\hat{G}(s) + \ldots \quad (5.51)$$

where $\hat{\xi} \equiv \frac{1}{c_0^2} d(\underline{r},t) \frac{\partial^2}{\partial t^2}(\cdot)$, an operator, and $\hat{G}$ is given by Eq. (5.50).

Before proceeding to statistical solutions, we must surmount a difficulty. Both $\hat{G}$ and $\hat{\xi}$ are operators so that we may not rearrange elements in the terms of Eq. (5.51) indiscriminately. Discrimination here lies in separating the operator in $\hat{\xi}$ by first operating <u>before</u> rearrangement.[1] Note first that

$$\frac{\partial^2}{\partial t_1^2} p_u(1) \equiv \frac{\partial^2}{\partial t_1^2} G(10)(s(0))$$

$$= \frac{\partial^2}{\partial t_1^2} \int_{t_0} \int_{\underline{r}_0} \int_{\omega_0} d\omega_0 dt_0 d\underline{r}_0 \frac{e^{-j[\omega_0(t_1-t_0)-k_0|\underline{r}_1-\underline{r}_0|]}}{8\pi^2|\underline{r}_1-\underline{r}_0|} s(\underline{r}_0,t_0),$$

or

$$\frac{\partial^2}{\partial t_1^2} p_u(1) = \int_{t_0} \int_{\underline{r}_0} \int_{\omega_0} d\omega_0 dt_0 d\underline{r}_0 (-\omega_0^2) \frac{e^{-j[\omega_0(t_1-t_0)-k_0|\underline{r}_1-\underline{r}_0|]}}{8\pi^2|\underline{r}_1-\underline{r}_0|}$$

$$\cdot (s(\underline{r}_0,t_0)) \quad . \quad (5.52)$$

Using this intelligence in Eq. (5.51), we have

---

[1]See Sec. 6.1.5 for an alternate development by Middleton.

$$p = \sum_{n=0}^{\infty} (\hat{G}\hat{\xi})^{(n)}\hat{G}(s) = \sum_{n=0}^{\infty} \left(\hat{G}\,\frac{\mathbf{d}}{c_0^2}\,\frac{\partial^2}{\partial t^2}\right)^{(n)}\hat{G}(s)\ ,$$

or

$$p = \hat{G}(s) + \hat{G}(-1)\,\frac{d(1)}{c_0^2}\,\hat{G}(-\omega_0^2(s))$$

$$+ \hat{G}(-1)\,\frac{d(1)}{c_0^2}\,\hat{G}(12)(-\omega_1^2)\,\frac{d_2}{c_0^2}\,\hat{G}(-\omega_0^2 s) + \ldots$$

$$= \hat{G}(s) + \hat{G}(-1)d(1)\hat{G}(10)[-k_0^2 s]$$

$$+ \hat{G}(-1)d(1)\hat{G}(12)(-k_1^2)d(2)\hat{G}(20)[-k_0^2 s]$$

$$+ \hat{G}(-1)d(1)\hat{G}(12)(-k_1^2)d(2)\hat{G}(23)(-k_2^2)d(3)\hat{G}(30)[-k_0^2 s]$$

$$+ \ldots . \qquad (5.53)$$

Now define

$$\hat{G}'(n-1,n) \equiv -k_{n-1}^2\,\hat{G}(n-1,n) \qquad (5.54)$$

so that

$$p = \hat{G}(s) + \hat{G}d\hat{G}'(s) + \hat{G}d\hat{G}'d\hat{G}'(s) + \hat{G}d\hat{G}'d\hat{G}'d\hat{G}'(s) + \ldots .$$

We thus obtain

$$p = \hat{G}(s) + \hat{G}d \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(s) \qquad (5.55)$$

$$p = \hat{G}\left(s + d \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(s)\right). \qquad (5.56)$$

Some idea of the generalization introduced by Eq. (5.1) may be seen from the grouping of the exponential terms in Eq. (5.55). The second term, for instance, is

$$-\int_{\underline{r}_0}\int_{t_0}\int_{\omega_0}\int_{\underline{r}_1}\int_{t_1}\int_{\omega_1} d(\underline{r}_1,t_1)\,\frac{\omega_0^2}{c_0^2}\,s(\underline{r}_0,t_0)\,\frac{\exp j[\omega_1(t-t_1)-k_1|\underline{r}-\underline{r}_1|]}{8\pi^2|\underline{r}-\underline{r}_1|}$$

$$\cdot\,\frac{\exp j[\omega_0(t_1-t_0)-k_0|\underline{r}_1-\underline{r}_0|]}{8\pi^2|\underline{r}_1-\underline{r}_0|}\,d\underline{r}_0\,d\underline{r}_1\,dt_0\,dt_1\,d\omega_0\,d\omega_1$$

$$\equiv -\int_0\int_1 d(1)k_0^2 s\,\frac{\exp j[(\omega_1 t-(\omega_1-\omega_0)t_1-\omega_0 t_0]}{8\pi^2|\underline{r}-\underline{r}_1|}\,\frac{\exp j[k_1|\underline{r}-\underline{r}_1|+k_0|\underline{r}_1-\underline{r}_0|]}{8\pi^2|\underline{r}_1-\underline{r}_0|}\,d_0 d_1 . \tag{5.57}$$

In the harmonic case $\omega_0 = \omega_1 = \omega$ so that

$$\frac{\omega_0}{c_0^2}\exp j\,\omega_1(t) - (\omega_1-\omega_0)t_1 - \omega_0 t_0 - \frac{\omega_1}{c_0^2}|\underline{r}_1-\underline{r}_0| - \frac{\omega_0}{c_0}|\underline{r}_1-\underline{r}_0|$$

$$= \frac{\omega^2}{c_0^2}\exp j\,\omega(t-t_0) - \frac{\omega(|\underline{r}-\underline{r}_1| + |\underline{r}_1-\underline{r}_0|)}{c_0} \quad . \tag{5.58}$$

The effect is noticeable in both the temporal and spatial portions. We are, in fact, constructing a model with both spatial and temporal interactions.

The effect of defining G', by Eq. (5.54), is to allow this operator to commute and thus allow statistical solutions in terms of the moments $\langle d\,.\,.^n.\,.\,d\rangle$ similar to the harmonic case. It is principally because of this definition (original with this Thesis) that the broadband case becomes manageable.

5.2.2 Statistical Solutions

With the formal solution to Eq. (5.1) in hand, we now calculate moments. Statistically, it is these moments which constitute the solution.

Equation (5.55) is so arranged that ensemble averages may be taken,

$$\langle p\rangle = \langle \hat{G}(s) + \hat{G}d \sum_{0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(s)\rangle$$

$$= \hat{G}(s) + \hat{G}\langle d \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\rangle \hat{G}(s) \qquad (5.59)$$

Where the n^{th} term is $\hat{G}\hat{G}'\hat{G}'\ldots^{n}\ldots \hat{G}'\langle d\ldots^{n}\ldots d\rangle\hat{G}'(s)$ and $\langle d\ldots^{n}\ldots d\rangle \equiv \langle d(\underline{r}_1,t_1) \cdot d(\underline{r}_2,t_2)\ldots d(\underline{r}_n,t_n)\rangle$ is the (space-time) n^{th} order correlation of d.

A closed form (Dyson's Equation) may be obtained from the original solution. With $\hat{G}$ as in Eq. (5.50), we have

$$p = \hat{G}(s) + \hat{G}\,\frac{d}{c_0^2}\,\frac{\partial^2 p}{\partial t^2} \quad ,$$

and

$$\langle p\rangle = \hat{G}(s) + \hat{G}\,\left\langle \frac{d}{c_0^2}\,\frac{\partial^2}{\partial t^2}\, p\right\rangle \quad . \qquad (5.60)$$

Letting

$$\hat{Q}\langle p\rangle \equiv \left\langle \frac{d}{c_0^2}\,\frac{\partial^2}{\partial t^2}\, p \right\rangle \quad , \qquad (5.61)$$

define Q one has immediately Dyson's equation:

$$\boxed{\langle p\rangle = \hat{G}(s) + \hat{G}Q\langle p\rangle \quad .} \tag{5.62}$$

From Eq. (5.59), we have the identity

$$Q\langle p\rangle \equiv \langle d \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\rangle \hat{G}'(s) \quad . \tag{5.63}$$

Further, from $\hat{L}p = \hat{\xi}p + s$, $\hat{L}\langle p\rangle = Q\langle p\rangle + s$, we have

$$\hat{L}\langle p\rangle = Q\langle p\rangle + s \tag{5.64}$$

or

$$(\hat{L}-Q)\langle p\rangle = s \quad . \tag{5.65}$$

Given $\langle p\rangle$ and s, we may readily determine Q. From Eq. (5.65), it is obvious that the medium (described by the statistics of the d_i) is <u>not</u> thus uniquely defined. We see that Q plays the part of a fictitious but equivalent medium and Dyson's Equation is the description of the results,

$$\boxed{\langle p\rangle = \sum_{n=0}^{\infty} (\hat{G}Q)^{(n)}\, \hat{G}(s) \quad . \quad .} \tag{5.66}$$

The practical upshot of this observation is simply that, in the framework of Eq. (5.1), no amount of source/receiver statistics will definitely establish the characteristics of the medium.

Let us digress momentarily to consider approximations[1] to these infinite series.

---

[1]Approximations are discussed more extensively in Sec. 5.3 following.

Equations (5.63) and (5.66) provide immediately a set of approximations represented by truncations as in Table 5-2. These are the Born approximations, B(1). We will call these approximations, "of truncation order m."

m	approximation, $\hat{Q}\langle p\rangle$	approximate solution
0	0	$\langle\hat{p}_0\rangle = \hat{G}(s) = p_u$
1	$\langle d\rangle\hat{G}'(s)$	$\langle\hat{p}_1\rangle = p_u + \hat{G}\langle d\rangle\hat{G}'(s)$
2	$\hat{G}'\langle dd\rangle\hat{G}'(s)$	$\langle\hat{p}_2\rangle = p_u + \hat{G}\hat{G}'\langle dd\rangle\hat{G}'(s)$

Table 5-2. Approximations for Q Based on the Born Approximations, B(1)

Solutions for the m=1 case are given for the first, second, and fourth moments in Section 5.3 following.

Beyond these truncated solutions, another order of approximations is possible by means of selective summation, B(2). These arise from comparing Eqs. (5.66) and (5.59) directly. For instance, if in Eq. (5.66), we put $Q = -\langle d\rangle\omega_{n-1}^2$, we have

$$\langle p\rangle = \hat{G}(s) + \hat{G}\langle d\rangle \sum_{n=0}^{\infty} \hat{G}'\langle d\rangle^{(n)}G'(s) \qquad (5.67)$$

which holds for $\langle d\overset{n}{\ldots}d\rangle = \langle d\rangle\overset{n}{\ldots}\langle d\rangle$, an independent correlation structure. For this reason, we call the B(2) approximations, "of independent order n."

Specifically, for selected subsums we have Table 5-3.

n	approximation $\langle d^n p\rangle$	approximate solution
0	$d = 0$	$\langle p_0\rangle = \hat{G}(s) = p_u$
1	$\langle dp\rangle = \langle d\rangle\langle p\rangle$	$\langle \hat{p}_1\rangle = \hat{G}(s) + \hat{G}\langle d\rangle \sum_{n=0}^{\infty} (\hat{G}'\langle d\rangle)^{(n)}\hat{G}'(s)$
2	$\langle ddp\rangle = \langle dd\rangle p$	$\langle \hat{p}_2\rangle = \sum_{n=0}^{\infty} (\hat{G}'\hat{G}'\langle dd\rangle)^{(n)}[\hat{G}(s)+\hat{G}\langle d\rangle\hat{G}'(s)]$

Table 5-3. Approximations for $\langle d^n p\rangle$ Based on Selected Subsums B(2)

For $n \geq 1$, numerous decompositions are possible which are described in Appendix E. The example given in Eq. (5.67) is one of the n=1 cases for which Q may be directly expressed. Normal processes with zero means become one of the n=2 solutions since, in this case, all higher order correlations can be decomposed to the second order. The bilocal (Bourret) approximation is yet another selection from B(2), n=2, (B(22)).

It is obvious that $B(1) \subset B(2)$ in each order. It does not thus follow that B(2) solutions are more accurate than corresponding B(1) solutions. For example, if $\langle d^n\rangle = 0$, $n > 2$, then $\hat{G}(s) + \hat{G}\langle d\rangle\hat{G}(s) + \hat{G}\hat{G}'\langle dd\rangle\hat{G}'(s)$ is an exact solution, (B(12)), while

$$\langle p_{B(21)}\rangle = \hat{G}(s) + \hat{G}\langle d\rangle \sum_{n=0}^{\infty} (\hat{G}'\langle d\rangle)^{(n)}\hat{G}'(s) \qquad (5.68)$$

(a B(21) form) remains an approximation.

The second moment is $\langle p_1 p_2 \rangle = \langle p_1(\underline{r}_1, t_1) p_2(\underline{r}_2, t_2) \rangle$. Special cases arise when $\underline{r}_1 = \underline{r}_1$ or $t_1 = t_2$. From the solution for p, we have for p_2

$$p_2 = p_{u_2} + \hat{G}_2 d_2 \sum_{m=0}^{\infty} (\hat{G}_2' d_2)^{(n)} \hat{G}_2'(s_2) \ . \tag{5.69}$$

Thus, we write

$$\begin{aligned} p_1 p_2 = & \left(p_{u_1} + \hat{G}_1 d_1 \sum_{n=0}^{\infty} (\hat{G}_1' d_1)^{(n)} \hat{G}_1(s_1) \right) \\ & \cdot \left(p_{u_2} + \hat{G}_2 d_2 \sum_{m=0}^{\infty} (\hat{G}_2' d_2)^{(m)} \hat{G}_2'(s_2) \right) \\ = & \ p_{u_1} p_{u_2} + p_{u_1} \hat{G}_2 d_2 \sum_{m=0}^{\infty} (\hat{G}_2' d_2)^{(m)} \hat{G}_2'(s_2) \\ & + p_{u_2} \hat{G}_1 d_1 \sum_{n=0}^{\infty} (\hat{G}_1' d_1)^{(n)} \hat{G}_1'(s_1) \\ & + \left(\hat{G}_1 d_1 \sum_{n=0}^{\infty} (\hat{G}_1' d_1)^{(n)} \hat{G}_1(s_1) \right. \\ & \left. \cdot \ \hat{G}_2 d_2 \sum_{m=0}^{\infty} (\hat{G}_2' d_1)^{(m)} \hat{G}_2'(s_2) \right) , \end{aligned} \tag{5.70}$$

or

$$\begin{aligned} \langle p_1 p_2 \rangle = & \ p_{u_1} p_{u_2} + p_{u_1} \hat{G}_2 \langle d_2 \sum_{m=0}^{\infty} \hat{G}_2' d_2^{(n)} \rangle \hat{G}_2'(s_2) \\ & + p_{u_2} \hat{G}_1 \langle d_1' \sum_{n=0}^{\infty} \hat{G}_1' d_1^{(n)} \rangle \hat{G}_1'(s_1) \end{aligned}$$

$$+ \left\langle \left(\hat{G}_2 d_2 \sum_{m=0}^{\infty} (\hat{G}_2' d_2)^{(m)} \hat{G}_2'(s_2) \right) \right.$$

$$\left. \cdot \left(\hat{G}_1 d_1 \sum_{n} (\hat{G}_1' d_1)^{(n)} \hat{G}_1'(s_1') \right) \right\rangle \quad . \tag{5.71}$$

Recognizing from Eq. (5.63) that

$$\hat{G} \langle d\hat{G}' \sum_{n} (\hat{G}_1' d)^{(n)} \rangle \hat{G}'(s) = GQ\langle p \rangle, \; Q\langle p \rangle = \langle \ddot{dp} \rangle \; , \tag{5.72}$$

we see that Eq. (5.71) may be rendered

$$\langle p_1 p_2 \rangle = p_{u_1} p_{u_2} + p_{u_1} \hat{G}_2 Q_2 \langle p_2 \rangle + p_{u_2} \hat{G}_1 Q_1 \langle p_1 \rangle$$

$$+ \left\langle \left(\hat{G}_2 d_2 \sum_{m} (\hat{G}_2' d_2)^{(m)} \hat{G}_2'(s_2) \right) \left(\hat{G}_1 d_1 \sum_{n} (\hat{G}_1' d_1)^{(n)} \hat{G}_1'(s) \right) \right\rangle . \tag{5.73}$$

Noting again that

$$\hat{G} d \sum_{n} (\hat{G}' d)^{(n)} \hat{G}'(s) = \hat{G} \hat{\xi} p \tag{5.74}$$

and defining B as earlier,[1] we obtain

$$\langle p_1 p_2 \rangle = p_{u_1} p_{u_2} + p_{u_1} Q_2 \langle p_2 \rangle + p_{u_1} Q_1 \langle p_2 \rangle + \hat{G}_1 \hat{G}_2 B \langle p_1 p_2 \rangle \; . \tag{5.75}$$

Equation (5.71) may be used for second moments whenever the field variable d is statistically known. Equation (5.73) may be used in conjunction with approximations for $\hat{Q}$. An important aspect of these

---

[1] See Eq. (5.15).

developments lies in the ability to write the error equation immediately for any approximation chosen. For a given situation, direct experimental measurement of some portion of the error may thus be made (e.g., a direct test of the third order correlation).

One may appreciate that B is again a fictitious second-order medium descriptor in the manner of Q.

Now, we particularize the general fourth moment previously given (Eq. (5.21)) to Eq. (5.1) using the definition of $\hat{G}$ and $\hat{G}'$ previous. The following definitions hold:

$$\hat{Q}\langle p\rangle \equiv \langle \ddot{dp}\rangle \tag{5.76a}$$

$$B\langle(p_1p_2)\rangle \equiv \langle \ddot{d}_1\ddot{p}_2\ddot{d}_2\ddot{p}_2\rangle \tag{5.76b}$$

$$T\langle p_1p_3p_3\rangle \equiv \langle \ddot{d}_1\ddot{p}_1\ddot{d}_3\ddot{p}_2\ddot{d}_3\ddot{p}_3\rangle \tag{5.76c}$$

$$W\langle p_1p_2p_3p_4\rangle \equiv \langle \ddot{d}_1\ddot{p}_1\ddot{d}_2\ddot{p}_2\ddot{d}_3\ddot{p}_3\ddot{d}_4\ddot{p}_4\rangle \quad . \tag{5.76d}$$

We may thus put down the fourth moment immediately

$$\begin{aligned}\langle p_1p_2p_3p_4\rangle = {} & p_{u_1}p_{u_2}p_{u_3}p_{u_4} \\ & + p_{u_1}p_{u_2}p_{u_3}\hat{G}_4\hat{Q}_4\langle p_4\rangle + p_{u_1}p_{u_3}p_{u_4}\hat{G}_2\hat{Q}_2\langle p_2\rangle \\ & + p_{u_1}p_{u_2}p_{u_4}\hat{G}_3\hat{Q}_3\langle p_3\rangle + p_{u_2}p_{u_3}p_{u_4}\hat{G}_1\hat{Q}_1\langle p_1\rangle \\ & + p_{u_1}p_{u_2}\hat{G}_3(\hat{G}_4B_{34}\langle p_3p_4\rangle) + p_{u_1}p_{u_3}\hat{G}_2(\hat{G}_4B_{24}\langle p_2p_4\rangle) \\ & + p_{u_1}p_{u_4}\hat{G}_2(\hat{G}_3B_{23}\langle p_2p_3\rangle) + p_{u_2}p_{u_3}\hat{G}_1(\hat{G}_4B_{14}\langle p_1p_4\rangle)\end{aligned}$$

$$+ p_{u_2} p_{u_4} \hat{G}_1(\hat{G}_3 B_{13} \langle p_1 p_3 \rangle) + p_{u_3} p_{u_4} \hat{G}_1(\hat{G}_2 B_{12} \langle p_1 p_2 \rangle)$$

$$+ p_{u_1} \hat{G}_2(\hat{G}_3(\hat{G}_4 T_{234} \langle p_2 p_3 p_4 \rangle)) + p_{u_2} \hat{G}_1(\hat{G}_3(\hat{G}_4 T_{134} \langle p_1 p_3 p_4 \rangle))$$

$$+ p_{u_3} \hat{G}_1(\hat{G}_2(\hat{G}_4 T_{124} \langle p_1 p_2 p_4 \rangle)) + p_{u_4} \hat{G}_1(\hat{G}_2(\hat{G}_3 T_{123} \langle p_1 p_2 p_3 \rangle))$$

$$+ \hat{G}_1(\hat{G}_2(\hat{G}_3(\hat{G}_4 W \langle p_1 p_2 p_3 p_4 \rangle))) \qquad (5.77)$$

in the equivalent media. From the necessity of estimating $\hat{Q}$, to obtain an estimate of B, to obtain an estimate of T, etc., it can be readily appreciated that the error associated with approximations of the higher moments may rise dramatically.

One may, of course, calculate the higher order moments directly in terms of the field correlations available.

5.3 Approximations

The formulations of Sec. 5.2 are exact solutions of Eq. (5.1). It is easy to appreciate in light of the operational solutions that the series solutions will not usually be capable of simple, closed-form expressions. Because only the series form of the exact solution has been available heretofore, the literature has been mainly devoted to finding suitable approximations and determining their validity. This type of approximation we have labelled Type III (See Preface).

In this section we propose a new set of approximations which, when added to the approximations presently available, allows a canonical ordering. The complete set of possible approximations we label B. It is divided into two groups: truncated sums B(1) and selected

subsums B(2). Within each group are found various orders. In this designation, B(12) means a truncated sum of second order, while B(21) means a selected subsum of first order.

5.3.1 Truncated Sums, B(1)

The (first) Born approximation truncates the series solution

$$p = \sum_{n=0}^{\infty} (\hat{G}\hat{\xi})^{(n)}\hat{G}(s)$$

to term n=1. Thus, we have explicitly from Eq. (5.55)

$$p(\underline{r},t)_{B(11)} = p_u(\underline{r},t) - \int_{t_1}\int_{\underline{r}_1}\int_{\omega_1}\int_{t_0}\int_{\underline{r}_0}\int_{\omega_0} \frac{d\omega_1 d\omega_0 dt_1 dt_0 d\underline{r}_1 d\underline{r}_0}{8\pi^2}$$

$$\cdot \frac{d(1)}{c_0^2} \quad \frac{\omega_0^2 s(\underline{r}_0,t_0)}{|\underline{r}_1-\underline{r}_0||\underline{r}-\underline{r}_1|} \exp j\left(\omega_1\left(t-t_1 - \frac{|\underline{r}-\underline{r}|}{c_0}\right)\right.$$

$$\left. + \omega_0\left(t_1-t_0 - \frac{|\underline{r}_1-\underline{r}_2|}{c_0}\right)\right), \quad d(1) \equiv d(\underline{r}_1,t_1), \qquad (5.78)$$

where

$$p_u = \int_{t_0}\int_{\underline{r}_0}\int_{\omega_0} \frac{s(\underline{r}_0,t_0)}{8\pi^2|\underline{r}-\underline{r}_0|} \exp j\omega_0\left(t-t_0 - \frac{|\underline{r}-\underline{r}_0|}{c_0}\right) d\underline{r}_0 dt_0 d\omega_0 = \hat{G}(s). \qquad (5.79)$$

For economy of expression, we put

$$\hat{G}_1' \equiv \hat{G}_1'(ij) \equiv -\int_{t_j}\int_{\underline{r}_j}\int_{\omega_j} \frac{d\underline{r}_j d\omega_j dt_j}{8\pi^2|\underline{r}_j-\underline{r}_j|} (\cdot) \frac{\omega j}{c_0^2}$$

$$\cdot \exp j\omega_j\left(t_j-t_j - \frac{|\underline{r}_j-\underline{r}_j|}{c_0}\right) \quad ,$$

so that Eq. (5.78) becomes

$$p_{B(11)} = \hat{G}(s) + \hat{G}d(1)\hat{G}(s)$$

$$= \hat{G}(-,0)(s) + \hat{G}(-,1)d(1)\hat{G}'(1,0)(s). \qquad (5.80)$$

Taking ensemble averages gives

$$<p>_{B(11)} = p_u + \hat{G}(-1)<d(1)>\hat{G}'(10)(s) \qquad (5.81)$$

so that if $<d(1)> \equiv 0$ (i.e., everywhere at all times), $<p> = p_u$.

The second moment is here

$$<p_0p_1>_{B(11)} = < p_{u_0} + \hat{G}_0(-1)d_0(1)\hat{G}_0(10)(s_0)$$

$$\cdot \quad p_{u_1} + \hat{G}_1(-1)d_1(1)\hat{G}_1(10)(s_1) > . \qquad (5.82)$$

Thus, we obtain

$$<p_0p_1>_{B(11)} = p_{u_0}p_{u_1} + p_{u_0}<\hat{G}_1(-1)d_1(1)>\hat{G}_1(10)(s_1)$$

$$+ p_u<\hat{G}_0'(-1)d_0(1)>\hat{G}_0(10)(s_0)$$

$$+ < \hat{G}_1(-1)d_1(1)\hat{G}_1(10)(s_1) \quad \hat{G}_0(-1)d_0(1)\hat{G}_0(10)(s_0) >, \qquad (5.83)$$

or

$$<p_0p_1>_{B(11)} = -p_{u_0}p_{u_1} + <p_0>p_{u_1}$$

$$+ < \hat{G}_0(-1)d_0(1)\hat{G}_0'(10)(s_0) \quad \hat{G}_1(-1)d_1(1)\hat{G}_1'(10)(s_1) >, \qquad (5.84)$$

since, from Eq. (5.81)

$$\hat{G}_i'(-1)d_i(1)\hat{G}_i'(10)(s_i) = p_i - p_{u_i} \quad . \tag{5.85}$$

We emphasize here that the second order correlation is a space-time variable, i.e.,

$$\langle p_0 p_1 \rangle = \langle p(\underline{r}_0, t_0) p(\underline{r}_1, t_1) \rangle \tag{5.86}$$

for which special cases are obtained when $t_0 = t_1$ and $\underline{r}_0 = \underline{r}_1$.

Following Middleton [136], we further anatomize d into the following components:

$$d = d_s + d_v + d_B \quad , \tag{5.87}$$

where d_s = d(surface) is associated with the surface

d_v = d(volume) is associated with the volume

d_B = d(bottom) is associated with the bottom.

These components may generally be regarded as independent, except perhaps for the volume and surface at the moving surface layer.

Inserting this anatomy of d and taking ensemble averages, we have

$$\langle p \rangle_{B(11)} = p_u + \hat{G}(-1)\hat{G}'(10)[\langle d_s(1) \rangle + \langle d_v(1) \rangle + \langle d_B(1) \rangle](s). \tag{5.88}$$

We call Eq. (5.88) the M-form of solution (Eq. (5.81)). For this approximation, B(11), the mean contains no interaction between scattering classes.[1]

However, the second order correlation of the M-form does contain such interaction terms. From Eqs. (5.84) and (5.87), we calculate

$$\langle p_0 p_1 \rangle_{B(11)} = -p_{u_0} p_{u_1} + p_{u_0} \langle p_1 \rangle + p_{u_1} \langle p_0 \rangle$$

---

[1]This is sometimes called the first order scattering theory. It is assumed by the FOM model. See Chap. IV, Sec. 4.4 and ref. [138].

$$+ \hat{G}_1(-1)\Big(\hat{G}_0(-1)\hat{G}_0^{\dagger}(10)\hat{G}_1^{\dagger}(10)$$

$$\cdot s_0 s_1 \langle (d_{s_0} + d_{v_0} + d_{B_0})(d_{s_1} + d_{v_1} + d_{B_1})\rangle\Big) \quad . \qquad (5.89)$$

The RHS of Eq. (5.89) is established from Eq. (5.88), except for the last term. Here

$$d_{s_0} + d_{v_0} + d_{B_0}\ \ d_{s_1} + d_{v_1} + d_{B_1} = d_{s_0}d_{s_1} + d_{s_0}d_{v_1} + d_{s_0}d_{B_1}$$

$$+ d_{v_0}d_{s_1} + d_{v_0}d_{v_1} + d_{v_0}d_{B_1}$$

$$+ d_{B_0}d_{s_1} + d_{B_0}d_{v_1} + d_{B_0}d_{B_1}. \qquad (5.90)$$

In terms of interacting scatterers, this is a matrix of

SxS	SxV	SxB
VxS	VxV	VxB
BxS	BxV	BxB.

It is easy to see that for independent classes, the interaction terms SxV, VxS, VxB, etc., are second order for small mean values.

Since $d_{v_0}d_{s_1} = d_v(\underline{r}_0,t_0)d_s(\underline{r}_1,t_1) \neq d_s(\underline{r}_0,t_0)d_v(\underline{r}_1,t_1)$, VxS ≠ SxV generally.

If the surface, volume and bottom scattering are independent, we have

$$\langle d_0 d_1\rangle = \langle d_{s_0}d_{s_1}\rangle + \langle d_{s_0}\rangle\langle d_{v_1}\rangle + \langle d_{s_0}\rangle\langle d_{B_1}\rangle$$

$$+ \langle d_{v_0}\rangle\langle d_{s_1}\rangle + \langle d_{v_0}d_{v_1}\rangle + \langle d_{v_0}\rangle\langle d_{B_1}\rangle$$

$$+ \langle d_{B_0}\rangle\langle d_{s_1}\rangle + \langle d_{B_0}\rangle\langle d_{v_1}\rangle + \langle d_{B_0}\rangle\langle d_{B_1}\rangle \ . \qquad (5.91)$$

If, further, the surface volume and bottom have mutually orthogonal statistics, then Eq. (5.91) becomes

$$\langle d_0 d_1\rangle = \langle d_{s_0} d_{s_1}\rangle + \langle d_{v_0} d_{v_1}\rangle + \langle d_{B_0} d_{B_1}\rangle \ . \qquad (5.92)$$

The simple case of $\langle d_s\rangle = \langle d_v\rangle = \langle d_B\rangle = 0$ with independence produces

$$\langle p_0 p_1\rangle_{B(11)} = p_{u_1} p_{u_0} + \hat{G}_0(-1)\Big(\hat{G}_1(10)\hat{G}_0(-1)\hat{G}_1(10)$$

$$\cdot\ s_0 s_1 \langle d_{s_0} d_{s_1}\rangle + \langle d_{v_0} d_{v_1}\rangle + \langle d_{B_0} d_{B_1}\rangle\Big) \qquad (5.93)$$

for which no interaction between scattering modes takes place. Thus, even in this simple approximation, one must have independence <u>and</u> zero means to avoid interaction between scattering classes in the second order statistics. (We note that only <u>two</u> of the three mean scatters need be zero; e.g.,

$$\langle d_0 d_1\rangle = \langle d_{s_0} d_{s_1}\rangle + \langle d_{v_0} d_{v_1}\rangle + \langle d_{B_0} d_{B_1}\rangle \qquad (5.94)$$

if only $\langle d_{s_0}\rangle = 0 = \langle d_{B_0}\rangle$ even if $\langle d_v\rangle \neq 0$.

The fourth moment may likewise be calculated. We write

$$p_0 p_1 p_2 p_3 B(11) = \prod_{i=0}^{3} p_{u_i} + \hat{G}_i(-1)\hat{G}_i(10)\ s_i\Big(d_{s_i} + d_{v_i} + d_{B_i}\Big) \ . \qquad (5.95)$$

The expansion results in one deterministic term, $p_{u_0} p_{u_1} p_{u_2} p_{u_3}$,

and fifteen stochastic terms. Of these latter, four are of type A:

$$p_{u_0}p_{u_1}p_{u_2}\hat{G}_3(-1)\hat{G}_3'(10)[s_3(0)d_3(1)] ,$$

six are of type B:

$$p_{u_0}p_{u_1}\hat{G}_2(-1)\hat{G}_2'(10)\left(\hat{G}_3(-1)\hat{G}_3'(10)[s_2s_3(d_2d_3)]\right)$$

four of type C:

$$p_{u_0}\hat{G}_1(-1)\hat{G}_1'(10)\left(\hat{G}_2(-1)\hat{G}_2'(10)\left(\hat{G}_3(-1)\hat{G}_3'(10)[s_1s_2s_3(d_1d_2d_3)]\right)\right)$$

and one of type D:

$$\hat{G}_0(-1)\hat{G}_0'(10)\left(\hat{G}_1(-1)\hat{G}_1'(10)\left(\hat{G}_2(-1)\hat{G}_2'(10)\left(\hat{G}_3(-1)\hat{G}_3'(10)\right.\right.\right.$$
$$\left.\left.\left.[s_1s_2s_3s_0(d_1d_2d_3d_0)]\right)\right)\right) .$$

The reader should recall the distinction between $\hat{G}\hat{G}$ and $\hat{G}(\hat{G})$ made in Chapter I, Preliminaries.

Following the decomposition of the M form, each first order correlation (type A) may be broken into 3 terms; the second order correlation (type B) into $3^2=9$ terms; a third order correlation (type C) into $3^3=27$ terms and the last fourth order term (type D) into $3^4=81$ terms. The entire fourth moment expansion in this approximation results, then, in

$$1 + 4(3) + 6(3^2) + 4(3^3) + 3^4 = (1+3)^4 = 256 \text{ terms.}$$

Considerable reduction occurs for independent, zero-mean normal statistics. For this case, we obtain

$$\langle p_0 p_1 p_2 p_3 \rangle_{B(11)} = p_{u_0} p_{u_1} p_{u_2} p_{u_3}$$

$$+ p_{u_0} p_{u_1} \hat{G}_2(-1)\hat{G}_2'(10)\ \hat{G}_3(-1)\hat{G}_3'(10)$$

$$\cdot\ s_2 s_3 \left(\langle d_{s_2} d_{s_3} \rangle + \langle d_{v_2} d_{v_3} \rangle + \langle d_{B_2} d_{B_3} \rangle \right)$$

$$+ p_{u_0} p_{u_2} \hat{G}_1(-1)\hat{G}_1'(10)\ \hat{G}_3(-1)\hat{G}_3'(10)$$

$$\cdot\ s_1 s_3 \left(\langle d_{s_1} d_{s_3} \rangle + \langle d_{v_1} d_{v_3} \rangle + \langle d_{B_1} d_{B_3} \rangle \right)$$

$$+ p_{u_0} p_{u_3} \hat{G}_1(-1)\hat{G}_1'(10)\ \hat{G}_2(-1)\hat{G}_2'(10)$$

$$\cdot\ s_1 s_2 \left(\langle d_{s_1} d_{s_2} \rangle + \langle d_{v_1} d_{v_2} \rangle + \langle d_{B_1} d_{B_2} \rangle \right)$$

$$+ p_{u_1} p_{u_2} \hat{G}_0(-1)\hat{G}_0'(10)\ \hat{G}_3(-1)\hat{G}_3'(10)$$

$$\cdot\ s_0 s_3 \left(\langle d_{s_0} d_{s_3} \rangle + \langle d_{v_0} d_{v_3} \rangle + \langle d_{B_0} d_{B_3} \rangle \right)$$

$$+ p_{u_1} p_{u_3} \hat{G}_0(-1)\hat{G}_0'(10)\ \hat{G}_2(-1)\hat{G}_2'(10)$$

$$\cdot\ s_0 s_2 \left(\langle d_{s_0} d_{s_2} \rangle + \langle d_{v_0} d_{v_2} \rangle + \langle d_{B_0} d_{B_2} \rangle \right)$$

$$+ p_{u_2} p_{u_3} \hat{G}_0(-1)\hat{G}_0'(10)\ \hat{G}_1(-1)\hat{G}_1'(10)$$

$$\cdot\ s_0 s_1 \left(\langle d_{s_0} d_{s_1} \rangle + \langle d_{v_0} d_{v_1} \rangle + \langle d_{B_0} d_{B_1} \rangle \right) \quad \cdots$$

$$+ \hat{G}_0(-1)\hat{G}_0^{\dagger}(10)\Big(\hat{G}_1(-1)\hat{G}_1^{\dagger}(10)\Big(\hat{G}_2(-1)\hat{G}_2^{\dagger}(10)\Big(\hat{G}_3(-1)\hat{G}_3^{\dagger}(10)$$

$$\cdot \Big[\langle d_{s_0} d_{s_1} d_{s_2} d_{s_3}\rangle + \langle d_{v_0} d_{v_1} d_{v_2} d_{v_3}\rangle + \langle d_{B_0} d_{B_1} d_{B_2} d_{B_3}\rangle$$

$$+ \langle d_{s_0} d_{s_1}\rangle\Big(\langle d_{v_2} d_{v_3}\rangle + \langle d_{B_2} d_{B_3}\rangle\Big)$$

$$+ \langle d_{s_0} d_{s_2}\rangle\Big(\langle d_{v_1} d_{v_3}\rangle + \langle d_{B_1} d_{B_3}\rangle\Big)$$

$$+ \langle d_{s_0} d_{s_3}\rangle\Big(\langle d_{v_1} d_{v_2}\rangle + \langle d_{B_1} d_{B_2}\rangle\Big)$$

$$+ \langle d_{s_1} d_{s_2}\rangle\Big(\langle d_{v_0} d_{v_3}\rangle + \langle d_{B_0} d_{B_3}\rangle\Big)$$

$$+ \langle d_{s_1} d_{s_3}\rangle\Big(\langle d_{v_0} d_{v_2}\rangle + \langle d_{B_0} d_{B_2}\rangle\Big)$$

$$+ \langle d_{s_2} d_{s_3}\rangle\Big(\langle d_{v_0} d_{v_1}\rangle + \langle d_{B_0} d_{B_1}\rangle\Big)\Big]\Big)\Big)\Big) \qquad (5.96)$$

or 34 terms. For the Gaussian case the four point correlations $\langle d_0 d_1 d_2 d_3\rangle$ may be further decomposed into two point correlations.

Similar, though correspondingly more complicated, formulations may be made for the B(1) approximations of higher order.

5.3.2 Selected Subsums, B(2)

The B(11) approximation contains only a single scattering, and so has proven useful in weakly homogeneous media. In a strongly turbulent situation, the error formed by omitting the multiple scattering may not be ignored. We thus turn to approximations based on selected sums, in

which all orders of scattering are represented. The first order of this set approximations is an original contribution of this Thesis. It also forms the basis for a canonical ordering of class B(2).

5.3.2.1 First Order Independent Solutions, B(21)

This approximation presumes that

$$\langle \xi p \rangle = \langle \xi \rangle \langle p \rangle \ . \tag{5.97}$$

Then Eq. (5.12) becomes

$$\langle p \rangle_{B(21)} = \hat{G}(s) + \hat{G}\langle \hat{\xi} \rangle \langle p \rangle \tag{5.98}$$

This is equivalent to $\hat{Q} = \langle \hat{\xi} \rangle$ for a mass operator. It may be noted that for $\langle \hat{\xi}(\underline{r},t) \rangle = 0$, this solution implies

$$\langle p \rangle = p_u$$

as in the first Born approximation.

If $\langle \xi \rangle \neq 0$ then

$$\langle p \rangle = \hat{G}(s) + \hat{G}\langle \hat{\xi} \rangle \hat{G}(s) + \hat{G}\langle \hat{\xi} \rangle \hat{G} \langle \hat{\xi} \rangle \hat{G}(s) + \ . \ . \ .$$

$$= p_u + \hat{G}\langle \hat{\xi} \rangle p_u + (\langle \hat{G}\hat{\xi} \rangle)^{(2)} p_u + \ . \ . \ .$$

$$= \sum_{n=0}^{\infty} (\langle \hat{G}\hat{\xi} \rangle)^{(n)} p_u \ . \tag{5.99}$$

Since the general solution is

$$\langle p \rangle = p_u + \hat{G}\langle \hat{\xi} \rangle p_u + \langle \hat{G}\hat{\xi}\hat{G}\hat{\xi} p_u \rangle + \ . \ . \ . \ ,$$

we see that this assumption holds if $\langle \hat{\xi}_1 \hat{\xi}_2 \ . \ . \ . \hat{\xi}_n \rangle = \langle \hat{\xi}_1 \rangle \langle \hat{\xi}_2 \rangle \ . \ . \ . \langle \hat{\xi}_n \rangle$

or statistical independence of all orders. For $\langle\hat{\xi}_i\rangle$ entirely impulsive, we have the condition for independent point scatterers.

In operational form, the first order independent situation for the mean is shown in Fig. 5-6.

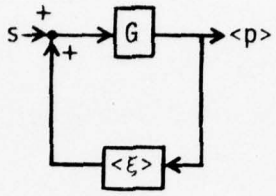
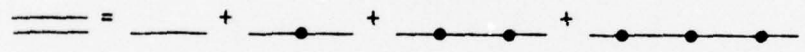


Fig. 5-6. Operational Solution for B(21) Mean

In terms of Feynman diagrams, this is

 (5.100)

which obviously extends the first Born solution but does not account for the correlation structure of the scatterers.

We emphasize that the operational form can be solved directly using recursive techniques. Thus, we have:

$$\begin{aligned} &\text{an initial solution:} && \langle p_0\rangle = p_u \,, \\ &\text{first iteration:} && \langle \hat{p}_1\rangle = p_u + G\langle\hat{\xi}\rangle p_u \,, \\ &\text{second iteration:} && \langle \hat{p}_2\rangle = p_u + G\langle\hat{\xi}\rangle G\langle\hat{\xi}\rangle p_u, \end{aligned} \tag{5.101}$$

so that given $\langle\hat{\xi}\rangle$, the empirical solution is straightforward.

For isotropic scattering, $\langle\hat{\xi}(\underline{r}_1,t_1)\rangle \equiv \langle\hat{\xi}(\underline{r}_2,t_2)\rangle$, we have

$$\langle(\hat{G}\hat{\xi})^{(n)}\rangle = \langle d^n\rangle\,(\hat{G}')^{(n)}, \tag{5.102}$$

so that Eq. (5.65), decomposed to form M, and using Eq. (5.55) becomes

$$\langle p\rangle_{B(21)} = \sum_{n=0}^{\infty} \langle (d_s + d_v + d_B)^n \hat{G}[\hat{G}']^{(n)} p_u \quad . \tag{5.103}$$

Here, obviously, the mean contains contributions from interacting classes.

Following the literature [55, p. 189, Eq. (2.46)], we press for an analytical solution. Using the usual approximation (Taylor series truncated to first term)

$$\hat{G}\langle\hat{\xi}\rangle\langle p\rangle \simeq \langle p\rangle\hat{G}'\langle d\rangle \ , \tag{5.104}$$

we have

$$\langle p\rangle_{B(21)} = \frac{p_u}{1-\hat{G}'\langle d\rangle} \quad . \tag{5.105}$$

This approximation may be readily evaluated. The Taylor's series truncation is typical of Class IV approximations mentioned in the preface. In space-time, the Taylor's series expansion of $p(\underline{r}_1,t_1)$ is

$$p(\underline{r}_1,t_1) = p(\underline{r},t) - (\underline{r}-\underline{r}_1)\cdot\nabla p(\underline{r},t)\Big|_r - (t-t_1)\frac{\partial}{\partial t_1} p(\underline{r},t)\Big|_t + \ . \ . \ . \tag{5.106}$$

The approximation used widely in the literature uses only the first term, that is

$$p(\underline{r}_1,t_1) = p(\underline{r},t) \tag{5.107}$$

so that $\hat{G}\hat{\xi}(\underline{r}_1)p(\underline{r}_1,t_1) \simeq \hat{G}\hat{\xi}(\underline{r}_1)p(\underline{r},t) = p(\underline{r},t)\ \hat{G}\hat{\xi}(\underline{r}_1)$. This approximation then allows algebraic to replace operational solutions. For Eq. (5.1), we have

$$\langle p(\underline{r},t)\rangle_{B(21)} = p_u(\underline{r},t)\left[1 - \int_{\underline{r}_0}\int_{t_0}\int_{\omega_0} \frac{\omega_0^2}{c_0^2}\,\frac{\langle d(\underline{r}_0,t_0)\rangle}{8\pi^2|\underline{r}-\underline{r}_0|}\right.$$

$$\left. \cdot\, e^{-j\omega_0\left(t-t_0-\frac{|\underline{r}-\underline{r}_0|}{c_0}\right)}\, d\underline{r}_0\, dt_0\, d\omega_0\right]^{-1} . \qquad (5.108)$$

Decomposing to form M, we have further,

$$\langle p(\underline{r},t)\rangle_{B(21)} = p_u(\underline{r},t)\left[1 - \int_{\underline{r}_0}\int_{t_0}\int_{\omega_0} \frac{\omega_0^2}{c_0^2}\,\frac{\langle d_s(\underline{r}_0,t_0)+d_v(\underline{r}_0,t_0)+d_B(\underline{r}_0,t_0)\rangle}{8\pi^2|\underline{r}-\underline{r}_0|}\right.$$

$$\left. \cdot\, e^{j\omega_0\left(t-t_0-\frac{|\underline{r}-\underline{r}_0|}{c_0}\right)}\, d\underline{r}_0\, dt_0\, d\omega_0\right]^{-1} . \qquad (5.109)$$

In this case (as contrasted with the Born approximation) interaction between scattering classes occurs in the mean. For example, if p_s, p_v and p_{s+v} denote the surface only, volume only, and combined effects on the received pressure, we have,

$$[(p_s+p_v) - p_{s+v}] \equiv p_u\left(\frac{1}{1-\hat{G}'(d_s)} + \frac{1}{1-\hat{G}'(d_v)} - \frac{1}{1-\hat{G}'(d_s)-\hat{G}'(d_v)}\right) \qquad (5.110)$$

$$= p_u[(1-\hat{G}'d_v)(1-\hat{G}'d_s-\hat{G}'d_v) + (1-\hat{G}d_s)(1-\hat{G}'d_s-\hat{G}'d_v)$$

$$- (1-\hat{G}'d_s) - (1-\hat{G}'d_v)][(1-\hat{G}'d_s)(1-\hat{G}'d_v)(1-\hat{G}'d_s-\hat{G}'d_v)]^{-1}$$

$$= \frac{(1-\hat{G}'d_v)(\hat{G}'d_s+\hat{G}'d_v) - (1-\hat{G}'d_s)(\hat{G}'d_s+\hat{G}'d_v)}{(1-\hat{G}'d_s)(1-\hat{G}'d_v)(1-\hat{G}'d_s-\hat{G}'d_v)}\, p_u$$

$$= \frac{(\hat{G}'d_v)^2 + 2(\hat{G}'d_v)(\hat{G}'d_s) + (\hat{G}'d_s)^2 - 2(\hat{G}'d_s+\hat{G}'d_v)}{(1-\hat{G}'d_s)(1-\hat{G}'d_v)(1-\hat{G}'d_s-\hat{G}'d_v)}\, p_u . \qquad (5.111)$$

For $|G'd_v|$ and $|G'd_s| \ll 1$

$$[(p_s+p_v) - p_{s+v}] \simeq -2(\hat{G}'d_s+\hat{G}'d_v)\, p_u \tag{5.112}$$

or

$$\frac{p_s+p_v}{p_{s+v}} \simeq \frac{2-(\hat{G}'d_s+\hat{G}'d_v)}{2+\hat{G}d_s+\hat{G}'d_v} \tag{5.113}$$

Similar error forms may be calculated for the other two combinations.

