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NUMERICAL SIMULATION OF A LINEAR FILTER.(U)
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DEFENSE RESEARCH LABORATORY

THE UNIVERSITY OF TEXAS
AUSTIN, TEXAS 78712

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(6) NUMERICAL SIMULATION OF A LINEAR FILTER

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ABSTRACT

This paper investigates certain properties of a method which calculates a sample of a stationary Gaussian random process with a specified power spectral density function. The study determines to what degree this method simulates a linear filter. Also included are correlation analyses of equi-distributed sequences which are used in the method.

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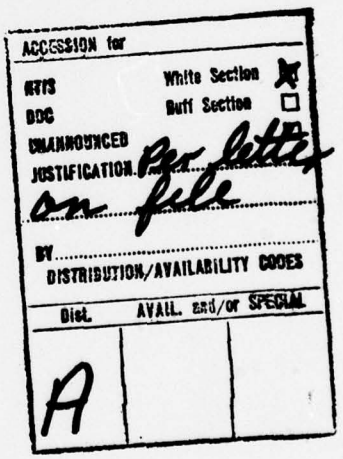

I. INTRODUCTION

For r a random variable, a collection of functions of time $x(t) = x_r(t)$ is called a random process. If for every finite collection of times $t_1 < \cdots < t_n$, the random variables $x(t_1), x(t_2), \cdots, x(t_n)$ have a multivariate Gaussian distribution, the process is called Gaussian. The process is called stationary if, for any increment Δt, the random variables $x(t_i + \Delta t)$ and $x(t_i)$ have the same joint distribution. A stationary Gaussian random process is called a time series if t represents values of time.[1]

Mr. Joel N. Franklin proposed a method for computing a time series with given statistical properties.[2] Franklin's method constructs a sample of a stationary Gaussian random process with a given power spectral density function. This technique requires the use of a sequence of random numbers. Walter Matuska wrote a computer program for the Control Data Corporation 3200 computer which produces these random numbers.[3] Matuska also wrote a computer program GAUSSIAN using the output of the random number generator which applies Franklin's technique in computing a time series with a

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[1] J. N. Franklin, "Numerical Simulation of Stationary and Nonstationary Gaussian Random Processes," SIAM Review 7, No. 1, p. 68 (January, 1965).

[2] Ibid., pp. 68 - 80.

[3] W. A. Matuska, Jr., and G. S. Innis, "Generation of a Stationary Gaussian Random Process with a Specified Power Spectral Density Function," Defense Research Laboratory, The University of Texas, Acoustical Report A-258 (6 July 1966), pp. 12 - 16.

specified spectrum.[4] Still another program SPECT[5] uses this time series as input and computes the power spectral density function.

The applications of Franklin's theory are varied. One problem is that of detecting a known signal in various noise backgrounds. The method presented in this paper generates a time series which has the same power spectrum as the time series obtained from a specified noise background. To this time series can be added the time series for a signal with known power spectrum. By varying the power spectral density function of the background, the effect of different noise backgrounds upon the detection technique can be studied.[6] This method can also be used in the theory of optimum linear prediction and filtering. The theory concerns the designing of engineering systems that either project into the future by information obtained in the past or recover desired signals which have been distorted by random noise. These systems can be applied to communications, meteorological forecasting, and economic analysis. An example is the problem of designing an optimum filter to smooth a perturbed message (signal plus noise). Separate and independent information about a second perturbed message whose signal portion is related to the first is then employed. Franklin's method can be used to obtain a time series with known power spectral density function for both perturbed signals. Then using known signal relationships, the time series

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[4] Ibid., pp. 20 - 29.

[5] G. E. Ellis and R. L. Boston, "Program SPECT: Power Spectral Analysis of a Random Process Using the CDC 1604 Computer," Defense Research Laboratory, The University of Texas, Austin, Texas, Acoustical Report A-252 (December 1965).

[6] Matuska, op. cit., p. 3.

of the noise can be obtained.[7] One further application is the study of smoothing techniques such as hamming and hanning windows in which it is of interest to know how the minor lobes of a power spectral density function are affected by these smoothing techniques. This can be accomplished by constructing time series for which the distances between the major and minor lobes of their power spectra vary. The relative sizes of these lobes can also be varied; therefore, the effect of a smoothing technique on a given power spectral density function can be studied.[8]

This paper uses the outputs of the aforementioned computer programs to determine the degree to which Franklin's method is a linear filter. The task is partially accomplished with the use of a computer program FOURXFRM written for the Control Data 3200 computer. This program computes the Fourier transform of a given time series. The paper also includes correlation analyses which are performed to determine the degree to which the random number generator produces independent sequences.

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[7] J. S. Bendat, Principles and Applications of Random Noise Theory (John Wiley and Sons, New York, 1958), pp. 201 - 206.

[8] R. B. Blackman and J. W. Tukey, The Measurement of Power Spectra (Dover Publications, Inc., New York, 1958), pp. 33 - 37.

II. FRANKLIN'S METHOD - A FILTER

To compute a sample of a stationary Gaussian random process by Franklin's Method, the autocorrelation function or the power spectral density function must be known. The autocorrelation function is defined as

$$R(t_1,t_2) = E\left[x(t_1)\ x(t_2)\right] \quad ,$$

where $E\left[x_i\right]$, the expected value operator, is defined as

$$E\left[x_i\right] = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} x_i$$

if the limit exists.[9] If $x(t)$ is a time series, then the autocorrelation function can be written as

$$R(\tau) = E\left[x(t)\ x(t-\tau)\right] \quad .$$

If $V(\omega)$ denotes the power spectral density function for a given frequency ω, then $V(\omega)$ is defined by first defining

$$C(\omega) = \frac{\left| \int_{-T}^{T} x(t)e^{-i\omega t}dt \right|}{T}$$

for a time series $x(t)$, $|t| \leq T$. Then let

$$P_T(\omega_c,\ x,\ \Delta\omega) = \int_{\omega_c - \frac{\Delta\omega}{2}}^{\omega_c + \frac{\Delta\omega}{2}} C(\omega)d\omega \quad ,$$

where $P_T(\omega_c,\ x,\ \Delta\omega)$ measures the average power for a range of frequencies

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[9] Franklin, op. cit., p. 68

of width $\Delta\omega$ and centered at ω_c. Then $V(\omega)$ can be defined

$$V(\omega) = \lim_{\Delta\omega \to o} \left\{ \frac{\lim_{T\to\infty} P_T(\omega_c, x, \Delta\omega)}{\Delta\omega} \right\} .$$

If $x(t)$ is a stationary random process, $V(\omega)$ can be shown to be

$$V(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} \, d\tau \qquad (2.1)$$

if the integral exists.[10] That is, the power spectral density function is the Fourier transform of the autocorrelation function.

If $x(t)$ is the input time series to a linear filter and $y(t)$ is the output time series, then they are related by the convolution

$$y(t) = \int_{-\infty}^{t} g(t-s)\, x(s)\, ds \quad . \qquad (2.2)$$

If x is equal to the Dirac delta function, then $y(t)$ is the response of the linear filter to a delta function input and equals $g(t)$. It can then be shown that the power spectral density functions $V_y(\omega)$ and $V_x(\omega)$ of the output and input are related by the equation

$$V_y(\omega) = |G(\omega)|^2 V_x(\omega) \quad , \qquad (2.3)$$

where $G(\omega)$ is the Fourier transform of $g(t)$.[11]

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[10] Bendat, op. cit., pp. 44, 66 - 67.

[11] W. B. Davenport and W. L. Root, An Introduction to the Theory of Random Signals and Noise (McGraw-Hill, New York, 1958), pp. 182 - 183, 225 - 227.

