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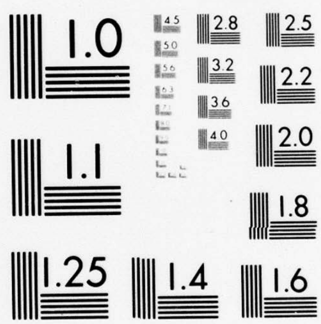

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OF LIMITING RELATIONS BETWEEN MEAN EXPONENTIAL AND CHEBYSHEV S APPROXIMATIONS

by

Ye. Ya. Remez

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OF LIMITING RELATIONS BETWEEN MEAN EXPONENTIAL AND CHEBYSHEV S APPROXIMATIONS

(First Announcement)

by Ye, Ya. Remez

In some of my previous publications, beginning 1940 (1),(2), (3) I have proposed a method of successive quadratic approximations for constructing generalized polynomials, which deviate least from null, in the meaning of the principle of the smallest m^{th} degrees. This method which,permitting monotonous formation for very much diverse cases, works at unusually general conditions and is based in its fundamental algorithm on some extreme principles, which meet the nature of the problem itself. I could view this method as some step in finding a general method to solve the most interesting and known by its difficulty problem of actual construction of solutions of Chebyschev's analogous problem (the smallest uniform deviation from null), since the theoretical possibility of limiting transition from the mean exponential approximations to Chebyschev's itself, at a condition of the continu -- ity of the examined function, was estableshed (4), (5) by Polya and Jackson. [1]

I started this work as a further step in the indicated direction; in this work I put to a precise experiment a complex of questions as to the study of the convergence speed of Polya-Jackson limiting transition and some of its interpolating modifications, taking as a measure its speed of decrease at $m \to \infty$ of the difference between the value of the uniform approximation , which is given by the solution of the mean exponential approximation and the value of the best (in Chebyschev's meaning) approximation. In this , it is discovered that, in general, the basic factor, which affects the theoretical order of the value of this difference , is a module of the continuity of these functions which form the examined polynomials of the minimized deviation from null.

The first two paragraphs have introductory character, however, the theorems which are included in them are, perhaps, mentioned the first time in such a general expression.

The in # 3-8 of this work raised quastion is investigated for the most important specific classes of function , in the next three paragraphs (9-11), general investigation for any class of functions,whose module of continuity is subordinate to a certain non- equality of a general form, is given. In all the cases, in addition to the established order of infinitely small values, which are investigated, an asymptotic evaluation of corresponding coefficients or their enough approximated limits at $m \to \infty$ is added.

In # 12, a theorem, which is analogous with its formulation to the Lebega (13) theorem on the order of the best approximation of any form of continuous function (not subjected to further structural conditions) by means of rational polynomials of an unlimitedly increasing degree, is established for the given circle of questions. In connection with this, it should be noted, that as the results in # 3-11 have certain analogy (in the essence of the question - separated enough and partial) with the known results of Lebega (13), Bernstein (15) and Jackson (16) which determine the dependence of the decrease speed of the best approximation value $E_n[f(x)]$ on the more or less regularity of local structural characteristics of the continuous function $f(x)$, then, on the other hand, in the investigated here by us area have no room the inverse correlations similar to those that are considered inversed by the theorems of Bernstein (15),(14).

The last paragraphs, which form the second part of this work and which shall be printed in the next issue of this collection, shall contain investigations of an analogous complex of problems for a corresponding interpolating problem. Here shall be studied a uniform approximation, which is given in the whole segment (a, b) by the solutions of the mean exponential approximation problem with a variable finite multiple of points of the same segment. Comparison of the results of this part with the previous work will enable us to make some conclusions, which have direct relation to the application of my mentioned above method of consecutive approximations.

The results of the whole investigation conducted for a specific function of one real variable allow direct approximations for other cases too.

1. First, we shall take note of some basic facts which are applicable to the approximation problem according to the smallest m^{th} degrees ($m > 1$).

Taking the problem in general form, which encompases as very special cases both problems investigated in further paragraphs (3-12 and 13-15), let E denote a given point number of a positive (finite) measure in some abstract space (in the meaning of Fresh (7), (8)), wherein for some positive number system $\{e\}$ there is designated a completely positive non-negative measure μe. We shall assign a letter x to all of the various elements of the (point) number. Further, let $v_0(x)$, $v_1(x), \ldots, v_n(x)$ be $n+1$ ($n > 0$) of given (real and complex) numerical functions of a point expressed simply, measuring (μ), with finite $\int_E |v_i(x)|^m d\mu$ $(i=0, 1, \ldots, n)$ and linearly independent of a number E. The last condition in this case means the existance of a fixed non-negative number $g < \mu E$, such that the subset $E_1 \subset E$ of these points, in which any generalized polinimial $\sum_{i=0}^{n} c_i v_i(x)$ $\left(\sum_{i=0}^{n} |c_i| > 0\right)$ of the investigated system of the function $\{v_i(x)\}$ is nullified, always satisfies the condition $\mu E_1 \leq g$).

The investigated "problem of the mean exponential approximation" (with a given index m) is based on determination of the numerical values of the coefficients $c_1, c_2, \ldots, c_n$ of an "assumed" generalized polynomial

$$\phi(x) = v_0(x) + P(x) = v_0(x) + \sum_{i=1}^{n} c_i v_i(x) \qquad (1)$$

according to the minimization of the value

$$\left[\frac{1}{\mu E}\int_E |\phi(x)|^m d\mu\right]^{\frac{1}{m}} = \delta_m[\phi] = \delta_m(c_1, c_2, \ldots, c_n). \qquad (2)$$

Unity of the problem solution, according to the given conditions[3], is determined easily by a corresponding expension of the proof method used by Jackson [4]). As to the existance of a solution, as it easily satisfied excluding dependent function, it is secured even without introducing of the condition of linear dependence of the functions $v_i(x)$.

The following assertion should also be a direct significance for investigations of further paragraphs of the given work.

<u>Theorem</u>. If at least one of the assumed polynomials $\phi^*(x) = v_0(x) + \sum_{i=1}^{n} c_i^* v_i(x)$ is limited almost everywhere in a set E (where the integral of $|v_0(x)|^m, \ldots, |v_n(x)|^m$ are assumed finite for all of the investigated values of m, then the coefficients c_{m1}, c_{m2} .. ,c_{mn} of the generalized polynomial $\phi(x) = \phi_m(x)$, which is the solution of the above mentioned problem of the mean exponential approximation , are uniformly limited for all of the in-

vestigated values of $m > 1$.

The proof is obtained exeptionally simply on the basis of such a general lemma established by me and already applied to some questions of functions' approximations in my previous works (9), (2), (3).

Let L denote the largest of the coefficients' modules of a generalized polynomial $\Omega(x) = \sum_{i=0}^{n} c_i v_i(x)$ of a system of measuring (μ) and linearly independent functions $\{v_i(x)\}$:

$$L = \max\{|c_0|, |c_1|, \ldots, |c_n|\} = L[\Omega] = L(c_0, c_1, \ldots, c_n). \tag{3}$$

Then to each positive $\varepsilon > 0$ (that deas not exceed the difference $\mu E - g$) one may add such $\lambda = \lambda_\varepsilon > 0$, that the inequality $|\Omega(x)| \leq \eta$ cannot be realized on any point set $E_1 \subset E$ of the measure $\mu E_1 > g + \varepsilon$, if $\eta < L\lambda$. More accurately, under λ_ε one can understand the largest of the numbers λ, which satisfy the indicated condition.

<u>The proof of the theorem.</u> On the basis of the quoted lemma, if any fixed $\varepsilon > 0$ ($\varepsilon < \mu E - g$), is given, then the inequality $|\Phi_m(x)| > L_m\lambda_\varepsilon$, where $L_m = L\{\Phi_m\}$, will be realized on a point set $E_1 \subset E$, the measure of which equals at least $\mu E - g - \varepsilon$. Therefore,

$$\int_E |\Phi_m(x)|^m d\mu > (L_m\lambda_\varepsilon)^m(\mu E - g - \varepsilon). \tag{4}$$

It is clearly, that putting

$$\text{vrai max } |\Phi^*(x)| = M,$$

we should have

$$\left[\frac{1}{\mu E}(L_m\lambda_\varepsilon)^m(\mu E - g - \varepsilon)\right]^{\frac{1}{m}} < \delta_m[\Phi^*] \leq \left(\frac{1}{\mu E} \cdot M^m \cdot \mu E\right)^{\frac{1}{m}} = M,$$

that is

$$L_m < \frac{M}{\lambda_\varepsilon}\left(\frac{\mu E}{\mu E - g - \varepsilon}\right)^{\frac{1}{m}} < \frac{M \cdot \mu E}{\lambda_\varepsilon(\mu E - g - \varepsilon)} \quad \text{for } m > 1, \tag{5}$$

whereby the proof is completed.

§2. Each time when all $n+1$ of functions $v_0(x)$, $v_1(x), \ldots, v_n(x)$ are limited in a usual meaning (with keeping of other conditions §1), side by side with the previous problem acquires a meaning the problem of the best (uniform) approximation by the Chebyshev's principle, which rely already in minimization on the " value of uniform approximation" (that measures the uniform deviation of the permissible generalized polynomial $\Phi(x)$ from null).

$$\sup_{x \in E} |\Phi(x)| = \sup_{x \in E} \left| v_0(x) + \sum_{i=1}^{n} c_i v_i(x) \right| = \delta_0[\Phi] = \delta_0(c_1, c_2, \ldots, c_n). \tag{6}$$

The lemma, quated in §1 has as its evident result unlimited increase of the value at $L[\Phi] \to \infty$, whereof it is easy to derive, the usual way, the fact of the existance of at least one solution of the investigated Chebyshev's problem 5). As to the unity of the solution, it is not assured by the given conditions, however, for the next investigations of the given work it has no essential meaning anyhow, 6).

As it was said in the introduction , the basic object of our further investigations is the exploration of the speed of convergence of the known, connected with the names of Polia-Jackson-Julia, limiting transition from the meam exponential approximations to Chebyshev's approximations (§3-12) and some analogous interpolating algorithms (§13-15) which allow more effective realization.

It easy to understand , that on the basis of these general conditions, at which the above both problems of approximation were formulated by us (§1 and the given paragraph), it is impossible to establish a connection between them, which is characterized by an interesting for us theorem of Polya-Jackson (5),(4), according to which

$$\lim_{m\to\infty} \delta_0[\Phi_m] = \delta_0[\Phi_0] = \rho \qquad (7)$$

where Φ_0 indicated a solution of the minimization problem of the value (6). To establish such a connection it is necessary to subordinate the functions of $v_i(x)$ and the set **E** to additional conditions of such a kind that condition

$$\sup_{x\in E} |\Phi(x)| = \operatorname{vrai}\max_{x\in E} |\Phi(x)|. \qquad \text{(Condition A)}$$

is realized.