Because the first order independent solution is a subset of the general solution, these considerations lead to a simple criterion for the validity of the first Born approximation:

The Born approximation is valid only when the interaction of scattering classes is not perceivable in the mean statistics.

In many cases, this condition can be easily validated experimentally.

Second Moment of the First Order, Independent Solution B(21)

It is well known that specification of the first moment does not generally specify the higher moments. Hence, the first order independent solution is generally compatible with any equivalent B medium.

For the approximation

$$B\langle p_1 p_2\rangle \equiv \langle \hat{\xi}_1 \hat{\xi}_2 p_1 p_2\rangle \simeq \langle \hat{\xi}_1 p_1\rangle\langle \hat{\xi}_2 p_2\rangle = Q_1\langle p_1\rangle Q_2\langle p_2\rangle, \tag{5.114}$$

we have

$$\langle p_1 p_2\rangle_{B(21)} = p_{u_1} p_{u_2} + p_{u_1}\hat{G}_2 Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1 Q_1\langle p_1\rangle + \hat{G}_1(\hat{G}_2 Q_1 Q_2\langle p_1\rangle\langle p_2\rangle)$$

$$= p_{u_2}(p_{u_1} + \hat{G}_1 Q_1\langle p_1\rangle)' + (\langle p_{u_1}\rangle + \hat{G}_1 Q_1\langle p_1\rangle)\hat{G}_2 Q_2\langle p_2\rangle$$

$$= p_{u_2}\langle p_1\rangle + \langle p_1\rangle\hat{G}_2Q_2\langle p_2\rangle$$

$$= \langle p_1\rangle(\langle p_{u_2}\rangle + \hat{G}_2Q_2\langle p_2\rangle)$$

$$= \langle p_1\rangle\langle p_2\rangle \ . \qquad (5.115)$$

So, as we might expect, insistence on independence leads to independent solutions. Experimentally, this is a simple test to conduct, and provides some immediate information on the correlation structures of the medium.

If $B = \langle\hat{\xi}_1\hat{\xi}_2\rangle = \langle\hat{\xi}_1\rangle\langle\hat{\xi}_2\rangle$, then we have only

$$\langle p_1p_2\rangle_{B(21)} = p_{u_1}p_{u_2} + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle$$

$$+ \hat{G}_1(\hat{G}_2Q_1Q_2\langle p_1p_2\rangle) \ , \qquad (5.116)$$

so that the second-order correlation structure may be interpreted in terms of the first.

We examine this latter in M-form. Writing Eq. (5.116), accordingly, as

$$\langle p_1p_2\rangle_{B(21)} = p_{u_1}p_{u_2} + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle$$

$$+ \hat{G}_1\langle d_{s_1}+d_{v_1}+d_{B_1}\rangle(\hat{G}_2'\langle d_{s_2}+d_{v_2}+d_{B_2}\rangle\langle p_1p_2\rangle) \qquad (5.117)$$

and using again a truncated Taylor's expansion, we have

$$\langle p_1p_2\rangle = \frac{p_{u_1}p_{u_2} + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle}{1 - \hat{G}_1'(\hat{G}_2'\ \langle d_{s_1}+d_{v_1}+d_{B_1}\rangle\langle d_{s_2}+d_{v_2}+d_{B_2}\rangle)} \ . \qquad (5.118)$$

Again, the nature of the independence is clearly noticeable. Moreover, volume interface interactions can occur (even with $\langle d_v \rangle \equiv 0$).

Fourth Moment of the First Order Independent Solution

As with the lower moments, the fourth moment is not determined solely by lower moments. Hence, the first order independent solution is generally compatible with any equivalent W medium. A very large number of possibilities exist.[1] We shall deal only with the lowest order independence. If we assume independence of the scatterers (and independence in the lower order moments), i.e.,

$$W\langle p_1 p_2 p_3 p_4 \rangle = \langle \hat{\xi}_1 \rangle \langle \hat{\xi}_2 \rangle \langle \hat{\xi}_3 \rangle \langle \hat{\xi}_4 \rangle \langle p_1 p_2 p_3 p_4 \rangle \qquad (5.119)$$

then again we use a truncated Taylor's expansion to obtain

$$\langle p_1 p_2 p_3 p_4 \rangle = \frac{p_{u_1} p_{u_2} p_{u_3} p_{u_4} + \cdots + p_{u_4} \hat{G}_1 \hat{G}_2 \hat{G}_3 T_{123} \langle p_1 p_2 p_3 \rangle}{1 - \hat{G}_1' (\hat{G}_2' (\hat{G}_3' (\hat{G}_4' \langle d_1 \rangle \langle d_2 \rangle \langle d_3 \rangle \langle d_4 \rangle)))} . \qquad (5.120)$$

It becomes apparent that a rule of thumb for $\langle \Pi p_i \rangle$ is

$$\langle \Pi p_i \rangle \quad \frac{\Pi p_{u_i}}{1 - \Pi_i (G_i' \langle d_i \rangle)} . \qquad (5.121)$$

It is easy to see that the generalization of this rule is

$$\langle \Pi p_i \rangle \quad \frac{\Pi p_{u_i}}{1 - \langle \Pi_i (\hat{G}_i' d_i \rangle)} \qquad (5.122)$$

[1] See Appendix E.

which now embodies the correlation structure $\langle d_1 d_2 \ldots d_n \rangle$.

Since the moments are available in experimental data and the d_i are presumably given, this rule of thumb can be checked conveniently for insight into the correlation structure.

5.3.2.2 Second Order Independent Solutions, B(22)

This approximation presumes

$$\langle \hat{\xi}_1 \hat{\xi}_2 p \rangle = \langle \hat{\xi}_1 \hat{\xi}_2 \rangle \langle p \rangle \quad . \tag{5.123}$$

From the basic solution

$$p = \hat{G}(s) + \hat{G}\hat{\xi}p$$

we easily derive

$$p = \hat{G}(s) + \hat{G}\hat{\xi}\hat{G}(s) + \hat{G}\hat{G}\hat{\xi}\hat{\xi}p \quad . \tag{5.124}$$

Thus,

$$\langle p \rangle = \hat{G}(s) + \hat{G}\langle\hat{\xi}\rangle\hat{G}(s) + \hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}p\rangle \quad . \tag{5.125}$$

The second order independent solutions approximate Eq. (5.125) by

$$\langle p \rangle_{B(21)} = \hat{G}(s) + \hat{G}\langle\hat{\xi}\rangle\hat{G}(s) + \hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\rangle\langle p\rangle \quad . \tag{5.126}$$

This gives

$$\hat{Q} = \langle\hat{\xi}\rangle \frac{p_u}{\langle p\rangle} + \hat{G}\langle\hat{\xi}\hat{\xi}\rangle \tag{5.127}$$

so that here, $\hat{Q}$ absorbs the second order field correlations. Equation (5.126) may be expanded to

$$\langle p\rangle_{B(21)} = \sum_{n=0}^{\infty} (\hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\rangle)^{(n)}[\hat{G}(s) + \hat{G}\langle\hat{\xi}\rangle G(s)]$$

$$= \hat{G}(s) + \hat{G}\langle\hat{\xi}\rangle\hat{G}(s) + \hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\rangle\hat{G}(s) + \hat{G}\hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\rangle\langle\hat{\xi}\rangle\hat{G}(s)$$

$$+ \hat{G}\hat{G}\hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\rangle\langle\hat{\xi}\hat{\xi}\rangle\hat{G}(s) + \ldots \qquad (5.128)$$

which compares to the exact solution

$$\langle p\rangle = \hat{G}(s) + \hat{G}\langle\hat{\xi}\rangle\hat{G}(s) + \hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\rangle\hat{G}(s) + \hat{G}\hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\hat{\xi}\rangle\hat{G}(s)$$

$$+ \hat{G}\hat{G}\hat{G}\hat{G}\langle\hat{\xi}\hat{\xi}\hat{\xi}\hat{\xi}\rangle\hat{G}(s) + \ldots . \qquad (5.129)$$

We see that this solution only allows correlation structure up to the second order; the comparison between Eqs. (5.128) and (5.129) justifies our naming these solutions "second order independent."

In operational form, the second order independent solution for the first moment is shown in Fig. 5-7.

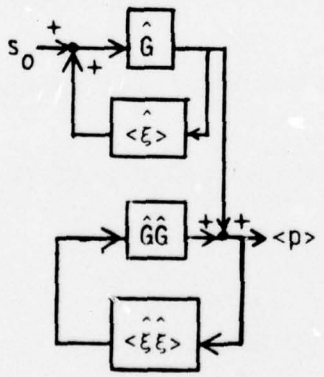
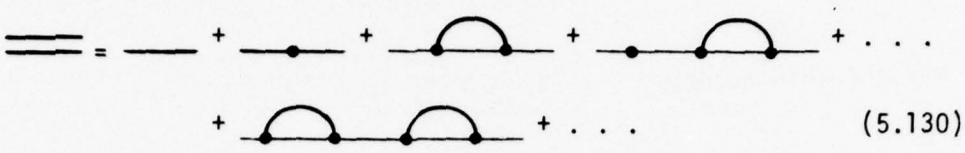


Fig. 5-7. Operational Solution for B(22) Mean

In terms of Feynman Diagrams, this solution is



(5.130)

This pattern accounts for the second order correlation structure of the scatterers.[1] It also, obviously, extends the second order truncation series B(12) and generalizes the first order independent solutions.

While the operational forms may be implemented directly, analytical expressions may be obtained only approximately. We note that for $\langle\xi\rangle = 0$, the second order independent solution is the bilocal (Bourret approximation). Bourret (1962) called the basic assumption $\langle\xi_1\xi_2 p\rangle = \langle\xi_1\xi_2\rangle\langle p\rangle$, "local statistical independence."

Particular solutions depend on the actual field correlation structure, often assumed homogeneously exponential (e.g., $B_0^2 e(-|\underline{r}-\underline{r}_0|/a)$) [189]. Rather than examine the particular solutions, we develop the moment structure for arbitrary $\langle d_1 d_2\rangle$.

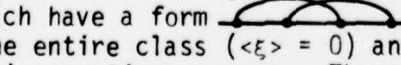
1. Means: B(22)

Putting, as before,[2]

$$\hat{G}\hat{G}\langle\hat{\xi}_1\hat{\xi}_2\rangle\langle p\rangle \simeq \langle p\rangle\hat{G}\hat{G}\langle d_1 d_2\rangle \tag{5.131}$$

we can write Eq. (5.125) as

---

[1]Tatarskii [188] and others [54] distinguish a subclass of "strong" correlations which have a form  . This subclass is a generator for the entire class ($\langle\xi\rangle = 0$) and allows a factorization from which Dyson's equation emerges. These "strong" correlations have no identity with numerically large physical correlation structures which are determined by the medium and not by formalism.

[2]Cf. Eq. (5.104) and the Taylor's approximation.

$$\langle p\rangle = \frac{\hat{G}(s) + \hat{G}\langle d\rangle\hat{G}'(s)}{1 - \hat{G}'\hat{G}'\langle d_1 d_2\rangle} \quad . \tag{5.132}$$

In M-form, this becomes

$$\langle p\rangle = \frac{\hat{G}(s) + \hat{G}\langle d_s + d_v + d_B\rangle\hat{G}'(s)}{1 - \hat{G}_1'\hat{G}_2'\langle (d_{s_1} + d_{v_1} + d_{B_1})(d_{s_2} + d_{v_2} + d_{B_2})\rangle} \quad . \tag{5.133}$$

Here, not only do we find interaction between scattering classes, but correlated interaction as well. For many applications, however, the statistics of the interfaces are independent of the volume statistics. Since $\langle d_{s_1} d_{v_2}\rangle \neq \langle d_{v_1} d_{s_2}\rangle$ in general, S x V ≠ V x S, generally.

2. The Second Moment of the Second Order Independent Solution, B(22)

We have from the exact solution, (5.15),

$$\langle p_1 p_2\rangle = p_{u_1} p_{u_2} + p_{u_1}\hat{G}_2 Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1 Q_1\langle p_1\rangle + \hat{G}_1(\hat{G}_2\langle\hat{\xi}_1\hat{\xi}_2 p_1 p_2\rangle) \quad . \tag{5.134}$$

Let $B\langle p_1 p_2\rangle \equiv \langle\hat{\xi}_1\hat{\xi}_2 p_1 p_2\rangle$ define B; then, approximating as before[1]

$$\langle p_1 p_2\rangle \simeq \frac{p_{u_1} p_{u_2} + p_{u_1}\hat{G}_2 Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1 Q_1\langle p_1\rangle}{1 - \hat{G}_1'(\hat{G}_2' B_{12})} \quad . \tag{5.135}$$

This latter strongly resembles, in form, the first order independent solution Eq. (5.118), although Q here is differently defined, and second

[1]Cf. Eq. (5.104) and the Taylor's approximation.

order correlations are evident. Indeed, if $B_{12} = Q_1 Q_2 \rightarrow \langle\hat{\xi}_1\rangle\langle\hat{\xi}_2\rangle$, then we recover Eq. (5.118).

Following this lead, we evolve a set of approximations for Q.

a) exact: $$\hat{Q}\langle p\rangle \equiv \langle\hat{\xi}p\rangle \tag{5.136}$$

b) first order: $$Q_1\langle p\rangle = \langle\hat{\xi}\rangle\langle p\rangle \tag{5.136a}$$

c) second order: $$\hat{Q}_2\langle p\rangle = \langle\hat{\xi}\rangle\, p_u + \langle\hat{\xi}\hat{\hat{G}}\hat{\xi}\rangle\langle p\rangle \tag{5.136b}$$

d) third order: $$\hat{Q}_3\langle p\rangle = \langle\hat{\xi}\rangle\, p_u + \langle\hat{\xi}\hat{\hat{G}}\hat{\xi}\rangle\, p_u + \langle\hat{\xi}\hat{\hat{G}}\hat{\xi}\hat{\hat{G}}\hat{\xi}\rangle\langle p\rangle \tag{5.136c}$$

so that

e) $$Q\langle p\rangle = \lim_{n\to\infty} \sum_{m=0}^{n=1} \langle\hat{\xi}(\hat{\hat{G}}\hat{\xi})^{(m)}\rangle\langle p_u\rangle + \langle\hat{\xi}(\hat{\hat{G}}\hat{\xi})^{(n)}\rangle\langle p\rangle \,. \tag{5.137}$$

In M-form, we have for Eq. (5.135)

$$\langle p_1 p_2\rangle = \frac{p_{u_1} p_{u_2} + p_{u_1}\hat{G}_2\hat{Q}_2\langle p_2\rangle + p_{u_2}\hat{G}_1\hat{Q}_1\langle p_1\rangle}{1 - \hat{G}_1\hat{G}_2\langle\left(d_{s_1}+d_{v_1}+d_{B_1}\right)\left(d_{s_2}+d_{v_2}+d_{B_2}\right)\rangle} \,. \tag{5.138}$$

Of course, the decomposition using B may not be homogeneous. Valid in one part of the field, it may bring on large errors elsewhere. The point is, however, that in these equivalent media, the equivalence is determined by the input/output relationships, so that conditions of experimental verification of the medium need not enter. These types of problems (as shown by the operational solutions) are essentially estimation problems.[1]

[1]See, for example, [216], Chap. 17.

3. The Fourth Moment, Second-Order Independent Solution B(22)

This moment is straightforward, but tedious, depending on assumptions about Q, B and T.

The rule of thumb, Eq. (5.109), may be extended for this case typically to

$$\langle p_1 p_2 p_3 p_4 \rangle = \frac{p_{u_1} p_{u_2} p_{u_3} p_{u_4}}{1 - \hat{G}_1'(\hat{G}_2'(\hat{G}_3'(\hat{G}_4'\langle d_1 d_2\rangle\langle d_3 d_4\rangle)))} \quad . \tag{5.139}$$

The B(22) approximation shows that the many possible correlation structures bring, each one, a new type of approximation in the higher moments. We might write Eq. (5.139) differently for $\langle d_1 d_2 d_3 d_4\rangle = \langle d_1 d_3\rangle\langle d_2 d_4\rangle$, etc. The full formal complexity of the correlation orders (and of the approximations they accompany) may be judged from Appendix E.

5.3.3 Comparison of Solutions

If we compare the mean solutions, we see that

$$\langle p\rangle_{B(11)} = p_u + \hat{G}'\langle d\rangle p_u \tag{5.140}$$

$$\langle p\rangle_{B(21)} = p_u/(1 - \hat{G}'\langle d\rangle) \tag{5.141}$$

$$\langle p\rangle_{B(22)} = (p_u + \hat{G}'\langle d\rangle p_u)/(1 - \hat{G}_1'(\hat{G}_2'\langle d_1 d_2\rangle)) \tag{5.142a}$$

$$= \langle p\rangle_{B(11)}/(1 - \hat{G}_1'(\hat{G}_2'\langle d_1 d_2\rangle)) \tag{5.142b}$$

so that the relative errors can be established, i.e.,

$$\langle p\rangle_{B(11)} - \langle p\rangle_{B(21)} = \frac{\hat{G}'\langle d\rangle p_u - p_u\hat{G}'\langle d\rangle - (\hat{G}'\langle d\rangle)(\hat{G}'\langle d\rangle p_u)}{1 - \hat{G}'\langle d\rangle}, \tag{5.143}$$

$$\langle p\rangle_{B(11)} - \langle p\rangle_{B(22)} = \frac{-\langle p\rangle_{B(11)}\hat{G}'\hat{G}'\langle d_1 d_2\rangle}{1 - \hat{G}'\hat{G}'\langle d_1 d_2\rangle} \tag{5.144}$$

and

$$\langle p\rangle_{B(21)} - \langle p\rangle_{B(22)} = \frac{p_u\hat{G}'\langle d\rangle - p_u\hat{G}'(\hat{G}'\langle d_1 d_2\rangle) + (\hat{G}'\langle d\rangle)(\hat{G}'\langle d\rangle p_u)}{(1 - \hat{G}'\langle d\rangle)(1 - \hat{G}'\hat{G}'\langle d_1 d_2\rangle)}. \tag{5.145}$$

While $\langle d_1 d_2\rangle \neq 0$ (correlated scattering) implies an error in the Born approximation, B(11), it does not follow that $\langle d_1 d_2\rangle = 0$ validates B(11), as may be seen from Eq. (5.143). Note, also, that for $\langle d\rangle = 0$, $\langle p\rangle_{B(11)} = \langle p\rangle_{B(21)} = p_u$. For correlated scattering to second order, Eq. (5.144) supplies a measure of B(11) error. By re-writing Eq. (5.144) as

$$\langle p\rangle_{B(11)} - \langle p\rangle_{B(22)} = \frac{\langle p\rangle_{B(11)}}{1 - \dfrac{1}{\hat{G}'\hat{G}'\langle d_1 d_2\rangle}}, \tag{5.146}$$

it is easy to see that the error depends on the magnitude of $\langle d_1 d_2\rangle$. In particular as $\langle d_1 d_2\rangle \to 0$, $\langle p\rangle_{B(22)} = \langle p\rangle_{B(11)}$.

The second moment solutions obtained here contain

$$a \equiv -p_{u_1}p_{u_2} + p_{u_2}\langle p_1\rangle + p_{u_1}\langle p_2\rangle \tag{5.147}$$

and are given by

$$\langle p_1 p_2 \rangle_{B(11)} = a + \hat{G}_1 \hat{G}_2 \langle d_1 d_2 \rangle p_{u_1} p_{u_2} , \tag{5.148}$$

$$\langle p_1 p_2 \rangle_{B(21)} = \frac{a}{1 - (\hat{G}_1' \langle d_1 \rangle)(\hat{G}_2' \langle d_2 \rangle)}, \tag{5.149}$$

and

$$\langle p_1 p_2 \rangle_{B(22)} = \frac{a}{1 - \hat{G}_1' (\hat{G}_2' \langle d_1 d_2 \rangle)} . \tag{5.150}$$

The second moment relative errors may thus be expressed as

$$\langle p_1 p_2 \rangle_{B(11)} - \langle p_1 p_2 \rangle_{B(21)} = \left[\hat{G}_1' \hat{G}_2' \langle d_1 d_2 \rangle p_{u_1} p_{u_2} - a(\hat{G}_1 \langle d_1 \rangle)^2 \right.$$

$$\left. + (\hat{G}_1 \langle d_1 \rangle)^2 \hat{G}_1' \hat{G}_2' \langle d_1 d_2 \rangle p_{u_1} p_{u_2} \right] \left[1 - (\hat{G}_1' \langle d_1 \rangle)^2 \right]^{-1} ; \tag{5.151}$$

$$\langle p_1 p_2 \rangle_{B(11)} - \langle p_1 p_2 \rangle_{B(22)} = \left[\hat{G}_1' \hat{G}_2' \langle d_1 d_2 \rangle p_{u_1} p_{u_2} - a \hat{G}_1' (\hat{G}_2' \langle d_1 d_2 \rangle) \right.$$

$$\left. - \hat{G}_1' (\hat{G}_2' \langle d_1 d_2 \rangle) \quad \hat{G}_1' \hat{G}_2' \langle d_1 d_2 \rangle p_{u_1} p_{u_2} \right]$$

$$\cdot \left[1 - \hat{G}' (\hat{G}_2' \langle d_1 d_2 \rangle) \right]^{-1} ; \tag{5.152}$$

and

$$\langle p_1 p_2 \rangle_{B(21)} - \langle p_1 p_2 \rangle_{B(22)} = \frac{a (\hat{G}_1 \langle d_1 \rangle)^2 - \hat{G}_1 (\hat{G}_2 \langle d_1 d_2 \rangle)}{(1 - (\hat{G}_1' (\langle d \rangle))^2)(1 - \hat{G}_1' (\hat{G}_2' \langle d_1 d_2 \rangle))} . \tag{5.153}$$

We see that the quantity, a, figures in the error. This, in turn, shows that an error in the first moment propagates to the second

moment. It follows that a B(11) solution invalid in the mean will generally be invalid in the second moment also.

Again, for the B(11) solution to hold in the second moment, it is necessary but <u>not</u> sufficient that $\langle d_1 d_2 \rangle \equiv 0$.

5.4 Poisson Statistics

It sometimes happens that our concern centers on a domain of independent, discrete scatterers. For these cases we develop a Poisson model.

We suppose a first order independent solution with

$$\xi_i = - \sum_1^n \frac{\omega^2}{c_0^2} d_i \delta(\underline{r}_i - \underline{r}) \tag{5.154}$$

where now n is a random variable.

Thus, we can write, using Eq. (5.55)

$$\langle p \rangle = p_u - \hat{G} \left\langle \sum_{i=0}^n d_i \delta(\underline{r}_i - \underline{r}_1) k_0^2 \, p_u(1) \right\rangle$$

$$+ \hat{G} \left\langle \sum_{i=0}^n d_i \delta(\underline{r}_i - \underline{r}_1) k_0^2 \hat{G} \sum_{j=0}^m d_j \delta(\underline{r}_j - \underline{r}_2) k_1^2 p_u(2) \right\rangle + \ldots . \tag{5.155}$$

We may interpret

$$\sum_{i=0}^n \hat{G} d_i \delta(\underline{r}_i - \underline{r}_1) k_0^2 p_u(1)$$

in two ways: $n = \infty$ and d_i is a random variable over $\underline{r}_i$; or d_i is deterministic but n ($< \infty$) is a random variable. In this latter interpretation, we can deal with the problem as a random sum. Assuming n

is independent of $x_i \equiv \hat{G} d_i \delta(\underline{r}_i - \underline{r}_1) k_0^2 p_u(1)$, each of which has an identical value, we get

$$< \sum_n x_i > = \hat{G} d_i \delta(\underline{r}_i - \underline{r}_1) k_0^2 p_u(1) < n > . \qquad (5.156)$$

Further precision can be made by assuming n poisson, with parameter ρ, so that for the first truncation approximation, B(11), we have

$$< p > = p_u - \rho d_i \frac{e^{-j(\omega t - \omega t_0 - k|\underline{r} - \underline{r}_i|)} p_u(\underline{r}_i)}{4\pi |\underline{r} - \underline{r}_i|} . \qquad (5.157)$$

The parameter d_i acts as the average scattering strength; the parameter ρ is the density of scatterers (average number of scatterers/unit volume). The extension to classes of similar scatterers separated in space is immediate.[1]

The third term in Eq. (5.143) leads to a form of the type

$$\int_{\underline{r}_2} \int_{\underline{r}_1} \sum_{i=0}^{n} d_i \frac{\delta_i(\underline{r}_1 - \underline{r}_i) k_1^2 e^{-jk_1|\underline{r} - \underline{r}_1|}}{4\pi |\underline{r} - \underline{r}_1|} \sum_{j=0}^{m} d_j \frac{\delta(\underline{r}_2 - \underline{r}_j) k_1^2 e^{-jk_2|\underline{r}_1 - \underline{r}_2|}}{4\pi |\underline{r} - \underline{r}_2|} p_u(\underline{r}_2)$$

which becomes (harmonic case),

$$\sum_{j=0}^{m} \sum_{i=0}^{n} d_i \frac{e^{-jk_o|\underline{r} - \underline{r}_i|}}{4\pi |\underline{r} - \underline{r}_i|} d_j \frac{e^{-jk_o|\underline{r}_i - \underline{r}_j|}}{4\pi |\underline{r}_i - \underline{r}_j|} p_u(\underline{r}_j), \; k_o = \frac{\omega_o}{c_o} . \qquad (5.158)$$

Letting a first random sum run over m, we have (with identical scatterers in a small region far from source and receiver), for Eq. (5.158)

---

[1]For a fuller account of this development, see [138].

$$< \sum_{i=0}^{n} d_i \frac{e^{-jk_o|\underline{r}-\underline{r}_i|}}{4\pi|\underline{r}-\underline{r}_i|} d_j \frac{e^{-jk_o|\underline{r}_i-\underline{r}_j|}}{4\pi|\underline{r}_i-\underline{r}_j|} p_u(\underline{r}_j)><m> , \quad (5.159)$$

a function of $\underline{r}_i$ and m. If further decomposition can be made, we can obtain

$$d_i d_j \frac{e^{-jk_o|\underline{r}-\underline{r}_i|}}{4\pi|\underline{r}-\underline{r}_i|} \frac{e^{-jk_o|\underline{r}_i-\underline{r}_j|}}{4\pi|\underline{r}_i-\underline{r}_j|} p_u(\underline{r}_j) <m><n> \quad (5.160)$$

where $|\underline{r}_i-\underline{r}_j|$ acts like a correlative distance (average distance between coupled scatterers). If $<m> = <n> = \rho$ then

$$<p> = p_u - \rho d_i \frac{e^{-jk_o|\underline{r}-\underline{r}_i|}}{4\pi|\underline{r}-\underline{r}_i|} p_u(\underline{r}_i)$$

$$+ \rho^2 d_i d_j \frac{e^{-jk_o|\underline{r}-\underline{r}_i|}}{4\pi|\underline{r}-\underline{r}_i|} \frac{e^{-jk_o|\underline{r}_i-\underline{r}_j|}}{4\pi|\underline{r}_i-\underline{r}_j|} p_u(\underline{r}_i). \quad (5.161)$$

The rapidity of convergence of Eq. (5.161) can thus be seen to depend on

$$\rho d_i \frac{e^{-jk|\underline{r}_j-\underline{r}_i|}}{4\pi|\underline{r}_j-\underline{r}_i|} \quad (5.162)$$

under these circumstances.

5.5 Summary

Historically, the randomness of the medium was first accommodated by a truncated sum, the first Born approximation, B(11), (which gave adequate results in weakly inhomogeneous media). Where this initial

approximation failed, solutions were pursued by selectively summing the series solution (Eq. (5.12a)). Most work assumed harmonic radiation.[1]

Our exposition has followed the usual path of the literature: (1) describing the random process, (2) obtaining moments in series form, and (3) obtaining approximations.

By beginning with broadband radiation, and by using techniques developed for the general solution (Chap. III) we have been able to extend the solutions of the literature. In particular, not only has the random process been modeled, (Eq. 5.23), but the moments (first, second, and fourth) have also been derived explicitly, (Sec. 5.1.1.1 - 5.1.1.4). Moreover a canonical representation of approximations, B, has been elaborated, (Sec. 5.3). This representation allows the particular approximations of the literature to be more readily appreciated. In terms of the preface, these approximations belong to class III and IV.

In addition to extending the literature, our methods have elucidated several conundrums. We mention the problem of convergence (of the series solution and approximations), (Sec. 5.1.2); the extension of Feynman diagrams to interfaces, (Sec. 5.1.4); the criterion for validity of the first Born approximation, (Sec. 5.3.2.1); the "mass" operator, Q, (Sec. 5.1.2).

Perhaps the most notable innovation in this chapter is the interpretation of the mass operator, $\hat{Q}$, as a first moment equivalent medium (Sec. 5.1.2). This allows not merely extensions to higher order in a

[1]See Chap. IV.

simple way but more importantly, opens a new avenue for exact solutions to Eq. (5.1).

The original items of this chapter are listed below.

I. Solutions

A. Broadband solution to Eq. (5.1) - Eq. (5.55).

B. Operational solution to Eq. (5.1) - (Sec. 5.1.2).

1. The broadband feedforward operator, $\hat{G}$ - Eq. (5.50).

C. First, second and fourth moments of the broadband solution - (Secs. 5.1.1.1 - 5.1.1.4).

D. Feynman Diagrams extended to interfaces - (Fig. 5-5).

E. Higher Order Moments - Eq. (5.22).

II. Approximations

A. A canonical decomposition of approximations - (Sec. 5.3).

B. The first, second and fourth moments for the first Born broadband approximation B(11) - (Sec. 5.3.1).

C. The first order independent approximations B(21) - (Sec. 5.3.2.1).

1. The operational solution of B(21) mean - (Fig. 5-6).

2. The first, second and fourth moments of B(21) - (Sec. 5.3.2.1).

D. The extension of the Tatarskii and Bourret solutions to non-zero mean, arbitrary distributions (non-isotropic), B(22) - (Sec. 5.3.2.2).

1. The operational solution of the B(22) mean - (Fig. 5-7).

2. The first, second, and fourth moments of B(22) - (Sec. 5.3.2.2).

E. A canonical approximation for the mass operator, $\hat{Q}$, - (Sec. 5.3.2.2).

F. A rule of thumb for approximating higher moments - Eq. (5.122).

G. Definition of G', which allows operator commutativity - Eq. (5.54).

III. Insights

A. Moment Equivalent Media - (Sec. 5.1.2).

1. A new method for obtaining moment solutions - (Sec. 5.1.2).

B. The canonical extension to second and higher moments without using the Bethe-Salpeter expression. - (Sec. 5.1.1.2).

C. The possible correlation structure - (Appendix E).

D. Misgivings on $\hat{G}$, $\delta(\rho)$, and $g\hat{L}p$. - (Sec. 5.2.1 and Appendixes D and H).

E. A complete set of moment differential equations - (Sec. 5.1.3).

F. Criterion for the validity of the first Born approximation, B(11) (Sec. 5.3.2.1).

G. Algebraic solutions by Taylor's series approximations (Class IV) - Eq. (5.105)

H. Elucidation of the convergence problem - (Sec. 5.1.2).

Several of our results have an experimental value. For instance, the ratio $\Pi p_{u_i} / \langle \Pi p_i \rangle$ may be easily calculated; it allows a quick measure of turbulence (see Eq. 5.122). The criterion for B(11) should likewise prove useful.

We have also introduced in this chapter the M-form (Eq. (5.2)) of Sec. 5.1 and occasionally discussed solutions in this form. This form introduces the surface-volume interactions and opens the way to solutions practically impossible in the classical view. This is the problem of moving and stochastic boundaries to which we now turn.

VI. SURFACE-VOLUME INTERACTION

6.0 Introduction

In the ocean, the most conspicuous scatterer of sound is the dynamic surface of the sea. Because of the large difference between the densities of air and water (and their corresponding propagation velocities), this surface acts as a sharp discontinuity which scatters sound strongly, both randomly and coherently.

Being conspicuous, surface scattering has attracted a large number of investigators. We mention the recent work of Middleton [136], McDonald and his associates [129, 215], and the earlier work of Medwin [131] and Gulin [79-84]. An excellent survey is given in Fortuin [71].

A large complex of problems falls into this purview. We may consider the surface over a homogeneous ocean, ($\nabla c=0$), over an inhomogeneous one, ($\nabla c\neq 0$), over one with discrete scatterers, with weak or strong scatterers, localized or not, interacting diversely. These many conditions correspond to the great diversity of the ocean (atmosphere) itself, and we should be presumptuous to take a particular case for more than its worth. Yet, the general formulations thus far developed provide general lines to the subject, so that particular cases may be seen in better perspective.

Our purpose in this Chapter is to develop the solutions of Chapter V for media with stochastic boundaries, particularly the moving surface of the sea. We shall examine and derive some recent (classical) results and shall as usual pay attention to the assumptions needed. Our

results will, for the most part, extend the classical results by the broadband formulation of Chapter V and because of the insights derived from the operational solution.

This purpose, in turn, will serve two other purposes of the overall thesis: (1) to study carefully class III and class IV errors[1] and (2) to determine criteria for the importance of surface-volume coupled scattering.

We begin with the surface, (Sec. 6.1), continue by considering an example of lossy volume propagation, (Sec. 6.2), and finish with a simple example of coupled scattering, (Sec. 6.4). Along the way we will consider a new model, (Sec. 6.1.5), soon to be published, which is especially relevant to scattering in media with stochastic boundaries.

6.1 Surface Scattering

We first consider the case where other than surface scattering is negligible. For convenience, we will restrict our attention to the known solutions of Eq. (5.1). In the general case, Eq. (1.5) characterizes the surface pressure interaction. As appropriate, typical cases will be drawn.

Our considerations here are restricted to

$$d = d_S + d_V + d_B = d_s , \qquad d_V = d_B = 0, \tag{6.1}$$

that is, surface scattering over a homogeneous ocean any point of whose dynamic surface is $\underline{r}_s$. Equation (5.55) becomes

---

[1]See Preface

$$p = p_u + \hat{G}d \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(s) \qquad (6.2)$$

$$\text{where } d = \begin{cases} 0 & \text{in homogeneous ocean} \\ d' & \text{in the atmosphere.} \end{cases}$$

Then $\hat{G} = \int_t \int_\omega \int_{\underline{r}'} dt d\omega d\underline{r}' \, \frac{(\cdot)e^{j(\omega t - k|\underline{r}'-\underline{r}|)}}{4\pi|\underline{r}'-\underline{r}|}$; that is, the integration is over the upper volume $\{\underline{r}'\}$.[1]

Let us bring the above considerations more precisely to the surface. Figure 6-1 describes the geometry.

Using a generalization of the divergence theorem[2] (to account for atmospheric discontinuities), put (in a non-unique way)

$$\nabla\cdot\left(\underline{d}'_s \frac{e^{-jk\rho'}}{4\pi\rho'} \hat{G}'(s)\right) \equiv d'\left(\sum(\hat{G}'d')^{(n)}\hat{G}'(s)\right) \frac{e^{-jk\rho'}}{4\pi\rho'}, \quad \rho' = |\underline{r}'-\underline{r}| \qquad (6.3)$$

for some $\underline{d}'_s$ (to be determined). We thus obtain

$$p = p_u + \int_{\underline{r}_s} \int_t \int_\omega e^{+j\omega(t-t_0)} \, \underline{d}'_s\cdot\hat{n} \, \frac{e^{-jk\rho}}{8\pi^2\rho} G'(s) \, d\underline{r}_s d\omega dt \qquad (6.4)$$

$$\equiv p_u + \hat{G}_s \, \underline{d}'_s\cdot\hat{n} \, \hat{G}'(s) \, . \qquad (6.4a)$$

Equation (6.4a) defines the operator $\hat{G}_s$, which implies an integration over the surface $\{\underline{r}_s\}$. The vector $\underline{d}'_s$ has dimensions of length.

---

[1] In this way atmospheric occurrences, such as layering, independent acoustic sources, etc., which affect underwater reception are accounted for. Most constructions of the literature omit these aspects, Cf. [193] pp. 193-199. An analogy here can be made to three dimensional transmission lines.

[2] See Sec. 1.2.1.2

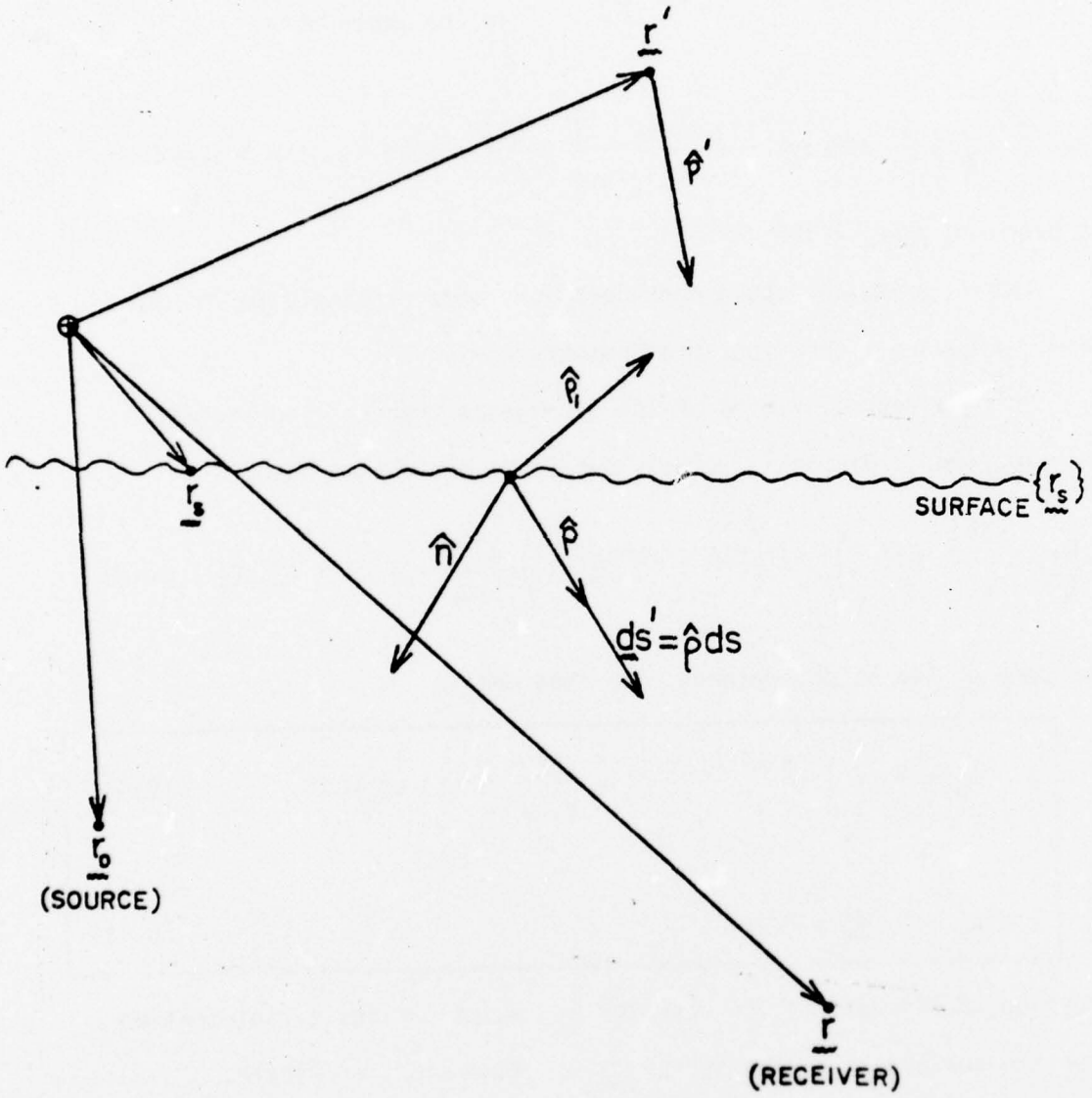


Fig. 6-1. Geometry (Eqs. (6.4), (6.9))

Equation (6.4) says that the effect of the atmosphere is summarized completely in the boundary.

Something of $\underline{d}'_s$ may be learned from the differentiation indicated in (6.3).

$$\nabla\cdot\left(\underline{d}'_s \frac{e^{-jk\rho}}{4\pi\rho}\hat{G}'(s)\right) = \frac{e^{-jk\rho}}{4\pi\rho}\left[\underline{d}'_s\cdot\nabla\hat{G}'(s)+\hat{G}'(s)(\nabla\cdot\underline{d}'_s-(\frac{1}{\rho}+jk)\hat{\rho}\cdot\underline{d}'_s)\right], \quad (6.5)$$

so that

$$d'\sum_{n=0}^{\infty}(\hat{G}'d')^{(n)}\hat{G}'(s) = \underline{d}'_s\cdot\nabla\hat{G}'(s)+\hat{G}(s)[\nabla\cdot\underline{d}'_s-(\frac{1}{\rho}+jk)\hat{\rho}\cdot\underline{d}'_s]. \quad (6.6)$$

Equation (6.6) gives $\underline{d}'_s$ in terms of known values and operations. The LHS shows the effect of multiple scattering in the atmosphere. It should be noted that by the divergence theorem we need only seek $\underline{d}'_s(\underline{r}_s)$, that is, we need only those values of $\underline{d}'_s$ <u>at the surface</u>.

Because Eq. (6.3) allows specification of $\underline{d}'_s$ in an arbitrary direction, we may choose for convenience the receiver direction, i.e., $\underline{d}'_s \equiv d'_s\hat{\rho}$. Then Eq. (6.6) becomes

$$d'\sum_{n=0}^{\infty}(\hat{G}'d')^{(n)}\hat{G}'(s) = d'_s\hat{\rho}\cdot\nabla\hat{G}'(s)+\hat{G}'(s)[\nabla\cdot\hat{\rho}d'_s-(\frac{1}{\rho}+jk)d'_s]. \quad (6.6a)$$

Usually $\hat{\rho}d'_s \equiv \underline{d}'_s$ is slowly varying spatially close to the surface ($\nabla\cdot\hat{\rho}d_s \simeq 0$), so that

$$d'_s \simeq \frac{d'\sum_{n=0}^{\infty}(\hat{G}'d')^{(n)}\hat{G}'(s)}{\hat{\rho}\cdot\nabla\hat{G}'(s)-\hat{G}'(s)(\frac{1}{\rho}+jk)}. \quad (6.6b)$$

Under the assumptions, $\underline{d}_s \cdot \nabla \hat{G}'(s) = 0 = \nabla \cdot \underline{d}_s$

$$d_s' = \frac{-d'}{\frac{1}{\rho} + jk} \sum (\hat{G}'d')^{(n)} \hat{G}'(s) \simeq \frac{-d_s}{jk} \tag{6.7}$$

where $d_s \equiv d' \sum (\hat{G}'d')^{(n)} \hat{G}(s)$ and for a receiver far removed from the surface (i.e., $\rho^{-1} \simeq 0$). Under these assumptions Eq. (6.4) reduces to

$$p = p_u + \int_{\underline{r}_s} \int_{t_0} \int_{\omega} \hat{\rho} \cdot \hat{n} \, \frac{e^{j[\omega(t-t_0)-k\rho]}}{8\pi^2 \rho} \left(\frac{d_s}{jk} \right) k^2 p_u(\underline{r}_s) d\underline{r}_s \, dt_0 \, d\omega \ . \tag{6.8}$$

One immediately notes a <u>first</u> power dependence on frequency.[1]

When less severe assumptions are desired, we may simplify from

$$\hat{G}'(s) = - \hat{G}' \left(\frac{1}{\rho_1} + jk \right) \hat{\rho}_1 s \ , \tag{6.9}$$

where $\hat{\rho}_1 = \frac{\underline{r}_s - \underline{r}_0}{|\underline{r}_s - \underline{r}_0|}$, a unit vector. For localized sources distant from the surface ($\rho_1^{-1} \to 0$), Eq. (6.6) becomes

$$d_s' = \frac{d' \sum (\hat{G}'d')^{(n)} \hat{G}'(s)}{j\hat{\rho} \cdot \hat{\rho}_1 \hat{G}'(ks) - jk\hat{G}'(s)} \ , \tag{6.10}$$

and for the harmonic case we get

$$d_s' = \frac{d' \sum (\hat{G}'d')^{(n)} \hat{G}'(s)}{jk_0 (\hat{\rho}_1 \cdot \hat{\rho} - 1) \, \hat{G}'(s)} \ . \tag{6.11}$$

For simplicity (and conformity tc the literature) we shall use Eqs. (6.7) and (6.10) in the sequel.

