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Assumed stress hybrid finite element methods are developed for linear fracture analysis of three dimensional solids and for fracture of metals under high temperature creep conditions. For the 3-D fracture analysis a series of special elements with hexagonal shape have been derived. These elements include the asymptotic singular behavior at the crack tip front. For the fracture analysis of metals at high temperature, a method has been developed to analyze plane problems with growing cracks. From a comprehensive literature survey made for both analytical and experimental studies of crack growth under creep conditions,

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) it is concluded that a possible numerical method is to incorporate the concept of damage parameter by Kachanov. Preliminary development in incorporating such parameter in the finite element creep analysis has been made.

FOREWORD

The research was conducted by the Aeroelastic and Structures Research Laboratory, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts under Contract No. F49620-77-C-0084 with the Air Force Office of Scientific Research, Bolling AFB, DC. The first Technical Monitor of this research work was Mr. William J. Walker and currently it is monitored by Lt. Col. Joseph D. Morgan, III.

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SECTION 1

INTRODUCTION

Since March 1977, the Aeroelastic and Structures Research Laboratory of the Massachusetts Institute of Technology has conducted research studies under the sponsorship of the Air Force Office of Scientific Research on finite element methods for linear fracture analysis of three-dimensional solids and for fracture of metals under high temperature creep conditions. The general thread of this research program is the application of the assumed stress hybrid finite element model. For the 3-D fracture analyses a series of special elements for hexagonal shape have been derived that can represent the asymptotic singular behavior at the crack front of 3-D solids. For the fracture analyses of metals under high temperature creep conditions the research effort has been concentrated on the development of methods of creep analyses of plane problems with initial cracks and for gradual crack extension. From a comprehensive literature survey made for both analytical and experimental studies of crack extension under creep conditions it is concluded that a possible numerical method for crack extension study under creep conditions is to incorporate the concept of damage parameter by Kachanov. Preliminary development in incorporating such parameter in the finite element creep analysis has been made.

Portions of results obtained under this research program, in particular, those in 3-D fracture analysis have already been documented either as interim reports submitted to the Air Force Office of Scientific Research or in the form of technical papers published in proceedings of technical conferences. A list of such publications is given in Section 2 of this report.

In the appendix a separate document on the study of finite element analyses for crack growth in metals under creep conditions is presented. Summaries of research findings of this research program are given in Section 3.

SECTION 2

LIST OF PUBLICATIONS UNDER THE PRESENT PROGRAM

1. Moriya, K., "Hybrid Crack Elements for Three-Dimensional Solids and Plate Bending", ASRL TR 191-1, Submitted to AFOSR in September 1977.

2. Lee, S.W. and Pian, T.H.H., "Improvement of Plate and Shell Finite Elements by Mixed Formulations", AIAA J., Vol. 16, No. 1, January 1978, pp. 29-34, Paper presented at AIAA/ASME 18th Structures, Structural Dynamics and Materials Conference, San Diego, California, March 21-23, 1977.

3. Pian, T.H.H. and Lee, S.W., "Creep and Viscoplastic Analysis by Assumed Stress Hybrid Finite Elements", Proc. International Conference on Finite Elements in Nonlinear Solid and Structural Mechanics, Edited by P.G. Bergan et al., TAPIR Press, Norwegian Institute of Technology, Trondheim, Vol. 2, 1977, pp. 807-822.

4. Rhee, H.C., Atluri, S.N., Moriya, K. and Pian, T.H.H., "Hybrid Finite Element Procedures for Analyzing Through Flaws in Plates in Bending", Transactions of 4th International Conference on Structural Mechanics in Reactors Technology, Paper No. M2/4, San Francisco, California, Aug. 15-19, 1977.

5. Pian, T.H.H. and Moriya, K., "Three Dimensional Crack Element by Assumed Stress Hybrid Model", Proceedings of the 14th Annual Meeting of the Society of Engineering Science, Bethlehem, Pennsylvania, Nov. 14-16, 1977, pp. 913-917.

6. Pian, T.H.H. and Moriya, K., "Three Dimensional Fracture Analysis by Assumed Stress Hybrid Elements", Numerical Methods in Fracture Mechanics, Proceedings of the First International Conference held at the University College Swansea, Jan. 9-13, 1978, pp. 363-373.

7. Lee, S.W., "Finite Element Methods for Reduction of Constraints and Creep Analyses", Ph.D. Thesis, Department of Aeronautics and Astronautics, M.I.T., Feb. 1978.

SECTION 3

SUMMARY OF RESEARCH FINDINGS

This research program consists of two separate studies in the area of finite element analyses in fracture mechanics.

3.1 Evaluation of Special 3-D Crack Elements Based on an Assumed Stress Hybrid Model

Special crack front elements have been developed and have been assembled into superelements for direct evaluation of stress intensity factors K_I, K_{II}, and K_{III} of arbitrarily shaped three-dimensional cracks. The formulation is based on the assumed stress hybrid finite element model. The assumption of stresses and boundary displacements contains asymptotically exact terms. The stress-free condition over the crack surface and the displacement compatibility across interelement boundaries are completely satisfied. The superelements are compatible with most existing finite element computer programs.

Numerical results for commonly used fracture test specimens (singe edge crack specimen, center crack specimen, double edge crack specimen and compact tension specimen), an embedded penny-shaped crack, a semi-circular surface flaw, and a quarter-circular corner flaw were obtained.

A superelement has also been developed directly for the analysis of bending and shearing stress intensity factors, K_B and K_S, of thin plates with a through-the-thickness crack subjected to out-of-plane bending. The particular approach is also based on the hybrid element concept, for which the assumed stresses satisfy both equilibrium and compatibility conditions. Poisson-Kirchoff's thin plate theory and the complex variable technique are used. Numerical example includes the problem of pure cylindrical bending of a thin plate with finite width and centrally located through-the-thickness crack. Extensions of the formulation have been made to bending analyses of an anisotropic plate with through-the-thickness crack, an isotropic plate with wedge shaped notch, and a bi-material plate with through-the-thickness crack located parallel or normal to the interface.

A technical report for this work has been submitted to OSR for approval.

3.2 Development of Methods for Predicting Crack Growth Rate under Creep Conditions

From an intensive literature survey on investigations of static creep crack growth problems it is concluded that experimental studies lead to contradictory results when correlated with crack growth rate da/dt. The relations obtained from the experiments are all empirical ones which are not correlated with the creep behavior and creep rupture data of the material. Furthermore, none of these empirical relations provides a criterion to predict the crack growth incubation time which, for many test conditions, may occupy a large portion of the total life. Many theoretical studies have been made to explain qualitatively the nature of crack growth under creep condition. Most of these works are based on a model similar to the Dugdale model in elastic-plastic fracture analysis in that the creep region is limited to a very small region at the crack tip while the rest of the solid remains elastic. The most direct theory for creep rupture is the concept of damage parameter that was suggested by Kachanov [1]. The material constants associated with the damage parameter can be determined from creep rupture tests of uniaxial specimens under different stresses. The concept of damage parameter has been included in the formulation of an assumed stress finite element method for creep analysis using either explicit or implicit integration scheme.

An incremental finite element method for analyzing gradual crack propagation has been developed. The method has been investigated by solving a center-cracked panel under elastic condition and under creep condition with given crack extension rate using assumed stress hybrid element.