The power spectral density $V_x(\omega)$ of a random process $x(t)$ is defined to be one for all real ω. Such a random process is known as white noise. If such a process is obtained then from Eq. (2.3),

$$V(\omega) = V_y(\omega) = |G(\omega)|^2 \quad . \tag{2.4}$$

It has been shown by Davenport and Root[12] that if

$$0 < V(\omega) < \infty, \qquad V(\omega) = V(-\omega), \qquad \text{and } V(\omega) \to 0 \text{ as } \omega \to \pm\infty, \tag{2.5}$$

and if $V(\omega)$ can be represented as a quotient of two polynomials in ω, then

$$V(\omega) = \left| \frac{P(i\omega)}{Q(i\omega)} \right|^2 \tag{2.6}$$

for ω real. P and Q are polynomials with real coefficients, where the degree of P is less than the degree of Q and where the zeros of $Q(z)$ lie in the halfplane $\mathrm{Re}(z) < 0$. Now $G(\omega)$ can be defined

$$G(\omega) = \frac{P(i\omega)}{Q(i\omega)} \quad ; \tag{2.7}$$

if D denotes the differential operator, $\frac{d}{dt}$,

$$y(t) = \frac{P(D)}{Q(D)}\, x(t) \quad . \tag{2.8}$$

The next step in obtaining the specified time series is to solve the differential equation

$$Q(D)\, \Phi(t) = x(t) \quad , -\infty < t < \infty \quad . \tag{2.9}$$

Once Eq. (2.9) has been solved, the solution of $\Phi(t)$ is used to obtain $y(t)$ by combining the derivatives of $\Phi(t)$ of order lower than the degree of Q

$$y(t) = P(D)\, \Phi(t) \quad . \tag{2.10}$$

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[12] Ibid., pp. 232 - 234, 375 - 376.

To obtain a solution to Eq. (2.10), a white sequence must first be generated.[13] If the white sequence is denoted x_n, a sequence w_n (called the w-sequence) of independent samples from a Gaussian distribution is constructed

$$w_{2n-1} = (-2 \ln x_{2n-1})^{\frac{1}{2}} \cos 2\pi x_{2n} \quad , \text{ and}$$

$$w_{2n} = (-2 \ln x_{2n-1})^{\frac{1}{2}} \sin 2\pi x_{2n}$$

for $n = 1, 2, 3, \cdots$. When this w-sequence is used in the solution of (2.9), n-dimensional vectors are formed from successive elements of the sequence

$$w^{(o)} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \quad , w^{(1)} = \begin{bmatrix} w_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ w_{2n} \end{bmatrix} \quad , w^{(2)} = \begin{bmatrix} w_{2n+1} \\ \cdot \\ \cdot \\ \cdot \\ w_{3n} \end{bmatrix} \quad \cdots \quad ,$$

where n is the degree of Q. As can be seen from this definition, n elements of the w-sequence are required for each vector defined. For this reason the shaped time series has only $\left(\frac{1}{n}\right)$ times the number of points as the input time series. Here it is of interest to consider the unpublished work of Mr. Joe England. For the same w-sequence England defined the vectors

$$w^{(o)} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} \quad , w^{(1)} = \begin{bmatrix} w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_{n+1} \end{bmatrix} \quad , w^{(2)} \begin{bmatrix} w_3 \\ \cdot \\ \cdot \\ \cdot \\ w_{n+2} \end{bmatrix} \quad \cdots \quad .$$

[13] Matuska, Op. cit., pp. 12 - 16.

Using this definition the shaped time series is reduced by only (n-1) points, and there seems to be no weakening of the power of the method when these vectors are used.

Numerically there does not exist a white noise sequence with power spectral density exactly one. A correlation analysis of these "white" sequences is presented in the Appendix.

The next step is that of investigating properties of Franklin's method as a linear filter. For any linear filter let $x(t)$ be the input time series and $y(t)$ the output time series such that they are related as in Eq. (2.2).

Theorem I

If $x(t) = 0$ for $t \neq 0$ and

$$\int_{-\infty}^{\infty} x(t)dt = 1 \quad ,$$

then $y_x(t) = h(t)$, the impulse response. If $x(t)$ is any input, then

$$y_x(t) = \int_{-\infty}^{\infty} x(t)\ h(\tau-t)\ dt \quad .$$

Theorem II

If $V_{y_x}(\omega)$ is the power spectral density function of $y_x(t)$ and $V_x(\omega)$ is the power spectral density function of $x(t)$, then

$$V_{y_x}(\omega) = |H(\omega)|^2 V_x(\omega) \quad ,$$

where $H(\omega)$ is equal to the Fourier transform of $h(t)$.

The following question now must be answered. Does

$$|H(\omega)|^2 = \left|\frac{P(i\omega)}{Q(i\omega)}\right|^2$$

when considered in Franklin's terminology?

The first step toward obtaining the answer to the question is to measure the impulse response of a particular filter. This is accomplished by use of Matuska's program GAUSSIAN.[14] From the program for an input time sequence with a Gaussian distribution and for a certain ratio of polynomials, a resultant time series is obtained. For the purposes of this paper the input sequence was replaced by an impulse sequence defined to be zero for all t except for one point at which it is defined to be one. With this sequence as input, the resultant time series approximates the impulse response.

From the two theorems presented it can be seen that if

$$|H(\omega)|^2 = \left|\frac{P(i\omega)}{Q(i\omega)}\right|^2 \quad ,$$

then $|H(\omega)|^2$ must equal the power spectral density function for the filter. The power spectrum is obtained from the program SPECT.[15] This equivalence is shown using

$$V_{y_x}(\omega) = |H(\omega)|^2 V_x(\omega)$$

obtained from Theorem II and Eq. (2.6). For a white sequence $V_x(\omega)$ is defined as one; therefore

$$V_{y_x}(\omega) = \left|\frac{P(i\omega)}{Q(i\omega)}\right|^2 = |H(\omega)|^2 \quad .$$

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[14] Ibid., pp. 20 - 29.

[15] Ellis, op. cit.

III. COMPUTER PROGRAMS

This discussion concerns three programs written for the Control Data 3200 computer. GAUSSIAN was written by W. Matuska to shape a w-sequence into a time series with a given power spectral density function.[16] The input w-sequence is a sequence of independent samples from a Gaussian distribution. The Gaussian distribution is defined from a "white sequence;" that is, an equidistributed sequence γ_n on the range $0 < \gamma_n < 1$. The given power spectral density function $V(\omega)$ of the output time series must be a rational function satisfying the conditions (2.5). One other input to GAUSSIAN is the time increment Δt at which samples of the specified time series are to be calculated. The resultant time series is buffered out on tape to be used as input to the program SPECT.[17]

The program SPECT computes the autocorrelation function of the time series shaped by GAUSSIAN. Then the power spectral density function (which is the Fourier transform of the autocorrelation function) is computed. The autocorrelation function is plotted linearly against time, and the power spectral density function is plotted against frequency on a log-log plot.

In addition to use as it was written, GAUSSIAN was also modified to obtain the impulse response for a particular ratio of polynomials. This was accomplished simply by redefining the input w-sequence to be an impulse sequence. The impulse sequence is defined to be zero everywhere except for one t.

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[16] Matuska, op. cit., pp. 20 - 29.

[17] Ellis, op. cit.

Another program FOURXFRM was written to compute the Fourier transform of the impulse response as obtained from GAUSSIAN. The program buffers the data from a magnetic tape and computes the Fourier transform for a given increment of time Δt and for a specified frequency range and frequency increment.