The essence of such a condition is obvious. If, for example, we take a set **E** over a segment (0,1) of the number axis, as $v_0(x)$, the known function from an example of Dirichle, equal 1 for rational x and 0 for irrational, and as $v_1(x)$ a constant equal unity, and investigate the minimization problems of values (2) and (6) for polynomials $\Phi(x) = v_0(x) + c$, then we shall have $\Phi_m(x) = v_0(x)$ for each m; $\Phi_0(x) = v_0(x) - \frac{1}{2}$,

Thus, here $\delta_0[\Phi_m] = 1$, $\delta_0[\Phi_0] = \frac{1}{2}$ and the relation (7) has no place at all.

On the other hand, we shall convince ourselves without any difficulty as to the justification of such an assertation, which is, evidently, a generalization of the Polya and Jackson theorem.:

Theorem. The correlation Polya-Jackson (7) is always realized, when at general conditions, indicated in the beginning of this paragraph, has a place also a condition A.

Proof. According to the theorem #1 and considering continuous dependence of the value (6) on the coefficients of the polynomial, it is enough to be convinced that each polynomial $\Phi(x)$, which is limiting to any sequence $\{\Phi_{m_i}(x)\}$, where $m_i \to \infty$ at $i\to\infty$, is one of the polynomials (of the smallest uniform deviation from null) $\Phi_0(x)$. However, if we assume the opposite and take that $\delta_0[\Phi]$ supercedes the value $\delta_0[\Phi_0] = \rho$ by a some additive value 3ε, then, according to the condition A $|\Phi(x)|$ will supercede $\rho + 2\varepsilon$ in all points of a certain subset $E_1 \subset E$ of the value $\mu E_1 = \sigma > 0$, according to the same subset $|\Phi_{m_i}(x)|$ will supercede $\rho + \varepsilon$ at $i > i_1$, however, this brings to a contradiction, since at large enough values of the index i (and at the same time m_i) the inequality $\delta_{m_i}[\Phi_{m_i}] \leq \delta_{m_i}[\Phi_0]$ will be disturbed:

$$\{\delta_{m_i}[\Phi_{m_i}]\}^{m_i} \geq (\rho+\varepsilon)^{m_i}\frac{\sigma}{\mu E}, \quad \{\delta_{m_i}[\Phi_0]\}^{m_i} \leq \rho^{m_i},$$

$$\left(1+\frac{\varepsilon}{\rho}\right)^{m_i} > \frac{\mu E}{\sigma} \quad \text{for large enough } m_i.$$

From the proved theorem, it follows that when an abstract space #1 is given some topological characteristics. as, for example, metric , then the application of the Polya-Jackson theorem may be assured, if it is assumed somehow less than continuity: to characterize a case of a euclide space of one or more dimentions, it is enough to demand that the polynomials $\Phi(x)$ have in each point $x \in$ **E**, so to speak, a partial continuity (a weakened form of known characteristic of an "approximating" or "asymptotic" continuity), that is a continuity with respect to some subset which in each proximity of this poit has a measure greater than null. It is understood, that we put aside a question as to the uniformity or non-uniformity of the above partial continuity in a set **E**.

Especially, if , for example, **E** is (open) interval (a, b) of a numerical line, then to establish an application of the Polya-Jackson theorem it is more than enough to assume a onesided (say , right hand) continuity of all n+1 (limited and linearly independent) functions $v_i(x)$ at each point.

However, for a more accurate investigation of this work, in which will be discussion not about an establishment of the fact itself of the congruence of the Polya-Jackson limiting transition, but about an investigation of the convergence velocity, it necessary to introduce a condition of a real and, mainly, uniform continuity.

In our further investigations x will always indicate a numerical argument.

For concreteness, we shall conduct further investigations according to a case of real functions of one real argument. The broadening of the results over cases of various complex or real functions of one or more numerical arguments is not difficult.

§3. Thus, x will further indicate a real number which passes through numerical quantity E, which in the beginning will be some given segment (a, b) having length of b-a=l. Under $v_0(x)$, $v_1(x)$,..., $v_n(x)$ we shall understand real, continuous, and linearly independent functions in the segment (a, b).

But this is not enough: to have something more than the establishment of the fact of the Polya-Jackson convergence process, it is necessary to further define the conditions, that are set on the structural characteristcs of the $v_i(x)$ function. Really, the established by us further (§12) theorem, analogous to the known theorem of Lebeg about the order of the best approximation of continuous functions by rational polynomials [9]), shows that also Polya-Jackson's process may converge as slowly as it pleases, if one understands under $v_i(x)$ (i=0, 1,...,n) completely arbitrarely given continuous functions.

We shall first investigate this question with the assumption that the given n+1 functions $v_i(x)$ satisfy the Lipshitz condition of a certain order of τ $(0<\tau<1)$ on the segment (a,b).

$$|v_i(x_2) - v_i(x_1)| < k_i \cdot |x_2 - x_1|^{\tau} \quad (i=0, 1, \ldots, n). \tag{8}$$

Without limiting the generalization, we shall assume that the order τ is the same for all n+1 of the given functions; otherwise we would have marked by τ the smallest of n+1 numbers $\tau_0, \tau_1, \ldots \tau_n$.

The above formulated two general problems of the mean exponential and the Chebyshev's approximation are taking the following form:

$$\delta_m[\Phi] = \delta_m(c_1, \ldots, c_n) = \left[\frac{1}{b-a}\int_a^b |\Phi(x)|^m dx\right]^{\frac{1}{m}} = \min \; (m>1) \tag{9}$$

and correspondingly

$$\delta_0[\Phi] = \delta_0(c_1, \ldots, c_n) = \max_{a \le x \le b} |\Phi(x)| = \min , \tag{10}$$

where as previously

$$\Phi(x) = v_0(x) + \sum_{i=1}^{n} c_i v_i(x).$$

Let accordingly to the designations of the previous paragraphs

$$\Phi_m(x) = v_0(x) + \sum_{i=1}^{n} c_{mi} v_i(x) \tag{11}$$

shall be the solution for problem (9) (the only for each fixed m) and (any) solution for problem (10)[10]. Keeping futher the sign ρ for surely [redacted] positive " magnitude of the best approximation" in problem (10) (12)

$$\delta_0[\Phi_0] = \varrho, \tag{13}$$

let

$$\delta_0[\Phi_m] = \max_{a \le x \le b} |\Phi_m(x)| = \varrho(1+2a_m) = \varrho(1+2a), \tag{14}$$

where not-subtractive number $2a_m = 2\alpha$ characterizes, obviously , the relative worsening of the uniform approximation value at a substitute of $\Phi_0(x)$ for Further , to simplify the notations, we shall mostly write a without an index m.

The Polya-Jackson theorem (that is in Jackson's formulation (4)) determined the fact of value a going [redacted] to zero [redacted] a at $m \to \infty$ in the case of any continuous function $v_i(x)$". Here, wanting to apprize the convergence speed of the Polya-Jackson process, we shall take as a measure the order of its decrease a at $m \to \infty$.

We shall note , first of all, that according to the proved theorem in §1 about the uniform limitation of the coefficients $c_{m1}, c_{m2}, \ldots c_{mn}$ of the $\Phi_m(x)$ polynomial at $m \to \infty$, all $\Phi_m(x)$ based on the conditions (8) will satisfy the Lipshitz condition of the same order τ with a constant coefficient k (independent from m)

$$|\Phi_m(x_2) - \Phi_m(x_1)| \leqslant k|x_2 - x_1|^\tau \quad (m > 1), \tag{15}$$

where

$$k \leqslant k_0 + \sum_{i=1}^{n} k_i H_i, \quad H_i = \sup_m |c_{mi}|. \tag{15"}$$

Let us determine $h > 0$ from the term

$$kh^\tau = a\varrho \quad (a = a_m), \tag{16}$$

wherefrom

$$h = \left(\frac{\varrho}{k} a\right)^{\frac{1}{\tau}} = Ca^{\frac{1}{\tau}} \quad \left[C = \left(\frac{\varrho}{k}\right)^{\frac{1}{\tau}}\right].$$

Assuming $Ca^{\frac{1}{\tau}} < l$, which certainly has a room for very large $m \geqslant m_0$, from the condition

$$\delta_m[\Phi_m] < \delta_m[\Phi_0]$$

we obtain

$$\varrho^m(1+a)^m \cdot Ca^{\frac{1}{\tau}} < \varrho^m l, \tag{17}$$

wherefrom

$$(1+a)^m < \frac{l}{C} \cdot \left(\frac{1}{a}\right)^{\frac{1}{\tau}}.$$

Taking logarithms from both sides of this inequality and designating (usually , with the assumption that $a_m \neq 0$ [12])

$$\frac{\log(1+a)}{a} = \frac{\log(1+a_m)}{a_m} = \sigma_m, \tag{18}$$

we have

$$m\sigma_m a < \log\frac{l}{C} + \frac{1}{\tau}\log\frac{1}{a}.$$

Noting that according to the uniform limitation of the coefficients of the polynomial $\Phi_m(x)$ for all investigated values of $m > 1$, tha value a also limited from above for all these meanings and that , on the other hand, $\sigma_m \to 1$ at $a_m \to 0$, we shall have

$$\inf_m \sigma_m = \inf_m \frac{\log(1+a_m)}{a_m} = \sigma > 0. \tag{20}$$

In addition

$$\sigma_m < 1, \quad \lim_{m \to \infty} \sigma_m = 1. \tag{21}$$

For all meanings of $m \geq m_0$, together with (19) has a place

$$m\sigma a < \log\frac{l}{C} + \frac{1}{\tau}\log\frac{1}{a},$$

wherefrom

$$ma < C' + \frac{1}{\tau\sigma}\log\frac{1}{a} \quad \left(C' = \frac{1}{\sigma}\log\frac{l}{C}\right). \qquad (22)$$

Let us put now

$$a = q_m\frac{\log m}{m}, \text{ де } q_m = \frac{am}{\log m} = \frac{a_m \cdot m}{\log m}. \qquad (23)$$

then from (22) we shall have

$$q_m \log m < C' + \frac{1}{\tau\sigma}\left(\log\frac{1}{q_m} + \log m - \log\log m\right),$$

$$q_m < \frac{1}{\tau\sigma} + \frac{C' - \frac{1}{\tau\sigma}\log\log m}{\log m} + \frac{\frac{1}{\tau\sigma}\log\frac{1}{q_m}}{\log m}. \qquad (24)$$

From here, it is easily to see, that $\sqrt{q_m} < \frac{1}{\tau\sigma}$ for very large m. (25).

Really, when $q_m \le 1$, this statement is safe, since $\frac{1}{\tau\sigma} > 1$. When $q_m > 1$, the inequality (25) is also safe, in so far as for large enough values of m, not only the second but also the third term on the right hand side (24) is negative.