A particular experiment by Haigh on crack propagation under creep condition has been chosen as the example problem to exercise various options in finite element analysis. The experiment consisted of a wedge opening load (WOL) specimen of Cr-Mo-V steel under constant load and the results include time histories of the displacement at the loading point and of the crack extension. By an explicit finite element scheme the time histories of stress and displacement distribution and of the damage parameter have been estimated.

The resulting incubation period is only about one-sixth of that indicated by the experiments. It should be pointed out that the material behavior modeled for this analysis is not an accurate one since it was obtained by very limited experimental data. To correlate results by computational and experimental studies it is necessary to obtain adequate data from creep rupture tests.

Detailed description of this research task is given in the appendix of this report.

REFERENCE

1. Kachanov, L.M., "Rupture Time under Creep Conditions", Problems of Continuum Mechanics, Edited by J.R.M. Radok, SIAM, 1961, pp. 202-218.

APPENDIX A

FINITE ELEMENT ANALYSIS FOR CRACK GROWTH IN METALS UNDER CREEP CONDITIONS

A.1 Introduction

Among the most urgent research topics in structures and materials technology for air breathing engines of high speed aircrafts is the life prediction of structural parts with notches at elevated temperature under fatigue and creep conditions. The problem of high temperature fatigue of metals has been studied by many institutions in the United States and abroad [1*,2]. Such studies are generally restricted to experimental investigations of the number of cycles to failure. A more basic research of the creep fracture problem is the study of the rate of crack extension under creep conditions. For such a problem both analytical and experimental studies have been made and different fracture mechanics parameters have been used to correlate the test results. But no agreement can be made among the various investigators. The objective of the present research is to use the finite element method to obtain the time history of stress and strain distributions around the tip of a slowly propagating crack and to see whether there is a validity of correlating the crack growth behavior with any of the suggested fracture mechanics parameters. During the course of study, a comprehensive literature survey was made for both analytical and experimental studies of slow crack growth under creep conditions. Among the suggested methods for creep crack growth is one based on the creep damage concept proposed by Kachanov. Subsection A.2 of this appendix is a review of literature on the creep crack growth problem.

Subsection A.3 is an outline of the assumed stress hybrid method for creep analyses taking the damage parameter into account. During the earlier period of this research program the finite element creep analyses was formulated by an explicit integration scheme. However, it was found that when the damage parameter is included in the analysis, an explicit integration scheme will require the use of very small time increments in order to maintain numerical

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*The references in this appendix are listed at the end of the appendix.

stability. Also, one of the main advantages of an explicit integration scheme in ordinary finite element creep analysis is its maintaining of the same global stiffness matrix for different time increments. For a crack growth study such advantage no longer exists because the global stiffness matrix will have to be modified at each time step. In the present formulation, thus, both explicit and implicit schemes are included.

Subsection A.4 is a description of the method used to simulate slow crack growth by finite element methods and Subsection A.5 presents some preliminary numerical results of the present creep crack growth study. Finally, in Subsection A.6 some extensions of the present study are proposed.

A.2 Review of Literature on Creep Crack Growth

A.2.1 Experimental Investigations

A review and analysis of earlier experimental works of static crack growth at high temperatures were made by Haigh in 1975 [3]. In a more recent review on high temperature crack growth given by Sadananda and Shahinian [4], the static creep crack growth is also discussed. It can be seen in Table A.1 that experimental investigations have been made in the United Kingdom, United States and Japan. These studies have covered different materials including aluminum alloys, low alloy steel, stainless steel, and superalloys and different specimen geometries.

Experimental studies have shown that the process of creep crack growth usually consists of an incubation period and a growth period [21]. Depending on the material and the configurations of test specimens and the loading magnitudes, the incubation period during which creep damage develops ahead of the crack may represent a substantial part of the life of the specimen. Nevertheless most of the research studies have concentrated only on the growth phase with the aim of establishing a correlation between the rate of crack growth da/dt and certain parameters. The most commonly used parameters for such correlations are: (1) the stress intensity factor, (2) the net sections stress, (3) the crack opening displacement or the relative displacement between two points at the two sides of the crack tip, and (4) the J integral or C^* parameter.

As shown in Table A.1, the parameter which was used most for correlating creep crack growth data is the stress intensity factor K. The relation between the crack growth rate da/dt and the stress intensity factor K has been presented as straight lines on a log-log plot, i.e.

$$\frac{da}{dt} = AK^m \tag{A.1}$$

Haigh [3] and Koterazawa and Mori [24], however, have pointed out that A and m may be considered as constants only for specimens of a given geometric shape, while for different geometries the results appear as a family of approximately parallel lines. Many experimental data show that the net cross section stress is a better parameter for correlating the crack growth rate [10,15,25]. Again, it is also clear that such correlation is only applicable to a very limited range or materials and configurations. In analyzing large amounts of data, Haigh and Pilkington, et al. [21,18] correlated the crack growth rate with the rate of increasing relative displacement across the crack in the form of

$$\frac{da}{dt} = C\left(\frac{d\delta}{dt}\right)^n \tag{A.2}$$

where C is a constant, δ is the crack tip displacement, and n is a constant slightly smaller than unity. This might well be a very pertinent factor for the crack growth rate. However, it is not a value that can be determined ahead of time in order to be used to estimate the crack growth rate.

The C^* integral or $\dot{J}$ integral [30] is obtained directly from the Eshelby-Cherepanov-Rice J integral by introducing strain rate instead of strain such that

$$C^* = \int_\Gamma \left[W(\dot{\varepsilon}_{ij})dy - T_i \frac{\partial \dot{u}_i}{\partial x} ds \right] \tag{A.3}$$

where

$$W(\dot{\varepsilon}_{ij}) = \int_0^{\dot{\varepsilon}_{ij}} \sigma_{ij} d\dot{\varepsilon}_{ij} \tag{A.4}$$

T_i = traction vector

$\dot{u}_i$ = displacement rate

$\dot{\varepsilon}^e_{ij}$ = elastic strain rate

$\dot{\varepsilon}^c_{ij}$ = creep strain rate

Γ = integration path drawn counter-clockwise from a point on a crack surface to the opposite point on the other crack surface.

The C^* integral is path independent. But for the evaluation of the integral in Eq. (A.4) one must neglect the elastic part in the strain rate. Otherwise, the integral has no meaning. But the rigid-creep assumption eliminates the time element in the problem. Before the initiation of the crack growth, C^* will remain constant. Therefore, it is impossible to relate C^* to the experimentally observed crack growth incubation time. Moreover during the growth stage, the contribution from the elastic part is substantial and cannot be ignored.

Thus, except for the existence of a mathematical analogy with the J integral, it appears that the C^* integral cannot be used as a fracture criterion. Landes and Begley [30] and Nikbin et al. [12,13] had to apply a very complicated procedure to obtain the values of C^* from experimental data. Koterazawa and Mori [24] estimated the values of $\dot{J}$ by an approximate relation that $\dot{J}$ is proportional to the product of net section stress σ_{net} and the rate of elongation of a gage length $\dot{\delta}$. These researchers and Harper and Ellison [31] all indicate that the correlation of the crack growth rate with C^* is much better than the correlation with K and σ_{net}. But, they all conclude that the correlation becomes poor in the early state of crack growth or at a lower value of C^*.