Letting $f(t)$ be a time series, it can be represented as a Fourier series as follows:[18]

$$f(t) = \left(\frac{1}{\pi}\right) \int_0^\infty [A(\omega)\cos(\omega t) + B(\omega)\sin(\omega t)]d\omega \quad ,$$

where

$$A(\omega) = \int_{-\infty}^{\infty} \cos(\omega t)\ f(t)dt \quad , \text{ and}$$

$$B(\omega) = \int_{-\infty}^{\infty} \sin(\omega t)\ f(t)dt \quad .$$

Let t_1 and t_2 be two consecutive points in time; then the contribution to $A(\omega)$ from this interval is

$$\Delta A(\omega) = \int_{t_1}^{t_2} \cos(\omega t) \left[f(t_1) + \frac{f(t_2) - f(t_1)}{t_2 - t_1}\ (t-t_1)\right] dt$$

$$= \frac{f(t_2) - f(t_1)}{t_2 - t_1} \int_{t_1}^{t_2} t\ \cos(\omega t)\ dt + \frac{f(t_1)t_2 - t_1 f(t_2)}{t_2 - t_1} \int_{t_1}^{t_2} \cos(\omega t)\ dt$$

$$= \frac{f(t_2) - f(t_1)}{(t_2 - t_1)\omega^2} \left[\cos(\omega t_2) - \cos(\omega t_1)\right] + \frac{f(t_2)\sin(\omega t_2) - f(t_1)\sin(\omega t_1)}{\omega} .$$

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[18] H. Herbert Howe, "Fourier Analysis of Nonperiodic Pulses on Automatic Computers," U. S. Department of Commerce, National Bureau of Standards Report 5018 (3 October 1956), pp. 9 - 15.

For two consecutive intervals if the contributions to $A(\omega)$ are added, $\left[\frac{f(t_2)}{\omega}\right]\sin(\omega t_2)$ for the first interval cancels with $\left[\frac{f(t_1)}{\omega}\right]\sin(\omega t_1)$ for the second interval. Hence the sine terms drop out with the exception of the first and the last values of $f(t)$. Therefore, if t_o represents the initial value of t and t_n the last value of t,

$$A(\omega) = \frac{f(t_n)\sin(\omega t_n)}{\omega} + \sum_{t_1}^{t_{n-1}} \frac{\Delta f\, \Delta\cos(\omega t)}{\omega^2 \Delta t} \quad .$$

Likewise,

$$B(\omega) = \frac{f(t_o) - f(t_n)\cos(\omega t_n)}{\omega} + \sum_{t_1}^{t_{n-1}} \frac{\Delta f\, \Delta\sin(\omega t)}{\omega^2 \Delta t} \quad .$$

The previously described method relies on the assumption that the function $f(t)$ is linear between any two consecutive points.

Now that $A(\omega)$ and $B(\omega)$ are known, the Fourier transform of $f(t)$ can be defined

$$F(\omega) = \sqrt{A^2(\omega) + B^2(\omega)}$$

for ω, the angular frequency.

The $\Delta\sin\,\omega t$ and $\Delta\cos\,\omega t$ were computed using a recursive method suggested by R. W. Hamming.[19] Since the time interval is invariant, $\Delta\sin(\omega t_m)$ can be written $\Delta\sin(\omega m\Delta t)$, and $\Delta\cos(\omega t_m)$ can be written $\Delta\cos(\omega m\Delta t)$.

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[19] R. W. Hamming, Numerical Methods for Scientists and Engineers (McGraw-Hill Book Company, Inc., 1962), pp. 72 - 73.

The first step in the recursion is to define

$$V_o = 0 \quad ,$$

$$V_1 = 1 \quad ,$$

$$V_m = [2\cos(\omega t)]V_{m-1} - V_{m-2} \quad (m = 2,3, \cdots) \quad ,$$

and then show that

$$V_m = \frac{\sin(m\omega t)}{\sin(\omega t)} \quad .$$

Since the above equation is true for m = 0 and 1, it is sufficient to show

$$[2\cos(\omega t)]V_{m-1} - V_{m-2} = \frac{2\cos(\omega t)\sin[(m-1)\omega t]}{\sin(\omega t)} - \frac{\sin[(m-2)\,\omega t]}{\sin(\omega t)}$$

$$= \frac{\sin(m\omega t) + \sin[(m-2)\omega t] - \sin[(m-2)\omega t]}{\sin(\omega t)}$$

$$= \frac{\sin(m\omega t)}{\sin(\omega t)} = V_m \quad .$$

Similarly it can also be seen that

$$\cos(m\omega t) = [\cos(t)]V_m - V_{m-1} \quad .$$

One further consideration when computing a Fourier transform is that of the Nyquist or folding frequency. Given a sampling rate Δt, the sampling theorem requires

$$2\omega\Delta t < 1 \quad ,$$

where ω is the largest angular frequency in the time series. As will be seen in the following section, computation of the Fourier transform for frequencies near ω has an undesirable effect on the transform.

Input for FOURXFRM

DATA CARD

DT,FREQ1,FREQEND, DELTAF,IFPRINT Y 4F10.5,I5 (FORMAT)

INPUT TAPE FORMAT

N

Y(1) ... Y(N)

N

Y(1) ... Y(N)

.

.

.

N

Y(1) ... Y(N)