---

[1][194] p. 199 Eq. (6.24), gives,

$$p_s = \frac{jk \, B \, f(\theta) \, R \, e^{-jk(\rho_1+\rho_2)}}{2\pi\rho_1\rho_2} \int_{\underline{r}_s} D_0 \, e^{2j(\alpha x+\beta y+\gamma z)} d\underline{r}_s .$$

6.1.1 The Average Surface

The ocean surface, $\underline{r}_s$, is best regarded as a random process [105]. If we adopt this point of view, $p_u(\underline{r}_s,t)$ and $d'_s(\underline{r}_s,t)$ likewise become random processes and, further, the operator $\hat{G}_s$ of Eq. (6.4a) implies integration over a random surface. The problems of computing and of accomodating these variations will be greatly reduced by referencing the quantities to the mean or average surface.

Let us decompose the surface into

$$\underline{r}_s = \underline{r}_{\bar{s}} + \underline{z}(\underline{r}_{\bar{s}}) \tag{6.12}$$

where $\underline{r}_{\bar{s}}$ is the mean surface and $\underline{z}(\underline{r}_{\bar{s}})$ the deviation about the mean surface. Thus, $\underline{r}_s$ has a normal projection to $\underline{r}_{\bar{s}}$ through $z(\underline{r}_{\bar{s}})$ as in Fig. 6-2.

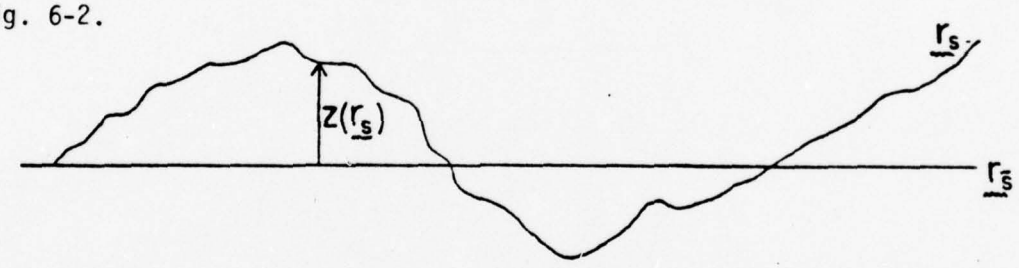


Fig. 6-2. The Average Surface, $\underline{r}_{\bar{s}}$

We assume the surface gentle enough to make this projection one to one so that the surface Jacobian, $J(\underline{r}_s,\underline{r}_{\bar{s}}) = dA(\underline{r}_s)/dA(\underline{r}_{\bar{s}})$, exists and has an inverse. Surfaces not so gentle (say, breaking waves) are often independent noise sources and so fall outside the context of Eq. (5.1) (though not of the general solution).

Many experiments have shown the process $\underline{z}$ to be centered on $\underline{r}_{\bar{s}}$ and normally distributed, as might be expected given the wind-generated gravitational nature of most ocean waves. Conversely, $\underline{r}_{\bar{s}}$ is an extremely slow function of time, mostly influenced by tides.

First, consider the incident wave,

$$p_u(\underline{r}_s) = p_u(\underline{r}_s + \underline{z}(\underline{r}_s))$$

$$= \int_{\underline{r}_0} \frac{s\left(\underline{r}_0, t_1 - \dfrac{|\underline{r}_s + \underline{z}(\underline{r}_s) - \underline{r}_0|}{c_0}\right) d\underline{r}_0}{4\pi|\underline{r}_s + z(\underline{r}_s) - \underline{r}_0|}, \qquad (6.13)$$

in the most arbitrary case (see Appendix F). At any appreciable distance from the source, $|\underline{r}_s + \underline{z}(\underline{r}_s) - \underline{r}_0| \approx |\underline{r}_s - \underline{r}_0|$, or

$$p_u(\underline{r}_s, t_1) = p_u(\underline{r}_s, t_1 + \tau_s) \qquad (6.14)$$

from which we get

$$\tau_s \approx \frac{|\underline{r}_s - \underline{r}_0| - |\underline{r}_s - \underline{r}_0| + \underline{z}(\underline{r}_s, t)}{c_0}, \qquad (6.15)$$

so that $\tau_s = \tau_s(\underline{r}_s, z(t))$. Rewriting, we have

$$p_u(\underline{r}_s, t_1) = \int_{\underline{r}_0} \frac{s\left(\underline{r}_0, t_1 + \tau_s(t) - \dfrac{|\underline{r}_s - \underline{r}_0|}{c_0}\right) dr_0}{4\pi|\underline{r}_s - \underline{r}_0|}. \qquad (6.16)$$

From the known statistics of the surface elevation z, one can calculate the statistics of τ_s and so for $p_u(\underline{r}_s)$ (the latter in terms of the mean surface).

The exact solution for τ_s can be found from trigonometry (given the homogeneous case considered here). From Fig. 6-3 we see that

$$\xi = \frac{z\sin\left(\frac{\pi}{2} - \theta_s + \frac{\phi}{2}\right)}{\sin\left(\frac{\pi}{2} + \frac{\phi}{2}\right)} = c_0\tau_s \qquad (6.17)$$

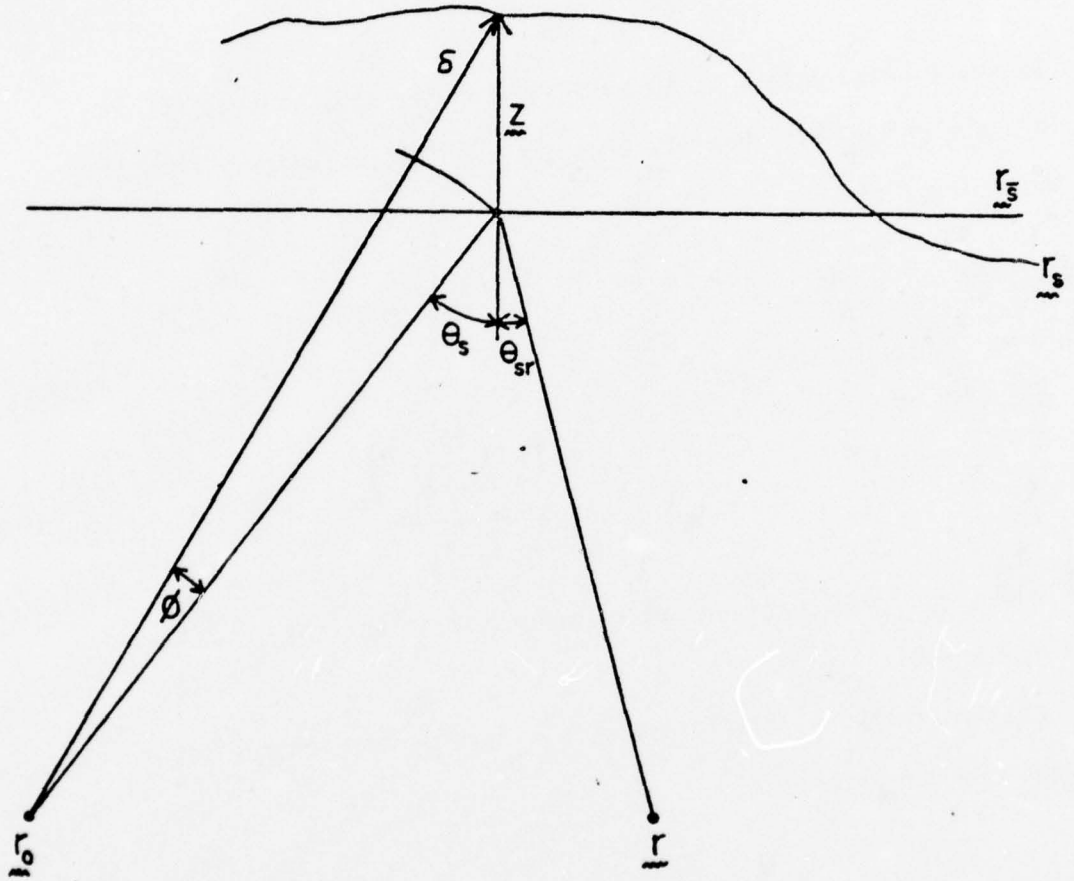


Fig. 6-3. Surface Delay Due to Source

Although this is essentially a nonlinear relationship in z (because $\emptyset = \emptyset(z)$, $\emptyset$ is usually so small ($\approx |z|/|\underline{r}_s - \underline{r}_0|$) it may, except for sources practically at the surface of monstrous seas, be safely neglected. We thus obtain

$$\tau_s(t) \doteq \frac{z(t)\cos\theta_s}{c_0}, \tag{6.18}$$

the estimate being always somewhat less than the true value. For Eq. (6.18), the probability density of τ_s, $f(\tau_s)$ follows quickly from $f(z)$. In particular, if

$$f(z) = \left(\sqrt{2\pi}\ \sigma_z\right)^{-1} \exp\left(-\frac{z^2}{2\sigma_z^2}\right), \tag{6.19}$$

then $\sigma_\tau = \sigma_z \cos\theta/c_0$ and

$$f(\tau_s) = \frac{c_0 \exp\left(\dfrac{-c_0^2\tau_s^2}{2\sigma_z^2\cos^2\theta_s}\right)}{\sqrt{2\pi}\ \sigma_z \cos\theta_s}, \tag{6.20}$$

using elementary manipulations with random variables ($\cos\theta_s/c_0$ is constant for a given $\underline{r}_s$).

Equation (6.18) similarly yields a simple expression for second moments. We can write

$$E(z_1 z_2) \equiv R_{z_1 z_2} = E\left(\frac{c_0^2 \tau_{s_1}\tau_{s_2}}{\cos\theta_{s_1}\ \cos\theta_{s_2}}\right), \quad E(z) = 0$$

$$= \frac{c_0^2 E\left(\tau_{s_1}\tau_{s_2}\right)}{\cos\theta_{s_1}\cos\theta_{s_2}}, \tag{6.21}$$

so that

$$R_{\tau_{s_1}\tau_{s_2}} = \frac{\cos\theta_{s_1}\ \cos\theta_{s_2}}{c_0^2} R_{z_1 z_2}. \tag{6.20}$$

If $R_{z_1 z_2}$ has a directional dependence (say $\exp-\underline{\alpha}\cdot|\underline{r}_{s_1} - \underline{r}_{s_2}|$), then the phase contribution from $p_u(\underline{r}_{s_1})p_u(\underline{r}_{s_0})$ will likewise depend on $\underline{\alpha}$.

Particular note should be made of the angle θ_s. For small grazing angles ($\theta_s \to \pi/2$), $\tau_s \to 0$ and $p_u(\underline{r}_s) \to p_u(\underline{r}_s)$ independent of the condition of the surface. This leads to a natural isolation of the "low grazing angle" geometry. But caution is advisable because a small grazing angle referred to the source may not imply a small grazing angle referred to the receiver.

Next consider from Eq. (6.4), the random process,

$$\underline{d}'_s\cdot\hat{n} = d'_s\hat{\rho}\cdot n(t). \tag{6.23}$$

The quantity d'_s is little affected by the random surface, (the local variation of $\hat{\rho}\cdot\hat{\rho}_1$ in Eq. (6.10) being very small), so our concern centers on $\hat{\rho}\cdot\hat{n}$. The unit vector $\hat{n}$ is a direct observable of the surface, with statistical distribution, $f(\hat{n}(\underline{r}_s))$. Statistically this corresponds to the distribution of slopes defined by the stochastic surface. For a flat surface,[1] the surface defines only one normal, $(\hat{n}=\hat{z})$, but as the surface grows more energetic, the distribution of slopes changes in a somewhat predictable way [105].

---

[1]For a flat surface $f(\hat{n}) = \delta(\hat{n}-\hat{z})$. Equation (6.4) becomes, for a unit harmonic point source,

$$p = p_u \; -jk_o \int_{\underline{r}_s} \left(\frac{\underline{d}_s \, e^{-jk_o(\rho_1+\rho)}}{16\pi^2\rho_1\rho} \right) d\underline{r}_s \quad e^{j\omega_0 t}$$

which shows the virtual image and phase reversal of an image plane.

Finally, we must deal with the operator

$$\hat{G}_s = \int_{\underline{r}_s} \int_{t_o} \int_{\omega_o} (\cdot) \frac{e^{j(\omega_0(t-t_0) - k_0|\underline{r}-\underline{r}_s|)}}{8\pi^2|\underline{r}-\underline{r}_s|} d\underline{r}_s dt_o d\omega_o \qquad (6.24)$$

The surfaces $\underline{r}_s$ and $\underline{r}_{\bar{s}}$ are related by the surface Jacobian J_s, so that we may reduce the operator Eq. (6.24) to an operator over the mean surface, $\hat{G}_{\bar{s}}$ by

$$\hat{G}_{\bar{s}} = \int_{r_{\bar{s}}} \int_{t_o} \int_{\omega_o} (\cdot) \frac{e^{j(\omega_0(t-t_0) - k_0|\underline{r} - \underline{r}_{\bar{s}} - \underline{z}|)}}{8\pi^2|\underline{r} - \underline{r}_{\bar{s}} - \underline{z}|} J_s d\underline{r}_{\bar{s}} dt_o d\omega_o \qquad (6.25)$$

$$= \hat{G}_s \, J_s e^{j\omega\tau_{sr}} \, . \qquad (6.26)$$

As before, Eq. (6.18), the exponent can be interpreted as a time delay

$$\tau_{sr} = \frac{z\cos\theta_r}{c_0} \, , \qquad (6.27)$$

with Gaussian distribution

$$f(\tau_{sr}) = \frac{c_0 \exp\left(\frac{-c_0^2\tau_{sr}^2}{2\sigma_z^2\cos^2\theta_r}\right)}{\sqrt{2\pi}\,\sigma_z\cos\theta_r} \, . \qquad (6.28)$$

The Jacobian, J_s, must equal or exceed unity. A suitable function is

$$J = -\frac{1}{\hat{z}\cdot\hat{n}} \qquad (6.29)$$

which gives J=1 for a horizontal surface and J=∞ for a vertical surface. This choice ignores cusps and other wave deformations which must be accounted for more carefully.

The transformation of Eq. (6.8) to the mean surface may thus be expressed under the above assumptions, (Eqs. (6.18), (6.19), (6.27), and (6.29)), as

$$p = p_u + \int_{\omega_0}\int_{t_0}\int_{\underline{r}_s} \frac{jk_o\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}}\, d_{\underline{s}}\, \frac{e^{j\omega_0\tau_{sr}}\, e^{j(\omega_0 t - k_o\rho)}}{8\pi^2\rho}\, p_u(\underline{r}_s, t_0+\tau_s)\, d\omega_0 dt_0 d\underline{r}_s \tag{6.30}$$

6.1.2 Locally and Globally Coupled Scattering

Before passing to moments, one should note that Eq. (6.30) contains all the interaction of the surface "globally."[1] The scattering orders covered by this global approach may be elucidated by developing the following definition of d_s instead of Eq. (6.3), namely

$$\nabla\cdot\left(d_{\underline{s}} \sum_{n=0}^{\infty} (\hat{G}'d')^{(n)}\, \hat{G}'(s)\frac{e^{-jk\rho}}{4\pi\rho}\right) \equiv \frac{e^{-jk_o\rho}}{4\pi\rho}\, d' \sum_{n=0} (\hat{G}'d')^{(n)}\hat{G}'(s). \tag{6.31}$$

Usually this approach is not desirable since only the compounded result is of direct interest. The actual physical coupling usually surpasses observation.

Our development gives the equivalent surface filter directly. If we put

$$p = \hat{G}(s) + \hat{H}_s(s) \tag{6.32}$$

[1]Private conversation; concept due D. Middleton.

where H_s defines the surface filtering effect, then from Eq. (6.30)

$$H_s = \int_{\omega_0} \int_{t_0} \int_{\underline{r}_s} -k_o^2 \, (\underline{d}_s \cdot \hat{n}) \frac{e^{-jk\rho}}{4\pi\rho} \, J\hat{G}_s(s) \, d\omega_0 \, dt_0 d\underline{r}_s \quad , \tag{6.33}$$

or

$$\boxed{H_s = \int_{\omega_0} \int_{t_0} \int_{\underline{r}_s} \frac{(\underline{d}_s) j k_o \hat{\rho} \cdot \hat{n}}{\hat{z} \cdot \hat{n}} \frac{e^{j\omega_0 \tau_{sr}} e^{j(\omega_0 t - k_o \rho)}}{8\pi^2 \rho} \, p_u(\underline{r}_s, t_0 + \tau_s) d\omega_0 d\underline{t}_0 d\underline{r}_s \, .} \tag{6.34}$$

The stochastic character of the filter is seen to depend on $\hat{n}$, τ_{sr}, and τ_s, as expected.

6.1.3 Solutions

Using Eq. (6.30) we may proceed directly to moments (which constitute one form of statistical solution). Accordingly, for the mean pressure we obtain

$$\langle p \rangle = p_u$$

$$+ \int_{\omega_0} \int_{t_0} \int_{\underline{r}_s} jk_o d_s \left\langle \frac{\hat{\rho} \cdot \hat{n}}{\hat{z} \cdot \hat{n}} e^{j\omega_0 \tau_{sr}} \, p_u(\underline{r}_s, t_0 - \tau_s) \right\rangle \frac{e^{j(\omega_0 t - k_o \rho)}}{8\pi^2 \rho} \, d\omega_0 dt_0 d\underline{r}_s \, . \tag{6.35}$$

For simplicity, suppose a point source. Then

$$p_u(\underline{r}_s) = \frac{e^{j(\omega_0(t+\tau_s) - k_o |\underline{r}_s - \underline{r}_o|)}}{8\pi^2 |\underline{r}_s - \underline{r}_o|} = \frac{e^{j\omega_0 \tau_s} e^{j(\omega_0 t - k_o |\underline{r}_s - \underline{r}_o|)}}{8\pi^2 |\underline{r}_s - \underline{r}_o|} \tag{6.36}$$

$$= e^{j\omega_0 \tau_s} p_u(\underline{r}_s) \, . \tag{6.37}$$

We see that a simple phase shift is induced whose statistics are given by Eq. (6.20). Now, from Eq. (6.35) we can write

$$p = p_u + \int_{\omega_0} \int_{t_0} \int_{\underline{r}_s} jk d\underline{s} \left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} e^{j\omega_0(\tau_{sr}+\tau_s)} \right\rangle p_u(\underline{r}_s) \frac{e^{j(\omega_0 t - k_0 \rho)}}{8\pi^2\rho} d\omega_0 dt_0 \, d\underline{r}_s, \tag{6.38}$$

or

$$\langle p \rangle = p_u + \hat{G}_{\underline{s}} \left[jk_0 d\underline{s} \left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} e^{j\omega_0(\tau_{sr}+\tau_s)} \right\rangle p_u(\underline{r}_s) \right], \tag{6.39}$$

where $\hat{G}_{\underline{s}}$ is defined as

$$\hat{G}_{\underline{s}} \equiv \int_{\omega_0} \int_{t_0} \int_{\underline{r}_s} (\cdot) \frac{\omega_0^2}{c_0^2} \frac{e^{j[\omega_0(t-t_0) - k_0|\underline{r}-\underline{r}_s|]}}{8\pi^2|\underline{r}-\underline{r}_s|} d\omega_0 dt_0 d\underline{r}_s. \tag{6.40}$$

If surface heights can be considered independent of slopes, i.e., $f(\hat{n},z) = f(\hat{n})f(z)$, we may put

$$\left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} e^{j\omega_0(\tau_{sr}+\tau_s)} \right\rangle = \left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} \right\rangle \left\langle e^{j\omega_0(\tau_{sr}+\tau_s)} \right\rangle \tag{6.41}$$

$$= \left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} \right\rangle \tag{6.42}$$

because

$$\tau_s + \tau_{sr} = \frac{z}{c_0}(\cos\theta_s + \cos\theta_{sr}), \tag{6.43}$$

and $\langle z \rangle = 0$. The angle, θ_{sr}, is the angle between the vectors $-\hat{t}$ and $\underline{r}-\underline{r}_s$ (see Fig. 6-3).

We thus obtain the following results for the mean pressure field,

$$\langle p \rangle = p_u + \int_{\omega_0} \int_{t_0} \int_{\underline{r}_s} jk_o d\underline{s} \left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} \right\rangle p_u(\underline{r}_s) \frac{e^{j(\omega_o t - k_o \rho)}}{8\pi^2 \rho} d\omega_0 dt_o d\underline{r}_s \quad , \tag{6.44}$$

$$= p_u + \hat{G}_{\underline{s}} \; jk_o d\underline{s} \left\langle \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} \right\rangle p_u(\underline{r}_s) \quad . \tag{6.45}$$

The result in Eq. (6.45) is given in terms of the distributions of surface normals. If, instead, we choose as observable the distribution of slopes, further insight may be achieved. Suppose at $\underline{r}_s$, we choose an orientation $\hat{x}$, $\hat{y}$, $\hat{z}$ as in Fig. 6-4.

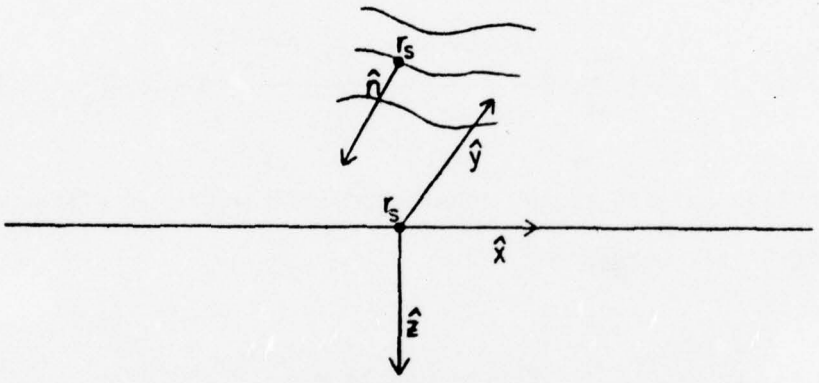


Fig. 6-4. Coordinate System for the Mean Surface

Then,

$$\hat{n} \equiv \frac{\partial \underline{r}_s}{\partial x} \otimes \frac{\partial \underline{r}_s}{\partial y} \tag{6.46}$$

where here $\otimes$ signifies the vector cross product, and

$$\frac{\partial \underline{r}_s}{\partial x} = \lim_{\Delta x \to 0} \left(\frac{\underline{r}(x+\Delta x) - \underline{r}(x)}{\Delta x} \right) = m_x \frac{\partial \underline{\hat{r}}}{\partial x}. \tag{6.47}$$

$$= m_x \quad a\hat{x} - b\hat{z}$$

$$= b(\hat{x} - m_x \hat{z}) \quad . \tag{6.48}$$

where $m = b/a$ for some b.

Likewise $\frac{\partial \underline{r}_y}{\partial y} = b'(\hat{y} - m_y \hat{z})$ for some b'. Then

$$\hat{n} = \frac{\partial \underline{r}_s}{\partial x} \otimes \frac{\partial \underline{r}_y}{\partial y} = \frac{\hat{x} \otimes \hat{y} - m_y \hat{x} \otimes \hat{z} - m_x \hat{y} \otimes \hat{z}}{|\hat{x} \otimes \hat{z} - m_y \hat{x} \otimes \hat{z} - m_x \hat{y} \otimes \hat{z}|} \tag{6.49}$$

or

$$\hat{n} = \frac{\hat{z} + m_x \hat{x} + m_y \hat{y}}{|\hat{z} + m_x \hat{x} + m_y \hat{y}|} \quad . \tag{6.50}$$

Now the ratio $\hat{\rho} \cdot \hat{n} / \hat{z} \cdot \hat{n}$ may be expressed as

$$\frac{\hat{\rho} \cdot \hat{n}}{\hat{z} \cdot \hat{n}} = \frac{\hat{\rho} \cdot (\hat{z} + m_x \hat{x} + m_y \hat{y})}{\hat{z} \cdot (\hat{z} + m_x \hat{x} + m_y \hat{y})} \tag{6.51}$$

$$= \hat{\rho} \cdot (\hat{z} + m_x \hat{x} + m_y \hat{y}) \quad . \tag{6.52}$$

For this rendition Eq. (6.34) becomes

$$p = p_u + \int\int\int_{\omega_0 t_0 \underline{r}_s} \frac{jk_o d_{\underline{s}} \hat{\rho}\cdot(\hat{z}+m_x\hat{x}+m_y\hat{y})}{8\pi^2\rho} e^{j\omega_o\left(\tau_s+\tau_{sr}+\tau-\frac{\rho}{c_o}\right)} p_u(\underline{r}_s) d\omega_o d\tau_o d\underline{r}_s \quad (6.53)$$

which has a first moment

$$\langle p\rangle = p_u + \hat{G}_{\underline{s}}\ jk_o d_{\underline{s}} \left[\hat{\rho}\cdot\hat{z} + \hat{\rho}\cdot\hat{x}\langle m_x\rangle + \hat{\rho}\cdot\hat{y}\langle m_y\rangle\right] p_u(\underline{r}_s) \ . \quad (6.54)$$

The term $\hat{\rho}\cdot\hat{z}$ is the specular term (being identical to a flat surface where $\hat{n} = \hat{z}$). The terms $m_x\ \hat{\rho}\cdot\hat{x}$ and $m_y\ \hat{\rho}\cdot\hat{y}$ account for the random fluctuations. Whenever $\langle m_x\rangle = \langle m_y\rangle = 0$ over the mean surface, then

$$\langle p\rangle = p_u + \hat{G}_{\underline{s}} \left[jk_o d_{\underline{s}}\hat{\rho}\cdot\hat{z}\right] p_u(\underline{r}_s) \ . \quad (6.55)$$

The second moment may be pursued in an analogous way. For the assumptions leading to Eq. (6.53) we can write

$$p_1 p_2 = \left(p_{u1} + \hat{G}_{\underline{s}1} jk_o (d_{\underline{s}})\hat{\rho}_1(\hat{z}+m_{x1}\hat{x}+m_{y1}\hat{y})\ \frac{e^{j\omega_o(\tau_{s1}+\tau_{sr1})}}{8\pi^2\rho_1}\ p_u(r_{s1})\right)$$

$$\cdot \left(p_{u2} + \hat{G}_{\underline{s}2} jk_o (d_{\underline{s}})\hat{\rho}_2(\hat{z}+m_{x2}\hat{x}+m_{y2}\hat{y})\ \frac{e^{j\omega_o(\tau_{s2}+\tau_{sr2})}}{8\pi^2\rho_2}\ p_u(\underline{r}_{s2})\right) \quad (6.56)$$

$$= p_{u1}p_{u2} + p_{u2}\hat{G}_{\underline{s}1} jk_o (d_{\underline{s}})\hat{\rho}_1(\hat{z}+m_{x1}\hat{x}+m_{y1}\hat{y})\ e^{j\omega_o(\tau_{s1}+\tau_{sr1})}\ p_u(\underline{r}_{s_1})$$

$$+ p_{u1}\,\hat{G}_{\bar{s}2}jk_o(d_{\bar{s}})\hat{\rho}_2\cdot(\hat{z}+m_{x2}\hat{x}+m_{y2}\hat{y})\,e^{j\omega_o(\tau_2+\tau_{sr2})}\,p_u\,(\underline{r}_{s2})$$

$$- \hat{G}_{\bar{s}1}\,\hat{G}_{\bar{s}2}k_1k_2(d_s)^2p_u(\underline{r}_{s1})p_u(\underline{r}_{s2})\,e^{j[\omega_1(\tau_{s1}+\tau_{sr1})+\omega_2(\tau_{s2}+\tau_{sr2})]}$$

$$\cdot[(\hat{\rho}_1\cdot\hat{z})\,(\hat{\rho}_2\cdot\hat{z}) + (\hat{\rho}_1\cdot\hat{z})(\hat{\rho}_2\cdot(m_{x2}\hat{x}+m_{y2}\hat{y})) + (\hat{\rho}_2\cdot\hat{z})\cdot(\hat{\rho}_1(m_{x1}\hat{x}+m_{y1}\hat{y}))$$

$$+ m_{x1}m_{x2}\,(\hat{\rho}_1\cdot\hat{x})(\hat{\rho}_2\cdot\hat{x}) + m_{x1}m_{y2}(\hat{\rho}_1\cdot\hat{x})(\hat{\rho}_2\cdot\hat{y})$$

$$+ m_{y1}\,m_{x1}(\hat{\rho}_1\cdot\hat{y})(\hat{\rho}_2\cdot\hat{x}) + m_{y1}m_{y2}(\hat{\rho}_1\cdot\hat{y})(\hat{\rho}_2\cdot\hat{y})] \quad . \tag{6.57}$$

If surface slopes are independent of surface height, and $\langle m_x\rangle = \langle m_y\rangle = 0$, then Eq. (6.57) reduces to

$$\langle p_1p_2\rangle = p_{u1}p_{u2} + p_{u2}\hat{G}_{\bar{s}1}jk(d_{\bar{s}})(\hat{\rho}_1\cdot\hat{z})p_u(\underline{r}_{\bar{s}1}) + p_{u1}\hat{G}_{\bar{s}2}jk(d_{\bar{s}})(\hat{\rho}_2\cdot\hat{z})p_u(\underline{r}_{\bar{s}2})$$

$$- \hat{G}_{\bar{s}1}\;\hat{G}_{\bar{s}2}\,(d_{\bar{s}})^2k_1k_2\,p_u(\underline{r}_{\bar{s}1})\,p_u\,(\underline{r}_{\bar{s}2})$$

$$\cdot[(\hat{\rho}_1\cdot\hat{z})(\hat{\rho}_2\cdot\hat{z}) + \langle m_{x1}m_{x2}\rangle(\rho_1\cdot\hat{x})(\hat{\rho}_2\cdot\hat{x}) + \langle m_{x1}m_{y2}\rangle(\hat{\rho}_1\cdot\hat{x})(\hat{\rho}_2\cdot\hat{y})$$

$$+ \langle m_{y1}m_{x2}\rangle(\hat{\rho}_1\cdot\hat{y})(\hat{\rho}_2\cdot\hat{x}) + \langle m_{y1}m_{y2}\rangle(\hat{\rho}_1\cdot\hat{y})(\hat{\rho}_2\cdot\hat{y})] \quad . \tag{6.58}$$

It often happens that slopes are correlated only over short distances, so that

$$\langle m_{x1}\,m_{x2}\rangle = f(|x_1 - x_2|). \tag{6.59}$$

For a completely uncorrelated surface, the correlation of the field, p, becomes

$$\langle p_1 p_2 \rangle = p_{u1} p_{u2}$$

$$+ p_{u2}\, \hat{G}_{s1}\, jk(\underline{d}_s)(\hat{\rho}_1 \cdot \hat{z})\, p_u(\underline{r}_{s1}) + p_{u1}\, \hat{G}_{s2}\, jk(\underline{d}_s)(\hat{\rho}_2 \cdot \hat{z})\, p_u(\underline{r}_{s2})$$

$$- \hat{G}_{s1}\, (\hat{G}_{s2}\, k_1 k_2\, (\underline{d}_s)^2\, p_u(\underline{r}_{s1})\, p_u(\underline{r}_{s2})\, [\hat{\rho}_1 \cdot \hat{z}][\hat{\rho}_2 \cdot \hat{z}] \quad . \qquad (6.60)$$

In this case the phase delay due to the surface statistics do not appear. If, however, the surface is correlated, then, in contrast to the first moment, (Eq. (6.55)), these delays appear in the result.

When $\langle m \rangle \neq 0$, as one might expect for a wind-driven sea, one finds surface influences even in $\langle p \rangle$.

If we now compare Eq. (6.53) with the classical formulation[1]

$$p_s \simeq \frac{jk_o B f(\theta)\, R\, e^{jk_o(\rho_1+\rho_2)}}{2\pi\rho_1\rho_2} \iint_{-\infty}^{\infty} D_o\, e^{2j(dx+\beta y+\gamma z)}\, dydx \quad ,$$

it can be seen that Eq. (6.53) is broadband (versus narrowband), is not restricted to a point source, uses an exact aperture (versus D_o), and a rather complicated aperture function (versus average values). Moreover, as mentioned earlier, the aperture function depends explicitly not only on the received pressure at the surface but also on atmospheric effects.

---

[1][194], Eq. (6.24).

6.1.4 Spectral Response

We have noted[1] that Eq. (5.1) is linear, so that a tone of frequency ω_0 will be received at the same frequency. Equation (6.30) shows, however, that the received pressure will be a stochastic process more or less random depending on the randomness of the surface. Consequently the received pressure will grow progressively decorrelated (ensemble-wise) from the source as the scattered wave, given by

$$p_s = \hat{G}_{\underline{s}} jk_o \frac{\hat{\rho}\cdot\hat{n}}{\hat{z}\cdot\hat{n}} d_{\underline{s}} e^{i\omega_o \tau_{sr}(t)} p_u(r_s, t_0+\tau_s(t)) \qquad (6.61)$$

dominates the unscattered wave, p_u.

This may be shown directly by taking the temporal Fourier transform $\hat{F}_t$ of Eq. (6.30) which gives

$$\hat{F}_t[p(\underline{r},t)] = p(\underline{r},\omega') = F_t[p_u] + \int_{\omega_0}\int_{t_0}\int_{\underline{r}_s} jk_o d_{\underline{s}} \frac{e^{-jk\rho}}{8\pi^2\rho}$$

$$\frac{\hat{\rho}\cdot\hat{n}(\omega')}{\hat{z}\cdot\hat{n}(\omega')} * p_u(\underline{r}_s, t_o+t_2(\omega')) * e^{j\omega_o \tau_{sr}(\omega')} * \delta(\omega'-\omega_o)\, d\omega_o\, dt_o\, dr_{\underline{s}}. \qquad (6.62)$$

The integral contains the random phase contributed by τ_{sr} and τ_s. Moreover, because $\hat{n}$ is random, these random phases are also randomly weighted. In narrowband radiation, the received spectrum will be broadened through the convolutions with resulting stochastic phase and amplitude. In broadband radiation, each spectral element will again be broadened, again with stochastic phase and amplitude.[2]

[1] See Sec. 5.0.

[2] For a comprehensive account of these effects, see [135], especially Eq. (56).

Now consider the first moment, namely Eq. (6.55)

$$\langle p \rangle = p_u + \hat{G}_s[jk_o d_s \hat{\rho}\cdot\hat{z}\ p_u(r_s)] \quad .$$

For narrowband radiation the mean result is the interference result between p_u and $p_s = \hat{G}_s[jk_o d_{\hat{\rho}}\cdot\hat{z}\ p_u(r_s)]$. In other words, the random phase and amplitudes will have zero means about the deterministic interference effect. Whenever $\langle m_x \rangle \neq 0 \neq \langle m_y \rangle$, as in Eq. (6.54), the situation changes because of an interference of altered amplitudes. In the more general case of Eq. (6.35), the mean result contains interference not only an altered amplitude but also of altered phase.

It will not always be possible to reformulate these effects analytically into random phase (i.e., $a=a_0 e^{j\emptyset}$) because the random magnitude of the interfering p_s must be accommodated.

For broadband radiation each spectral element, $\delta(\omega'-\omega_0)$, undergoes similar changes, except that the "interference" may take place with different frequencies due to path delays and the emitted signal. For the linear case of Eq. (5.1), then the medium induces a kind of "linear modulation." The detected signal, in addition, will contain the envelope frequencies, which may fall outside the transmitted band.

6.1.5 An Alternate Method

The characteristics of surface scattering described above are given in a somewhat different model[1] by Middleton, who has extended the

[1]To be published. The material for this section has been privately communicated to the author in conversation and notes by Middleton. Cf. also [136, 137].

earlier FOM solutions (Sec. 4.4) to include multiple scattering and model interactions by canonically modeling the local scattering mechanisms. Middleton begins with the equation

$$(\nabla^2 - \frac{1}{c_0}\frac{\partial^2}{\partial t^2} - \hat{g}_M)\, p = s_o \,, \tag{6.64}$$

which may be regarded as extending Eq. (5.1), (see Sec. 2.4). The scattering operator $\hat{g}_M$ is now defined to embody a large number of observed effects:

$$\hat{g}_M \equiv \int_Z dN(\underline{Z}|\underline{r}_s,t)\, \delta(\underline{r}-\underline{r}') \int_{\tau=\infty}^{\infty} h_M(t',t/r',Z_s) D_o(\underline{r}',t')(\cdot)r,t-t^{dt} \tag{6.65}$$

where $D_o=1$ for interfaces usually and becomes $\frac{d}{c_0^2}\frac{\partial^2}{\partial t^2}$, here, for volumes. Here Z signifies a radiation event space x spatial parameter variation, etc., ($\equiv \underline{Z}_R \times \underline{Z}_s$) ($Z_R$ is an active scatterer in the same region); h_M is a (linear) stochastic filter associated with each active scatterer (at $\underline{r}'$), and $dN(\underline{Z})$ is a random point process acting like counting functional (i.e., counts the number of differential scattering elements (discrete or continuous) which are activated in a physical region).

A key feature of this model is the resolution of the $dN(\underline{Z})$ into an independent hierarchical structure, by the following definitions.

$$dN = \langle dN^o \rangle_{Rad} + \sum_{k=1}^{\infty} dN^{(k)}, \quad dN^{(k)} \equiv dN^{(k)} - \langle dN^{(k)} \rangle_R \tag{6.66}$$

where these $dN^{(k)}$ represent statistically independent random point processes. Physically this decomposition may be thought of as decomposing the scattering event into a "continuum" $\langle dN^o \rangle$ and a set of discrete (single point, double points, triple point, etc.) coupled

elements which produce corresponding scattering events. Thus the scattering structure at the point of radiation may be said to be "anatomized."

The stochastic filter h_M is likewise given a definite form[1]

$$h_M = \gamma_0(\underline{Z}_s)\ \delta(\tau - t_d|\underline{Z}) \tag{6.67}$$

where γ_0 is the intrinsic "cross-section" or strength of the scatterer (which embodies both the local boundary condition and any associated statistics) and t_d is the dynamic random path delay, including doppler.[2]

With these canonical definitions and results (e.g., specific structures for γ_0, τ_d, etc.) a general canonical model has been developed which includes "classical" results (contained in k=0 (e.g., the $\langle dN^0 \rangle$), the FOM model (k=1), and higher order or multiple scatter effects, not given in most other models).

The solution to Eq. (6.64) proceeds by the usual iteration, to produce the expression

$$p = \sum_{n=0}^{\infty} (\hat{G}\ \hat{g}_M)^{(n)}\ [s_0]\ . \tag{6.68}$$

The correlation structure may now be immediately addressed because both $\hat{G}$ and $\hat{g}_M$ are integral operators.[3] It may readily be seen that the Middleton model will account for the so-called speckle effect (k=1) (which is readily observed with highly coherent propagation) whereas

---

[1]This form characterizes the so called basic point scatter model (BPSM).

[2]See [135].

[3]In contrast to $\hat{\xi} = \frac{d}{c\ell} \frac{\partial^2}{\partial t^2}$ of Eq. (5.1). Therefore individual terms of Eq. (6.68) will not generally equal the terms of Eq. (5.55).

the classical formulation does not.[1] Also one of the most important aspects of the model turns on its handling of interfaces, which are seen as specialized types of scatters on a point by point basis. Multiple scattering and the associated medium correlations are naturally modelled in the higher order terms of the series.

In addition and more specifically with this formulation, it is also possible to specify the statistics of $dN^{(k)}$, $dN^{(k)}{}_{k_M}$, and associated Green's functions, as well as to include surfaces, volume x interface interactions, etc. The new method also provides a physical interpretation of the formal Feynman diagram elements in the context of strong scattering, among many other new results.

There is (almost universally) in the literature[2] an identification of the terms in the series solution with quasi-physical phenomenon (e.g., the B(11) approximation with the mean single scattered wave). The complete identification of the series solution and the operational solution, shows, however, that this is not the case. The terms in the series are only "overshoot" terms naturally arising as the loop stabilizes, and are physically meaningful only to the extent they approach the stabilized value. From this point of view the Middleton model restructures the usual series to bring the more meaningful physical events into the lower orders.

6.2 Volume Reverberation (An example)

We now turn to an example of volume scattering which attempts to account for losses. In addition to extending our earlier solutions, it

---

[1] See note to Eq. (6.8).

[2] See [55], p. 122.

is especially our purpose here to demonstrate how the methods developed in this Thesis provide insight into problems of this type.

T.A. Moroz [146] has published an extension of Eq. (4.35) which we use for illustrating volume reverberation. Her wave equation is

$$\left(\nabla^2 - \frac{\gamma}{c_0}\frac{\partial}{\partial t} - \frac{(d+1)}{c_0^2}\frac{\partial^2}{\partial t^2}\right) p = 0 \qquad (6.69)$$

where γ, the absorption per unit of path, has dimensions of inverse length. Its inclusion is meant to model a relaxation phenomenon. Under the assumption of small correlation distances, in a statistically Gaussian homogeneous and isotropic medium, for a point source radiator, Moroz obtains

$$\langle p(\underline{r},\omega)\rangle = \frac{1}{4\pi|\underline{r} - \underline{r}_0|} \exp\left(j\frac{\omega_0}{c_0}\langle\sqrt{d_e(\omega)}\rangle - \gamma\right)|\underline{r} - \underline{r}_0| \ , \qquad (6.70)$$

where d_e, the effective complex refractive index is given by

$$d_e = \sqrt{1 + \langle d\rangle} + \frac{\omega^2}{2\sqrt{1+\langle d\rangle}c^2}\int_0^\infty \tau B(\tau)d\tau + \frac{j\omega^3}{2c^3}\int_0^\infty \tau^2 B(\tau)d\tau \qquad (6.71)$$

where now $B(\tau)$ is the temporally varying spatial correlation. The terms in $\langle d_e\rangle$ describe a phase aberration and an attenuation loss due to the scattering (distinguishable from absorption loss due to γ). Equation (6.70) is, in fact, an approximation[1] to Eq. (4.35), excepting the factor γ. The Moroz treatmen t presumes γ constant in space and time.

Let us attempt to reformulate the general equation into Eq. (6.74) below. In the general equation, Eq. (1.28), if $\underline{b}_3$ is taken with

---

[1]See [54], p. 140.

∇p, it becomes a pure loss term. We thus have, neglecting other terms,

$$\frac{d^2}{dt^2}(p|\underline{r}) + c^2\nabla\cdot(-\nabla p - \underline{b}_3) = 0 , \tag{6.72}$$

or

$$\nabla^2 p + \nabla\cdot\underline{b}_3 - \frac{1}{c^2}\frac{d^2}{dt^2}(p|\underline{r}) = 0 . \tag{6.73}$$

Writing the hydrodynamic derivative as a partial, and using the relationship $c_0^2 \equiv c^2(1 + d)$, we have

$$\nabla^2 p + \nabla\cdot\underline{b}_3 - \frac{(1+d)}{c_0^2}\frac{\partial^2}{\partial t^2} p = 0 , \tag{6.74}$$

giving, by comparison with Eq. (6.69)

$$-\gamma \frac{\partial p}{\partial t} = c_0\nabla\cdot\underline{b}_3 . \tag{6.75}$$

For $\nabla\cdot b_3 \equiv -a\frac{\partial p}{\partial t}$ for some a, that is, a frequency dependent loss, we have

$$\gamma = c_0 a . \tag{6.76}$$

The solution in this case is given by

$$p = \sum_{n=0}^{\infty} (\hat{G}[-k_0^2 d + ja\omega_0])^{(n)}\hat{G}(s) \tag{6.77}$$

and

$$\langle p\rangle = \hat{G}(s) + \hat{G}Q\langle p\rangle \tag{6.78}$$

where $\hat{G}$ is given by Eq. (5.50) and

$$Q\langle p\rangle = \langle d \frac{\partial^2}{\partial t^2} p + a \frac{\partial p}{\partial t} \rangle = \langle d \frac{\partial^2}{\partial t^2} p\rangle + \langle a \frac{\partial p}{\partial t} \rangle \tag{6.79}$$

$$= Q_1\langle p\rangle + Q_2\langle p\rangle = [Q_1 + Q_2]\langle p\rangle \ . \tag{6.80}$$

The solution is thus given in terms of two first moment equivalent media, one for scattering and the other for absorption. These two media interact, i.e.,

$$\langle p\rangle = \sum_{n=0}^{\infty} (\hat{G}Q_1 + \hat{G}Q_2)^{(n)} p_u \tag{6.81}$$

This interaction may be seen by rewriting Eq. (6.81) as

$$[1 - \hat{G}Q_2]\langle p\rangle = p_u + \hat{G}Q_1\langle p\rangle \ . \tag{6.82}$$

The RHS of Eq. (6.82) has a solution, namely[1]

$$\langle p\rangle_s \equiv p_u + \hat{G}Q_1\langle p\rangle = \frac{1}{4\pi|\underline{r} - \underline{r}_0|} \exp\left(j \frac{\omega}{c_0} \langle d_e(\omega)\rangle|\underline{r} - \underline{r}_0|\right), \tag{6.83}$$

which, under the conditions stipulated, is the unattenuated solution of Bourret and Tatarskii, namely an approximation of Eq. (4.35). The solution of Eq. (6.82) is thus given by

$$\langle p\rangle = \sum_{n=0}^{\infty} (\hat{G}Q_2)^{(n)} \langle p_s\rangle \ . \tag{6.84}$$

The Moroz solution,

---

[1]See Eq. (6.70) for $\gamma = 0$.

$$\langle p \rangle_M = e^{-\gamma |\underline{r}-\underline{r}_0|} \langle p_s \rangle \tag{6.85}$$

can thus be seen as an independent and uncoupled approximation of Eq. (6.84), i.e., consists of the following approximations:

$$\langle p \rangle = \sum_{n=0}^{\infty} (GQ_2)^{(n)} \langle p \rangle_s \quad , \tag{6.84}$$

$$\simeq \langle p \rangle_s \sum (GQ_2)^{(n)}, \qquad \text{(approximation 1)}$$

$$\simeq \langle p \rangle_s \, e^{-\gamma |\underline{r}-\underline{r}_0|} = \langle p \rangle_M \, , \qquad \text{(approximation 2).} \tag{6.85}$$

Equation (6.84) on the other hand, applies to situations where the loss is random or coupled to the scattering. Operationally, Eq. (6.78) has, for solution the process of Fig. 6-5.