A.2.2 Theoretical Prediction of Crack Growth Behavior

Most of the existing theories for creep crack growth are based on idealized models similar to the Dugdale model that has been used to determine the size of the plastic zone at the tip of a crack. In the corresponding model here, the creep behavior is assumed to be confined only to an array of edge dislocation along the plane of the crack, while the remaining region is considered elastic. The strain rate $\dot{\varepsilon}$ along the line ahead of the crack is determined from the creep law with elastic strain neglected. Both Vitek [32] and Riedel [33] have obtained nonlinear integro-differential equations for the problem, and the time history of the crack opening displacement can be evaluated. In determining the crack incubation time and crack growth rate, Vitek has to assign a value for the critical crack opening displacement (COD) and a value for the effective length of the creep region. Both values are to be determined from ad hoc crack growth experiments. In Riedel's result both the crack growth incubation period and the crack growth rate da/dt can be expressed as functions of stress intensity factor K_I. But he claims that further experimental evidence is needed to prove the validity of his model. The Dugdale model has also been used in connection with Kachanov's damage parameter concept which will be described in detail in the next subsection.

Another type of creep crack growth theory is based on a void growth model. This model attempts to find a relation between the crack growth rate and the growth and coalescence of voids in front of the pre-existing crack. For example, Raj and Ashby [34,35] examine the mechanisms of void nucleation and growth which lead to fracture on a microscopic level. Dimelfi and Nix [36] analyze a growth of a series of cavities growing by power law creep in the elastic crack tip stress field. Due to the basic assumption, the resulting approximate solution gives a functional relation between the crack growth rate and the elastic stress intensity factor. The application of this model requires an estimation of the spacing and size of the initial cavities which are, of course, functions of applied stress level. Most recently, Sadananda [37] proposed a micromechanism for crack growth based on an analysis of the grain boundary diffusion.

A.2.3 Kachanov's Damage Parameter Concept and Applications to Creep Crack Growth

The damage parameter concept was proposed by Kachanov [38] to represent macroscopically the acceleration in material deterioration during the tertiary phase of a creep test under constant stress conditions. For example, when the creep strain rate $\dot{\varepsilon}^c$ of a metal in the primary and secondary phase under uniaxial tension stress σ is expressed in the form,

$$\dot{\varepsilon}^c = f(\sigma, t) \tag{A.5}$$

it can be modified to include tertiary creep phase by

$$\dot{\varepsilon}^c = f\left(\frac{\sigma}{1-\omega}, t\right) \tag{A.6}$$

where ω is the damage parameter which ranges from zero to unity. The rate of change of the damage parameter can also be expressed as a function of $\sigma/(1-\omega)$, in the form of

$$\dot{\omega} = g\left(\frac{\sigma}{1-\omega}, t\right) \tag{A.7}$$

In the case of steady state or secondary creep that can be represented by a power law relation, the modified creep strain law is then

$$\dot{\varepsilon}^c = \dot{\varepsilon}_0 \left[\frac{\sigma}{\sigma_0 (1-\omega)}\right]^n \tag{A.8}$$

where $\dot{\varepsilon}_0$ and σ_0 are the reference creep strain and uniaxial stress respectively, and the rate of change of the damage parameter is

$$\dot{\omega} = B\left[\frac{\sigma}{\sigma_0 (1-\omega)}\right]^m \tag{A.9}$$

Thus, full damage occurs when ω becomes unity, i.e. when the creep strain rate becomes infinite. It is also clear that for most metals for which m and n are much larger than unity, the acceleration of material damage will be confined only to a short period prior to rupture of the material. Extension of the damage parameter concept to creep rupture under multiaxial loading conditions has been investigated by many experimentalists. A paper by Leckie and Hayhurst [39] summarizes the experimental evidence, and analytical expressions are proposed for the rate of creep strain components and for the rate of change of the damage parameter ω. Basically, experiments have shown that under multi-axial loading conditions, creep strain components for various metals follow the general rules for plastic strain components in plasticity. However, in estimating the rupture time or the degree of damage, the reference stress will be dependent on the mechanism of creep deterioration of the individual materials. Thus, for example, in materials such as steel and aluminum alloys, for which the damage is related to grain boundary slide, it is expected that the multi-axial rupture behavior would be shear dependent. On the other hand, in a material such as copper for which the creep damage is a result of vacancy diffusion, it is expected that the multi-axial rupture behavior is governed by the maximum principal stress components. Torsion-tension experimental results of copper and aluminum alloys [40], in fact, have confirmed these conjectures. The strain rates in the tertiary phase thus must be represented, in general, by two reference stresses: $\bar{\sigma}$ which is the octahedral shear stress and $\bar{\Sigma}$ which may be either the shear stress or the maximum principal stress. When the creep behavior of a material follows a power law, the constitutive equations may be written in the form

$$\frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_0} = \frac{\left(\frac{\bar{\sigma}}{\sigma_0}\right)^n}{(1-\omega)^n} \frac{\partial \bar{\sigma}}{\partial \sigma_{ij}} \tag{A.10}$$

and

$$\dot{\omega} = B\left[\frac{\bar{\Sigma}}{\sigma_0(1-\omega)}\right]^m \tag{A.11}$$

For steel and aluminum alloys, $\bar{\sigma}$ and $\bar{\Sigma}$ are, of course, identical.

The damage parameter concept can also be extended to primary creep under a time hardening law. In that case the uniaxial creep strain curve is expressed as

$$\varepsilon^c = F(\sigma, \omega)\, g(t) \tag{A.12}$$

When reduced time $\tau = g(t)$ is introduced, the creep strain rate may be represented by

$$\frac{d\varepsilon^c}{d\tau} = F(\sigma, \omega) \tag{A.13}$$

This is of the same form as the creep strain rate under steady state creep. Similarly, the damage parameter rate with respect to the reduced time has the same form as Eq. (A.9) if $\dot{\omega}$ is replaced by $d\omega/d\tau$.

Applications of Kachanov's concept to creep crack growth problems are again limited to the ideal case of the Dugdale model. Kachanov [41], in fact, proposed to assign the yield stress as the stress at the crack edge and estimate the plastic zone size based on the Dugdale model. For the uniaxial creep relation which includes the damage parameter, it is a simple matter to calculate the time required to rupture this plastic zone; hence, the rate of crack growth can be calculated. This simplified theory neglects the stress relief due to creep deformation and the degree of damage, in the region ahead of the plastic zone. This will make an overly conservative estimation of the incubation period but an under-estimation of the succeeding crack growth rate. To [42] has applied the same approach but considers the gradual increase in damage ahead of the crack tip plastic zone. He has obtained an analytical solution for da/dt vs. K. It is approximately a straight line on a log-log plot. In To's analysis the stress redistribution due to creep was again ignored and under the assumption of perfectly plastic material, the stress in the plastic zone is maintained as the limiting yield stress. In actuality the stress should be reduced to a value lower than this value and the incubation time should be increased. Further improvement in the analysis is to include the strain hardening behavior under high temperature conditions. Unfortunately, such information is in general not provided in the literature. For example, in the data provided by Haigh [21], the only information on plastic behavior is the yield stress defined by the 0.2% permanent strain.

A.3 Creep Analysis by an Assumed Stress Hybrid Method Taking Damage Parameter into Account

The finite element analysis program [43] developed earlier in this research project is related to material behaviors that are classified as primary and/or secondary creep. The finite element model used is the assumed stress hybrid model. The present section is an extension of the earlier formulation to include the Kachanov damage parameter.

The existing creep analysis program is based on an explicit time integration scheme with the time increment limited by numerical stability. For example, the time increment in the analysis must be reduced when the element stress and/or the creep rate increases. But in the crack problem the stresses near the crack tip are very high. This means the time increments must be very small. This drawback is further amplified when the damage parameter is included, since the creep strain rate will accelerate rapidly when the damage parameter ω is approaching unity.