END-OF-FILE

```
      PROGRAM FOURXFRM
      DIMENSION Y(2500), C(300), DELTACOS(2500), DELTASIN(2500),
     1          V(2500), F(300)
      PI = 3.141592654
      NMAX = 0
      JF = 0
      NL40 = 0
      V(1) = 0.0
      V(2) = 1.0
      READ 15, DT, FREQ1, FREQEND, DELTAF, IFPRINT Y
      FRQ = FREQEND + 1.0E-08
      DO 100 I = 1,2500
  100 Y(I) = 1.0
  101 IF (JF .EQ. 1) 108, 90
   90 BUFFER IN (1,1) (N,N)
      N = N
  102 GO TO (102,103,108,104), UNITSTF(1)
  103 NMIN = NMAX + 1
      NMAX = NMIN + N - 1
      IF (NMAX .GE. 2500) 91, 105
   91 NMAX = 2500
      JF = 1
      GO TO 105
  104 PRINT 10
      GO TO 9999
  105 BUFFER IN (1,1) (Y(NMIN),Y(NMAX))
  106 GO TO (106,101,107,104), UNITSTF(1)
  107 PRINT 11
      GO TO 9999
  108 PRINT 12
      NOY = NMAX
      DO 94 I = 1,NOY
      IF (ABSF(Y(I)) .LT. 1.0E-40  .AND. Y(I) .NE. 0.0) 92, 94
   92 NL40 = NL40 + 1
      IF (NL40 .GE. 10) 93, 94
   93 IZERO = I
      GO TO 95
   94 CONTINUE
   95 DO 96 I = IZERO,NOY
      Y(I) = 0.0
   96 CONTINUE
      IF (IFPRINTY .EQ. 1) 109, 110
  109 PRINT 13
      PRINT 14, (Y(I), I = 1,NOY,500)
      PRINT 12
      PRINT 16
  110 J = 1
      FREQ = FREQ1
      NM1 = NMAX - 1
  112 A = 0.0
      B = 0.0
      W = 2.0*PI*FREQ
      SINE = SINF(W*DT)
      COSINE = COSF(W*DT)
      COSPROD = (COSINE - 1.0)*(COSINE + 0.5)
      ONEMCOS = 1.0 - COSINE
      TOCOSM1 = 2.0*COSINE - 1.0
```

```
      CONS = 1.0/(W*W*DT)
      M = 0
      A = A + W*DT*SINF(W*NM1*DT)*Y(NMAX)
      B = B + W*DT*(Y(1) - Y(NMAX)*COSF(W*NM1*DT))
  113 M = M + 1
      IF (M .EQ. 1) 114, 115
  114 DELTACOS(M) = -ONEMCOS
      DELTASIN(M) = SINE
      GO TO 118
  115 IF (M .EQ. 2) 117, 116
  116 V(M) = 2.0*V(M-1)*COSINE - V(M-2)
  117 DELTASIN(M) = SINE*(V(M)*TOCOSM1 - V(M-1))
      DELTACOS(M) = 2.0*V(M)*COSPROD + V(M-1)*ONEMCOS
  118 A = A + DELTACOS(M)*(Y(M+1)-Y(M))
      B = B + DELTASIN(M)*(Y(M+1)-Y(M))
      IF (M.EQ. NM1) 119, 113
  119 A = A*CONS
      B = B*CONS
      C(J) = SQRTF(A*A + B*B)
      F(J) = FREQ
      FREQ = FREQ + DELTAF
      IF (FREQ .LT. FRQ) 121, 122
  121 J = J + 1
      GO TO 112
  122 CMIN = C(1)
      CMAX = C(1)
      IMIN = 1
      DO 128 I = 1,J
      IF (C(I) .GT. CMAX) 125, 126
  125 CMAX = C(I)
      GO TO 128
  126 IF (C(I) .LT. CMIN) 127, 128
  127 CMIN = C(I)
      IMIN = I
  128 CONTINUE
      DO 1029 I = 1,J
      C(I) = C(I)*C(I)
      C(I) = C(I)/(CMAX*CMAX)
 1029 CONTINUE
      DO 1030 I = 1,J
      PRINT 17, F(I), C(I)
 1030 CONTINUE
      CMAX = 1.0
      CMIN = C(IMIN)
      PRINT 12
      CALL MINMAX (F(1),F(J),IFMIN,IFMAX)
      NOF = IFMAX - IFMIN
      CALL MINMAX(CMIN, CMAX, ICMIN, ICMAX)
      NOC = ICMAX - ICMIN
      IF (NOC .GT. 5) 129, 130
  129 YHEIGHT = 1.0
      GO TO 131
  130 YHEIGHT = 2.0
  131 CALL LAX(YHEIGHT,NOF,NOC)
      CON1 = IFMIN*2.0
      CON2 = 0.43429*YHEIGHT
      CON3 = ICMIN*YHEIGHT
      LPEN = 3
```

```
      DO 132 K = 1,J
      XX = 0.86858*ALOG(F(K)) - CON1
      YY = CON2*ALOG(C(K)) - CON3
      CALL PLOT(XX,YY,LPEN)
  132 LPEN = 2
      CALL PLOT(NOF*2.0,0.0,-3)
   10 FORMAT (30H PARITY ERROR ON MAGNETIC TAPE)
   11 FORMAT (19H EOF IN WRONG PLACE)
   12 FORMAT (1H1)
   13 FORMAT (13X,28HLISTING OF DATA POINTS, Y(I) )
   14 FORMAT (6E20.10)
   15 FORMAT(4F10.5,I5)
   16 FORMAT (10X,4HFREQ,10X,12HFOUR. TRANS./)
   17 FORMAT (E14.4,3X,E19.5)
 9999 PRINT 12
      END

      SUBROUTINE MINMAX (XMIN, XMAX, IMIN, IMAX)
      I = 0
    1 IF (XMIN .LT. 10.0**(-I)) 2,3
    2 I = I + 1
      GO TO 1
    3 IF (I .EQ. 0) 4,5
    4 IF (XMIN .GT. 10.0**I) 6,7
    6 I = I + 1
      GO TO 4
    7 IMIN = I - 1
      GO TO 8
    5 IMIN = -1
    8 I = IMIN
   10 IF (XMAX .GT. 10.0**I) 9,11
    9 I = I + 1
      GO TO 10
   11 IMAX = I
      END

      SUBROUTINE LAX (YHEIGHT,NOF,NOC)
      CALL PLOT(3.0,-12.0,-3)
      CALL PLOT (0.0,0.5,-3)
      Y = 0.0
      DO 50 I = 1,NOF
      A = 10.0**(I-1)
      DO 51 J = 1,10
      X = J
      X = 0.86858*ALOG(X*A)
   51 CALL SYMBOL(X,Y,0.1,3,0.0,-2)
   50 CONTINUE
      CALL PLOT (0.0,0.0,3)
      X = 0.0
      CON = 0.43429*YHEIGHT
      DO 52 I = 1,NOC
      A = 10.0**(I-1)
      DO 53 J = 1,10
      Y = J
      Y = CON*ALOG(Y*A)
   53 CALL SYMBOL(X,Y,0.1,3,0.0,-2)
   52 CONTINUE
      END
```

Glossary of Terms in FOURXFRM

A	Accumulative storage location for sine terms
B	Accumulative storage location for cosine terms
C(I)	Fourier transform
CMAX	Maximum Fourier transform
CMIN	Minimum Fourier transform
COSINE	$\cos(\omega\Delta t)$
COSPROD	$[\cos(\omega\Delta t)-1]\cdot[\cos(\omega\Delta t) + 0.5]$
DELTACOS(I)	$\cos[(N+1)\omega\Delta t] - \cos[N\omega\Delta t]$
DELTAF	Frequency increment
DELTASIN(I)	$\sin[(N+1)\omega\Delta t] - \sin[N\omega\Delta t]$
DT	Time increment
F(I)	Frequencies for which Fourier transform is computed
FREQ1	Initial frequency for which Fourier transform is computed
FREQEND	Last frequency for which Fourier transform is computed
ICMAX	Maximum power of ten of Fourier transform range
ICMIN	Minimum power of ten of Fourier transform range
IFMAX	Maximum power of ten of frequency range
IFMIN	Minimum power of ten of frequency range
IFPRINT Y	Parameter to determine whether or not to print input time series
LAX	Subroutine to plot logarithmic axes
LPEN	Parameter used in plot routine
MINMAX	Subroutine to determine maximum and minimum powers of ten for a sequence

NM1	Number of points of input time series minus one
NMAX	Changing parameter determining maximum subscript for each block of buffered data
NMIN	Changing parameter determining minimum subscript for each block of buffered data
NOC	Number of powers of ten within which all Fourier transforms fall
NOF	Number of powers of ten within which all frequencies fall
NOY	Total number of points of input time series
ONEMCOS	$1.0 - \cos(\omega\Delta t)$
SINE	$\sin(\omega\Delta t)$
TOCOSM1	$2.0\cos(\omega\Delta t) - 1.0$
V(I)	Parameter used in recursion for DELTACOS(I) and DELTASIN(I)
W(I)	Angular frequencies
Y(I)	Time Series
YHEIGHT	Parameter used in plot routine

IV. EXAMPLE RUNS AND RESULTS

Four examples are now presented. For each example the power spectral density function $V(\omega)$ and the time interval Δt must be known. The program GAUSSIAN is then used to obtain an approximation to the impulse response. FOURXFRM is next used to obtain the Fourier transform of the impulse response. The square of the transform is plotted against frequency on a log-log plot. The examples used are those presented by Matuska.[20]

Because of the manner in which terms of the w-sequence are used in GAUSSIAN to form vectors for the solution of the differential equations (2.9) and (2.10), the impulse time series was formed in different ways for each example. If the degree of the polynomial $Q(i\omega)$ in Eq. (2.6) was n, then the impulse of magnitude one was placed at $w(n+1)$, $w(n+2)$, $\cdots$, $w(2n)$ and the impulse response computed for the impulse at each of these times. The impulses could have been placed at $w(kn+1)$, $w(kn+2)$, $\cdots$, $w(kn+n)$, $k > 0$, with the same resultant impulse responses. It is seen that n approximations to the impulse response were computed for each of the four examples.