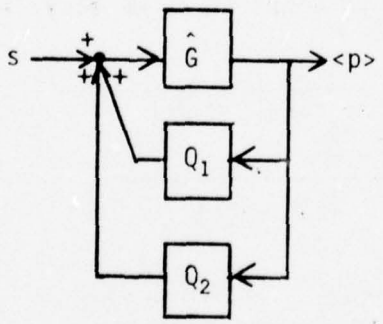


Fig. 6-5. Operational Solution with Absorption, Eq. (6.78)

The Moroz solution, in contrast, is given schematically in Fig. 6-6.

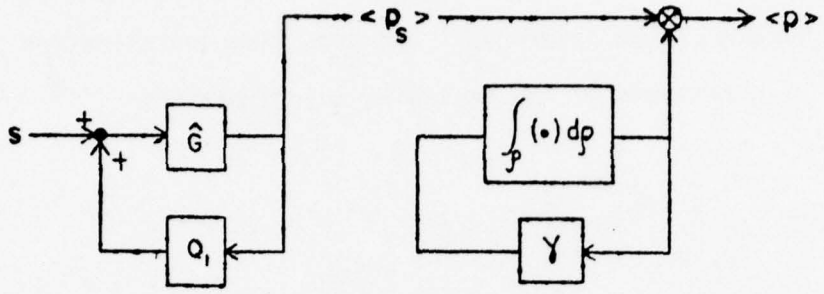


Fig. 6-6. Operational Solution of Moroz Formula, Eq. (6.85)

Using a Taylor Series expansion[1] ($[1 - \hat{G}Q_2]\langle p \rangle \simeq \langle p \rangle [1 - \hat{G}Q_2]$), we can obtain an approximate solution of Eq. (6.84) in the usual manner. Thus we write

$$\langle p \rangle \simeq \frac{\langle p_s \rangle}{1 - \hat{G}Q_2} \quad . \tag{6.86}$$

For the harmonic case, with constant a, we have, therefore

$$\langle p(\underline{r}, \omega_o) \rangle \quad \frac{\langle p \rangle_s}{1 - \int_{r_o} \frac{j\omega_o a e^{-jk_o|\underline{r} - \underline{r}_o|}}{4\pi|\underline{r} - \underline{r}_o|} d\underline{r}_o} \tag{6.87}$$

$$= \frac{\langle p \rangle_s}{1 - j\omega_o a \int_{r_o} \frac{e^{-jk_o|\underline{r} - \underline{r}_o|}}{4\pi|\underline{r} - \underline{r}_o|} d\underline{r}_o} \tag{6.88}$$

[1]See Chap. V, Sec. 5.3.2.1.

$$= \frac{k_0(k_0 + j\gamma)}{k_0^2 + \gamma^2} \langle p_s \rangle \quad . \tag{6.89}$$

The step from Eq. (6.88) to Eq. (6.89) is instructive and relevant to class IV errors. The integral

$$I = \int_r \frac{e^{-jk_0\rho}}{4\pi\rho} \, d\underline{r} \tag{6.90}$$

is in the ordinary sense unstable. This can be seen by transforming to spherical coordinates where

$$\int_\rho \int_\theta \frac{e^{-jk_0\rho}}{4\pi\rho} \, d\underline{r} = \int_\rho \frac{4\pi\rho^2 e^{-jk_0\rho}}{4\pi\rho} \, d\rho \ ,$$

so that

$$I = \int_{=0}^{\infty} e^{-jk} \quad d$$

This happens because the assumed function which is included in the kernel of I is the constah t function f=1, which does not satisfy a radiation condition.[1]

The problem, however, must physically obey the radiation condition, so that we are entitled to reimpose such a condition where it has been removed artificially by the Taylor's approximation. In Eq. (6.91),

[1]See also App. H for an explanation of these effects.

the integral may be evaluated directly.[1]

$$I = \frac{1}{(-jk_0)^2} e^{-jk_0\rho}[-jk_0\rho-1]\Big|_0^\infty = \frac{e^{-jk_0\rho}}{k_0^2}[jk_0\rho+1]\Big|_0$$

$$= -\frac{1}{k_0^2} + \lim_{\rho\to\infty} \frac{e^{-jk_0\rho}}{k_0^2}[jk_0\rho+1] = -\frac{1}{k_0^2}\ , \qquad (6.92)$$

upon imposing the radiation condition. It can be seen that the limit is indeterminate (unstable). Introducing the result of Eq. (6.92) into Eq. (6.88) gives

$$\langle p\rangle = \frac{\langle p_s\rangle}{1-\frac{j\omega_0 a}{k_0^2}} = \frac{k_0\langle p_s\rangle}{k_0-j\gamma} \qquad (6.93)$$

remembering that $a = \gamma/c_0$. Equation (6.89) follows immediately.

---

[1]Indirectly $\hat{F}_r\,(\nabla^2+k_0^2)\,\frac{e^{-jk_0\rho}}{4\pi\rho} = 1 = (-k^2+k_0^2)\int_{\underline{r}} \frac{e^{-jk_0\rho}}{4\pi\rho} e^{j\underline{k}\cdot\underline{r}}\,d\underline{r}$ from Eq. (5.36). The result follows for $\underline{k}=0$. Alternatively one may use a convergence factor $e^{-a\rho}$, where $I = \lim_{a\to 0} \frac{e^{-(jk_0+a)\rho}}{(jk_0+a)^2}(-jk_0-a-1) =$

$$-\frac{1}{k_0^2} + \lim_{a\to 0}\lim_{\rho\to 0} \frac{e^{-jk_0\rho}e^{-a\rho}}{(jk_0+aj)^2}(-jk_0-a-1) \to -\frac{1}{k_0^2}\ .$$

The solution, Eq. (6.89) reduces to $\langle p_s \rangle$ whenever $\gamma=0$. For a fixed γ, however, the solution is physically meaningless as k_o approaches zero (no propagation at low frequencies) or as k_o approaches infinity (no attenuation at high frequencies). Indeed such a conclusion may be reached directly from Eq. (6.69) by comparing the relative strengths of γ versus $\frac{1}{c_o}\frac{\partial}{\partial t} \rightarrow \omega_o/c_o = k_o$. We are thus led to conclude that the initial problem statement of Eq. (6.69) is incorrect, or that γ must be specified for each frequency.

We have seen[1] that the attenuation modeled by Eq. (6.69) is not the most general. The Moroz formulation, as can be seen from Eq. (6.69) is a temporal relaxation, which we have found in Chap. II to be usually a small effect.

A radiation relaxation for the stipulated conditions may be similarly modeled by setting

$$\underline{b} \equiv a \frac{\partial}{\partial t} \nabla p \tag{6.94}$$

from Eq. (2.35) giving the following differential equations

$$(1 + a \frac{\partial}{\partial t})\nabla^2 p - \frac{(1+d)}{c_o^2} \frac{\partial^2}{\partial t^2} p = 0 . \tag{6.95}$$

The identical procedure followed above gives

$$\langle p(\underline{r},\omega) \rangle = \frac{1 + ja\omega_o}{1 + a^2\omega_o^2} \langle p_s \rangle \quad . \tag{6.96}$$

Equation (6.96) shows a second power dependence on frequency and exhibits a low pass characteristic. As we have noted earlier, attenuation

---

[1]See Sec. 2.5.

is a physical process dependent on the material properties (chemical, etc.) of the medium. The basic general equations can give no more than mechanical insight, which for the ocean points mainly to a radiative relaxation.

6.3 Surface x Volume

The literature of scattering in random media has listed reasonable solutions for volume scattering and surface scattering separately.[1] Most experimental evidence indicates the received effect is largely the simple sum of both effects. It is physically intuitive, however, that some coupled scattering occurs. We propose, in this section, to obtain an estimate for this effect.

The theory advanced to this point permits a frontal attack on interactive coupling of interface and volume scattering (S x V). Problems of this kind we may expect to be more complicated than either surface or volume scattering separately.

Consider, for example, an inhomogeneous, but deterministic ocean, i.e., one in which $d_v(\underline{r},t) \neq 0$ is known. The case of pure refraction provides an illustration. Here, the simple trigonometric formulas of Sec. 6.1 no longer hold, or rather must be modified by the refraction. Furthermore, the refractive medium allows surface coupling of various orders over widely separated areas of the surface. So even when no random scattering occurs in the medium, the solutions may grow much more complicated -- so much so that purely analytical solutions may be impossible. For the case of simple refraction, one might come to

---

[1]As indicated by the references.

solutions using ray theory combined with the surface scattering of Sec. 6.1 suitably modified.[1] As a generalization, we can say that these solutions depend even more heavily on the particular geometry.

In the context of Eq. (5.1) problems dealing with refraction can be stated simply. Here

$$d = d_s + d_v ,$$

where d_v is deterministic and related to the index of refraction. In particular, for $c^2 \equiv c_0^2(1+\varepsilon)$, $\varepsilon(r,t) = -d(1+d)^{-1}$. From its definition $c^2(1+d) \equiv c_0^2$, d is a real variable with range $(-1,\infty)$. This gives ε a range of $(-\infty,\infty)$. The range is influenced by the somewhat arbitrary choice of c_0. For purely horizontal profiles a typical situation is shown in Fig. 6-7.

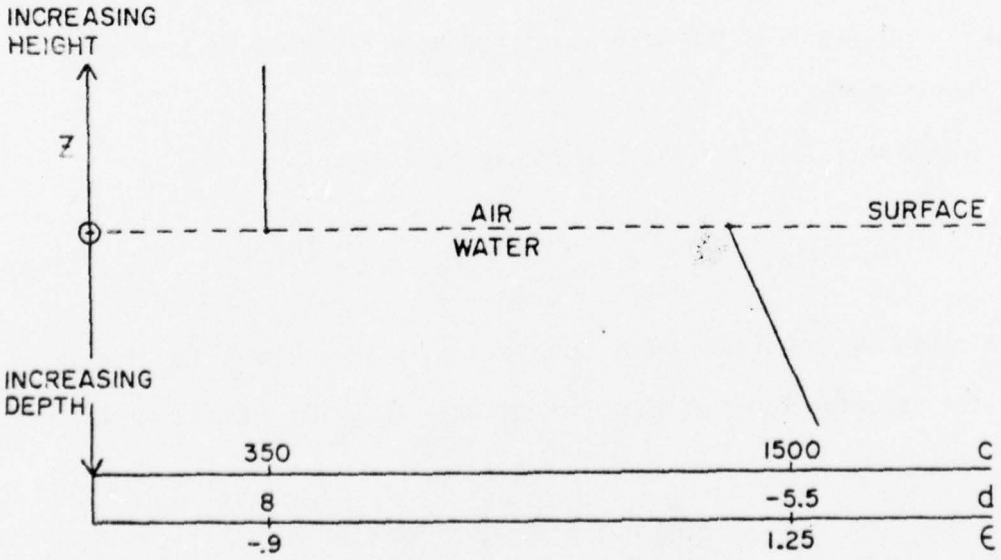


Fig. 6-7. Typical c, d, ε for Ocean and Atmosphere (c_0 = 1000)

[1]Also Cf. [138], Parts III, IV.

The large discontinuity at the surface produces the surface effect described in Sec. 6.1.

The solution to deterministic refraction is formally given by

$$p = p_u + \hat{G}\,\frac{-\varepsilon}{1+\varepsilon}\sum_{n=0}^{\infty}\hat{G}'\,\frac{-\varepsilon}{1+\varepsilon}^{(n)}\hat{G}'(s)\,, \qquad (6.97)$$

where the integrals are taken over the whole unbounded volume.

In this section, we pursue more closely the general problem of random scattering over a scattering surface, a subject begun in Sec. 5.3.1. The context of the solutions remains Eq. (5.1); following the literature the solutions are based on Taylor's approximation.

6.3.1 Solutions

Solutions for the stochastic case are given in terms of moments. We shall advance from the more exact operational forms to less exact but closed forms.

Equation (5.55) for the S x V case is given by

$$p = \hat{G}(s) + \hat{G}(d_s + d_v)\sum_{n=0}^{\infty}(\hat{G}'(d_s + d_v))^{(n)}\hat{G}'(s) \qquad (6.98)$$

which contains the interaction terms formally in the term $(d_s + d_v)^n$.

The solution for the first moment of p is given operationally in Fig. 6-8.

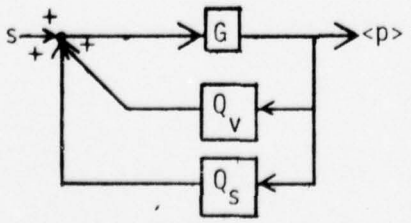


Fig. 6-8. First Moment Solution of S x V

where Q_s is the equivalent upper medium and Q_v is the equivalent lower medium.

To obtain closed solutions, we use the approximations B(11), B(21) and B(22) of Sec. 5.3. For the B(11) approximation we have

$$\langle p \rangle_{B(11)} = p_u + \hat{G}\langle (d_s + d_v) \rangle p_u = \langle p_s \rangle_{B(11)} + \langle p_v \rangle_{B(11)} - p_u \,, \qquad (6.99)$$

where no interactions occur. The second moment is

$$\begin{aligned}
\langle p_1 p_2 \rangle_{B(11)} &= - p_{u_1} p_{u_2} + p_{u_1} \langle p_2 \rangle + p_{u_2} \langle p_1 \rangle \\
&\quad + \hat{G}_1' \left(\hat{G}_2' \langle (d_{s_1} + d_{v_1})(d_{s_2} + d_{v_2}) \rangle \, p_{u_1} p_{u_2} \right) \\
&= p_{u_1} p_{u_2} + \langle p_{s_1} p_{s_2} \rangle + \langle p_{v_1} p_{v_2} \rangle + \langle p_{s_1} p_{v_2} \rangle + \langle p_{v_1} p_{s_2} \rangle \\
&\quad - p_{u_1} (\langle p_{s_2} \rangle + \langle p_{v_2} \rangle) - p_{u_2} (\langle p_{s_1} \rangle + \langle p_{v_1} \rangle) \\
&= -p_{u_1} p_{u_2} + \langle p_{s_1} p_{s_2} \rangle + \langle p_{v_1} p_{v_2} \rangle \\
&\quad + \left(\hat{G}_1' \langle d_{v_1} \rangle p_{u_1} \right) \left(\hat{G}_2' \langle d_{s_2} \rangle p_{u_2} \right) \\
&\quad + \left(\hat{G}_2' \langle d_{v_2} \rangle p_{u_2} \right) \left(\hat{G}_1' \langle d_{s_1} \rangle p_{u_1} \right) . \qquad (6.100)
\end{aligned}$$

Each item in Eqs. (6.99) and (6.100) can be separately obtained from methods used in Sections 6.1 and 6.2.

For B(21), Eq. (5.105), we have for the mean

$$\langle p \rangle_{B(21)} = \frac{p_u}{1 - \hat{G}' \langle d_s + d_v \rangle} = \frac{p_u}{(1 - \hat{G}' \langle d_s \rangle) - \hat{G}' \langle d_v \rangle} \,, \qquad (6.101)$$

so that the volume and surface interact. The second moment is

$$\langle p_1 p_2 \rangle_{B(21)} = \frac{-p_{u_1} p_{u_2} + p_{u_1} \langle p_2 \rangle + p_{u_2} \langle p_1 \rangle}{1 - \hat{G}_1^{\prime}(\hat{G}_2^{\prime}(\langle d_{s_1} \rangle + \langle d_{v_1} \rangle)(\langle d_{s_1} \rangle + \langle d_{v_2} \rangle))} \tag{6.102}$$

where we have assumed again independence of scattering classes. Separate items of Eqs. (6.101) and (6.102) may be found from earlier considerations.

For B(22), Eq. (5.132), we have

$$\langle p \rangle_{B(22)} = \frac{\langle p \rangle_{B(11)}}{1 - \hat{G}_1^{\prime} \hat{G}_2^{\prime} \langle (d_{s_1} + d_{v_1})(d_{s_2} + d_{v_2}) \rangle} ,$$

$$= \frac{\langle p \rangle_{B(11)}}{1 - \hat{G}_1^{\prime} \hat{G}_2^{\prime} \langle d_{s_1} d_{s_2} \rangle - \hat{G}_1^{\prime} \hat{G}_2^{\prime} \langle d_{v_1} d_{v_2} \rangle - 2\hat{G}_1^{\prime} \hat{G}_2^{\prime} \langle d_s \rangle \langle d_v \rangle} , \tag{6.103}$$

so that interaction is again evident in the mean. The second moment from Eq. (5.135) is

$$\langle p_1 p_2 \rangle_{B(22)} = \frac{- p_{u_1} p_{u_2} + p_{u_1} \langle p_2 \rangle + p_{u_2} \langle p_1 \rangle}{1 - \hat{G}_1^{\prime}(\hat{G}_2^{\prime} \langle d_{s_1} d_{s_2} \rangle) - \hat{G}_1^{\prime}(\hat{G}_2^{\prime} \langle d_{v_1} d_{v_2} \rangle) - 2\hat{G}_1^{\prime}(\hat{G}_2^{\prime} \langle d_s \rangle \langle d_v \rangle)} \tag{6.104}$$

where we have once more assumed independence of scattering classes.

In addition to these "pure" forms, mixtures are possible. One may have, for instance, a B(11) solution for the volume and a B(21) solution for the surface, etc. The mixtures provide additional solutions and demonstrate how the medium influences not merely the numerical result but also the form of the solution.

As we shall see in Chapter VII, this is analogous to using different communication channels.[1]

Equations (6.95) to (6.104) show how the result may be considered in terms of scattering classes each of which may be considered separately in the manner of Sections 6.1 and 6.2.

6.3.2 Criteria

One purpose of this Thesis is to establish the relative effects of cross-coupled scattering. The preceding section now allows this to be done formally. The reference, for the mean, is approximation B(11), Eq. (6.99), in which no cross-coupling is evident. We thus define

$$m_1 \equiv \frac{\langle p\rangle - \langle p\rangle_{B(11)}}{\langle p\rangle} = 1 - \frac{\langle p\rangle_{B(11)}}{\langle p\rangle} \quad . \tag{6.105}$$

The parameter, m_1, is thus the relative contribution of the cross-coupled scattering in the mean.

From its definition, it is clear that m_1 depends on the exact mean, and not on its approximations. Thus m_1 may be established directly from experiment.

For this Thesis, however, we make do with analytic approximations to $\langle p\rangle$. For B(11), $m_1=0$. For B(21), allowing the usual Taylor's approximation on $\hat{G}'(dp_u)$, gives

$$m_1(B(21)) = 1 - \frac{p_u(1+\hat{G}'\langle d\rangle)}{p_u/(1-\hat{G}'\langle d\rangle)} = 1-(1-(\hat{G}'\langle d\rangle^2)) = [\hat{G}'(\langle d_s\rangle+\langle d_v\rangle)]^2. \tag{6.106}$$

[1] See Sec. 7.4.

When $\langle d_v \rangle = 0$, $m_1 = [\hat{G}'\langle d_s \rangle]^2$, clearly a second order effect for surface scattering much smaller than the scattered wave.

For B(22), we have

$$m_1(B(22)) = \hat{G}_1'\hat{G}_2'\langle d_{s_1} d_{s_2} \rangle + \hat{G}_1'\hat{G}_2'\langle d_{v_1} d_{v_2} \rangle + 2\hat{G}_1'\hat{G}_2'\langle d_{s_1} \rangle \langle d_{v_2} \rangle \tag{6.107}$$

as may be seen from Eq. (6.103). When $\langle d_v \rangle = 0$, $m_1(B(22))$ still depends on the correlation of both surface and volume scattering. Equations (6.106) and (6.107) provide simple criteria for the effect of cross-scattering in the mean.

Even for B(11), the second moment contains evidence of interaction. If we suppress the interaction, the reference becomes (from Eq. (6.100)),

$$\begin{aligned} \langle p_1 p_2 \rangle_{ref} &\equiv - p_{u_1} p_{u_2} + p_{u_1} \langle p_2 \rangle_{B(11)} + p_{u_2} \langle p_1 \rangle_{B(11)} \\ &+ \hat{G}_1'(\hat{G}_2'(\langle d_{s_1} d_{s_2} \rangle + \langle d_{v_1} d_{v_2} \rangle) p_{u_1} p_{u_2}) \\ &= p_{u_1} p_{u_2} + \hat{G}_1'\langle d_s + d_v \rangle p_{u_1} + \hat{G}_2'\langle d_s + d_v \rangle p_{u_2} \\ &+ \hat{G}_1'(\hat{G}_2'(\langle d_{s_1} d_{s_2} \rangle + \langle d_{v_1} d_{v_2} \rangle) p_{u_1} p_{u_2}) \ . \end{aligned} \tag{6.108}$$

Analogously to Eq. (6.105), define

$$m_2 \equiv \frac{\langle p_1 p_2 \rangle - \langle p_1 p_2 \rangle_{ref}}{\langle p_1 p_2 \rangle} \ . \tag{6.109}$$

For B(11), using Eq. (6.100) we get

$$m_2(B(11)) = \hat{G}_1'(\hat{G}_2'(\langle d_{s_1} d_{v_2}\rangle + \langle d_{v_1} d_{s_2}\rangle)p_{u_1}p_{u_2})$$

$$\left[-p_{u_1}p_{u_2} + p_{u_2}\langle p_1\rangle + p_{u_1}\langle p_2\rangle + \hat{G}_1'(\hat{G}_2'(\langle d_{s_1} d_{s_2}\rangle\right.$$

$$\left. + \langle d_{v_1} d_{v_2}\rangle + \langle d_{s_1} d_{v_2}\rangle + \langle d_{s_2} d_{v_1}\rangle)p_{u_1}p_{u_2})\right]^{-1} . \quad (6.110)$$

Thus, one may safely neglect the cross-coupled scattering in the second moment provided $\langle p\rangle = \langle p\rangle_{B(11)}$ and $\langle d_s\, d_v\rangle = \langle d_s\rangle\langle d_v\rangle = 0$. This may often be the case with weak volume scattering.

For B(21), we get, from Eq. (6.102)

$$m_2(B(21)) = \left[p_u(\langle p_2\rangle_{B(21)} - \langle p_2\rangle_{B(11)}) + p_{u_2}(\langle p_1\rangle_{B(21)} - \langle p_1\rangle_{B(11)})\right.$$

$$- \hat{G}_1'(\hat{G}_2(\langle d_{s_1} d_{v_2}\rangle + \langle d_{s_2} d_{v_1}\rangle))p_{u_1}p_{u_2}$$

$$+ \left(-p_{u_1}p_{u_2} + p_{u_2}\langle p_1\rangle_{B(11)} + p_{u_2}\langle p_1\rangle_{B(11)}\right.$$

$$\left.\left. + \hat{G}_1'(\hat{G}_2'(\langle d_{s_1} d_{v_2}\rangle + \langle d_{s_2} d_{v_1}\rangle)p_{u_1}p_{u_2})\right)\left(\hat{G}_1'(\hat{G}_2'\langle d_1\rangle\langle d_2\rangle)\right)\right]$$

$$\cdot \left[-p_{u_1}p_{u_2} + p_{u_1}\langle p_2\rangle + p_{u_2}\langle p_1\rangle\right]^{-1} . \quad (6.111)$$

For B(22) we likewise find that

$$m_2(B(22)) = \left[p_{u_1}(\langle p_2\rangle_{B(22)} - \langle p_2\rangle_{B(11)}) + p_{u_2}(\langle p_1\rangle_{B(22)} - \langle p_1\rangle_{B(11)})\right.$$

$$- \hat{G}_1^{\dagger}(\hat{G}_2^{\dagger}(\langle d_{s_1} d_{v_2}\rangle + \langle d_{s_2} d_{v_1}\rangle)p_{u_1}p_{u_2})$$

$$+ \; -p_{u_1}p_{u_2} + p_{u_1}\langle p_2\rangle_{B(11)} + p_{u_2}\langle p_1\rangle_{B(11)}$$

$$+ \; \hat{G}_1^{\dagger}(\hat{G}_2^{\dagger}(\langle d_{s_1} d_{v_2}\rangle + \langle d_{s_2} d_{v_1}\rangle)p_{u_1}p_{u_2})$$

$$\cdot \; \hat{G}_1^{\dagger}(\hat{G}_2^{\dagger}\langle (d_{s_1}+d_{v_1})\,(d_{s_2}+d_{v_2})\rangle)\Big]$$

$$\cdot \; [-p_{u_1}p_{u_2} + p_{u_1}\langle p_2\rangle + p_{u_2}\langle p_1\rangle]^{-1} . \qquad (6.112)$$

While Eqs. (6.110) to (6.112) provide complicated criteria, the resulting formulas clearly contain the assumptions governing the approximation. So as was to be expected, the "correct" assumption depends entirely on the medium. Thus, the question: "Under what conditions can cross-coupling between scattering classes be ignored?" presupposes an anterior question: "Which approximation is valid?" For B(11) propagation, interaction does not appear in the first moment, but may be significant in the second moment according to Eq. (6.110). For B(21) and B(22) propagation, interaction appears in both first and second moments, and is important according to Eqs. (6.106), (6.107), (6.111) and (6.112).

6.3.3 The Global Filter

The argument of Section 6.1.2 can be used to define a global surface and a global volume filter, i.e.,

$$\hat{H}_s = \frac{d_s}{1 - \hat{G}'d_s} \tag{6.113}$$

and

$$\hat{H}_v = \frac{d_v}{1 - \hat{G}'d_v} \quad . \tag{6.114}$$

The global filter for the entire coupled propagation can be modelled similarly as

$$\hat{H}_{S \times V} = \frac{d_s + d_v}{1 - \hat{G}'d_s - \hat{G}'d_v} \tag{6.115}$$

$$= \frac{d_s}{1 - \hat{G}'d_s - \hat{G}'d_v} + \frac{d_v}{1 - \hat{G}'d_s - \hat{G}'d_v}$$

$$= \hat{H}_s \left(\frac{1 - \hat{G}'d_s}{1 - \hat{G}'d_s - \hat{G}'d_v} \right) + \hat{H}_v \left(\frac{1 - \hat{G}'d_v}{1 - \hat{G}'d_s - \hat{G}'d_v} \right)$$

$$= \hat{H}_s \sum_{n=0}^{\infty} \left(\frac{\hat{G}'d_v}{1 - \hat{G}'d_s} \right)^{(n)} + \hat{H}_v \sum_{n=0}^{\infty} \left(\frac{\hat{G}'d_s}{1 - \hat{G}'d_v} \right)^{(n)} . \tag{6.116}$$

From input/output data, $\hat{H}_{S \times V}$ may be estimated directly. In general, the surface and volume global filter interact.

6.4 An Example

Understanding of these many models and approximations may be aided by an example. Suppose a harmonic point source located at $(0,0,z_s)$ (z downward), a slab of independent scatterers of uniform density ρ

(each of scattering strength d") located between z_1 and z_2 in an otherwise homogeneous medium under a Gaussian random surface ($\sigma_x = \sigma_x$), as in Fig. 6-9.

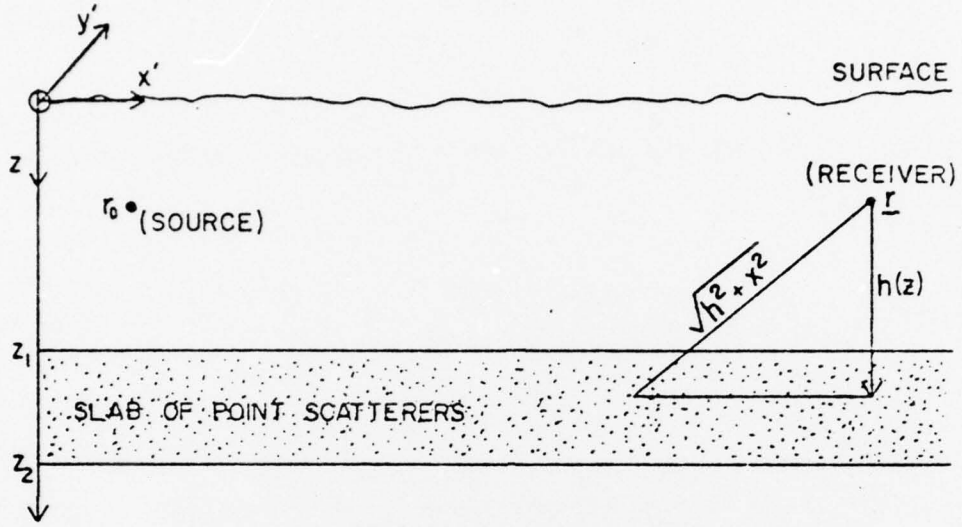


Fig. 6-9. Example Geometry

We will calculate $m_1(B(2l)) = [\hat{G}'(\langle d_s \rangle + \langle d_v \rangle)]^2$. (Cf. Eq. (6.106)).

Supposing $\langle d_v \rangle = d''\rho\delta(\underline{r} - \underline{r}_v)$, then we have

$$\hat{G}'\langle d_v \rangle = -d''\rho k_0^2 \int_{z_1}^{z_2} \int_{y'=-\infty}^{\infty} \int_{x'=-}^{\infty} \frac{e^{-jk_0|\underline{r}-\underline{r}_v|}}{4\pi|\underline{r}-\underline{r}_v|} \, d\underline{r}_v \, . \qquad (6.117)$$

Changing to cylindrical coordinates (centered a $\underline{r}$ with axis $\hat{z}$) and putting $|\underline{r}-\underline{r}_v| \equiv (h^2(z) + x^2)^{\frac{1}{2}}$, we obtain

$$\hat{G}'\langle d_v \rangle = \frac{-d''\rho k_0^2 2\pi}{4\pi} \int_0^{\infty} \int_{h_0}^{h_1} \frac{e^{-jk_0(h^2+x^2)^{\frac{1}{2}}}}{(h^2+x^2)^{\frac{1}{2}}} \, dxdz \, . \qquad (6.118)$$

Applying a change in variable, $y=(h^2+x^2)^{\frac{1}{2}}$ gives us,

$$\hat{G}'\langle d_v\rangle = \frac{-d''\rho k_o^2}{2} \int_h^\infty \int_{h_0}^{h_1} e^{-jk_o y}\, dydz ,$$

$$= \frac{-d''\rho k_o^2}{2} \int_{h_0}^{h_1} \frac{e^{-jk_o y}}{-jk_o}\Bigg|_h^\infty dz ,$$

$$= \frac{d''\rho k_o^2}{2} \frac{e^{-jk_o h(z)}}{-k_o^2}\Bigg|_{h_0}^{h_1} ,$$

or finally

$$\hat{G}'\langle d_v\rangle = \frac{d''\rho}{2} \left(e^{-jk_o h_1} - e^{-jk_o h_0}\right) , \tag{6.119}$$

where we have used a radiation condition.[1] It should be noted that the use of the Taylor's approximation has altered the spherical spreading to a plane wave from the layer. Moreover, the result, Eq. (6.119), may be used only for tenuous slabs, because as $\rho\to\infty$, $\hat{G}'\langle dv\rangle\to\infty$, as well. Thus, questions of applicability immediately arise, together with the need for experimental justification.

For $\hat{G}'\langle d_s\rangle$, let us use Eq. (6.54), where

$$\hat{G}'\langle d_s\rangle = \hat{G}_{\bar{s}}\, jk_o d_{\bar{s}} \left[\hat{\rho}\cdot\hat{z} + \hat{\rho}\cdot(\hat{x}+\hat{y})\langle m\rangle\right] \tag{6.120}$$

---

[1] This must be reimposed because of the initial use of the Taylor's approximation. The use of a convergence factor gives

$$\frac{e^{-jk_o y}}{-jk_o}\Bigg|_h^\infty - \lim_{a\to 0} \frac{e^{-jk_o y} e^{-ay}}{(-jk_o - a)}\Bigg|_h^\infty = \frac{-e^{-jk_o h}}{-jk_o} + \lim_{a\to 0}\lim_{y\to\infty} \frac{e^{-jk_o y} e^{-ay}}{-jk_o - a} \to \frac{-e^{-jk_o h}}{-jk_o} .$$

See also Appendix H.

for $\langle m_x \rangle = \langle m_y \rangle = \langle m \rangle$. Writing Eq. (6.120) in terms of cylindrical coordinates gives

$$\hat{G}'\langle d_s \rangle = jk_o \int_{h_2}^{\infty} \bar{d_s}[\cos\theta + \langle m \rangle \sin\theta] e^{-jk\rho} d\rho$$

$$\simeq jk_o \left(\bar{d_s} \frac{1}{2}(1 + \langle m \rangle)\right) \frac{e^{-jk_o h_2}}{+jk_o}$$

$$= \frac{\bar{d_s}}{2}(1 + \langle m \rangle)\, e^{-jk_o h_2}, \qquad (6.121)$$

where h_2 is the distance from the source to the mean surface. We see again that the Taylor's approximation has altered the spherical spreading to a plane wave from the surface.

If now we neglect the phase contributions (and thus neglect possible mean interference effects which will vary from point to point), we may write from Eq. (6.106)

$$m_1(B(21)) = \left[\frac{\bar{d_s}}{2}(1 + \langle m \rangle) + \frac{d''}{2}\rho\right]^2 \qquad (6.122)$$

as the relative effect on the mean received pressure of the S x V inter-mode scattering in this case.[1] It can be seen to depend on the roughness of the surface ($\langle m \rangle$) and the density of the slab (ρ). Taking as reasonable values[2] for $\bar{d_s} = 19/20$ and for $d'' = 1/4\pi$ gives Fig. 6-10.

---

[1]The choice of m(B(21)) is indicated in this case because the slab is composed of independent point scatterers distributed tenuously. The surface, moreover, has been given no correlation structure. This amounts to saying the surface is spatially incoherent.

[2]For $\bar{d_s} = d'\Sigma (\hat{G}'d')^{(n)}\hat{G}'(s)$, Eq.(6.7). In our example $d' = \frac{(1500)^2}{(330)^2} \simeq$ 19. The operation of the loop is to bring $\bar{d_s} \leq 1$ (as it must be physically); that is the loop operation has the effect of making $d'(1-d'+(d')^2-(d')^3\ldots) \rightarrow \frac{d'}{1+d'} \simeq \frac{19}{20}$, which we have chosen. For d", we have chosen a spherical wave emanation, i.e., a ratio of input to output equal to $1/4\pi$.

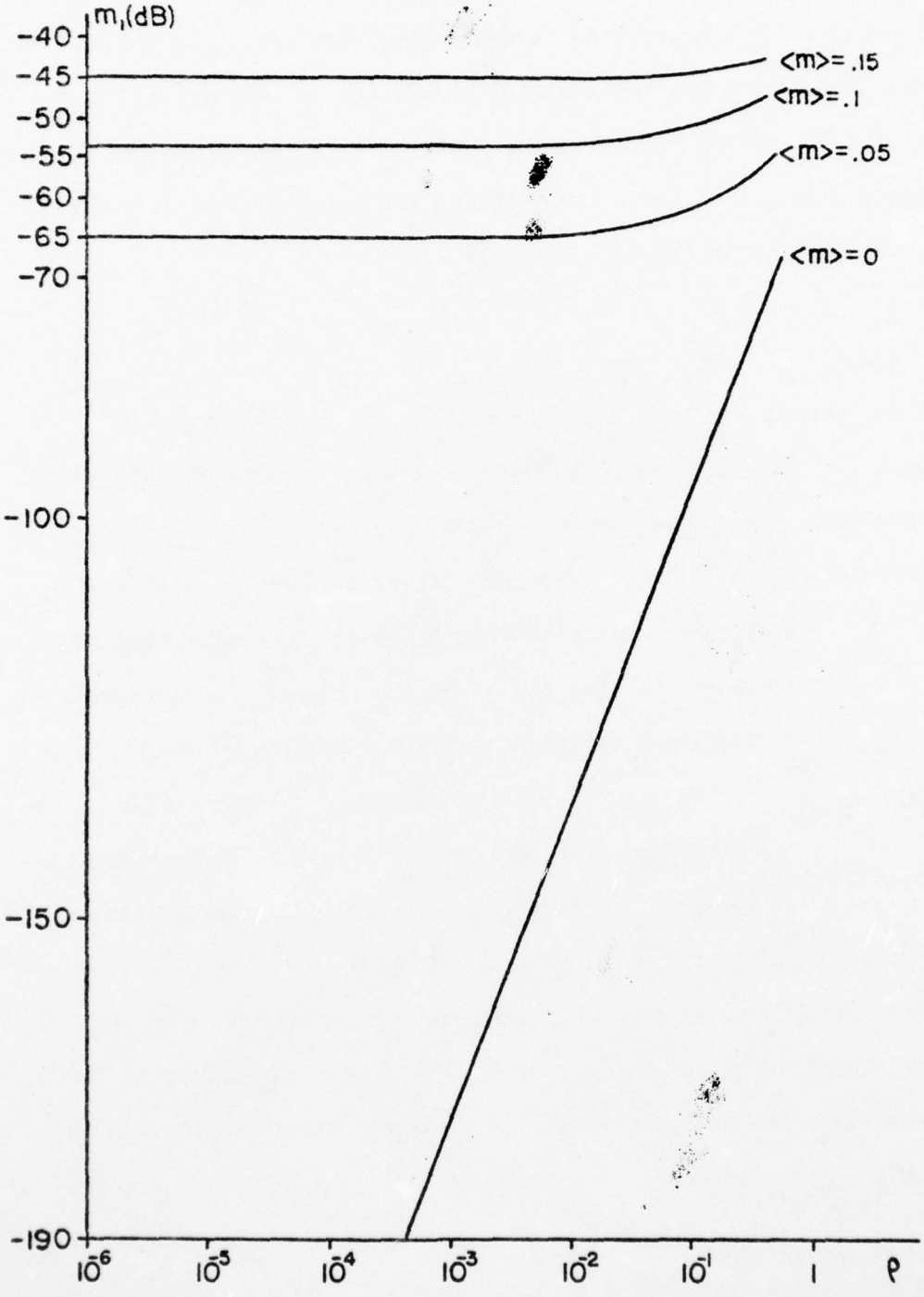


Fig. 6-10. Relative Effect of S x V Scattering to Mean Received Pressure (Example)

Granting the approximations, the results corroborate the experimental evidence that S x V scattering is negligible over most conditions. The average wave slope <m>, for example, should rarely exceed the Stokes limit of 1/7. In the volume layers of fish are acknowledged the most conspicious source of volume scattering, and except for schools their density should generally fall below 10^{-2}.

6.5 Summary

This chapter has developed the formal results of Chapter V, (especially Eq. (5.55)) by specifying (although rather broadly) physical attributes most likely to be found in the ocean and atmosphere. Primary among these latter is the presence of a bounding interface. Section 6.1 is devoted to applying Eq. (5.55) to the random surface. The treatment involves the application of the generalized divergence theorem (Sec. 1.2.1.2) and readily obtains a broadband generalization of the classical result, Eq. (6.8). Moreover, our results may be cast into a form which gives the specular term and the influence of the wave slopes (see Eq. (6.54)). The filtering effect due to the surface is given in Eq. (6.33); the spectral response is given in Sec. 6.1.3.

For the volume, we analyze (using the insights of earlier chapters) a recent formulation, Eq. (6.70), Cf. [146], of scattering in a lossy medium. The analysis shows how the formulation can be extended to show the combined effect of scattering and loss (Fig. 6-5 and Eq. (6.90)). This shows that the original formulation cannot account for the main observed loss in the oceans. We give, consequently, a formulation, Eq. (6.96), derived from a different differential equation, Eq. (6.95), which has the correct attributes.

The final sections of this Chapter are devoted to obtaining criteria for the interactive coupling of interface and volume scattering. Our results are obtained in Eqs. (6.106), (6.107) and (6.110) to (6.112). Finally, a numerical example shows that our formulations give results which agree with observation. While our orientation has favored propagation of sound in the sea, our results are equally applicable, in principle, to the atmosphere.

Thus this Chapter illustrates, in specific ways, the power of our methods: (1) in elucidating classical results; (2) in extending classical results; and (3) in obtaining results not heretofore available.

We list below the items considered original either in method, result, or both, of this Chapter. With respect to the surface only, these are:

1. The effect of atmospheric occurrences on the propagation of underwater sound, (Sec. 6.1)
2. Extension of the classical surface reflection over a stochastic surface, (Eq. (6.8))
3. The surface filter formulation, (Eq. (6.34))
4. The surface response giving the specular scattered terms, (Eqs. (6.54)-(6.60))
5. The effect of the stochastic surface on the received spectrum. (Sec 6.1.3)

For volumes, we have:

1. The extension of Moroz's formulation for scattering in a lossy medium, (Eq. (6.93))
2. A formulation giving a correct frequency dependence of scattering in a lossy medium. (Eq. (6.96))

For combined effects S x V:

1. Propagation in a deterministic medium with boundaries. This result, while not demonstrated, should generalize solutions by ray theory. (Eq. (6.93))
2. The partition of equivalent media, (Fig. 6-8)
3. The S x V Solutions, (Eqs. (6.99) - (6.104))
4. The criteria, m_1, m_2, (Eqs. (6.106, 6.107, 6.110-6.112))
5. The global filter, (Sec. (6.33))
6. The result that, under a large variety of circumstances, the effect of intermode scattering on the mean should be less than -40 dB. (Fig. 6-10)

We mention finally the examination, in a qualitative way, of the effect of the Taylor's approximation (Class IV) on the solutions,[1] (Sec. 6.4) and our brief discussion of the new Middleton scattering models (Sec. 6.1.5).

[1]See also Figs. 7-10 and 7-11.

VII. CONNECTIONS

7.0 Introduction

To the extent the preceding theory and models can claim some generality, they may be expected to approach formulations specific to other fields. The medium, for instance, may be considered for its ability to transmit information so that one may naturally ask for its "capacity." In this chapter, we show a few of these connections without attempting either a comprehensive survey or a profound development. Our point of view will be initially restricted to Eq. (5.1).

$$\left[\nabla^2 - \frac{1}{c^2(\underline{r},t)}\frac{\partial^2}{\partial t^2}\right] p = s(\underline{r}_o, t_o) \quad .$$

7.1 Filters

For a fixed source-receiver combination, the medium has some aspects of a filter: it attenuates the signal; it introduces delays and phase changes. Furthermore, it can be exhibited in feedback forms similar to eigenvalue representation of a linear filter. To pursue the analogy, the exact solution of Eq. (5.1) may be expressed by Fig. 7-1 or Fig. 7-2.

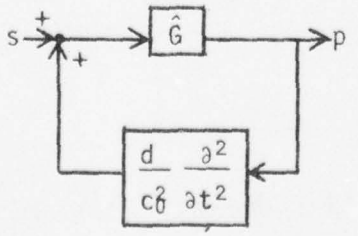


Fig. 7-1. Transmission as a Filter (I), Eq. (5.1)

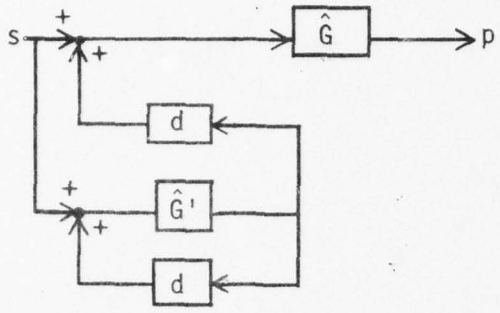


Fig. 7-2. Transmission as a Filter (II), Eq. (5.55)

The feedback portion of Fig. 7-1 is a stochastic operator, so that although s and $\hat{G}$ are deterministic, the output is a random process. The feedback, operator $\frac{d}{c_0^2}\frac{\partial^2}{\partial t^2}$, is medium-related; $\hat{G}$ is geometry-related. We have thus a different temporal filter for each geometrical configuration (including source-receiver locations) and each scattering medium. Since the feedback is stochastic, the analogy must relate to a stochastic filter.

The stochastic filter has meaningful representation in terms of its moments as these moments relate to statistical outputs. Analogous to the temporal filter, solutions generally will not be available in closed form unless suitable approximations are made. In the case of Eq. (5.1), the exact first moment solution is shown operationally in Fig. 7-3 in terms of the usually unknown equivalent medium Q.

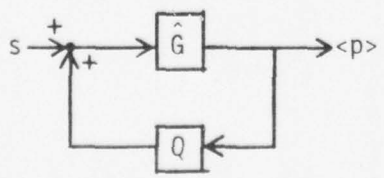


Fig. 7-3. First-Moment Filter, Eq. (5.62)

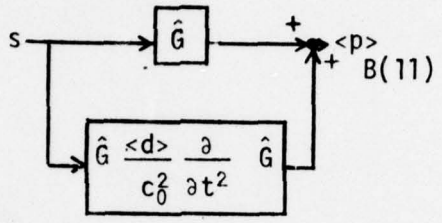
AD-A065 780 NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CONN NEW --ETC F/G 20/1
A GENERAL THEORY OF ACOUSTIC PROPAGATION WITH APPLICATIONS TO S--ETC(U)
JAN 78 J R BRETON

UNCLASSIFIED NUSC-TR-5871 NL

4 OF 5
AD
A065780

Equation (5.62), (Fig. 7-3), which preserves the feedback form may be approximated by a feed-forward form as shown in Fig. 7-4, which can be seen to be the B(11) approximation.

(a):



or

(b):

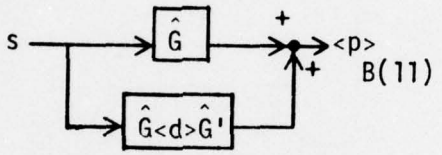
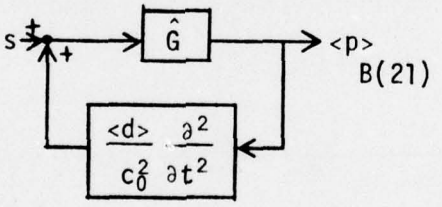


Fig. 7-4. Feed-Forward Approximation B(11), Eq. (5.81)

The B(21) approximation relates directly to the mean medium, as indicated in Fig. 7-5 below.