However, if only the inequality (25) is established, from there directly comes the limitation of the expression $\frac{a_m}{\log m}$ for all meanings of $m \geq m_1$, where m_1 is any number > 1. Especially, taking for example, $m_1 = 2$, we shall have [12])

$$a = a_m \leq \frac{c \log m}{m} \text{ для всіх (for) } m \geq 2, \qquad (26)$$

where

$$c = \sup_{m \geq 2}\frac{a_m m}{\log m} = \sup_{m \geq 2} q_m.$$

The otained inequality (26), which gives ua the necessary appraisal of the congruence order of the Polya-Jackson approximation process for the investigated function $v_i(x)$, which satisfy the Lipshitz's condition, shows that

$$a_m = O\left(\frac{\log m}{m}\right) \text{ at } m \to \infty \qquad (26')$$

We see, that the appearance of the obtained inequality (26) does not depend even on the indicator τ in the Lipshitz's condition (8), (15), even though that closer evaluation of the coefficient q_m in (23) shows already , unquestionably, the dependence on the structural parameter, and a weaker, at large m, dependence on k ; if at large m one introduces the magnitude of σ_m into (22)-(24), instead of its lower limit σ, then, according to (21) , we shall get the ralation (beside of (25)

$$\overline{\lim_{m \to \infty}}\left(a_m : \frac{\log m}{m}\right) \leq \frac{1}{\tau}. \qquad (27)$$

see page 9 for omission → We shall further (§5) see, that the obtained evaluation (26') of the decrease order of the magnitude a cannot generally be improved, even if the conditions as to the regularity of the $v_i(x)$ function is strengthened, introducing into the investigation classes of the p-multiple differentiating functions etc., and finally, the analytical functions and even the class of whole rational functions; although, on the other hand, it is easy to construct individual examples, which pertain to continuous functions of any little regulating structure, where the order $\left(\frac{\log m}{m}\right)$ is substituted by $\left(\frac{1}{m}\right)$ or even where $a_m = 0$ for all meanings of m.

Further it is not difficult to prove that the right side can be replaced by $\frac{1}{\tau(2-\gamma}$ in this inequality when there is a sufficiently small fixed $\gamma>0$, if the factor on the right side of 1 is introduced into (16), and correspondingly, in the left side of (17) $(1+\alpha)^m$ is replaced by $[1+(2-\gamma)\alpha]^m$. Finally, by removing the sufficiently small γ, we obtain;

$$\overline{\lim_{m\to\infty}}\left(a_m : \frac{\log m}{m}\right) \leqq \frac{1}{2\tau}.$$

4. First, before we shall proceed with proving of the statement , formulated at the end of the last paragraph, we shall prove two lemmas.

Lemma 1. Let f(x) be continuous non-abating function $[f(0)=0,\ f(l)\leq 1]$ on the segment $(0,l)$ and sharply increasing at least at some part of the given segment. Further, let δ denote any small but fixed positive number , which we shall regard as smaller than the number l. Then

$$\lim_{m\to+\infty}\frac{\int_0^l [1-f(x)]^m\,dx}{\int_0^\delta [1-f(x)]^m\,dx}=1. \tag{28}$$

Proof. Setting, for the sake of abbreviation.

$$J_a^b=\int_a^b [1-f(x)]^m\,dx, \tag{29}$$

denoting by ε any supplementary small positive number, we have

$$J_0^l=J_0^\delta+J_\delta^{\delta+\varepsilon\delta}+J_{\delta+\varepsilon\delta}^l. \tag{30}$$

It obvious, that $J_\delta^{\delta+\varepsilon\delta}<\varepsilon J_0^\delta$ at each $m>0$. Further from the expression

$$J_0^\delta>\delta[1-f(\delta)]^m\ ;\ J_{\delta+\varepsilon\delta}^l<l[1-f(\delta+\varepsilon\delta)]^m,$$

it is easy to conclude that

$$\frac{J_{\delta+\varepsilon\delta}^l}{J_0^\delta}<\frac{l}{\delta}\left[\frac{1-f(\delta+\varepsilon\delta)}{1-f(\delta)}\right]^m<\varepsilon \quad \text{at large enough } m>0$$

In this way, for large enough positive meanings of m , we shall have

$$J_0^\delta<J_0^l<J_0^\delta(1+2\varepsilon), \tag{31}$$

wherefrom , considering that ε is small enough, comes out a relation (28) which we had to prove.

Lemma II [14]). Let k,λ and δ denote any positive numbers, which are subordinate to the condition of $\lambda\delta^k\leq 1$. Then

$$\lim_{m\to+\infty}\left[m^{\frac{1}{k}}\int_0^\delta (1-\lambda x^k)^m\,dx\right]=\frac{\Gamma\left(\frac{1}{k}\right)}{k\lambda^{\frac{1}{k}}}. \tag{32}$$

Proof. On the basis of the known double inequality

$$\frac{u}{1+u}<\log(1+u)<u\quad(-1<u<\infty) \tag{33}$$

we first have

$$I_\delta=m^{\frac{1}{k}}\int_0^\delta (1-\lambda x^k)^m\,dx=m^{\frac{1}{k}}\int_0^\delta e^{-\frac{\lambda m x^k}{1-\theta\lambda x^k}}\,dx \tag{34}$$

$$[0<\theta=\theta(x)<1].$$

Introducing an auxiliary small enough number $\eta>0$ and understanding that $I_{(\eta\delta)}$ is the result of the sustitution of δ on $\eta\delta$ in (34), on the the basis of the previous lemma we have

$$\lim_{m\to\infty}\frac{I_\delta}{I_{(\eta\delta)}}=1. \tag{35}$$

Denoting further

$$\frac{\lambda}{1-\lambda(\eta\delta)^k}=\Lambda_\eta, \tag{36}$$

we shall have

$$m^{\frac{1}{k}}\int_0^{\eta\delta} e^{-\Lambda_\eta m x^k}\,dx<I_{(\eta\delta)}<m^{\frac{1}{k}}\int_0^{\eta\delta} e^{-\lambda m x^k}\,dx. \tag{37}$$

The first and the third member of this double inequality (37) , at $m\to+\infty$ aim correspondingly to

$$\frac{1}{\Lambda_\eta^{\frac{1}{k}}}\int_0^\infty e^{-z^k}\,dz=\frac{\Gamma\left(\frac{1}{k}\right)}{k\Lambda_\eta^{\frac{1}{k}}}\ ;\ \frac{1}{\lambda^{\frac{1}{k}}}\int_0^\infty e^{-z^k}\,dz=\frac{\Gamma\left(\frac{1}{k}\right)}{k\lambda^{\frac{1}{k}}}. \tag{38}$$

Wherefrom

$$\varliminf_{m\to\infty} I_{(\nu,\lambda)} \geqslant \frac{\Gamma\left(\frac{1}{k}\right)}{k\Lambda_\eta^{\frac{1}{k}}}, \quad \varlimsup_{m\to\infty} I_{(\nu,\lambda)} \leqslant \frac{\Gamma\left(\frac{1}{k}\right)}{k\lambda^{\frac{1}{k}}}, \tag{39}$$

and, in view of (35), we simultaneously have

$$\varliminf_{m\to\infty} I_\lambda \geqslant \frac{\Gamma\left(\frac{1}{k}\right)}{k\Lambda_\eta^{\frac{1}{k}}}, \quad \varlimsup_{m\to\infty} I_\lambda \leqslant \frac{\Gamma\left(\frac{1}{k}\right)}{k\lambda^{\frac{1}{k}}}. \tag{40}$$

Freeing ourselves from small enough η, we see that

$$\varliminf_{m\to\infty} I_\lambda = \varlimsup_{m\to\infty} I_\lambda = \lim_{m\to\infty} I_\lambda = \frac{\Gamma\left(\frac{1}{k}\right)}{k\lambda^{\frac{1}{k}}}, \tag{41}$$

which is what we wanted to prove.

Attention. It is easy to see, that the limiting relation (32) will remain valid also in the case when λx^k under the integral sign is sustituted by $\lambda[1+\varepsilon(x)]x^k$, where, keeping the conditions of positiveness and monotonousness of the expression $1-\lambda[1+\varepsilon(x)]x^k$, it is assumed that $\varepsilon(x)\to 0$ when $x\to 0$. Really, in this the numbers Λ_η and λ_η in the correlations (37)-(40) will change only infinitely little, and at $\eta\to 0$ the end result (41) will remain unchanged.

§5. Going back to the question which we referred to at the end of § 3, we shall analyze the problems (9) and (10) according to a simple example of polynomials

$$\Phi(x) = x^r - c \quad (r > 1) \tag{42}$$

on a segment (0, 1) (r - a given number) and prove their asymptotic equality

$$a \sim \frac{1-\frac{1}{r}}{4}\frac{\log m}{m}, \text{ that is } \lim_{m\to\infty}\left(a : \frac{1-\frac{1}{r}}{4}\frac{\log m}{m}\right) = 1, \tag{43}$$

whereby the impossibility of improving the evaluation (26') will generally be established, even for a class of whole rational polynomials.

Considering the selfevident asymptotic equality

$$\frac{\log m}{m} \sim \frac{\log(m-1)}{m-1} \tag{44}$$

we may allow ourselves , on account of simplification of further explanations, to find the evaluation for a_{m+1} instead of a_m.

It is evident that, for a given example, $\rho = \frac{1}{2}$ (see (13)), and in the expression $\Phi_{m+1}(x) = x^r - c$ the value $c = c_{m+1}$ will certainly be infinitely close to ½; application of a general method of analysis of infinitely small to a minimum-problem

$$\{\delta_{m+1}(\Phi)\}^{m+1} = \int_0^1 |x^r - c|^{m+1}dx = \int_0^{c^{\frac{1}{r}}} (c-x^r)^{m+1}dx + \int_{c^{\frac{1}{r}}}^1 (x^r-c)^{m+1}dx = \min \tag{45}$$

leads, as it is easily seen, to a condition

$$\int_0^{c^{\frac{1}{r}}} (c-x^r)^m dx = \int_{c^{\frac{1}{r}}}^1 (x^r-c)^m dx,$$

or , substituting on the right hand side x= 1 - y:

$$\int_0^{c^{\frac{1}{r}}} (c-x^r)^m dx = \int_0^{1-c^{\frac{1}{r}}} (1-c-r[1+\varepsilon(y)]y)^m dy; \tag{46}$$

or else

$$c^m\int_0^{c^{\frac{1}{r}}}\left(1-\frac{1}{c}x^r\right)^m dx=(1-c)^m\int_0^{(1-c)^{\frac{1}{r}}}\left\{1-\frac{r}{1-c}[1+\varepsilon(y)]y\right\}^m dy. \tag{47}$$

Applying to both integrals the asymptotic evaluation of the Lemma 11, with respect to which considering and denoting by $\varepsilon_1, \varepsilon_2$ two values, which are dependent on m and which are infinitely small at $m\to\infty$, we obtain

$$c^m\frac{1}{m^{\frac{1}{r}}}\frac{\Gamma\left(\frac{1}{r}\right)}{r\left(\frac{1}{c}\right)^{\frac{1}{r}}}(1+\varepsilon_1)=(1-c)^m\cdot\frac{1}{m}\cdot\frac{1}{\left(\frac{r}{1-c}\right)}(1+\varepsilon_2), \tag{48}$$

wherefrom we further have

$$\left(\frac{1-c}{c}\right)^m=m^{1-\frac{1}{r}}\frac{c^{\frac{1}{r}}}{1-c}\Gamma\left(\frac{1}{r}\right)\frac{1+\varepsilon_1}{1+\varepsilon_2}=(2m)^{1-\frac{1}{r}}\Gamma\left(\frac{1}{r}\right)(1+\varepsilon_3), \tag{49}$$

where ε_3 denotes a new dependent on m infinitely small value.