The theoretical power spectral density function of example one is given to be

$$V_1(\omega) = \frac{9\omega^2 + 1}{\omega^4 - 6\omega^2 + 25} \quad ,$$

which has zeros at the points (0, 1/3) and (0, -1/3) and poles at the points r, $\bar{r}$, $-r$, $-\bar{r}$ for $r = (2,1)$. Therefore the condition $0 < V_1(\omega) < \infty$ holds for

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[20] Matuska, op. cit., pp. 31 - 35.

all real ω. Also $V_1(\omega) = V_1(-\omega)$ and $V_1(\omega) \to 0$ as $\omega \to \pm\infty$ for ω real. With these three conditions satisfied $V_1(\omega)$ can be represented in the form of Eq. (2.6)

$$V_1(\omega) = \left| \frac{3(i\omega) + 1}{(i\omega)^2 + 2(i\omega) + 5} \right|^2 ,$$

where $(i\omega)^2 + 2(i\omega) + 5$ has zeros at $(-1,2)$ and $(-1,-2)$. Both of these zeros lie in the halfplane with $\mathrm{Re}(i\omega) < 0$.

The theoretical power spectral density function of example two is given to be

$$V_2(\omega) = \frac{\omega^4 - 70\omega^2 + 1369}{\omega^6 + 14\omega^4 + 49\omega^2 + 36} .$$

It can be shown that the conditions (2.5) hold such that $V_2(\omega)$ can be represented

$$V_2(\omega) = \left| \frac{(i\omega)^2 + 2(i\omega) + 37}{(i\omega)^3 + 6(i\omega)^2 + 11(i\omega) + 6} \right|^2 .$$

For example three the theoretical power spectral density function is

$$V_3(\omega) = \frac{400\omega^2 + 1}{\omega^6 + 14\omega^4 + 49\omega^2 + 36} ,$$

and as before it is seen that

$$V_3(\omega) = \left| \frac{20(i\omega) + 1}{(i\omega)^3 + 6(i\omega)^2 + 11(i\omega) + 6} \right|^2 .$$

The theoretical power spectral density function of example four is given by[21]

$$V_4(\omega) = 4Ak \frac{\omega^2 + (k^2 + c^2)}{\omega^4 + 2(k^2-c^2)\omega^2 + (k^2+c^2)^2} ,$$

where A = 2149.2, k = 0.658, and c = 2.71. $V_4(\omega)$ can be insured to have the representation

$$V_4(\omega) = \left| \frac{b_1(i\omega) + b_2}{a_1(i\omega)^2 + a_2(i\omega) + a_3} \right|^2 ,$$

where $b_1 \cong 75.210999197$, $b_2 \cong 53.521755647$, $a_1 = 1.0$, $a_2 = 1.316$, and $a_3 = 0.506405$.

For each of these four examples the square of the Fourier transform of the impulse response was found to agree quite well with the theoretical power spectral density function. There were, however, two points at which discrepancies did occur. One of the points was at the extremely low frequencies, assuming that the peak frequency did not occur there. For those examples, namely two and four, at which the peak frequency was the lowest frequency, there was no discrepancy between the theoretical power spectral density function and the square of the Fourier transform of the impulse response. In examples one and three the square of the Fourier transform of the impulse response, for the impulse at different points, bracketed the power spectral density function at these low frequencies.

The other discrepancy occurred at the highest frequencies. As discussed earlier, the frequency range is bounded above by $(\frac{1}{2} \Delta t)$ where Δt

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[21] Ibid., p. 34.

is the time interval between points of the input time series. In all examples the maximum frequency for which the Fourier transform was computed was equal to $\frac{1}{2}\Delta t$. This fact caused a folding back of the Fourier transform in such a manner as to cause the square of the Fourier transform of the impulse response and the theoretical power spectral density function to disagree in a small manner.

For plotting purposes both the power spectral density function and the square of the Fourier transform of the impulse response were normalized to have a maximum value of one. The plots of these four examples are shown in Figs. 1 through 4.