The alternative implicit time integration scheme have been suggested for creep analysis in connection with the conventional assumed displacement method [44-46]. The basic idea is to express the creep strain increments in terms of not only the state of stress but also the stress increments. The formulation in this section, thus, also covers the implicit time integration scheme.

An assumed stress hybrid finite element can be derived by the modified complementary energy principle or by the Hellinger-Reissner principle when the assumed stresses are in equilibrium within each element [47]. For plane stress, plane strains and three-dimensional solids it is easy to construct compatible shape functions for element displacements, hence, the Hellinger-Reissner principle is also a suitable approach. For creep analyses, the Hellinger-Reissner principle can be extended by including the creep strains as initial strains, and by expressing the variational functional π_R in terms of stress and displacement rate. For the present finite element formulation the incremental approach is adopted, hence, it is most convenient to express the functional π_R in terms of stress and displacement increments and the variational principle is stated as follows:

$$\pi_R = \sum_n \Big[\int_{V_n} \Big\{ -\frac{1}{2} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S}\, \Delta\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma}^T (\underset{\sim}{D}\Delta\underset{\sim}{u}) - \Delta\underset{\sim}{\sigma}^T \Delta\underset{\sim}{\varepsilon}^c - \Delta\underset{\sim}{F}^T \Delta\underset{\sim}{u} \Big\} dv - \int_{S_{\sigma_n}} \Delta\underset{\sim}{\bar{T}}^T \Delta\underset{\sim}{u}\, dS \Big] \qquad \text{(A.14)}$$

where

$\Delta\underset{\sim}{\sigma}$ = stress increment vector

$\Delta\underset{\sim}{u}$ = displacement increment vector

$\underset{\sim}{S}$ = elastic coefficient matrix

$\underset{\sim}{D}\Delta\underset{\sim}{u}$ = strain increment vector in terms of displacement increments

$\Delta\underset{\sim}{\varepsilon}^c$ = creep strain increment

$\Delta\underset{\sim}{\bar{F}}$ = applied body force increment

$\Delta\underset{\sim}{\bar{T}}$ = increment of applied boundary traction

V_n = volume of the nth element

S_{σ_n} = surface of the nth element over which tractions are prescribed

The creep strain rate may be expressed in terms of the state of stress $\underset{\sim}{\sigma}$ for the steady state creep condition or in terms of either stresses and time or stresses and creep strains for the transient creep condition. When the damage parameter ω is included the creep rate is also a function of ω. Thus, the creep strain rate may be written, in general, as

$$\dot{\underset{\sim}{\varepsilon}}^c = \underset{\sim}{f}(\underset{\sim}{\sigma}, \underset{\sim}{\varepsilon}^c, \omega, t) \tag{A.15}$$

The functional π_{mc} for the corresponding modified complementary energy principle is

$$\pi_{mc} = \sum_n \Big[\int_{V_n} \left(\frac{1}{2} \Delta\underset{\sim}{\sigma}^T \underset{\sim}{S} \Delta\underset{\sim}{\sigma} + \Delta\underset{\sim}{\sigma}^T \Delta\underset{\sim}{\varepsilon}^c \right) dV - \int_{\partial V_n} \Delta\underset{\sim}{T}^T \Delta\underset{\sim}{\tilde{u}}\, dS + \int_{S_{\sigma_n}} \Delta\underset{\sim}{\bar{T}}^T \Delta\underset{\sim}{u}\, dS \Big] \tag{A.16}$$

where ∂V_n = the entire boundary of the nth element

$\Delta\underset{\sim}{\tilde{u}}$ = boundary displacement increments

In these variational principles the variables are $\Delta\underset{\sim}{\sigma}$ (and the corresponding $\Delta\underset{\sim}{T}$), $\Delta\underset{\sim}{u}$ or $\Delta\underset{\sim}{\tilde{u}}$ while $\Delta\underset{\sim}{\varepsilon}^c$ is assumed to be prescribed. In the finite element formulation the element stress and displacement increments are assumed in terms of unknown stress parameters $\Delta\underset{\sim}{\beta}$ and nodal displacement increments $\Delta\underset{\sim}{q}$ respectively. For example when π_R is used, one expresses

$$\Delta\underset{\sim}{\sigma} = \underset{\sim}{P}\,\Delta\underset{\sim}{\beta} \tag{A.17}$$

$$\Delta\underset{\sim}{u} = \underset{\sim}{A}\,\Delta\underset{\sim}{q} \tag{A.18}$$

and from which the increments in boundary tractions and strains are

$$\Delta\underset{\sim}{T} = \underset{\sim}{R}\,\Delta\underset{\sim}{\beta} \tag{A.19}$$

and

$$\underset{\sim}{D}\,\Delta\underset{\sim}{u} = \underset{\sim}{B}\,\Delta\underset{\sim}{q} \tag{A.20}$$

Here the assumed stress increments are made to satisfy the equilibrium condition. Substituting Eqs. (A.17)-(A.20) into Eq. (A.14), one obtains

$$\pi_R = \sum_n \Big\{ -\tfrac{1}{2}\Delta\underset{\sim}{\beta}^T \underset{\sim}{H}\,\Delta\underset{\sim}{\beta} + \Delta\underset{\sim}{\beta}^T \underset{\sim}{G}\,\Delta\underset{\sim}{q} - \Delta\underset{\sim}{\beta}^T \Delta\underset{\sim}{J}^c - \Delta\underset{\sim}{q}^T \Delta\underset{\sim}{Q}_m \Big\} \tag{A.21}$$

where

$$\underset{\sim}{H} = \int_{V_m} \underset{\sim}{P}^T \underset{\sim}{S}\,\underset{\sim}{P}\,dV \tag{A.22}$$

$$\underset{\sim}{G} = \int_{V_m} \underset{\sim}{P}^T \underset{\sim}{B}\,dV \tag{A.23}$$

$$\Delta\underset{\sim}{J}^c = \int_{V_m} \underset{\sim}{P}^T \Delta\underset{\sim}{\varepsilon}^c\,dV \tag{A.24}$$

$$\Delta \underset{\sim}{Q}_m = \int_{V_m} \underset{\sim}{A}^T \underset{\sim}{F} \, dV + \int_{S_{\sigma m}} \underset{\sim}{A}^T \bar{\underset{\sim}{T}} \, dS \tag{A.25}$$

For each individual element one can set to zero the variation of π_R with respect to $\Delta\beta$ and obtain a set of equations relating $\Delta\beta$ to Δq, i.e.

$$\underset{\sim}{H} \, \Delta\underset{\sim}{\beta} = \underset{\sim}{G} \, \Delta\underset{\sim}{q} - \Delta\underset{\sim}{J}^c \tag{A.26}$$

In constructing the time integration scheme the increments of $\underset{\sim}{\varepsilon}^c$ and $\underset{\sim}{\omega}$ at time t_m are given by

$$\Delta\underset{\sim}{\varepsilon}_m^c = \left[(1-\theta)\, \dot{\underset{\sim}{\varepsilon}}_m^c + \theta\, \dot{\underset{\sim}{\varepsilon}}_{m+1}^c \right] \Delta t \tag{A.27}$$

and

$$\Delta\omega_m = \left[(1-\theta)\, \dot{\omega}_m + \theta\, \dot{\omega}_{m+1} \right] \Delta t \tag{A.28}$$

where for the explicit scheme using the Euler method $\theta=0$, and for an implicit scheme $1/2<\theta<1$.