We see, that $\frac{1-c}{c}>1$; $c<\frac{1}{2}$. Thus, evidently, $\delta_0[\Phi_{m+1}]=1-c$. From (14) at $p=\frac{1}{2}$ we have $q=\frac{1}{2}$, wherefrom $c=\frac{1}{2}-a$, $1-c=\frac{1}{2}+a$.
The equality (49) takes a form

$$\left(\frac{1+2a}{1-2a}\right)^m=(2m)^{1-\frac{1}{r}}\Gamma\left(\frac{1}{r}\right)(1+\varepsilon_3). \tag{50}$$

Taking a logarithm and denoting (see (18)) by $\sigma'=\sigma'_m$, $\sigma''=\sigma''_m$ two values, which go to the limit 1 at $m\to\infty$, and by σ their mean arithmetic value, we have

$$m\sigma'\cdot 2a+m\sigma''\cdot 2a\equiv m\sigma\cdot 4a=\left(1-\frac{1}{r}\right)\log m(1+\varepsilon_4), \tag{51}$$

where $\varepsilon_4\to 0$ at $m\to\infty$. Therefrom, finally

$$\lim_{m\to\infty}\frac{4am}{\left(1-\frac{1}{r}\right)\log m}=1,\quad a\sim\frac{1-\frac{1}{r}}{4}\frac{\log m}{m},$$

by which the proof is completed.

§ 6. Now, we shall dwell, the second time, on a class of function which satisfies on a given segment a condition [15])

$$|F(x_2)-F(x_1)|\leqslant\frac{\text{const}}{\log\frac{1}{|x_2-x_1|}}\quad(a\leqslant x_1,\ x_2\leqslant b;\ |x_2-x_1|<1), \tag{52}$$

which we shall name a weakened condition of Dini; this class includes, as a component part, that class of function which satisfies the general condition of Dini (Dini-Lipschitz):

$$\lim_{\delta\to 0}\left[\omega(\delta)\cdot\log\frac{1}{\delta}\right]=\lim_{\delta\to 0}\left\{\max_{|x_2-x_1|\leqslant\delta}|F(x_2)-F(x_1)|\cdot\log\frac{1}{\delta}\right\}=0. \tag{53}$$

Thus, let the analyzed by us n + 1 functions $v_i(x)$ satisfy on a segment (a, b) correspondingly the inequality

$$|v_i(x_2)-v_i(x_1)|\leqslant\frac{k_i}{\log\frac{1}{|x_2-x_1|}}\quad(i=0,1,\dots,n). \tag{54}$$

Similarly, as in §3, we are convinced that there is such a positive constant k, independent from m, that

$$|\Phi_m(x_2)-\Phi_m(x_1)|\leqslant\frac{k}{\log\frac{1}{|x_2-x_1|}}\quad(m>1)^{16)}. \tag{55}$$

Denoting at $a\neq 0$ a number h from the condition

$$k:\log\frac{1}{h}=a\varrho,\ \text{whence}\ \log\frac{1}{h}=\frac{k}{a\varrho}\quad(a=a_m), \tag{56}$$

we, as in §3, shall obtain a correlation analogous to (17)

$$\varrho^m (1+a)^m h < \varrho^m l, \tag{57}$$

from which, taking a logarithm and using again the value (18), we shall obtain

$$m\sigma_m a < \log l + \frac{k}{a\varrho} = \frac{k}{a\varrho}(1+\varepsilon_m). \tag{58}$$

Thus, inasmuch as $\sigma_m \to 1, \eta_m \to 0$ at $m \to \infty$, we have

$$a\sqrt{m} < \sqrt{\frac{k}{\varrho\sigma_m}(1+\eta_m)} = \sqrt{\frac{k}{\varrho}}(1+e_m) \quad (\lim_{m\to\infty} e_m = 0), \tag{59}$$

wherefrom directly

$$a = O\left(\frac{1}{\sqrt{m}}\right). \tag{60}$$

§7. We have to be convinced now, that the obtained evaluation (60) cannot be made better in the general case of fuction $v_i(x)$, which satisfy the weakened condition of Dini (54). For this reason we shall first establish two lemmas.

Lemma I. Let be analyzed a class of continuous functions denoted over some segment of length l and characterized by this condition, that their modul of continuity $\omega(\delta)$ satisfies certain inequality of the form

$$\omega(\delta) \leq K\bar{\omega}(\delta) \quad (0 < \delta \leq l), \tag{61}$$

where K=const. > 0, and $\bar{\omega}(x)$ is increasing on a segment $(0,l)$ continuous function, which is annuled at $x=0$ and which has a non-increasing [17] derivative $\bar{\omega}'(x)$ on a (open) interval $(0,l)$. Then, the function itself $\bar{\omega}'(x)$ belongs to the analyzed class, at which its modul of continuity is exactly $\bar{\omega}(\delta)$.

Proof. Let us take an arbitrary number $\delta < l$.
Analyzing the difference $r(x) = \bar{\omega}(x+\delta) - \bar{\omega}'(x)$ we have

$$r'(x) = \bar{\omega}'(x+\delta) - \bar{\omega}'(x) \leq 0 \tag{62}$$

Thus, $r(x)$ appears as non-increasing function over a segment $0 \leq x < l-\delta$ and, therefore, at x=0 it reaches its greatest value $r(0) = \bar{\omega}(\delta) - \bar{\omega}(0) = \bar{\omega}(\delta)$. On the other hand, at $0 < h \leq \delta$ we shall have similarly $\max_{0 \leq x \leq l-h} |\bar{\omega}(x+h) - \bar{\omega}(x)| = \bar{\omega}(h) \leq \bar{\omega}(\delta)$,

with which the proof of the lemma is completed.

Attention I. It easy to be convinced, that the deduction of the lemma will remain in force even after we widen the definition of the function $\bar{\omega}(x)$ over a larger segment $(0, L)$, setting $\bar{\omega}(x) = \bar{\omega}(l)$ at $l \leq x \leq L$.

Attention II. On the basis of the known characteristics of the function, the derivatives of which have a limited variation [18], the deduction of the lemma will remain in force if, keeping the conditions $\bar{\omega}(0)=0$, one assumes only that any of the four derivative numbers of a continuous function $\bar{\omega}(x)$ is positive and does not increase over the interval $(0,l)$ (or, the same in the given case, one of the two singlesided derivatives, right hand or left hand).

Lemma II. The value of the integral

$$I_\delta = \int_0^\delta \left(1 - \frac{\lambda^2}{\log\frac{1}{x}}\right)^m dx = I(\delta, \lambda, m), - \tag{63}$$

where λ and δ are some two given positive constants, subordinate to the condition of $\delta \leq e^{-\lambda k}$ for large enough $m > m_1$ satisfies the double inequality

$$e^{-2\lambda(1+\epsilon)^2\sqrt{m}}(1-\epsilon) < I_\delta < 3\lambda\sqrt{m}\, e^{-2\lambda\sqrt{m}}, \tag{64}$$

where ϵ denotes an arbitrarily taken positive number.

Proof. The same as in proving the lemma II § 4, we have first

$$I_\delta = (1+x) I_{(\eta\delta)} = (1+x)\int_0^{\eta\delta} e^{-\frac{\lambda^2 m}{\log\frac{1}{x}}\left(1-\frac{\theta\lambda^2}{\log\frac{1}{x}}\right)} dx, \tag{65}$$

where $0 < \theta = \theta(x) < 1$, η denotes an arbitrarily designated auxiliary real fraction 1, finally[19], $|x| < \epsilon$ for large enough values of $m > m' = m_{\epsilon,\eta}$. Considering ϵ being taken arbitrarely, we shall subordinate here η to the condition of

$$1 - \frac{\lambda^2}{\log\frac{1}{\eta\delta}} > \frac{1}{(1+\epsilon)^2}. \tag{66}$$

In this way, at $m > m'$ we have

$$(1-\epsilon)\int_0^{\eta\delta} e^{-\frac{\lambda^2(1+\epsilon)^2 m}{\log\frac{1}{x}}} dx < I_\delta < \int_0^{\delta} e^{-\frac{\lambda^2 m}{\log\frac{1}{x}}} dx \text{ [20]}. \tag{67}$$

We shall find the lower and upper limits for the value of the integral

$$J_{H,\Lambda} = \int_0^H e^{-\frac{\Lambda^2 m}{\log\frac{1}{x}}} dx \quad (\Lambda > 0;\ 0 < H < 1) \tag{68}$$

for large values of m.

Substitution of $\log\frac{1}{x} = z$ gives

$$J_{H,\Lambda} = \int_{\log\frac{1}{H}}^{+\infty} e^{-\frac{\Lambda^2 m}{z} - z} dz. \tag{69}$$

Having noted that

$$\max_{z>0} e^{-\frac{\Lambda^2 m}{z} - z} = e^{-2\Lambda\sqrt{m}} \text{ at } z = \Lambda\sqrt{m}, \tag{70}$$

evidently we have

$$J_{H,\Lambda} < \int_0^{\infty} e^{-\frac{\Lambda^2 m}{z} - z} dz < \int_0^{2\Lambda\sqrt{m}} e^{-2\Lambda\sqrt{m}} dz + \int_{2\Lambda\sqrt{m}}^{\infty} e^{-z} dz =$$
$$= 2\Lambda\sqrt{m}\, e^{-2\Lambda\sqrt{m}} + e^{-2\Lambda\sqrt{m}} = (1 + 2\Lambda\sqrt{m})\, e^{-2\Lambda\sqrt{m}}. \tag{71}$$

For large enough values of m, that is, such that exceed each of the two values of $\frac{1}{\Lambda^2}$ and $\left(\frac{\log\frac{1}{H}}{\Lambda}\right)^2$, in the obtained expression (71) for the upper limit of the integral the multiplier $(1 + 2\Lambda\sqrt{m})$ may be substituted by $3\Lambda\sqrt{m}$, on the other hand, we shall have for the evaluation of the integral also the lower limit

$$J_{H,\Lambda} > \int_{\Lambda\sqrt{m}}^{\infty} e^{-\Lambda\sqrt{m} - z} dz = e^{-2\Lambda\sqrt{m}}. \tag{72}$$

Thus, for the mentioned large enough values of m, we shall have the searched for two-sided evaluation

$$e^{-2\Lambda\sqrt{m}} < J_{H,\Lambda} < 3\Lambda\sqrt{m}\, e^{-2\Lambda\sqrt{m}}. \tag{73}$$

Going back to the previously established double inequality (67), we see that it is enough to sustitute in it the first member of its lower limit according to (73) at $\Lambda = \lambda(1+\epsilon)$, $H = \eta\delta$, and the third member by its upper limit according to the same correlation

(73) at $\Lambda = \lambda$, $H = \delta$, to end the proof of the lemma.