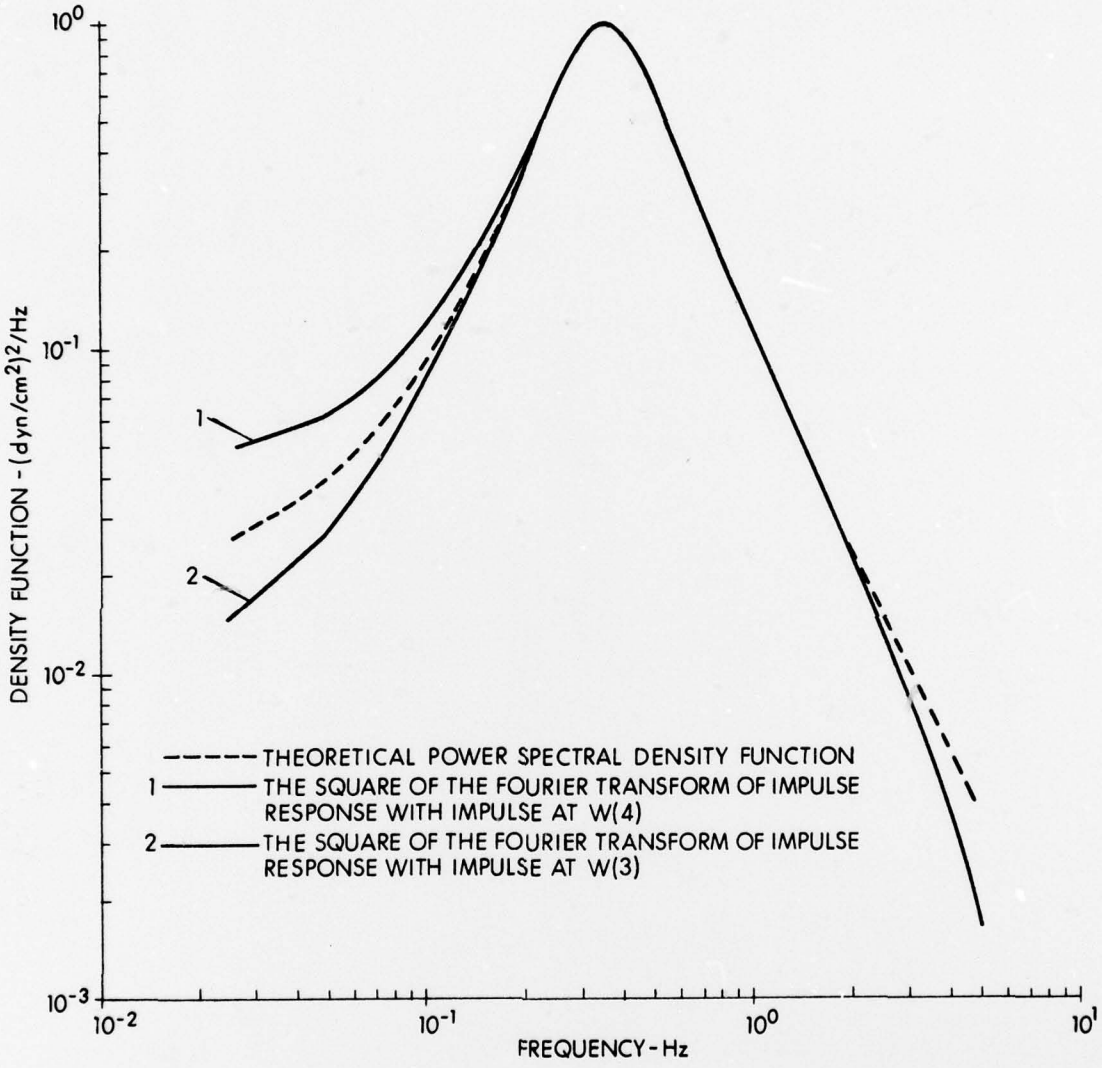

FIGURE 1
EXAMPLE 1

$$V_1(\omega) = \frac{9\omega^2 + 1}{\omega^4 - 6\omega^2 + 25}$$

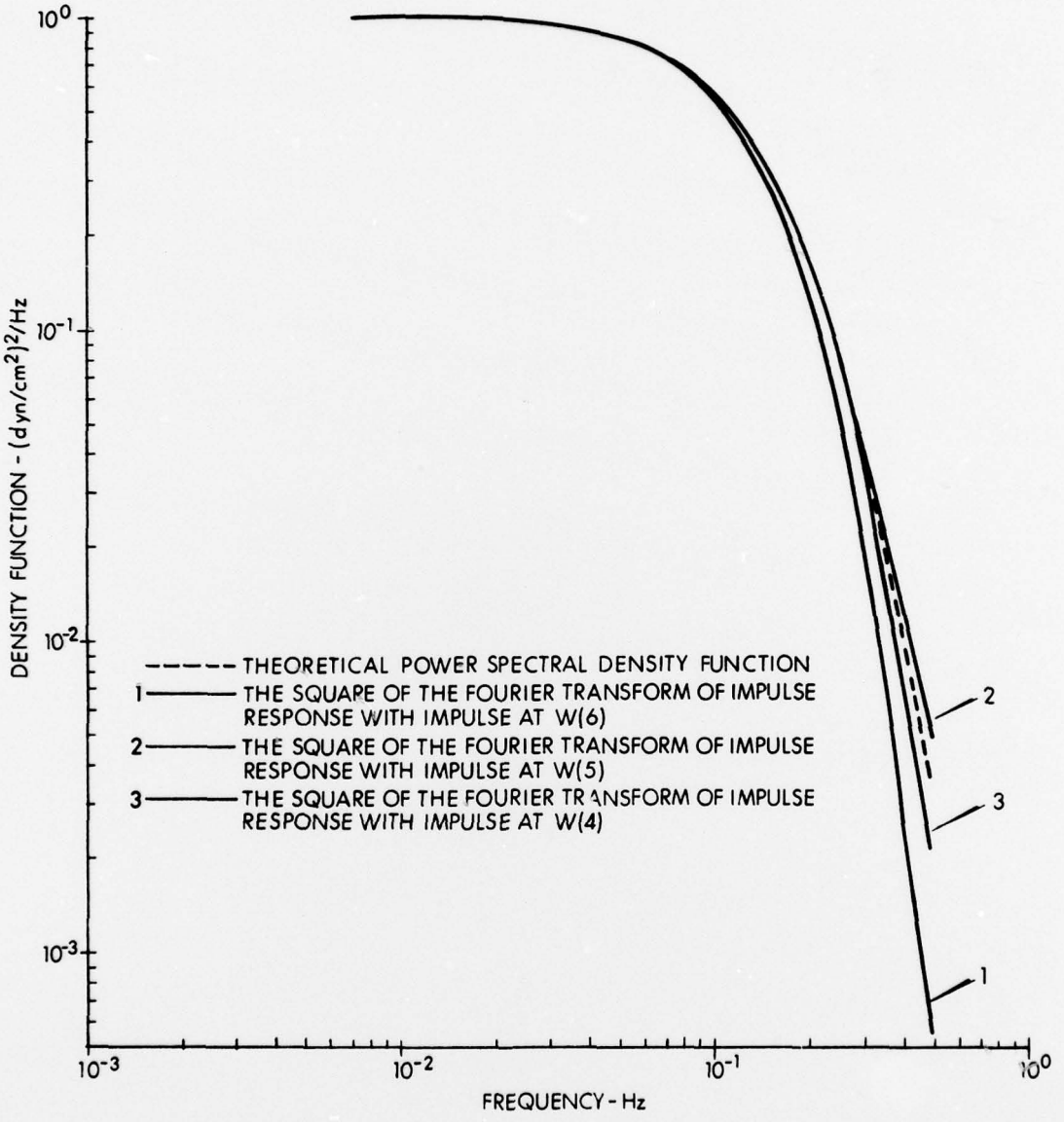

FIGURE 2
EXAMPLE 2

$$V_2(\omega) = \frac{\omega^4 - 70\omega^2 + 1369}{\omega^6 + 14\omega^4 + 49\omega^2 + 36}$$

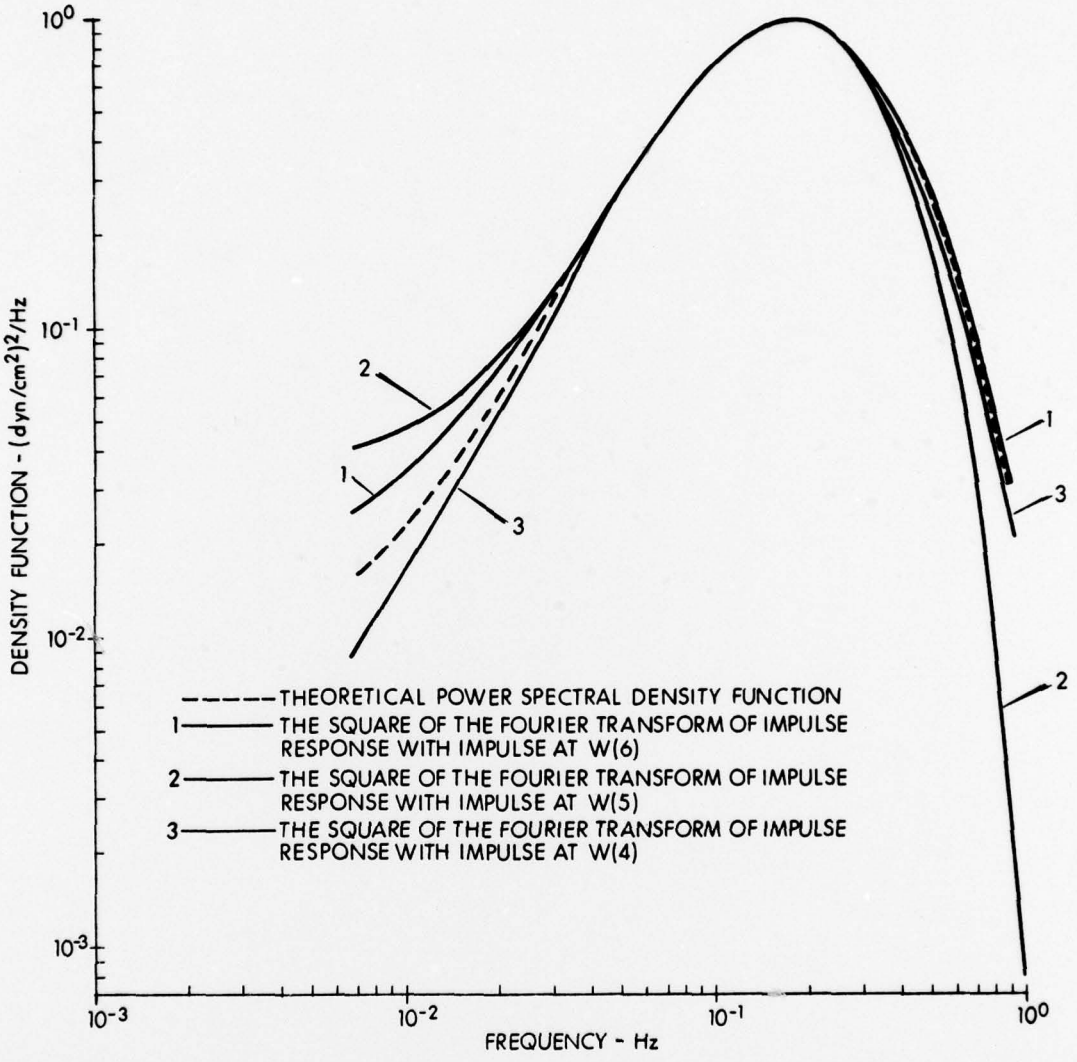

FIGURE 3
EXAMPLE 3

$$V_3(\omega) = \frac{400\omega^2 + 1}{\omega^6 + 14\omega^4 + 49\omega^2 + 36}$$