If the creep rate and the damage rate can be expressed as

$$\dot{\underset{\sim}{\varepsilon}}^c = \underset{\sim}{f}(\underset{\sim}{\sigma}, \omega, t) \tag{A.29}$$

and

$$\dot{\omega}^c = g(\underset{\sim}{\sigma}, \omega, t) \tag{A.30}$$

one can write

$$\dot{\underset{\sim}{\varepsilon}}_{m+1}^c = \dot{\underset{\sim}{\varepsilon}}_m^c + \underset{\sim}{F}_m \, \Delta\underset{\sim}{\sigma}_m + \underset{\sim}{F}_m' \, \Delta\omega_m \tag{A.31}$$

and

$$\dot{\omega}_{m+1} = \dot{\omega}_m + \underset{\sim}{M}_m \, \Delta\underset{\sim}{\sigma}_m + M_m' \, \Delta\omega_m \tag{A.32}$$

where

$$\underset{\sim}{F}_m = \left(\frac{\partial \underset{\sim}{f}}{\partial \underset{\sim}{\sigma}}\right)_m \qquad \underset{\sim}{F}_m' = \left(\frac{\partial \underset{\sim}{f}}{\partial \omega}\right)_m$$

$$\underset{\sim}{M}_m = \left(\frac{\partial g}{\partial \underset{\sim}{\sigma}}\right)_m \qquad M_m' = \left(\frac{\partial g}{\partial \omega}\right)_m \tag{A.33}$$

Then, from Eqs. (A.17), (A.27), (A.28), (A.31) and (A.32) one can write $\Delta\varepsilon_m^c$ and $\Delta\omega_m$ as

$$\Delta\underset{\sim}{\varepsilon}_m^c = \underset{\sim}{f}_m \Delta t + \theta \Delta t \left(\underset{\sim}{F}_m \underset{\sim}{P} \Delta\underset{\sim}{\beta} + \underset{\sim}{F}_m' \Delta\omega_m\right) \tag{A.34}$$

$$\Delta\omega_m = g_m \Delta t + \theta \Delta t \left(\underset{\sim}{M}_m \underset{\sim}{P} \Delta\underset{\sim}{\beta} + M_m' \Delta\omega_m\right) \tag{A.35}$$

It is seen that $\Delta\omega_m$ can be solved from Eq. (A.35), i.e.

$$\Delta\omega_m = \frac{1}{1-\theta\Delta t M_m'}\left(g_m \Delta t + \theta \Delta t \underset{\sim}{M}_m \underset{\sim}{P} \Delta\underset{\sim}{\beta}\right) \tag{A.36}$$

and by substituting into Eq. (A.34),

$$\Delta\underset{\sim}{\varepsilon}_m^c = \left(\underset{\sim}{f}_m + \frac{\theta\Delta t\, g_m \underset{\sim}{F}_m'}{1-\theta\Delta t M_m'}\right)\Delta t + \theta\Delta t\left(\underset{\sim}{F}_m + \frac{\theta\Delta t \underset{\sim}{F}_m' \underset{\sim}{M}_m}{1-\theta\Delta t M_m'}\right)\underset{\sim}{P}\Delta\underset{\sim}{\beta} \tag{A.37}$$

In the explicit scheme by Euler's method, $\theta=0$ and

$$\Delta\underset{\sim}{\varepsilon}_m^c = \underset{\sim}{f}_m \Delta t \tag{A.38}$$

By substituting Eq. (A.37) into Eq. (A.24) and then into Eq. (A.26) one obtains

$$\left(\underset{\sim}{H} + \underset{\sim}{H}_\theta\right)\Delta\underset{\sim}{\beta} = \underset{\sim}{G}\Delta\underset{\sim}{q} - \Delta\underset{\sim}{\bar{J}}^c \tag{A.39}$$

where

$$\underset{\sim}{H}_\theta = \theta \Delta t \int_{V_m} \underset{\sim}{P}^T \left(\underset{\sim}{F}_m + \frac{\theta \Delta t \, \underset{\sim}{F}_m' \, \underset{\sim}{M}_m}{1 - \theta \Delta t \, M_m'} \right) \underset{\sim}{P} \, dv \tag{A.40}$$

$$\Delta \underset{\sim}{\bar{J}}^c = \int_{V_m} \underset{\sim}{P}^T \left(\underset{\sim}{f}_m + \frac{\theta \Delta t \, g \, \underset{\sim}{F}_m'}{1 - \theta \Delta t \, M_m'} \right) \underset{\sim}{P} \, dv \tag{A.41}$$

Taking $\delta\pi_R = 0$ with respect to $\Delta \underset{\sim}{q}$ in Eq. (A.21),

$$\sum_m \left(\underset{\sim}{G}^T \Delta \underset{\sim}{\beta} - \Delta \underset{\sim}{Q}_m \right) = 0 \tag{A.42}$$

Solving for $\Delta \underset{\sim}{\beta}$ from Eq. (A.39) and substituting into Eq. (A.42) then yields

$$\sum_m \underset{\sim}{k}_m \Delta \underset{\sim}{q} = \sum_m \left(\Delta \underset{\sim}{Q}_m + \Delta \underset{\sim}{Q}_m^c \right) \tag{A.43}$$

where

$$\underset{\sim}{k}_m = \underset{\sim}{G}^T \left(\underset{\sim}{H} + \underset{\sim}{H}_\theta \right)^{-1} \underset{\sim}{G} \tag{A.44}$$

$$\Delta \underset{\sim}{Q}_m^c = \underset{\sim}{G}^T \left(\underset{\sim}{H} + \underset{\sim}{H}_\theta \right)^{-1} \Delta \underset{\sim}{\bar{J}}^c \tag{A.45}$$

It is seen that $\underset{\sim}{H}_\theta$, therefore $(\underset{\sim}{H} + \underset{\sim}{H}_\theta)^{-1}$, is not a symmetric matrix. If the term involving $(\Delta t)^2$ is neglected $\underset{\sim}{H}_\theta$ will be symmetric.

The element matrices $\underset{\sim}{k}_n$, $\Delta \underset{\sim}{Q}_n$, $\Delta \underset{\sim}{Q}_n^c$ are assembled as global matrices $\underset{\sim}{K}$, $\Delta \underset{\sim}{Q}$, and $\Delta \underset{\sim}{Q}^c$ such that

$$\underset{\sim}{K} \Delta \underset{\sim}{q} = \Delta \underset{\sim}{Q} + \Delta \underset{\sim}{Q}^c \tag{A.46}$$

which can be solved to obtain the increments of nodal displacements.

A.4 Simulation of Slow Crack Growth by Finite Element Methods

The simplest method for considering the propagation of a crack in the finite element method is the gradual releasing of nodes along the crack line [48]. In such a method the crack propagation must cover the width of one or more elements during a given time increment. Thus, in practice, the elements used should be simple ones such as 3-node triangles and 4-node quadrilaterals in the 2-D problems and the element size must be very small. An alternative method [48] is to use elements with a side node the position of which is shifted forward to simulate the crack propagation. Typical elements to be used are 8-node quadrilateral elements. In the node shifting technique, the total crack extension is limited only over the width of one element. Thus, for a large crack extension it is necessary to combine the node shifting with the node releasing technique. In this section the procedure used for the node releasing technique is derived in connection with the assumed stress hybrid model.