§ 8. We now may prove the statement, said at the beginnig of the previous paragraph. For this reason we shall analyze the problems (9), (10) according to the examples of the polynomials

$$\Phi(x) = f(x) - c, \tag{74}$$

where the function $f(x)$ is established over the segment $(0, l)$ as follows:

$$f(x) = \begin{cases} \dfrac{1}{\log \frac{1}{x}} & \text{при } 0 < x < e^{-2}\text{)} \\ \dfrac{1}{2} & \text{при } e^{-2} \le x \le 1 \\ 0 & \text{при } x = 0 \end{cases} \tag{75}$$

при = at

On the basis of the lemma I § 7 and referred to it Attention I, for the function $f(x)$ the continuity module $\omega(\delta)$ has by its expression the function $f(\delta)$ itself, and which means , that it surely satisfies the weakened condition of Dini (54). We shall prove the assimptotic equality for the polynomials (74)

$$a \sim \frac{1}{\sqrt{m}}, \tag{76}$$

and in this way, the impossibility of improving the evaluation (60) in a regular case of the analyzed function case will be established.

Adhering, if possible, to the general plan of thoughts applied in § 5, in the analyzis of the value $c = c_{m+1}$ of the parmeter c in the now observed expression $\Phi(x) = \Phi_{m+1}(x)$, we are coming , first of all , to the condition

$$\int_0^{c'} [c - f(x)]^m dx = \int_{c'}^{1} [f(x) - c]^m dx \quad \left(c' = e^{-\frac{1}{c}}\right), \tag{77}$$

or

$$c^m I_1 = \left(\frac{1}{2} - c\right)^m I_1, \tag{78}$$

where

$$I_1 = \int_0^{c'} \left(1 - \frac{\frac{1}{c}}{\log \frac{1}{x}}\right)^m dx, \tag{79}$$

then as

$$I_1 = \int_{c'}^{e^{-2}} \left[\frac{f(x) - c}{\frac{1}{2} - c}\right]^m dx + \int_{e^{-2}}^{1} dx = 1 - e^{-2} + \int_0^{e^{-2} - c'} [\eta(y)]^m dy, \tag{80}$$

where (after application of sustitution $x = e^{-2} - y$) for abbriviation it was set

$$\frac{\dfrac{1}{\log \dfrac{1}{e^{-2} - y}} - c}{\dfrac{1}{2} - c} = \eta(y). \tag{81}$$

The last integral in (80) , as it is easy to see, permits the applicaton of the lemma I § 4. The integral is infinitely small at $m \to \infty$, and assymtotically equal to the integral

$$\int_0^{\delta} [\eta(y)]^m dy$$

at any small value of δ . Thus, denoting by ε any arbitrary and further affixed positive number, at large enough values of $m > m_1(\varepsilon)$ we shall have for the value of I_2 a double inequality

$$1 - e^{-2} < I_1 < (1 - e^{-2})\ (1 + \varepsilon). \tag{82}$$

As to the integral I_1, on the basis of the proved Lemma II, # 7 (8), in which is necessay to set $\lambda = \frac{1}{\sqrt{c}}$, we may assume, that for large enough $m > m_2(\varepsilon) \geq m_1(\varepsilon)$, together with (82) also such double inequality is realized:

$$(1-\varepsilon)e^{-2(1+\varepsilon)\sqrt{\frac{m}{c}}} < I_1 < 3\sqrt{\frac{m}{c}}\, e^{-2\sqrt{\frac{m}{c}}}. \tag{83}$$

Taking in account, in addition to this, that $c \to p = \frac{1}{4}$ at $m \to \infty$ and that in this $c > \frac{1}{4}$, as shows the correlation

$$\left(\frac{c}{\frac{1}{2}-c}\right)^m = \frac{I_2}{I_1}, \tag{78''}$$

in which I_2 remains larger than $1-e^{-2}$, at the same time as $I_1 \to 0$ at $m \to \infty$, for $m > m_3(\varepsilon) \geq m_2(\varepsilon)$ we may substitute the correlation (83) by

$$(1-\varepsilon)e^{-4(1+\varepsilon)\sqrt{m}} < I_1 < 6\sqrt{m}\, e^{-4\sqrt{m}(1-\varepsilon)}. \tag{84}$$

Substituting now in (78') c= ¼ + ½ 22) and noting (Comp. (51)), that

$$\left(\frac{c}{\frac{1}{2}-c}\right)^m = \left(\frac{\frac{1}{4}+\frac{1}{2}\alpha}{\frac{1}{4}-\frac{1}{2}\alpha}\right)^m = \left(\frac{1+2\alpha}{1-2\alpha}\right)^m = e^{m\alpha \cdot 4\alpha}; \lim_{m\to\infty} \alpha = 1 \tag{85}$$

and that because of large enough $m > m_4(\varepsilon) \geq m_3(\varepsilon)$ we shall also have

$$4\alpha m(1-\varepsilon) < \log\left(\frac{c}{\frac{1}{2}-c}\right)^m < 4\alpha m(1+\varepsilon), \tag{86}$$

taking logarithm and using the inequalities (86), (82) and (84), we obtain from the equality (78') such inequalities for all $m > m_4(\varepsilon)$:

$$4\alpha m(1+\varepsilon) > \log(1-e^{-2}) - \log(6\sqrt{m}) + 4\sqrt{m}(1-\varepsilon), \tag{87}$$

$$4\alpha m(1-\varepsilon) < \log[(1-e^{-2})(1+\varepsilon)] - \log(1-\varepsilon) + 4\sqrt{m}(1+\varepsilon). \tag{88}$$

Setting up these two inequalities, immidiately shows that the relation $\frac{4\alpha m}{4\sqrt{m}} = \alpha\sqrt{m}$ for large enough values of m appears limited from both sides by two expressions having any value close to one. Therefore,

$$\lim_{m\to\infty}(\alpha\sqrt{m}) = 1, \quad \alpha \sim \frac{1}{\sqrt{m}}, \tag{89}$$

which is what we wanted to prove.

9. Let now, as in lemma I # 7, be more generally analyzed a certain class of functions continuous over some segments of the length l, module of continuity of which $\omega(\delta)$ satisfies the condition

$$\omega(\delta) \leq K\bar{\omega}(\delta); \tag{61}$$

where K = const > 0, and $\bar{\omega}(x)$ is the given, continuous, increasing function, the derivative of which or at least one of the derivative numbers of Dini does not increase (what in reality means an existance of at least right hand and left hand derivatives $D_d\bar{\omega}(x)$, $D_g\bar{\omega}(x)$ which do not increase), at which $\bar{\omega}$ (0) = 0. This function may be defined either over the whole segment (0,l) or over a smaller segment (0,1), if in our analysis will play a roll only the characteristics of the continuity module for any small values of δ .

Let us denote by $\chi(x)$ a function, which is reversed to $\bar{\omega}(x)$. The function $\chi(x)$, defined over the segment (0, $\bar{\omega}(l_1)$) is also increasing, but which already has a ~~non-dec se-~~

non-decreasing derivative, which is equivalent to the non-decreasing left and right hand derivative.

Analyzing the experimental problems (9), (10), correspondingly to the generalized polynomials (1) of any fixed systems of linearly independent functions $v_i(x)$ $(i=0, 1, \ldots, n)$ belonging to the given class, and studying the dependence of the quantity α (see (14)), on m at large values of m, we, exacly as we have done in §5 and §6, may also in this case take as a setting off point the inequality

$$\varrho^{m}(1+\alpha)^{m}\chi\left(\frac{\varrho\alpha}{k}\right) < \varrho^{m} l. \tag{90}$$

Here, the parameter k, analogous to the coefficient of Lipschitz in the inequality (15), may be defined more accurately by the condition

$$k = \sup_{m \geq m_0} k^{(m)} = [1 + \varepsilon^{(m_0)}] \overline{\lim_{m\to\infty}}\, k^{(m)}, \tag{91}$$

where $k^{(m)}$ denotes a minimum value of k for a givem $m>1$, at which the condition (61) is realized for the module of continuity of the polynomial $\Phi_m(x)$ over a certain defined interval $0<\delta<k^l$, $(k_1^l \leq k^l)$; m_0 is any large, but fixed number.

Abbreviating (90) on ϱ^m and taking a logarithm by using the correlations (18) (21), we have

$$m\sigma_m\alpha < \log l + \log\frac{1}{\chi\left(\frac{\varrho\alpha}{k}\right)} = (1+x)\log\frac{1}{\chi\left(\frac{\varrho\alpha}{k}\right)}$$

$$(x \to 0 \text{ at } m \to \infty),$$

otherwise saying:

$$m(1+r_1) < \frac{\log\frac{1}{\chi\left(\frac{\varrho\alpha}{k}\right)}}{\alpha}\ (r_1 \to 0 \text{ at } m \to \infty). \tag{92}$$

Let us investigate more closely the equation

$$M = \frac{\log\frac{1}{\chi(A)}}{A}. \tag{93}$$

At $0<A<\bar{\omega}(l_1) - \chi(A) < 1$, its right hand side represents a decreasing function of A; at $A \to 0$ the function increases monotonically and continuously, going to $+\infty$. It is understood, that at

$$M > 0, \frac{\log\frac{1}{l_1}}{\bar{\omega}(l_1)} \tag{94}$$

it has a single value for A, which is completely defined function of M.

$$A = \varphi(M) \tag{95}$$

A very real characteristics of this function may be described by such a correlation

$$\varphi(pM) = \left(\frac{1}{p}\right)^{\theta}\varphi(M),$$
$$[0 < \theta = \theta(M, p) < 1] \tag{96}$$

which is justified for any positive number p, which does not take the argument (pM) over the limits of the region (94) permitted for the values of M. This means that, at any variation of the argument M, the function $\varphi(M)$ varies in that direction in which varies the magnitude $\frac{1}{M}$, however, in all cases slower than the last one, in the meaning of the relative variation.

The proof of the correlation (96) may be obtained directly from the observation, that, for example, when M is multipled by $p > 1$, the value $A=\varphi(M)$ decreases (as is immediately comes from the above mentioned character of the monotonous variation of the right hand of the equation (93)), however, because at this the product AM, which equals $\log \frac{1}{\chi(A)}$, it increases.