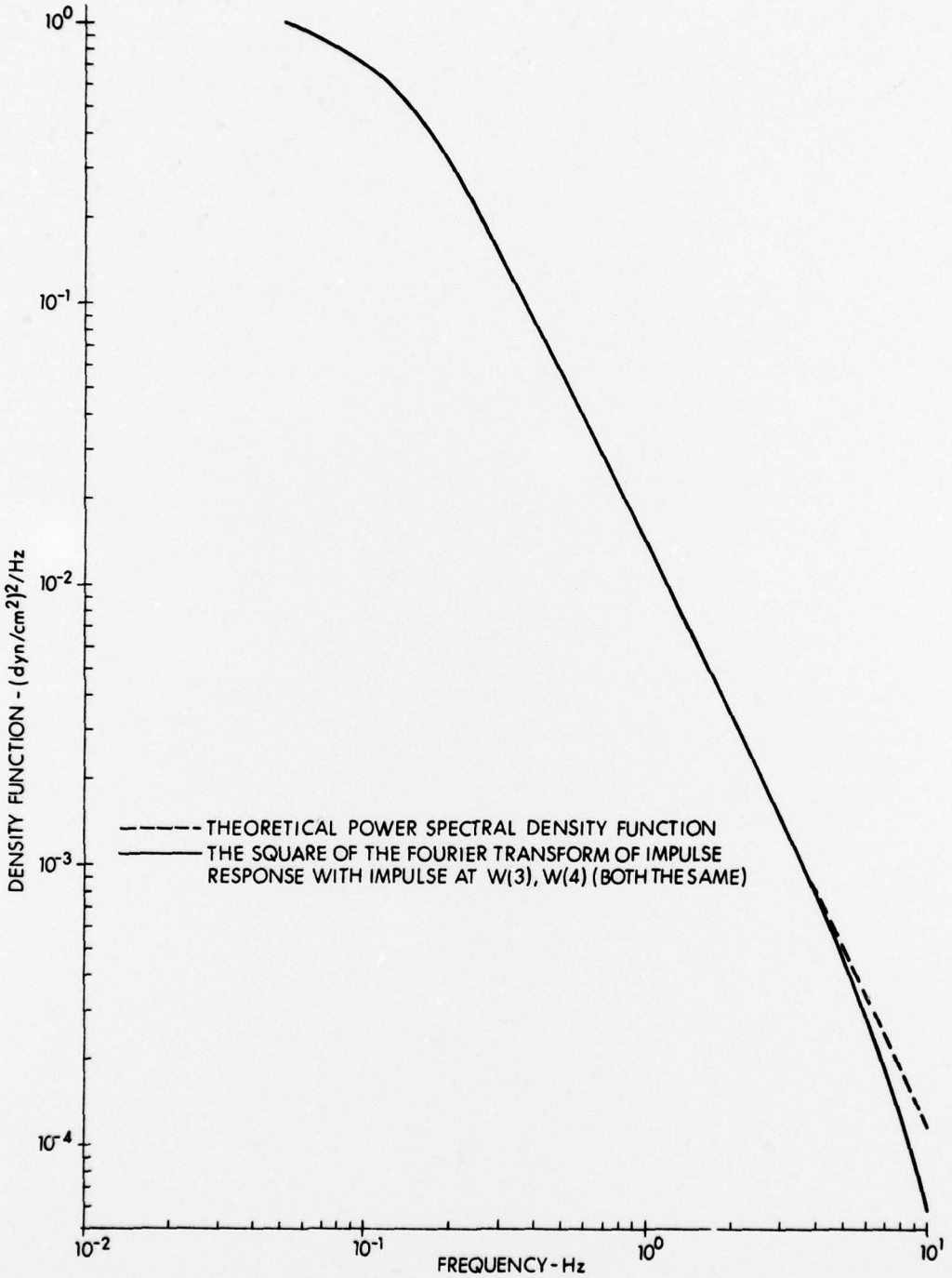

FIGURE 4
EXAMPLE 4

$$V_4(\omega) = 4Ak \frac{\omega^2 + (k^2 + c^2)}{\omega^4 + 2(k^2 - c^2)\omega^2 + (k^2 + c^2)^2}$$

APPENDIX

The white noise sequence used in solving the differential equations of Franklin's method was generated by taking the fractional part of

$$\theta^n \qquad n = 1, 2, 3, \cdots ,$$

where θ is a transcendental number greater than one. The fractional part of a number Φ is represented $\{\Phi\}$, for example $\{3.14\} = 0.14$. Franklin has shown that a sequence generated in this manner is completely equidistributed for almost all $\theta > 1$.[22]

Because of the finite word length of a digital computer the "transcendental" number is not actually transcendental. Statistical tests were made on various white sequences, and in all cases they were found to be equidistributed with a high confidence percentage.

A question of interest then arises. How much correlation is there between sequences generated by two close transcendental numbers? To answer this question a program was written to obtain the cross-correlation function between $\{\theta^n\}$ and $\{(\theta+\epsilon)^n\}$, for $n = 1, 2, 3, \cdots, N$ and for ϵ arbitrarily small, yet still within the machine capability. The program called CORRELAT computes the cross-correlation coefficients for ±10% of N lags, where N is the total number of points in the white sequence.

No theorem was proved concerning this cross-correlation analysis. Indeed no theorem could be proved because of the fact that $\{\theta+\epsilon\}$ might well be zero for the machine capacity.

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[22] Joel N. Franklin, "Deterministic Simulation of Random Processes," Math. Comp. 17, (1963), pp. 47 - 50.

An example run, with the cross-correlation function shown in Fig. 5, was chosen as follows:

$$\theta = 2.71828183 \cong e \quad ,$$

$$\epsilon = 1.0 \times 10^{-8} \quad ,$$

$$N = 4266 \quad .$$

As shown in Fig. 5, there was no significant correlation for any of the lags considered.