To account for a change in internal or external boundary conditions of a solid during a given time interval, the functionals given in Eqs. (A.14) and (A.16) must be modified. Consider, for example, during the interval between t and $t+\Delta t$ a crack surface extends from Σ to $\Sigma+\Delta\Sigma$. First, consider the state of stress σ at t and a virtual displacement corresponding to that during this time increment. The statement of Principle of Virtual Work then is,

$$\int_V \underset{\sim}{\sigma}^T (\underset{\sim}{D}\, \delta \underset{\sim}{u})\, dv - \int_{S_\sigma} \underset{\sim}{\bar{T}}^T \delta \Delta \underset{\sim}{u}\, ds - \int_{\Delta\Sigma^{(a)}} \underset{\sim}{\bar{T}}^{(a)T} \delta \Delta \underset{\sim}{u}^{(a)} ds - \int_{\Delta\Sigma^{(b)}} \underset{\sim}{\bar{T}}^{(b)T} \delta \Delta \underset{\sim}{u}^{(b)} ds = 0 \tag{A.47}$$

Here, for simplicity, body forces are assumed absent. The superscripts (a) and (b) are referred to the two surfaces created by the crack. Now, based on the state of stress at $t+\Delta t$ the statement of virtual work is

$$\int_V (\underset{\sim}{\sigma}^T + \Delta \underset{\sim}{\sigma}^T)\, \delta(\underset{\sim}{D} \Delta \underset{\sim}{u})\, dv - \int_{S_\sigma} (\underset{\sim}{\bar{T}}^T + \Delta \underset{\sim}{T}^T)\, \delta \Delta \underset{\sim}{u}\, ds = 0 \tag{A.48}$$

since under this situation the surfaces $\Delta\Sigma^{(a)}$ and $\Delta\Sigma^{(b)}$ are traction free. By combining Eqs. (A.47) and (A.48), the statement of the Principle of Virtual Work for the incremental problem becomes

$$\int_V \Delta\underset{\sim}{\sigma}^T \delta(D\Delta\underset{\sim}{u})\,dV - \int_{S_\sigma} \Delta\underset{\sim}{T}^T \Delta\underset{\sim}{u}\,dS + \int_{\Delta\Sigma^{(a)}} \overline{\underset{\sim}{T}}^{(a)T} \delta\Delta\underset{\sim}{u}^{(a)}\,dS \qquad \text{(A.49)}$$

$$+ \int_{\Delta\Sigma^{(b)}} \overline{\underset{\sim}{T}}^{(b)T} \delta\Delta\underset{\sim}{u}^{(b)}\,dS = 0$$

In comparison with the statement of virtual work for conventional incremental problems it is recognized that the last two terms are the additional terms to account for the crack extension during the time increment. The corresponding expressions for π_R and π_{mc} will be those obtained by adding the $\pm\left[\int_{\Delta\Sigma^{(a)}} \underset{\sim}{T}^{(a)T} \Delta\underset{\sim}{u}^{(a)}\,dS + \int_{\Delta\Sigma^{(b)}} \underset{\sim}{T}^{(b)T} \Delta\underset{\sim}{u}^{(b)}\,dS\right]$ respectively, in Eqs. (A.14) and (A.16).

An incremental finite element method for crack propagation has been developed based on (a) the release of the node which is located at the crack tip and (b) the advance of the crack for the distance of one element. A center-cracked panel shown in Figure 1 was analyzed as an elastic-creep problem neglecting both plasticity and damage parameters. Four-node rectangular elements were used. A hypothetical incubation period of 100 hours was used for the present calculation. After 100 hours the crack was assumed to propagate with the rate of 0.008 in/hr. Figure 1 shows the displacements normal to the crack line before and after the opening of the two nodes. Since the panel has been allowed to creep for 100 hours before the opening of the nodes, stresses near the crack tip have been relaxed to values lower than elastic solutions. These lower stresses are responsible for the fact that the displacements are smaller than the elastic solutions near the released nodes.

A.5 Results of Creep Crack Growth Studies

(1) The incremental and explicit finite element method has also been extended to problems for combined plasticity and creep including the damage parameter. In an attempt to correlate the result of finite element analysis to experimental results, a particular experiment by Haigh [21] on crack growth under creep condition has been chosen as the example problem. The experiment consists of a wedge opening load (WOL) specimen under constant tension load. Haigh provides rather limited experimental data on the relation of creep strain vs. time at different stress levels, the rupture time at different stress levels, and the crack growth history. In the finite element analysis, the panel is modeled by 103 triangular elements and 68 rectangular elements, the smallest around the crack tip having a dimension of 0.375 mm. The numerical solution for the stress distribution near the crack tip is shown in Fig. 2. It indicates some stress relief due to creep and damage in the region ahead of the crack tip. However, during the time span covered in the calculation, the stress in the element closest to the crack tip has not been reduced by any appreciable amount. As a result in this element the values of the damage parameters ω are almost identical to those obtained by Kachanov's approximate method [41] which assumes no stress relief due to creep and damage. For this case the crack incubation time calculated by Kachanov's method is 117 hours which is much smaller than the experimentally observed value of 750 hours. Thus at this moment, one can only claim that the result of the present analysis can only estimate the incubation time to a right order of magnitude. It is felt that when the material behavior can be modeled more accurately using adequate experimental data, there will be considerable adjustment in the finite element results.

(2) In order to compare the explicit and implicit schemes in creep analysis, a simple statically indeterminate three-bar structure

(Fig. 3) is used as an example. Under a constant tension load, the bar AB is more heavily loaded than the other two bars. Thus, this structure actually simulates the problem of tension load of a panel with a center crack for which the stress near the crack tip is higher than that away from the crack tip. In both cases the creep action is to even the stress distribution. For the analysis of the 3-bar truss, the creep and damage behavior of the material is based on that of Cr-Mo-V steel used in Haigh's experiment [21]. The time histories of the stresses in these two bars are given in Fig. 3. When the explicit scheme is used, the time increments must be reduced progressively. In fact, it is not possible to extend the calculation to the point where $\omega=1$. For the implicit solution there is no restriction to the choice of time increments and the solution can be carried to the point where $\omega=1$. It is seen that the stresses in these bars become relaxed; but when the damage is nearing completion, the stress in bar AB reduces further. This is an indication that for the crack growth problem there will also be appreciable stress relief at the crack tip region.

(3) A method for finite element analysis for elastic-plastic creep behavior under finite deformation has been formulated. The key in this study is the representation of the incremental constitutive relation. For this relation a pair of conjugate kinematical and dynamical variables, i.e. deformation rate and Kirchhoff stress tensors, are chosen. This is justified because in the case of a uniaxial tensile test, the deformation rate is equivalent to the rate of natural strain and the Kirchhoff stress is the engineering stress multiplied by the elongation ratio. The present formulation is based on an additive decomposition of deformation rate tensor into elastic, plastic, and creep components which is equivalent to the multiplicative decomposition of deformation gradient. In order to avoid the effects of rotation of the material, i.e. to be frame-indifferent, the elastic component of deformation rate is assumed to be linearly proportional to the Jaumann rate, instead of the

usual material rate of the Kirchhoff stress. For plastic deformation, a type of inequality similar to that of Drucker's is postulated for the plastic component of deformation rate and the Jaumann rate of Kirchhoff stress. Although this postulation is not a universal one, it does guarantee a kind of material stability. By assuming a Mises loading function defined in Kirchhoff stress space and the linear relation between plastic component of deformation rate and Jaumann rate of Kirchhoff stress, the normality condition and the associated flow rule can be derived. The same flow rule can then be applied to creep deformation. In the present study, Norton's power law is adopted for steady state creep. The finite element formulation is based on four node hybrid stress elements. The damage parameter is also included in the formulation.