(a):



or

(b):

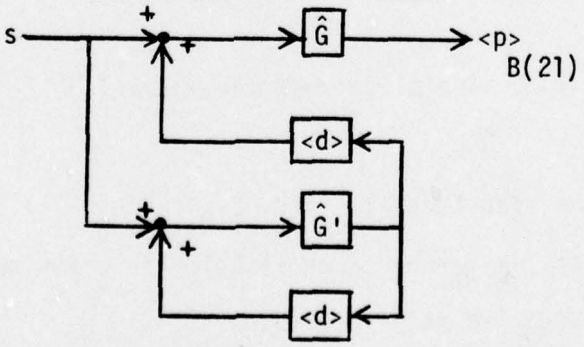


Fig. 7-5. The B(21) Approximation, Eq. (5.98)

This approximation preserves the feed-back characteristic.

The B(22) approximation contains both feed-forward and feed-backward elements as shown in Fig. 7-6.

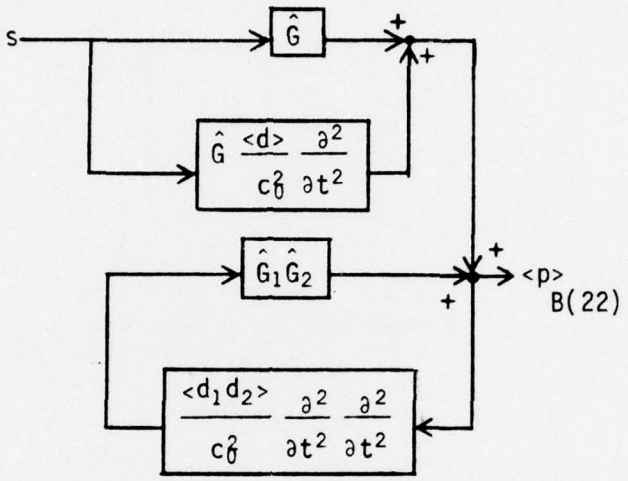


Fig. 7-6. The B(22) Approximation (I), Eq. (5.125)

The form using $\hat{G}'$ can be decomposed into many different forms, a possible one is shown in Fig. 7-7.

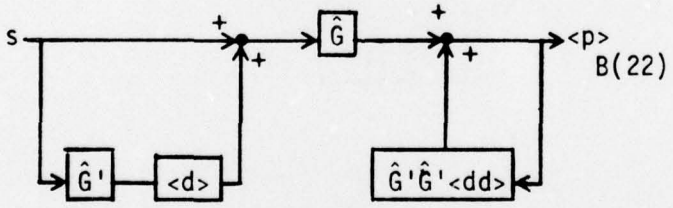


Fig. 7-7. The B(22) Approximation (II), Eq. (5.125)

Since the structures of Figs. 7-5 through 7-7 do not yield closed forms generally, a further approximation is often made in the literature, i.e., Fig. 7-8 and Fig. 7-9.

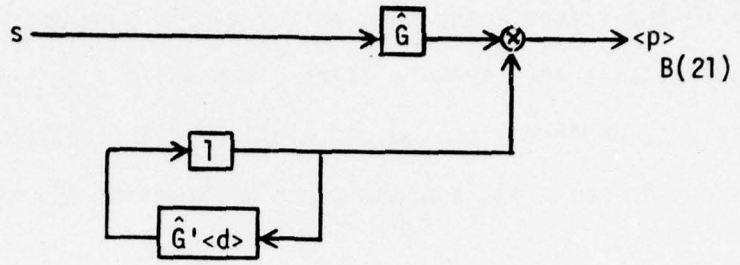


Fig. 7-8. Taylor's Approximation B(21), Eq. (5.105)

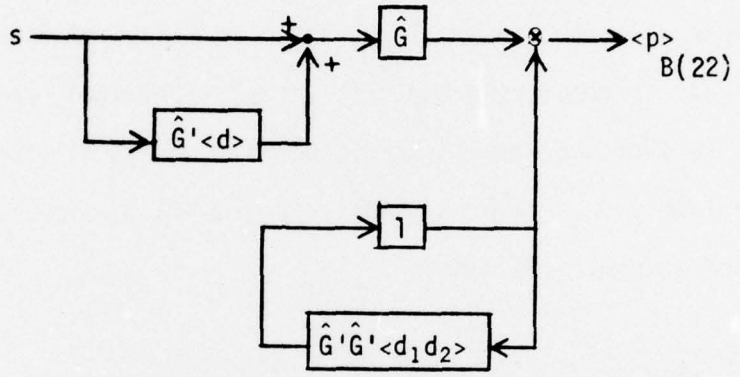


Fig. 7-9. Taylor's Approximation B(22), Eq. (5.132)

The Taylor's approximation, using only the first term of a Taylor's expansion of p, has little to recommend it except the ability to achieve closed solutions (after suitable approximations of the feedback loop). It introduces, on the negative side, spurious sources and non-linearities (through multiplication) and raises questions about the convergence of the feedback (i.e., $\hat{G}'\langle d\rangle$ and $\hat{G}'\hat{G}'\langle d_1 d_2\rangle$ must be less than unity).

In short, the medium as represented in Eq. (5.1) (with given geometry), acts as a stochastic, space-variable filter. Temporal change in the medium, source, or receiver adds, furthermore, a temporal

dimension to the filter. The mean output can be described in terms of a non-stochastic, space-variable filter, for which a set of canonical approximations, dependent on the correlation of the active scatterers, can be made. An analogous statement can be made for the higher order moments.

The diagrams of this section may be regarded as the beginnings of a space-time filter theory, analogous to the temporal filters of conventional circuit theory. The particular virtue of these filters lies in their accommodating explicitly the distributed nature of the medium.[1] As such they should be useful in modeling multipath, fluctuations (volume and surface related), etc., which physically characterize distributed transmission media.

7.2 Control Theory and Estimation

The operational forms of Figs. 7-1 to 7-9 suggest analogous formulations in control theory. Indeed, the terms "feed-forward" and "feed-backward" used with the B(11) and B(21) approximations (Figs. 7-3 and 7-4) come from classical control theory. Again, the relationship of filter theory to optimal control theory is well known.[2]

In terms of control theory, the acoustic source is the "control function," the medium is the plant, the output is $p(\underline{r},t)$. In contrast to the usual treatment, all these quantities are now space-time functions rather than purely temporal ones, and the "plant" is stochastic rather than deterministic. In the context of Eq. (5.1), the stochastic output arises from the stochastic plant rather than from a noise input.

[1] See [217].

[2] See [216], Chapter 15, among others.

The analogy in form can be striking. Consider the simple feedback system of Eq. (7.1), shown in Fig. 7-10,

$$\frac{dp}{dt} \equiv \dot{p} = ap + s \quad , \qquad \underline{a} \text{ constant} \tag{7.1}$$

with the well known solution,

$$p(t) = p(t_0)e^{(at - at_0)} + \int_{t_0}^{t} e^{(at - at_1)} s(t_1)dt_1 . \tag{7.2}$$

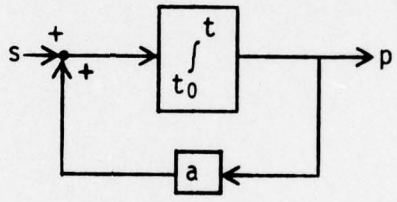


Fig. 7-10. A Simple Temporal Feedback Loop, Eq. (7.1)

If we apply recursive methods to Eq. (7.2), we obtain upon initially integrating Eq. (7.2)

$$\int_{t_0}^{t} \dot{p}dt_1 = p(t) - p(t_0) = \int_{t_0}^{t} (ap + s)dt_1$$

so that

$$p(t) = p(t_0) + \int_{t_0}^{t} (ap + s)dt_1 . \tag{7.3}$$

Substituting recursively gives

$$p(t) = p(t_0) + \int_{t_0}^{t} a\, p(t_0) + \int_{t_0}^{t} (ap + s)dt_2 \; + s\, dt_1 ,$$

$$= p(t_0) + \int_{t_0}^{t} ap(t_0)dt_1 + \int_{t_0}^{t} a \int_{t_0}^{t_1} (ap)dt_1dt_2$$

$$+ \int_{t_0}^{t} \int_{t_0}^{t} as\, dt_1dt_2 + \int_{t_0}^{t} sdt_1 ,$$

$$= p(t_0) + a(t - t_0)p(t_0) + \int_{t_0}^{t} \int_{t_0}^{t_1} a^2pdt_1dt_2$$

$$+ \int_{t_0}^{t} \int_{t_0}^{t_1} as\, dt_1dt_2 + \int_{t_0}^{t} sdt_1 \quad . \tag{7.4}$$

Continuing gives

$$p(t) = p(t_0)\left(1 + (at - at_0) + \frac{a^2(t - t_0)^2}{2!} + \ldots\right)$$

$$+ \int_{t_1}^{t} a \int_{t_0}^{t_1} a \ldots \int_{t_0}^{t_\infty} (ap + s)dt_1 \ldots dt_\infty$$

$$+ \int_{t_0}^{t} sdt_1 + \int_{t_0}^{t} \int_{t_0}^{t_1} asdt_1dt_2 + \ldots$$

$$+ \int_{t_0}^{t} a \int_{t_0}^{t_1} a \int \ldots a \int_{t_0}^{t_\infty} sdt_1dt_2 \ldots dt_\infty \quad . \tag{7.5}$$

The first term of the RHS is easily recognized as $p(t_0)e^{a(t - t_0)}$. The second term must vanish for convergence. (Note here the influence of a^n.) The remaining terms contain s and are equal to

$$\int_{t_0}^{t} e^{(at - a\tau)} s(\tau) d\tau \ .$$

This can be seen for the first iteration by

$$\int_{t_0}^{t} a \int_{t_0}^{t_1} at_2 dt_2 dt_1 = a(t - t_1) \int_{t_0}^{t_1} s(t_2) dt_2 \Big|_{t_0}^{t_1}$$

$$+ \int_{t_0}^{t} (at - a\tau) s(\tau) d\tau = \int_{t_0}^{t} (at - a\tau) s(\tau) d\tau \ , \qquad (7.6)$$

using integration by parts. We thus obtain the known solution, Eq. (7.2).

The first term (RHS) of Eq. (7.2) is called the "zero-input" response. It can be obtained by stimulating the system with an impulse. The second term is called the "zero-state" response. It constitutes the source's contribution to the output. If the source is harmonic, we may speak of transient and steady state conditions. These correspond directly to experiments possible in both conventional control and acoustical situations.

Equation (7.5) corresponds nicely to the series form of the general solution or to the solutions given in Chapters V and VI. It is easy to see that these solutions are extensions of the usual control formulation, i.e.,

$$\int_{t_0}^{t} (\cdot) dt_1 \rightarrow \hat{G}$$

$$a \rightarrow \frac{d(\underline{r},t)}{c_0^2} \frac{\partial^2}{\partial t^2} \qquad (7.7)$$

in the case of Eq. (5.1). Explicitly, the temporal integral is extended to a space-time integral operator and $\underline{a}$ is extended to the medium. As in control theory, closed solutions like Eq. (7.2) are available in only a small number of cases; the majority of cases are adequately analyzed by specifying an operational solution.

Two further points are noted. If we consider the source-receiver (perhaps with beam patterns) as points in the field, the transmission process becomes a single input/single output system and may be described by a stochastic transfer function, $\hat{T}$. Several forms can be set: $\hat{T}(\underline{r},s)$; $\hat{T}(\underline{k},s)$, $\hat{T}(\underline{r}_1,t_1;\underline{k},s)$, etc., depending on the transforms employed. These transforms, then, formally define the stochastic space-

Again, the problem of determining Q, the equivalent medium (mass operator) is a type of system identification.[1] Given experimentally <p> and s, Q may be determined by the scheme given in Fig. 7-11.

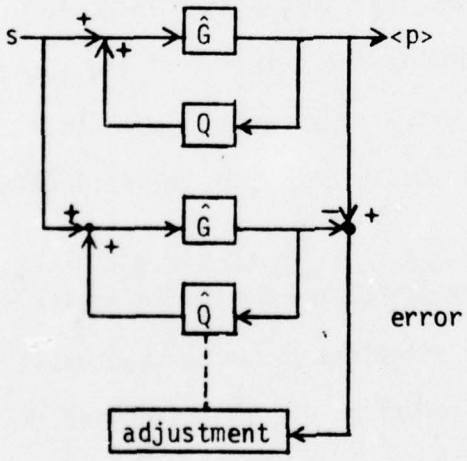


Fig. 7-11. Operational Method for Obtaining Q

[1]One reference among many: A. V. Balakrishnan, and V. Peterka, 1969; "Identification in Automatic Control Systems," Automatica 5(817).

In the figure, Q^+ denotes an estimate of Q. While optimum adjustment schemes are known for special cases (Wiener filters, for example), even suboptimum schemes (gradients, for another example) may be used to drive the error to zero. When this is accomplished, $Q^+ = Q$. In experimental situations, this technique may often be available for suggesting allowable approximations.

7.3 Noise

The models of Chapters V and VI, as mentioned earlier,[1] do not incorporate independent noise sources. The propagation is more or less correlated with the source depending on the strength, nature and geometry of the scatterers. This is typical of reverberation, often called "correlated noise," whereas it is more precisely a decorrelated signal.

The general theory, on the other hand, does incorporate independent as well as dependent noise sources. From Eq. (1.27), we can discern two types: "noise only" and "signal and noise."

The "noise only" types are:

a) $$\underline{f} \cdot \nabla(p_0 + p_1) \tag{7.8a}$$

b) $$-\frac{\partial^2}{\partial t^2}((p_0 + p_1)|\underline{r}) \tag{7.8b}$$

The "signal and noise" types are:

c) $$\left(\frac{d}{dt}(\ln\rho c^2)|\underline{r}\right)\left(\frac{d(p|\underline{r})}{dt} + \frac{d((p_0 + p_1)|\underline{r})}{dt}\right) \tag{7.9a}$$

---

[1]See Chapter V, Introduction.

$$d)\quad \underline{v}\cdot\frac{d}{dt}\left((\nabla p + \nabla(p_0 + p_1))|\underline{r}\right) \tag{7.9b}$$

$$e)\quad \left(-\underline{v}\nabla\cdot\underline{v} - \frac{d}{dt}(\ln\rho c^2)|\underline{r}\;\underline{v} + \frac{\nabla\cdot\underline{T}_2}{\rho}\right)\cdot\left(\nabla p_0 + \nabla(p_1 + p_2)\right) \tag{7.9c}$$

$$f)\quad \rho c^2\nabla\cdot\left(\frac{\nabla\cdot\underline{T}}{\rho}\right) \tag{7.9d}$$

$$g)\quad \rho c^2\nabla\cdot\left(\underline{f} - \underline{v}\nabla\cdot\underline{v} + \nabla\cdot\frac{(\underline{T}_0 + \underline{T}_1)}{\rho}\right). \tag{7.9e}$$

The "noise-only" types are generated by turbulent and background temporal change, Eq. (7.8b), and spatial gradients acting in a force field, Eq. (7.8a). The "signal and noise" types arise from a large number of circumstances (See Chapter II), but are all to some extent (sometimes minimal) influenced by the acoustic (signal) pressure field.

The "noise-only" type may be brought readily into the context of Eq. (5.1). We should then have

$$\left(\nabla^2 - \frac{(1+d)}{c_0^2}\frac{\partial^2}{\partial t^2}\right)p = n(\underline{r},t)$$

$$\equiv -\frac{d^2}{dt^2}\left((p_0 + p_1)|\underline{r}\right) + \underline{f}\cdot\nabla(p_0 + p_1)\,. \tag{7.10}$$

The solutions of Chapters V and VI then carry over directly from

$$\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\;p = n(\underline{r},t) + \frac{d}{c_0^2}\frac{\partial^2}{\partial t^2}(p) + s(\underline{r}_0,t_0), \tag{7.11}$$

or

$$p = \hat{G}(s+n) + \hat{G}d\sum_{m=0}^{\infty}(\hat{G}'d)^{(m)}\hat{G}'(s+n), \tag{7.12}$$

$$= \hat{G}(s) + \hat{G}d \sum_{m=0}^{\infty} (\hat{G}'d)^{(m)}\hat{G}'(s) + \hat{G}(n) + \hat{G}d \sum_{m=0}^{\infty} (\hat{G}'d)^{(m)}\hat{G}'(n),$$

$$\equiv (p_u + P_s) + (p_n + p_{ns}) . \qquad (7.13)$$

Thus, in this formulation, the received pressure is made up of four components:

p_u - the unscattered signal $\equiv \hat{G}(s)$,

p_s - the scattered signal $\equiv \hat{G}d \sum_{m=0}^{\infty} (\hat{G}'d)^{m}\hat{G}'(s)$,

p_n - the unscattered noise $\equiv \hat{G}(n)$,

p_{ns} - the scattered noise $\equiv \hat{G}d \sum_{m=0}^{\infty} (\hat{G}'d)^{(m)}\hat{G}(s)$. (7.13a)

Because the noise is independent and additive, no coupling between these components occurs.

Experimentally, we have $p_n + p_{ns}$ usually available as background noise, so that

$$p - (p_n + p_{ns}) = p_u + p_s \qquad (7.14)$$

may sometimes be formed into a criterion for determining the effect of gross movements of the medium on the transmission. (Note that many of the "signal and noise" types contain particle velocity terms, Cf. Eq. (7.9).)

For second moment characterization, the received signal will often contain more elements than the four output elements of conventional

detector theory:[1] s x s, s x n, n x s, and n x n. We may characterize the transmission output as $(p_u + p_s + p_n + p_{ns}) \times (p_u + p_s + p_n + p_{ns})$ yielding sixteen types of cross products. The additional products arise from the distributed scattering mechanism which influences both the signal (p_s) and the noise (p_{ns}). These factors play a significant role for signal detection in reverberation.

7.4 Communication Theory

Consider the acoustic transmission[2] from source to sensor, as a communication channel, T. The following may be stated:[3]

1) In the context of the Langevin Equation, Eq. (5.1), T is linear.
2) *T causes a phase and amplitude randomization.*
3) T has "memory."
4) T is *not* measure preserving.
5) T is stochastic. (7.15)

To show the first statement in Eq. (7.15), let

$$p_1 = \hat{G}_1(s_1) + \hat{G}_1\hat{\xi}_1 p_1 \equiv T(s_1) \tag{7.16}$$

and

$$p_2 = \hat{G}_2(s_2) + \hat{G}_2\hat{\xi}_2 p_2 \equiv T(s_2) \ . \tag{7.17}$$

---

[1]See [140], Chap. 13.

[2]Specifically excluded are all modulation and demodulation schemes. We are concerned here with the "medium" transmission only.

[3]Considerations derived from [140], Chap. 3.

Then for $s_1 = s_{t_1}\,\delta(\underline{r}-\underline{r}_0)$ and $s_2 = s_{t_2}\delta(\underline{r}-\underline{r}_0)$, we have

$$p(\underline{r}) \equiv p_1(\underline{r}) + p_2(\underline{r}) = \hat{G}(s_1 + s_2) + \hat{G}\hat{\xi}(p_1 + p_2) \qquad (7.18)$$

or

$$T(s_1) + T(s_2) = T(s_1 + s_2) \qquad (7.19)$$

where we assume $\underline{r}_0$, $\underline{r}$ are now identical for T_1 and T_2. In other words, the linearity is purely temporal. If either the sources or receivers are separated, then we have

$$p_1 + p_2 = \hat{G}_1(s_1) + \hat{G}_2(s_2) + \hat{G}_1\hat{\xi}_1 p_1 + \hat{G}_2\hat{\xi}_2 p_2 \ , \qquad (7.20)$$

$$= T(s_1) + T(s_2),\ \hat{\xi}_1 \neq \hat{\xi}_2;\ \hat{G}_1 \neq \hat{G}_2$$

$$\neq T(s_1 + s_2). \qquad (7.21)$$

In other words, the point-to-point channel is linear; channels involving multiple point sources or receivers are not linear, but may be looked upon as a "packet" of linear channels. In the context of Eq. (5.1), the channels do not interact.

The second statement of Eq. (7.13) is shown in Chapter VI, Sec. 6.1.3 and comes from the numerous paths the medium allows.

The third statement of Eq. (7.13) comes from the finite propagation velocity.

The fourth statement of Eq. (7.13) means that input probability measures may be distorted in the output. In the cases treated earlier, a harmonic signal (i.e., a spectrally impulsive probability) is distorted in the output.

Finally, T is stochastic, from the stochastic nature of the scatterers considered over an ensemble of media.

The acoustic channel is thus, (for Eq. (5.1)), a linear channel which produces phase and amplitude randomization. In this context, we may put precisely:

$$\hat{T} = \hat{G}(\cdot) + d\sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(\cdot) \tag{7.22}$$

from Eq. (5.55) where the stochastic nature of $\hat{T}$ is evident.

If $\hat{T}$ is linear, what is the impulse function of the channel? Suppose

$$p(\underline{r},t) = \int_0^{\infty} \hat{H}(t-\tau|\underline{r})s(\tau)d\tau$$

$$= \int_0^{\infty}\int_{\underline{r}_0}\int_{\omega_0} \left(\frac{e^{j(\omega_0(t-\tau) - k_0|\underline{r}-\underline{r}_0|)}}{8\pi^2|\underline{r}-\underline{r}_0|} s(\tau) \right.$$

$$\left. + \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'s(\tau_n) \right) d\tau\, d\underline{r}_0 d\omega_0$$

$$= \int_0^{\infty} \frac{e^{j(\omega_0(t-\tau) - k_0|\underline{r}-\underline{r}_0|)}}{8\pi^2|\underline{r}-\underline{r}_0|}$$

$$\cdot \left[(\cdot) + \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(\cdot)\right] d\tau\, d\underline{r}_0 d\omega_0 \tag{7.23}$$

so that

$$\hat{H}(t-\tau|\underline{r}) = \int_{r_0}\int_{\omega_0} \frac{e^{j(\omega_0(t-\tau) - k_0|\underline{r}-\underline{r}_0|)}}{8\pi^2|\underline{r}-\underline{r}_0|} \cdot (\cdot) + \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)}\hat{G}'(\cdot)\ d\underline{r}_0 d\omega_0 \ . \tag{7.24}$$

This model does not contain line spectra, so that where such exist, they must derive from the source or from an effect not modeled by Eq. (5.1). The general equation, Eq. (1.27), is, of course, non-linear even in the simplest cases (see Appendix A).

The second moment characterization of the channel may be written directly from Eq. (5.75) as

$$\langle p(t_1,\omega_1)p(t_2,\omega_2)\rangle = p_{u_1}p_{u_2} + p_{u_1}\hat{G}_2Q_2\langle p_2\rangle + p_{u_2}\hat{G}_1Q_1\langle p_1\rangle + \hat{G}_1(\hat{G}_2B\langle p_1p_2\rangle) \tag{7.25}$$

where

$$\hat{G}_i \equiv \int_{t_n}\int_{\underline{r}_n} \frac{e^{-j\omega_i(t_i-t_n) - k|\underline{r}-\underline{r}_n|}}{4\pi^2|\underline{r}-\underline{r}_n|}(\cdot)\ dt_n d\underline{r}_n;\ k_i = \frac{\omega_i}{c_0} \tag{7.26}$$

so that p_u and $G\,\hat{Q}\langle p_u\rangle$ are functions of ω.

A word about channel capacity, defined as the maximum rate at which information may be passed through the channel (subject to possible constraints). Arguments based on Eq. (5.1) may easily lead to erroneous conclusions. For Eq. (5.1), if d = 0, the channel capacity,

C, is infinite -- that is, any amount of information, H, at any rate, C, transmitted by the source, s, can be ideally decoded. For the simple case of homogeneous transmission we have (Cf. Fig. 7-12),

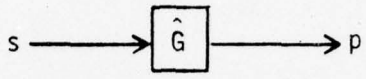


Fig. 7-12. Homogeneous Transmission

or

$$p = \frac{1}{4\pi|\underline{r} - \underline{r}_0|} e^{j(\omega_0 t - k_0|\underline{r} - \underline{r}_0|)}$$

$$= \frac{s\left(t - \frac{\omega_0|\underline{r} - \underline{r}_0|}{c_0}\right)}{4\pi|\underline{r} - \underline{r}_0|} \qquad (7.27)$$

for a harmonic point source. Moreover, the channel has infinite bandwidth.

Suppose now that $d \neq 0$, but deterministic and known. Then, with p known, we get from Eq. (5.10)

$$s = \hat{L}p - \hat{L}\hat{G}\hat{\xi}p \qquad (7.28)$$

so that a decoding scheme (shown in Fig. 7-13)

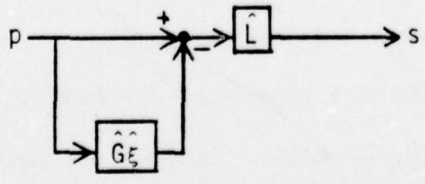


Fig. 7-13. Decoding Scheme for Transmission Through Deterministic Media

again recovers the signal. These are no inherent bandwidth limitations, and the scattering channel likewise has infinite capacity.

The problem of transmitting information in the context of Eq. (5.1) then, comes not from the capacity of the channel, but from the decoding scheme. Even with independent noise, the infinite bandwidth still suggests a channel with infinite capacity. If, as is always the case, our knowledge of $\hat{\xi}$ is incomplete, then our ability to extract information from the channel will be impaired. Still, to speak of channel capacity in this context implies that the physical capacity of the channel depends on <u>our</u> knowledge of the scattering medium.

We thus concentrate on the decoding problem. The source may be uncovered from the mean received statistics, provided the equivalent medium $\hat{Q}$ is known. The procedure is shown schematically in Fig. 7-14.

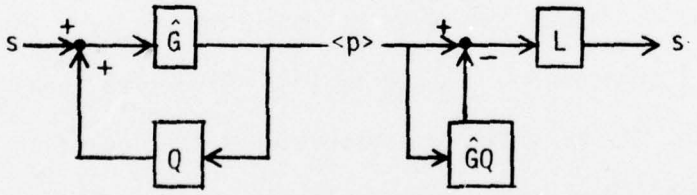


Fig. 7-14. Decoding Scheme for Transmission Through Stochastic Media

The errors, $Q - Q^{+}$, as well as $\langle p \rangle - \langle \hat{p} \rangle$, thus inherently determine the throughput. Figure 7-11 and the B(2) approximations of Chapter V[1] are thus seen as important elements in acoustic communications.

Often, Q may be susceptible to canonical decompositons, <u>i.e.</u>, in the M form, $Q = Q_s + Q_v + Q_B$ defining equivalent surface, volume and

---

[1]See Sec. 5.3.2.

bottom media. Whenever one of these is constant (e.g., Q_B) or slowly changing, the dynamic problem of estimation may be reduced.

We have assumed here a channel defined from a fixed transmission point to a fixed source point. A second receiver point defines a second channel with another equivalent medium which must be estimated to establish a second decoder, etc.

A realistic ocean channel is, of course, both noisy and frequency limited. In the most general cases a variety of frequency limiting mechanisms exists. We may expect these to be associated with losses and relaxation effects (see Sec. 2.5).

But the decoding problem remains.

7.5 Summary

This chapter shows some ways by which the analysis developed in Chapters I-VI connects with other fields. The operational solutions, in particular, may be viewed as space-time filtering of an input (source) signal (Sec. 7.1). Alternatively they may be considered space-time feedback loops with close analogies to control theory and statistical estimation (Sec. 7.2). This viewpoint clarifies the series approximations (B(1) and B(2)) and Taylor's approximation (Figs. 7-4 to 7-9). The connection with control theory suggests existing methods of "system identification" (Fig. 7-17) for determining equivalent media operationally.

In Sec. 7.3 we extend the model described by Eq. (5.1) by including the independent noise sources identified in the general equation, Eq. (1.27). The methods employed earlier are then used (1) to produce a solution for the noisy case (Eq. 7.27) and (2) to show how the

transmission output differs significantly from the outputs of conventional detector theory (Sec. 7.3). Lastly, we describe the transmission process based on the model of Eq. (5.1) in terms of the channel's information capacity. We are led to conclude that as a description of a communication channel the wave equation may easily lead to erroneous results. For the Eq. (5.1) model we give the channel explicitly as a transform, (Eq. (7.22)), its impulse function (Eq. (7.24)), and its second moment characterization (Eq. (7.25)). In the context of Eq. (5.1) we observe that the problem of transmitting information comes not from the medium but from our inability to formulate suitable decoding schemes. Such schemes are suggested in Fig. 7-14 and the following discussion.

The original contribution of this chapter stems naturally from the entire exposition of the Thesis. With the important exception of the scattering process modeled as a feedback loop, the various interdisciplinary connections cited in this Chapter have been recognized before, e.g., the notion of the medium as an acoustic communications channel, etc., but they have not been systematically identified in the context of the general approach developed in this Thesis. The new contributions here are thus largely the specific results obtained. We mention:

1. The filter forms (Fig. 7-1 to 7-9).
2. The extension of the control theory formulations to four dimensional communication channels.
3. Estimating the first moment equivalent medium, Q, as a type of system identification problem (Fig. 7-11).
4. The specific form of the transmission operator, $\hat{T}$ (Eq. (7.22)).

5. The specific form of the impulse response of the transmission channel described by the Langevin equation, Eq. (5.1). (Eq. (7.24))
6. The specific second moment characterization of the channel (Eq. (7.25))
7. An interpretation of the decoding problem (Sec. 7.4).

VIII. SUMMARY

This Thesis develops an acoustical theory sufficiently broad to order the models of the literature. By so doing, we have gained insight into conventional models, have obtained new methods of solution, and have arrived at specific solutions beyond those currently available.

The first stage (theoretical development) is covered in Chapters I-IV which demonstrate that the wave-like solutions and formulations of the literature are encompassed by a suitably general hydrodynamic theory. Chapter III, in particular, gives the general solution, unavailable before, to the adiabatic hydrodynamic equations in an arbitrary medium. Chapters I and II catalog the approximations inherent in the models current in the past thirty (or more) years. Chapter II describes the various physical circumstances which limit the validity of the current models and gives quantitative measures (criteria) for their applicability. Thus, this stage may be viewed as the completion of a broad earlier effort which sought to expand the applicability of the wave equation.

We have developed a theory (logically consistent from axioms), but harbor little illusion that we have fully illuminated its foundations. These latter, we show, are based in an extended divergence theorem and an extended feedback theorem.

In the second stage (Chapters V-VII), which turns on the first, we obtain new insights and solutions to the broadband form of the

stochastic wave equation, Eq. (5.1). "Operational" solutions, employing feedback loops (employed in the general solution) give exact solutions. Moreover, these latter show that the multiple scattering models of the literature are not so much physical as "operational." In particular the question of convergence of these models is thus given new light by distinguishing the form of a solution from its stability.

The approximations (historically necessary for specific results) to the multiple scattering models are formally completed by means of the canonical ordering of "B" approximations in Chapter V. These approximations may be readily appreciated using Feynman diagrams or the operational forms given in Chapter VII.

Scattering from interfaces, contained canonically in the formulation of Chapter V, is more directly addressed in Chapter VI, which in addition to extending classical solutions, gives a criterion for the hitherto unsolved surface-volume scattering interactions (S x V). The finding, that for many circumstances this effect amounts to less than 40 dB of non-interacting scattering, agrees well with experiment.

Lastly, in Chapter VII, the Thesis shows how its extended results connect with other disciplines: we discuss the acoustic medium as a multi-dimensional filter, as a control system, and as a communications channel.

While not a primary purpose of the Thesis, experiments which are suggested by its formulations have been briefly and concisely stated.

Classification of approximations is the unifying theme of the Thesis. These approximations are found to consist of four generic types or classes. The first and most pervasive class we call hydrodynamic. This class results from a confusion of the hydrodynamic

derivatives with those of the calculus. The second class we call differential meaning the approximations made in simplifying the governing differential equation. The third class we call truncation/selection meaning the approximation made in the (open) series solution. The fourth class we call analytical meaning the approximations required to obtain a solution in closed form. This latter often uses only the first term of a Taylor's series.

Finally, we give the major new results of the Thesis:

1. The general solution of adiabatic acoustic propagation.
2. Operational solutions (FF-BT).
3. A listing of conditions and circumstances for the validity of the usual propagation models.
4. The exact operational solution to the broadband scalar wave. Separation of the solution into purely geometrical and purely medium operators.
5. A canonical decomposition of approximate solutions to the broadband scalar wave equations, a new category of approximations B(21).
6. Criteria (a) for the validity of the first Born approximation, (b) for the effect of surface-volume interactions.
7. Interdisciplinary connections between scattering theory and statistical communication theory and control theory.

A more detailed listing of new results is given in the Preface.

Appendix A

THE HYDRODYNAMIC PLANE WAVE

Trusdell mentions [48] that a Navier-Stokes medium is unable to support a physical wave. If such be the case, we should like to know to what extent the physical "wave" looks like the classical wave. In this appendix we answer the question: What is the hydrodynamic propagation of a "plane" wave in a hydrodynamic medium which is initially homogeneous?

To achieve an answer we must generate the hydrodynamic wave. In the medium (infinite by assumption) let us assume an infinite piston source moving spatially sinusoidally in time in the direction $\hat{x}$. That is, if we set the rest position of the piston at x=0, its subsequent position is given by

$$\underline{r}(x=0,t) = a \sin(\omega_o t)\ \hat{r}$$

where $\underline{a}$ is the piston's maximum displacement.

Now conceive another (inertialess) piston, initially at rest at x, which moves so as to maintain the motion generated by the first piston. The second piston (which does no work in the ideal, homogeneous case), has an assumed motion

$$\underline{r}(\underline{x},t) = (x + a \sin(\omega_o t - k_o x))\ \hat{r}, \qquad k_o = \frac{\omega_o}{c_o} = \frac{2\pi}{\lambda_o} \tag{A.1}$$

for fixed initial rest position x. Notice that the rest position, x, is not the position $\underline{r}(x,t=0)$ since

$$\underline{r}(x,t=0) = (x + a \sin(-k_o x))\ \hat{r} \neq x\hat{r}$$

in general.

Let us denote by x the "rest" position; by "initial" position we mean $\underline{r}\ (x,t=0)$.

This is somewhat cumbersome. However, by agreeing to this convention we avoid considering transient conditions. We thus use two epochs: (1) a relaxed state (before any motion) during which we tag each particle with its position $\underline{x}$ (this being a 1-1 correspondence) and (2) a subsequent time when we observe a steady state plane wave propagating as described.

By an abuse of language we characterize all the particles having identical motion in one plane by the single variable $\underline{x}$(or $\underline{r}$) which may be regarded as the distance between the zero rest plane and the plane in question. We thus write $\underline{r}$ as r and drop the unit vector designation.

The motion described by Eq. (A.1) indicates that the particles in the rest plane x, at some differing time, all have a motion identical to the source except for a translation.

Having described thus the motion of the medium, we need to find the particle velocity, dynamic density, dilatation rate, and finally the hydrodynamic pressure.

Before so doing, however, it will be important to make two observations.

No small confusion has resulted from the awkwardness of the usual notation. The symbol $\frac{\partial}{\partial r}$ is used with two distinct (and sometimes contradictory) meanings:

a) $\frac{\partial}{\partial \alpha} \equiv$ the partial derivative with respect to α, all other independent variables held constant (the partial derivative of the calculus).

b) $\frac{\partial}{\partial \alpha} \equiv$ the hydrodynamic derivative with respect to α, the spatial position held constant (Eulerian or fixed-space derivative).

For a) $\frac{\partial r}{\partial r} = 1$;

for b) $\frac{\partial r}{\partial r} = 0$.

Similar confusion surrounds the material (Lagrangian) derivative, $\frac{d}{d\alpha}$, and its counterpart.

Clarity requires a distinction. To avoid the confusion we introduce the notation

$\frac{d}{dt}(\cdot\, r) \equiv$ the temporal derivative, holding the spatial coordinate, r, fixed (Eulerian, fixed space derivative).

$\frac{d}{dt}(\cdot\, x) \equiv$ the temporal derivative, holding the material coordinate, (x), fixed. (Lagrangian, or material derivative).

$\frac{d}{dr}(\cdot\, t) \equiv$ the spatial derivative with time fixed. Since time is an (and usually the only) independent variable, this quantity is often equal to $\frac{\partial}{\partial r} \equiv$ the spatial derivative, holding all other independent variables fixed.

$\nabla \cdot, \nabla \equiv$ the three dimensional spatial derivatives.

Euler's rule in this notation becomes

$$\frac{d}{dt}(\cdot\, x) = \frac{d}{dt}(\cdot\, r) + \underline{v} \cdot \nabla(\cdot).$$

A careful consideration of the above notation contrasted with the usage in most textbooks, will give some appreciation of the consternation of most students.[1] As often as possible, in the sequel, we will calculate all derivatives, a procedure not only instructive but salutary for analysis.

Our second observation comments on the facile expression, too often found, that one may alternate between spatial, (r), and material, (x), coordinates by simply inverting the expression r(x). The reader is invited to invert

$$r(x) = x + a \sin(\omega_0 t - k_0 x)$$

simply. He will soon find that there is no simple closed form. However he tries, he must solve a transcendental equation. Needing the solution, however, we will obtain one (unfortunately complicated).

Put

$$\hat{G} = -a \sin(\omega_0 t - k_0(\cdot)), \qquad \hat{G} \text{ an operator} \qquad (A.2)$$

and write

$$r(x) = (1-\hat{G})[x].$$

Then by a powerful theorem, (the feedback theorem)

$$x(r) = \sum_{n=0}^{\infty} (\hat{G})^{(n)}[r]. \qquad (A.3)$$

The operator, $\hat{G}^{(n)}$ denotes a nested operator i.e.,

$$\hat{G}^{(2)}[r] = -a \sin [\omega_0 t + k_0 a(\sin(\omega_0 t - k_0 r))].$$

---

[1]See [147] pp. 235-239.

It is easy to see that while the position of a particle, r(x), follows a sinusoidal motion (as it must to follow the transducer), the motion of particles passing through a given point, x(r), do not.

The fluid we describe thermodynamically as

$$\frac{d(p|x)}{dt} = c_o^2 \frac{d(\rho|x)}{dt} \quad , \; c_o^2 \text{ constant} \tag{A.4}$$

We insist furthermore that mass be conserved,[1] i.e.,

$$\frac{d(\rho|x)}{dt} = - \rho \nabla \cdot \underline{v} \quad . \tag{A.5}$$

We return now to the calculation of the hydrodynamic variables. First let us determine $\underline{v}$ the particle velocity.

The particle velocity is the velocity the particle undergoes, i.e.,

$$v = \frac{d(\underline{r}|x)}{dt} = \frac{d}{dt} \left(\; x + a \sin(\omega_o t - k_o x) \; | \; x \right)$$

$$v = a \, \omega_o \cos(\omega_o t - k_o x) \quad . \tag{A.6}$$

The associated derivatives are the following:

$$\frac{d(r|r)}{dt} = 0$$

$$\frac{d(x|x)}{dt} = 0 \quad .$$

The derivative $\frac{d(x|r)}{dt}$ can be easily calculated:

---

[1] See Chapter I. Sec. 1.2.1.1

$$\frac{d(r|r)}{dt} = 0 = \frac{d}{dt}\ (x + a\ \sin(\omega_o t - k_o x)\ |\ r),$$

$$= \frac{d(x|r)}{dt} + a\ \cos(\omega_o t - k_o x)\ \frac{d}{dt}\ ((\omega_o t - k_o x)\ |\ r)$$

$$= \frac{d(x|r)}{dt} + a\ \cos(\omega_o t - k_o x)\ \left[\omega_o - k_o\ \frac{d(x|r)}{dt}\right],$$

$$\text{or} \quad \frac{d(x|r)}{dt} = -\frac{a\ \omega_o\ \cos(\omega_o t - k_o x)}{1 - k_o\ a\ \cos(\omega_o t - k_o x)}\ . \tag{A.7}$$

The derivatives $\nabla \cdot x = \frac{\partial x}{\partial r}$ (for plane wave motion) can be obtained from

$$\frac{\partial r}{\partial r} = 1 = \frac{\partial}{\partial r}\ (x + a\ \sin(\omega_o t - k_o x)),$$

$$= \frac{\partial x}{dr} + \ -k_o\ a\ \cos(\omega_o t - k_o x)\ \frac{\partial x}{\partial r},$$

$$\text{or} \quad \frac{\partial x}{\partial r} = \frac{1}{1 - k_o\ a\ \cos(\omega_o t - k_o x)}\ , \tag{A.8}$$

so that for the plane wave, we have

$$\frac{d(x|r)}{dt} = -\underline{v}\ \nabla \cdot \underline{x} = \frac{-d(r|x)}{dt}\ \ \frac{\partial x}{\partial r}\ . \tag{A.9}$$

The conclusion may be reached immediately from Euler's Rule since $\frac{d(x|x)}{dt} = 0$ and $|\nabla \underline{x}| = |\nabla \cdot \underline{x}|$ for a plane wave.

Having determined the particle velocity, $\underline{v}$, and associated derivatives, let us now turn to the calculation of the dynamic density, ρ. We fix on the fact that ρ is a <u>material</u> property of the medium.

In a homogeneous fluid, with unidirectional motion, the density can be given by

$$\frac{\rho_0}{\rho} = \lim_{\Delta x \to 0} \frac{r(x+\Delta x) - r(x)}{\Delta x} \quad ,$$

when ρ_0 is the density of the fluid at rest (not equal to $\rho(t=0)$).

Since

$$r(x+\Delta x) = x + \Delta x + a \sin(\omega_0 t - k_0(x+\Delta x)) \quad ,$$

$$\frac{\rho_0}{\rho} = \lim_{\Delta x \to 0} \frac{\Delta x + a[\sin(\omega_0 t - k_0 x)(\cos(-k_0 \Delta x) - \Delta x) + \cos(\omega_0 t - k_0 x)\sin(-k_0 \Delta x)]}{\Delta x} \quad .$$

By L'Hopital's Rule, this gives

$$\frac{\rho_0}{\rho} = 1 - k_0 a \cos(\omega_0 t - k_0 x) \quad ,$$

and

$$\rho(t=0) = \rho_0/(1 - k_0 \cos(k_0 x)),$$

or

$$\rho = \rho_0/(1 - k_0 a \cos(\omega_0 t - k_0 x)) = \rho_0 \frac{\partial x}{\partial r} \quad . \qquad \text{(A.10)}$$

Equation (A.10) describes the density of the particle x; we may use Eq. (A.3) again to obtain $\rho(\underline{r},t)$. We see that the particle does not undergo sinusoidal changes in density; neither does the density at a point in space.

Next turn to the dilatation rate, $\nabla \cdot \underline{v}$. We will calculate this important quantity two ways. The first from basic principles, derives

from the definition

$$\nabla\cdot\underline{v} = \lim_{\Delta r\to 0} \frac{v(r+\Delta r) - v(r)}{\Delta r} \tag{A.11}$$

where v(r) is the velocity of the particle which happens to be at r. This notion is easily confused with

$$\lim_{r_1\to r_2} \frac{v(r(x_1)) - v(r(x_2))}{r(x_1) - r(x_2)} = 0 \neq \nabla\cdot\underline{v} ,$$

or with,

$$\frac{d(v|t)}{dx} = \frac{\partial v}{\partial x} = k_o\, a\, \omega_o \sin(\omega_o t - k_o x) \neq \nabla\cdot\underline{v} .$$

From first principles we get (for a plane wave)

$$\nabla\cdot\underline{v} = \frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\; a\,\omega_o \cos(\omega_o t - k_o \sum_{n=0}^{\infty} \hat{G}^{(n)}[r]) ,$$

$$= a k_o \omega_o (\sin(\omega_o t - k_o x)) \frac{\partial}{\partial r} \sum_{n=0}^{\infty} \hat{G}^{(n)}(\underline{r}) ,$$

so that

$$\nabla\cdot\underline{v} = \frac{k_o\omega_o\, a \sin(\omega_o t - k_o x)}{1 - k_o\, a \cos(\omega_o t - k_o x)} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} , \tag{A.12}$$

after some interesting manipulations.

Because the subtlety of these derivations may leave the reader unconvinced, we give an alternate (and simpler) proof of Eq. (A.12).

Using the continuity equation, $\frac{d(\rho|x)}{dt} + \rho\nabla\cdot\underline{v} = 0$, we may write

$$\nabla\cdot\underline{v} = -\frac{1}{\rho}\frac{d(\rho|x)}{dt} = -\frac{d}{dt}\ln(\rho|x) \ .$$

Knowing ρ, we may thus compute $\nabla\cdot\underline{v}$. Hence from Eq. (A.10)

$$\ln\rho = \ln\rho_0 - \ln\left(1 - k_o\, a\cos(\omega_o t - k_o x)\right)$$

and $$\frac{d}{dt}\ln(\rho|x) = \frac{-k_o\omega_o\, a\sin(\omega_o t - k_o x)}{1 - k_o\, a\cos(\omega_o t - k_o x)} \ ,$$

so that $$\nabla\cdot\underline{v} = \frac{k_o\omega_o\, a\sin(\omega_o t - k_o x)}{1 - k_o\, a\cos(\omega_o t - k_o x)}$$

as before.

The derivation of ρ and $\nabla\cdot v$ from first principles is given here for cogency of argument. For simplicity one may calculate $\nabla\cdot\underline{v}$ directly from $\underline{v}$.

In this case, one has

$$\nabla\cdot\underline{v} = \frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\ a\omega_o\cos(\omega_o t - k_o x) \ = k_o\omega_o\, a\sin(\omega_o t - k_o x)\,\frac{\partial x}{\partial r}$$

$$= \frac{k_o\omega_o\, a\sin(\omega_o t - k_o x)}{1 - k_o\, a\cos(\omega_o t - k_o x)}$$

directly. The calculation for ρ follows quickly.

$$\frac{d}{dt}\ln(\rho|x) = -\frac{k_o\omega_o\, a\sin(\omega_o t - k_o x)}{1 - k_o\, a\cos(\omega_o t - k_o x)} = -\nabla\cdot\underline{v}$$

so that upon material integration,

$$\ln\rho - \ln\rho_0 = -\ln(1 - k_o\, a\cos(\omega_o t - k_o x)) + \ln(1),$$

or $$\rho = \frac{\rho_0}{1 - k_o\, a\cos(\ _o t - k_o x)}$$

as before.