The proved characteristic quality of the function $\varphi(M)$ finds further a series of important examples, from which two we shall mention immediately.

First, let

$$A = \bar{\varphi}(M) = \varphi[M(1+\varepsilon)] \tag{97}$$

denote a solution of the equation, which differs from the equation (93) only by infinitely small relative change of the left hand side:

$$M(1+\varepsilon) = \frac{\log \frac{1}{\chi(A)}}{A}, \tag{98}$$

where ε denotes any function of M, which goes to null at $M \to \infty$. From the proven correlation (96), it immediately comes that

$$\frac{\bar{\varphi}(M)}{\varphi(M)} = \frac{\varphi[M(1+\varepsilon)]}{\varphi(M)} = \frac{1}{(1+\varepsilon)^{\theta}} \quad (0 < \theta < 1) \tag{99}$$

which means

$$\lim_{M \to \infty} \frac{\bar{\varphi}(M)}{\varphi(M)} = 1, \text{ тобто } \bar{\varphi}(M) \sim \varphi(M). \tag{100}$$

Secondly, going back to (92), let

$$\bar{\alpha} = \Psi(m) \tag{101}$$

denotes a solution of the equation

$$m(1+\eta) = \frac{\log \frac{1}{\chi\left(\frac{\varrho\bar{\alpha}}{k}\right)}}{\bar{\alpha}} \text{ або } \frac{k}{\varrho} m(1+\eta) = \frac{\log \frac{1}{\chi\left(\frac{\varrho\bar{\alpha}}{k}\right)}}{\frac{\varrho\bar{\alpha}}{k}} \tag{102}$$

$$[\eta = \eta(m) \to 0 \text{ при (at) } m \to \infty];$$

it gives, as it is easily understood, the upper limit of interesting for us quantity α, which satisfies the inequality (92) and is accurately defined by the correlation (14)[23]).

It is directly clear, that

$$\frac{\varrho\bar{\alpha}}{k} = \varphi\left[\frac{k}{\varrho} m(1+\eta)\right],$$

wherefrom

$$\bar{\alpha} = \Psi(m) = \frac{k}{\varrho}\varphi\left[\frac{k}{\varrho} m(1+\eta)\right]^{24)}. \tag{104}$$

or including again the correlation (96),

$$\bar{\alpha} = \Psi(m) = \frac{k}{\varrho}\left[\frac{\varrho}{k(1+\eta)}\right]^{\theta}\varphi(m) = \left(\frac{k}{\varrho}\right)^{1-\theta}(1+\eta)^{-\theta}\varphi(m). \tag{104'}$$

Thus, we have obtained that result, that the upper limit $\bar{\alpha} = \Psi(m)$, which is established by the here analyzed method for the value of α exactly defied by the correlation (14), has at $m \to \infty$ the same order of smallness for all systems of the function $\{v_i(\chi)\}$ with the continuity module subordinate the given condition (61); it accurately coincides with the order of smallness of the function $\varphi(m)$, the form of which is denoted by the eqaulity (93) and, thus, depends only on the presentation of the major function $\omega(\delta)$ in the condition (61).

§ 10. We may now really supplement general conclusions of the previous paragraph, showing that in the class of function (61), one can always point out a system of functions $\{v_i(x)\}$ over a segment $(0, l)$, for which the exact order of smallness, at $m \to \infty$, of the value of α, defined by the correlation (14) (and not only its upper limit $\bar{\alpha}$) coicides with order of smallness $\varphi(m)$, wherefrom comes out the impossibility of making better the established by us evaluation

$$\alpha = O[\varphi(m)] \tag{105}$$

in a general case of function belonging to the given class (61).

For this reason we shall investigate the polynomials

$$\Phi(x) = f(x) - c \tag{106}$$

denoting the function $f(x)$ by conditions: [25])

$$f(x) = \begin{cases} \bar{\omega}(x) & \text{at } 0 \leq x \leq l_0 \\ \bar{\omega}(l_0) = \mu & \text{at } l_0 < x \leq l \end{cases} \quad \left(l_0 = \min\left\{l_1, \frac{l}{2}\right\}\right). \tag{107}$$

Having in mind, as in # 8 and 5 , we bring together the investigation of the meaning of $c = c_{m+1}$ and with it connected number $a = \frac{c-\rho}{2\rho} = \frac{c - \frac{1}{2}\mu}{\mu}$ to study the following equality, which with the value of $c = c_{m+1}$ should exactly satisfy:

$$\int_0^{c'} [c - f(x)]^m dx = \int_{c'}^{l} [f(x) - c]^m dx \quad [c' = \chi(c)] \tag{108}$$

or

$$c^m I_1 = (\mu - c)^m I_2, \tag{109}$$

where

$$I_1 = \int_0^{c'} \left[1 - \frac{\bar{\omega}(x)}{c}\right]^m dx, \tag{110}$$

$$I_2 = (l - l_0)(1 + x) \quad [x = x(m) > 0, \lim_{m \to \infty} x = 0]. \tag{111}$$

Giving any number

$$r = 1 + \delta, \tag{112}$$

where δ is an arbitrarily small but fixed positive number, we further have (for large enough m)

$$I_1 = \int_0^{\chi(rca)} \left[1 - \frac{\bar{\omega}(x)}{c}\right]^m dx + \int_{\chi(rca)}^{c'} id < \int_0^{\chi(rca)} dx + \int_{\chi(rca)}^{c'} (1 - ra)^m dx <$$
$$< \chi(rca) + c'e^{-ram}. \tag{113}$$

Substituting from (111) and (113) in (109) and simultaneously substituting

$$c = \mu\left(\frac{1}{2} + a\right), \quad \mu - c = \mu\left(\frac{1}{2} - a\right). \text{ [redacted]}$$

we obtain

$$\left(\frac{1 + 2a}{1 - 2a}\right)^m [\chi(rca) + c'e^{-ram}] > l - l_0. \tag{114}$$

We shall note now, that denoting by σ (comp(51)) any number, which depends on m and which goes to the limit at $m \to \infty$, we have

$$\left(\frac{1 + 2a}{1 - 2a}\right)^m \cdot c'e^{-ram} = e^{4a\sigma m} \cdot c' \cdot e^{-(1+\delta)am} = c'e^{am(4\sigma - 1 - \delta)}. \tag{115}$$

It is easily to see, that the last expression goes to null at $m \to \infty$, if in it the exponent indicator at e goes to $(-\infty)$. To be convinced in this, it is enough already a coarse [redacted] evaluation of the value of the product am . That is, noting that at general conditions set by us on $\bar{\omega}(x)$, we shall have (at $0 < x < l_0$):

$$\frac{\bar{\omega}(x)}{x} \geq \frac{\mu}{l_0} \,^{[26]}, \quad 1 - \frac{\bar{\omega}(x)}{c} \leq 1 - \frac{\mu x}{cl_0} < 1 - \frac{x}{l_0},$$

wherefrom

wherefrom by the lemma II § 4 (including the idea applied at its proof), we have $I_1 < \frac{l_0}{m}$, we may easily be convinced, substituting $\frac{I_0}{m}$ instead of I_1 in (109), that

$$\alpha_m > \frac{\delta}{4} m(1-\delta_1) \qquad (116)$$

for large enough m, understanding by δ any small fixed positive number. From (116) our statement comes directly.

If we denote by ε' the expression (115), the infinity of smallness of which we have proved, then the correlation (114) takes a form

$$\left(\frac{1+2\alpha}{1-2\alpha}\right)^m > \frac{1-I_0-\varepsilon'}{\chi(rc\alpha)} \qquad (117)$$

and, further, taking a logarithm and introducing the same number $\alpha \to 1$:

$$4\alpha\sigma m > \log(1-I_0-\varepsilon') + \log\frac{1}{\chi(rc\alpha)} \qquad (118)$$

or

$$4\alpha m(1+\varepsilon'') > \log\frac{1}{\chi(rc\alpha)} \quad (\varepsilon'' = \varepsilon''_m \to 0 \text{ at } m \to \infty). \qquad (119)$$

Otherwise

$$\frac{4}{rc} m(1+\varepsilon'') > \frac{\log\frac{1}{\chi(rc\alpha)}}{rc\alpha}. \qquad (120)$$

Wherefrom, remembering the basic characteristics of the function $\varphi(M)$ defined by the equation (93), we shall immediately conclude, that

$$rc\alpha = rc\alpha_{m+1} > \varphi\left[\frac{4}{rc} m(1+\varepsilon'')\right]$$

or replacing m+1 by m :

$$\alpha = \alpha_m > \frac{1}{rc}\varphi\left[\frac{4}{rc}(m-1)(1+\varepsilon'')\right] = \psi(m,\delta) \quad (\delta = r-4). \qquad (121)$$

Setting up in such a way obtained lower limit for $\alpha = \alpha_m$ with the previously established upper limit $\psi(m)$ (104) of the same value at k=1 for the given class, we have

$$\psi(m,\delta) = \frac{1}{\varrho}\varphi\left[\frac{m(1+\eta)}{\varrho}\right]\cdot\frac{1}{r}\cdot\frac{\varrho}{c}\left[\frac{rcm(1+\eta)}{4\varrho(nr-1)(1+\delta'')}\right]^{\theta} =$$
$$= \psi(m)\cdot\frac{1}{4+\delta}\cdot\left(\frac{\varrho}{c}\right)^{1-\theta}\left[\frac{\left(1+\frac{\delta}{4}\right)(1+\eta)}{\left(1-\frac{1}{m}\right)(1+\varepsilon'')}\right]^{\theta} \quad (0<\theta<1). \qquad (122)$$

From this, remembering that $c \to \rho$, it is easily to understand that, setting small enough fixed number $\delta = \delta_\varepsilon > 0$ in correspondence to the arbitrarely given $\varepsilon > 0$, for large enough $m \geq m_\varepsilon$ we shall have

$$\psi(m,\delta) > \left(\frac{1}{4}-\varepsilon\right)\psi(m). \qquad (123)$$

This inequality proves that the here obtained by us lower limit $\psi(m,\delta)$ for $\alpha = \alpha_m$ has at $m \to \infty$ the same order of diminutivity as the previously established upper limit $\alpha = \psi(m)$, that is such, that accurately coincides, according to (104'), with the order of diminutivity $\varphi(m)$ also the quantity α has, understandingly, the same order in diminutivity, in which we wanted to convince ourselves [27]).

§ 11. Le us note now, that at more partial assignment of function $\omega(x)$, it appears possible, taking individually the investigation for all classes of functions $v_i(x)$, somewhat bring closer the multiplier $\frac{1}{4}-\varepsilon$ in (23) to unity - or on account of the decrese of quantity ψ (m), or by increasing the quantity $\psi(m,\delta)$.