A.6 Suggested Further Studies

It is apparent that the present study of static creep crack growth is just a preliminary attempt. For a deeper understanding of this problem a combined computational and experimental approach is required. The implicit finite element method should be implemented into a computer code for the analysis of crack extension under creep conditions based on Kachanov's damage parameter concept. The numerical solution should be verified by experimental results.

Further extensions of the research effort should include a study of the effect of unloading and of possible crack closure and an investigation of the effect of finite strain at the crack tip in the finite element solution of crack growth under creep conditions.

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TABLE A.1

EXPERIMENTAL INVESTIGATION OF STATIC CREEP CRACK GROWTH IN METALS

Investigators (Country)	Year Published	Material	Temperature (°C)	Specimen Type*	Best Parameter to Correlate Crack Growth Data	Reference No.
Popp & Coles (US)	1969	Inconel 718	538	CC	K	5
Silverns, Price Neate (UK)	1970,1973 1977	Steel Cr-Mo-V	565	SEN(T)	K	6-9
Harrison & Sander (US)	1971	Steel Cr-Mo-V	538	CC	σ_{Net}	10
Kenyon, Webster, Turner, Nikbin (UK)	1973,1974 1976,1977	Aluminum Alloy RR58	100-200	DCB (Const.K)	K C*	11-14
Taira & Ohtani (Japan)	1973,1976	Steel Cr-Mo-V	600	NRB, DEN	σ_{Net}	15 16
Ellison & Walton (UK)	1973	Steel Cr-Mo-V	565	SEN(T)	K,t	17
Pilkington & Jones (UK)	1974,1977 1978	Steel Cr-Mo-V	530-570	SEN(B)	d(COD)/dt	18-20
Haigh (UK)	1975,1977	Steel Cr-Mo-V	550	WOL	d(COD)/dt	21,22
		Stainless Steel 316 Weld	538	WOL	d(COD)/dt	
Koterazawa et al. (Japan)	1976 1977	Stainless Steel 304	650	DEN SEN(T) CC. NRB	K(for low stress level) σ_{Net}, $\dot{J}$	23,24

TABLE A.1 (Concluded)

Investigators (Country)	Year Published	Material	Temperature (°C)	Specimen Type*	Best Parameter to Correlate Crack Growth Data	Reference No.
Nicholson et al. (UK)	1975 1976	Stainless Steel 316	600-850	DEN	σ_{Net}	25,26
Landes & Begley (US)	1976	Discaloy	647	CC, WOL	C*	30
Shaninian & Sadananda (US)	1977	Inconel 718	650	CT	K	27
		Udimet 700	850	CT	K	28
		Stainless Steel 308	593	CT	K	29

*Types of Specimen used in experimental investigations:

1. CC - Center Cracked
2. CT - Compact Tension
3. DCB - Double Cantilever Beam
4. DEN - Double edge Notch
5. NRB - Notched Round Bar
6. SEN (T or B) - Single Edge Notch (tension of bending)
7. WOL - Wedge Opening Load