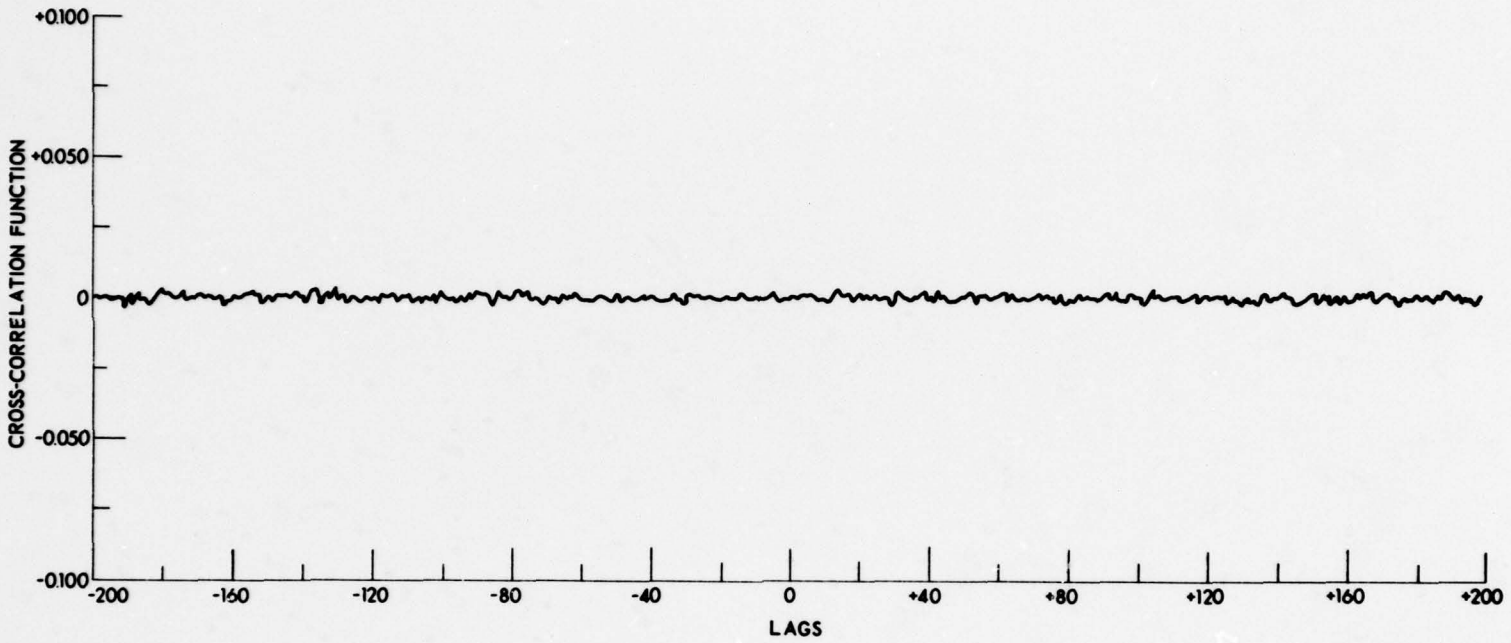

FIGURE 5
CROSS CORRELATION OF $\{e^n\}$ AND $\{(e+10^{-8})^n\}$

Input for CORRELAT

DATA CARDS

IEOF	I2 (Format)
ML, INC	2I5

INPUT TAPES FORMAT

IBUF

X(1) . . . X(IBUF)

IBUF

X(1) . . . X(IBUF)

.

.

.

IDEC

X(1) . . . X(IDEC)

END-OF-FILE

```
      PROGRAM CORRELAT
      DIMENSION X(4270), Y(4270),AM(402), C(402), IX(4), IY(4), M(400)
      IDEC = 0
      ITEST = 0
      CORRJ = 0.0
      K = 0
      ISTOP = 0
      JSTOP = 0
      XAVE = 0.0
      YAVE = 0.0
      DO 260 J = 5,6
      NEOF = 0
      READ 10, IEOF
      IF (IEOF .EQ. 0) 260, 80
   80 IIEOF = 2*IEOF
   90 ITEST = 0
      BUFFER IN (J,1) (IBUF,IBUF)
  100 GO TO (100,140,130,110), UNITSTF(J)
  110 PRINT 12
      GO TO 9999
  130 NEOF = NEOF + 1
      IF (NEOF .EQ. IIEOF) 260, 90
  140 IBUF = IBUF
      IF (IBUF .LT. 500) 141, 142
  141 ITEST = 1
  142 BUFFER IN (J,1) (X(1),X(IBUF))
  150 GO TO (150,160,160,110), UNITSTF(J)
  160 IF (ITEST .EQ. 0) 90, 170
  170 BUFFER IN (J,1) (IDEC,IDEC)
  180 GO TO (180,200,190,110), UNITSTF(J)
  190 NEOF = NEOF + 1
      GO TO 170
  200 BUFFER IN (J,1) (L,L)
  210 GO TO (210, 230, 230, 110), UNITSTF(J)
  230 BUFFER IN (J,1) (M(1),M(400))
  232 GO TO (232,90,90,110), UNITSTF(J)
  260 CONTINUE
  265 BUFFER IN (6,1) (I,I)
  270 GO TO (270, 280, 280, 110), UNITSTF(6)
  280 IF (I .EQ. 500) 300, 290
  290 ISTOP = 1
  300 K1 = 500*K + 1
      I = I
      KI = 500*K + 1
      BUFFER IN (6,1) (X(K1),X(KI))
  310 GO TO (310, 320, 320,110), UNITSTF(6)
  320 BUFFER IN (5,1) (J,J)
  330 GO TO (330, 340, 340, 110), UNITSTF(5)
  340 IF (J .EQ. 500) 360, 350
  350 JSTOP = 1
  360 KK1 = 500*K + 1
      J = J
      KKI = 500*K + J
      BUFFER IN (5,1) (Y(KK1),Y(KKI))
  370 GO TO (370,380,380,110), UNITSTF(5)
```

```
380 K = K + 1
    IF (ISTOP .EQ. 0  .AND. JSTOP .EQ. 0) 265,390
390 IF (I-J) 400, 400, 410
400 N = I + (K-1)*500
    GO TO 420
410 N = J + (K-1)*500
420 DO 430 K = 1,N
    XAVE = XAVE + X(K)
    YAVE = YAVE + Y(K)
430 CONTINUE
    XAVE = XAVE/N
    YAVE = YAVE/N
    PRINT 17, XAVE,YAVE
    DO 440 K = 1,N
    X(K) = X(K) - XAVE
    Y(K) = Y(K) - YAVE
440 CONTINUE
    READ 11, ML, INC
    MM = 2*IABS(ML)/INC  + 1
    DO 490 L = 1,MM
    ML = ML + INC
    IF (ML) 450, 450, 460
450 NN = N + ML
    K = 1
    GO TO 470
460 NN = N
    K = ML + 1
470 DO 480 I = K,NN
    J = I - ML
    CORRJ = CORRJ + X(I)*Y(J)
480 CONTINUE
    C(L) = CORRJ/(N-IABS(ML))
    CORRJ = 0.0
490 CONTINUE
    AM(1) = -200.0
    DO 500 I = 2,MM
500 AM(I) = AM(I-1) + 1.0
    PRINT 13, MM
    PRINT 14, (C(J),J = 1,MM)
    IX(1) = 4HLAG
    IY(1) = 4HC
    CALL PLOT (3.0,-12.0,-3)
    CALL PLOT (0.0,0.5,-3)
    MM1 = MM + 1
    MM2 = MM + 2
    C(MM1) = -0.100
    C(MM2) = 0.025
    AM(MM1) = -200.0
    AM(MM2) = 20.0
    CALL AXIS (0.0,0.0,IX,-4,20.0,0.0,-200.0,20.0,0.1,1.0,1)
    CALL AXIS (0.0,0.0,IY,4,8.0,90.0,-0.10,0.025,0.1,1.0,1)
    CALL LINE (AM,C,MM,1,0,0,0.08,1,0.0)
 10 FORMAT (I2)
 11 FORMAT (16I5)
 12 FORMAT (1H1,21H PARITY ERROR ON TAPE)
 13 FORMAT (35X16H CORRELATION FOR,I4,5H LAGS//)
 14 FORMAT (6(4XF12.8))
 15 FORMAT (2(4XE17.10))
```

```
   16 FORMAT (1H1)
   17 FORMAT (13H X AVERAGE = ,E17.10,13H Y AVERAGE = ,E17.10)
 9999 PRINT 16
      END
```

Glossary of Terms in CORRELAT

C(I)	Cross-correlation coefficient for lag I
CORRJ	Temporary storage for correlation coefficients
IBUF	Number of points in each block of input tape
IDEC	Number of points in last block of input tape
IEOF	Number of sequences on tape prior to sequence of interest
INC	Number of points skipped between each lag
ITEST	Parameter to determine location of data on input tape
ISTOP	Parameter to determine whether all of the X(I) sequence has been buffered from tape
IX(I)	Abscissa label for plots
IY(I)	Ordinate label for plots
JSTOP	Parameter to determine whether all of the Y(I) sequence has been buffered from tape
ML	Number of lags
NEOF	Number of end-of-file's on tape prior to sequence of interest
X(I)	Input sequence
XAVE	Average of X(I) for lag I
Y(I)	Input sequence
YAVE	Average of Y(I) for lag I

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