If, in the first place, we take the class of function $v_i(x)$, which satisfies the condition of Lipschitz of any order τ $(0<\tau\leq 1)$, then we shall have

$$\varphi(M) = \frac{1}{\tau}\frac{\log M}{M}(1+\varepsilon_M) \quad (\lim_{M\to\infty}\varepsilon_M = 0). \qquad (124)$$

Really, substituting $A = \frac{1}{\tau}\frac{\log M}{M}$ in the equation (93), which takes a form

$$M = \frac{\log\left(\frac{1}{A}\right)^{\frac{1}{\tau}}}{A},$$

we obtain

$$\frac{\log\frac{1}{\chi(A)}}{A} = \frac{\frac{1}{\tau}(\log M - \log\log M + \log\tau)}{\frac{1}{\tau}\frac{\log M}{M}} = M(1+\eta_M)\ [\lim_{M\to\infty}\eta_M = 0]. \tag{125}$$

Which means that

$$\frac{1}{\tau}\frac{\log M}{M} = \varphi[M(1+\eta_M)] = (1+\eta_M)^{-\tau_M}\varphi(M) = \frac{\varphi(M)}{1+\varepsilon_M}, \tag{126}$$

by which (124) is established.

But if we here,in the outcoming inequality (90), take on the left hand side $\chi\left(\frac{\gamma\alpha}{k}\right)$, instead of $\chi\left(\frac{\mu\alpha}{k}\right)$, and correspondingly $[1+(2-\gamma)\alpha]^m$ instead of $(1+\alpha)^m$, at any small fixed γ, then we shall see, as it was noted in §3, that after setting free the correlation (27') from any small γ, it transforms into (27). This means lowering twice the asymptotic expression of the upper limit ψ (m) for $\alpha=\alpha_m$ and the exchange of the multiplier $\frac{1}{1}-\varepsilon$ on $\frac{1}{2}-\varepsilon$ in (23).

An anoalogous effect may be obtained for classes of functions $v_i(x)$, which satisfy " the weakened condition of Dini order ν " (ν=2,3,4,...):

$$\omega(\delta) \leqslant K\bar{\omega}_\nu(\delta) = \frac{K}{\underbrace{\log\log\ldots\log}_{\nu}\frac{1}{\delta}} = \frac{K}{\log^{(\nu)}\frac{1}{\delta}}\left|\delta<\delta_\nu,\ \log^{(\nu)}\frac{1}{\delta_\nu}=0\right|^{**)}, \tag{127}$$

however, taking already in this case in (90) $\chi\left[\frac{(2-\gamma)\varrho\alpha}{k}\right]$ and correspondingly $(1+\gamma\alpha)^m$ at arbitrarily small fixed γ, with the following freeing from it, which permits, as it is easily checked, also in this case to decrease twice the asymptotic expression of the upper limit $\psi(m) \sim \frac{k}{\varrho}\varphi\left(\frac{k}{\varrho}m\right)$ for $\alpha=\alpha_m$ which is given by the formula (104).

It should be enough here to mention the introduction of the expression $\Phi(M)$ itself for the class (127). Substituting

$$A = \frac{1}{\log^{(\nu-2)}[\log M - \log^{(\nu)}M]}$$

in the equation (93), which in the given case has an expression

$$M = \frac{e^{e^{\cdot^{\cdot^{\cdot^{\frac{1}{A}}}}}}}{A},$$

we shall obtain

$$\frac{\log\frac{1}{\chi(A)}}{A} = \frac{M\cdot\log^{(\nu-2)}[\log M - \log^{(\nu)}M]}{\log^{(\nu-1)}M} = M(1+\eta_M)\quad(\lim_{M\to\infty}\eta_M=0), \tag{125'}$$

wherefrom, as above, we conclude:

$$\varphi(M) = \frac{(1+\eta_M)^{\nu_M}}{\log^{(\nu-2)}[\log M - \log^{(\nu)}M]} = \frac{1+\varepsilon_M}{\log^{(\nu-1)}M}\quad(\lim_{M\to\infty}\varepsilon_M=0). \tag{128}$$

Taking, finally, the class of functions $v_i(x)$, which satisfies the weakened condition of Dini of the first order (52) [29], we note, that the application of an analogous acceptance does not even lead to the reduction of the upper limit ψ(m) for $\alpha=\alpha_m$, since in this case $\gamma=1$ appears to be the optimum value of the parameter γ. Instead here, as shows the result of the investigation in § 8, the assymptotic expression of the lower limit for $\alpha = \alpha_m$ which is derived from (121), appears in the formula (89) raised twice, and this again, as in all the cases previously investigated, means an exchange of the multiplier $1/4 - \varepsilon$ on $1/2 - \varepsilon$ in (123).

It may be assumed, that the multiplier ($1/2 - \varepsilon$), which remains for all investigated classes of functions, is defined not by the possibility of further reduction of the upper limit ψ (m) for $\alpha = \alpha_m$ but by a peculiarity of the used by us individual costructions of polynomials of a special type $\phi(x) = f(x) - c$, owing to the equality (108) together with the inequality (90), where the maximum value of the subintegrated function on the right hand side equals not ρ^m (as in the inequality $\delta_m[\phi_m] \leq \delta_m[\phi_0]$), which leads to (90), but $[\rho(1-2\alpha)]^m$.

12. Using the investigated methods applied in the previous paragraphs, we may now, without any difficulties , give the proof of the theorem noted in § 3, which appears as a direct analogy to the corresponding theorem of Lebeg as to the best approximation of the continuous functions by rational polynomials at truly different condition of infinitely increasing coefficient of a polynomial.

Theorem. Let it be given an arbitrary positive function h(m), which is denoted for $m \geq 1$ [30]) and which goes to null at $m \to \infty$. Then, one may indicate a system of continuous functions $\nu_i(x)$ over a corresponding segment, for which the quantity $\alpha = \alpha_M$, which is defined by the formula (14), satisfies the correlation

$$\overline{\lim_{m \to \infty}} \frac{\alpha_m}{h(m)} = \infty \text{ [31])}. \tag{129}$$

Proof. It is enough for us , rather, to keep in mind these cases when $h(m) > \frac{1}{m}$ for large enough m , since in the opposite case the fairness of the theorem is established immediately by means of first-best example constructed in § 5, 8, 10. Not to change the essence of the question, we,obviously,may assume the function h (m) to be limited and further - to introduce to the investigation , instead of h (m), the function

$$\bar{h}(m) = \frac{1}{m} + \sup_{M \geq m} h(M),$$

which is monotonically decreased and certainly larger, than $\frac{1}{m}$ for all of the investigated values of $m \geq 1$. Finally, it is completely obviously that it is enough to prove the correlation (129) for the values of whole numbers of the argument m, and only such values we shall further have in mind.

We shall set

$$\psi(m) = \sqrt{\bar{h}(m)} \tag{130}$$

and introduce for investigation a continuous function f(x), defined over the segment (0,2) by such conditions:

$$r_l = e^{-\sum_{\nu=1}^{l-1} \nu\psi(\nu)} \quad (l \geq 2),\ r_1 = 1,$$

further:

$$f(r_l) = e^{-\psi(l-1)} \quad (l \geq 2),\ f(r_1) = 0 \text{ [32])}$$

and,finally, setting f(x) to be linear over each of the segments $[r_{l+1}, r_l]$, we add the conditions:

$$f(0) = 1; \; f(x) = 0 \text{ at } 1 \leqslant x \leqslant 2.$$

Evaluating the integral

$$J_{m-1} = \int_0^2 |f(x)|^{m-1} dx = \sum_{l=1}^{\infty} \int_{r_{l+1}}^{r_l} [f(x)]^{m-1} dx \quad (m \geqslant 2),$$

we have

$$J_{m-1} < \sum_{l=1}^{\infty} [f(r_{l+1})]^{m-1} \cdot r_l = e^{-(m-1)\psi(1)} \cdot 1 + e^{-(m-1)\psi(2)} \cdot e^{-1\cdot\psi(1)} + \ldots +$$

$$+ e^{-(m-1)\psi(l)} \cdot e^{-\sum_{\nu=1}^{l-1} \nu\psi(\nu)} + \ldots + e^{-(m-1)\psi(m)} \cdot e^{-\sum_{\nu=1}^{m-1} \nu\psi(\nu)} +$$

$$+ e^{-(m-1)\psi(m+1)} \cdot e^{-\sum_{\nu=1}^{m} \nu\psi(\nu)} + e^{-(m-1)\psi(m+2)} \cdot e^{-\sum_{\nu=1}^{m+1} \nu\psi(\nu)} + \ldots <$$

$$< me^{-(m-1)\psi(m)} + \sum_{j=0}^{\infty} e^{-\sum_{\nu=m-1}^{m+j} \nu\psi(\nu)} <$$

$$< me^{-(m-1)\psi(m)} + e^{-(m-1)\psi(m)} \cdot \sum_{j=0}^{\infty} e^{-\sum_{\nu=m}^{m+j} \nu^{-}} <$$

$$< e^{-(m-1)\psi(m)} \left[m + \sum_{j=0}^{\infty} e^{-(j+1)} \right] < (m+1) e^{-(m-1)\psi(m)} =$$

$$= e^{-(m-1)\left[\psi(m) - \frac{\log(m+1)}{m-1}\right]},$$

wherefrom

$$J_{m-1} < e^{-(m-1)\psi(m)(1+\varepsilon)} \quad (\varepsilon = \varepsilon_m \to 0 \text{ at } m \to \infty). \tag{131}$$

By means of this auxiliary function f(x) further proof of the theorem is obtained in the following way.

Setting

$$1 - f(x) = \vartheta(x),$$

we introduce for investigation the polinomials defined over the segment (0,2)

$$\Phi(x) = f(x) - c = 1 - \vartheta(x) - c,$$

with regards to which we are investigating the problems (9), (10). Thinking, as in § 5, 8, (10), but without a change of α_m to α_{m+1}, we shall have for the definition of $c = c_m$ for large values of m($0 < c_m < 1$, $c_m \to \rho = \frac{1}{2}$ at $m \to \infty$) the equation:

$$\int_0^{c'} [f(x) - c]^{m-1} dx = \int_{c'}^{2} [c - f(x)]^{m-1} dx \quad [f(c') = c] \tag{132}$$

or

$$(1-c)^{m-1} I = c^{m-1}(1 + \eta) \quad (\lim_{m\to\infty} \eta = 0), \tag{132'}$$

where

$$I = \int_0^{c'} \left[\frac{1 - \vartheta(x) - c}{1 - c}\right]^{m-1} dx = \int_0^{c'} \left[1 - \frac{\vartheta(x)}{1-c}\right]^{m-1} dx < \int_0^{c'} [1 - \vartheta(x)]^{m-1} dx =$$

$$= \int_0^{c'} [f(x)]^{m-1} dx < J_{m-1} < e^{-(m-1)\psi(m)(1+\varepsilon)}.$$

Substituting in (132), together with the obtained limit for I, the expression $c = \frac{1}{2} - \alpha$, $1 - c = \frac{1}{2} + \alpha$ $(\alpha = \alpha_m)$, we shall come to the correlation

$$\left(\frac{1+2\alpha}{1-2\alpha}\right)^{m-1} e^{-(m-1)\psi(m)(1+\varepsilon)} > 1 + \eta$$

or, taking a logarithm and introducing the value δ→1, as in (51):

wherefrom

$$4a\alpha(m-1) > \log(1+\eta) + (m-1)\,\psi(m)\,(1+\varepsilon),$$

$$4\alpha(m-1) > (m-1)\,\psi(m)\,(1+\eta') \qquad (\lim_{m\to\infty}\eta'=0),$$

$$\alpha = \alpha_m > \frac{1+\eta'}{4}\,\psi(m),$$

Thus

$$\frac{\alpha_m}{h(m)} > \frac{1+\eta'}{4}\,\frac{\sqrt{h(m)}}{h(m)} = \frac{1+\eta'}{4\sqrt{h(m)}}. \qquad (133)$$

$$\lim_{m\to\infty}\frac{\alpha_m}{h(m)} = \infty,$$

by which the proof of the theorem is concluded.