With the density known, we may calculate the pressure directly from the equation of state

$$\frac{d(p|x)}{dt} = c_o^2 \frac{d(\rho|x)}{dt} .$$

Thus $$p_2(t) - p_2(t=0) = c_o^2(\rho(t) - \rho(t=0))$$

or

$$p_2(t) - p_2(t=0) = \rho_0 c_o^2 \left(\frac{1}{1 - k_o\, a \cos(\omega_o t - k_o x)} - \frac{1}{1 - k_o\, a \cos(k_o x)} \right).$$

We have thus[1]

$$p_2(x,t) = \rho_0 c_o^2/(1 - k_o\, a \cos(\omega_o t - k_o x)) + b, \qquad (A.13)$$

$$p_2(x,t=0) = \rho_0 c_o^2/(1 - k_o\, a \cos(k_o x)) + b, \qquad (A.14)$$

$$p_2(\text{rest}) = \rho_0 c_o^2 + b, \qquad (A.15)$$

for some constant b.

Finally the acoustic pressure $p = p_2(x,t) - p_2(\text{rest})$ or

$$p(x,t) = \frac{\rho_0 c^2\, k_o\, a \cos(\omega_o t - k_o x)}{1 - k_o\, a \cos(\omega_o t - b_o x)} . \qquad (A.16)$$

All the relevant hydrodynamic variables may now be summarized as in Tables A-1 to A-3, Plane Wave Parameters. We check finally that the fundamental equation is satisfied:

$$\frac{d(p|x)}{dt} + \rho c^2 \nabla \cdot v = 0$$

---

[1]For b=0, this solution gives $p/\rho = p_o/\rho_o$, so that (in terms of an ideal gas), the solution is exact for a fluid with parameter $\gamma = 1$, $(c^2 \equiv c_o^2)$.

Since here,

$$\frac{d(p|x)}{dt} = \frac{-\rho_o c_o^2 \omega_o k_o \, a \sin(\omega_o t - k_o x)}{[1 - k_o \, a \cos(\omega_o t - k_o x)]^2} ,$$

$$\rho = \rho_0/(1 - k_o \, a \cos(\omega_o t - k_o x)),$$

and

$$\nabla \cdot \underline{v} = \frac{\omega_o k_o \, a \sin(\omega_o t - k_o x)}{1 - k_o \, a \cos(\omega_o t - k_o x)} ,$$

the fundamental equation reduces to

$$\frac{d(p|x)}{dt} + \rho c^2 \nabla \cdot v = \frac{-\rho_o c_o^2 \omega_o k_o \, a \sin(\omega_o t - k_o x)}{[1 - k_o \, a \cos(\omega_o t - k_o x)]^2}$$

$$+ \frac{\rho_o c_o^2}{[1 - k_o \, a \cos(\omega_o t - k_o x)]} \quad \frac{\omega_o k_o \, a \sin(\omega_o t - k_o x)}{[1 - k_o \, a \cos(\omega_o t - k_o x)]} = 0$$

so that these solutions indeed solve the hydrodynamic equations.

Graphs for the important variables are also given. The first graph, (Fig. A-1, Position Graph), gives a graphical solution of position at t = 0 and allows the transition between x and r coordinates. The other graphs (Figs. A-2 to A-5) are given in material coordinates. It should be noted that the pressure, density, and dilatation rate are <u>not</u> sinusoidal, in contrast to particle position and velocity, which are (in material coordinates only). In spatial coordinates, not one of these parameters is sinusoidal. Moreover the phase velocity is seen to be a function of the <u>material</u>, rather than the spatial, coordinates.

The "error" of the classical form, can be easily expressed. Supposing the classical form to be written as

$$\hat{p}_2(\underline{r},t) = \rho_0 c_0^2 \, (1 + k_0 \, a \, \cos(\omega_0 t - k_0 r)) + b$$

(as usual for the <u>total</u> pressure), then noting that

$$\text{error} = e_2 = p_2 - \hat{p}_2$$

$$= \rho_0 c_0^2 \left(\sum_{n=2}^{\infty} (k_0 a)^n \cos^n(\omega_0 t - k_0 x) + k_0 a \, (\cos(\omega_0 t - k_0 x) - \cos(\omega_0 t - k_0 r)) \right) = p - \hat{p} \quad . \tag{A.17}$$

Thus the approximation

$$\frac{\rho_0 c^2}{1 - k_0 \, a \, \cos(\omega_0 t - k_0 x)} \simeq \rho_0 c^2 (1 + k_0 \, a \, \cos(\omega_0 t - k_0 r))$$

depends on the value of $k_0 a$, and may be suspect for large k_0 (high frequency) or large <u>a</u> (high power).

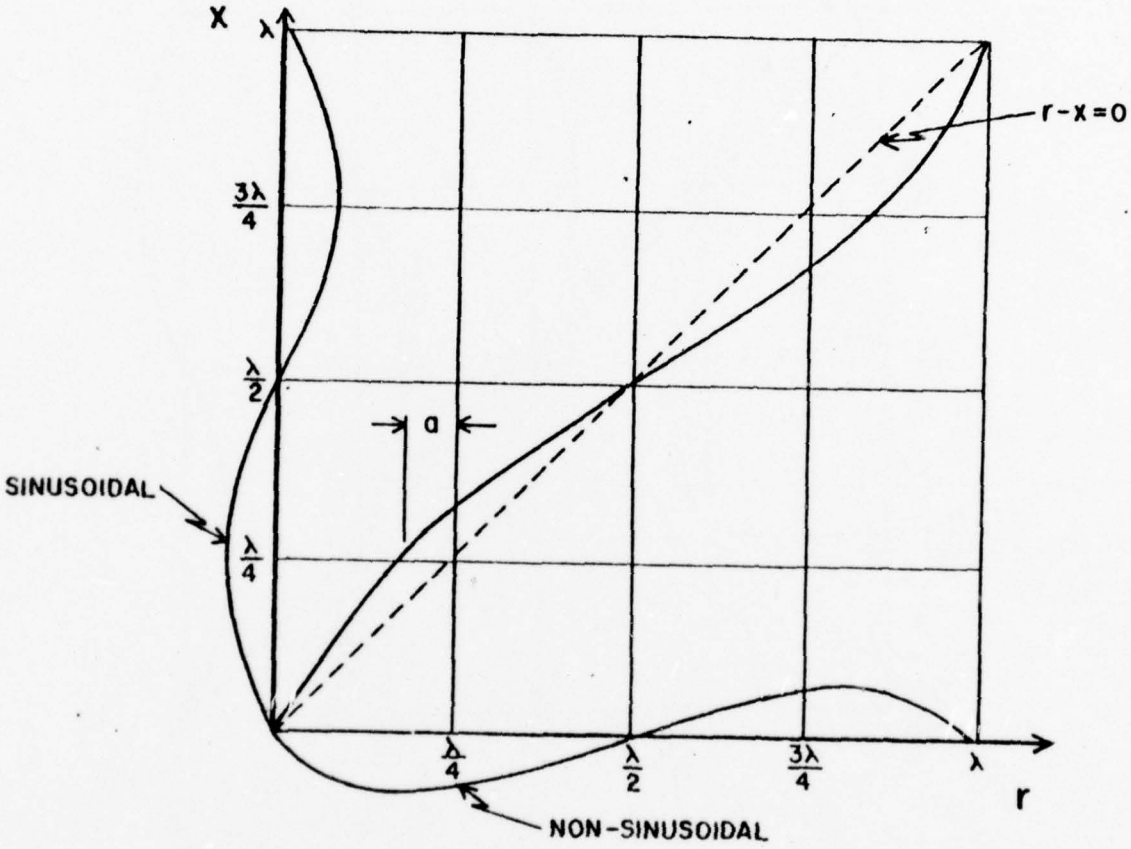


Fig. A-1. Position Graph

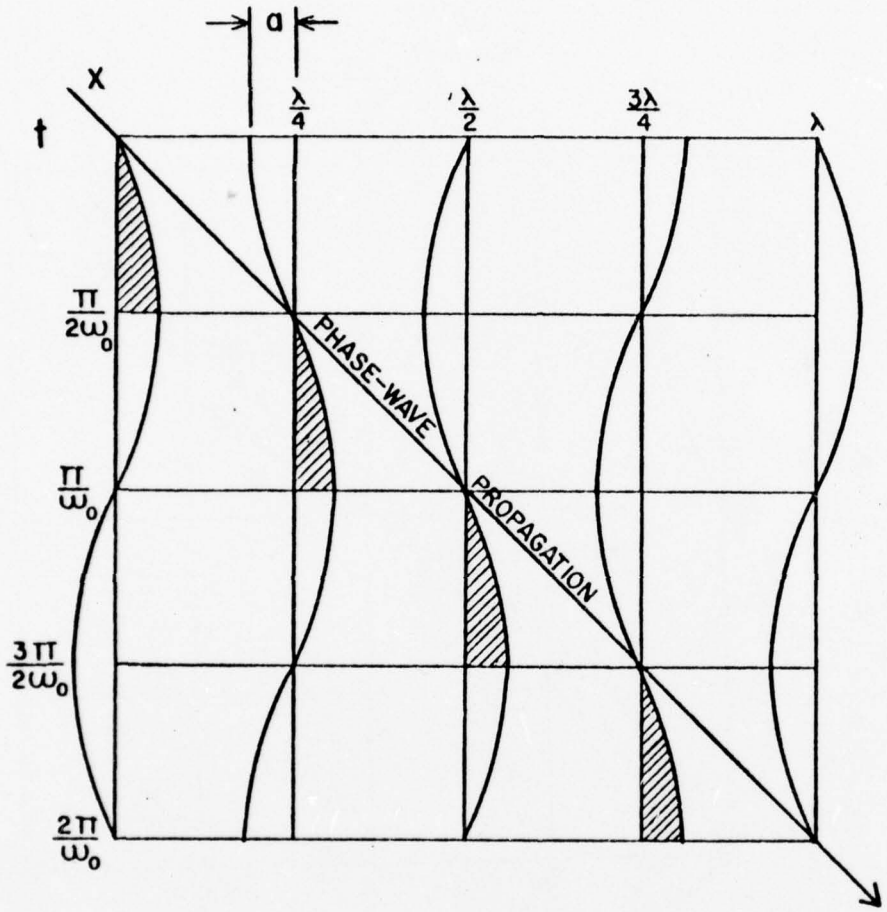


Fig. A-2. Particle Position $r = x+a \sin(\omega_0 t - k_0 x)$

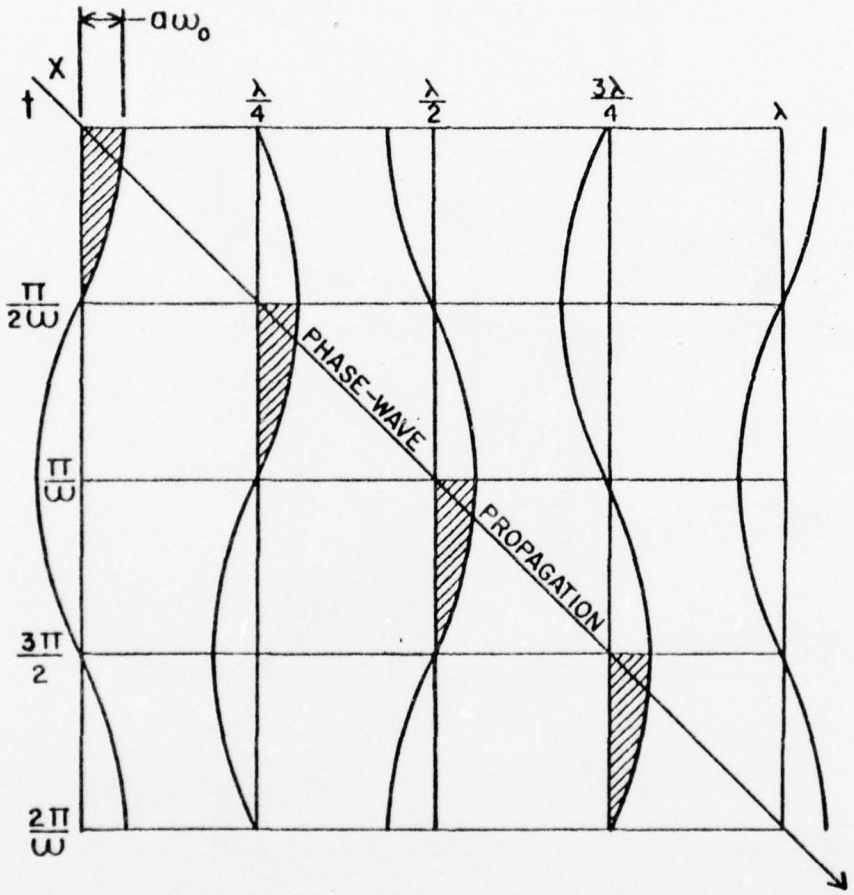


Fig. A-3. Particle Velocity, $v = a\omega_o \cos(\omega_o t - k_o x)$

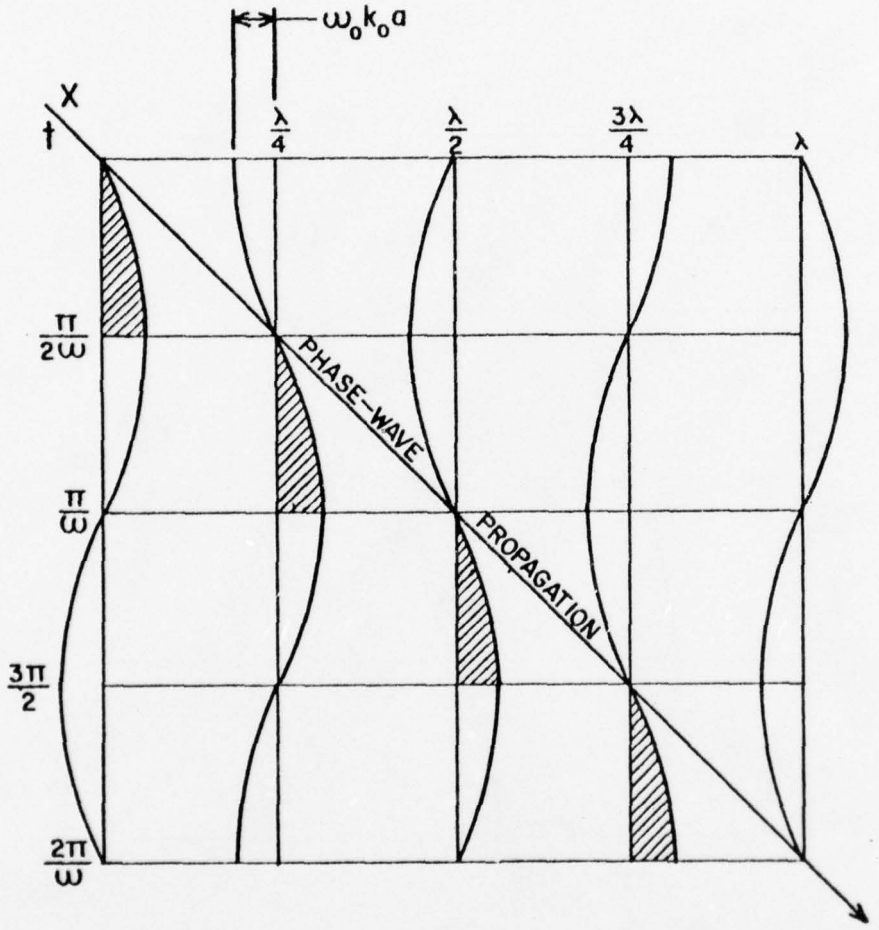


Fig. A-4. Dilatation Rate, $\nabla.\underline{v}$

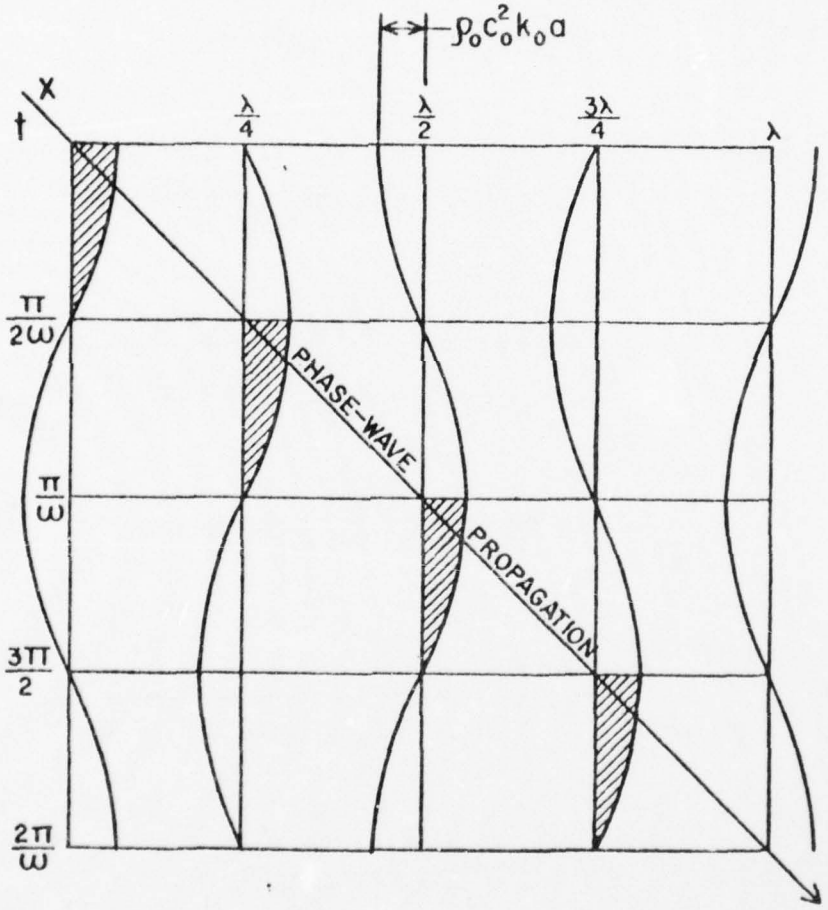


Fig. A-5. Acoustic Pressure, p

position

$$r(x) = x + a \sin(\omega_o t - k_o x)$$

velocity

$$\frac{d(r|r)}{dt} = 0$$

$$\frac{d(x|r)}{dt} = \frac{-a\ \omega_o \cos(\omega_o t - k_o x)}{1 - k_o\ a \cos(\omega_o t - k_o x)}$$

acceleration

$$\frac{d(v|r)}{dt} = \frac{a\omega_o^2 \sin(\omega_o t - k_o x)}{1 - k_o\ a \cos(\omega_o t - k_o x)}$$

density

$$\rho = \rho_0/(1 - k_o\ a \cos(\omega_o t - k_o x(r)))$$

$$\frac{d(\rho|r)}{dt} = \frac{-\rho_0 \omega_o k_o\ a \sin(\omega_o t - k_o x)}{[1 - k_o\ a \cos(\omega_o t - k_o x)]^3}$$

pressure

$$p_2 = \frac{\rho_0 c_o^2}{1 - k_o\ a \cos(\omega_o t - k_o x)} + b$$

$$\frac{d(p_2|r)}{dt} = \frac{-\rho_0 c_o^2 \omega_o k_o\ a \sin(\omega_o t - k_o x)}{[1 - k_o\ a \cos(\omega_o t - k_o x)]^3}$$

$$p = \frac{\rho_0 c^2 k_o\ a \cos(\omega_o t - k_o x)}{1 - k_o\ a \cos(\omega_o t - k_o x)}$$

TABLE A-1. Plane Wave Parameters (Fixed-Space)

position $$x(\underline{r}) = \sum_{n=0}^{\infty} \hat{G}^{(n)}[r],\ \hat{G} = -a \sin(\omega_0 t - k_0(\cdot))$$

velocity $$\underline{v} = \frac{d(r|x)}{dt} = a\ \omega_0 \cos(\omega_0 t - k_0 x)$$

$$\frac{d(x|x)}{dt} = 0$$

acceleration $$\frac{d(v|x)}{dt} = -a\ \omega_0^2 \sin(\omega_0 t - k_0 x)$$

density $$\rho = \rho_0/(1 - k_0\ a \cos(\omega_0 t - k_0 x))$$

$$\frac{d(\rho|x)}{dt} = \frac{\rho_0 k_0 \omega_0\ a \sin(\omega_0 t - k_0 x)}{[1 - k_0\ a \cos(\omega_0 t - k_0 x)]^2}$$

pressure $$p_2 = \frac{\rho_0 c_0^2}{1 - k_0\ a \cos(\omega_0 t - k_0 x)} + b$$

$$\frac{d(p|x)}{dt} = \frac{\rho_0 c_0^2 k_0 \omega_0\ a \sin(\omega_0 t - k_0 x)}{[1 - k_0\ a \cos(\omega_0 t - k_0 x)]^2}$$

$$\nabla \cdot \underline{T} = -\rho_0 a \omega_0^2 \sin(\omega_0 t - k_0 x)/(1 - k_0\ a \cos(\omega_0 t - k_0 x))$$

$$p = \frac{\rho_0 c_0^2\ k_0\ a \cos(\omega_0 t - k_0 x)}{[1 - k_0\ a \cos(\omega_0 t - k_0 x)]}$$

TABLE A-2. Plane Wave Parameters (Material)

velocity

$$\frac{d(x|t)}{dr} = \frac{1}{1 - k_o\, a\, \cos(\omega_o t - k_o x)} = \nabla \cdot \underline{x}$$

acceleration

$$\nabla \cdot \underline{v} = \frac{\omega_o k_o\, a\, \sin(\omega_o t - k_o x)}{1 - k_o\, a\, \cos(\omega_o t - k_o x)}$$

density

$$\nabla \rho = \frac{\rho_0 k_o^2\, a\, \sin(\omega_o t - k_o x)}{[1 - k_o\, a\, \cos(\omega_o t - k_o x)]^3}$$

pressure

$$\nabla p = \frac{\rho_0 c_o^2 k_o^2\, a\, \sin(\omega_o t - k_o x)}{[1 - k_o\, a\, \cos(\omega_o t - k_o x)]}$$

$$= \frac{\rho_0 \omega_o^2\, a\, \sin(\omega_o t - k_o x)}{[1 - k_o\, a\, \cos(\omega_o t - k_o x)]^3}$$

TABLE A-3. Plane Wave Parameters (Variational)

Appendix B

COMPUTABILITY

Introduction

The exact solution to the wave equation,

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) p = s, \tag{B.1}$$

in an arbitrary medium with boundaries is given by Eq. (5.55)

$$p = \hat{G}\left(s + d \sum_{n=0}^{\infty} (\hat{G}'d)^{(n)} \hat{G}'(s)\right) \tag{B.2}$$

where $c^2(1 + d) \equiv c_0^2$, a reference speed and where,

$$\hat{G} = \int_{\omega_0} \int_{t_0} \int_{\underline{r}_0} (\cdot) \frac{e^{j(\omega_0(t - t_0) - k_0|\underline{r}-\underline{r}_0|)}}{8\pi^2|\underline{r}-\underline{r}_0|} \, d\underline{r}_0 \, d\omega_0 \, dt_0 \tag{B.3}$$

and $\hat{G}' = -k_0^2\hat{G}$, $k_o^2 = \omega_o^2/c_o^2$.

Equation (B.2) may be rewritten in operational form shown in Fig. B-1.

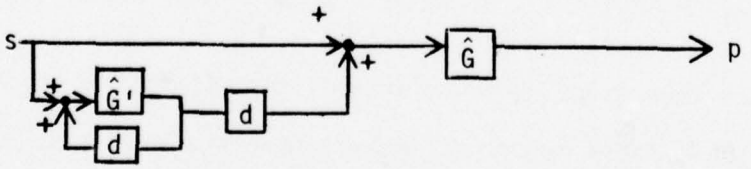


Fig. B-1. Operational Solution of Eq. (5.55)

Here the feed-forward operators, $\hat{G}$, $\hat{G}'$ are purely geometrical, whereas the feedback parameter, d, is purely medium-related.

The ability to solve the system of Fig. B-1, being the solution to wave propagation in any arbitrary medium with boundaries (described by $d(\underline{r},t)$), is necessarily the solution to, and generalization of, a large body of problems involving acoustic propagation.

For random media, moment analogues are known. (Cf. Eq. (5.26), Fig. 5-3.)

Problem

Though, in form, the system of Fig. B-1 is reminiscent of control theory formulations, some differences need noting. Both $\hat{G}$ and d are multidimensional. The process may be sequenced for each operation of $\hat{G}$:

pass no.	$\hat{p}$	
1:	$\hat{p}_1(\underline{r},t) = \hat{G}(s)$	=The unscattered wave,
2:	$\hat{p}_2(\underline{r},t) = \hat{G}(s) + \hat{G}d\hat{G}'(s)$	+single scattering
3:	$\hat{p}_3(\underline{r},t) = \hat{G}(s) + \hat{G}d\hat{G}'d\hat{G}'(s) + \hat{G}d\hat{G}'(s)$	+double scattering
(m)	$p_m(r,t) = \hat{G}(s) + \hat{G}d \sum_{n=0}^{m} (\hat{G}'d)^{(n)}\hat{G}'(s)$	+ mth scattering, etc.

The process thus gives us the effect of the mth scattering directly as $\hat{p}_{m+1} - \hat{p}_m$. Surfaces and other interfaces are given by the discontinuities defined by $d = (c_0^2/c^2) - 1$. For example, at the air/water interface $d = (1500)^2/(330)^2 - 1 = 19.7$, nominally. When exact surface conditions are known $d = d(\underline{r},t)$ may directly incorporate the effects (temporal and spatial) of the undulations. When, as most often, only statistical knowledge is available, probabilistic measures must be used.

To compute the system of Fig. B-1 the following must be done:

(a) Evaluate a five dimensional integral, $\hat{G}(\cdot)$;

(b) Store a four dimensional medium variable, d.

Neither represents a theoretical problem. From an analog viewpoint, the memory device for (b) may not exist.

From a digital viewpoint, the requirement first to store and secondly to implement the integration brings problems of sampling rates, of approximations and of capacity. Of sampling rates, we must expect some analogy to the sampling theorem to hold (i.e., spatial sampling must occur twice or more than the spatial frequency of interest) with attendant "aliasing" problems. Of approximation, we may expect errors to be multiplied in the five fold integration. Of capacity, we should dearly love to know, for the algorithm is, as yet, an unexplored calculation.

Appendix C

DIMENSIONS

Appreciation of the hydrodynamic equations can be aided materially by an awareness of dimensional statements. The first order general Eq. (1.22) is a statement of power density. Its second order companion, Eq. (1.27), is a statement about temporal change of power density. Alternate forms using $\nabla^2 p$ are statements about pressure flux.[1]

Table C-1 is a table of dimensional relationships.

---

[1]Van Nostrand's Scientific Encyclopedia (Fourth Ed.) gives three meanings for "flux."

"1. A quantity proportional to the surface integral of the normal (perpendicular) for field intensity over a given area...

$$\text{Flux} = K \int_S F_n dS,$$

where...K is the constant of proportionality between the field and the flux density...
2. A term which denotes the volume or mass of fluid or particles transferred across a given area perpendicular to the direction of flow in a given time.
3. An aid to melting."

Our meaning here is the first, where

$$\text{Pressure Flux} \equiv \lim_{V \to x} \frac{1}{V} \int_S \nabla p \cdot \hat{n} dS = \nabla^2 p.$$

TABLE C-1. DIMENSIONAL RELATIONSHIPS

Element	Physical Dimension	Spatial Dimension	Name
p	F/L^2	1	pressure
ρ	M/L^2	1	density
c	L/T	1	wavefront propagation speed
$\underline{c}$	L/T	3	wavefront propagation velocity
$\underline{v}$	L/T	3	particle velocity
$\underline{f}$	F/M $(= L/T^2)$	3	(body) force/unit mass
T	F/L^2	3x3	stress tensor (pressure)
Operators			
$\frac{d}{dt}$	$\frac{1}{T}$	-	temporal frequency
∇	$\frac{1}{L}$	-	spatial frequency
$\frac{d^2}{dt^2}$	$\frac{1}{T^2}$	-	temporal change in temporal frequency
∇^2	$\frac{1}{L^2}$	-	spatial change in spatial frequency

TABLE C-1. DIMENSIONAL RELATIONSHIP (Cont'd)

Element - Combinations	Physical Dimension	Spatial Dimension	Name
$p\underline{v}$	$\frac{1}{L^2}\ \frac{FL}{T}$	3	power/area
$\underline{c}\cdot\underline{c} = c^2$	L^2/T^2	1	velocity squared
ρc^2	F/L^2	1	pressure
Element - operator combinations			
$\nabla\cdot\underline{v}$	$1/T$	1	temporal frequency
$\frac{dp}{dt}$	$\frac{1}{T}\ \frac{FL}{L^3}$	1	power density
$\frac{d\rho}{dt}$	$M/L^3T = \frac{TF}{L^4}$	1	pressure-seconds/area
$\frac{d\underline{v}}{dt}$	L/T^2	3	particle acceleration
∇p	F/L^3	3	spatial change in pressure
$\nabla\cdot T$	F/L^3	3	spatial change in pressure

TABLE C-1. DIMENSIONAL RELATIONSHIPS (Cont'd)

Element - operator combinations	Physical Dimension	Spatial Dimension	Name
(Cont'd)			
$\frac{d^2p}{dt^2}$	$\frac{1}{T}\ \frac{1}{L^3}\ \frac{FL}{T}$	1	temporal change in power density
$\nabla^2 p$	$\frac{F}{L^2}\ \frac{1}{L^2}$	1	pressure flux
$\underline{v}\cdot\nabla p$	$\frac{1}{T}\frac{1}{T}\frac{F}{L^2} = \frac{FL}{T}\ \frac{1}{L^3}$	1	power density
$\rho c^2 \nabla\cdot\underline{v}$	$\frac{M}{L^3}\frac{L^2}{T^2}\frac{1}{L}\frac{L}{t} = \frac{1}{L^3}\ \frac{ML}{T^2}\ \frac{L}{T}$	1	power density
$\nabla\cdot(\underline{v}p)$	$\frac{1}{L^3}\ \frac{FL}{T}$	1	power density
$\frac{d}{dt}(\ln\rho c^2 \mid \underline{r})$	$\frac{1}{T}$	1	frequency
$\underline{v}\cdot\nabla\underline{v}$	$\frac{L}{T^2}$	3	acceleration

TABLE C-1. DIMENSIONAL RELATIONSHIPS (Cont'd)

Element - operator combinations	Physical Dimension	Spatial Dimension	Name
(Cont'd)			
$\frac{\nabla \cdot \underline{T}}{\rho}$	$\frac{F}{M}$	3	acceleration
$\left(\frac{d}{dt}(\ln \rho c^2 \mid \underline{r})\right) \underline{v}$	L/T^2	3	acceleration
$c^2 \nabla$	L/T^2	3	acceleration
$c^2 \nabla \ln \rho$	L/T^2	3	acceleration
$\nabla \ln \rho$	$1/L$	3	spatial frequency
$\left(\frac{d}{dt}(\ln \rho c^2 \mid \underline{r})\right)\left(\frac{d}{dt}(p \mid \underline{r})\right)$	$\frac{1}{T}\ \frac{1}{L^3}\ F\ \frac{L}{T}$	1	temporal change in power density
$\rho c^2 \nabla \cdot \underline{f}$	$\frac{1}{T}\ \frac{1}{L^3}\ F\ \frac{L}{T}$	1	temporal change in power density
$\rho c^2 \nabla \cdot (\underline{v} \cdot \nabla \underline{v})$	$\frac{1}{T}\ \frac{1}{L^3}\ F\ \frac{L}{T}$	1	temporal change in power density

TABLE C-1. DIMENSIONAL RELATIONSHIPS (Cont'd)

Element - operator combinations		Physical Dimension	Spatial Dimension	Name
(Cont'd)				
$\rho c^2 \nabla \cdot \frac{\nabla \cdot \underline{T}}{\rho}$		$\frac{1}{T} \frac{1}{L^3} F \frac{L}{T}$	1	temporal change in power density
Operators				
$\hat{G}$	(5.52)	L^2	-	a feed forward operator
$\hat{G}'$	(5.52)	1	-	a feed forward operator
$\hat{\xi}$		$1/L^2$	1	a feed backward operator
d		1	1	a feed backward parameter
Q		$1/L^2$	1	a feed backward parameter

Appendix D

INVERTING $(\nabla^2 + k_0^2)p$

We want to solve

$$\hat{L}\, p(\underline{r}_0,\underline{r}') = [\nabla^2 + k_0^2][p(\underline{r}_0,\underline{r}')]$$

$$= \hat{\varepsilon}(\underline{r}_0)p(\underline{r}_0,\underline{r}') + s(\underline{r}'), \tag{D.1}$$

or symbolically

$$\hat{L}p = \hat{\varepsilon}p + s,$$

which describes the situation at $p(r_0)$ due to radiation from $\underline{r}'$ transmitted through a medium characterized by $\hat{\varepsilon}$. The solution is $p(\underline{r},\underline{r}')$ for some designated point, $\underline{r}$.

Observe (Chapter V, Sec. 5.2.1) that

$$[\nabla_\rho^2 + k_0^2]\,\frac{e^{-jk_0|\underline{r} - \underline{r}_0|}}{4\pi|\underline{r} - \underline{r}_0|} = \delta(\underline{r} - \underline{r}_0), \tag{D.2}$$

where

$$\nabla_\rho^2 \equiv \frac{\partial^2(\cdot)}{\partial^2|\underline{r} - \underline{r}_0|} + \frac{2(\cdot)}{|\underline{r} - \underline{r}_0|}$$

for the given function. If we can find and inverse $\hat{G} = (\hat{L}^{-1})$ then the solution is a straightforward iteration. (Cf. Sec. 5.0.)

From Eq. (D.2), we have

$$\int_{\underline{r}_0} \left(p(\underline{r}_0)[\nabla_\rho^2 + k_0^2] \; \frac{e^{-jk_0|\underline{r} - \underline{r}_0|}}{4\pi|\underline{r} - \underline{r}_0|} \right) d\underline{r}_0$$

$$= \int_{\underline{r}_0} p(\underline{r}_0)\delta(\underline{r} - \underline{r}_0)d\underline{r}_0 = p(\underline{r}) \tag{D.3}$$

which is inversion of sorts, but scarcely useful in Eq. (D.1). None of the terms in Eq. (D.3) is available from Eq. (D.1). (Remember that $[\nabla^2 + k_0^2]p(\underline{r}_0)$ is a function.)

The assertion of the literature[1] is:

given

$$[\nabla_\rho^2 + k_0^2]p(\underline{r}_0,\underline{r}') = \xi(\underline{r}_0)p(\underline{r}_0) + s(\underline{r}'), \tag{D.4}$$

and

$$[\nabla_\rho^2 + k_0^2]g(\underline{r},\underline{r}') = \delta(\underline{r} - \underline{r}'), \tag{D.5}$$

then

$$p(\underline{r},\underline{r}') = \int_{\underline{r}_0} g(\underline{r},\underline{r}_0)s(\underline{r}')d\underline{r}_0$$

$$+ \int_{\underline{r}_0} g(\underline{r},\underline{r}_0)\hat{\xi}(\underline{r}_0)p(\underline{r}_0,\underline{r}')d\underline{r}$$

$$= \int_{\underline{r}_0} g[\hat{\xi}p + s] \; . \tag{D.6}$$

[1] See [55], Eq. (2.3), p. 128.

In Eq. (D.5)

$$g(\underline{r},\underline{r}') = \frac{e^{-jk_0|\underline{r} - \underline{r}'|}}{4\pi|\underline{r} - \underline{r}'|}$$

using the argument of the text. The LHS of Eq. (D.6) arises by multiplying the LHS of Eq. (D.4) by $g(\underline{r},\underline{r}_0)$ and integrating over the field points $\underline{r}_0$. For the LHS, this means $\int_{\underline{r}_0} g\hat{L}p$. But, in general,

$$g\hat{L}p \neq p\hat{L}g \ .$$

In other words, the contention of the literature is true provided

$$\int_{\underline{r}_0} (g\hat{L}p - p\hat{L}g)d\underline{r}_0 = 0 \qquad (D.7)$$

Now, for any realization of p which is continuous and has the proper derivatives, we have

$$\int_{\underline{r}_0} [g(\nabla^2 + k_0^2)p - p(\nabla^2 + k_0^2)g]d\underline{r}_0$$

$$= \int_{\underline{r}_0} (g\nabla^2 p - p\nabla^2 g)d\underline{r}_0 = \int_{S(r_0)} \left(g\frac{\partial p}{\partial n} - p\frac{\partial g}{\partial n}\right) dS \qquad (D.8)$$

by Green's theorem, where $s(r_0)$ is the surface at infinity and $\partial/\partial n$ is the normal to that surface ($=\hat{n}\cdot\nabla(\cdot)$). On this surface, both g and $\partial g/\partial n$ are zero. Even so, we must postulate a radiation condition, that the surface integral does, indeed, equal zero. (There obviously exist arbitrary functions, p, which overpower $\partial g/\partial n$, for example.)

With this radiation postulate, we have

$$\int_{\underline{r}_0} g\hat{L}p = \int_{\underline{r}_0} p\hat{L}g = p(r), \qquad \text{for } \hat{L} = (\nabla^2 + k_0^2) \; . \tag{D.9}$$

Again, to summarize: to invert $\hat{L} = (\nabla^2 + k_0^2)$ using the function $e^{-jk_0|\underline{r} - \underline{r}_0|}/4\pi|\underline{r} - \underline{r}_0|$, we require that

1) p has first and second partial derivatives, all single valued and continuous in the volume (a regularity condition),

2) $\int_{S(\underline{r}_0)_{\to\infty}} \left(g \frac{\partial p}{\partial n} - p \frac{\partial g}{\partial n} \right) dS = 0$ (a radiation condition).

The restrictions of an unbounded medium and continuity may be relaxed if we allow p to be a generalized function (whose main property is that derivatives always exist in a generalized sense). At the discontinuities, a residual will occur which must be reflected in the statistics of $\hat{\varepsilon}$.

Further restraints on the invertibility of g are given in Appendix H.

Appendix E

CORRELATION DECOMPOSITION

Given a random variable, a, over a field, r, we extend the usual notion of a correlation to k-space, $(r_1 x r_2 x r_3 x r_4 \ldots . x r_k)$, as a particular functional given by $c_k(a) = \langle a(r_1)a(r_2) \ldots . a(r_k)\rangle$. We call $c_k(a)$ the k_{TH} order correlation function of the random variation a. The point of view adopted here underscores the correlation structure as completely defining the process; that is, for an arbitrary correlation structure of all orders, there exists a random process exhibiting the structure. A general study of correlation structures will then be a general study of random processes.

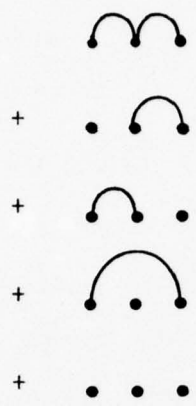
It is known that for some distributions $c_k(a)$ may be decomposed into suborders. The even order correlations of zero-mean normal distributions, for instance, can be decomposed into sums of products of second order correlations.[1] Again, the possible third order correlation structure admits only five possibilities. These are:

$$
\begin{aligned}
c_3(a) = \langle a(r_1)a(r_2)a(r_3)\rangle &\equiv \langle 1,2,3\rangle \\
+ \langle a(r_1)\rangle\langle a(r_2)\ a(r_3)\rangle &\equiv \langle 1\rangle\langle 2,3\rangle \\
+ \langle a(r_1)a(r_2)\rangle\langle a(r_3)\rangle &\equiv \langle 1,2\rangle\langle 3\rangle \\
+ \langle a(r_1)a(r_3)\rangle\langle a(r_2)\rangle &\equiv \langle 1,3\rangle\langle 2\rangle \\
+ \langle a(r_1)\rangle\langle a(r_2)\rangle\langle a(r_3)\rangle &\equiv \langle 1\rangle\langle 2\rangle\langle 3\rangle
\end{aligned}
\qquad \text{(E.1)}
$$

---

[1] See [140], p. 343.

In terms of Feynman diagrams, these are



(E.2)

The symbol "+" in Eq. (E.1) and Eq. (E.2) may not refer to an algebraic operation. The correlation may be any one, any combination or all of the "addends." Thus, "+" represents something between algebraic and logical summation. If $\langle a(r)\rangle = 0$, then only $\langle a(r_1)a(r_1)a(r_2)\rangle$ is possible.

We seek to determine the number of these possible correlation structures for arbitrary k. Our approach is to divide the problem into two parts: (1) to inquire concerning the substructure in k, and (2) to fix the number of correlation possibilities in the substructure. In the example of $c_3(a)$, we see that there are decompositions containing first order correlations; others containing second order correlations and one containing a third order correlation. Our two step process is: first, to find the n structure of k and then to find the correlation possibility in each separate n structure element. The n = 1 structure will contain all possible correlation structures containing $\langle a(r_i)\rangle$. The n = 2 structure will contain all correlation structures with $\langle a(r_i)a(r_j)\rangle$ but <u>no</u> n = 1 structure, and so forth. Returning to the $c_3(a)$ example, we designate the n-structure in the following way:

3 structure - (3)

2 structure - (none)

1 structure - (1,2),(1,1,1).

It is clear that the n-structures of k are simply the unique additive factors of the number k. In the example, 3 = 3+0 or 3 = 2+1 or 3 = 1+1+1. For this condition, the order, then, is not important, i.e., (1,2) ≡ (2,1).

We seek, thus, the number of additive factors, b(k), of an integer k > 0 where an additive factor is a positive integer n > 0 such that n+m = k, where m is also a positive integer. We sometimes shorten "additive factor" to simply "factor."

To begin, let us define the set S($\underline{n}$;m) as the set of factorizations of k having the form ($\underline{n}$,S(m)) where S(m) is the set of all additive factors of m. Here, $\underline{n}$ is a vector notation. For example,

$$S(\underline{1};3) = (1,3), (1,2,1), (1,1,1,1)$$

where

$$S(3) = (3), (2,1), (1,1,1)$$

Secondly, define the difference operators backwardly as:

$$E\ b(k) \equiv b(k-1)$$

$$\Delta\ b(k) = b(k) - b(k-1) = (1-E)(b(k))$$

$$\Delta_j b(k) = b(k) - b(k-j) = (1-E^j)\ b(k).$$

We give the solution in recursive form by showing that

$$b(k+1) = \sum_{i=0}^{k} E^i \prod_{j=0}^{i} \Delta_j b(k), \qquad (E.3)$$

where

$$\prod_{o}^{i} \Delta_j b(k) = \Delta \Delta_1 \Delta_2 \ldots \Delta_j b(k-1)$$

$k = 1,2,3,\ldots$.

To be precise, we deal with two different functions:

$$b_s: \; S \to N$$

and

$$b_k: \; N \to N$$

where the first is a set function (actually a measure), and the second is a function over the natural integers, N. Both functions, except for one exception, are identical in their ranges; but they have different domains. The exception concerns the domain of b(0) and is reflected in the problem of initializing the induction. In particular, we have

$$b_s(\emptyset) = 0,$$

$$b_s(s(0)) = 0,$$

$$b_s(S(k;0)) = 1,$$

For any $m > 0$, $b_s(s(m)) = b_k(m)$, so that the ambiguity is trivial except for $k = 0$. In the example given for k=6, b(S(1,2,;3))=b(3)=3.

Similarly, $b(24,3;3) = b(3) = 3$ since $S(24,3;3) = \{(24,3;3), (24,3;2,1)$ $(24,3;1,1,1)\}$, where $k = 30$. For initialization, we see that $b(1) = 1$ and, since zero is excluded as a factor, $b(0) = 0$.

Further note that for a sequence of sets $S_0, S_1, \ldots S_k$

$$S_0 \overline{S_1}\ \overline{S_2}\ \overline{S_3} \ \ldots \overline{S_k} = S_0 - S_0(S_1 + S_2 + \ldots S_k)$$

$$+ S_0(S_1S_2, + S_1S_3 + S_2S_3 + \text{all doubles})$$

$$+ S_0 \text{ (All triples)}$$

$$+ \ldots .$$

$$+ S_0 \text{ (All n-tuples)}$$

$$+ \ldots .$$

$$+ S_0S_1S_2 \ldots S_k. \qquad \text{(E.4)}$$

Here, the overbar denotes the complement; multiplication, logical intersection; addition, logical union; and subtraction, the residual set $(S_0\text{-}S_i = S_0 \quad \overline{S}_i)$. The results can be shown by mathematical induction. The induction will be shown to be valid for $k \geq 1$ generating the pseudo sequence $1, 1, 2, 3, 5, 7, 11, \ldots b(k), \ldots$.

The actual sequence desired is $0, 1, 2, 3, 5, 7, 11, \ldots b(k)$. The difficulty arises from the ambiguity between the set and integer functions noted earlier. The induction will thus be correct for all k except $k = 0$ where we know $b(0) = 0$. Our initialization is thus

$$b(0) = 1,$$

$$b(1) = 1,$$

$$b(k) = 0,\ k < 0. \qquad \text{(E.5)}$$

For k = 1, the proposition states:

$$b(1) = E^0 \circ b(1) = b(0) = 1,$$

$$b(2) = E^0 \circ b(1) + E^1 \circ b(1)$$

$$= b(1) + E^1(B(1) - b(0)) = b(1) + b(0) = 2, \qquad \text{(E.6)}$$

so that the induction initializes the pseudo sequence correctly. Next, assuming the proposition

$$b(k) = \sum_{i=0}^{k-1} E^i \prod_{j=0}^{i} j\ b(k-1),$$

we show that

$$b(k+1) = \sum_{i=0}^{k} E^i \prod_{j=0}^{i} j\ b(k).$$

We first partition S(k+1), the set of all additive factors of k+1, in the following way.

$$S(k+1) = S(1;k) + S(2;k-1)\ \overline{S(1;k)} + \ldots .$$

$$+ (S(i;k-1+1)\ \overline{S(1;k)}\ \overline{S(2;k-1)} \ldots .$$

$$\ldots . \overline{S(i-1,\ k-i+2)} + \ldots$$

$$+ (S(k+1;0)\ \overline{(S(i;k)}\ \overline{S(2;k-1)}\ \ldots\ \overline{S(k;1)}),$$

where we emphasize that the logical summands are disjoint. For example,

$S(1;k)\ S(2;k-1)\ \overline{S(1;k)} = \emptyset$ since $A\bar{A} = \emptyset$ for any set A.