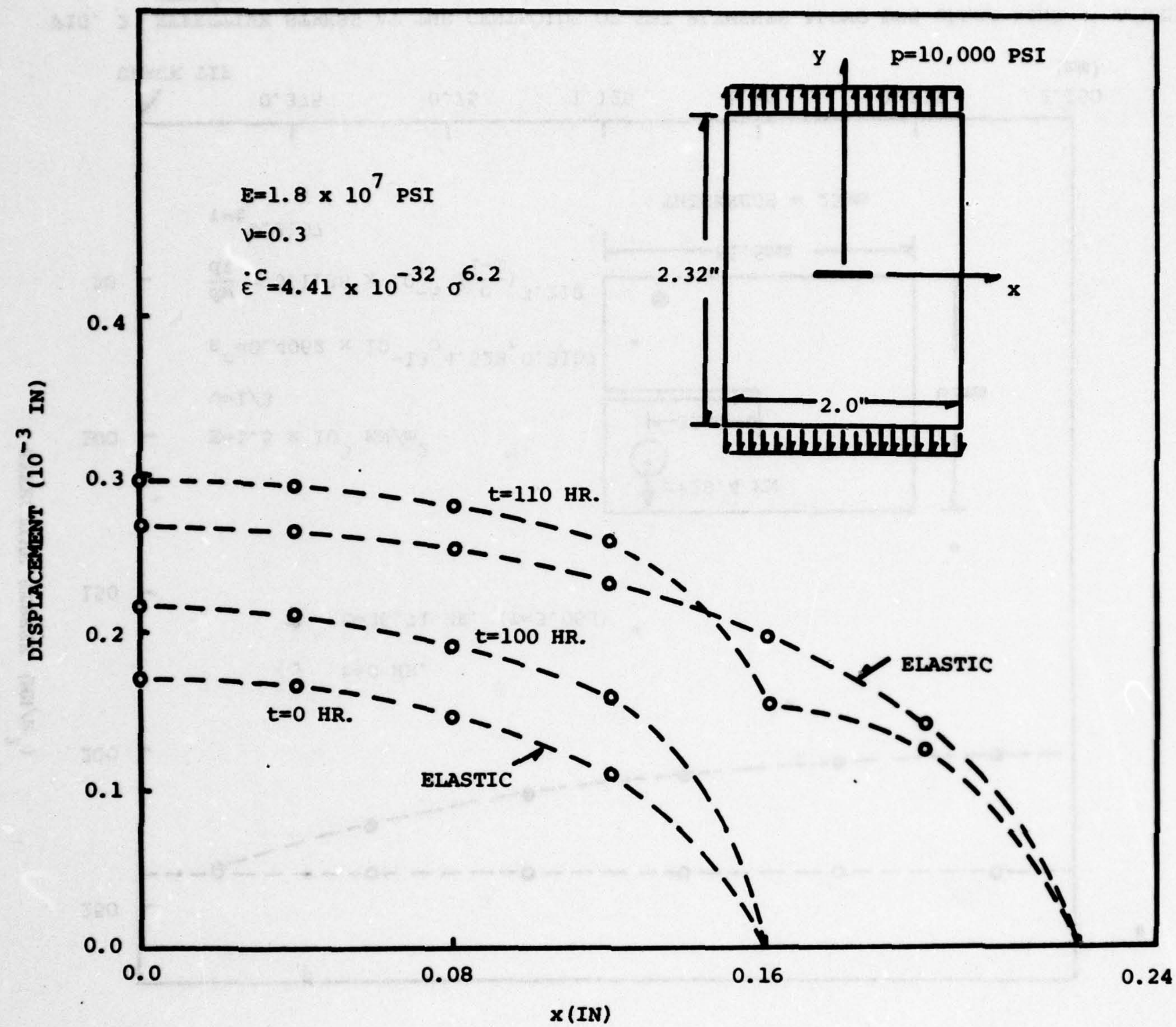

FIG. 1 DISPLACEMENT NORMAL TO THE CRACK SURFACE OF A CENTER CRACKED PANEL

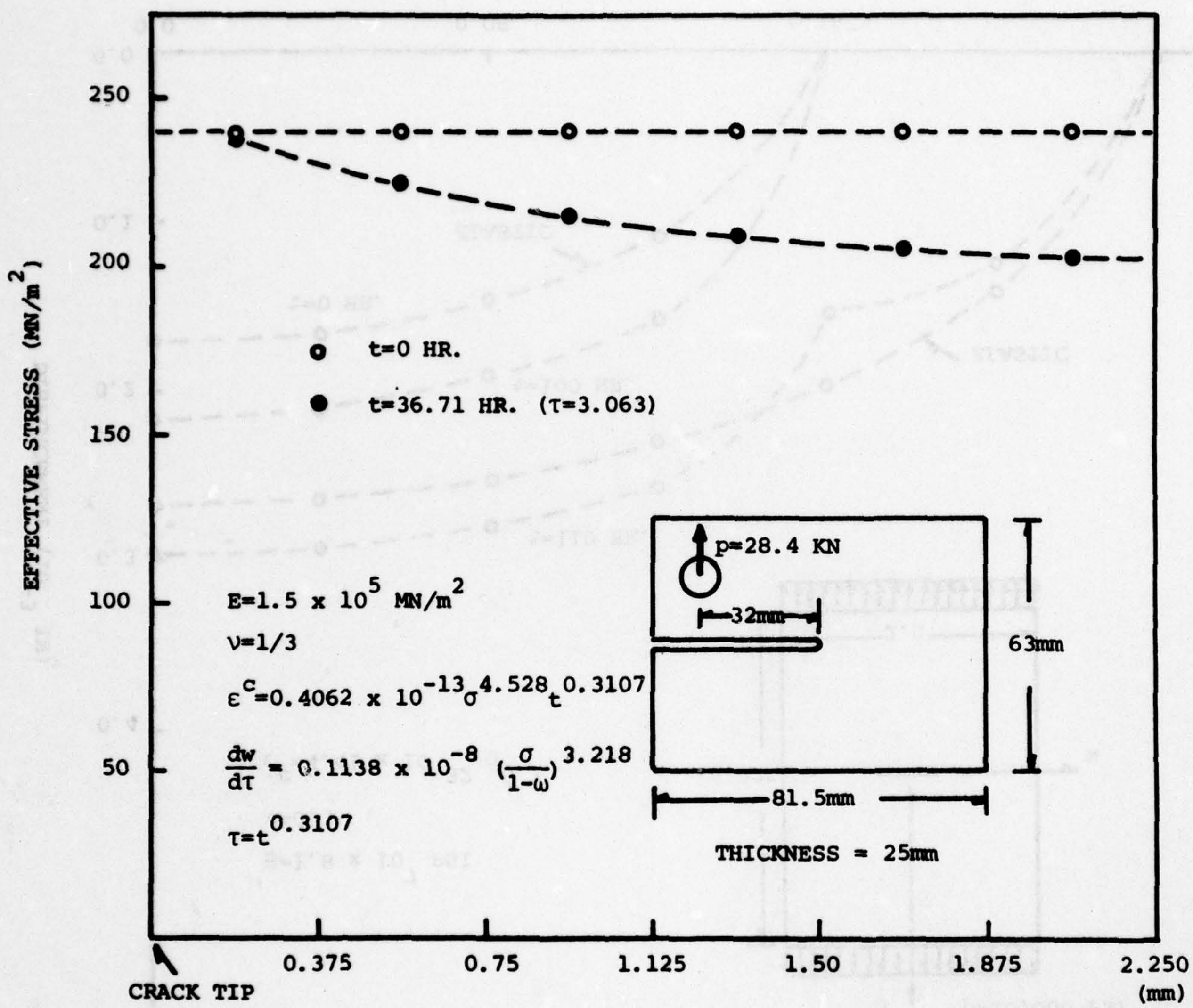

FIG. 2 EFFECTIVE STRESS AT THE CENTROIDS OF THE ELEMENTS ALONG THE CRACK LINE -- A WEDGE OPENING LOAD SPECIMEN

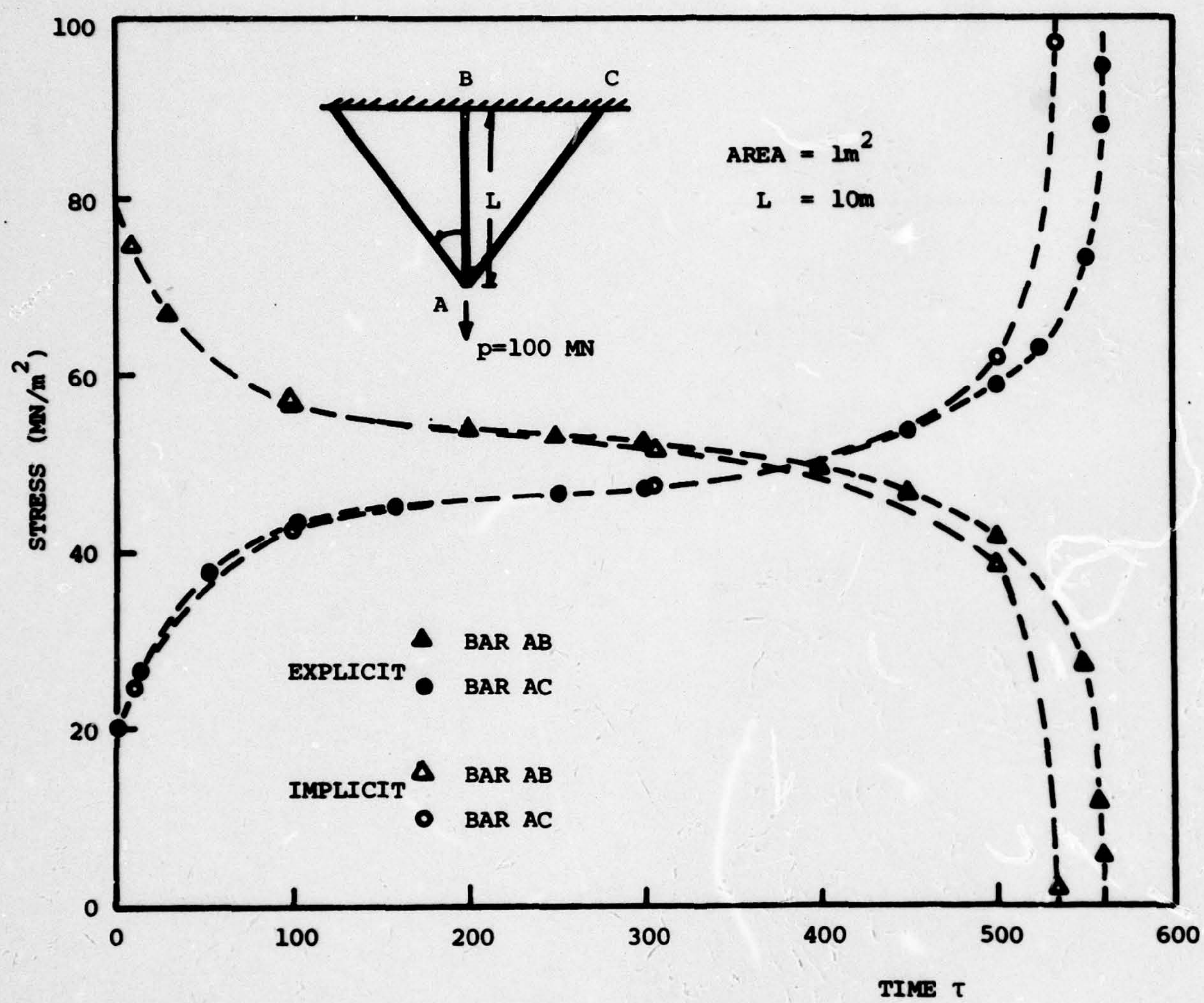

FIG. 3 STRESS VS. TIME -- A THREE BAR PROBLEM