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1) The investigations in the mentioned works of Polya and Jackson had an existential character and did not contain any effective ways to the problem of constructing a solution of the problem of mean exponential approximation itself, which by its nature belongs to nonlinear as in the Chebyschev's problem.

However, Acad. S. Bernschtein in his review presentation (6) at the I National symposium of matematians (1930, Kharkiv) already pointed out, by setting up the problem, on the problem of searching for general ways to work out the questions of approaching the factual determination of Chebyschev's approximating expressions on the basis of usage of analogous process of Polya-Jackson.

2) As to the meaning of this condition compare my article (3)

3) For this, strictly speaking, would have been enough the condition of linear independence of n functions $v_1(x), \ldots, v_n(x)$. However, in a case of linear dependence of the function $v_o(x)$ from the rest of the functions $v_i(x)$ the investigated problem looses its interest.

4) See (4), # 6 . As to the idea, applied to the analogous aim of Polya (5) then it gives something more (compare our quoted work (3), # 2), that is the proof of the singularity of the stationary point of the problem, for this it requires some additional conditions.

5) However, this fact may be established completely independent from the assumption about the measure of the function $v_i(x)$ (and also without the requirement for their linear independence, as it comes directly from the general investigations of the section VIII (# 23, 24)) of my monograph (10).

6) If one explains the problem of value minimization (6) as a problem of approximation of function $v_o(x)$ by means of a generalized polynomial - $P(x) = -\sum_{i=1}^{n} c_i v_i(x)$ of the given system of functions $v_i(x), \ldots, v_n(x)$ within the set E, then, as it is known (Tonelli (17), Haar (11)), even in classical problems of this kind, which pertain to the Chebyschev's approximation, let us say, continuous function of two or more independent variables (in any continuous area of a corresponding number of measures) by means of rational polynomials or finite trigonometric sums, the unity of solution is not secured. The same we have in one dimentional area in the case of the Chebyschev's approximation of the continuous functions by means of generalized polynomials of an arbitrarily given system of continuous and linearly independent functions of one independent measure.

However, these negative conclusions in any case do not predict a question as to the synonimous or not-synonimous solution of the Chebyschev's problem in the indicated cases at one or onother individully given function $v_o(x)$. We do not intend to limit the frame of our investigation here by such questions for which a known unity of solution pointed out by Khaar Kharom) (11), may be analyzed as a necessary or in which it is assumed that it is realized. Some general thoughts which may be applied to the class of questions, which we are touching here, is in the section VIII of my quoted monogram (10), and also in my paper (12).

7) Julia in series of his works, beginning in 1926 (18), pointed out a series of interesting

examples of the investigated limiting transition to the problems in a complex area and real area of several measures.

8) This question was put by Acad. S.N. Bernstein at the session of the mathematics section of Academy of Science in Moscow (1944).

9) See (13), p. 110-112. Also (14), page 42-43.

10) As it is known,(For example see my above quated monograph (10), the problem (10) has either one or infinite number of solutions. In the last case under (X) we may understand any one of the solutions.

11) Above in # 2, we noted the generalization of this statement.

12) In the case of $a_m - 0$ the value q_m in (Q1) also nullifies , and the resultant conclusions (96) (97) , understandably, retain the power.

13) To encompase all the meanings of $m > 1$, it would have been enough instead of (23), to use the substitute $a = q_m \frac{\log(m+1)}{m}$

14) The statement analogous to this lemma, is in the memuar of H. Hahn (19), # 11. However, our method of proving this lemma permits various generalizations (compare for example, the below # 7 , lemma II) , which clearly comes out beyond the limits of the application of thoughts of Hahn.

15) We, clearly, would have had completely equivalent condition,if we would have required the proof of the inequality (52) for the values $|x_2 - x_1|$ as small as you choose.

16) Studying $a = a_m$ at $m \to \infty$, one may more acurately determine the parameter k such as it was done below in # 9 by means of the condition (91).

17) Functions

17) Функции $\bar{\omega}(\delta) = \delta^r (0 < r < 1)$, $\frac{1}{\log \frac{1}{\delta}}$, $\frac{1}{\log \log \frac{1}{\delta}}$, etc. or $\frac{1}{\left(\log \frac{1}{\delta}\right)^s}$ $(s > 0)$

etc. satisfy the condition of non-increasing $\bar{\omega}'(\delta)$: the first at $0 < \delta < \infty$, the rest over small enough interval $(0, l)$. One shoud underline at this, that in such questions as these that are investigated in this work, play a roll only the characteristics of continuity module for any small values of δ .

We shall note here, to the subject matter, that the thoughts of further lemma II and § 8 are very easily widened in the case of $\bar{\omega}(\delta) = \frac{1}{(\log \frac{1}{\delta})^3}$ $(0 < \delta < \infty)$ (comp. 29) in # 11).

18) See , for example, memuar Ch. de la Vallee-Poussin (20), n 46, and also the course of the analysis of the same aouthor GTTI, 1933, Vol. I n n 112-113.

19) Compare (35). Here we support ourselves on the result of the lemma I #4.

20) It should be accouted for that the equality (65) is justified without the multiplier $(1 + x)$, if replacing η by 1.

21) It is easy to prove, that over this interval is relaized the condition of lemma I of

the previous paragraph as to the non-increase of the derivative χ f(x).

22) From the equality $\delta_0[\varphi_{m+1}] = \varrho(1+2a)$, where in the given case, clearly,

23) $\delta_0[\varphi_{m+1}] = \quad = c, \ \varrho = \frac{1}{4}$, или $c = \frac{1}{4}(1+2a) = \frac{1}{4} + \frac{1}{2}a.$

$$1 + \eta = \sigma_m : \left[1 + \frac{\log l}{\log \frac{1}{\chi\left(\frac{am}{k}\right)}} \right], \text{ где } \sigma_m = \frac{\log(1+am)}{am}. \qquad 103$$

Thus, η is some defined function of m, which, however, is expressed by the exact definition of the value $a = a_m$ itself. It is, however, enough to know here, that $\eta \to 0$ at $m \to \infty$, if in (102) we completely reject infinitely close to unity value $(1+\eta)$, then the solution of the such obtained equation, which will not contain η, will (compare (100) and (104)) by assymptotically equal ψ(m).

24) This permits immediatel to write in a clear form the assymptotic equality $\psi(m) \sim \frac{h}{\varrho}\varphi\left(\frac{k}{\varrho}m\right)$, mentioned in the previous input.

25) According to lemma I #7 and bacouse of it, the function f(x) and polynomials (106) belong to the investigated class, if their module of continuity equals exactly $\omega(\delta)$ for $\delta \le l_0$.

26) This is very easily obtained from the condition of non-increasing of the derivative a $\omega'(x)$ accordingly - non-increasing of the one sided derivatives $D_d\omega(x), D_g\omega(x)$.

27) Function $\varphi(M)$ and (95), which played the basic role in the series of last investigations, being a solution of the equation (93), is defined completely by function χ and therefore, in the end conclusion, the duty of the major function $\omega(\delta)$ in the structural condition (61).

It is easily to see, that in reverse, knowing the function $\psi(M)$, which corresponds to some $\omega(x)$, one may easily reproduce $\chi(x)$, and thus $\omega(x)$; denoting by g(x) a function which is reversed to g(x) we have :

$$\frac{\log \frac{1}{\chi(A)}}{A} = g(A),$$

wherefrom $\chi(x) = e^{-xg(x)}.$

It is not difficult to establish necessary and sufficient conditions, at which the ahead given function (which is defined for infinitely large positive argument and goes monotonically to null) may paly a role g(x) for some ω(x), which has in the interval $(0, l)$ (negative) second derivative: denoting $xg(x) = G(x)$, где $g(x)$ — the function, reversed to φ (x), we have one of the three really equivalent among themselves variants of the searched conditions :

$$\text{(I)} \begin{cases} 1)\ G(x) \to +\infty \text{ monotonically at } x \to +0, \\ 2)\ G'(x) < 0 \text{ (equality is excluded)} \\ 3)\ G''(x) \leqslant [G'(x)]^2 \end{cases} \text{ or (II) } \begin{cases} G'(x) \leqslant -\frac{1}{x} \\ G''(x) \leqslant [G'(x)]^2 \end{cases} \text{ або (III) } \begin{cases} G'(+0) = -\infty \\ G''(x) \leqslant [G'(x)]^2 \end{cases}$$

Using the obtained conditions, it is possibly to construct various examples of concrete functions $\varphi(x)$

defining for each of them a corresponding structural class of functions $\varphi(x)$ and in this class - of a certain form of polynomial $\phi(x)$ (106), for which the order of diminutivity $a = a_m$ at $m \to \infty$ coincides exacly with the order of diminutivity $\rho(m)$.

28) In small enough interval $(0, \delta_1') \subset (0, \delta_1)$ each of the function $\omega_i(\delta)$ has a non-increasing derivative, as it is required in the lemma I #7 and in general investigations of the previous § 10.

29) Here, as it is easily to prove, we obtain exactly $\eta(M) = \frac{1}{\sqrt{M}}$.
In more general terms, one may ~~experienc e~~ experience directly that at $\omega(\delta) = \frac{1}{\left(\log \frac{1}{\delta}\right)^s}$ $(0 < s < \infty)$

we have exactly $\eta(M) = M^{-\frac{s}{s+1}}$. In this way we obtain a whole scale of inbetween cases betwee (124) and (128) from r=2. On these and the like investigated here we shall not stop here more acurately (compare conclusion (17) in # 7)

30) Because of convenience we include also the value of m=1.

31) The sign lim here, in reality, may be substituted by a more defined general sign lim. One could easily be convinced by means of a nonsignificant modification of further thoughts.

32) It is clear, that

$0 < r_l < e^{-\sum_{i=1}^{l-1} \gamma_i} \to 0$ при $l \to \infty$, then as $\lim f(r_l) = 1$, when $\lim_{m \to \infty} \psi(m) = 0$.

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