Next, using the logical identity, Eq. (E.4), we get

$$S(k+1) = S(1;k)$$
$$+ S(2;k-1) - S(2;k-1)\ S(1;k)$$
$$+ S(3;k-2) - S(3;k-2)\ (S(1;k) + S(2;k-1))$$
$$+ S(3;k-2)(S(1;k)S(2;k-3))$$
$$+ \ldots .$$
$$+ S(i;k-i+1)\ (1-U_i(1) + U_i(2) - U_i(3) + \ldots \pm U_i(i))$$
$$+ S(k+1;0)\ (1-U_k(1) + U_k(2) + \ldots . \pm U_k(k)),$$

where

$U_k(j)$ = the union of all combinations, j, at a time of the k designated sets.

Next, note that $S(2;k-1)\ S(1;k) = S(1,2;k-2)$; that is, the set of factors which contain both a one and a two may be written $S(1,2;k-2)$. The generalization is obvious, so that with our rule that $b(S(\underline{n};k)) = b(k)$, we can count each of the partitioned blocks of $S(k+1)$. We thus have, relying on the fact that $S(k+1)$ has finite cardinality,

$$b(S(k+1)) = b(k+1) = b(k)$$
$$+ b(k-1) - b(k-2)$$

$$
\begin{aligned}
&+ b(k-2) - b(k-3) - b(k-4) + b(k-5) \\
&+ b(k-3) - b(k-4) - b(k-5) - b(k-6) + b(k-7) \\
&+ b(k-7) + b(k-8) - b(k-9) \\
&+ b(k-4) - b(k-5) - b(k-6) - b(k-7) - b(k-8) \\
&\quad + b(k-7) + b(k-8) + b(k-9) + b(k-10) + b(k-11) \\
&\quad - b(k-10) - b(k-11) - b(k-12) - b(k-13) \\
&\quad + b(k-14) \\
&\quad + \text{-------} \\
&\quad + b(S(k+1;0)) \\
&\quad = b(k) \\
&\quad + E\Delta b(k) \\
&\quad + E^2\Delta\Delta b(k) \\
&\quad + E^3\Delta\Delta\Delta b(k) \\
&\quad + \ldots . \\
&\quad + E^k\Delta\Delta \ldots \Delta b(k). \qquad \text{(E.7)}
\end{aligned}
$$

It is worth noting that the last term is always $b(0) = 1$.

The ambiguity mentioned previously between the set function, b_s, and the integral function, b_k, is seen to be pervasive. The set $S(k+1;0)(1-U_k(1)) \ldots \pm U_k(k) = S(k+1;0) = \{(k+1)\}$, since $\{(k+1)\}$ is

disjoint from any of the combination sets. Obviously, $b(S(k+1;0)) = 1$. Also, $E^k \Delta \Delta \ldots \Delta b(k) = E^k b(k) = b(0) = 1$, since $b(k) = 0$, $k < 0$. The proposition is thus established.

The additive structure for the first few numbers is given in table E-1 where the structure refers to all factors containing the given number but no smaller integers. For example, for k = 10, the factorization (2,2,2,2,2) would be contained in the two structure, but (2,2,2,2,1,1) would not since it would be included in the one structure.

Looking upon the table as a matrix, M, we have

$\underline{b}(k) = \underline{1}[M]$, where $\underline{1} = (1,1,1, \ldots)$, M is the extended table and $\underline{b}(k)$ is the vector of $b(k)$ values. It is simple to see that

$\underline{b}(k) = \underline{1}[I + M'] = \underline{1} + \underline{1}[M']$

or that $\underline{b}(k) - \underline{1} = \underline{1}[M']$

where $M' = M - I$, and I is the (matrix) identity. While this point of view would seem artificial initially, it resolves the ambiguity mentioned above.

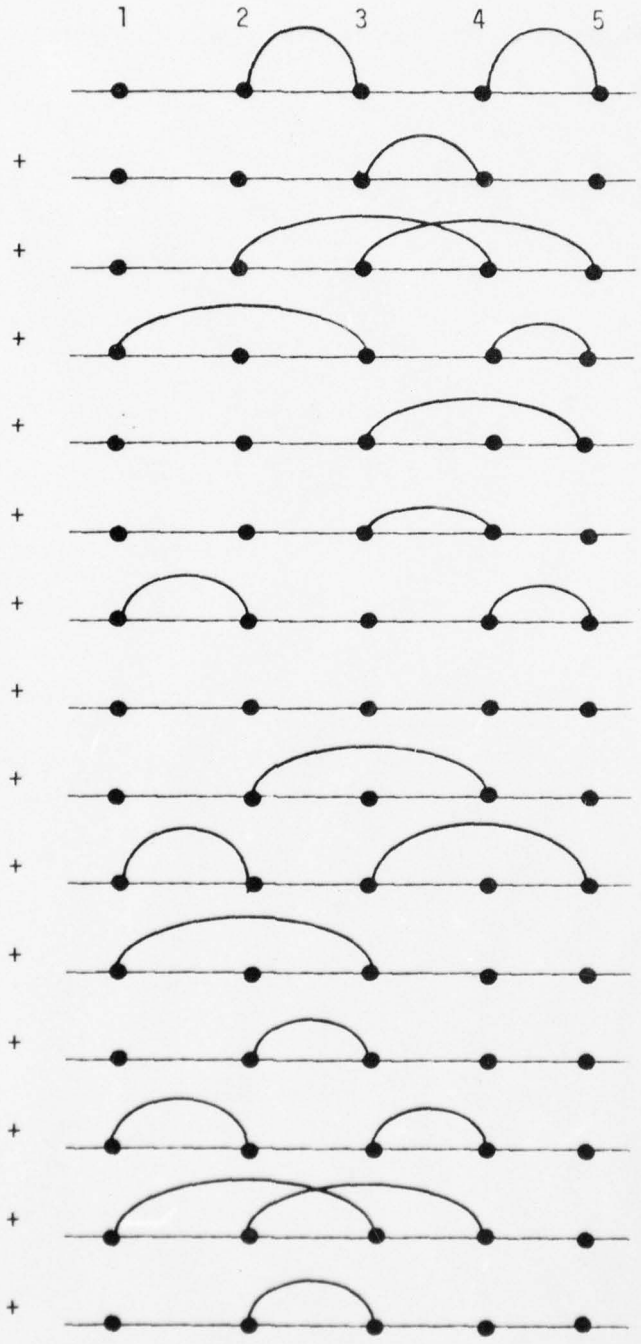
TABLE E-1. The n-Structure of k

Structure n \ Integer = k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176
2	0	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34
3	0	0	1	0	0	1	1	1	2	2	3	4	5	6	9	10
4	0	0	0	1	0	0	0	1	1	1	1	2	2	3	3	5
5	0	0	0	0	1	0	0	0	0	1	1	1	1	1	2	2
6	0	0	0	0	0	1	0	0	0	0	0	1	1	1	1	1
7	0	0	0	0	0	0	1	0	0	0	0	0	0	1	1	1
8	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1
9	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
16	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
b(k)	1	2	3	4	5	7	11	15	22	30	42	56	77	101	135	231

We have thus completed the first part of the inquiry, i.e., to establish the n structure of k. The second part, namely to determine the number of correlation possibilities for a given element in the n structure is readily determined. Consider an example. One element of the 1 structure of 5 is (2,2,1). The possible correlations by simple enumeration are:

$$\langle a(r_1)a(r)_2a(r_3)a(r_4)a(r_5)\rangle_{(2,2,1)} = \langle 1\rangle \langle 2,3\rangle \langle 4,5\rangle$$
$$+ \langle 1\rangle \langle 2,5\rangle \langle 3,4\rangle$$
$$+ \langle 1\rangle \langle 2,4\rangle \langle 3,5\rangle$$
$$+ \langle 2\rangle \langle 1,3\rangle \langle 4,5\rangle$$
$$+ \langle 2\rangle \langle 1,4\rangle \langle 3,5\rangle$$
$$+ \langle 2\rangle \langle 1,5\rangle \langle 3,4\rangle$$
$$+ \langle 3\rangle \langle 1,2\rangle \langle 4,5\rangle$$
$$+ \langle 3\rangle \langle 1,4\rangle \langle 2,5\rangle$$
$$+ \langle 3\rangle \langle 1,5\rangle \langle 2,4\rangle$$
$$+ \langle 4\rangle \langle 1,2\rangle \langle 3,5\rangle$$
$$+ \langle 4\rangle \langle 1,3\rangle \langle 2,5\rangle$$
$$+ \langle 4\rangle \langle 1,5\rangle \langle 2,3\rangle$$
$$+ \langle 5\rangle \langle 1,2\rangle \langle 3,4\rangle$$
$$+ \langle 5\rangle \langle 1,3\rangle \langle 2,4\rangle$$
$$+ \langle 5\rangle \langle 1,4\rangle \langle 2,3\rangle$$

Where the right hand side is abbreviated, i.e., <1> <2,3> <4,5> ≡ $\langle a(r_1)\rangle\langle a(r_2)(ar_2)\rangle\langle a(r_4)a(r_5)\rangle$. There are thus fifteen possible correlation decompositions of this one element, (2,2,1) in the one structure of 5. In terms of Feynman diagrams, this decomposition is given as



The general case may now simply be stated. The number of possible decompositions of an element (p,q,r, . . .) can be seen to be a simple combinational problem: the number of ways, b_k, of choosing categories out of a k elements. For k = mp + tq + . . ., this is

$$b_k(\overbrace{p,p,\ldots p}^{m \text{ terms}},\ \overbrace{q,q,\ldots q}^{t \text{ terms}},\ \ldots)$$

$$= \frac{k!}{m!(p!)^m t!(q!)^t} \qquad (E.8)$$

for k = 5

$$b_5(2,2,1) = \frac{5!}{2!(2!)^2} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2!2!2} = 15, \text{ as noted previously.}$$

We have thus determined the potential correlation decomposition for an arbitrary element, $c_k(a)$.

Appendix F

THE UNSCATTERED WAVE: $p_u(\underline{r},t)$

Sources differ as being distributed or not in space and time. We set down in this appendix, the influence of the source on the unscattered wave p_u. Our purpose is to generate the forms that allow non-point sources in the developments of Chapters V - VII. From Eq. (5.50) we write for a general source $s(\underline{r}_o,t_o)$

$$p_u(1) = \hat{G}(s) = \int_{t_0}\int_{\underline{r}_0}\int_{\omega_0} d\underline{r}_0 dt_0 d\omega_0 \frac{e^{j\omega_0\left(t_1-t_0-\frac{|\underline{r}_1-\underline{r}_0|}{c_0}\right)}}{8\pi^2|\underline{r}_1-\underline{r}_0|}(s(\underline{r}_0,t_0))$$

$$= \int_{t_0}\int_{\underline{r}_0}\frac{d\underline{r}_0 dt_0 \delta\left(t_1-t_0-\frac{|\underline{r}_1-\underline{r}_0|}{c_0}\right)s(\underline{r}_0,t_0)}{4\pi|\underline{r}_1-\underline{r}_0|}$$

$$= \int_{\underline{r}_0}\frac{s\left(r_0,t_1-\frac{|\underline{r}_1-\underline{r}_0|}{c_0}\right)d\underline{r}_0}{4\pi|\underline{r}_1-\underline{r}_0|}. \qquad (F.1)$$

If s is a point source, i.e.,

$$s \equiv \delta(\underline{r}_0)s_t(t),$$

Eq. (F.1) reduces to

$$p_u(1) = \frac{s_t\left(t_1-\frac{|\underline{r}_1|}{c_0}\right)}{4\pi|\underline{r}_1|}. \qquad (F.2)$$

If

$$s_t = s_0 e^{j\omega_0 t},$$

Eq. (F.1) further reduces to

$$p_u(1) = s_0 \frac{e^{j\omega_0\left(t_1 - \frac{r_1}{c_0}\right)}}{|4\pi \underline{r}_1|} . \qquad (F.3)$$

Equation (F.1) can be used for distributed sources which vary in time and frequency. This represents the most general case here. Equation (F.2) is used for point sources broadcasting arbitrary signals. Equation (F.3) represents a harmonic point source.

If $\hat{G}(\omega_0)$ is misapplied to the source frequency, ω_1, then the formulation rectifies the error. Suppose $s = \delta(r_0 - r_0')\, e^{j\omega t_0}$, $\omega \neq \omega_0$. Then

$$\hat{G}(s) = \int_{\underline{r}_0'} \int_{t_0} \int_{\omega_0} \frac{d\underline{r}_0' d\omega_0 dt_0}{8\pi^2 |\underline{r}_1 - \underline{r}_0'|} e^{j\omega_0\left((t_1 - t_0) - \frac{|\underline{r}_1 - \underline{r}_0'|}{c_0}\right)} e^{j\omega_1 t_0}\, \delta(r_0 - r_0'),$$

$$= \int_{t_0} \int_{\omega_0} \frac{dt_0 d\omega_0}{8\pi^2 |\underline{r}_1 - \underline{r}_0|} e^{j\omega_0 t_1} e^{jt_0(\omega_0 - \omega_1)} e^{-j\frac{\omega_0}{c_0}|\underline{r}_1 - \underline{r}_0|},$$

$$= \int_{\omega_0} \frac{d\omega_0}{4\pi |\underline{r}_1 - \underline{r}_0|} e^{j\omega_0 t_1}\, \delta(\omega_1 - \omega_0)\, e^{j\frac{\omega_0}{c_0}|\underline{r}_1 - \underline{r}_0|},$$

$$= \frac{e^{j\left(\omega_1 t_1 - \frac{\omega_1}{c_0}|\underline{r}_1 - \underline{r}_0|\right)}}{4\pi|\underline{r}_1 - \underline{r}_0|} = \frac{e^{j\left(\omega_1 t_1 - k_1|\underline{r}_1 - \underline{r}_0|\right)}}{4\pi|\underline{r}_1 - \underline{r}_0|} . \qquad (F.4)$$

Thus $\hat{G}$ is self-correcting for the harmonic case.

For broadband signals (point source at $\underline{r}_0'$), we have

$$\hat{G}(s) = \int_{\omega_0}\int_{t_0}\int_{\underline{r}_0} \frac{e^{-j\,\omega_0\left((t - t_0) - \frac{|\underline{r} - \underline{r}_0|}{c_0}\right)}}{4\pi|\underline{r} - \underline{r}_0|} \delta(\underline{r}_0 - \underline{r}_0') \int_{\omega} s(\omega)\, e^{j\omega t} d\omega$$

$$= \frac{1}{4\pi|\underline{r} - \underline{r}_0'|} \int_{\omega_0} e^{-j\omega_0(t - \frac{|\underline{r} - \underline{r}_0'|}{c_0}} s(-\omega_0)\, d\omega_0 . \qquad (F.5)$$

For a distributed source

$$\hat{G}(s) = \int_{\omega_0}\int_{t_0}\int_{\underline{r}_0} \frac{e^{-j\omega_0\left((t - t_0) - \frac{|\underline{r} - \underline{r}_0|}{c_0}\right)}}{4\pi|\underline{r} - \underline{r}_0|} \int_{\omega} s(\underline{r}_0, \omega)\, d\underline{r}_0\, d\omega\, d\omega_0\, dt_0 . \qquad (F.6)$$

Our results are summarized in Table F-1.

TABLE F-1

The Unscattered Wave: $p_u = \hat{G}(s)$

time \ space	distributed source	point source
arbitrary signal	$\int_{\underline{r}_0} \frac{s\left(\underline{r}_0, t - \frac{\lvert \underline{r} - \underline{r}_0 \rvert}{c_0}\right) d\underline{r}_0}{4\pi \lvert \underline{r} - \underline{r}_0 \rvert}$	$\frac{s\left(\underline{r}_0, t - \frac{\lvert \underline{r} - \underline{r}_0 \rvert}{c_0}\right)}{4\pi \lvert \underline{r} - \underline{r}_0 \rvert}$
harmonic signal	$\int_{\underline{r}_0} \frac{e^{j\omega_0\left(t - \frac{\lvert \underline{r} - \underline{r}_0 \rvert}{c}\right)} d\underline{r}_0}{4\pi \lvert \underline{r} - \underline{r}_0 \rvert}$	$\frac{s_0 e^{j\omega_0\left(t - \frac{\lvert \underline{r} - \underline{r}_0 \rvert}{c_0}\right)}}{4\pi \lvert \underline{r} - \underline{r}_0 \rvert}$

Source at $\underline{r}_0$ is $s(\underline{r}_0, t_0)$

Receiver is at $(\underline{r}, t)$

Appendix G
EIGENVALUES

The study of eigenvalues (and associated eigenvectors) often provides considerable insight into a system's structure. For linear systems (such as those considered in Chap. V) they indicate whether or not the system is stable. For this reason, we append a discussion of this subject relative to the formulations of Chapter V.

We note first that an eigenvalue, λ, is defined relative to some operator A. For the simple system,

$$\frac{dx}{dt} \equiv \dot{x} = Ax \tag{G.1}$$

the eigenvectors are those vectors for which the operator may be replaced by a scalar

$$Ax_{\lambda} = \lambda x_{\lambda} . \tag{G.2}$$

For Eq. (G.1), the non-trivial solutions of Eq. (G.2) must satisfy the equation

$$\text{Det}\,[\lambda I - A_x] = 0 .$$

The vector field of which x is an element must likewise be specified.

The simple system Eq. (G.1) is inappropriate for the <u>linear</u> broadband acoustic case presented in this Thesis. In fact, more than one class of eigenvalue may be represented. We present two subsequently.

For Eq. (5.1) we may write

$$p = \hat{G}s + \hat{G}\hat{\xi}p = \sum_{n=0}^{\infty} (\hat{G}\hat{\xi})^{(n)}\hat{G}(s) = \hat{A}(s) \tag{G.3}$$

Note the <u>difference in form</u> vis-a-vis Eq. (G.1): the output p is not the derivative of s; the operator $\hat{A}$, while linear, is not a simple matrix multiplication, $\hat{A}s$.

In the context of Eq. (5.1), p and s are complex numbers. However, the reference operator, $\hat{A}$, may be variously described. In the first method we reference $\hat{A}$ one dimensionally.

<u>Method 1</u>. If we can find a λ_s such that $p = \lambda_s s$, then let us call λ, an eigenvalue of $\hat{A}$. In this view

$$\hat{A}: \; \mathbb{C} \rightarrow \mathbb{C} \quad , \qquad \mathbb{C} \equiv \text{complex numbers}$$

which is to say the specified vector field is the one dimensional field of complex numbers. For the specific eigenvectors, the operator $\hat{A}$ may be replaced by λ_s . Then the following hold:

1. For every non-zero, bounded s and p there is a unique eigenvalue.
2. The eigenvalue is a random process which includes surface, volume and bottom effects (M formulation).
3. Convergence is assured whenever $|\lambda| < \infty$.
4. λ_s is independent of $|s|$, but dependent on its position and temporal character.
5. The eigenspace space of λ is $\mathbb{C}$, the whole space.
6. $\hat{A}$ has spectral representation, $\hat{A} \rightarrow \lambda$.

The first conclusion follows from the fact that p and s are complex numbers, i.e.,

$$\hat{A}(s) = \lambda_s(\underline{r},t)s(\underline{r}_0,t_0) = p(\underline{r},t), \qquad (G.4)$$

so that

$$\lambda_s(\underline{r},t) = \frac{p(\underline{r},t)}{s(\underline{r}_0,t_0)} . \tag{G.5}$$

The second conclusion follows from the first, that is

$$\lambda_s(\underline{r},t) = \frac{\hat{G}(s)}{s} + \frac{\hat{G}\hat{\xi}p}{s} = \lambda_u+\lambda \tag{G.6}$$

$$= \frac{\hat{G}(s)}{s} + \frac{\hat{G}(\hat{\xi}_S+\hat{G}\hat{\xi}_V+\hat{G}\hat{\xi}_B)p}{s} = \lambda_u+\lambda_S+\lambda_V+\lambda_B \tag{G.7}$$

in M-form, where, since $\hat{\xi}$ is a random variable, $\lambda = \lambda_S+\lambda_V+\lambda_B$ is also random.

Thirdly, for a bounded input, s, p must be bounded whenever $|\lambda|<\infty$.

Fourthly, if $s = cs$, c a constant complex number, then we have

$$\sum_{n=0}^{\infty} (\hat{G}\hat{\xi})^{(n)}\hat{G}(cs) = cp = c\lambda s = \lambda(cs), \tag{G.8}$$

so that a constant amplitude or phase change in the source does not change the eigenvalue (as we must expect of a linear system). However, Eq. (G.6) shows λ dependent on spatial and temporal conditions.

The fifth proposition follows from the above argument also because c being a complex number, s, is any complex number too.

The fifth proposition follows from the above argument also because c being a complex number, s, is any complex number too.

The final proposition is the trivial breakdown $\hat{A}(\underline{r},t)=(\underline{1})\lambda(\underline{r},t)$.

The results may be considered mathematically trivial since they depend entirely on the solution, p. This should not be too surprising because in starting with the solution Eq. (G.3), the utility of determining eigenvalues must be considerably diminished.

We end the discussion of method 1 with the following observations:

a) For $\hat{\xi} \equiv 0$ and s harmonic,

$$\lambda(\underline{r},t) = \frac{e^{-jk_0|\underline{r}-\underline{r}_0|}}{|\underline{r}-\underline{r}_0|} e^{j\omega_0 t};$$

b) If s is distributed, i.e., $s = \int_{\underline{r}_0} s(\underline{r}_0,t_0)d\underline{r}_0$,

then s is still a complex number and propositions 1-6 still hold.

c) If independent noise is added to Eq. (5.1), then

$$\begin{aligned}\lambda(s+n) = p(\underline{r},t) &= \sum (\hat{G}\hat{\xi})^{(n)}\hat{G}(s+n) \\ &= \hat{G}s+\sum(\hat{G}\hat{\xi})^{(n)}G(s)+\hat{G}n+\sum_{m=1}^{\infty}(\hat{G}\hat{\xi})^m\hat{G}(n) \\ &= \lambda_u+\lambda_s+\lambda_n+\lambda_{ns} \qquad (G.9)\end{aligned}$$

so that each independent source, whether noise or signal, contributes to the eigenvalue a deterministic and a random part, (see also Eq. (7.27)).

d) As a function of time, we have ($\underline{r},\underline{r}_0$ fixed)

$$\lambda(t) = \frac{p(\underline{r},t+t_1)}{s(\underline{r}_0,t_0+t_1)} = \lambda_u(t) + \lambda_s(t) \qquad (G.10)$$

and

$\lambda_u(t)$ depends only on s, but $\lambda_s = \sum_{n=1}^{\infty} (\hat{G}\hat{\xi}(t))^{(n)}\hat{G}(s)/s(r_0,t_0+t_1)$

depends on both the source and the moving scattering medium, $\hat{\xi}(\underline{r},t)$.

The second method references λ to a different operator, which in this case has two dimensions.

Method 2. We may attempt a state variable representation of Eq. (5.1).

$$\frac{\partial^2 p}{\partial t^2} - c^2\nabla^2 p = 0, \qquad c^2 = c^2(\underline{r},t) . \qquad \text{(G.11)}$$

Put

$$x_1 = p$$
$$x_2 = \dot{p};$$

then Eq. (G.11) can be expressed as

$$\dot{\underline{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c^2\nabla^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} s \qquad \text{(G.12)}$$

$$p = y = [1\ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

analogous to

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$
$$\underline{y} = C\underline{x} + D\underline{u},$$

so that

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ c^2\nabla^2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = [1,0]\ ,\ D = 0\ . \qquad \text{(G.13)}$$

The difference between this approach and modern control theory lies in $\hat{A}(t)$, which is more than a simple matrix (with operator $c^2(\underline{r},t)\nabla^2$).

We should like to know

$$\text{Det}[\lambda I - \hat{A}] = \text{Det}\begin{bmatrix} \lambda & -1 \\ -c^2\nabla^2 & \lambda \end{bmatrix} = \lambda^2 - c^2\nabla^2 = 0 \qquad (G.14)$$

but this is mostly meaningless. If however $\nabla^2 \to -k_o^2$, then

$$\lambda = \pm jck_o = \pm \frac{jc_o k_o}{1+d} \quad . \qquad (G.15)$$

The system then acts like an oscillator with two poles on the imaginary axis, as in the diagram, Fig. G-1.

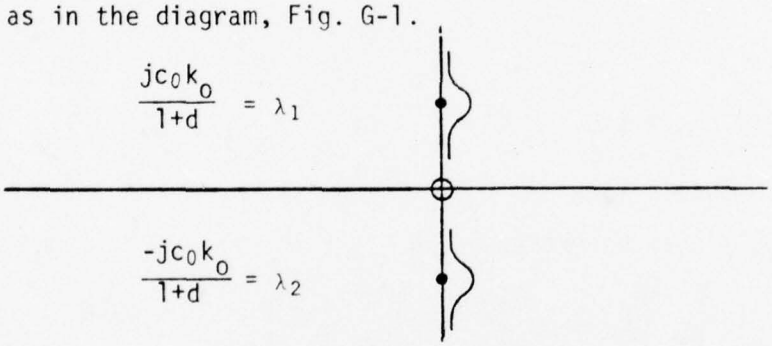


Fig. G-1. Eigenvalue Representation of Harmonic Propagation

The randomness of $c = c_o/(1+d)$ gives the λ_i a random displacement along the imaginary axis.

In the more general case, the notion of eigenvalues tends to break down, as may be seen from the Laplace representation of

$$\dot{x}(t) = \hat{A}(t)x(t)$$
$$sX(s) = \hat{A})*x(s) \qquad (G.16)$$

To the extent A has "memory," Det [sI-A(s)*] will be frustrating.[1]

---

[1]See [222] Sec. 3.9.

We may nevertheless press the state variable approach further. Suppose $x_1(t)$ and $x_2(t)$ are solutions to

$$\dot{\underline{x}} = \hat{A}\underline{x} \tag{G.17}$$

with $\hat{A}$ of Eq. (G.13) and $\underline{x}_1(t_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\underline{x}_2(t_0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

In general, closed solutions will not be available, but operational solutions may be obtained by setting the output of the system in Fig. G-2.

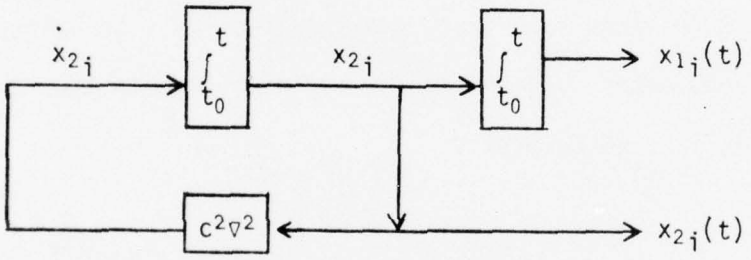


Fig. G-2. Operational Solution - State Variable Form

Of course, $c^2\nabla^2$ is a random feedback operator, so that $\underline{x}_i = \begin{bmatrix} x_{1i} \\ x_{2i} \end{bmatrix}$ is a realization of a random vector.

Now construct the fundamental solution matrix, $u = [\underline{x}_1, \underline{x}_2]$. Then, on any realization, we have

$$\dot{U}(t) = A(t)U(t) \tag{G.18}$$

and

$$x(t) = U(t)x(t_0). \tag{G.19}$$

For an input $s(t)$, we can write,

$$x(t) = U(t)U^{-1}(t_0)\, x(t_0) + \int_{t_0}^{t} U(t)U^{-1}(\tau)B(\tau)s(\tau)d\tau. \tag{G.20}$$

The matrix $U(t)U^{-1}(\tau)$ is often called the transition matrix, $\emptyset(t,\tau)$.

It may be shown that

$$\emptyset(t,t_0) = I_n + \int_{t_0}^{t} A(t_1)dt_1 + \int_{t_0}^{t} A(t_1) \int_{t_0}^{t_1} A(t_2)dt_1 dt_2 + \ldots. \tag{G.21}$$

redolent of the series solutions in Chaps IV - VI.

If an eigenvector λ exists, then from Eq. (G.19), we may write

$$\dot{U}(t) = \lambda(t)U(t)$$

which provides a criterion for $\lambda(t)$ (and thus x_λ also).

These considerations illustrate that analysis by eigenvalues have only limited application and provide no more information than the analytical solutions of Chapter V.

Appendix H

CONVERGENCE, INVERTIBILITY, THE RADIATION CONDITION, CAUSALITY AND TAYLOR'S APPROXIMATION

We have used, as in the literature

$$(\nabla^2 + k_o^2)\,\frac{e^{-jk_o\rho}}{4\pi\rho} = \delta(\underline{\rho}) \tag{H.1}$$

but not without misgiving. To be generally useful we should have

$$\int_{\underline{r}} f(\rho)\,(\nabla^2 + k_o^2)\,\frac{e^{-jk_o\rho}}{4\pi\rho}\,d\underline{r} = f(0) \tag{H.2}$$

We have pointed out[1] that

$$(\nabla^2 + k_0^2)\,\frac{e^{-jk\rho}}{4\pi\rho} = \begin{cases} 0 & \rho \neq 0 \\ \infty & \rho = 0 \end{cases} . \tag{H.3}$$

There are, however, many "functions" that are everywhere zero except for an unbounded value at the origin, e.g., $\{\frac{d^n}{d\rho^n}\,[8(\underline{\rho})]\}$ and the sums, differences and products of such a set.

Secondly, the conclusions of Appendix D affect our use of the operator. The use of the Taylor's approximation,

$$\hat{GQ}\langle\rho\rangle \simeq \langle\rho\rangle\,\hat{GQ} = \langle\rho\rangle\hat{GQ}[1] \tag{H.4}$$

---

[1]Sec. 5.2.1.

effectively replaces a physically "correct" function by $f(\underline{r}) = 1$, which neither satisfies the wave equation nor tends to zero at infinity. It should not be surprising, then, to find that solutions based on this approximation yield plane waves, or that the solutions do not converge (are unstable at infinity). One may recover a "physcial" solution by insisting on convergence. This may be done (1) by simply ignoring the activity at infinity, (2) by the use of convergence factors (which essentially introduces a bounding function to represent the remainder term in the Taylor's series), or (3) by making use of Eq. (H.1). Each of these methods is illustrated in the example of Sec. 6.4.

Beyond these considerations, others intrude. Consider, for example the function

$$\hat{f}(\underline{r}) = \frac{e^{+jk_0\rho}}{4\pi\rho}, \tag{H.5}$$

which obeys the wave equation, tends to zero at infinity, but which unfortunately is physically meaningless. Equation (H.5) represents the wave with source density zero at infinity which converges (focuses) on the point $\underline{r}_0$. Thus it does not satisfy a causality condition. These functions may be eliminated by insisting on a "radiation" condition, namely

$$\lim_{\rho\to\infty} \rho\left(\frac{\partial}{\partial\rho} + jk_0\right)p = 0, \tag{H.6}$$

so that only waves travelling toward infinity get considered.

The full complexity of the contention in Eq. (H.1) may be appreciated from Fourier transforms. First consider

$$\hat{F}_r\left[\frac{e^{-jk_o\rho}}{4\pi\rho}\right] = \int_{\underline{r}} \frac{e^{-jk_o\rho}}{4\pi\rho}\, e^{j\underline{k}\cdot\underline{r}}\, d\underline{r} \tag{H.7}$$

$$= e^{-j\underline{k}\cdot\underline{r}_0} \int_f \frac{e^{-jk\rho}}{4\pi\rho}\, e^{j\underline{k}\cdot\underline{\rho}}\, d\underline{\rho}\,, \quad \underline{\rho} = (\underline{r} - \underline{r}_0)$$

$$\rho = |\underline{r} - \underline{r}_0|\,. \tag{H.8}$$

In spherical coordinates Eq. (H.7) becomes,

$$e^{j\underline{k}\cdot\underline{r}_0}\hat{F}_r\left[\frac{e^{-jk_o\rho}}{4\pi\rho}\right] = \int_0^{\pi} d\phi \int_0^{2\pi} d\theta \int_0^{\infty} d\rho\, \rho^2 \sin\phi\, \frac{e^{-jk_o\rho}}{4\pi\rho}\, e^{j\underline{k}\cdot\underline{\rho}}\, d\rho \tag{H.9}$$

$$= \int_0^{\infty} \rho\, e^{-j(k_o-k)\rho}\, d\rho$$

$$= \frac{e^{-j(k_o-k)\rho}}{[-j(k_o-k)]^2}\,[-j(k_o-k)\,\rho - 1]\Big|_0^{\infty}$$

$$= \frac{1}{[-j(k_o-k)]^2} + \lim_{\rho\to\infty}\left[\frac{-e^{-j(k_o-k)\rho}}{[-j(k_o-k)]^2} + \frac{\rho e^{-j(k_o-k)\rho}}{-j(k_o-k)}\right]. \tag{H.10}$$

Now by analogy with the well-known relation,

$$\delta(t) = \lim_{\alpha\to\infty} \frac{\sin(\alpha t)}{\pi t}$$

or

$$\delta'(t) = \lim_{\alpha\to\infty}\left(\frac{\alpha\cos\alpha t}{\pi t} - \frac{\sin\alpha t}{\pi t^2}\right),$$

the limit in Eq. (H.10) is

$$\lim_{\rho\to\infty}\left(\frac{\rho e^{-j(k_o-k)\rho}}{-j(k_o-k)} - \frac{e^{-j(k_o-k)\rho}}{[-k(k_o-k)]^2}\right) = \delta_k'(k_o-k) \ . \tag{H.11}$$

Thus Eq. (H.9) becomes

$$\hat{F}_r\left[\frac{e^{-jk_o\rho}}{4\pi\rho}\right] = \left(\frac{1}{[-j(k_o-k)]^2} + \delta_k'(k_o-k)\right) e^{-j\underline{k}\cdot\underline{r}_0} \ . \tag{H.12}$$

We obtain immediately from Eq. (H.12),

$$\hat{F}_r\left[(\nabla^2+k_o^2)\frac{e^{-jk_o\rho}}{4\pi\rho}\right] = (-k^2+k_o^2)\,\hat{F}_r\left[\frac{e^{-jk_o\rho}}{4\pi\rho}\right] e^{-j\underline{k}\cdot\underline{r}_0}$$

$$= \left(\frac{k^2-k_o^2}{(k-k_o)^2} + (k_o^2-k^2)\delta_k'(k_o-k)\right) e^{-j\underline{k}_o\underline{r}_0} \tag{H.13}$$

The component $(k_o^2-k^2)\delta_k'(k_o-k)$, represents activity only at $k = k_o$ in $\underline{k}$ space ($\rho = \infty$ in $\underline{r}$ space) and is not identically zero.[1] In fact, we have

$$(k_o^2-k^2)\delta_k'(k_o-k) = -(k_o+k)\delta(k_o-k) \tag{H.14}$$

where the reader will note the reversal in sign of k_o.

The first component of Eq. (H.13), $(k^2-k_o^2)/(k-k_o)^2$, is plotted in Fig. H-1. The reader will note the difficulty in the region $k=(0,k_o)$, and the <u>selective</u> applicability of this function as an impulsive function at $k=k_o$ and as $k\to\infty$.

---

[1]This may be obtained as a corollary to theorem 9, p. 21 in [219].

While this analysis is original with this Thesis, the reader may consult [51] p. 510 which gives a proof for Eq. (H.1) based on selected integration contours in the complex plane. The problem has likewise been treated by earlier scholars [220] and [221] p. 485, but in the context of homogeneous propagation. In the latter Stratton gives four conditions for the uniqueness of solutions to the wave equation:

1. the sources are located in a finite region of space.
2. a homogeneous boundary condition, $\alpha p + \beta \frac{\partial p}{\partial n} = 0$.
3. $\rho p < M$, as $\rho \to \infty$ (regularity condition).
4. $\lim_{\rho \to \infty} \rho\left(\frac{\partial p}{\partial \rho} + jk\rho\right) = 0$ (radiation condition).

In our case, these notions need to be extended to include the results of an extended divergence theorem (Sec. 1.2.1.2).

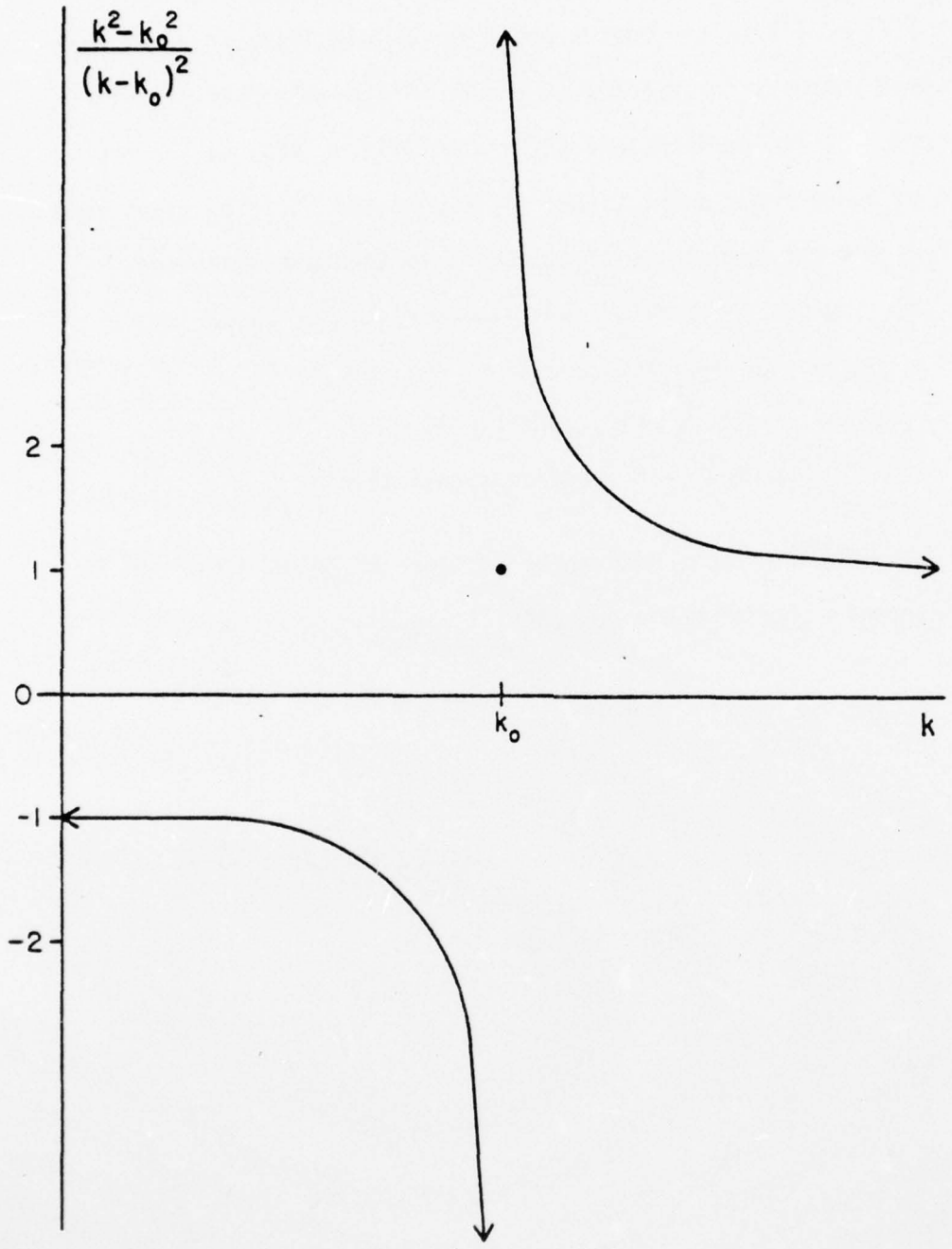


Fig. H-1. Plot of $\frac{k^2-k_o^2}{(k-k_o)^2}$ versus k

REFERENCES

1. Adams, S.L., Doubek, J.W., 1977. Frequency coherence and time coherence in random multipath channels. JASA 62:286-294.
2. Adams, S.L., Jôbst, W.J., 1977. Estimation and Interpretation of acoustic cw fluctuation spectra. JASA 61:675-681.
3. Aiken, R.T., 1969. Propagation from a point source in a randomly refracting medium. Bell Systems Technical Journal: 1129-1165.
4. Anderson, G., Gocht, R., Sirota, D., 1964. Spreading loss in an inhomogeneous medium. JASA 36:140-145.
5. Andreev, N.N., Rusakov, I.G., 1934. The acoustics of moving medium. Moscow: GTTI.
6. Aris, R., 1962. Vectors, Tensors, and the Basic Equations of Fluid Mechanics. New York: Prentice-Hall.
7. Barabanenkov, Yu. N., Kravtsov, Yu. A., Rytov, S.M., and Tatarskii, V.I., 1971. Status of the theory of propagation of waves in a randomly inhomogeneous medium, Uspekhi 1B, 551-575.
8. Barrick, D.E., Headrick, J.M., Bogle, R.W., Crombie, D.D., 1974. "Sea Backscatter at HF, Interpretation and Utilization of the Echo," Proc IEEE 62:673-680.
9. Bartlett, M.S., 1966. An Introduction to Stochastic Processes. Cambridge Univ. Press.
10. Bassanini, P., Cercignani, C., Sernagiotto, F., and Tirini, G., 1967. Scattering of waves by a medium with strong fluctuations of refractive index. Rad. Sci.2:1-18.

REFERENCES (Cont'd)

11. Baxter, L. and Yang, C.Y., 1975. Fluctuations of narrow-band sound amplitudes from long-range transmissions in the deep Atlantic Ocean. JASA 58:595-603.

12. Beckmann, P., 1967. Probability in Communication Engineering, New York: Harcourt.

13. Beckmann, P., 1965. Signal Degradation in Radio Beams. Rad. Sci. 1:69:692-640.

14. Beckmann, P. and Spizzichino, A., 1963. The Scattering of Electromagnetic Waves from Rough Surface. London: Pergamon.

15. Beran, M.J. and McCoy, J.J., 1974. Propagation of radiation from a finite beam through an anisotropic medium. JASA 56:1667-1672.

16. Beyer, R.T., 1974. Non-Linear Acoustics. Dept. of Navy, Naval Ships Systems Command.

17. Beyer, R.T., Letcher, S.V., 1969. Physical Ultrasonics. New York: Academic Press.

18. Bixon, M., Zwanzig, R., 1969. Boltzmann-Langevin equation and hydrodynamic fluctuations. Phys. Rev. 187:267.

19. Blackstock, D.T., 1975. Basic Research in Non-Linear Acoustics. NTIS AD-A015296. 16 pp.

20. Blackstock, D.T., 1966. Connection between Fay and Fubini Solutions for Sound waves of Finite Amplitudes. JASA 39:1019-1026.

21. Blackstock, D.T., 1962. Propagation of Plane Sound Waves of Finite Amplitude in Nondissipative Fluids. JASA 34:9-30.

22. Blokhintsev, D., 1946. The propagation of sound in an inhomogenous and moving medium. JASA 18:322-334.

REFERENCES (Cont'd)

23. Bogdanoff, J.L., Kozin, F., 1962. Moments of the outputs of linear random systems. JASA 34:1063-1066.

24. Born, M., Wolf, E., 1964. Principles of Optics, New York: MacMillan.

25. Boyd, M.L., Deavenport, R.L., 1973. Forward and Specular Scattering from a Rough Surface: Theory and Experiment. JASA 53:791-801.

26. Bourret, R.C., 1962. Stochastically perturbed fields, with applications to wave propagation in random media, Nuovo Cimento 26:1-31.

27. Bremmer, H., 1963. "Scattering by a perturbed continuum" in Electromagnetic Theory and Antennas, E.C. Jorden, Ed. London: Pergamon.

28. Bremmer, H., 1973. General remarks concerning theories dealing with scattering and diffraction in random media. Rad. Sci. 8:511-534.

29. Brown, E.H. and Clifford, S.F., 1976. On the attenuation of sound by turbulence. JASA 60:788-794.

30. Brown, E.H., 1974. Turbulent spectral broadening of backscattered acoustic pulses. JASA 56:1398-1406.

31. Brown, M.V. and Frisk, G.V., 1974. Frequency smearing of sound forward-scattered from the ocean surface. JASA 55:744-749.

32. Brown, M.V., 1969. Intensity fluctuation in reflections from the ocean surface. JASA 46:196-204.

33. Brown, M.V. and Ricard, J., 1960. Fluctuations in surface-reflected pulsed CW arrivals. JASA 32:1551-1554.

REFERENCES (Cont'd)

34. Brown, W.P., 1967. "Coherent field in a random medium: Effective refractive index," Proc. Symp. on Modern. Opt., Polytechnic Press of the Polytechnic Inst. of Brooklyn.

35. Brown, W.P., 1967. Propagation in a random medium: Cumulative effect of weak inhomogeneities, IEEE Trans. AP-15, 81-89.

36. Brownlee, L.R., Sagav F.H. and Ash, D.E., 1974. Measurement of wave propagation in a weakly inhomogeneous medium - the sea. JASA 56:1059-1070.

37. Burnett, T.D., 1974. Acoustic variability in the ocean. JASA 56: 817-823.

38. Carrier, G.F., 1970. Stochastically driven dynamical systems, J. Fluid Mech. 44:249.

39. Chen, Y.M., 1964. Wave propagation in inhomogeneous and discontinuous random media, J. Math. Phys. 43:314-324.

40. Chernov, L., 1967. Wave Propagation in a Random Medium. New York: Dover. (reprint of [41])

41. Chernov, L.A., 1960. Wave Propagation in a Random Medium. New York: McGraw-Hill, (English Transl.).

42. Chernov, L.A., 1958. The acoustics of a moving medium. Sov. Phys. Acoust. 4:311.

43. Chow, Pau-Lui, 1972. Application of functions - space integrals to problems in wave propagation in a random media. J. Math. Phys. 13:1224-1236.

44. Chow, T.J., 1974. Scattering of elastic waves in an inhomogeneous solid. JASA 56:1049-1051.

REFERENCES (Cont'd)

45. Chuprov, S.D., 1966. On the observation of a sound signal in the presence of internal waves. Izv. Acad Sci. USSR Atm. Oceanic Phys. 2:334-335.

46. Clay, C.S. and Medwin, H., 1970. Dependence of spatial and temporal correlations of forward-scattered underwater sound on the surface statistics. JASA 47:1412-1418.

47. Clay, C.S., 1960. Fluctuation of Sound reflected from the sea surface. JASA 32:1547-1551.

48. Coleman, B.D., Gurtin, M.E., Herrera, I. and Truesdell, C., 1965. Wave Propagation in Dissipative Materials, New York: Springer-Verlag.

49. Cox, D.R. and Miller, H.D., 1965. The Theory of Stochastic Processes, London: Mathuen.

50. Cramer, H., 1951. Mathematical Method of Statistics. Princeton, Princeton U. Press.

51. Cushing, J.T., 1975. Applied Mathematical Analysis for Physical Scientists. New York: Wiley.

52. Dashen, R., 1977. Path Integrals for Waves in Random Media. SRI TR JSP-76-1.

53. DeFerrari, H. and Leung, R., 1975. Spectrum of phase fluctuations caused by multipath interference. JASA 58:604-607.

54. Dence, D. and Spence, J.E., 1971. Dyson's Equation for a medium possessing uniaxial anisotropic fluctuations, Elect. Letrs. 7, 304-305.

REFERENCES (Cont'd)

55. Dence, D., Spence, J.E., 1973. "Wave Propagation in Random Anisotropic Media" in Probabilistic Method in Applied Mathematics Ed. by Bharucha-Reid, New York: Academic Press. Vol. 3 pp. 121-181.

56. deWolf, D.A., 1967. Multiple scattering in a random continuum, Rad. Sci. 2: 1379-1392.

57. Dorsey, N.E., 1948. Properties of Ordinary Water Substance. New York: Reinhold.

58. Dyer, I., Shepard, G.W., 1977. Amplitude and Phase Fluctuation periods for long-range propagation in the ocean. JASA 61:937-42.

59. Dyer, I., 1970. Statistics of Sound Propagation in the Ocean. JASA 48:337-345.

60. Eby, E.S. and Einstein, L.T., 1965. General Spreading-Loss Expression. JASA 37:933(L).

61. Eckart, C., 1953. The Scattering of Sound from the Sea Surface. JASA 25:566-570.

62. Fante, R., 1975. Electromagnetic beam propagation in turbulent media. Proc. IEEE 63:1669-1692.

63. Fauré, P., 1964. Theoretical Model of Reverberation Noise. JASA 36:259-266.

64. Fisher, F.H., Simmons, V.P., 1977. Sound Absorption in Sea Water. JASA 61:538-565.

65. Fitelson, M.M., 1976. Broadside bearing response pattern of a line array to a cylindrical wave in a random medium. JASA 59:525-527.

REFERENCES (Cont'd)

66. Fitelson, M.M., 1975. The correlation function of the field between arbitrarily oriented points for a plane wave in a random medium. JASA 58:679-683.

67. Fitelson, M.M., 1974. Coherence Function for Cylindrically Spreading Waves in a Random Medium. JASA 56:53-57.

68. Flatté, S.M., Dashen, R., Munk, W.H., Zachariasen, F., 1977. Sound Transmission through a Fluctuating Ocean. SRI TR JSR-76-39.

69. Foldy, L.L., 1945. The multiple scattering of waves: I, Phys. Rev. 67:107-119.

70. Fortuin, L., 1973. The Sea Surface as a Random Filter for Underwater Sound Waves. ISBN 90 21240096, The Netherlands UITGEVERN WALTMAN-Delft. (Dissertation)

71. Fortuin, L., 1970. Survey of Literature on Reflection and Scattering of Sound Waves at the Sea Surface. JASA 47:1209-1228.

72. Frisch, U., 1968. " Wave propagation in random media," in Probabilistic Methods in Applied Mathematics, (A.T. Bharucha-Reid, ed.), New York, Academic Press, Vol. 1, pp. 75-198.

73. Fuks, I.M., 1974. Spectral width of signals scattered by a disturbed sea surface. Sov. Phys. Acoust. 20:276-281.

74. Furutsu, K., 1963. On the statistical theory of electromagnetic waves in a fluctuating medium, J. Res. Nat. Bur. Stds. Sect. D. 67, 303-323.

75. Garrett, C. and Munk, W.H., 1975. Space-Time Scales of Internal Waves. J. Geophys. Research. 80:291-299.

REFERENCES (Cont'd)

76. Gel'fand, I.M. and Shilov, G.E., 1968. Generalized Functions. trans by M.D. Friedman et al. New York: Academic Press., Vol. 2.

77. Glotov, V.P. and Lysanov, Y.P., 1965. Coherent reflection of sound from an ocean surface containing resonance scatters. Sov. Phys. Ac. 10:360-364.

78. Gracheva, M., Gurvich, A. and Kallistrova, M., 1970. Dispersion of strong atmospheric fluctuations in the intensity of laser radiation. Rad. Phys. Q.E. 13:40-42.

79. Gulin, E.P., 1975. Statistical characteristics of the quadrature components of sound signals reflected from a perturbed sea surface. Sov. Phys. Ac. 21:721-731.

80. Gulin, E.P., 1970. Correlation of the wave field scattered by the water surface waves. Izv. VUZ. Rad. 13:401.

81. Gulin, E.P., 1967. Frequency space - time correlation of the wave field scattered by an uneven surface. Trudy. Akust. Inst. No. 2. 49-70.

82. Gulin, E.P., 1963. Amplitude and Phase Fluctuation of a sound wave reflected from a sinusoidal surface. Sov. Phys. Ac. 8: 223-227.

83. Gulin, E.P. and Malyshev, K.I., 1963. Statistical Characteristics of Sound Signals reflected from the undulating sea surface. JASA 35:240-244.

84. Gulin, E.P., 1962. Amplitude and Phase fluctuation of a sound wave reflected from a statistically uneven surface. Sov. Phys. Ac. 8:135-140.

REFERENCES (Cont'd)

85. Gumowski, I and Mira, C., 1968. Optimization in Control Theory and Practice, Cambridge.

86. Harper, E.Y. and Labianca, F.M., 1975. Scattering of sound from a point source by a rough surface progressing over an isovelocity ocean. JASA 58:349-364.

87. Harper, E.Y. and Labianca, F.M., 1975. Perturbation theory for scattering of sound from a point source by a moving rough surface in the presence of refraction. JASA 57:1044-1051.

88. Hayes, M., 1974. Simple universal connections between energy velocities in anisotropic elastic media. JASA 56:1-3.

89. Hinton, P.L., 1970. Nonequilibrium theory of fluid fluctuations. Phys. Fluids 13:857-866.

90. Horton, C.W., Mitchell, S.K., Barnard, G.R., 1967. Model Studies on the Scattering of Acoustic Waves from a Rough Surface. JASA 41:635-643.

91. Horton, C.W. and Muir, T.G., 1967. "Theoretical Studies on the Scattering of Acoustic Waves from a Rough surface." JASA 41: 627-634.

92. Hucter, T.F. and Bolt, R.H., 1955. Sonics. New York: Wiley.

93. Hunt, F.V., 1963. "Propagation of Sound in Fluids," in American Institute of Physics Handbook, 2nd ed.

94. Huygens, C., 1678. Traité de la Lumière, published in Leyden, 1690.

95. Ishimaru, P., 1977. Theory and Application of Wave Propagation and Scattering in Random Media. Proc. IEEE. 65:1030-1061.

AD-A065 780 NAVAL UNDERWATER SYSTEMS CENTER NEW LONDON CONN NEW --ETC F/G 20/1
A GENERAL THEORY OF ACOUSTIC PROPAGATION WITH APPLICATIONS TO S--ETC(U)
JAN 78 J R BRETON

UNCLASSIFIED NUSC-TR-5871 NL

5 OF 5
AD
A065780

END
DATE
FILMED
5-79
DDC

REFERENCES (Cont'd)

96. Jôbst , W.J. and Vmets, T., 1974. Mathematical Model for volume reverberation:experiment and simulation. JASA 55:227-235.

97. Karal, F.C. and Keller, J.B., 1964. Elastic, electromagnetic and other waves in a random medium, J. Math. Phys. 5, 537-547.

98. Karavainikov, V.N. and Chuklov, V.A., Amplitude and phase fluctuations in a spherical wave. Sov. Phys. Ac. 19:539-541.

99. Keller, J.B., 1968. "A survey of the theory of wave propagation in continuous random media," Proc. Symp Turbulence Fluids and Plasmas, Polytechnic Press of the Polytechnic Inst. of Brooklyn, pp. 131-142.

100. Keller, J.B., 1962. "Wave propagation in random media," Proc. Symp. Appl. Math. 13:227-246.

101. Keller, J.B., 1960. Stochastic equations and wave propagation in random media, Proc. Symp. Appl. Math. 16, 145-170.

102. Kellogg, O.D., 1929. Foundations of Potential Theory. New York: Springer.

103. Kelly, G.E., Lewis, H.B., 1971. Hydrodynamics fluctuations. Phys. Fluids. 14:1925-1931.

104. Kennedy, R.M., 1969. Phase and Amplitude Fluctuations in Progating through a Layered Ocean. JASA 46:737-745.

105. Kinsman, B., 1965. Wind Waves, New York:Prentice-Hall.

106. Korn, G.A. and Korn, T.M., 1961. Mathematical Handbook for Scientists and Engineers. New York: McGraw-Hill.

107. Konrady, J.A., 1974. Scattering of sound from time-varying irregular free surfaces. JASA 56:1687-1694.

REFERENCES (Cont'd)

108. Kraichnan, R.H., 1962. The closure problem of turbulence theory. Proc. Symp. Appl. Math. Providence, R.I., AM Math Soc., 13:199-225.

109. Kraichnan, R.H., 1961. Dynamics of non-linear stochastic systems. J. Math. Phys. 2:124-148.

110. Kraichnan, R.H., 1953. The scattering of sound in a turbulent medium. JASA 25:822 (Abstract).

111. LaGrange, J.L., 1760. "Nouvelles recherches sur la nature et la propagation du son," Miscellanea Tauriniensis t. II p. 45.

112. Langevin, P., 1908. Sur la théorie du mouvement Brownien. C.R. Acad. Sci. Paris. 146:530.

113. Lax, M., 1966. Classical Noise. Non-linear markov processes. Rev. Mod. Phys. 38:359-379.

114. Lax, M., 1966. Langevin Methods. Rev. Mod. Phys. 38:541-566.

115. Lax, M. 1951. Multiple Scattering of Waves, Rev. Mod. Phys. 23:287-310.

116. Liebermann, L., 1951. The effect of temperature inhomogeneities in the Ocean on the Propagation of Sound. JASA 23:563-570.

117. Liebowitz, M.A., 1963. Statistical behavior of linear systems with randomly varying parameters. J. Math Phys. 4:852-858.

118. Lighthill, M.J., 1962. Sound Generated Aerodynamically. Proc. Roy. Soc. Ser. A. 267:147-182.

119. Lighthill, M.J., 1956. "Viscosity Effects in sound waves of finite amplitude," Survey in Mechanics. Cambridge University Press.

REFERENCES (Cont'd)

120. Lin, J.C. and Ishimaru, A., 1974. Multiple Scattering of waves by a uniform random distribution of discrete isotropic scatters. JASA 56:1695-1700.

121. Lindsay, R.B., 1960. Mechanical Radiation. New York: McGraw-Hill.

122. Lysanov, Y.P., 1973. Scattering of low-frequency sound by ocean surface waves. Sov. Phys. Ac. 19:359-361.

123. Lysanov, Y.P., 1969. Mean Coefficient of reflection from an uneven surface bounding an inhomogeneous medium. Sov. Phys. Ac. 15:393-400.

124. Lysanov, Y.P., 1966. Average decay law in a surface sound channel with an uneven boundary. Sov. Phys. Ac. 12:425.

125. Lysanov, Y.P., 1965. Theoretical Foundations of Underwater Acoustics Moscow. Mosk. Inst. Radio elektroniki, Gernoi Electromekaniki.

126. Mano, K., 1967. Classification of wave phenomena and its relation to coherence theory, IEEE Trans. AP-15:21-24.

127. Marsh, H.W., 1963. Sound reflection and scattering from the sea surface. JASA 35:240-244.

128. Mattuck, R., 1965. A Guide to Feynman Diagrams in the Many Body Problem. New York: McGraw-Hill.

129. McDonald, J.F., Tuteur, F.B., Zornig, J.G., 1975. Spatial inter-frequency correlation effects in a surface-scatter channel. JASA 59:1284-1293.

130. Medwin, H., 1974. Acoustic fluctuation due to microbubbles in the near-surface ocean. JASA 56:1100-1104.

131. Medwin, H., 1967. Specular scattering of underwater sound from a wind driven surface. JASA 41:1485-1495.

REFERENCES (Cont'd)

132. Mellen, R.H. and Browning, D.G., 1977. Variability of low-frequency sound absorption in the ocean: pH dependence. JASA 61:704-706.

133. Mellen, R.H., Browning, D.G. and Ross, J.M., 1974. Attenuation in randomly inhomogeneous sound channels. JASA 56:80-82.

134. Melton, D.R. and Horton, C.W., 1970. Importance of the Fresnel correction in scattering from a rough surface. JASA 47:290-303.

135. Middleton, D., 1977. Doppler Effects for Randomly Moving Scatterer and Platforms. JASA 61:1231-1250.

136. Middleton, D., 1975. "A new approach to scattering problems in random media." ONR Report, Code 222. Republished in 1977, Multivariate Analysis - IV. ed. by T.R. Krishnaialy, New York: North Holland Publishing Co.

137. Middleton, D., 1974. Characterization of active underwater acoustic channels. Parts I, II, Technical Report ARL-TR 74-61, Applied Research Laboratories, University of Texas at Austin.

138. Middleton, D., 1967. A Statistical Theory of Reverberation and Similar First-Order Scattered Field, I and II, PGIT IEEE 3:372-414. 1969, Correction 1-15; 372. 1972, Parts III and IV, IEEE PGIT & 18:35-90.

139. Middleton, D., 1965. Topics in Communication Theory. New York: McGraw-Hill.

140. Middleton, D., 1960. An Introduction to Statistical Communication Theory. New York: McGraw-Hill.

141. Mintzer, D., 1953. Wave propagation in a randomly inhomogeneous medium I. JASA 25:922-927.

REFERENCES (Cont'd)

142. Mintzer, D., 1953. Wave propagation in a randomly inhomogeneous medium II. JASA 25:1107-1111.

143. Mintzer, D., 1954. Wave propagation in a randomly inhomogeneous medium III. JASA 26:186-190.

144. Monin, A.S., "Characteristics of the scattering of sound in a turbulent atmosphere." Sov. Phys. Ac. 4:370.

145. Mooers, C.N.K., 1973. Sound Speed Perturbations due to internal tides. JASA 53:333A.

146. Moroz, T.A., 1974. "The Effect of Strong Fluctuations of the Refractive Index of the Medium on the Statistical Marine Reverberation." in Seventh All-Union Symposium "Methods of Representation and Instrument Analysis of Random Processes and Fields" Taganrog, 1974. Trans. by JPRS L/6178, 9 July 1976.

147. Morse, P.M. and Ingard, K.U., 1968. Theoretical Acoustics. New York: McGraw-Hill.

148. Nefedov, L.M., 1975. Pulse transit-time fluctuations in the deep ocean. Sov. Phys. Ac. 20:393-4.

149. Neklyudov, V.I. and Chuprov, S.D., 1974. Spatial and frequency correlation of the amplitude fluctuations of sound signals reflected from the ocean surface. Sov. Phys. Ac. 19:255-9.

150. Neubert, J.A., 1977. Experimental Agreement of stochastic ray-theory relations. JASA 64:326-335.

151. Neubert, J.A. and Lumley, J.L., 1970. Derivation of the Stochastic Helmholtz equation for sound propagation in a turbulent fluid. JASA 48:1212-1218.

REFERENCES (Cont'd)

152. Nichols, R.H. and Sayer, C.E., 1977. Frequency-frequency correlations of ocean ambient noise levels. JASA 61:1188-1190.

153. Nichols, R.H. and Senko, A., 1974. Amplitude fluctuations of low-frequency underwater acoustic pulses reflected from the ocean surface. JASA 55:550-554.

154. Nichols, R.H. and Young, H.J., 1967. Fluctuations in Low-Frequency Acoustic Propagation in the Ocean. JASA 43:716-722.

155. Nichols, R.H. and Young, H.J., 1966. Fluctuations in Low-Frequency Acoustic Propagation in the Ocean. JASA 43:716-722.

156. Noll, W., 1955. On the continuity of solid and fluid states. J. Rational Mech. Anal.: 4:13-81.

157. Obukhov, A.M., 1953. "Effect of weak atmospheric irregularities on the propagation of sound and light." Izv. AN SSSR, Geophiz. 17, 2:155-165.

158. Ol'shevskii, V.V., 1967. Characteristics of Sea Reverberation. Trans. by Consultants Bureau Plenum: New York.

159. Parkins, B., 1967. Scattering from the time-varying surface of the Ocean. JASA 42:1262-1267.

160. Phillips, O.M., 1966. The Dynamics of the Upper Ocean, Cambridge.

161. Piercy, J.E., Embleton, T. and Sutherland, L., 1977. Review of noise propagation in the atmosphere. JASA 61:1403-1418.

162. Pochapsky, T.E., 1972. Internal Waves and Turbulence in the Deep Ocean. J. Phys. Oceanog. 2:96-103.

163. Poisson, 1808. Memoire sur la theorie du son. Journal de L'école polytechnique. 7:319.

REFERENCES (Cont'd)

164. Polyanskaya, V.A. and Kharat'yan, E.G., 1975. Sound field in the ocean in the presence of high frequency external waves. Sov. Phys. Ac. 21:270-275.

165. Polyanskaya, V.A., 1974. Influence of high-frequency internal waves on the sound field of a point source. Sov. Phys. Ac. 20:55-59.

166. Porter, R.P., Spindel, R.C., 1977. Low-frequency acoustic fluctuations and internal gravity waves in the ocean. JASA 61:943-958.

167. Porter, R.P., Spindel, R.C. and Jaffee, R.J., 1974. Acoustic internal wave interaction at long ranges in the ocean. JASA 56: 1426-1436.

168. Prokhorov, A.M., Bunkin, F.V., Gochelashvily, K.S., Shishov, V.I., 1975. Laser Irradiance Propagation in Turbulent Media. Proc. IEEE 63:790-811.

169. Rasmussen, R.A. and Head, N.E., 1976. Characteristics of high-frequency sea reverberation and their application to turbulence measurement. JASA 59:55-61.

170. Rayleigh, 1878. Theory of Sound, London: MacMillan (1940 reprint).

171. Ribner, H.S., 1958. Note on Acoustic Energy flow in a moving medium. University of Toronto Inst. Geophysics, TN21.

172. Rogers, H., 1967. Theory of Recursive Functions and Effective Computability. NY: McGraw-Hill.

173. Rosenblatt, M., 1962. Random Processes. New York, Oxford U. Press.

REFERENCES (Cont'd)

174. Rytov, S.M., 1937. Izv. Akad. Nauk SSSR. Fiz. 2:223.

175. Samuels, J.C., 1960. On the Stability of Random Systems and the Stabilization of Deterministic Systems with Random Noise. JASA 32:594-601.

176. Sanford, T.B., 1974. Observations of strong current shears in the deep ocean and some implications on sound rays. JASA 56:1118-1121.

177. Scharf, L.L. and Swarts, R.L., 1974. Acoustic scattering from a stochastic sea surface. JASA 55:242-257.

178. Schulkin, M. and Marsh, H.W., 1962. Sound absorption in sea water. JASA 34:864-865.

179. Shvachko, R.F., 1967. Sound fluctuation and random inhomogeneities Sov. Phys. Ac. 13:93-97.

180. Shvachko, R.F., 1967. Sound fluctuations and random inhomogeneities in the ocean. Sov. Phys. Ac. 13:119-125.

181. Skudrzyk, E., 1971. The Foundation of Acoustics: Basic Mathematics and Basic Acoustics. New York: Springer-Verlag.

182. Smirnov, G.E. and Tonakanov, O.S., 1961. Fluctuation in hydroacoustic pulse signals on reflection from a water surface on which waves are present. Sov. Phys. Ac. 6:480-487.

183. Smith, P.W., 1974. Averaged sound transmission in range-dependent channels. JASA 55:1197-1204.

184. Spindel, R.C. and Schultheiss, 1972. Two dimensional probability structure of wind driven waves. JASA 52:1065-1068.

185. Stanford, G.E., 1974. Low frequency fluctuations of a CW signal in the ocean. JASA: 55:968-977.

REFERENCES (Cont'd)

186. Stokes, G.G., 1884. On a difficulty in the theory of sound. Phil. Mag. Nov.

187. Takács, L., 1960. Stochastic Processes: Problems and Solutions. London: Methuen, New York: Wiley.

188. Tatarskii, V.I., 1971. The Effects of the Turbulent Atmosphere on Wave Propagation, trans. by IPST staff (1974 available from U.S. Dept. of Commerce, NTIS).

189. Tatarskii, V., 1967. Wave Propagation in a Turbulent Medium. New York: Dover (reprint of [191]).

190. Tatarskii, V.I. and Gertsenshtein, M.E., 1963. Propagation of Waves in a Medium with Strong Fluctuations of the Refraction Index. Soviet Physics JETP 17: 458-463.

191. Tatarskii, V.I., 1961. Wave Propagation in a Turbulent Medium. New York: McGraw-Hill.

192. Thompson, R.J., 1974. Ray acoustic intensity in a moving medium I & II. JASA: 55:729-737.

193. Thorp, W.H., 1965. Deep Ocean Sound Attenuation in the sub and low kilocycle-per-second region. JASA 38:648-654.

194. Tolstoy, I. and Clay, C.S., 1966. Ocean Acoustics: Theory and Experiment in Underwater Sound. New York: McGraw-Hill.

195. Truesdell, C. and Toupin, R., 1960. "The classical field theories," in Handbuch der Physik Bd III, New York: Gordon and Breach.

196. Tuteur, F.B., 1976. Underwater acoustic scatter channels with several bounces. JASA: 60:840-843.

REFERENCES (Cont'd)

197. Vollerner, N.F. and Slyadnev, V.A., 1975. Statistical modeling of sea reverberation. Sov. Phys. Ac. 21:399.

198. Volovov, V.I., Lysanov, Y.P., Sechkin, V.A., 1974. Vertical correlation of the amplitude fluctuations of sound signals scattered by an uneven surface. Sov. Phys. Ac. 20:225-228.

199. Von Misês, R. and Friedricks, R.O., 1971. Fluid Dynamics. New York: Springer.

200. Waterman, P.C., 1969. New formulation of Acoustic Scattering. JASA 45:1417-1429.

201. Watson, J.G., Siegmann, W.L., Jacobson, M.J., 1977. Acoustically relevant statistics for stochastic internal wave model. JASA: 61:716-726.

202. Weinberg, N.L., Clark, J.G. and Flanagan, R.P., 1974. Internal tidal influence on deep ocean acoustic-ray propagation. JASA 56: 447-458.

203. Wenzel, 1975. Propagation of transients in a random medium. J. Math. Phys. 16:44-49.

204. Wenzel, A. and Keller, J.B., 1971. Propagation of Acoustic Waves in a Turbulent Medium. JASA 50:911-920.

205. Westervelt, P.J., 1976. Absorption of sound by sound. JASA 59:760-764.

206. Westervelt, P.J., 1957. Scattering of sound by sound. JASA 29: 934-935.

207. Westervelt, P.J., 1957. Scattering of sound by sound. JASA 29: 199-203.

REFERENCES (Cont'd)

208. Whitmarsh, D.C., 1957. Forward scattering of sound in the sea and its correlation with the temperature microstructure. JASA 29: 1124-1143.

209. Williams, R.E. and Battestin, 1976. Time coherence of acoustic signals transmitted over resolved paths with deep ocean. JASA 59:312-328.

210. Williams, R.E., 1970. Stability of ocean propagated acoustic signals. JASA 48:92.

211. Wilson, W.D., 1960. Equation of the speed of sound in sea water. JASA 32:1357(L).

212. Yaglom, A.M., 1962. An Introduction to the Theory of Stationary Random Functions, Englewood, Cliffs, NJ: Prentice Hall.

213. Zel'dis, V.I., Liekin, I.A., Rozenberg, A.D. and Ruskevich, V.G., Amplitude characteristics of sound signals scattered by water waves. Sov. Phys. Ac. 19:118-122.

214. Zipfel, G.G., Jr., 1974. Scattering of Sound from a random time-varying rough surface. JASA 56:22-30.

215. Zornig, J.G. and McDonald, J.F., 1974. Direct measurement of surface-scatter channel coherence by impulse probing. JASA 56: 1205-1211.

SUPPLEMENTAL REFERENCES

216. Brogan, W. L., 1974. Modern Control Theory, New York: Quantum.

SUPPLEMENTAL REFERENCES (Cont'd)

217. Middleton, D., and Breton, J. R., 1977. Stochastic channels as generalized communications networks, IEEE Cat. No. 77 CH 1277-3 IT, given at 1977 International Symposium on Information Theory.

218. Balakrishnan and Peterka, V., 1969. Identification in automatic control systems, Automatica 5:817.

219. Lighthill, M. J., 1959. Introduction to Fourier Analysis and Generalized Functions, Cambridge University Press.

220. Sommerfeld, 1912. Jahresber. deut. mat. 21:326.

221. Stratton, J. A., 1941. Electromagnetic Theory. New York: McGraw-Hill.

222. DeRusso, P. M., Roy, R. J. and Close, C. M., 1965. State Variables for Engineers, New York: Wiley.

223. Twersky, V., 1973. Multiple scattering by a periodic line of obstacles, JASA 53:96-112.

224. Twersky, V., 1968. Scattering by discrete random media in Proc. Symp. on Turbulence and Plasmas. New York: Poly. Tech. Inst. of Brooklyn, 143-161.

225. Twersky, V., 1967. Theory and microwave measurements of higher statistical moments of randomly scattered fields, in Electromagnetic Scattering. New York: Gordon and Breach, 579-695.

JOURNAL ABBREVIATIONS

Elect. Ltrs:	Electronics Letters, Institute of Electrical Engineers, P. O. Box 8, Southgate House, Stevengate, Herts. SG1 1 HQ England.
Izv. VUZ Rad:	Izvestiya Vysshikh Uchebnykh Zavedenii Radio-elektronika Politekhnichnyi Instytut., Kiev. Kiev, U.S.S.R.
JASA:	Journal of the Acoustical Society of America, American Institute of Physics, 335 E 45 St., New York, NY 10017.
J. Math. Phys.:	Journal of Mathematical Physics, American Institute of Physics, 335 E 45th St., New York, NY 10017.
Phys. Review:	Physical Review, the American Physical Society, American Institute of Physics, 335 E 45 th St., New York, NY 10017.
Phys. Fluids:	Physics of Fluids, American Institute of Physics, 335 E 45th St., New York, NY 10017.
Proc. IEEE:	Institute of Electrical and Electronic Engineers, Proceedings, 345 E 49th St., New York, NY 10017.
Proc. Roy. Soc.:	Proceedings of the Royal Society of London, The Royal Society, 6 Carlton House Terrace, London SWIY 5AG England.
Rad. Phys. Q. E.:	Radio Physics and Quantum Electronics (Eng trans. of Izvestiga VUZ Radiofisika) Plenum Publishing Corp. Consultants Bureau, 227 W 17th St., New York, NY 10011.

JOURNAL ABBREVIATIONS (Cont'd)

Rad. Sci.:	Radio Science, American Institute of Physics, 335 E 45th St. New York, NY 10017.
Rev. Mod. Phys.:	Reviews of Modern Physics, American Institute of Physics, 335 E 45th St., New York, NY 10017.
Sov. Phys. Ac.:	Soviet Physics - Acoustics (English translation of Akusticheskii Zhurnal), American Institute of Physics, 335 E 45th St., New York, NY 10017.
SRI-TR:	Stanford Research Institute - Technical Report, Menlo Park, CA 94025.
Uspekhi:	Soviet Physics - Uspekhi (English translation of Uspekhi Fisicheskikh Nauk.) American Institute of Physics, 335 E 45th St., New York, NY 10017.

External Distribution

Mr. Bob Adams
Naval Research Laboratory
4555 Overlook Avenue, SW
Washington, D.C. 20375

Dr. V. C. Anderson
Scripps Institution of Oceanography
University of California
La Jolla, CA 92037

Dr. Alf Andreassen
OP-095T
The Pentagon, Room 5D560
Washington, D.C. 20350

Dr. A. V. Balakrishnan
UCLA
Dept. of Information & Computing Science
Los Angeles, Calif.

Mr. George L. Boyer
Office of Naval Research
Code 222
800 N. Quincy Street
Arlington, VA 22217

Dr. Gerald Carruthers
P.O. Box 1925
Main Post Office
Washington, D.C. 20013

TRACOR
6500 Tracor Avenue
Austin, Texas 78721
ATTN: Dr. D. T. Plemons
Dr. L. M. Deuser

Dr. R. H. Clarke
Imperial College of Science & Technology
Department of Electrical Engineering
Exhibition Road
London SW7 2BT ENGLAND

Mr. Randall G. Cook
Advanced Research Projects Agency
Tactical Technology Office
1400 Wilson Boulevard
Arlington, VA 22209

Mr. R. Cooper
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

CDR J. Ballou
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

Dr. H.F. Bezdek
Program Director
Code 460
NORDA
Bay St. Louis, MS 39529

Dr. Joel Bengston
Institute for Defense Analysis
400 Army Navy Drive
Arlington, VA 22202

Mr. R. Buchal
Office of Naval Research
Acoustic Environmental Support
Detachment
Arlington, VA 22217

Dr. Roger F. Dashen
Institute for Advanced Study
Princeton, N.J. 08540

Mr. John A. DeSanto
Naval Research Laboratory
Code 8160
Washington, D.C. 20375

Dr. Ruth Davis
ODDR&E
The Pentagon, Room 3E114
Washington, D.C. 20301

Prof. I. Dyer
Massachusetts Institute of Technology
Department of Ocean Engineering
Cambridge, MA 02139

Dr. F. Fisher
Scripps Institution of Oceanography
University of California/San Diego
La Jolla, CA 92037

Dr. R.M. Fitzgerald
Naval Research Laboratory
Department of the Navy
Washington, D.C. 20375

Commander
Naval Electronics Systems Command
Naval Electronics Systems Command Hdqts.
Washington, D. C. 20360
ATTN: CAPT H. Cox, PME 124

Dr. Marvin L. Goldberger
Department of Physics
Princeton University
Princeton, NJ 08540

Dr. Ralph Goodman
Technical Director
NORDA, Code 110
Bay St. Louis, MS 39529

Mr. G. R. Hamilton
Director, Ocean Research Office
NORDA, Code 400
Bay St. Louis, MS 39529

Dr. John Brackett Hersey
Assistant for Oceanography
Chief of Naval Research
Naval Research Laboratory
Washington, D.C. 20390

Dr. T. Horwath
Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217

Dr. J. Keller
Courant Institute
251 Mercer Street
New York, NY 10012

Mr. Theo Kooij
ARC Director
Advanced Research Projects Agency
ARPA Research Center, Unit 1
Moffett Field, CA 94035

Prof. J. Lumley
Department of Electrical Engineering
Cornell University
Ithaca, N. Y. 14853

Prof. H. Medwin
Naval Postgraduate School
Department of Physics
Monetery, CA 93940

Dr. Stanley M. Flatte
360 Moore Street
Santa Cruz, CA 95060

Dr. Benjamin Huberman
Asst. Director for National Security
International & Space Affairs
Office of Science & Technical Policy
New Executive Office Bldg., Rm. 3019
Washington, D.C. 20506

Mr. B. Hurdle
Naval Research Laboratory
4555 Overlook Avenue, SW
Washington, D.C. 20375

Dr. William J. Hurley
Center for Naval Analysis
1401 Wilson Boulevard
Arlington, VA 22209

Dr. David Hyde
OASN (RE&S)
The Pentagon, Room SE777
Washington, D.C. 20350

Dr. William John Jobst
Research Scientist
Palisades Geophysical Institute
615 SW Second Avenue
Miami, FL 33130

Dr. C. W. Horton, Sr.
University of Texas
Austin, Texas 78712

Dr. William A. Nierenberg
Scripps Institution of Oceanography
University of California
La Jolla, CA 92037

Prof. J. F. MacDonald
Dept of Electrical Engineering
R. P. I.
Troy, New York

Mr. Richard R. Rojas
Assoc. Director of Research for
Oceanography
Naval Research Laboratory
Washington, D.C. 20390

Dr. Paul H. Moose
Naval Ocean Systems Center
San Diego, CA 92132

Dr. Walter H. Munk
9530 La Jolla Shores Drive
La Jolla, CA 92037

Dr. Frederik Zachariasen
452-48, Department of Physics
California Institute of Technology
Pasadena, CA 91109

Naval Ordnance Laboratory
Acoustics Division
White Oak
Silver Spring, MD 20910
ATTN: Dr. Zaka Slawsky

Dr. Harry DeFerrari
Univerity of Miami
Rosenstiel School of Marine and
Atmospheric Sciences
Miami, FL 33149

Dr. Melvin J. Jacobson
Rensselaer Polytechnic Institute
Troy, New York 12181

Dr. David Middleton (5 cys)
127 East 91st Street
New York, NY 10028

Dr. Donald W. Tufts
University of Rhode Island
Kingston, RI 02881

Raytheon Company
Research Division
Seyon Building
Waltham, MA 02154
ATTN: Dr. James A. Mullen

Department of Electrical Engineering
University of Washington
1013 N.E. 40th Street
Seattle, WA 98195
ATTN: Dr. P. Ishimaru
Dr. D. W. Lytle

Prof. Peter Schultheiss
Prof. F. B. Tuteur
Yale Univerity
New Haven, CT 06520

Dr. F. Spiess
University of California
Scripps Institution of Oceanography
Marine Physical Laboratory
San Diego, CA 92152

Dr. R. Spindel
Woods Hole Oceanographic Institution
Woods Hole, MA 02543

Institute for Acoustical Research
Miami Division of the Palisades
Geophysical Institute
615 SW Second Avenue
Miami, FL 33130
ATTN: Mr. M. Kronengold
Dr. J. Clark
Dr. C. Kimball

Dr. C. N. K. Mooers
University of Miami
Rosenstiel School of Marine and
Atmospheric Sciences
10 Rickenbacker Causeway
Miami, FL 33149

Director
Naval Research Laboratory
Department of the Navy
Washington, D.C. 20390
ATTN: Dr. John Munson
James Headrick

Dr. R. D. Turner
Office of the Director of Defense
Research and Engineering, C3-I
The Pentagon, Room 3E274
Washington, D.C. 20301

Harvard University
Division of Engineering and Applied
Physics
Pierce Hall
Cambridge, MA 02138
ATTN: Prof. Harvey Brooks
Prof. R. W. P. King

Department of Geology
Florida State University
Tallahassee, FL 35306
ATTN: Prof. Ivan Tolstoy

Institute for Telecommunication Science
Office of Telecommunications
U.S. Department of Commerce
325 Broadway
Boulder, CO 80302
ATTN: Mr. Douglass Crombie

Bell Telephone Laboratories, Inc.
Whippany, NJ 07981
ATTN: Dr. B. E. Parkins
Dr. Frank LaBianca
Dr. Edward Harper

Department of Information Sciences
The University of Hawaii
Honolulu, Oahu, HI 96822
ATTN: Dr. Norman Abramson
Dept. of Oceanography

ARL
University of Texas
Austin, Texas 78712
ATTN: Prof. D. Blackstock
Prof. G. Carrier

The John Hopkins University
33rd and Charles Streets
Baltimore, MD 21218
ATTN: Prof. O. M. Phillips
Prof. Blair Kinsman

The Catholic University of America
6220 Michigan Avenue, N.E.
Washington, D.C. 20017
ATTN: Dr. H. M. Uberall, Dept. of Physics
Dr. T. I. Smits, Dept. of Elec. Engineering
Dr. Frank Andrews, Dept. of Elec. Engineering

Dr. P. Stocklin
25 Ledgeway Court
Norwich, CT 06360

Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139
ATTN: Prof. R.S. Kennedy, Dept. of Elec. Engineering
Prof. Harry L. Van Trees, Dept. of Elec. Engineering
Prof. Eric Mollo-Christiansen, Dept. of Meteorology

Department of Applied Physics and Information Science
University of California, San Diego
La Jolla, CA 92037
ATTN: Dr. C. W. Helstrom
Dr. S. O. Rice

Department of Electrical Engineering
University of New Mexico
Albuquerque, NM 87106
ATTN: Dr. D. P. Peterson

Wave Propagation Laboratory (R-45)
NOAA, Department of Commerce
Boulder, CO 80302
ATTN: Dr. Vernon E. Derr
Dr. D. E. Barrick
Dr. Freeman Hall
Dr. Steven Clifford
Dr. Edmund Browne

Chief of Naval Research
Department of the Navy
Arlington, VA 22217
ATTN: A.O. Sykes (Code 222) (2 cys)
Robert F. Obrochta (Code 222)
Dr. Roy D. Gaul

Department of Geology and Geophysics
Geophysical and Polar Research Center
The University of Wisconsin
1215 West Dayton Street
Madison, WI 53706
ATTN: Dr. C. S. Clay

Department of Electrical Engineering
University of Michigan
500 South State Street
Ann Arbor, MI 48103
ATTN: Prof. T. G. Birdsall

Department of Statistics
University of North Carolina
Chapel Hill, NC 27514
ATTN: Dr. Charles R. Baker

D. P. Vasholz
Naval Coastal Systems Laboratory
Panama City, FL 32407

Prof. Charles Polk
Department of Electrical Engineering
University of Rhode Island
Kingston, RI 02881

Prof. John Spence
Department of Electrical Engineering
University of Rhode Island
Kingston, RI 02881

Electrical Engineering Department
Engineering Quadrangle
Princeton University
Princeton, NJ 08540
ATTN: Prof. Stuart Schwartz
Prof. John Thomas

University of Rhode Island
Physics Department
Kingston, RI 02881
ATTN: Prof. Scott Desjardins
Prof. Charles Kaufmann

Dr. Richard Hoglund
Pentagon, Room 5E787
Washington, D.C.

COMASWFORSIXTHFLT
COMNAVSURFLANT
COMSUBLANT
COMSUBDEVRONTWELVE
Chief of Naval Research (ONR)
ONR-102-OS
ONR-427
ONR-220
ONR-102
ONR-480
ONR-481
ONR-485
ONR-212
ONR-222
ONR-230
ONR-486
CNM MAT-00

NOSC
San Diego, CA
ATTN: Dr. James L. Stewart, Code 714
Dr. Homer Bucker, Code 5311
Mr. Melvin Pedersen, Code 714
Dr. Jack Northrop, Code 5311
Dr. Richard White, Code 6354
Dr. Jerry Neubert, Code 5311
Mr. Larry Stotts, Code 8114
Dr. Wilbur Watson, Code 714
Dr. Benjamin L. McGlassey
Dr. Eugene Cooper, Code 013
Dr. Harry Schinck, Code 71
Dr. Manuel Gonzales, Code 7122
Dr. Yosh. Igaraski, Code 6B52
Dr. Chuck Schellner, Code 5324

NAVSEA
Dr. John Huth (03C)
Beatrice Orleans (03F)
Washington, D.C. 20362

Dr. Gene Gleissner
NSRDC
Carderock, MD

CAPT Charles G. Darrell
NORDA
NSTL Sta., Miss. 39529

Prof. Osman K. Mawardi
Case Western Reserve Univ.
University Circle
Cleveland, Ohio 44106

COMNAVSUFPAC
COMSUBPAC
Asst. Sec. Navy for Research & Dev.
Chief of Naval Operations CNO OP-02
OP-03EG
OP-090
OP-095
OP-098
OP-03
OP-22
OP-35
OP-941
OP-946
OP-951
OP-951D
OP-951E
OP-953

MAT-03B
MAT-031
MAT-032
MAT-034
Special Project Off. (PM-1) SP-20
Special Project Off. (PM-1) SP-24
ASW Proj. Off. (PM-4) ASW-122
ASW Proj. Off. (PM-4) ASW-111
ASW Proj. Off. (PM-4) ASW-10
ASW Proj. Off. (PM-4) ASW-23
ASW Proj. Off. (PM-4) ASW-24
ASW SPO (PM-4) ASW-53
Ocean Engineering (MAT-03621)
Surface Ships Proj. Off. (PM-18)
Naval Research Laboratory
NRL, USRD
NORDA
OCEANAV
NAVELECSYSCOM PME-117
NAVELECSYSCOM PME-124
NAVELECSYSCOM ELEX 320
NAVSEASYSCOM
SEA-03C
SEA-032
SEA-034
SEA-04G2
SEA-06G
SEA-06H
SEA-06H1
SEA-06H1-1
SEA-06H1-2
SEA-06H1-3
SEA-06H1-4
SEA-06H2
SEA-09G32
SEA-660
SEA-660C
SEA-660E
SEA-92
DARPA

OP-955
OP-96
OP-37
OP-981
OP-981F
OP-981G1
OP-981H
OP-981J
OP-982F
OP-983
OP-983F
DDR&E (G. Cann)
DWTNSRDC (ANNA)
DWTNSRDC (CARD)
NRL, AESD
SUBASE LANT
NAVOCEANO
NAVOCEANO Code 02
NAVOCEANO Code 037
NAVOCEANO Code 6100
NAVOCEANO Code 3400
NAVOCEANO Code 6200
NAVOCEANO Code 6130
DTNSRDC
NAVCOASTSYSLABSEA-0341
NAVOCEANSYSCEN
NAVOCEANSYSCEN Code 6565
NAVSEC SEC-6034
NAVWARCOL
APL/UW, Seattle
ARL/PENN State
ARL, Univ. of Texas
Center for Naval Analyses
DDC, Alexandria (10)
Marine Physical Lab, Scripps
NOAA/ERL
National Research Council (Committee Undersea Warfare)
Woods Hole Oceanographic Institution

Internal Distribution

Dr. C. N. Pryor	Code 01
Dr. W. A. Von Winkle	10
F. G. Weigle	101
R. M. Dunlap	101
Dr. E. S. Eby	101
Dr. A. J. Van Woerkom	101
Dr. D. M. Viccione	101
W. L. Clearwaters	20
A. S. L'Etoile	201
H. J. Wilms	201
Dr. D. J. Bordelon	202
G. E. Everly	211
T. G. Bell	22
K. Y. Korolenko	2301
R. J. Konley	234
J. Webster	24
C. A. Spero	30
R. L. Moore	301
R. W. Hasse, Jr.	31
Dr. F. J. Kingsbury	31
A. Donn Cobb	3102
A. W. Ellinthorpe	3103
H. A. Freese	3103
B. F. Cole	3104
C. L. Brown	311
Dr. D. N. Connors	311
T. R. Cummings	311
Dr. L. Goodman	311
J. Gorman	311
D. G. Browning	312
Dr. R. C. Deavenport	312
Dr. F. R. DiNapoli	312
W. R. Schumacher	312
Dr. D. Wood	3123
Dr. P. B. Abraham	313
Dr. P. G. Cable	313
Dr. E. L. Green	313
Dr. M. Green	313
Dr. R. H. Mellen	313
Dr. R. F. Santopietro	313
Dr. P. Shajenko	313
Dr. A. H. Nuttall	313
Dr. R. L. Gordon	314
Dr. J. S. Papadakis	314
Dr. R. A. Saenger	314
Dr. H. Weinberg	314
Dr. J. E. Atkinson	315
S. A. Dzerovych	315
Dr. W. K. Fischer	315
Dr. T. V. McAndrew	315
Dr. N. L. Owsley	315
Dr. S. H. Ko	316
Dr. C. H. Sherman	316
Dr. W. A. Strawderman	316
Dr. R. S. Woollett	316
E. C. Gannon	3164
W. L. Konrad	3164
Dr. M. B. Moffett	3164
J. W. Kyle	32
H. E. Fridge	3202
H. Weaver	321
J. H. Caywood	3212
J. E. Geary	322
D. M. White	322
Dr. R. C. Elswick	323
Dr. H. H. Schloemer	323
F. D. Manganelli	324
Dr. R. F. Croteau	3241
R. Read	3242
P. F. Radics	3292
D. Walters	33
R. E. Baline	3301
R. L. Boivin	331
M. M. Hundt	3312
A. L. Goodman	3314
B. Deligeorges	332
A. A. Filippini	333
W. C. Hay, Jr.	334
J. J. Hanrahan	335
J. Merrill	34
Dr. D. Dence	3401
Dr. P. R. Bannister	341
Dr. D. E. Fessenden	342
D. A. McQueeney	344
Dr. J. C. Hassab	35201
C. S. Soliozy	38
Dr. R. A. Rubega	3803
D. L. Nichols	40
M. Berger	40
A.D. Carlson	401
Dr. A. J. Kalinowski	401
Dr. R. G. Kasper	401
Dr. H. L. Pond	401
M. A. Tucchio	401
W. J. Prysner	41
W. A. Andersen	411

H. G. Lipsett	411
G. R. Akers	4142
Dr. J. R. Breton	4142 (25)
J. Dorman	4142
R.O. Carson	4311
J. G. Navin	434
M. J. Goldstein	4424
Dr. P. Scully-Power	101
Dr. A. Quazi	313

New London

Code 4532 (2)
4522 (Library) (3)

Newport

Code 4555 (Storage) (10)